TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 12, December 2018, Pages 8739–8756
https://doi.org/10.1090/tran/7395
Article electronically published on September 13, 2018

ON BODIES WITH CONGRUENT SECTIONS BY CONES OR NON-CENTRAL PLANES

N. ZHANG

Abstract. Let K and L be two convex bodies in $\mathbb{R}^3$, such that their sections by cones $\{x \in \mathbb{R}^3 : x \cdot \xi = t|x|\}$ or non-central planes with a fixed distance from the origin are directly congruent. We prove that if their boundaries are of class C^2, then K and L coincide.

1. Introduction and main results

This paper is motivatied by the following problem (see, for example, the book of R. J. Gardner "Geometric tomography" [2, Page 289]).

Problem 1.1. Suppose that $2 \le k \le n-1$ and that K and L are star bodies in $\mathbb{R}^n$ such that the section $K \cap H$ is congruent to $L \cap H$ (see Figure 1) for all $H \in G(n,k)$. Do K and L coincide up to a reflection only?

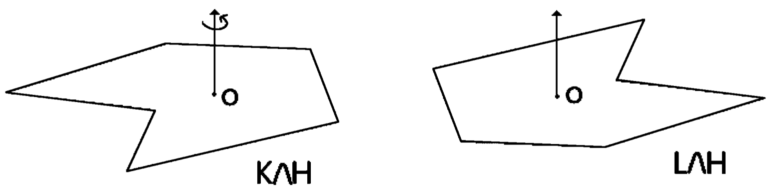

Figure 1. Congruent sections by central planes.

Here, $K \cap H$ being congruent to $L \cap H$ means that there exists an orthogonal transformation φ in H such that $\varphi(K \cap H)$ is a translate of $L \cap H$. The answer is affirmative in the case when $K \cap H$ is a translate of $L \cap H$ for every H (see Gardner [2, Theorem 7.1.1] and Ryabogin [8]). If $K \cap H$ is a rotation of $L \cap H$ for each H and $k = 2$, Ryabogin [7] gave an affirmative answer. For the higher dimension, some partial results were obtained by Alfonseca, Cordier, and Ryabogin in [1] and Myroshnychenko and Ryabogin in [6]. Several other results can be found in the book of Golubyatnikov [3]. In general, this problem is still open. Below we study two versions of this problem.

Problem 1.2. Let $K, L \subset \mathbb{R}^n$ be star bodies and $t \in (0,1)$. Assume that for every $\xi \in S^{n-1}$ there is a rigid motion ϕ_ξ such that $K \cap C_t(\xi) = \phi_\xi(L \cap C_t(\xi))$ (see Figure 2). Does it follow that $K = L$?

Received by the editors May 8, 2017, and, in revised form, July 22, 2017.
2010 *Mathematics Subject Classification.* Primary 52A20, 52A38.
Key words and phrases. Convex body, congruent sections, unique determination.

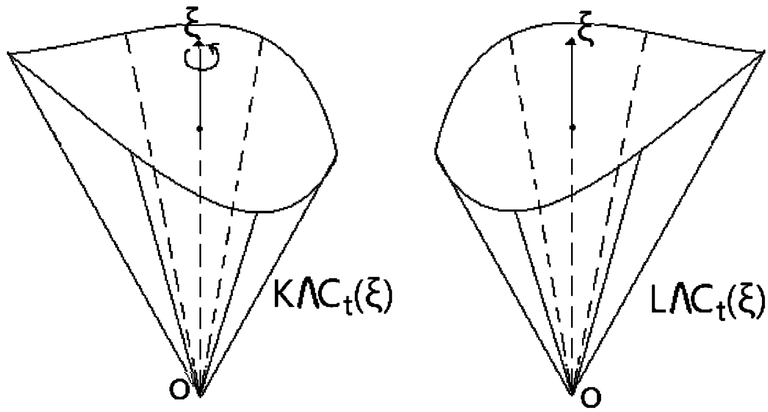

FIGURE 2. Congruent sections by cones.

Here, for $t \in (0,1)$, we define

$$C_t(\xi) := \{x \in \mathbb{R}^n : \langle x, \xi \rangle = t|x|\}$$

to be a cone in the direction of ξ. For some special values of t, Problem 1.2 has an affirmative answer (cf. Schneider [10]; see also Sacco [9] for details).

Problem 1.3. Let $K, L \subset \mathbb{R}^n$ be convex bodies containing a ball B in their interiors. Assume that for every $\xi \in S^{n-1}$ there is a rigid motion ϕ_ξ such that $K \cap (\xi^\perp + t\xi) = \phi_\xi(L \cap (\xi^\perp + t\xi))$ (see Figure 3). Does it follow that $K = L$?

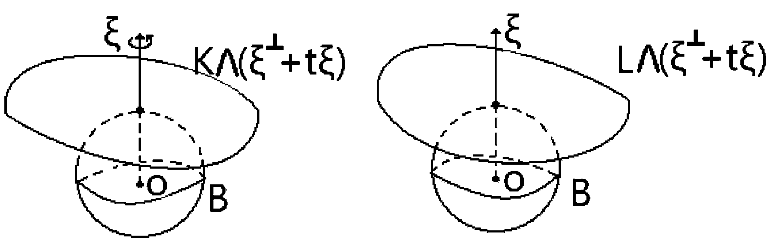

FIGURE 3. Congruent sections by non-central planes.

Here and below, B is the ball with centre at the origin of radius t, $t > 0$.

In this paper, we solve Problem 1.2 in $\mathbb{R}^3$ in the class of C^2 star bodies, i.e., star bodies with C^2 boundaries.

Theorem 1.4. *Let $f, g \in C^2(S^2)$ and $t \in (0,1)$. Assume that for every $\xi \in S^2$ there is a rotation ϕ_ξ around ξ such that*

$$f(\phi_\xi(\theta)) = g(\theta)$$

for all $\theta \in S^2 \cap (\xi^\perp + t\xi)$. Then $f = g$.

The case $t \in (0,1)$ is more difficult than the case $t = 0$, for, in general, there is no injectivity of the corresponding Spherical Radon transform. And the smoothness of the function is necessary when creating disjoint C^2 level sets of the function.

As a corollary of Theorem 1.4, we get a positive answer to a version of Problem 1.2.

Corollary 1.5. *Let $K, L \subset \mathbb{R}^3$ be C^2 star bodies and $t \in (0,1)$. Assume that for every $\xi \in S^{n-1}$ there is a rotation ϕ_ξ around ξ such that $K \cap C_t(\xi) = \phi_\xi(L \cap C_t(\xi))$. Then $K = L$.*

We also solve a version of Problem 1.3 in $\mathbb{R}^3$.

Theorem 1.6. *Let $K, L \subset \mathbb{R}^3$ be C^2 convex bodies containing a ball B in their interiors. Assume that for every $\xi \in S^2$ there is a rotation ϕ_ξ around ξ such that $K \cap (\xi^\perp + t\xi) = \phi_\xi(L \cap (\xi^\perp + t\xi))$. Then $K = L$.*

2. Proofs of main results

For a unit vector $\xi \in S^2$, we define an open ball on S^2 with centre at ξ to be

$$B_\epsilon(\xi) := \{\theta \in S^2 : \|\theta - \xi\| < \epsilon\},$$

where $\|\cdot\|$ is the Euclidean distance. We also define $\phi_\xi = \phi_{\xi,\alpha} \in \mathrm{SO}(3)$ to be the rotation around ξ by an angle α in the counterclockwise direction. Namely, for any $\theta \in S^2$,

$$\phi_{\xi,\alpha}(\theta) = \theta\cos(\alpha\pi) + (\xi \times \theta)\sin(\alpha\pi) + \xi\langle\xi,\theta\rangle(1 - \cos(\alpha\pi)),$$

where $\xi \times \theta$, $\langle\xi,\theta\rangle$ are usual vector and scalar products in $\mathbb{R}^3$.

2.1. **Congruent sections by cones. Auxiliary results.** In order to prove the theorem, we assume the opposite, that is, by Lemma 2.5, there exists an $x \in S^2$ such that $f(x) \neq g(x)$ and $\nabla_{S^2} f(x) \neq 0$, which gives a neighbourhood of x on S^2 containing local level sets of f. By the C^2 smoothness of f and the implicit function theorem, those level sets are a collection of disjoint C^2 curves. Then, the $\sqrt{2-t}$-distance is parallel to the sets of those C^2 curves forming a subset of S^2 with non-empty interior, whose intersection with Ξ_{con} is not empty. By Lemmas 2.4 and 2.6, the level set $\Theta_\tau \cup \Lambda(\theta)$ of f is not C^2 curve for some τ; a contradiction. (See Figure 4.)

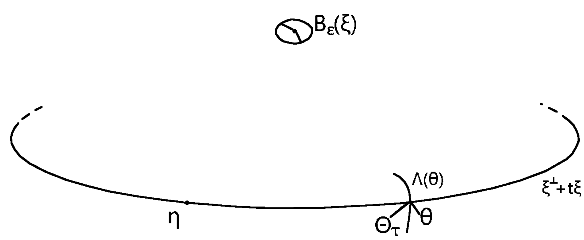

FIGURE 4. The level set $\Theta_\tau \cup \Lambda(\theta)$ is not C^2 curve.

The details of the proof are organized as follows.

Definition 2.1. Let f, g, t be as in Theorem 1.4. Define the following three subsets of S^2:

$$\Xi_0 = \{\xi \in S^2 : f(\theta) = g(\theta) \ \ \forall \theta \in S^2 \cap (\xi^\perp + t\xi)\};$$
$$\Xi_n = \{\xi \in S^2 : f(\phi_{\xi,\frac{2}{n}}(\theta)) = f(\theta) \ \ \forall \theta \in S^2 \cap (\xi^\perp + t\xi)\}, \quad n = 2, 3, \ldots;$$
$$\Xi_{\mathrm{con}} = S^2 \backslash (\Xi_0 \cup (\bigcup_{n=2}^{\infty} \Xi_n)).$$

Lemma 2.2. *Ξ_n are closed for all $n = 0, 2, 3, \ldots$.*

Proof. First, for any pair of directions ξ_1 and ξ_2, we define the map ψ_{ξ_1,ξ_2}.

If ξ_1 is not parallel to ξ_2, then consider the great circle passing through ξ_1, ξ_2, which intersects $\xi_1^\perp + t\xi_1$ and $\xi_2^\perp + t\xi_2$ at θ_{11}, θ_{12} and θ_{21}, θ_{22}, respectively. The directions θ_{11}, θ_{12} are chosen in such a way that the triple $\theta_{11}, \theta_{12}, \xi_1 \times \xi_2$ has a positive orientation. The same is assumed to hold for θ_{21}, θ_{22}. For any point $\theta \in S^2 \cap (\xi_1^\perp + t\xi_1)$, there exists $\phi_{\xi_1,\alpha} \in \mathrm{SO}(3)$, such that $\theta = \phi_{\xi_1,\alpha}(\theta_{11})$. We define $\psi_{\xi_1,\xi_2}(\theta) := \phi_{\xi_2,\alpha}(\theta_{21})$.

If ξ_1 is parallel to ξ_2, we define $\psi_{\xi_1,\xi_2}(\theta) = \theta$. Note that for any $\theta \in S^2 \cap (\xi_1^\perp + t\xi_1)$, since $\xi_2 \times \theta_{21} = \xi_1 \times \theta_{11}$ and $\langle \xi_2, \theta_{21} \rangle = \langle \xi_1, \theta_{11} \rangle = t$, we have

$$\begin{aligned}
&\|\psi_{\xi_1,\xi_2}(\theta) - \theta\| = \|\phi_{\xi_2,\alpha}(\theta_{21}) - \phi_{\xi_1,\alpha}(\theta_{11})\| \\
&= \|\theta_{21}\cos(\alpha\pi) + (\xi_2 \times \theta_{21})\sin(\alpha\pi) + \xi_2\langle \xi_2, \theta_{21}\rangle(1 - \cos(\alpha\pi)) \\
&\quad - \theta_{11}\cos(\alpha\pi) - (\xi_1 \times \theta_{11})\sin(\alpha\pi) - \xi_1\langle \xi_1, \theta_{11}\rangle(1 - \cos(\alpha\pi))\| \\
&= \|\theta_{21}\cos(\alpha\pi) + t\xi_2(1 - \cos(\alpha\pi)) - \theta_{11}\cos(\alpha\pi) - t\xi_1(1 - \cos(\alpha\pi))\| \\
&\le \|\theta_{21}\cos(\alpha\pi) - \theta_{11}\cos(\alpha\pi)\| + \|t\xi_2(1 - \cos(\alpha\pi)) - t\xi_1(1 - \cos(\alpha\pi))\| \\
&\le \|\theta_{21} - \theta_{11}\| + 2\|\xi_1 - \xi_2\| \\
&= 3\|\xi_1 - \xi_2\|
\end{aligned}$$

and

$$\phi_{\xi_2,\beta}(\psi_{\xi_1,\xi_2}(\theta)) = \psi_{\xi_1,\xi_2}(\phi_{\xi_1,\beta}(\theta)) \quad \text{for any } \beta.$$

Given a sequence $\xi_i \in \Xi_0$ with $\lim_{i\to\infty} \xi_i = \xi$, define ψ_{ξ,ξ_i} as similar to the above. For any $\theta \in S^2 \cap (\xi^\perp + t\xi)$, we have

$$\begin{aligned}
&|f(\theta) - g(\theta)| \\
&\le |f(\theta) - f(\psi_{\xi,\xi_i}(\theta))| + |f(\psi_{\xi,\xi_i}(\theta)) - g(\psi_{\xi,\xi_i}(\theta))| + |g(\psi_{\xi,\xi_i}(\theta)) - g(\theta)| \\
&= |f(\theta) - f(\psi_{\xi,\xi_i}(\theta))| + |g(\psi_{\xi,\xi_i}(\theta)) - g(\theta)|.
\end{aligned}$$

As $\xi_i \to \xi$, $\psi_{\xi,\xi_i}(\theta) \to \theta$; hence, by continuity of f and g,

$$|f(\theta) - g(\theta)| = 0 \quad \forall \theta \in S^2 \cap (\xi^\perp + t\xi),$$

which implies $\xi \in \Xi_0$. Now we prove the closeness of Ξ_0.

Similarly, given a sequence $\xi_i \in \Xi_n$ with $\lim_{i\to\infty} \xi_i = \xi$, for any $\theta \in S^2 \cap (\xi^\perp + t\xi)$, we have

$$\begin{aligned}
&|f(\phi_{\xi,\frac{2}{n}}(\theta)) - f(\theta)| \\
&\le |f(\phi_{\xi,\frac{2}{n}}(\theta)) - f(\phi_{\xi_i,\frac{2}{n}}(\psi_{\xi,\xi_i}(\theta)))| + |f(\phi_{\xi_i,\frac{2}{n}}(\psi_{\xi,\xi_i}(\theta))) - f(\psi_{\xi,\xi_i}(\theta))| \\
&\quad + |f(\psi_{\xi,\xi_i}(\theta)) - f(\theta)| \\
&= |f(\phi_{\xi,\frac{2}{n}}(\theta)) - f(\psi_{\xi,\xi_i}(\phi_{\xi,\frac{2}{n}}(\theta)))| + |f(\psi_{\xi,\xi_i}(\theta)) - f(\theta)|.
\end{aligned}$$

As $\xi_i \to \xi$, $\psi_{\xi,\xi_i}(\theta) \to \theta$; hence, by continuity of f,

$$|f(\phi_{\xi,\frac{2}{n}}(\theta)) - f(\theta)| = 0 \quad \forall\theta \in S^2 \cap (\xi^\perp + t\xi),$$

which implies $\xi \in \Xi_n$. □

Lemma 2.3. *Suppose that for some $\xi \in S^2$ there exists $\alpha \in \mathbb{Q}$ such that $f(\phi_{\xi,\alpha}(\theta)) = f(\theta)$ $\forall\theta \in S^2 \cap (\xi^\perp + t\xi)$. Then, there exists $n \geq 2$, such that $\xi \in \Xi_n$.*

Proof. Let us write $\alpha = \frac{p}{q}$, where p and q are coprime integers. It is sufficient to show $\frac{2}{n} = m\frac{p}{q} + 2l$ for some $m, n, l \in \mathbb{Z}$. Indeed, this would imply that

$$f(\phi_{\xi,\frac{2}{n}}(\theta)) = f(\phi_{\xi,m\frac{p}{q}+2l}(\theta)) = f(\phi_{\xi,m\frac{p}{q}}(\theta)) = f(\theta).$$

But, since p, q are coprime, there exist $k, r \in \mathbb{Z}$, such that $pk + qr = 1$. If we set $n = q$, then

$$\frac{2}{n} = \frac{2(pk + qr)}{q} = 2k\frac{p}{q} + 2r.$$ □

Now, we define

$$\lambda(\xi) := \{\alpha \in [0, 2) : f(\phi_{\xi,\alpha}(\theta)) = g(\theta) \;\; \forall\theta \in S^2 \cap (\xi^\perp + t\xi)\}.$$

In the case when $\xi \in \Xi_n$, $n \geq 2$, $\lambda(\xi)$ is a multi-valued function; on the other hand, if $\xi \in \Xi_{\text{con}}$, $\lambda(\xi)$ is a single-valued function; otherwise, if $\alpha, \beta \in \lambda(\xi)$ with $\alpha \neq \beta$,

$$f(\phi_{\xi,\alpha}(\theta)) = g(\theta) = f(\phi_{\xi,\beta}(\theta)) \;\; \forall\theta \in S^2 \cap (\xi^\perp + t\xi),$$

implying

$$f(\phi_{\xi,\alpha-\beta}(\theta)) = f(\theta) \;\; \forall\theta \in S^2 \cap (\xi^\perp + t\xi).$$

If $\alpha - \beta$ is irrational, then $f(\theta) \equiv C \equiv g(\theta)$ $\forall\theta \in S^2 \cap (\xi^\perp + t\xi)$, which means $\xi \in \Xi_0$; a contradiction. If $\alpha - \beta$ is rational, then by Lemma 2.3, $\xi \in \Xi_n$; a contradiction.

Lemma 2.4. *Let f, g, t be as in Theorem* 1.4. *Then Ξ_{con} is open and $\lambda(\xi)$ is a continuous function on Ξ_{con} if $\Xi_{\text{con}} \neq \emptyset$.*

Proof. If $\Xi_{\text{con}} = \emptyset$, then Ξ_{con} is open. Now assume that Ξ_{con} is not open. There exists $\xi \in \Xi_{\text{con}}$, such that, for any $i \in \mathbb{N}$, there exists $\xi_i \in B_{\frac{1}{i}}(\xi) \cap \Xi_{n_i}$ for some n_i. If there are infinitely many ξ_i that belong to Ξ_0, then $0 \in \lambda(\xi)$, that is, $\xi \in \Xi_0$; a contradiction. If there are infinitely many ξ_i, for which $n_i \neq 0$, then $\lambda(\xi_i)$ is a multi-valued function. Thus there exists $\alpha_i \in \lambda(\xi_i)$, such that $|\alpha_i - \lambda(\xi)| > \varepsilon$ for some $\varepsilon > 0$. By compactness of $[0, 2]$, there exists a subsequence ξ_{i_k}, such that $\lim_{k\to\infty} \alpha_{i_k} = \alpha$, where $|\alpha - \lambda(\xi)| \geq \varepsilon$. Set $\psi_{\xi,\xi_{i_k}}$ similarly to the ones defined in Lemma 2.2.

Then for any $\theta \in S^2 \cap (\xi^\perp + t\xi)$,

$$\begin{aligned}
&|f(\phi_{\xi,\lambda(\xi)}(\theta)) - f(\phi_{\xi,\alpha}(\theta))| \\
&\leq |f(\phi_{\xi,\lambda(\xi)}(\theta)) - g(\theta)| + |g(\theta) - g(\psi_{\xi,\xi_{i_k}}(\theta))| \\
&\quad + |g(\psi_{\xi,\xi_{i_k}}(\theta)) - f(\phi_{\xi_{i_k},\alpha_{i_k}}(\psi_{\xi,\xi_{i_k}}(\theta)))| + |f(\phi_{\xi_{i_k},\alpha_{i_k}}(\psi_{\xi,\xi_{i_k}}(\theta))) - f(\phi_{\xi,\alpha_{i_k}}(\theta))| \\
&= |g(\theta) - g(\psi_{\xi,\xi_{i_k}}(\theta))| + |f(\psi_{\xi,\xi_{i_k}}(\phi_{\xi,\alpha_{i_k}}(\theta))) - f(\phi_{\xi,\alpha}(\theta))|.
\end{aligned}$$

As $k \to \infty$, we have $\psi_{\xi,\xi_{i_k}}(\theta) \to \theta$ and $\psi_{\xi,\xi_{i_k}}(\phi_{\xi,\alpha_{i_k}}(\theta)) \to \phi_{\xi,\alpha}(\theta)$; hence, by continuity of f and g,

$$|f(\phi_{\xi,\lambda(\xi)}(\theta)) - f(\phi_{\xi,\alpha}(\theta))| = 0 \;\; \forall\theta \in S^2 \cap (\xi^\perp + t\xi),$$

implying

$$|f(\phi_{\xi,\alpha}(\theta)) - g(\theta)| = 0 \ \ \forall\theta \in S^2 \cap (\xi^\perp + t\xi);$$

a contradiction.

For the continuity, since $\lambda(\xi)$ is a single-valued function when $\xi \in \Xi_{\rm con}$, consider a sequence $\{\xi_i\}_{i=1}^\infty \in \Xi_{\rm con}$, such that $\Xi_{\rm con} \ni \xi = \lim_{i\to\infty} \xi_i$. By compactness of $[0,2]$, there exists a subsequence $\{\xi_{i_k}\}_{k=1}^\infty$, such that $\alpha = \lim_{k\to\infty} \lambda(\xi_{i_k})$. Then for any $\theta \in S^2 \cap (\xi^\perp + t\xi)$,

$$\begin{aligned}
&|f(\phi_{\xi,\alpha}(\theta)) - g(\theta)| \\
&\le |f(\phi_{\xi,\alpha}(\theta)) - f(\phi_{\xi_{i_k},\lambda(\xi_{i_k})}(\psi_{\xi,\xi_{i_k}}(\theta)))| + |f(\phi_{\xi_{i_k},\lambda(\xi_{i_k})}(\psi_{\xi,\xi_{i_k}}(\theta))) - g(\psi_{\xi,\xi_{i_k}}(\theta))| \\
&quad + |g(\psi_{\xi,\xi_{i_k}}(\theta)) - g(\theta)| \\
&= |f(\phi_{\xi,\alpha}(\theta)) - f(\phi_{\xi_{i_k},\lambda(\xi_{i_k})}(\psi_{\xi,\xi_{i_k}}(\theta)))| + |g(\psi_{\xi,\xi_{i_k}}(\theta)) - g(\theta)|.
\end{aligned}$$

As $k \to \infty$, we have $\phi_{\xi_{i_k},\lambda(\xi_{i_k})}(\psi_{\xi,\xi_{i_k}}(\theta)) \to \phi_{\xi,\alpha}(\theta)$ and $\psi_{\xi,\xi_{i_k}}(\theta) \to \theta$; hence, by continuity of f and g,

$$|f(\phi_{\xi,\alpha}(\theta)) - g(\theta)| = 0 \ \ \forall\theta \in S^2 \cap (\xi^\perp + t\xi),$$

that is, $\lambda(\xi) = \alpha$. If $\{\lambda(\xi_i)\}_{i=1}^\infty$ has another subsequence with a different limit $\beta \neq \alpha$, then $\{\alpha, \beta\} \subset \lambda(\xi)$, contradicting the fact that $\xi \in \Xi_{\rm con}$. ☐

Lemma 2.5. *Let f, g, t be as in Theorem* 1.4. *Then either $\{\theta \in S^2 : f(\theta) = g(\theta)\} = S^2$ or the set*

$$\{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap [\{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\} \cup \{\theta \in S^2 : \nabla_{S^2} g(\theta) \neq 0\}]$$

is not empty.

Here, ∇_{S^2} is the spherical gradient, that is, for a function f on S^2,

$$(\nabla_{S^2} f)(x/|x|) = \nabla(f(x/|x|)), \quad x \in \mathbb{R}^3/\{0\},$$

where $f(x/|x|)$ is the 0-degree homogeneous extension of the function f to $\mathbb{R}^3/\{0\}$ and ∇ is the gradient in the ambient space $\mathbb{R}^3$.

Proof. First, the set $\{\theta \in S^2 : f(\theta) \neq g(\theta)\}$ is not the whole sphere; otherwise without loss of generality let $f(\theta) < g(\theta)$. Then

$$\begin{aligned}
\int_{S^2\cap(\xi^\perp+t\xi)} g(\theta)\, d\theta &= \int_{S^2\cap(\xi^\perp+t\xi)} f(\phi_\xi(\theta))\, d\theta \\
&= \int_{S^2\cap(\xi^\perp+t\xi)} f(\theta)\, d\theta < \int_{S^2\cap(\xi^\perp+t\xi)} g(\theta)\, d\theta.
\end{aligned}$$

Now assume

$$\{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap [\{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\} \cup \{\theta \in S^2 : \nabla_{S^2} g(\theta) \neq 0\}] = \emptyset.$$

Then

$$\{\theta \in S^2 : f(\theta) \neq g(\theta)\} \subset \{\theta \in S^2 : \nabla_{S^2} f(\theta) = 0\} \cap \{\theta \in S^2 : \nabla_{S^2} g(\theta) = 0\}.$$

Since $f, g \in C^2(S^2)$, the set

$$\Upsilon_0 := \{\theta \in S^2 : \nabla_{S^2} f(\theta) = 0\} \cap \{\theta \in S^2 : \nabla_{S^2} g(\theta) = 0\}$$

is closed and f and g are constant in any connected subset of Υ_0.

Assume there exists $x \in \{\theta \in S^2 : f(\theta) \neq g(\theta)\}$. Choose the largest connected open neighbourhood $\mathcal{N}_x$ of x in $\{\theta \in S^2 : f(\theta) \neq g(\theta)\}$ $\forall\theta \in S^2$. Then the closure of $\mathcal{N}_x$ is in Υ_0 and the boundary of $\mathcal{N}_x$ is a subset of $\{\theta \in S^2 : f(\theta) = g(\theta)\}$, which implies $C_1 = f = g = C_2$ in the closure of $\mathcal{N}_x$; a contradiction. Hence, $\{\theta \in S^2 : f(\theta) \neq g(\theta)\} = \emptyset$. □

Note that for $\xi \in \Xi_{\text{con}}$, since Ξ_{con} is open, there exists an $\epsilon > 0$, such that $B_\epsilon(\xi) \subset \Xi_{\text{con}}$. Then, for any point $\theta \in S^2 \cap (\xi^\perp + t\xi)$, we set $\eta = \phi_{\xi,-\lambda(\xi)}(\theta)$ and $\xi \in \eta^\perp + t\eta$ since $\langle \xi, \eta \rangle = t$; hence, $(\eta^\perp + t\eta) \cap B_\epsilon(\xi)$ is not empty. Thus, we define the curve

$$\Lambda(\theta) := \bigcup_{\zeta \in (\eta^\perp + t\eta) \cap B_\epsilon(\xi)} \phi_{\zeta,\lambda(\zeta)}(\eta)$$

passing through θ (see Figure 5).

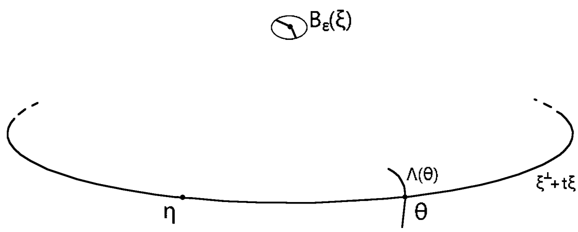

FIGURE 5. The construction of $\Lambda(\theta)$.

We set $(\xi^\perp + t\xi)_+ := \{x \in \mathbb{R}^3 : \langle x, \xi \rangle \geq t\}$ and $\operatorname{int}((\xi^\perp + t\xi)_+) := \{x \in \mathbb{R}^3 : \langle x, \xi \rangle > t\}$.

Lemma 2.6. *Let f, g, t be as in Theorem 1.4 and $\xi \in \Xi_{\text{con}}$. If $\lambda(\xi) \neq 1$, then*

$$\Lambda(\theta) \cap S^2 \cap \operatorname{int}((\xi^\perp + t\xi)_+) \neq \emptyset.$$

Proof. Without loss of generality, we can assume that $0 < \lambda(\xi) < 1$. The other case $1 < \lambda(\xi) < 2$ is similar. Since $\xi \in \Xi_{\text{con}}$ and $0 < \lambda(\xi) < 1$, by Lemma 2.4 there exists $0 < \iota < 1/2$ and a ball $B_\epsilon(\xi) \subset \Xi_{\text{con}}$ such that $\iota \leq \lambda(\zeta) \leq 1 - \iota$ for any $\zeta \in B_\epsilon(\xi)$.

Now take any $\theta \in S^2 \cap (\xi^\perp + t\xi)$ and define $\eta = \phi_{\xi,-\lambda(\xi)}(\theta)$. We set $\zeta = \phi_{\eta,\alpha}(\xi)$ for some small $\alpha > 0$ and $\omega = \phi_{\zeta,\lambda(\zeta)}(\eta)$. Then we have

$$\begin{aligned}
\zeta \times \eta &= \phi_{\eta,\alpha}(\xi) \times \eta \\
&= (\xi \cos(\alpha\pi) + (\eta \times \xi) \sin(\alpha\pi) + \eta\langle \eta, \xi \rangle (1 - \cos(\alpha\pi))) \times \eta \\
&= \xi \times \eta \cos(\alpha\pi) + (\xi \langle \eta, \eta \rangle - \eta \langle \xi, \eta \rangle) \sin(\alpha\pi) \\
&= \xi \times \eta \cos(\alpha\pi) + (\xi - t\eta) \sin(\alpha\pi)
\end{aligned}$$

and

$$\begin{aligned}
\langle \xi, \zeta \rangle &= \langle \xi, \phi_{\eta,\alpha}(\xi) \rangle \\
&= \langle \xi, \xi \cos(\alpha\pi) + (\eta \times \xi) \sin(\alpha\pi) + \eta \langle \eta, \xi \rangle (1 - \cos(\alpha\pi)) \rangle \\
&= \cos(\alpha\pi) + t^2 (1 - \cos(\alpha\pi)). \qquad (1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \xi, \omega\rangle - t &= \langle \xi, \phi_{\zeta,\lambda(\zeta)}(\eta)\rangle - t\\
&= \langle \xi, \eta\cos(\lambda(\zeta)\pi) + (\zeta\times\eta)\sin(\lambda(\zeta)\pi) + t\zeta(1-\cos(\lambda(\zeta)\pi))\rangle - t\\
&= t\cos(\lambda(\zeta)\pi) + \langle \xi, \xi\times\eta\cos(\alpha\pi) + (\xi - t\eta)\sin(\alpha\pi)\rangle \sin(\lambda(\zeta)\pi)\\
&\quad + t(1-\cos(\lambda(\zeta)\pi))(\cos(\alpha\pi) + t^2(1-\cos(\alpha\pi))) - t\\
&= t\cos(\lambda(\zeta)\pi) + (1-t^2)\sin(\alpha\pi)\sin(\lambda(\zeta)\pi)\\
&\quad + t(1-\cos(\lambda(\zeta)\pi))(\cos(\alpha\pi) + t^2(1-\cos(\alpha\pi))) - t\\
&= (1-t^2)\sin(\alpha\pi)\sin(\lambda(\zeta)\pi) + t(1-\cos(\lambda(\zeta)\pi))(t^2-1)(1-\cos(\alpha\pi))\\
&= (1-t^2)(\sin(\alpha\pi)\sin(\lambda(\zeta)\pi) - t(1-\cos(\lambda(\zeta)\pi))(1-\cos(\alpha\pi)))\\
&\geq (1-t^2)(\sin(\alpha\pi)\sin(\iota\pi) - t(1-\cos((1-\iota)\pi))(1-\cos(\alpha\pi)))\\
&> 0 \quad \text{for sufficiently small } \alpha.
\end{aligned}$$

To show

$$\sin(\alpha\pi)\sin(\iota\pi) - t(1-\cos((1-\iota)\pi))(1-\cos(\alpha\pi)) > 0$$

for sufficiently small $\alpha > 0$, we used that for $a, b > 0$, $x > 0$ sufficiently small, and $h(x) = a\sin x - b(1-\cos x)$,

$$h'(x) = a\cos x - b\sin x > 0$$

and $h(0) = 0$.

Hence, $\omega \in \Lambda(\theta) \cap S^2 \cap \operatorname{int}((\xi^{\perp} + t\xi)_+)$. □

Proof of Theorem 1.4. Assume the set $\{\theta \in S^2 : f(\theta) \neq g(\theta)\}$ is not empty. By Lemma 2.5, we have

$$\{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap [\{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\} \cup \{\theta \in S^2 : \nabla_{S^2} g(\theta) \neq 0\}] \neq \emptyset.$$

Without loss of generality, we can choose

$$x \in \{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap \{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\},$$

and therefore, there exists an open ball

$$B_\epsilon(x) \subset \{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap \{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\}.$$

By the implicit function theorem (see [5, Section I-5]), the collection of local level sets of f, $\mathcal{L}(f) := \{\Theta_\tau\}_{a<\tau<b}$, is a collection of disjoint C^2 curves, where $\Theta_\tau := \{\theta \in S^2 : f(\theta) = \tau\} \cap B_\epsilon(x)$. Here, $a = \inf_{\theta \in B_\epsilon(x)} f(\theta)$ and $b = \sup_{\theta\in B_\epsilon(x)} f(\theta)$.

For curves $\{\Theta_\tau\} \subset S^2$, consider their geodesic curvature $k_g(\cdot)$ (see [4, Section 17.4] for details). If for every $\eta \in \Theta_\tau$ and $\Theta_\tau \in \mathcal{L}(f)$, we have $k_g(\eta) = 0$, then each Θ_τ belongs to some great circle. Choose one of these great circles. It divides S^2 into two hemispheres. Fix one of these hemispheres and denote it by S^2_+. Consider all circles of the form $S^2 \cap (\xi^\perp + t\xi)$ that are tangent to the curves Θ_τ and $\xi \in S^2_+$. Denote by Σ the set of such directions ξ.

Now consider the case when for some $\tau \in (a, b)$, there exists a $\theta \in \Theta_\tau$, such that $k_g(\theta) \neq 0$. Then by C^2 smoothness of f, there exists a smaller neighbourhood of x, which we will again denote by $B_\epsilon(x)$, and a collection of level sets in $B_\epsilon(x)$, such that $k_g(\eta) \neq 0$ for any $\eta \in \Theta_\tau$ and $a < \tau < b$. For each point $\eta \in \Theta_\tau$, consider the great circle which is tangent to Θ_τ at η. Then $\{\Theta_\tau\}_{a<\tau<b}$ lie on one side of their tangent great circle. For each τ and each $\eta \in \Theta_\tau$ consider a circle $S^2 \cap (\xi^\perp + t\xi)$

that is tangent to Θ_τ at η and lies on the other side with respect to the tangent great circle. Let Σ be the set of such directions ξ (see Figure 6).

Note that for each Θ_τ, these $\xi \in \Sigma$ form a parallel set of Θ_τ on S^2, i.e., the envelope of a family of circles on S^2 with centres on Θ_τ and of radius $\sqrt{2-2t}$.

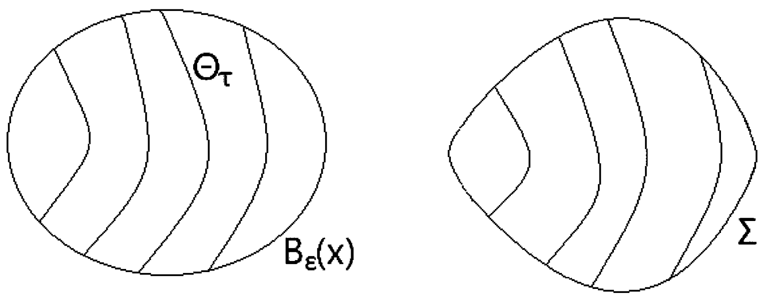

FIGURE 6. The construction of Σ.

Hence, the set Σ is a union of such curves and thus contains non-empty interior. We claim $\Xi_{\text{con}} \cap \operatorname{int}(\Sigma) \neq \emptyset$; otherwise, $\operatorname{int}(\Sigma) \subset \Xi_0 \cup (\bigcup_{n=2}^\infty \Xi_n)$, but $\operatorname{int}(\Sigma) \cap \Xi_0 = \emptyset$, since $B_\epsilon(x) \subset \{\theta \in S^2 : f(\theta) \neq g(\theta)\}$. Hence, $\operatorname{int}(\Sigma) \subset \bigcup_{n=2}^\infty \Xi_n$, which implies that $(\bigcup_{n=2}^\infty \Xi_n) \cap \operatorname{int}(\Sigma)$ contains non-empty interior. By the Baire category theorem and Lemma 2.2, there exists some $k \in \mathbb{N}$, such that $\Xi_k \cap \operatorname{int}(\Sigma)$ contains non-empty interior.

Now assume $\xi \in \operatorname{int}(\Xi_k \cap \Sigma)$. Then, there exists $\delta > 0$ such that $B_\delta(\xi) \subset \operatorname{int}(\Xi_k \cap \Sigma)$. For any $\theta \in \xi^\perp + t\xi$, we have

$$f(\eta) = f(\theta) \quad \forall \eta \in \Lambda_\xi(\theta) := \bigcup_{\zeta \in (\theta^\perp + t\theta) \cap B_\delta(\xi)} \phi_{\zeta, \frac{2}{k}}(\theta)$$

and

$$f(\omega) = f(\theta) \quad \forall \omega \in \Delta_\xi(\theta) := \bigcup_{\eta \in \Lambda_\xi(\theta)} \bigcup_{\vartheta \in (\eta^\perp + t\eta) \cap B_\delta(\xi)} \phi_{\vartheta, -\frac{2}{k}}(\eta).$$

Let us show that $\Delta_\xi(\theta)$ has non-empty interior. Note that for any $\eta \in \Lambda_\xi(\theta)$, by equation (1) we have

$$\langle \theta, \eta \rangle = \cos(2\pi/k) + t^2(1 - \cos(2\pi/k)) =: \varsigma(t),$$

where $-1 < \varsigma(t) < 1$. If $\varsigma(t) = 0$, then $\Lambda_\xi(\theta) \subset S^2 \cap \theta^\perp$. Fix $\eta \in \Lambda_\xi(\theta)$; then for each

$$\omega \in \bigcup_{\vartheta \in (\eta^\perp + t\eta) \cap B_\delta(\xi)} \phi_{\vartheta, -\frac{2}{k}}(\eta),$$

by equation (1) we have

$$\langle \omega, \eta \rangle = \varsigma(t) = 0,$$

which means $\bigcup_{\vartheta \in (\eta^\perp + t\eta) \cap B_\delta(\xi)} \phi_{\vartheta, -\frac{2}{k}}(\eta)$ is a curve passing through θ and contained in $S^2 \cap \eta^\perp$. Since $\Lambda_\xi(\theta)$ is a continuous curve, by changing η we see that $\Delta_\xi(\theta)$ has the shape of a sand dial, which we will refer to as a $\bowtie$ shape.

If $0 < \varsigma(t) < 1$, then

$$\Lambda_\xi(\theta) \subset S^2 \cap (\theta^\perp + \varsigma(t)\theta).$$

Fix $\eta \in \Lambda_\xi(\theta)$; then $\bigcup_{\vartheta\in(\eta^\perp+t\eta)\cap B_\delta(\xi)} \phi_{\vartheta,-\frac{2}{k}}(\eta)$ gives a curve passing through θ and contained in $S^2\cap(\eta^\perp+\varsigma(t)\eta)$. Observe that for different $\eta\in\Lambda_\xi(\theta)$, we have different curves $\bigcup_{\vartheta\in(\eta^\perp+t\eta)\cap B_\delta(\xi)} \phi_{\vartheta,-\frac{2}{k}}(\eta)$ with the only common point θ. Since these curves change continuously, the set $\Delta_\xi(\theta)$ again has a $\bowtie$ shape.

If $-1<\varsigma(t)<0$, use the same argument to show that $\Delta_\xi(\theta)$ has a $\bowtie$ shape. Therefore, $\Delta_\xi(\theta)$ is a set with non-empty interior on S^2.

Now to reach a contradiction, assume that f is not constant on $S^2\cap(\xi^\perp+t\xi)$. Then f takes on infinitely many values and so there are infinitely many disjoint sets $\Delta_\xi(\theta)$ with $m(\Delta_\xi(\theta))=\nu>0$, where ν is a number independent of $\theta\in S^2\cap(\xi^\perp+t\xi)$, which is impossible. Here m is the Hausdorff measure on S^2. On the other hand, if f is a constant on $S^2\cap(\xi^\perp+t\xi)$, then $\xi\in\Xi_0$, which contradicts $\operatorname{int}(\Sigma)\cap\Xi_0=\emptyset$. Thus, we have proved $\Xi_{\text{con}}\cap\operatorname{int}(\Sigma)\neq\emptyset$.

Now assume that for every $\xi\in\Xi_{\text{con}}\cap\operatorname{int}(\Sigma)$, we have $\lambda(\xi)=1$. Then, there exists $\delta>0$ such that $B_\delta(\xi)\subset\Xi_{\text{con}}\cap\operatorname{int}(\Sigma)$ and $\lambda(\zeta)=1$ for any $\zeta\in B_\delta(\xi)$. For any $\theta\in\xi^\perp+t\xi$, we have

$$g(\eta)=f(\theta)\quad\forall\eta\in\Lambda_\xi(\theta):=\bigcup_{\zeta\in(\theta^\perp+t\theta)\cap B_\delta(\xi)}\phi_{\zeta,1}(\theta)$$

and

$$f(\omega)=g(\eta)=f(\theta)\quad\forall\omega\in\Delta_\xi(\theta):=\bigcup_{\eta\in\Lambda_\xi(\theta)}\ \bigcup_{\vartheta\in(\eta^\perp+t\eta)\cap B_\delta(\xi)}\phi_{\vartheta,1}(\eta).$$

Following the same argument as above, we have that $\Delta_\xi(\theta)$ is a set with non-empty interior on S^2. Therefore, f is a constant on $\xi^\perp+t\xi$; otherwise, if f takes on infinitely many values, then there are infinitely many disjoint sets $\Delta_\xi(\theta)$, where $m(\Delta_\xi(\theta))=\nu>0$; a contradiction. But if f is a constant on $\xi^\perp+t\xi$, then $\xi\in\Xi_0$, which contradicts $\xi\in\Xi_{\text{con}}$.

Finally, assume that there exists $\xi\in\Xi_{\text{con}}\cap\operatorname{int}(\Sigma)$ such that $\lambda(\xi)\neq 1$. Then by Lemma 2.4 there exists a neighbourhood $B_\epsilon(\xi)\subset\Xi_{\text{con}}\cap\operatorname{int}(\Sigma)$ and $\theta\in\Theta_\tau\in\mathcal{L}(f)$ for some τ, such that $S^2\cap(\xi^\perp+t\xi)_+\cap\Theta_\tau=\theta$. On the other hand, by Lemma 2.6

$$\mathit{\Lambda}(\theta)=\bigcup_{\zeta\in(\eta^\perp+t\eta)\cap B_\epsilon(\xi)}\phi_{\zeta,\lambda(\zeta)}(\eta),\quad\text{where }\eta=\phi_{\xi,-\lambda(\xi)}(\theta),$$

gives a curve such that $\mathit{\Lambda}(\theta)\cap S^2\cap\operatorname{int}((\xi^\perp+t\xi)_+)\neq\emptyset$ and $f(\omega)=f(\theta)$ for any $\omega\in\mathit{\Lambda}(\theta)$. Thus, $\mathit{\Lambda}(\theta)\cup\mathit{\Theta}_\tau$ must be a level set of f at value τ but it is not a 1-manifold; a contradiction.

Therefore, $\{\theta\in S^2 : f(\theta)\neq g(\theta)\}=\emptyset$. □

2.2. **Congruent sections by non-central planes. Auxiliary results.** We will use ideas of Section 2.1. However, some proofs will be different.

Definition 2.7. Let K, L, B be as in Theorem 1.6. Define the following three subsets of S^2:

$$\Xi_0'=\{\xi\in S^2 : K\cap(\xi^\perp+t\xi)=L\cap(\xi^\perp+t\xi)\};$$
$$\Xi_n'=\{\xi\in S^2 : \phi_{\xi,\frac{2}{n}}(L\cap(\xi^\perp+t\xi))=L\cap(\xi^\perp+t\xi)\},\quad n=2,3,\ldots;$$
$$\Xi_{\text{con}}'=S^2\backslash(\Xi_0'\cup(\bigcup_{n=2}^{\infty}\Xi_n')).$$

Using the same argument as above, it is easy to prove the following lemmas.

Lemma 2.8. *Ξ'_n are closed for all $n = 0, 2, 3, \dots$.*

Proof. First, set $d_{\mathrm{H}}(\cdot,\cdot)$ to be the Hausdorff distance on sets in $\mathbb{R}^3$. Given a sequence $\xi_i \in \Xi'_0$ with $\lim_{i\to\infty} \xi_i = \xi$, we have

$$\begin{aligned}
&d_{\mathrm{H}}(K \cap (\xi^\perp + t\xi), L \cap (\xi^\perp + t\xi)) \\
&\le d_{\mathrm{H}}(K \cap (\xi^\perp + t\xi), K \cap (\xi_i^\perp + t\xi_i)) + d_{\mathrm{H}}(K \cap (\xi_i^\perp + t\xi_i), L \cap (\xi_i^\perp + t\xi_i)) \\
&\quad + d_{\mathrm{H}}(L \cap (\xi_i^\perp + t\xi_i), L \cap (\xi^\perp + t\xi)) \\
&= d_{\mathrm{H}}(K \cap (\xi^\perp + t\xi), K \cap (\xi_i^\perp + t\xi_i)) + d_{\mathrm{H}}(L \cap (\xi_i^\perp + t\xi_i), L \cap (\xi^\perp + t\xi)).
\end{aligned}$$

As $\xi_i \to \xi$, $d_{\mathrm{H}}(K\cap(\xi^\perp+t\xi), K\cap(\xi_i^\perp+t\xi_i)) \to 0$ and $d_{\mathrm{H}}(L\cap(\xi_i^\perp+t\xi_i), L\cap(\xi^\perp+t\xi)) \to 0$; hence, $d_{\mathrm{H}}(K \cap (\xi^\perp + t\xi), L \cap (\xi^\perp + t\xi)) = 0$, which implies $\xi \in \Xi'_0$. Now we prove the closeness of Ξ'_0.

Similarly, given a sequence $\xi_i \in \Xi'_n$ with $\lim_{i\to\infty} \xi_i = \xi$, we have

$$\begin{aligned}
&d_{\mathrm{H}}(\phi_{\xi,\frac{2}{n}}(L \cap (\xi^\perp + t\xi)), L \cap (\xi^\perp + t\xi)) \\
&\le d_{\mathrm{H}}(\phi_{\xi,\frac{2}{n}}(L \cap (\xi^\perp + t\xi)), \phi_{\xi_i,\frac{2}{n}}(L \cap (\xi_i^\perp + t\xi_i))) \\
&\quad + d_{\mathrm{H}}(\phi_{\xi_i,\frac{2}{n}}(L \cap (\xi_i^\perp + t\xi_i)), L \cap (\xi_i^\perp + t\xi_i)) \\
&\quad + d_{\mathrm{H}}(L \cap (\xi_i^\perp + t\xi_i), L \cap (\xi^\perp + t\xi)) \\
&= d_{\mathrm{H}}(\phi_{\xi,\frac{2}{n}}(L \cap (\xi^\perp + t\xi)), \phi_{\xi_i,\frac{2}{n}}(L \cap (\xi_i^\perp + t\xi_i))) \\
&\quad + d_{\mathrm{H}}(L \cap (\xi_i^\perp + t\xi_i), L \cap (\xi^\perp + t\xi)).
\end{aligned}$$

As $\xi_i \to \xi$, $d_{\mathrm{H}}(L \cap (\xi^\perp + t\xi), L \cap (\xi_i^\perp + t\xi_i)) \to 0$; hence,

$$d_{\mathrm{H}}(\phi_{\xi,\frac{2}{n}}(L \cap (\xi^\perp + t\xi)), L \cap (\xi^\perp + t\xi)) = 0,$$

which implies $\xi \in \Xi'_n$. ∎

Now we define

$$\lambda'(\xi) := \{\alpha : K \cap (\xi^\perp + t\xi) = \phi_{\xi,\alpha}(L \cap (\xi^\perp + t\xi))\}.$$

In the case when $\xi \in \Xi'_n$, $n \ge 2$, $\lambda'(\xi)$ is a multi-valued function; on the other hand, if $\xi \in \Xi'_{\mathrm{con}}$, $\lambda'(\xi)$ is a single-valued function; otherwise, if $\alpha, \beta \in \lambda'(\xi)$ with $\alpha \ne \beta$,

$$\phi_{\xi,\alpha}(L \cap (\xi^\perp + t\xi)) = K \cap (\xi^\perp + t\xi) = \phi_{\xi,\beta}(L \cap (\xi^\perp + t\xi))$$

implying

$$\phi_{\xi,\alpha}(L \cap (\xi^\perp + t\xi)) = \phi_{\xi,\beta}(L \cap (\xi^\perp + t\xi)).$$

If $\alpha - \beta$ is irrational, then $L\cap(\xi^\perp+t\xi)$ is a disk, which means $\xi \in \Xi'_0$; a contradiction. If $\alpha - \beta$ is rational, then by Lemma 2.3, $\xi \in \Xi'_n$; a contradiction.

Lemma 2.9. *Let K, L, B be as in Theorem* 1.6. *Then Ξ'_{con} is open and $\lambda'(\xi)$ is a continuous function on Ξ'_{con} if $\Xi'_{\mathrm{con}} \ne \emptyset$.*

Proof. If $\Xi'_{\mathrm{con}} = \emptyset$, then Ξ'_{con} is open. Now assume that Ξ'_{con} is not open. There exists $\xi \in \Xi'_{\mathrm{con}}$, such that, for any $i \in \mathbb{N}$, there exists $\xi_i \in B_{\frac{1}{i}}(\xi) \cap \Xi'_{n_i}$ for some n_i. If there are infinitely many ξ_i that belong to Ξ'_0, then $0 \in \lambda'(\xi)$. If there are infinitely many ξ_i for which $n_i \ne 0$, then $\lambda'(\xi_i)$ is a multi-valued function. Thus there exists $\alpha_i \in \lambda'(\xi_i)$, such that $|\alpha_i - \lambda'(\xi)| > \varepsilon$ for some $\varepsilon > 0$. By compactness of $[0, 2]$, there exists a subsequence ξ_{i_k}, such that $\lim_{k\to\infty}\alpha_{i_k}=\alpha$, where $|\alpha-\lambda'(\xi)|\geq\varepsilon$. Then

$$\begin{aligned}
&d_{\mathrm{H}}(\phi_{\xi,\lambda'(\xi)}(L\cap(\xi^{\perp}+t\xi)),\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi)))\\
&\leq d_{\mathrm{H}}(\phi_{\xi,\lambda'(\xi)}(L\cap(\xi^{\perp}+t\xi)),K\cap(\xi^{\perp}+t\xi))\\
&\quad+d_{\mathrm{H}}(K\cap(\xi^{\perp}+t\xi),K\cap(\xi_{i_k}^{\perp}+t\xi_{i_k}))\\
&\quad+d_{\mathrm{H}}(K\cap(\xi_{i_k}^{\perp}+t\xi_{i_k}),\phi_{\xi_{i_k},\alpha_{i_k}}(L\cap(\xi_{i_k}^{\perp}+t\xi_{i_k})))\\
&\quad+d_{\mathrm{H}}(\phi_{\xi_{i_k},\alpha_{i_k}}(L\cap(\xi_{i_k}^{\perp}+t\xi_{i_k})),\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi)))\\
&=d_{\mathrm{H}}(K\cap(\xi^{\perp}+t\xi),K\cap(\xi_{i_k}^{\perp}+t\xi_{i_k}))\\
&\quad+d_{\mathrm{H}}(\phi_{\xi_{i_k},\alpha_{i_k}}(L\cap(\xi_{i_k}^{\perp}+t\xi_{i_k})),\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi))).
\end{aligned}$$

As $k\to\infty$, we have $\xi_{i_k}\to\xi$ and $\alpha_{i_k}\to\alpha$; hence,

$$d_{\mathrm{H}}(\phi_{\xi,\lambda'(\xi)}(L\cap(\xi^{\perp}+t\xi)),\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi)))=0,$$

implying

$$K\cap(\xi^{\perp}+t\xi)=\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi));$$

a contradiction.

For continuity, since $\lambda'(\xi)$ is a single-valued function when $\xi\in\Xi'_{\mathrm{con}}$, consider a sequence $\{\xi_i\}_{i=1}^{\infty}\in\Xi'_{\mathrm{con}}$, such that $\Xi'_{\mathrm{con}}\ni\xi=\lim_{i\to\infty}\xi_i$. By compactness of $[0,2]$, there exists a subsequence $\{\xi_{i_k}\}_{k=1}^{\infty}$, such that $\alpha=\lim_{k\to\infty}\lambda'(\xi_{i_k})$. Then

$$\begin{aligned}
&d_{\mathrm{H}}(K\cap(\xi^{\perp}+t\xi),\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi)))\\
&\leq d_{\mathrm{H}}(K\cap(\xi^{\perp}+t\xi),K\cap(\xi_{i_k}^{\perp}+t\xi_{i_k}))\\
&\quad+d_{\mathrm{H}}(K\cap(\xi_{i_k}^{\perp}+t\xi_{i_k}),\phi_{\xi_{i_k},\lambda'(\xi_{i_k})}(L\cap(\xi_{i_k}^{\perp}+t\xi_{i_k})))\\
&\quad+d_{\mathrm{H}}(\phi_{\xi_{i_k},\lambda'(\xi_{i_k})}(L\cap(\xi_{i_k}^{\perp}+t\xi_{i_k})),\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi)))\\
&=d_{\mathrm{H}}(K\cap(\xi^{\perp}+t\xi),K\cap(\xi_{i_k}^{\perp}+t\xi_{i_k}))\\
&\quad+d_{\mathrm{H}}(\phi_{\xi_{i_k},\lambda'(\xi_{i_k})}(L\cap(\xi_{i_k}^{\perp}+t\xi_{i_k})),\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi))).
\end{aligned}$$

As $k\to\infty$, we have $\lambda'(\xi_{i_k})\to\alpha$ and $\xi_{i_k}\to\xi$; hence,

$$d_{\mathrm{H}}(K\cap(\xi^{\perp}+t\xi),\phi_{\xi,\alpha}(L\cap(\xi^{\perp}+t\xi)))=0,$$

that is, $\lambda'(\xi)=\alpha$. If $\{\lambda'(\xi_i)\}_{i=1}^{\infty}$ has another subsequence with a different limit $\beta\neq\alpha$, then $\{\alpha,\beta\}\subset\lambda'(\xi)$, contradicting $\xi\in\Xi'_{\mathrm{con}}$. □

For a convex body K, we define its radial function in the direction of θ to be

$$\rho_K(\theta):=\sup\{t:t\theta\in K\}.$$

Lemma 2.10. *Let K,L,B be as in Theorem 1.6. Then either $\{\theta\in S^2:\rho_K(\theta)=\rho_L(\theta)\}=S^2$ or the set*

$$\{\theta\in S^2:\rho_K(\theta)\neq\rho_L(\theta)\}\cap[\{\theta\in S^2:\nabla_{S^2}\rho_K(\theta)\neq0\}\cup\{\theta\in S^2:\nabla_{S^2}\rho_L(\theta)\neq0\}]$$

is not empty.

Proof. First, the set $\{\theta\in S^2:\rho_K(\theta)\neq\rho_L(\theta)\}$ is not the whole sphere; otherwise without loss of generality let $\rho_K(\theta)<\rho_L(\theta)\ \ \forall\theta\in S^2$. This means that K is strictly inside L. Then

$$\begin{aligned}\mathrm{vol}_{n-1}(K\cap(\xi^\perp+t\xi)) &= \mathrm{vol}_{n-1}(\phi_\xi(L\cap(\xi^\perp+t\xi)))\\ &= \mathrm{vol}_{n-1}(L\cap(\xi^\perp+t\xi)) > \mathrm{vol}_{n-1}(K\cap(\xi^\perp+t\xi)).\end{aligned}$$

Now assume

$$\{\theta\in S^2:\rho_K(\theta)\neq\rho_L(\theta)\}\\ \cap\left[\{\theta\in S^2:\nabla_{S^2}\rho_K(\theta)\neq 0\}\cup\{\theta\in S^2:\nabla_{S^2}\rho_L(\theta)\neq 0\}\right]=\emptyset.$$

Then

$$\{\theta\in S^2:\rho_K(\theta)\neq\rho_L(\theta)\}\\ \subset\{\theta\in S^2:\nabla_{S^2}\rho_K(\theta)=0\}\cap\{\theta\in S^2:\nabla_{S^2}\rho_L(\theta)=0\}.$$

Since $\rho_K,\rho_L\in C^2$, the set

$$\Upsilon_0':=\{\theta\in S^2:\nabla_{S^2}\rho_K(\theta)=0\}\cap\{\theta\in S^2:\nabla_{S^2}\rho_L(\theta)=0\}$$

is closed and ρ_K and ρ_L are constant in any connected subset of Υ_0'.

Assume there exists $x\in\{\theta\in S^2:\rho_K(\theta)\neq\rho_L(\theta)\}$. Choose the largest connected open neighbourhood $\mathcal{N}_x$ of x in $\{\theta\in S^2:\rho_K(\theta)\neq\rho_L(\theta)\}$. Then the closure of $\mathcal{N}_x$ is in Υ_0' and the boundary of $\mathcal{N}_x$ is a subset of $\{\theta\in S^2:\rho_K(\theta)=\rho_L(\theta)\}$, which implies $C_1=\rho_K=\rho_L=C_2$ in the closure of $\mathcal{N}_x$; a contradiction. Hence, $\{\theta\in S^2:\rho_K(\theta)\neq\rho_L(\theta)\}=\emptyset$. □

Note that for $\xi\in\Xi_{\mathrm{con}}'$, since Ξ_{con}' is open, there exists an $\epsilon>0$, such that $B_\epsilon(\xi)\subset\Xi_{\mathrm{con}}'$. Then, for any point $\theta\in S^2$ such that $\rho_L(\theta)\theta\in L\cap(\xi^\perp+t\xi)$, we set $\eta=\phi_{\xi,\lambda'(\xi)}(\theta)$. Thus, we define the curve

$$\Lambda'(\theta):=\{\phi_{\zeta,-\lambda'(\zeta)}(\eta):\zeta\in B_\epsilon(\xi)\text{ and }\rho_L(\theta)\eta\in\zeta^\perp+t\zeta\}$$

passing through θ (see Figure 7).

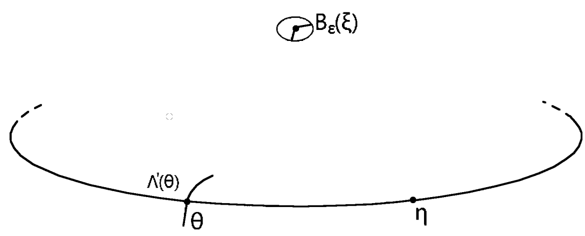

FIGURE 7. The construction of $\Lambda'(\theta)$.

Lemma 2.11. *Let K,L,B be as in Theorem* 1.6 *and $\xi\in\Xi_{\mathrm{con}}'$. If $\lambda'(\xi)\neq 1$, then*

$$\{\rho_L(\theta)u:u\in\Lambda'(\theta)\}\cap\mathrm{int}\,((\xi^\perp+t\xi)_+)\neq\emptyset.$$

Proof. Without loss of generality, we can assume that $0<\lambda'(\xi)<1$. The other case $1<\lambda'(\xi)<2$ is similar. Since $\xi\in\Xi_{\mathrm{con}}'$ and $0<\lambda'(\xi)<1$, by Lemma 2.9 there exists $0<\iota'<\frac{1}{2}$ and a ball $B_\epsilon(\xi)\subset\Xi_{\mathrm{con}}'$ such that $\iota'\leq\lambda'(\zeta)\leq 1-\iota'$ for any $\zeta\in B_\epsilon(\xi)$.

Now take any $\theta \in S^2$ such that $\rho_L(\theta)\theta \in L \cap (\xi^\perp + t\xi)$ and define $\eta = \phi_{\xi,\lambda'(\xi)}(\theta)$. We set $\zeta = \phi_{\eta,-\alpha}(\xi)$ for small $\alpha > 0$ and $\omega = \phi_{\zeta,-\lambda'(\zeta)}(\eta)$; then we have

$$\langle \zeta, \eta \rangle = \langle \phi_{\eta,-\alpha}(\xi), \eta \rangle = \langle \xi, \eta \rangle = \frac{t}{\rho_K(\eta)} = \frac{t}{\rho_L(\theta)} \in (0,1).$$

Hence, $\rho_L(\theta)\eta \in \zeta^\perp + t\zeta$,

$$\begin{aligned}
\zeta \times \eta &= \phi_{\eta,-\alpha}(\xi) \times \eta \\
&= (\xi \cos(-\alpha\pi) + (\eta \times \xi)\sin(-\alpha\pi) + \eta\langle \eta, \xi \rangle (1 - \cos(-\alpha\pi))) \times \eta \\
&= \xi \times \eta \cos(\alpha\pi) - (\xi \langle \eta, \eta \rangle - \eta \langle \xi, \eta \rangle) \sin(\alpha\pi) \\
&= \xi \times \eta \cos(\alpha\pi) - (\xi - \frac{t}{\rho_L(\theta)}\eta)\sin(\alpha\pi),
\end{aligned}$$

and

$$\begin{aligned}
\langle \xi, \zeta \rangle &= \langle \xi, \phi_{\eta,-\alpha}(\xi) \rangle \\
&= \langle \xi, \xi\cos(-\alpha\pi) + (\eta \times \xi)\sin(-\alpha\pi) + \eta\langle \eta, \xi \rangle (1 - \cos(-\alpha\pi)) \rangle \\
&= \cos(\alpha\pi) + \frac{t^2}{\rho_L^2(\theta)}(1 - \cos(\alpha\pi)). \qquad (2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\rho_L(\theta)\langle \xi, \omega \rangle - t \\
&= \rho_L(\theta)\langle \xi, \eta\cos(-\lambda'(\zeta)\pi) + (\zeta \times \eta)\sin(-\lambda'(\zeta)\pi) \\
&\quad + \frac{t}{\rho_L(\theta)}\zeta(1 - \cos(-\lambda'(\zeta)\pi))\rangle - t \\
&= t\cos(\lambda'(\zeta)\pi) - \rho_L(\theta)\langle \xi, \xi \times \eta \cos(\alpha\pi) - (\xi - \frac{t}{\rho_L(\theta)}\eta)\sin(\alpha\pi)\rangle \sin(\lambda'(\zeta)\pi) \\
&\quad + t(1 - \cos(\lambda'(\zeta)\pi))(\cos(\alpha\pi) + \frac{t^2}{\rho_L^2(\theta)}(1 - \cos(\alpha\pi))) - t \\
&= t\cos(\lambda'(\zeta)\pi) + \rho_L(\theta)(1 - \frac{t^2}{\rho_L^2(\theta)})\sin(\alpha\pi)\sin(\lambda'(\zeta)\pi) \\
&\quad + t(1 - \cos(\lambda'(\zeta)\pi))(\cos(\alpha\pi) + \frac{t^2}{\rho_L^2(\theta)}(1 - \cos(\alpha\pi))) - t \\
&= (1 - \frac{t^2}{\rho_L^2(\theta)})\rho_L(\theta)\sin(\alpha\pi)\sin(\lambda'(\zeta)\pi) \\
&\quad - t(1 - \cos(\lambda'(\zeta)\pi))(1 - \frac{t^2}{\rho_L^2(\theta)})(1 - \cos(\alpha\pi)) \\
&= (1 - \frac{t^2}{\rho_L^2(\theta)})(\rho_L(\theta)\sin(\lambda'(\zeta)\pi)\sin(\alpha\pi) - t(1 - \cos(\lambda'(\zeta)\pi))(1 - \cos(\alpha\pi))) \\
&\geq (1 - \frac{t^2}{\rho_L^2(\theta)})(\rho_L(\theta)\sin(\iota'\pi)\sin(\alpha\pi) - t(1 - \cos((1 - \iota')\pi))(1 - \cos(\alpha\pi))) \\
&> 0 \quad \text{for sufficiently small } \alpha > 0.
\end{aligned}$$

Hence, $\rho_L(\theta)\omega \in \{\rho_L(\theta)u : u \in \Lambda'(\theta)\} \cap \operatorname{int}((\xi^\perp + t\xi)_+)$. □

Proof of Theorem 1.6. Assume $\{\theta \in S^2 : \rho_K(\theta) \neq \rho_L(\theta)\} \neq \emptyset$. By Lemma 2.10, we have

$$\{\theta \in S^2 : \rho_K(\theta) \neq \rho_L(\theta)\} \cap [\{\theta \in S^2 : \nabla_{S^2}\rho_K(\theta) \neq 0\} \cup \{\theta \in S^2 : \nabla_{S^2}\rho_L(\theta) \neq 0\}] \neq \emptyset.$$

Without loss of generality, we can choose

$$x \in \{\theta \in S^2 : \rho_K(\theta) \neq \rho_L(\theta)\} \cap \{\theta \in S^2 : \nabla_{S^2}\rho_L(\theta) \neq 0\},$$

Therefore there exists an open ball

$$B_\epsilon(x) \subset \{\theta \in S^2 : \rho_K(\theta) \neq \rho_L(\theta)\} \cap \{\theta \in S^2 : \nabla_{S^2}\rho_L(\theta) \neq 0\}.$$

By the implicit function theorem, the collection of local level sets of ρ_L, $\mathcal{L}'(L) := \{\Theta'_\tau\}_{a'<\tau<b'}$, is a collection of disjoint C^2 curves, where $\Theta'_\tau := \{\theta \in S^2 : \rho_L(\theta) = \tau\} \cap B_\epsilon(x)$. Here, $a' = \inf_{\theta\in B_\epsilon(x)} \rho_L(\theta)$ and $b' = \sup_{\theta\in B_\epsilon(x)} \rho_L(\theta)$.

For curves $\{\Theta'_\tau\} \subset S^2$, consider their geodesic curvature $k_g(\cdot)$. If for every $\eta \in \Theta'_\tau$ and $\Theta'_\tau \in \mathcal{L}'(L)$, we have $k_g(\eta) = 0$, then each Θ'_τ belongs to some great circle. Choose one of these great circles. It divides S^2 into two hemispheres. Fix one of these hemispheres and denote it by S^2_+. Consider all circles of the form $S^2 \cap (\xi^\perp + \frac{t}{\tau}\xi)$ that are tangent to the curves Θ_τ and $\xi \in S^2_+$. Denote by Σ' the set of such directions ξ.

Now consider the case when for some $\tau \in (a', b')$, there exists a $\theta \in \Theta'_\tau$, such that $k_g(\theta) \neq 0$. Then by C^2 smoothness of ρ_L, there exists a smaller neighbourhood of x, which we will again denote by $B_\epsilon(x)$, and a collection of level sets in $B_\epsilon(x)$, such that $k_g(\eta) \neq 0$ for any $\eta \in \Theta'_\tau$ and $a' < \tau < b'$. For each point $\eta \in \Theta'_\tau$, consider the great circle which is tangent to Θ'_τ at η. Then $\{\Theta'_\tau\}_{a'<\tau<b'}$ lie on one side of their tangent great circle. For each τ and each $\eta \in \Theta'_\tau$ consider a circle $S^2 \cap (\xi^\perp + \frac{t}{\tau}\xi)$ that is tangent to Θ'_τ at η and lies on the other side with respect to the tangent great circle. Let Σ' be the set of such directions ξ.

Note that for each Θ'_τ, these ξ form a parallel set of Θ'_τ on S^2, which we denote by $\mathcal{C}'_\tau$. This is the envelope of a family of circles on S^2 with centres at Θ'_τ and of radius $\frac{t}{\tau}$. We claim that for different values of $\tau \in (a', b')$, the corresponding $\mathcal{C}'_\tau$ do not coincide. Otherwise, for some $\tau_1 \in (a', b')$, the envelope of a family of tangent planes of $\frac{1}{\tau_1}B$ at the points on $\frac{t}{\tau_1}\mathcal{C}'_{\tau_1}$ intersects S^2 along Θ'_{τ_1}, i.e.,

$$\Theta'_{\tau_1} = \text{Envelope}\left\{\bigcup_{\xi\in\mathcal{C}'_{\tau_1}} H_{\frac{t}{\tau_1}\xi, \frac{1}{\tau_1}B}\right\} \cap S^2,$$

where $H_{\frac{t}{\tau_1}\xi, \frac{1}{\tau_1}B}$ is the tangent plane to $\frac{1}{\tau_1}B$ at the point $\frac{t}{\tau_1}\xi$ and Envelope$\{\cdot\}$ is the envelope of a one-parameter family of curves. Hence, multiplying both sides by τ_1, we obtain

$$\partial L \supset \tau_1\Theta'_{\tau_1} = \text{Envelope}\left\{\bigcup_{\xi\in\mathcal{C}'_{\tau_1}} H_{t\xi, B}\right\} \cap \tau_1 S^2.$$

On the other hand, assume that for a different value $(a', b') \ni \tau_2 \neq \tau_1$, we have

$$\tau_2\Theta'_{\tau_2} = \text{Envelope}\left\{\bigcup_{\xi\in\mathcal{C}'_{\tau_1}} H_{t\xi, B}\right\} \cap \partial L.$$

However, Envelope$\{\bigcup_{\xi\in\mathcal{C}'_{\tau_1}} H_{t\xi,B}\}$ is a ruled surface (i.e., comprised of straight lines), which cannot intersect ∂L along two different curves. So $\mathcal{C}'_{\tau_1}$ and $\mathcal{C}'_{\tau_2}$ do not coincide. We conclude that by the continuity of $\mathcal{C}'_\tau$ with respect to τ, the set Σ' is a union of $\mathcal{C}'_\tau$ and thus contains non-empty interior.

Now we claim $\Xi'_{\mathrm{con}} \cap \mathrm{int}\,(\Sigma') \neq \emptyset$; otherwise, $\mathrm{int}\,(\Sigma') \subset \Xi'_0 \cup \big(\bigcup_{n=2}^\infty \Xi_n\big)$, but $\mathrm{int}\,(\Sigma') \cap \Xi'_0 = \emptyset$, since $B_\epsilon(x) \subset \{\theta \in S^2 : \rho_K(\theta) \neq \rho_L(\theta)\}$. Hence, $\mathrm{int}\,(\Sigma') \subset \bigcup_{n=2}^\infty \Xi'_n$, which implies that $\big(\bigcup_{n=2}^\infty \Xi'_n\big) \cap \mathrm{int}\,(\Sigma)$ contains non-empty interior. By the Baire category theorem and Lemma 2.8, there exists some $k \in \mathbb{N}$, such that $\Xi_k \cap \mathrm{int}\,(\Sigma')$ contains non-empty interior.

Assume $\xi \in \mathrm{int}\,(\Xi'_k \cap \Sigma')$. Then, there exists $\delta > 0$ such that $B_\delta(\xi) \subset \mathrm{int}\,(\Xi'_k \cap \Sigma')$. For any $\rho_L(\theta)\theta \in \xi^\perp + t\xi$,

$$\rho_L(\eta) = \rho_L(\theta) \quad \forall \eta \in \Lambda'_\xi(\theta) := \{\phi_{\zeta,\frac{2}{k}}(\theta) : \rho_L(\theta)\theta \in \zeta^\perp + t\zeta, \zeta \in B_\delta(\xi)\}$$

and

$$\begin{aligned}&\rho_L(\omega) = \rho_L(\eta) = \rho_L(\theta)\\ &\qquad \forall \omega \in \Delta'_\xi(\theta) := \{\phi_{\vartheta,-\frac{2}{k}}(\eta) : \eta \in \Lambda'_\xi(\theta),\ \rho_L(\theta)\eta \in \vartheta^\perp + t\vartheta,\ \vartheta \in B_\delta(\xi)\}.\end{aligned}$$

Let us show that $\Delta'_\xi(\theta)$ has non-empty interior. Note that for any $\eta \in \Lambda'_\xi(\theta)$, by equation (2) we have

$$\langle \theta, \eta\rangle = \cos(2\pi/k) + \frac{t^2}{\rho_L^2(\theta)}(1 - \cos(2\pi/k)) =: \varsigma'(t),$$

where $-1 < \varsigma'(t) < 1$. If $\varsigma'(t) = 0$, then $\Lambda'_\xi(\theta) \subset S^2 \cap \theta^\perp$. Fix $\eta \in \Lambda'_\xi(\theta)$; then for each

$$\omega \in \{\phi_{\vartheta,-\frac{2}{k}}(\eta) : \rho_L(\theta)\eta \in \vartheta^\perp + t\vartheta,\ \vartheta \in B_\delta(\xi)\},$$

by equation (2) we have

$$\langle \omega, \eta\rangle = \varsigma'(t) = 0,$$

which means $\{\phi_{\vartheta,-\frac{2}{k}}(\eta) : \rho_L(\theta)\eta \in \vartheta^\perp + t\vartheta,\ \vartheta \in B_\delta(\xi)\}$ is a curve passing through θ and contained in $S^2 \cap \eta^\perp$. Since $\Lambda'_\xi(\theta)$ is a continuous curve, by changing η we see that $\Delta'_\xi(\theta)$ has the shape of a sand dial, which we will refer to as a $\bowtie$ shape.

If $0 < \varsigma'(t) < 1$, then $\Lambda'_\xi(\theta) \subset S^2 \cap (\theta^\perp + \varsigma(t)\theta)$. Fix $\eta \in \Lambda'_\xi(\theta)$; then $\{\phi_{\vartheta,-\frac{2}{k}}(\eta) : \rho_L(\theta)\eta \in \vartheta^\perp + t\vartheta,\ \vartheta \in B_\delta(\xi)\}$ gives a curve passing through θ and contained in $S^2 \cap (\eta^\perp + \varsigma'(t)\eta)$. Observe that for different $\eta \in \Lambda'_\xi(\theta)$, we have different curves $\{\phi_{\vartheta,-\frac{2}{k}}(\eta) : \rho_L(\theta)\eta \in \vartheta^\perp + t\vartheta,\ \vartheta \in B_\delta(\xi)\}$ with the only common point θ. Since these curves change continuously, the set $\Delta'_\xi(\theta)$ again has a $\bowtie$ shape.

If $-1 < \varsigma'(t) < 0$, use the same argument to show that $\Delta'_\xi(\theta)$ has a $\bowtie$ shape. Therefore, $\Delta'_\xi(\theta)$ is a set with non-empty interior on S^2; hence, it is not a 1-manifold.

Now to reach a contradiction, assume that $\nabla_{S^2}\rho_L(\theta) = 0$ for every $\theta \in S^2$ such that $\rho_L(\theta)\theta \in L \cap (\xi^\perp + t\xi)$; then $L \cap (\xi^\perp + t\xi)$ is a disk, $\xi \in \Xi'_0$, which contradicts $\mathrm{int}\,(\Sigma') \cap \Xi'_0 = \emptyset$.

On the other hand, if $\nabla_{S^2}\rho_L(\theta) \neq 0$ for some point θ satisfying

$$\rho_L(\theta)\theta \in L \cap (\xi^\perp + t\xi),$$

then by the implicit function theorem, the level set of ρ_L passing through θ on S^2 is a 1-manifold; a contradiction. Thus, we have shown that $\Xi'_{\mathrm{con}} \cap \mathrm{int}\,(\Sigma') \neq \emptyset$.

Now assume that for every $\xi \in \Xi'_{\text{con}} \cap \operatorname{int}(\Sigma')$, $\lambda'(\xi) = 1$. Then, there exists $\delta > 0$ such that $B_\delta(\xi) \subset \operatorname{int}(\Xi'_{\text{con}} \cap \Sigma')$ and $\lambda'(\zeta) = 1$ for any $\zeta \in B_\delta(\xi)$. For any $\theta \in S^2$ such that $\rho_L(\theta)\theta \in \xi^\perp + t\xi$, we have

$$\rho_K(\eta) = \rho_L(\theta) \quad \forall \eta \in \Lambda'_\xi(\theta) := \{\phi_{\zeta,1}(\theta) : \rho_L(\theta)\theta \in \zeta^\perp + t\zeta, \zeta \in B_\delta(\xi)\}$$

and

$$\rho_L(\omega) = \rho_K(\eta) = \rho_L(\theta)$$
$$\forall \omega \in \Delta'_\xi(\theta) := \{\phi_{\vartheta,1}(\eta) : \eta \in \Lambda'_\xi(\theta),\ \rho_L(\theta)\eta \in \vartheta^\perp + t\vartheta,\ \vartheta \in B_\delta(\xi)\}.$$

Following the same argument as above, we have that $\Delta'_\xi(\theta)$ is a set with non-empty interior on S^2; hence, it is not a 1-manifold. If $\nabla_{S^2}\rho_L(\theta) = 0$, for every θ such that $\rho_L(\theta)\theta \in L \cap (\xi^\perp + t\xi)$, then $L \cap (\xi^\perp + t\xi)$ is a disk; a contradiction.

If $\nabla_{S^2}\rho_L(\theta) \neq 0$ for some point θ satisfying $\rho_L(\theta)\theta \in L \cap (\xi^\perp + t\xi)$, then by the implicit function theorem, the level set of ρ_L passing through θ on S^2 is a 1-manifold; a contradiction.

Finally, consider the case when there exists $\xi \in \operatorname{int}(\Xi'_{\text{con}} \cap \Sigma')$ such that $\lambda'(\xi) \neq 1$. Then, there exists a neighbourhood $B_\epsilon(\xi) \in \Xi'_{\text{con}} \cap \operatorname{int}(\Sigma')$ and $\theta \in \Theta'_\tau \in \mathcal{L}'(L)$ for some τ, such that $(\xi^\perp + t\xi)_+ \cap \Theta'_\tau = \theta$. On the other hand, by Lemma 2.11

$$\Lambda'(\theta) = \{\phi_{\zeta,-\lambda'(\zeta)}(\eta) : \zeta \in B_\epsilon(\xi) \text{ and } \rho_L(\theta)\eta \in \zeta^\perp + t\zeta\},$$

where $\eta = \phi_{\xi,\lambda'(\xi)}(\theta)$, is a curve such that $\{\rho_L(\theta)u : u \in \Lambda'(\theta)\} \cap \operatorname{int}((\xi^\perp + t\xi)_+) \neq \emptyset$ and $\rho_L(\vartheta) = \rho_L(\theta)$ for any $\vartheta \in \Lambda'(\theta)$; however, $\Lambda'(\theta) \cup \Theta'_\tau \subset \mathcal{L}'(L)$ must be a level set of ρ_L in $\mathcal{L}'(L)$ that is not a 1-manifold; a contradiction.

Therefore, $\{\theta \in S^2 : \rho_K(\theta) \neq \rho_L(\theta)\} = \emptyset$. □

Acknowledgments

I would like to express my gratitude to my supervisor, Dr. Vladyslav Yaskin, for fruitful discussions. Thanks are also due to the referee for the critical reading of the manuscript.

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School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, Hubei, People's Republic of China
Email address: nzhang2@ualberta.ca