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SHARP STRICHARTZ ESTIMATES FOR WATER WAVES SYSTEMS

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ABSTRACT. Water waves are well-known to be dispersive at the linearization level. Considering the fully nonlinear systems, we prove for reasonably smooth solutions the optimal Strichartz estimates for pure gravity waves and the semi-classical Strichartz estimates for gravity-capillary waves for both 2D and 3D waves. Here, by optimal we mean the gains of regularity (over the Sobolev embedding from Sobolev spaces to Hölder spaces) obtained for the linearized systems. Our proofs combine the paradifferential reductions of Alazard-Burq-Zuily with a dispersive estimate using a localized wave package type parametrix of Koch-Tataru.

1. INTRODUCTION

Water waves systems govern the dynamic of an interface between a fluid domain and the vacuum. It is well-known that these systems are dispersive; i.e., waves at different frequencies propagate at different speeds. For approximate models of water waves in certain regimes such as Kadomtsev-Petviashvili equations, Korteweg-de Vries equations, Schrödinger equations, wave equations, etc., dispersive properties have been extensively studied. For the fully nonlinear system of water waves, dispersive properties are however less understood.

On the one hand, in the global dynamic, dispersive properties have been considered in establishing the existence of global (or almost global) solutions for small, localized, smooth data by the works of Wu [26, 27], Germain-Masmoudi-Shatah [12, 13], Ionescu-Pusateri [17, 18], Alazard-Delort [5], and Ifrim-Tataru [15, 16]. On the other hand, in the local dynamic, dispersive properties and more precisely Strichartz estimates have been exploited in proving the existence of local-in-time solutions with rough, generic data initiated by the work of Alazard-Burq-Zuily [4] and then followed by de Poyferré-Nguyen [10, 11], and Nguyen [23]. Prior to these, a Strichartz estimate was proved for 2D gravity-capillary waves by Christianson-Hur-Staffilani [9]. It is worth noting that [9] allows overturning waves. Unlike the case of semilinear Schrödinger (wave) equations, water waves systems are quasilinear in nature, and thus how much regularity one can gain in Strichartz estimates depends also on the smoothness of solutions under consideration. In other words, in terms of dispersive analysis (for generic solutions), the nonlinear systems are not

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obviously dictated by their linearizations. In fact, the Strichartz estimates proved in [4, 11, 23] are nonoptimal compared to the linearized systems. We address in this paper the following problem:

At which level of regularity do solutions to the fully nonlinear systems of water waves obey the same Strichartz estimates as their linearizations?

Remark first that due to the systematic use of symbolic calculus in the framework of semi-classical analysis in [4, 11, 23] we were not able to reach sharp Strichartz estimates by simply adapting the method there to the case of sufficiently smooth solutions.

In this paper, we choose to work on the Zakharov-Craig-Sulem formulation of water waves, which is recalled now.

1.1. **The Zakharov-Craig-Sulem formulation of water waves.** We consider an incompressible inviscid fluid with unit density moving in a time-dependent domain

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Omega_t\},$$

where each Ω_t is a domain located underneath a free surface

$$\Sigma_t = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x)\}$$

and above a fixed bottom $\Gamma = \partial\Omega_t \setminus \Sigma_t$. We also assume that Ω satisfies

Condition (H)**.** Each Ω_t is the intersection of the half space

$$\Omega_{1,t} = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y < \eta(t, x)\}$$

and an open connected set Ω_2 containing a fixed strip around Σ_t; i.e., there exists $h > 0$ such that for all $t \in [0, T]$,

$$\{(x, y) \in \mathbf{R}^d \times \mathbf{R} : \eta(x) - h \leq y \leq \eta(t, x)\} \subset \Omega_2.$$

Assume that the velocity field v admits a potential $\phi : \Omega \to \mathbf{R}$, i.e., $v = \nabla\phi$. Using the Zakharov formulation, we introduce the trace of ϕ on the free surface

$$\psi(t, x) = \phi(t, x, \eta(t, x)).$$

Then $\phi(t, x, y)$ is the unique variational solution of

$$\Delta\phi = 0 \text{ in } \Omega_t, \quad \phi(t, x, \eta(t, x)) = \psi(t, x). \tag{1.1}$$

The Dirichlet-Neumann operator is then defined by

$$\begin{aligned} G(\eta)\psi &= \sqrt{1 + |\nabla_x\eta|^2}\Big(\frac{\partial\phi}{\partial n}\Big|_{\Sigma}\Big) \\ &= (\partial_y\phi)(t, x, \eta(t, x)) - \nabla_x\eta(t, x) \cdot (\nabla_x\phi)(t, x, \eta(t, x)). \end{aligned}$$

The *gravity-capillary waves* (see [20]) problem consists of solving the following system of (η, ψ):

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + g\eta + \sigma H(\eta) + \dfrac{1}{2}|\nabla_x\psi|^2 - \dfrac{1}{2}\dfrac{(\nabla_x\eta \cdot \nabla_x\psi + G(\eta)\psi)^2}{1 + |\nabla_x\eta|^2} = 0, \end{cases} \tag{1.2}$$

where σ is the surface tension coefficient and $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) = -\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right).$$

In the regime of large wavelengths, one can discard the effect of surface tension by taking $\sigma = 0$ in the system (1.3) to obtain the system of *pure gravity water waves*

$$\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g\eta + \frac{1}{2}|\nabla_x \psi|^2 - \frac{1}{2}\frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1+|\nabla_x \eta|^2} = 0. \end{cases} \tag{1.3}$$

The physical dimensions are $d = 1, 2$. For terminology, when $d = 1$ (respectively $d = 2$) we call (1.2), (1.3) the 2D (respectively 3D) waves systems. It is important to introduce the vertical and horizontal components of the trace of the velocity on Σ, which can be expressed in terms of η and ψ:

$$B = (v_y)|_\Sigma = \frac{\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi}{1+|\nabla_x \eta|^2}, \quad V = (v_x)|_\Sigma = \nabla_x \psi - B\nabla_x \eta. \tag{1.4}$$

We recall also that the Taylor coefficient defined by $\mathfrak{a} = -\frac{\partial P}{\partial y}\big|_\Sigma$ can be defined in terms of η, ψ, B, V only (see section 4.2 in [3]).

1.2. **Known results and main theorems.**

1.2.1. *Pure gravity water waves.* For the system (1.3) of pure gravity water waves, the only existent Strichartz estimate, to our knowledge, is [4], where the authors proved Strichartz estimates for rough solutions with a gain of

$$\mu = \frac{1}{24}- \ \text{ when } d = 1, \quad \mu = \frac{1}{12}- \ \text{ when } d \geq 2. \tag{1.5}$$

The starting point of this result is the symmetrization of (1.2) into a quasilinear paradifferential equation of the following form (see Appendix A for the paradifferential calculus theory and Theorem 2.2 below for a precise reduction statement):

$$(\partial_t + T_V \cdot \nabla + iT_\gamma)\, u = f \in L^\infty_t H^s_x, \qquad s > 1 + \frac{d}{2}, \tag{1.6}$$

where γ is a symbol of order $\frac{1}{2}$.

Let us now look at the linearization of (1.3) (take $g = 1$ and infinite depth) around the rest state $(0, 0)$:

$$\begin{cases} \partial_t \eta - |D_x|\psi = 0, \\ \partial_t \psi + \eta = 0, \end{cases}$$

which is equivalent to, after imposing $u := \eta + i|D_x|^{\frac{1}{2}}\psi$,

$$\partial_t u + i|D_x|^{\frac{1}{2}} u = 0. \tag{1.7}$$

For this Schrödinger-type dispersive equation we can prove classically the Strichartz estimates

$$\|u\|_{L^p W^{s-\frac{d}{2}+\mu_{opt},\infty}} \leq C(s, d)\, \|u(0)\|_{H^s}, \quad \begin{cases} \mu_{opt} = \frac{1}{8}, \ p = 4 \quad \text{if } d = 1, \\ \mu_{opt} = \frac{1}{4}-, \ p = 2 \quad \text{if } d \geq 2, \end{cases} \tag{1.8}$$

from which the estimates for the original unknowns η, ψ can be recovered. Our first result states that the fully nonlinear system (1.3) enjoys Strichartz estimates with the same gain as in (1.8) for solutions slightly smoother than the energy threshold in [3].

Notation 1.1. Denote

$$\mathcal{H}^s = H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d),$$
$$\mathcal{W}^s = W^{r+\frac{1}{2},\infty}(\mathbf{R}^d) \times W^{r+\frac{1}{2},\infty}(\mathbf{R}^d) \times W^{r,\infty}(\mathbf{R}^d) \times W^{r,\infty}(\mathbf{R}^d).$$

Theorem 1.2. *Let $d = 1, 2$ and consider a solution (η, ψ) of* (1.3) *on the time interval $I = [0, T]$, $T < +\infty$, such that Ω satisfies condition (H) (see section* 1.1) *and*

$$(\eta, \psi, B, V) \in C([0, T]; \mathcal{H}^s).$$

(*see* [3, Theorem 1.2]). *Define*

$$\begin{cases} s(d) = \frac{5}{3} + \frac{d}{2},\ \mu_{opt}(d) = \frac{1}{8},\ p(d) = 4 \quad \text{if } d = 1, \\ s(d) = 2 + \frac{d}{2},\ \mu_{opt}(d) = \frac{1}{4}-,\ p(d) = 2 \quad \text{if } d \geq 2. \end{cases}$$

Then for any $s > s(d)$ we have

$$(\eta, \psi, B, V) \in L^{p(d)}(I; \mathcal{W}^{s-\frac{d}{2}+\mu_{opt}(d),\infty}).$$

1.2.2. *Gravity-capillary waves.* Let us now look at the linearization of (1.2) (with infinite depth) around the rest state $(0, 0)$,

$$\begin{cases} \partial_t \eta - |D_x|\psi = 0, \\ \partial_t \psi - \Delta \eta = -g\eta, \end{cases}$$

or, equivalently, with $\Phi = |D_x|^{\frac{1}{2}}\eta + i\psi$,

$$\partial_t \Phi + i|D_x|^{\frac{3}{2}}\Phi = -ig\eta, \tag{1.9}$$

for which one can easily prove the following Strichartz estimates:

(1.10)

$$\|\Phi\|_{L^p W^{s-\frac{d}{2}+\mu_{opt},\infty}} \leq C(s, d)(\|\Phi(0)\|_{H^s} + g\|\eta\|_{H^s}), \quad \begin{cases} \mu_{opt} = \frac{3}{8},\ p = 4 \quad \text{if } d = 1, \\ \mu_{opt} = \frac{3}{4}-,\ p = 2 \quad \text{if } d \geq 2. \end{cases}$$

Note that (1.10) is valid for all $g \geq 0$. Turning to the nonlinear case, in high dimensions ($d \geq 2$) the geometry can be nontrivial, and hence trapping can occur. As a consequence, natural dispersive estimates expected are the ones constructed at small time-scales which are tailored to the frequencies. The propagator $e^{-it|D_x|^{\frac{3}{2}}}$ has the speed of propagation of order $|\xi|^{\frac{1}{2}}$. Hence, for time $|t| < |\xi|^{-\frac{1}{2}}$, we do not expect to encounter any problems due to the global geometry. This leads to the so-called *semi-classical Strichartz estimate.* This terminology appeared in [8] for a study of the Schrödinger equations on compact manifolds. To realize this heuristic argument, one multiplies both sides of (1.9) by $h^{\frac{3}{2}}$ with $h = 2^{-j}$, $j \in \mathbf{N}$, and makes a change of temporal variables $t = h^{\frac{1}{2}}\sigma$, $u(\sigma, x) = \Phi(h^{\frac{1}{2}}\sigma, x)$ to derive the semi-classical equation

$$h\partial_\sigma u + |hD_x|^{\frac{3}{2}} u = 0. \tag{1.11}$$

Then the optimal dispersive estimates for (1.11) imply the semi-classical Strichartz estimates for (1.9) with a loss of $\frac{1}{8}$ derivatives when $d = 1$ and $\frac{1}{4}$ derivatives when $d \geq 2$.

In [1] it was proved that if

$$(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}} \times H^s), \quad s > 2 + \frac{d}{2}, \tag{1.12}$$

then system (1.2) can be symmetrized into a single equation analogous to its linearization (1.9):

$$(\partial_t + T_V \cdot \nabla + iT_\gamma)\, u = f \in L^\infty H^s, \quad \gamma \in \Gamma^{\frac{3}{2}}, \tag{1.13}$$

from which the local wellposedness was obtain at this regularity level (1.12). Using this reduction, Alazard-Burq-Zuily [2] established, for 2D waves, the semi-classical Strichartz estimate at the threshold (1.12) and the classical (optimal) Strichartz estimate when $s > 5 + \frac{1}{2}$. We remark that in [1], the semi-classical gain is achieved due to the fact that after a *para change of variables* (see [2, Proposition 3.4]), the highest order term $T_\gamma u$ in (1.13) is converted into the simple Fourier multiplier $|D_x|^{\frac{3}{2}}$. Unfortunately, such a reduction cannot work for the 3D case, and hence the semi-classical Strichartz estimate in this case is much more difficult to establish, especially at the regularity level (1.12). In the present paper, we aim to investigate the semi-classical Strichartz estimate for (1.2) when $d \geq 2$, assuming that the solution is slightly smoother than (1.12) (1/2 derivatives). Our second result reads as follows.

Theorem 1.3. *Let $d \geq 2$ and $0 < T < \infty$. Consider a solution (η, ψ) of (1.2) on the time interval $I = [0, T]$ such that Ω satisfies condition (H) and*

$$(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)).$$

If $s > \frac{5}{2} + \frac{d}{2}$, then for every $\varepsilon > 0$, there holds

$$(\eta, \psi) \in L^2([0, T]; W^{s+1-\varepsilon-\frac{d}{2},\infty}(\mathbf{R}^d) \times W^{s+\frac{1}{2}-\varepsilon-\frac{d}{2},\infty}(\mathbf{R}^d)).$$

Remark 1.4. Our proof of Theorem 1.3 works equally for the 2D waves ($d = 1$) when $(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R}))$ with $s > \frac{5}{2} + \frac{1}{2}$. On the other hand, using the paracomposition reduction of Proposition 3.4 in [2] we can indeed improve the preceding regularity to $s > 2 + \frac{1}{2}$, which is the same as Theorem 1.1 in [2].

Remark 1.5. Theorem 1.3 still holds for capillary waves, i.e., $(\sigma, g) = (1, 0)$, because the reduction in Theorem 2.1 below is still valid for capillary waves, and the gravity term $g\eta$ appearing in (1.2) contributes to a lower order term.

1.2.3. *On the proof of the main results.* In [4, 11] the authors worked completely in the semi-classical formalism and proved dispersive estimates using the approximation WKB method. This allowed the authors to prove Strichartz estimates with nontrivial gains even for very rough backgrounds. However, we emphasize that with this method, we were not able to reach the classical or semi-classical level as in Theorems 1.2 and 1.3. The dispersive estimates for principally normal pseudo-differential operators in [19] require more regularity (C^2) of the symbols to control the Hamiltonian flow and apply the FBI transform technique. This allows us to obtain sharp dispersive estimates when the characteristic set of the symbol has maximal nonvanishing principal curvatures.

For the proof of our main results, we shall combine the paradifferential reduction in the works of Alazard-Burq-Zuily with the phase transform method in the work [19] of Koch-Tataru. Notice that the latter works effectively for operators of order 1 after renormalizing. For gravity-capillary waves (see (1.13)) the dispersive term has order $\frac{3}{2}$, and thus the semi-classical time-scale brings it to the one of order 1 and hence leads to the semi-classical Strichartz estimate in Theorem 1.3. For the pure gravity waves (1.6), one observes that the dispersive term iT_γ has order $\frac{1}{2}$, which is lower than that of the transport term $T_V \cdot \nabla$. Here, we follow [4], suppressing this transport term by straightening the vector field $\partial_t + T_V \cdot \nabla$ and then making another change of spatial variables to convert it to an operator of order 1. However, the new symbol then is not in the standard form $p(x,\xi)$ to apply phase transforms, and other technical issues appear. Thus, the proof of Theorem 1.2 requires much more care.

2. Preliminaries

2.1. **Symmetrization of system** (1.3)**.** We first recall the symmetrization of system (1.3) to a single quasilinear equation, performed in [1]. This reduction requires the following symbols:

- Symbols of the Dirichlet-Neumann operator

$$\lambda^{(1)} := \sqrt{(1+|\nabla\eta|^2)|\xi|^2 - (\nabla\eta\cdot\xi)^2},$$

$$\lambda^{(0)} := \frac{1+|\nabla\eta|^2}{2\lambda^{(1)}}\left\{\operatorname{div}(\alpha^{(1)}\nabla\eta) + i\nabla_\xi\lambda^{(1)}\cdot\nabla_x\alpha^{(1)}\right\}, \quad \alpha^{(1)} := \frac{\lambda^{(1)} + i\nabla\eta\cdot\xi}{1+|\nabla\eta|^2}.$$

- Symbol of the mean-curvature operator:

$$\ell^{(2)} := (1+|\nabla\eta|^2)^{-\frac{1}{2}}\left(|\xi|^2 - \frac{(\nabla\eta\cdot\xi)^2}{1+|\nabla\eta|^2}\right).$$

- Symbols using for symmetrization

$$q := \left(1+(\nabla_x\eta)^2\right)^{-\frac{1}{2}}, \quad p = \left(1+(\nabla_x\eta)^2\right)^{-\frac{5}{4}}|\xi|^{\frac{1}{2}} + p^{(-\frac{1}{2})},$$

  where $p^{(-\frac{1}{2})} := F(\nabla_x\eta,\xi)\partial_x^\alpha\eta$, with $|\alpha| = 2$ and $F \in C^\infty(\mathbf{R}^d\times\mathbf{R}^d\setminus\{0\};\mathbf{C})$ homogeneous of order $-\frac{1}{2}$ in ξ.
- Symbols in the symmetrized equation:

$$\gamma := \sqrt{l^{(2)}\lambda^{(1)}} = \left(\frac{|\xi|^2(1+|\nabla\eta|^2) - (\nabla\eta\cdot\xi)^2}{1+|\nabla\eta|^2}\right)^{\frac{3}{4}},$$

$$\omega := -\frac{i}{2}(\partial_\xi\cdot\partial_x)\sqrt{l^{(2)}\lambda^{(1)}}, \quad \omega_1 := \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}}\frac{\Re\lambda^{(0)}}{2}.$$

Define the *good unknown* $U := \psi - T_B\eta$. The following result was proved in [1].

Theorem 2.1 ([1, Corollary 4.9])**.** *Let $s > 2+\frac{d}{2}$ and let $(\eta,\psi) \in C^0([0,T]; H^{s+\frac{1}{2}}\times H^s)$ be a solution to* (1.2) *such that Ω satisfies condition* (*H*)*. The complex-valued function $u := T_p\eta + iT_qU$ then solves the paradifferential equation*

$$\partial_t u + T_V\cdot\nabla u + iT_{\gamma+\omega+\omega_1}u = f, \tag{2.1}$$

where there exists a nondecreasing function $\mathcal{F} : \mathbf{R}^+\times\mathbf{R}^+ \to \mathbf{R}^+$ independent of (η,ψ) such that

$$\|f\|_{L^\infty([0,T];H^s)} \le \mathcal{F}\left(\|(\eta,\psi)\|_{L^\infty([0,T];H^{s+\frac{1}{2}}\times H^s)}\right). \tag{2.2}$$

2.2. **Symmetrization of system** (1.2)**.** Define first the principle symbol of the Dirichlet-Neumann operator

$$\lambda = \Big(\big(1 + |\nabla\eta|^2\big)|\xi|^2 - \big(\xi \cdot \nabla\eta\big)^2 \Big)^{\frac{1}{2}}.$$

Next, set $\zeta = \nabla\eta$ and introduce

$$U_s := \langle D_x\rangle^s V + T_\zeta \langle D_x\rangle^s B, \quad \zeta_s := \langle D_x\rangle^s \zeta.$$

Theorem 2.2 ([3, Proposition 4.10])**.** *Let $s > 1 + \frac{d}{2}$ and let*

$$(\eta, \psi) \in C^0([0,T]; H^{s+\frac{1}{2}} \times H^s)$$

be a solution to (1.3) *such that condition* (H) *is fulfilled and the velocity trace is*

$$(B, V) \in C^0([0,T]; H^{s+\frac{1}{2}} \times H^s),$$

and there exists $c_0 > 0$ such that $\mathfrak{a}(t,x) \geq c_0$ for all $(t,x) \in [0,T] \times \mathbf{R}^d$. Then the complex-valued function

$$u := \langle D_x\rangle^{-s} \big(U_s - iT_{\sqrt{\mathfrak{a}/\lambda}}\zeta_s\big)$$

solves the paradifferential equation

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = f, \tag{2.3}$$

where $\gamma = \sqrt{\mathfrak{a}\lambda}$ and

$$\|\mathfrak{a} - g\|_{L^\infty([0,T];H^{s-\frac{1}{2}})} + \|f\|_{L^\infty([0,T];H^s)} \leq \mathcal{F}\left(\|(\eta,\psi)\|_{H^{s+\frac{1}{2}}}, \|(V,B)\|_{H^s} \right).$$

Remark 2.3. The change of variables $(\eta, \psi) \mapsto u$ in Theorem 2.1 and $(\eta, \psi, B, V) \mapsto u$ in Theorem 2.2 are essentially "invertible" in the sense that one can recover Sobolev estimates and Hölder estimates for (η, ψ, B, V) from those for u by virtue of the symbolic calculus for paradifferential operators contained in Theorem A.5.

2.3. **Para- and pseudo-differential operators.** Since the paradifferential setting is not suitable for proving dispersive estimates, we shall change it into the pseudo-differential setting, whose standard definitions are recalled here.

Definition 2.4.

(1) For any $m \in \mathbf{R}$, $0 \leq \delta_1, \delta_2, \ \rho \leq 1$, we denote by $S^m_{\rho,\delta_1,\delta_2}$ the class of all symbols $a(x,y,\xi) : (\mathbf{R}^d)^3 \to \mathbf{C}$ satisfying

$$\Big|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x,y,\xi)\Big| \leq C_{\alpha,\beta,\gamma}(1 + |\xi|)^{m+\delta_1|\alpha|+\delta_2|\beta|-\rho|\gamma|}.$$

The corresponding pseudo-differential operator is defined by

$$\mathrm{Op}(a)u(x) = \int_{\mathbf{R}^d} e^{i(x-y)\xi} a(x,y,\xi) u(y) dy d\xi.$$

When $a : (\mathbf{R}^d)^2 \to \mathbf{C}$ we consider it as a symbol in $S^m_{\rho,\delta_1,0}$ that does not depend on y and rename $S^m_{\rho,\delta_1,0} \equiv S^m_{\rho,\delta_1}$.

(2) For any symbol $a(x,\xi) \in S^m_{\rho,\delta}$ the Weyl quantization $\mathrm{Op}^w(a) \equiv a^w(x, D_x)$ is defined by $\mathrm{Op}^w(a)u(x) = \mathrm{Op}(b)u(x)$ with $b(x,y,\xi) := a(\frac{x+y}{2}, \xi) \in S^m_{\rho,\delta,\delta}$.

We shall later need to transform the operators $\mathrm{Op}(a)$ to $\mathrm{Op}^w(a)$. This is done by means of the following result, which can be easily deduced from [25, Proposition 0.3.A].

Proposition 2.5. *For any symbol $a \in S^m_{\rho,\delta}$ with $m \in \mathbf{R}$, $0 \le \delta < \rho \le 1$, there exists a symbol $b \in S^m_{\rho,\delta}$ such that $\mathrm{Op}^w(a) = \mathrm{Op}(b)$. Moreover, we have the following asymptotic expansion in the sense of symbolic calculus:*

$$b(x,\xi) \sim \sum_{|\alpha|\ge 0} \frac{(-i)^{|\alpha|}}{\alpha! 2^{|\alpha|}} \partial_x^\alpha \partial_\xi^\alpha a(x,\xi).$$

Remark that for all $\alpha \in \mathbf{N}^d$, $\partial_x^\alpha \partial_\xi^\alpha a(x,\xi) \in S^{m-(\rho-\delta)|\alpha|}_{\rho,\delta}$.

Now, let $a \in \Gamma^m_r$, $r > 0$, be a paradifferential symbol (see Definition A.3) and define

$$\forall j \in \mathbf{Z},\ \forall \delta > 0, \quad S_{j\delta}(a)(x,\xi) = \psi(2^{-j\delta} D_x) a(x,\xi), \tag{2.4}$$

the spatial regularization of the symbol a, where ψ is the Littlewood-Paley function defined in (A.1). We first prove a Bernstein's type inequality for $S_{j\delta}(a)$.

Lemma 2.6. *If $a \in \Gamma^m_\rho$, then for all $\alpha, \beta \in \mathbf{N}^d$, $|\alpha| \ge \rho$, there exists a constant $C_{\alpha,\beta}$ such that for all $(x,\xi) \in \mathbf{R}^{2d}$,*

$$|\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x,\xi)| \le C_{\alpha,\beta} 2^{j\delta(|\alpha|-\rho)} \|\partial_\xi^\beta a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)}.$$

Proof. If $|\alpha| = \rho$ the estimate is obvious by writing $\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)$ as a convolution of $\partial_x^\alpha \partial_\xi^\beta a$ with a kernel. Consider now the case $|\alpha| > \rho$. Recall the dyadic partition of unity (A.2): $1 = \sum_{k=0}^\infty \Delta_k$ where each Δ_k is spectrally supported in the annulus $\{2^{k-1} \le |\xi| \le 2^{k+1}\}$. Using this partition, we can write

$$\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x,\xi) = \sum_{k=0}^{+\infty} \Delta_k \partial_x^\alpha \psi(2^{-j\delta} D_x) \partial_\xi^\beta a(x,\xi) := \sum_{k=0}^{+\infty} u_k.$$

If $\frac{1}{2} 2^k \ge 2^{j\delta}$, then $\Delta_k \psi(2^{-j\delta} D_x) = 0$. Therefore

$$\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x,\xi) = \sum_{k=0}^{2+[j\delta]} u_k.$$

Now, introducing $\varphi_1(\xi) \in C_c^\infty(\mathbf{R}^d)$, supported in $\{\frac{1}{3} \le \xi| \le 3\}$ one has

$$u_k = 2^{k|\alpha|} \varphi_1(2^{-k} D_x) \psi(2^{-j\delta} D_x) \Delta_k \partial_\xi^\beta a(x,\xi).$$

Consequently,

$$\|u_k\|_{L^\infty(\mathbf{R}^d)} \le 2^{k|\alpha|} \|\Delta_k D_\xi^\beta a(\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \le C 2^{k|\alpha|} 2^{-k\rho} \|\partial_\xi^\beta a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)}.$$

It follows that

$$\|\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x,\xi)\|_{L^\infty(\mathbf{R}^d)} \le C \sum_{k=0}^{2+[j\delta]} 2^{k(|\alpha|-\rho)} \|D_\xi^\beta a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)}.$$

Finally, since $|\alpha| - \rho > 0$ we deduce that

$$\|\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x,\xi)\|_{L^\infty(\mathbf{R}^d)} \le C 2^{j\delta(|\alpha|-\rho)} \|\partial_\xi^\beta a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)},$$

which concludes the proof. □

We show in the next proposition that after localizing a distribution u in frequency, one can go from paradifferential operators to pseudo-differential operators when acting on u.

Proposition 2.7. *For every $j \in \mathbf{N}^*$, define*

$$R_j u := T_a \Delta_j u - S_{j-3}(a)(x, D_x)\Delta_j u.$$

Then $R_j u$ is spectrally supported (*see Definition* A.2) *in an annulus $\{c_1^{-1} 2^{j-1} \leq |\xi| \leq c_1 2^{j+1}\}$, and for every $\mu \in \mathbf{R}$ we have*

$$\|R_j u\|_{H^{\mu-m+r}(\mathbf{R}^d)} \leq C M_r^m(a) \|u\|_{H^\mu(\mathbf{R}^d)},$$

where the constants c_1, $C > 0$, are independent of a, u, j.

Proof. Recall first the definition (A.5) of $T_a u$, where we have $\varrho = 1$ on the support of φ_j (see Definition A.1) for any $j \geq 1$, so

$$R_j u = T_a \Delta_j u - S_{j-3}(a)(x, D_x)\varrho(D_x)\Delta_j u.$$

In the following proof we shall use the presentation of Métivier [22] on pseudo-differential and paradifferential operators. To be compatible with [22] we also abuse notation: by Γ_r^m we denote the class of symbols a satisfying the growth condition (A.3) for any $\xi \in \mathbf{R}^d$ and by M_0^m the semi-norm (A.4) where the supremum is taken over $\xi \in \mathbf{R}^d$.

(1) By definition (A.5) it holds that $T_a v = \mathrm{Op}(\sigma_a \varrho) v$, where $\mathrm{Op}(\sigma_a \varrho)$ denotes the classical pseudo-differential operator with symbol

$$\sigma_a(x,\xi)\varrho(\xi) = \chi(D_x, \xi) a(x,\xi)\varrho(\xi).$$

Hence $R_j u = \mathrm{Op}(a_j) u$ with

$$a_j(x,\xi) = \sigma_a(x,\xi)\varrho(\xi)\varphi_j(\xi) - S_{j-3}(a)(x,\xi)\varrho(\xi)\varphi_j(\xi).$$

Now, we write

$$a_j = \big(\sigma_a \varrho \varphi_j - a \varrho \varphi_j\big) + \big(a\varrho\varphi_j - S_{j-3}(a)\varrho\varphi_j\big) = a_j^1 + a_j^2.$$

Applying Proposition 5.8(ii) in [22] gives $a_j^1 \in \Gamma_0^{m-r}$ and (remark that $(\varphi_j)_j$ is bounded in Γ_r^0)

$$M_0^{m-r}(a_j^1) \leq C M_r^m(a\varrho\varphi_j) \leq C M_r^m(a\rho).$$

On the other hand, if we denote $b = a\varrho\varphi$, then $a_j^2(x,\xi) = b(x,\xi) - \psi_{j-3}(D_x,\xi) b(x,\xi)$. Taking into account the fact that $\operatorname{supp}\varphi_j \subset B(0, C2^j)$ we may estimate

$$\begin{aligned}
|a_j^2(x,\xi)| &\leq \sum_{k \geq j-2} |\Delta_j b(x,\xi)| \leq \sum_{k\geq j-2} 2^{-kr} \|b(\cdot,\xi)\|_{W^{r,\infty}} \\
&\leq C 2^{-jr} \|b(\cdot,\xi)\|_{W^{r,\infty}} = C2^{-jr} |\varphi_j(\xi)| \, \|a(\cdot,\xi)\varrho(\xi)\|_{W^{r,\infty}} \\
&\leq C(1+|\xi|)^{m-r} M_r^m(a\varrho), \quad \forall \xi \in \mathbf{R}^d.
\end{aligned}$$

Estimates for $|\partial_\xi^\alpha a_j^2|$ can be derived along the same lines. Thus, $a_j^2 \in \Gamma_0^{m-r}$ and hence $a_j \in \Gamma_0^{m-r}$; moreover

$$M_0^{m-r}(a_j) \leq C M_r^m(a\varrho).$$

(2) Property (A.7) implies in particular that

$$\mathfrak{F}_x(\sigma_a)(\eta,\xi) = 0 \quad \text{for } |\eta| \geq \varepsilon_2(1+|\xi|).$$

Here, we denote by $\mathfrak{F}_x$ the Fourier transform with respect to the spatial variable x.

On the other hand, by definition of the smoothing operator,

$$\mathfrak{F}_x S_{j-3}(a)(x,\xi)\varrho(\xi)\varphi_j(\xi) = \psi(2^{-(j-3)}\eta)\mathfrak{F}_x a(\eta,\xi)\varrho(\xi)\varphi(2^{-j}\xi),$$

which is vanishing if $|\eta| \geq \frac{1}{2}(1+|\xi|)$. Indeed, if either $|\xi| > 2^{j+1}$ or $|\xi| \leq 2^{j-1}$, then $\varphi(2^{-j}\xi) = 0$. Considering $2^{j-1} < |\xi| \leq 2^{j+1}$ then $|\eta| \geq \frac{1}{2}(1+|\xi|) > 2^{j-2}$ and thus $\psi(2^{-(j-3)}\eta) = 0$. We have proved the existence of $0 < \varepsilon < 1$ such that

$$\mathfrak{F}_x a_j(\eta, \xi) = 0 \quad \text{for } |\eta| \geq \varepsilon(1+|\xi|). \tag{2.5}$$

(3) By the spectral property (2.5) one can use Bernstein's inequalities (see [22, Corollary 4.1.7]) to prove that a_j is a pseudo-differential symbol in the class $S^{m-r}_{1,1}$. Then, applying [22, Theorem 4.3.5] we conclude that

$$\|R_j u\|_{H^{\mu-m+r}(\mathbf{R}^d)} = \|\mathrm{Op}(a_j)u\|_{H^{\mu-m+r}(\mathbf{R}^d)} \leq C M_0^{m-r}(a_j) \|u\|_{H^{\mu}(\mathbf{R}^d)}.$$

Finally, the Fourier transform of $R_j u$ reads

$$\mathfrak{F}(R_j u)(\xi) = \int_{\mathbf{R}^d} \mathfrak{F}_x(a_j)(\xi - \eta, \eta)\hat{u}(\eta) d\eta.$$

Using the spectral localization property (2.5) and the fact that $\mathfrak{F}_x(a_j)(\xi - \eta, \eta)$ contains the factor $\varphi_j(\eta)$ we conclude that the spectrum of $R_j u$ is contained in an annulus of size 2^j as claimed. □

2.4. **A result of Koch-Tataru.** In this section, we recall the dispersive estimates proved by Koch-Tataru [19] based on the technique of FBI transform on phase space. These estimates were established for the following class of operators.

Definition 2.8. For $\lambda > 1$, $m \in \mathbf{R}$, and $k = 0, 1, \ldots$ consider classes of symbols $p : T^*\mathbf{R}^d \to \mathbf{C}$, denoted by $\lambda^m S^k_\lambda$, which satisfy

$$\begin{aligned} \left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\right| &\leq c_{\alpha,\beta} \lambda^{m-|\beta|} \quad |\alpha| \leq k, \\ \left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\right| &\leq c_{\alpha,\beta} \lambda^{m+\frac{|\alpha|-k}{2}-|\beta|} \quad |\alpha| \geq k. \end{aligned} \tag{2.6}$$

The mentioned result reads as follows.

Proposition 2.9 ([19, Proposition 4.7])**.** *Let $p(\sigma, x, \xi) \in \lambda S^2_\lambda$ be a real symbol in (x, ξ), uniformly in $\sigma \in [0,1]$. Assume that p satisfies the following curvature condition:*

(**A**) *For each $(\sigma, x, \xi) \in [0,1] \times \mathbf{R}^d \times \mathcal{C}_\lambda$, $|\det \partial_\xi^2 p| \gtrsim \lambda^{-d}$, where $\mathcal{C}_\lambda = \{c^{-1}\lambda \leq |\xi| \leq c\lambda\}$.*

Denote by $S(\sigma, \sigma_0)$ the flow maps of $D_\sigma + \mathrm{Op}^w(p)$. Then for any $\chi \in S^0_\lambda$ such that for all $x \in \mathbf{R}^d$, $\chi(x, \cdot)$ compactly supported in $\mathcal{C}'_\lambda = \{c'^{-1}\lambda \leq |\xi| \leq c'\lambda\}$ with $1 < c' < c$, we have

$$\|S(\sigma, \sigma_0)(\chi(x, D_x)v_0)\|_{L^\infty} \lesssim \lambda^{\frac{d}{2}} |\sigma - \sigma_0|^{\frac{-d}{2}} \|v_0\|_{L^1} \qquad \forall \sigma,\ \sigma_0 \in [0,1].$$

Remark 2.10. In the statement of [19, Proposition 4.7], condition (**A**) is stated for $(x, \xi) \in B_\lambda := \{|x| \leq 1,\ |\xi| \leq \lambda\}$, and correspondingly, χ is supported in B. In addition, the usual quantization $\chi(x, D_x)$ is replaced by the Weyl quantization χ^w. However, one can easily inspect its proof to see that if (**A**) is fulfilled globally in x, then we have the above variant.

2.5. **Remarks on the symbolic calculus for $\lambda^m S^k_\lambda$.** Let $a \in \lambda^m S^k_m,\ k \geq 0$, and suppose that on the support of $a(x,\xi)$, $\lambda^{-1}|\xi| \sim 1$. It follows by definition of $\lambda^m S^k_\lambda$ that $a \in S^m_{1,\frac{1}{2}}$. Observe however that when $k \geq 1$, a behaves better than a general symbol in $S^m_{1,\frac{1}{2}}$. In this section we present some enhanced properties of $\lambda^m S^k_\lambda$ with $k \geq 1$.

First, we are concerned with the relation between the Weyl quantization and the usual quantization. According to Proposition 2.5, for $a \in S^m_{1,\frac{1}{2}}$ there holds

$$\mathrm{Op}^w(a) - \mathrm{Op}(a) = \mathrm{Op}(r), \quad r \in S^{m-\frac{1}{2}}_{1,\frac{1}{2}}.$$

In fact, we have

$$\mathrm{Op}^w(a) = \mathrm{Op}(\widetilde{a}), \quad \widetilde{a}(x,y,\xi) = a(\frac{x+y}{2},\xi)$$

and

$$\widetilde{a}(x,y,\xi) = a(x+\frac{y-x}{2},\xi) = a(x,\xi) + \frac{1}{2}\int_0^1 (\partial_x a)(x+s\frac{y-x}{2},\xi)ds\,(y-x).$$

It follows that

$$\mathrm{Op}^w(a) - \mathrm{Op}(a) = \mathrm{Op}(r), \tag{2.7}$$

with

$$r(x,y,\xi) = -\frac{i}{2}\int_0^1 (\partial_\xi\partial_x a)(x+s\frac{y-x}{2},\xi)ds. \tag{2.8}$$

We now show that in fact r is of order $m-1$ as in the case $a \in S^m_{1,0}$.

Lemma 2.11. *Let $a \in \lambda^m S^k_\lambda,\ k \geq 1$, satisfy $\lambda^{-1}|\xi| \sim 1$ on the support of $a(x,\xi)$. Then we have the relation (2.7)-(2.8) with $r \in S^{m-1}_{1,\frac{1}{2},\frac{1}{2}}$.*

Proof. For all $\alpha,\beta,\nu \in \mathbf{N}^d$ we have

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\nu r(x,y,\xi)| \leq \begin{cases} C_{\alpha,\beta,\nu}\lambda^{m-|\nu|-1}\mathbb{1}_{\lambda^{-1}|\xi|\sim 1}(\xi) & \text{if } |\alpha|+|\beta|+1 \leq k, \\ C_{\alpha,\beta,\nu}\lambda^{m+\frac{|\alpha|+1+|\beta|-k}{2}-|\nu|-1}\mathbb{1}_{\lambda^{-1}|\xi|\sim 1}(\xi) & \text{if } |\alpha|+|\beta|+1 \geq k. \end{cases}$$

Since $k \geq 1$,

$$\frac{|\alpha|+1+|\beta|-k}{2} \leq \frac{|\alpha|+|\beta|}{2}.$$

Consequently, it holds that

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\nu r(x,y,\xi)| \leq C_{\alpha,\beta,\nu}(1+|\xi|)^{(m-1)+\frac{|\alpha|+|\beta|}{2}-|\nu|} \quad \forall \alpha,\beta,\nu \in \mathbf{N}^d.$$

In other words, $r \in S^{m-1}_{1,\frac{1}{2},\frac{1}{2}}$. □

For the composition rule we prove the following lemma.

Lemma 2.12. *Let $p \in S^n_{1,0}$ and $a \in \lambda^m S^k_\lambda,\ k \geq 1$, satisfy $\lambda^{-1}|\xi| \sim 1$ on the support of $a(x,\xi)$. Then we have*

$$\mathrm{Op}(p) \circ \mathrm{Op}(a) - \mathrm{Op}(pa) = \mathrm{Op}(r)$$

with $r \in S^{n+m-1}_{1,\frac{1}{2}}$.

Proof. According to [25, Proposition 0.3.C], one has $\mathrm{Op}(p) \circ \mathrm{Op}(a) = \mathrm{Op}(b)$ with

$$b \sim \sum_{|\alpha| \geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x,\xi) \cdot \partial_x^\alpha a(x,\xi)$$

in the sense of symbol asymptotic. The general term in the above expansion belongs to $S^{n+m-\frac{|\alpha|}{2}}_{1,\frac{1}{2}}$; hence

$$(b - pa) - \sum_{|\alpha|=1} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x,\xi) \cdot \partial_x^\alpha a(x,\xi) \in S^{n+m-1}_{1,\frac{1}{2}}.$$

It then suffices to show that $c_\alpha := \partial_\xi^\alpha p(x,\xi) \cdot \partial_x^\alpha a(x,\xi) \in S^{n+m-1}_{1,\frac{1}{2}}$ for $|\alpha| = 1$ or, again, $\partial_x^\alpha a(x,\xi) \in S^m_{1,\frac{1}{2}}$ for $|\alpha| = 1$. The latter follows along the same lines as in the proof of Lemma 2.11. □

In the same spirit we have the following result on adjoint operators, taking into account [25, Proposition 0.3.B].

Lemma 2.13. *Let $a \in \lambda^m S^k_\lambda$, $k \geq 1$, satisfy $\lambda^{-1}|\xi| \sim 1$ on the support of $a(x,\xi)$. Then we have*

$$\mathrm{Op}^*(a) - \mathrm{Op}(\bar{a}) = \mathrm{Op}(r),$$

with $r \in S^{m-1}_{1,\frac{1}{2}}$ and $\bar{a}$ the complex conjugate of a.

Notation 2.14. Throughout this article, we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$, where C may depend on the coefficients of the equations under consideration. If the constant C involved has some explicit dependency, say, on some quantity μ, we emphasize this by denoting $A \lesssim_\mu B$.

3. Proof of Theorem 1.3

Throughout this section, the dimension d is greater than or equal to 2, and $s > \frac{5}{2} + \frac{d}{2}$ is a Sobolev index.

3.1. **Littlewood-Paley reduction.**

We shall prove Strichartz estimates for solution u to (2.1), which is a quasilinear paradifferential equation with time-dependent coefficients. Remark that since

$$(\eta, \psi) \in C^0([0,T]; H^{s+\frac{1}{2}} \times H^s),$$

we have

$$V \in C^0([0,T]; H^{s-1}), \quad \gamma(\cdot,\xi) \in C^0([0,T]; H^{s-\frac{1}{2}}).$$

The first step in our proof consists of localizing (2.1) at frequency 2^j using the Littlewood-Paley decomposition (cf. Definition A.1(1)). For every $j \geq 0$, the dyadic piece $\Delta_j u$ solves

$$(\partial_t + T_V \cdot \nabla + iT_{\gamma+\omega}) \Delta_j u = F_j^1, \tag{3.1}$$

with

$$F_j^1 := \Delta_j f - i\Delta_j(T_{\omega_1} u) + i\,[T_\gamma, \Delta_j]\, u + i\,[T_\omega, \Delta_j]\, u + [T_V, \Delta_j] \cdot \nabla u. \tag{3.2}$$

Remark that for each $j \geq 1$, $\Delta_j u$ is spectrally supported in $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. In view of Proposition 2.7 and the facts that $\gamma \in \Gamma^{\frac{3}{2}}_{\frac{3}{2}}$, $\omega \in \Gamma^{\frac{1}{2}}_{\frac{1}{2}}$, and $V \cdot \xi \in \Gamma^1_1$, equation (3.1) can be rewritten as

$$(\partial_t + S_{j-3}(V) \cdot \nabla + iS_{j-3}(\gamma)(x, D_x))\, \Delta_j u = F^2_j, \tag{3.3}$$

with

$$F^2_j := F^1_j + R_j; \tag{3.4}$$

$R_j u$ is spectrally supported in an annulus $\{c_1^{-1} 2^{j-1} \leq |\xi| \leq c_1 2^{j+1}\}$ and satisfies

$$\|R_j\|_{H^s} \lesssim \|u\|_{H^s}. \tag{3.5}$$

Next, we use (2.4) to smooth out the symbols by $\delta = \frac{1}{2}$ derivative.

Now, let $\frac{1}{2} < c_1 < c_2 < c_3$, $\mathcal{C}^k := \{(2c_k)^{-1} \leq |\xi| \leq 2c_k\}$, $k = 1, 2, 3$, and

$$\widetilde{\varphi} \in C^\infty, \quad \operatorname{supp} \widetilde{\varphi} \subset \mathcal{C}^3, \quad \widetilde{\varphi} \equiv 1 \text{ on } \mathcal{C}^2.$$

Then, equation (3.3) is equivalent to

$$\begin{aligned} \mathcal{L}_j \Delta_j u := \Big(&\partial_t + S_{(j-3)\frac{1}{2}}(V) \cdot \nabla \widetilde{\varphi}(2^{-j} D_x) \\ &+ iS_{(j-3)\frac{1}{2}}(\gamma)(x, D_x)\widetilde{\varphi}(2^{-j} D_x) + iS_{(j-3)\frac{1}{2}}(\omega)(x, D_x)\widetilde{\varphi}(2^{-j} D_x)\Big)\, \Delta_j u = F_j, \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} F_j = F^2_j + F^3_j &:= F^2_j + i\left(S_{(j-3)\frac{1}{2}}\gamma(x, D_x) - S_{j-3}\gamma(x, D_x)\right)\Delta_j u \\ &+ i\left(S_{(j-3)\frac{1}{2}}\omega(x, D_x) - S_{j-3}\omega(x, D_x)\right)\Delta_j u + \left(S_{(j-3)\frac{1}{2}}(V) - S_{j-3}(V)\right) \cdot \nabla \Delta_j u. \end{aligned} \tag{3.7}$$

3.2. **Semi-classical time-scale.** Observe that the highest order operator on the left-hand side of (3.3) has order $\frac{3}{2}$, which does not match the result in [19] that we want to apply. Therefore, we reduce it to an operator of order 1 by multiplying both sides by $h^{\frac{1}{2}}$, where

$$h := 2^{-j},$$

then making a change of temporal variables $t = h^{\frac{1}{2}}\sigma$. For this purpose, let us reset the symbols in this new time-scale:

$$\Gamma_h(\sigma, x, \xi) = h^{\frac{1}{2}} S_{(j-3)\frac{1}{2}}(\gamma)(h^{\frac{1}{2}}\sigma, x, \xi)\widetilde{\varphi}(h\xi), \tag{3.8}$$

$$\Omega_h(\sigma, x, \xi) = h^{\frac{1}{2}} S_{(j-3)\frac{1}{2}}(\omega)(h^{\frac{1}{2}}\sigma, x, \xi)\widetilde{\varphi}(h\xi), \tag{3.9}$$

$$V_h(\sigma, x, \xi) = h^{\frac{1}{2}} S_{(j-3)\frac{1}{2}}(V)(h^{\frac{1}{2}}\sigma, x) \cdot \xi\widetilde{\varphi}(h\xi). \tag{3.10}$$

Next, set

$$w_h(\sigma, x) = \Delta_j u(h^{\frac{1}{2}}\sigma, x), \quad G_h(\sigma, x) = -ih^{\frac{1}{2}} F_j(h^{\frac{1}{2}}\sigma, x).$$

Equation (3.6) is then equivalent to

$$(D_t + \Gamma_h(\sigma, x, D_x) + \Omega_h(\sigma, x, D_x) + V_h(\sigma, x, D_x))\, w_h(\sigma, x) = G_h(\sigma, x). \tag{3.11}$$

In what follows, we shall prove the classical Strichartz estimate for (3.11), from which the semi-classical Strichartz estimate for (3.6) follows.

We now replace the pseudo-differential operators in (3.11) by the corresponding Weyl operators using Proposition 2.5. Denote

$$R^1_h := \big(\operatorname{Op}(\Gamma_h) + \operatorname{Op}(\Omega_h)\big) - \operatorname{Op}^w(\Gamma_h), \qquad (3.12)$$
$$R^2_h := \operatorname{Op}(V_h) - \operatorname{Op}^w(V_h).$$

Let us note that $\omega = -\frac{i}{2}\partial_\xi \cdot \partial_x \gamma$. With this notation (3.11) becomes

$$\begin{aligned} L_h w_h(\sigma, x) &:= \left(D_t + \Gamma^w_h(\sigma, x, D_x) + V^w_h(\sigma, x, D_x)\right) w_h(\sigma, x) \\ &= G_h(\sigma, x) - R^1_h(\sigma, x) w_h(\sigma, x) - R^2_h(\sigma, x) w_h(\sigma, x). \end{aligned} \qquad (3.13)$$

3.3. **Classical Strichartz estimate for** (3.11) **(**$\Leftrightarrow$ (3.13)**).** In this step, we show that Proposition 2.9 can be applied to the real symbol

$$p_h := \Gamma_h + V_h.$$

Set $\lambda = h^{-1} = 2^j$. First, the characteristic set of γ has maximal (d) nonvanishing principal curvatures.

Proposition 3.1. *Let $\mathcal{C}$ be a fixed annulus in $\mathbf{R}^d$.*

(1) *There exists an absolute constant $C_d > 0$ such that with $c_0 = C_d(1 + \|\nabla \eta\|_{L^\infty(I\times\mathbf{R}^d)})$ we have*

$$\sup_{(t,x,\xi)\in I\times\mathbf{R}^d\times\mathcal{C}} \left|\det \partial^2_\xi \gamma(t, x, \xi)\right| \geq c_0. \qquad (3.14)$$

(2) *For any $0 < \delta \leq 1$ there exists $j_0 \in \mathbf{N}$ large enough such that*

$$\sup_{(t,x,\xi)\in I\times\mathbf{R}^d\times\mathcal{C}} \left|\det \partial^2_\xi S_{j\delta}(\gamma)(t, x, \xi)\right| \geq c_0. \qquad (3.15)$$

Proof. (1) For the proof of part (1), we refer to [4, Corollary 4.7]. Part (2) is a consequence of part (1), because $S_{j\delta}(\gamma)$ is a small perturbation of γ when j is large enough (see for instance [4, Proposition 4.5]). □

Lemma 3.2.

(1) *We have $\Gamma_h \in \lambda S^2_\lambda$, $V_h \in \lambda^{\frac{3}{4}} S^2_\lambda$, and hence $p_h \in \lambda S^2_\lambda$.*

(2) *There exists $h_0 > 0$ small enough such that for $0 < h \leq h_0$, the symbol p_h satisfies condition* (**A**) *in Proposition* 2.9 *with $\mathcal{C}_\lambda = \lambda\mathcal{C}^2$.*

Proof. (1) Since $\gamma \in W^{2,\infty}$ and $V \in W^{\frac{3}{2},\infty}$ in x, assertion (1) follows easily from Lemma 2.6 and the fact that on the support of $\widetilde{\varphi}(h\xi)$ we have $|\xi| \sim \lambda$.

(2) Let $\xi \in \lambda\mathcal{C}^2$. We have $\widetilde{\varphi}(h\xi) = 1$; hence $\partial^2_\xi V_h$ vanishes. On the other hand, noticing that γ is homogeneous of order $\frac{3}{2}$ in ξ, we have

$$\partial^2_\xi \Gamma_h(\sigma, x, \xi) = \partial^2_\xi \left(h^{\frac{1}{2}} S_{(j-3)\frac{1}{2}}(\gamma)(h^{\frac{1}{2}}\sigma, x, \xi)\right) = h\left(\partial^2_\xi S_{(j-3)\frac{1}{2}}(\gamma)\right)(h^{\frac{1}{2}}\sigma, x, h\xi).$$

Therefore, in view of (3.15), condition (**A**) is verified. □

Calling $S_h(\sigma, \sigma_0)$ the flow map of the evolution operator $L_h = D_\sigma + \operatorname{Op}^w(p_h)$ (see (3.13)), we have the following.

Lemma 3.3. *If v^0_h is spectrally supported in $\lambda\mathcal{C}^1 = \left\{(2c_1)^{-1}h^{-1} \leq |\xi| \leq 2c_1 h^{-1}\right\}$, then*

(i)

$$\left\|S_h(\sigma, \sigma_0) v^0_h\right\|_{L^\infty(\mathbf{R}^d)} \lesssim h^{-\frac{d}{2}} \left|\sigma - \sigma_0\right|^{-\frac{d}{2}} \left\|v^0_h\right\|_{L^1(\mathbf{R}^d)}$$

for all σ, $\sigma_0 \in [0, 1]$, and $0 < h \leq h_0$;

(ii) *with* $q > 2$ *and* $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$,

$$\left\|S_h(\cdot,0)v_h^0\right\|_{L^q([0,1],L^r)} \lesssim h^{\frac{-1}{q}} \left\|v_h^0\right\|_{L^2}.$$

Proof. In view of Lemma 3.2, (i) is a direct consequence of Proposition 2.9 if one chooses

$$\chi(\xi) \in C^\infty,\ \operatorname{supp}\chi \subset \{(2c_{1,2})^{-1}\lambda \le |\xi| \le 2c_{1,2}\lambda\},\ c_1 < c_{1,2} < c_2,\ \chi \equiv 1 \text{ in } \mathcal{C}^1.$$

For (ii) we remark that since $\mathrm{Op}^w(\Gamma_h)$ and $\mathrm{Op}^w(V_h)$ are self-adjoint, $S_h(\sigma,\sigma_0)$ is isometric in L^2. This combined with the dispersive estimate in (i) and a standard TT^* argument (see the abstract semi-classical Strichartz estimate in [28, Theorem 10.7]) yields (ii). □

Lemma 3.4. *For any* $\mu \in \mathbf{R}$*, the operators* $S_h(\sigma,\tau)$ *are bounded on* $H^\mu(\mathbf{R}^d)$ *uniformly in* $\tau,\sigma \in I$.

Proof. The proof proceeds using standard energy estimates. However, more care is required since we are not working on standard operators of the class $S^m_{1,0}$. Without loss of generality we assume that $\tau = 0$ and let $f(\sigma,x)$ be a solution of

$$(\partial_\sigma + i\,\mathrm{Op}^w(p_h))\, f(\sigma,x) = 0, \quad f(0,x) = f_0(x).$$

We first apply Lemma 2.11 to obtain

$$\mathrm{Op}^w(p_h) = \mathrm{Op}(p_h) + \mathrm{Op}(r_h), \quad r_h \in S^0_{1,\frac{1}{2},\frac{1}{2}}.$$

Then f solves the problem

$$(\partial_\sigma + i\,\mathrm{Op}(p_h) + i\,\mathrm{Op}(r_h))\, f(\sigma,x) = 0, \quad f(0,x) = f_0(x).$$

Let $\mu \in \mathbf{R}$ and set $f^\mu := \langle D_x\rangle^\mu f$. Then

$$\frac{d}{d\sigma}\left\|f^\mu\right\|^2_{L^2} = -i\langle(\mathrm{Op}(p_h) - \mathrm{Op}^*(p_h))\, f^\mu, f^\mu\rangle + 2\Re\langle F, f^\mu\rangle,$$

where

$$F := -i\left[\langle D_x, \mathrm{Op}(p_h)\rangle^\mu\right] f - i\langle D_x\rangle^\mu\,\mathrm{Op}(r_h) f.$$

According to Lemma 2.12, $[\langle D_x\rangle^\mu, \mathrm{Op}(p_h)] \in S^\mu_{1,\frac{1}{2}}$. This combined with the fact that $r_h \in S^0_{1,\frac{1}{2},\frac{1}{2}}$ gives

$$\left\|F\right\|_{L^2} \lesssim \left\|f\right\|_{H^\mu}.$$

On the other hand, since p_h is real, Lemma 2.13 implies that

$$\mathrm{Op}(p_h) - \mathrm{Op}^*(p_h) \in S^0_{1,\frac{1}{2}}.$$

Consequently,

$$\left\|\left(\mathrm{Op}(p_h^0) - \mathrm{Op}^*(p_h^0)\right) f^\mu\right\|_{L^2} \lesssim \left\|f^\mu\right\|_{L^2}.$$

Finally, we conclude using Gronwall's inequality that

$$\left\|f(\sigma)\right\|_{H^\mu} \lesssim \left\|f_0\right\|_{H^\mu} \quad \forall \sigma \in I.$$

□

Proposition 3.5. *If* w_h *is a solution to* (3.11) *with* $w_h(0) = w_h^0$ *and*

$$\operatorname{supp}\widehat{w_h},\ \operatorname{supp}\widehat{w_h^0} \subset \lambda\mathcal{C}^1,$$

then we have for any $\varepsilon > 0$,

$$\left\|w_h\right\|_{L^{2+\varepsilon}([0,1],W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} \lesssim_\varepsilon \left\|w_h^0\right\|_{H^s} + \left\|G_h\right\|_{L^1([0,1],H^s)} + h^{\frac{1}{2}}\left\|w_h\right\|_{L^1([0,1],H^s)}.$$

Proof. To simplify notation, we write $S_h(\sigma,\tau) = S(\sigma,\tau)$. If w_h is a solution to (3.11), it is also a solution to (3.13). By Duhamel's formula,

$$w_h(\sigma,0) = S_h(\sigma,0)w_h^0 + \int_0^\sigma S(\sigma,\tau)[G_h(\tau)]d\tau - \int_0^\sigma S(\sigma,\tau)[(R_h^1 w_h + R_h^2 w_h)(\tau)]d\tau.$$

Let us call (I) and (II), respectively, the first and the second integral on the right-hand side. Choosing c_1 large enough such that $\operatorname{supp}\widehat{G}_h \subset \lambda\mathcal{C}^1$, Lemma 3.3(ii) gives

$$\left\|S_h(\sigma,0)w_h^0\right\|_{L^q([0,1],L^r)} \lesssim h^{-\frac{1}{q}} \left\|w_h^0\right\|_{L^2}, \quad \|(I)\|_{L^q([0,1],L^r)} \lesssim h^{-\frac{1}{q}} \|G_h\|_{L^1([0,1];L^2)}.$$

For (II) we set

$$b_h^\alpha = \frac{(-i)^{|\alpha|}}{\alpha!2^{|\alpha|}}\partial_x^\alpha\partial_\xi^\alpha\Gamma_h(\sigma,x,\xi)\ |\alpha|\geq 2; \quad c_h^\alpha = \frac{(-i)^{|\alpha|}}{\alpha!2^{|\alpha|}}\partial_x^\alpha\partial_\xi^\alpha V_h(\sigma,x,\xi)\ |\alpha|\geq 1.$$

Since γ is $W^{2,\infty}$ in x, by applying Lemma 2.6 we have for $|\alpha|\geq 2$,

$$\left\|\partial_x^\mu\partial_\xi^\nu(\partial_x^\alpha\partial_\xi^\alpha\Gamma_h)\right\| \lesssim (1+|\xi|)^{1+\frac{|\mu|+|\alpha|-2}{2}-(|\nu|+|\alpha|)} \lesssim (1+|\xi|)^{-\frac{|\alpha|}{2}+\frac{|\mu|}{2}-|\nu|},$$

hence $b_h^\alpha \in S^{-\frac{|\alpha|}{2}}_{1,\frac{1}{2}}$. Similarly, as $V\in W^{1,\infty}$ it holds that $c_h^\alpha \in S^{-\frac{|\alpha|}{2}}_{1,\frac{1}{2}}$ for $|\alpha|\geq 1$. Taking $q>2$ and $\frac{2}{q}=\frac{d}{2}-\frac{d}{r}$, we claim that uniformly in $\tau\in[0,1]$,

$$\left\|S(\sigma,\tau)[(R_h^1 w_h)(\tau)]\right\|_{L^q_\sigma L^r_x} \lesssim h^{-\frac{1}{q}+1}\|w_h(\tau)\|_{L^2_x}. \tag{3.16}$$

Indeed, by the asymptotic expansion in Proposition 2.5,

$$R_h^1 = \sum_{|\alpha|=2}^{N-1}\mathrm{Op}(b_h^\alpha) + \mathrm{Op}(r_h^N), \qquad r_h^N \in S^{1-\frac{N}{2}}_{1,\frac{1}{2}} \quad \forall N\geq 3.$$

If we choose c_1 large enough, then each $\mathrm{Op}(b_h^\alpha w_h)(\tau)$ is spectrally supported in $\lambda\mathcal{C}^1$ (and so is w_h) so that Lemma 3.3(ii) can be applied to get

$$\|S(\sigma,\tau)[\mathrm{Op}(b_h^\alpha)w_h(\tau)]\|_{L^q_\sigma L^r_x} \leq h^{-\frac{1}{q}}\|\mathrm{Op}(b_h^\alpha)w_h\|_{L^2_x} \lesssim h^{-\frac{1}{q}+\frac{|\alpha|}{2}}\|w_h(\tau)\|_{L^2_x}. \tag{3.17}$$

For $\mathrm{Op}(r_h^N)$ we use the Sobolev embedding $H^{\frac{d}{2}}\hookrightarrow L^r$, $\forall r\in[2,+\infty)$, to estimate

$$\|S(\sigma,\tau)[\mathrm{Op}(r_h^N)w_h(\tau)]\|_{L^r_x} \lesssim \|S(\sigma,\tau)[\mathrm{Op}(r_h^N)w_h(\tau)]\|_{H_x^{\frac{d}{2}}}.$$

On the other hand, we know from Lemma 3.4 that $S(\sigma,\tau)$ is bounded from H^μ to H^μ uniformly in $\sigma,\ \tau\in[0,1]$ for all $\mu\in\mathbf{R}$. Hence

$$\|S(\sigma,\tau)[\mathrm{Op}(r_h^N)w_h(\tau)]\|_{L^r_x} \lesssim h^{-1+\frac{N}{2}-\frac{d}{2}}\|w_h(\tau)\|_{L^2_x}. \tag{3.18}$$

Choosing $N=N(d)$ large enough, we obtain the claim (3.16) from (3.17) and (3.18).

By the same argument, we have uniformly in $\tau\in[0,1]$,

$$\left\|S(\sigma,\tau)[(R_h^2 w_h)(\tau)]\right\|_{L^q_\sigma L^r_x} \lesssim h^{-\frac{1}{q}+\frac{1}{2}}\|w_h(\tau)\|_{L^2_x}.$$

Putting together the above estimates leads to

$$\|w_h\|_{L^qL^r} \leq h^{-\frac{1}{q}}\left(\left\|w_h^0\right\|_{L^2} + \|G_h\|_{L^1([0,1];L^2)} + h^{\frac{1}{2}}\|w_h\|_{L^1L^2}\right).$$

Taking $q = 2+\varepsilon$, then $h^{\frac{-1}{q}} \le h^{\frac{-1}{2}}$. We multiply both sides by $h^{-s+\frac{1}{2}}$ and use the frequency localization of w_h, w_h^0 to get

$$\|w_h\|_{L^{2+\varepsilon}W^{s-\frac{1}{2},r}} \lesssim_\varepsilon \left\|w_h^0\right\|_{H^s} + \|G_h\|_{L^1H^s} + h^{\frac{1}{2}}\|w_h\|_{L^1H^s}.$$

We write $s-\frac{1}{2} = (\frac{d}{2}-1+\varepsilon)+(s-\frac{d}{2}+\frac{1}{2}-\varepsilon)$ where $\frac{d}{2}-1+\varepsilon > \frac{d}{2}-1+\frac{\varepsilon}{2+\varepsilon} = \frac{d}{r}$. The Sobolev embedding $W^{s-\frac{1}{2},r} \hookrightarrow W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty}$ then concludes the proof. ☐

3.4. **Semi-classical Strichartz estimate for** (3.6)**.** From the preceding proposition, one deduces the corresponding Strichartz estimate for $u_j \equiv \Delta_j u$ as a solution of (3.6) via the change of temporal variables $w_h(\sigma,x) = u_j(h^{\frac{1}{2}}\sigma, x)$ as follows.

Corollary 3.6. *If u_j is a solution to* (3.6)*, i.e., $\mathcal{L}_j u_j = F_j$ with data u_j^0 and u_j, u_j^0, F_j spectrally supported in $2^j\mathcal{C}^1$, then u_j satisfies*

$$\|u_j\|_{L^{2+\varepsilon}(I_j;W^{s-\frac{d}{2}+\frac{3}{4}-\varepsilon,\infty})} \lesssim_\varepsilon \left\|u_j^0\right\|_{H^s} + \|F_j\|_{L^1(I_j;H^s)} + \|u_j\|_{L^1(I_j;H^s)},$$

with $I_j = [0, 2^{-\frac{j}{2}}] = [0, h^{\frac{1}{2}}]$ and $\varepsilon > 0$.

The next step consists of gluing the estimates on small time-scales above to derive an estimate on the whole interval of time $[0,T]$ at the price of losing $\frac{1}{4}$ derivatives.

Corollary 3.7. *If u_j is a solution to $\mathcal{L}_j u_j = F_j$ with data u_j^0 and u_j, u_j^0, F_j spectrally supported in $\mathcal{C}_h^1$, then u_j satisfies, with $I = [0,T]$ and $\varepsilon > 0$,*

$$\|u_j\|_{L^2(I;W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} \lesssim_\varepsilon \|F_j\|_{L^2(I;H^{s-\frac{1}{2}})} + \|u_j\|_{L^\infty(I,H^s)}.$$

Proof. Let $\chi \in C_0^\infty(0,2)$ be equal to one on $[\frac{1}{2},\frac{3}{2}]$. For $0 \le k \le [Th^{-\frac{1}{2}}]-2$ define

$$I_{j,k} = [kh^{\frac{1}{2}}, (k+2)h^{\frac{1}{2}}), \quad \chi_{j,k}(t) = \chi\Big(\frac{t-kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\Big), \quad u_{j,k} = \chi_{j,k}(t)u_j.$$

Then each $u_{j,k}$ is a solution to

$$\mathcal{L}_j u_{j,k} = \chi_{j,k}F_j + h^{-\frac{1}{2}}\chi'\Big(\frac{t-kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\Big)u_j, \quad u_{j,k}(t = kh^{\frac{1}{2}}) = 0,$$

from which we deduce by virtue of Corollary 3.6 that

$$\|u_{j,k}\|_{L^2(I_{j,k};W^{s-\frac{d}{2}+\frac{3}{4}-\varepsilon,\infty})} \lesssim_\varepsilon \|F_j\|_{L^1(I_{j,k};H^s)} + h^{-\frac{1}{2}}\|u_j\|_{L^1(I_{j,k};H^s)}.$$

Noticing that $\chi_{j,k} = 1$ on $\Big((k+\frac{1}{2})h^{\frac{1}{2}}, (k+\frac{3}{2})h^{\frac{1}{2}}\Big)$ we get

$$\begin{aligned}\|u_{j,k}\|_{L^2(((k+\frac{1}{2})h^{\frac{1}{2}},(k+\frac{3}{2})h^{\frac{1}{2}});W^{s-\frac{d}{2}+\frac{3}{4}-\varepsilon,\infty})} &\lesssim_\varepsilon \|F_j\|_{L^1(I_{j,k};H^s)} + h^{-\frac{1}{2}}\|u_j\|_{L^1(I_{j,k};H^s)} \\ &\lesssim_\varepsilon h^{\frac{1}{4}}\|F_j\|_{L^2(I_{j,k};H^s)} + h^{-\frac{1}{4}}\|u_j\|_{L^2(I_{j,k};H^s)}.\end{aligned}$$

Squaring both sides of the above inequality and then summing in $0 \le k \le [Th^{-\frac{1}{2}}] - 2 =: N_h$ yields, with $J_j := [\frac{1}{2}h^{\frac{1}{2}}, (N_h - \frac{1}{2})h^{\frac{1}{2}}]$,

$$\|u_j\|_{L^2(J_j;W^{s-\frac{d}{2}+\frac{3}{4}-\varepsilon,\infty})} \lesssim_\varepsilon h^{\frac{1}{4}}\|F_j\|_{L^2(I;H^s)} + h^{-\frac{1}{4}}\|u_j\|_{L^2(I;H^s)}$$

or, equivalently after multiplying by $h^{\frac{1}{4}}$,

$$\|u_j\|_{L^2(J_j;W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} \lesssim_\varepsilon \|F_j\|_{L^2(I;H^{s-\frac{1}{2}})} + \|u_j\|_{L^2(I;H^s)}. \tag{3.19}$$

Next, we note that Corollary 3.6 still holds with I_j replaced by any interval of length $ch^{\frac{1}{2}}$. Applying this to $J = [0, \frac{1}{2}h^{\frac{1}{2}}]$ gives

$$\begin{aligned}\|u_j\|_{L^{2+\varepsilon}(J;W^{s-\frac{d}{2}+\frac{3}{4}-\varepsilon,\infty})} &\lesssim_\varepsilon \|u_j^0\|_{H^s} + \|F_j\|_{L^1(J;H^s)} + \|u_j\|_{L^1(J;H^s)} \\ &\lesssim_\varepsilon \|u_j^0\|_{H^s} + h^{\frac{1}{4}} \|F_j\|_{L^2(J;H^s)} + h^{\frac{1}{4}} \|u_j\|_{L^2(J;H^s)}.\end{aligned}$$

After multiplying both sides by $h^{\frac{1}{4}}$, this implies that

$$\begin{aligned}\|u_j\|_{L^{2+\varepsilon}(J;W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} &\lesssim_\varepsilon \|u_j^0\|_{H^{s-\frac{1}{4}}} + \|F_j\|_{L^2(J;H^{s-\frac{1}{2}})} + \|u_j\|_{L^2(J;H^{s-\frac{1}{2}})} \\ &\lesssim_\varepsilon \|F_j\|_{L^2(J;H^{s-\frac{1}{2}})} + \|u_j\|_{L^\infty(J;H^{s-\frac{1}{4}})}.\end{aligned} \tag{3.20}$$

Similarly, on $J = [(N_h - \frac{1}{2})h^{\frac{1}{2}}, T]$ we also have

$$\|u_j\|_{L^{2+\varepsilon}(J;W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} \lesssim_\varepsilon \|F_j\|_{L^2(J;H^{s-\frac{1}{2}})} + \|u_j\|_{L^\infty(J;H^{s-\frac{1}{4}})}. \tag{3.21}$$

A combination of (3.19), (3.20), and (3.21) leads to

$$\|u_j\|_{L^2(I;W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} \lesssim_\varepsilon \|F_j\|_{L^2(I;H^{s-\frac{1}{2}})} + \|u_j\|_{L^\infty(I,H^s)}.$$

□

In the final step, we shall glue the estimates for $\Delta_j u$ over different frequency regimes to obtain an estimate for u, from which the corresponding estimates for (η, ψ) follow.

3.5. **Concluding the proof of Theorem 1.3.** If u is a solution to (2.1) with $u(0) = u^0$, then by (3.6), the dyadic piece $\Delta_j u$ is a solution to $L_j \Delta_j u = F_j$ with F_j given by (3.7). Applying Corollary 3.7 one gets

$$\|\Delta_j u\|_{L^2(I;W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} \lesssim_\varepsilon \|F_j\|_{L^2(I;H^{s-\frac{1}{2}})} + \|\Delta_j u\|_{L^\infty(I;H^s)}. \tag{3.22}$$

Recall that $F_j = F_j^1 + R_j + F_j^3$, where $\|R_j\|_{H^s} \le \|u\|_{H^s}$ (see (3.5)) and F_j^k are given by (3.2) and (3.7). Using the symbolic calculus Theorem A.5 one obtains without any difficulty that

$$\|F_j^1\|_{L^2 H^{s-\frac{1}{2}}} \lesssim \|u\|_{L^2(I;H^s)} + \|f\|_{L^2(I;H^{s-\frac{1}{2}})}.$$

For F_j^3 we consider for example

$$\begin{aligned}A_j &:= S_{(j-3)\frac{1}{2}}\gamma(x, D_x)\Delta_j u - S_{j-3}\gamma(x, D_x)\Delta_j u \\ &= \left(S_{(j-3)\frac{1}{2}}\gamma(x, D_x)\Delta_j u - \gamma(x, D_x)\Delta_j u\right) + (\gamma(x, D_x)\Delta_j u - S_{j-3}\gamma(x, D_x)\Delta_j u) \\ &= A_j^1 + A_j^2.\end{aligned}$$

More generally, let $a \in \Gamma_\rho^m$ be homogeneous of degree m in ξ. Using the spherical harmonic decomposition we can assume that $a(x, \xi) = b(x)c(\xi)$ with $b \in W^{\rho,\infty}$ and c homogeneous of order m. Then $S_{\delta j}(a)(x, \xi) = S_{\delta j}(b)(x)c(\xi)$ and

$$\|(S_{\delta j}(a)(x, D_x) - a(x, D_x))v\|_{L^2} \le \|S_{\delta j}(b) - b\|_{L^\infty} \|c(D_x)v\|_{L^2} \lesssim 2^{-\delta j \rho} \|v\|_{H^m}. \tag{3.23}$$

Since $\gamma \in \Gamma_2^{\frac{3}{2}}$ is homogeneous of order $\frac{3}{2}$ in ξ, the $H^{s-\frac{1}{2}}$-norm of A_j^1 can be bounded by

$$2^{j(s-\frac{1}{2})-\frac{1}{2}j2} \|\Delta_j u\|_{H^{\frac{3}{2}}} \lesssim \|u\|_{H^s},$$

while

$$\left\|A_j^2\right\|_{H^{s-\frac{1}{2}}} \lesssim 2^{j(s-\frac{1}{2})-2j} \left\|\Delta_j u\right\|_{H^{\frac{3}{2}}} \lesssim \|u\|_{H^{s-1}}.$$

The other terms in F_j^3 can be treated in the same fashion, and we obtain

$$\left\|F_j^3\right\|_{L^2 H^{s-\frac{1}{2}}} \lesssim \|u\|_{L^2(I;H^s)} + \|f\|_{L^2(I;H^s)}.$$

The estimate (3.22) then implies that

$$\|\Delta_j u\|_{L^2(I;W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} \lesssim_\varepsilon \|u\|_{L^\infty(I;H^s)}.$$

Gluing these estimates together leads to

$$\begin{aligned} \|u\|_{L^2(I;W^{s-\frac{d}{2}+\frac{1}{2}-2\varepsilon,\infty})} &\leq \sum_j 2^{-j\varepsilon} \|\Delta_j u\|_{L^2(I;W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon,\infty})} \\ &\lesssim_\varepsilon \|u\|_{L^\infty(I;H^s)} + \|f\|_{L^2(I;H^{s-\frac{1}{2}})}. \end{aligned} \tag{3.24}$$

Recall from Theorem 2.1 that $u = T_p\eta + iT_q(\psi - T_B\eta)$. From (3.24) one can use the symbolic calculus for paradifferential operators in Theorem A.5 to recover the corresponding estimates for (η, ψ) (cf. [1], [2]):

$$\|\eta\|_{L^2(I;W^{s-\frac{d}{2}+1-2\varepsilon,\infty})} + \|\psi\|_{L^2(I;W^{s-\frac{d}{2}+\frac{1}{2}-2\varepsilon,\infty})} \lesssim \mathcal{F}_\varepsilon\left(\|(\eta,\psi)\|_{L^\infty([0,T];H^{s+\frac{1}{2}}\times H^s)}\right),$$

where $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$. The proof of Theorem 1.3 is complete.

4. Proof of Theorem 1.2

We consider three parameters $\delta \in (0,1)$, $r_0 \in [0,1]$, $r_1 \in [0, \frac{1}{2}]$ to be determined later. Assume furthermore that

$$V \in L^\infty(I; W^{1+r_0,\infty}(\mathbf{R}^d)), \quad \gamma(\cdot,\xi) \in L^\infty(I; W^{\frac{1}{2}+r_1,\infty}(\mathbf{R}^d)). \tag{4.1}$$

4.1. **Littlewood-Paley reduction.** For every $j \geq 0$, the dyadic piece $\Delta_j u$ solves

$$(\partial_t + T_V \cdot \nabla + iT_\gamma)\,\Delta_j u = F_j \tag{4.2}$$

with

$$F_j^1 := \Delta_j f + i\,[T_\gamma, \Delta_j]\,u + [T_V, \Delta_j] \cdot \nabla u. \tag{4.3}$$

In view of Proposition 2.7, one has

$$(\partial_t + S_{j-3}(V) \cdot \nabla + iS_{j-3}(\gamma)(x, D_x))\,\Delta_j u = F_j^2 \tag{4.4}$$

with

$$F_j^2 := F_j^1 + R_j, \tag{4.5}$$

$R_j u$ being spectrally supported in an annulus $\{c_1^{-1}2^{j-1} \leq |\xi| \leq c_1 2^{j+1}\}$ and

$$\|R_j\|_{H^s} \lesssim \|u\|_{H^s}. \tag{4.6}$$

Now, let $\frac{1}{2} < c_1 < \cdots < c_5$, $\mathcal{C}^k := \{(2c_k)^{-1} \leq |\xi| \leq 2c_k\}$, $k = \overline{1,5}$, and

$$\widetilde{\varphi} \in C^\infty, \quad \operatorname{supp}\widetilde{\varphi} \subset \mathcal{C}^5, \quad \widetilde{\varphi} \equiv 1 \text{ on } \mathcal{C}^4.$$

Equation (4.4) is then equivalent to

$$\left(\partial_t + S_{(j-3)\delta}(V) \cdot \nabla + iS_{(j-3)\delta}(\gamma)(x, D_x)\widetilde{\varphi}(2^{-j}D_x)\right)\Delta_j u = F_j \tag{4.7}$$

with

$$F_j = F_j^2 + F_j^3 := F_j^2 + i\left(S_{(j-3)\delta}\gamma(x,D_x) - S_{j-3}\gamma(x,D_x)\right)\Delta_j u + \left(S_{(j-3)\delta}(V) - S_{j-3}(V)\right)\cdot\nabla\Delta_j u. \tag{4.8}$$

Let us define the operator corresponding to the homogeneous problem of (4.7):

$$\mathcal{L}_j := \partial_t + S_{(j-3)\delta}(V)\cdot\nabla + iS_{(j-3)\delta}(\gamma)(x,D_x)\widetilde{\varphi}(2^{-j}D_x). \tag{4.9}$$

To prove a Strichartz estimate for $\Delta_j u$ as a solution to (4.7), we shall first establish a "pseudo-dispersive estimate" for $\mathcal{L}_j$. Set

$$h := 2^{-j}, \quad \widetilde{h} := h^{\frac{1}{2}}.$$

4.2. **Straightening the vector field.** Following [4] we straighten the vector field $\partial_t + S_{(j-3)\delta}\cdot\nabla$ by considering the system

$$\begin{cases} \dot{X}_k(s) = S_{(j-3)\delta}(V_k)(s,X(s)), \quad 1\le k\le d, \quad X=(X_1,\dots,X_d),\\ X_k(0)=x_k. \end{cases} \tag{4.10}$$

Since $V\in L^\infty([0,T];L^\infty_x)$, system (4.10) has a unique solution on $I=[0,T]$, which shall be denoted for simplicity by $X(s,x;h)$ or even $X(s,x)$. Estimates on the flow $s\mapsto X(s,\cdot)$ are given in the next proposition.

Proposition 4.1. *For fixed (s,h) the map $x\mapsto X(s,x;h)$ belongs to $C^\infty(\mathbf{R}^d,\mathbf{R}^d)$. Moreover, for all $(s,h)\in I\times(0,1]$ we have*

$$\|(\partial_x X)(s,\cdot;h) - Id\|_{L^\infty(\mathbf{R}^d)} \le \mathcal{F}\big(\|V\|_{L^\infty([0,T];W^{1,\infty})}\big)|s|, \tag{4.11}$$

$$\|(\partial_x^\alpha X)(s,\cdot;h)\|_{L^\infty(\mathbf{R}^d)} \le \mathcal{F}_\alpha\big(\|V\|_{L^\infty([0,T];W^{1+r_0,\infty})}\big)h^{-\delta(|\alpha|-(1+r_0))}|s|, \quad |\alpha|\ge 2, \tag{4.12}$$

where $\mathcal{F}$, $\mathcal{F}_\alpha:\mathbf{R}^+\to\mathbf{R}^+$.

Proof. Here we follow the proof of [4, Proposition 4.10]. The improvement is due to the estimate

$$\left\|\partial_x^\beta S_{j\delta}(V)(s)\right\|_{L^\infty(\mathbf{R}^d)} \le C_\beta h^{-\delta(|\beta|-1-r_0)}\|V(s)\|_{W^{1+r_0,\infty}} \quad \forall|\beta|\ge 2, \tag{4.13}$$

which follows from Lemma 2.6.

(i) To prove (4.11) we differentiate with respect to x_l to obtain

$$\begin{cases} \dfrac{\partial \dot{X}_k}{\partial x_l}(s) = \displaystyle\sum_{q=1}^d S_{j\delta}\Big(\frac{\partial V_k}{\partial x_q}\Big)(s,X(s))\frac{\partial X_q}{\partial x_l}(s),\\ \dfrac{\partial X_k}{\partial x_l}(0) = \delta_{kl}, \end{cases}$$

from which we deduce that

$$\frac{\partial X_k}{\partial x_l}(s) = \delta_{kl} + \int_0^s \sum_{q=1}^d S_{j\delta}\Big(\frac{\partial V_k}{\partial x_q}\Big)(\sigma,X(\sigma))\frac{\partial X_q}{\partial x_l}(\sigma)\,d\sigma. \tag{4.14}$$

Setting $|\nabla X| = \sum_{k,l=1}^d |\frac{\partial X_k}{\partial x_l}|$ we obtain from (4.14)

$$|\nabla X(s)| \le C_d + \int_0^s |\nabla V(\sigma,X(\sigma))|\,|\nabla X(\sigma)|\,d\sigma.$$

Gronwall's inequality implies that

$$|\nabla X(s)| \leq \mathcal{F}(\|V\|_{L^\infty(I;W^{1,\infty})}) \quad \forall s \in I. \tag{4.15}$$

Coming back to (4.14) and using (4.15) lead to

$$\left|\frac{\partial X}{\partial x}(s) - Id\right| \leq \mathcal{F}(\|V\|_{L^\infty(I;W^{1,\infty})}) \int_0^s \|\nabla V(\sigma,\cdot)\|_{L^\infty(\mathbf{R}^d)}\, d\sigma \leq \mathcal{F}_1(\|V\|_{W^{1,\infty}})|s|.$$

(ii) We shall prove (4.12) for $|\alpha| = 2$ first and then prove by induction on $|\alpha|$ that the estimates

$$\|(\partial_x^\alpha X)(s;\cdot,h)\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}_\alpha(\|V\|_{W^{1+r_0,\infty}})h^{-\delta(|\alpha|-1-r_0)}$$

for $2 \leq |\alpha| \leq k$ imply (4.12) for $|\alpha| = k+1$.

Differentiating $|\alpha|$ times ($|\alpha| \geq 2$) system (4.10) and using Faà-di-Bruno's formula we obtain

$$\frac{d}{ds}\big(\partial_x^\alpha X\big)(s) = S_{j\delta}(\nabla V)(s, X(s))\partial_x^\alpha X + (1), \tag{4.16}$$

where the term (1) is a finite linear combination of terms of the form

$$A_\beta(s,x) = \partial_x^\beta\big(S_{j\delta}(V)\big)(s,X(s)) \prod_{i=1}^q \big(\partial_x^{L_i} X(s)\big)^{K_i},$$

where

$$2 \leq |\beta| \leq |\alpha|, \quad |L_i|, |K_i| \geq 1, \quad \sum_{i=1}^q |K_i| L_i = \alpha, \quad \sum_{i=1}^q K_i = \beta.$$

(1) When $|\alpha| = 2$, we have

$$A_\beta(s,x) = \partial_x^\beta\big(S_{j\delta}(V)\big)(s,X(s)) \prod_{i=1}^q \big(\partial_x^{L_i} X(s)\big)^{K_i}$$

with $|L_i| = 1$ and $|\beta| = |\alpha| = 2$. It then follows from (i) that

$$|\prod_{i=1}^q \big(\partial_x^{L_i} X(s)\big)^{K_i}| \leq \mathcal{F}(\|V\|_{L^\infty(I;W^{1,\infty})}) \quad \forall s \in I.$$

On the other hand, by (4.13)

$$\left\|\partial_x^\beta S_{j\delta}(V)(s)\right\|_{L^\infty(\mathbf{R}^d)} \leq C h^{-\delta(|\alpha|-1-r_0)} \|V(s)\|_{W^{1+r_0,\infty}}.$$

Consequently

$$\|(1)(s)\|_{L^\infty(\mathbf{R}^d)} \leq h^{-\delta(|\alpha|-1-r_0)} \mathcal{F}(\|V\|_{L^\infty(I;W^{1+r_0,\infty})}) \quad \forall s \in I.$$

This together with (4.16) and Gronwall's inequality yields (4.12) for $|\alpha| = 2$.

(2) Assuming now that (4.12) holds for $2 \leq |\alpha| \leq k$, we shall prove it for $|\alpha| = k+1$. Indeed, from (4.11) and the induction hypothesis it holds for any $1 \leq |\nu| \leq k$ that

$$\|(\partial_x^\nu X)(s,\cdot;h)\|_{L^\infty(\mathbf{R}^d)} \leq_\alpha \mathcal{F}\big(\|V\|_{L^\infty([0,T];W^{1+r_0,\infty})}\big)h^{-\delta(|\nu|-1)}|s|.$$

Because $|\beta| \geq 2$ and $|L_i| \geq 1$, using (4.13) and the preceding estimate we have

$$\begin{aligned}\|A_\beta(s,\cdot)\|_{L^\infty(\mathbf{R}^d)} &\leq \left\|\partial_x^\beta\big(S_{j\delta}(V)\big)(s,\cdot)\right\|_{L^\infty(\mathbf{R}^d)} \prod_{i=1}^q \left\|\partial_x^{L_i} X(s,\cdot)\right\|_{L^\infty(\mathbf{R}^d)}^{|K_i|} \\ &\leq Ch^{-\delta(|\beta|-1-r_0)}\|V(s,\cdot)\|_{W^{1+r_0,\infty}} h^{-\delta\sum_{i=1}^q |K_i|(|L_i|-1)}\mathcal{F}(\|V\|_{L^\infty(I;W^{1+r_0,\infty})}) \\ &\leq h^{-\delta(|\alpha|-1-r_0)}\mathcal{F}(\|V\|_{L^\infty(I;W^{1+r_0,\infty})}).\end{aligned}$$

As before, we conclude by (4.16) and Gronwall's inequality. □

In view of (4.11) the mapping $x \mapsto X(s,x;h)$ is a C^∞-diffeomorphism for any $s \in [0,T_0]$ if T_0 is small enough. This is not a restriction, for one can iterate the final estimate over time intervals of length T_0 which depends only on $\|V\|_{L^\infty([0,T];W^{1,\infty})}$.

Now, in (4.9) we first make the change of spatial variables

$$v_h(t,y) = u_j(t, X(t,y;h)) \tag{4.17}$$

so that

$$\big(\partial_t + S_{(j-3)\delta}(V)\cdot\nabla\big)\, u_j(t,X(t,y;h)) = \partial_t v_h(t,y). \tag{4.18}$$

Denoting

$$q_h(x,\xi) := S_{(j-3)\delta}(\gamma)(x,\xi)\widetilde{\varphi}(h\xi), \tag{4.19}$$

let us compute this dispersive term after the above change of variables. To this end, set

$$\begin{aligned} H(y,y') &= \int_0^1 \frac{\partial X}{\partial x}(\lambda y + (1-\lambda)y')\, d\lambda, \quad M(y,y') = \big({}^t H(y,y')\big)^{-1}, \\ M_0(y) &= \Big({}^t\Big(\frac{\partial X}{\partial x}(y)\Big)\Big)^{-1}, \quad J(y,y') = \Big|\det\Big(\frac{\partial X}{\partial x}(y')\Big)\Big|\,|\det M(y,y')|. \end{aligned} \tag{4.20}$$

Then,

$$(\mathrm{Op}(q_h)u_j)\circ X(y) = (2\pi)^{-d}\iint e^{i(X(y)-x')\cdot\eta} q_h(X(y),\eta)u_j(x')dx'd\eta.$$

Now, we make two changes of variables $x' = X(y')$ and $\eta = M(y,y')\zeta$ to obtain

$$(\mathrm{Op}(q_h)u_j)\circ X(y) = (2\pi)^{-d}\iint e^{i(y-y')\cdot\zeta} q_h\big(X(y), M(y,y')\zeta\big)J(y,y')v_h(y')\,dy'\,d\zeta.$$

Observe that the above pseudo-differential operator is still of order $\frac{1}{2}$. To change its order to 1, we make another change of spatial variables:

$$y = h^{\frac{1}{2}}z = \widetilde{h}z, \quad y' = \widetilde{h}z', \quad w_h(z') = v_h(\widetilde{h}z'), \quad \xi = \widetilde{h}\zeta, \tag{4.21}$$

so that

$$\begin{aligned}(\mathrm{Op}(q_h)u_j)\circ X(y) = (2\pi)^{-d}\iint e^{i(z-z')\cdot\xi} q_h\big(X(\widetilde{h}z), M\big(\widetilde{h}z,\widetilde{h}z'\big)\widetilde{h}^{-1}\xi\big) \\ \times J\big(\widetilde{h}z,\widetilde{h}z'\big)w_h(z')\,dz'\,d\xi.\end{aligned} \tag{4.22}$$

Summing up, with

$$p_h(z,z',\xi) := q_h\big(X(\widetilde{h}z), M\big(\widetilde{h}z,\widetilde{h}z'\big)\widetilde{h}^{-1}\xi\big)J\big(\widetilde{h}z,\widetilde{h}z'\big), \quad w_h(t,z) = u_j(t,X(t,\widetilde{h}z)) \tag{4.23}$$

it holds that

$$(\mathrm{Op}(q_h)u_j)\circ X(\widetilde{h}z) = \mathrm{Op}(p_h)w_h(z),$$

which, combined with (4.18) and (4.9), yields

(4.24)
$$(\mathcal{L}_j u_j)(t, X(t,\widetilde{h}z)) = (\partial_t + i\,\mathrm{Op}(p_h))\, w_h(t,z), \qquad w_h(t,z) = u_j(t, X(t,\widetilde{h}z)).$$

We have transformed the operator $\mathcal{L}_j$ of order $\frac{1}{2}$ into the right-hand side of (4.24), which has order 1.

4.3. **Approximation of the symbol p_h.** Observe that p_h depends on (z, z', ξ), which is not in the standard form to use the phase space transform in [19]. We will approximate p_h by some symbol depending only on (z,ξ). A general result can be found in [25, Proposition 0.3A]. However, we will inspect more carefully the smoothness of p_h to obtain better estimates for the error. To do this, we write as in (2.7)-(2.8)

$$\begin{aligned} p_h(z,z',\xi) &= p_h(z,z,\xi) + \int_0^1 \partial_{z'} p_h(z, z+s(z'-z), \xi) ds (z'-z) \\ &:= p_h^0(z,\xi) + r_h^0(z,z',\xi)(z'-z), \end{aligned}$$

where

$$p_h^0(z,\xi) = p_h(z,z,\xi) = q_h\big(X(\widetilde{h}z), M_0(\widetilde{h}z)\widetilde{h}^{-1}\xi\big). \tag{4.25}$$

On the other hand,

$$\mathrm{Op}(r_h^0.(z'-z))w(z) = -i\,\mathrm{Op}(r_h)w(z)$$

with

$$r_h(z,z',\xi) = \int_0^1 \partial_\xi \partial_{z'} p_h(z, z+s(z'-z), \xi) ds.$$

To simplify notation, we denote $[z,z']_s = z + s(z'-z)$ so that

$$r_h(z,z',\xi) = \int_0^1 \partial_\xi \partial_{z'} q_h\big(X(\widetilde{h}z), M\big(\widetilde{h}z, \widetilde{h}[z,z']_s\big)\widetilde{h}^{-1}\xi\big) J\big(\widetilde{h}z, \widetilde{h}[z,z']_s\big) ds. \tag{4.26}$$

Thus, $\mathrm{Op}(p_h) = \mathrm{Op}(p_h^0) - i\,\mathrm{Op}(r_h)$.

4.3.1. *The symbol p_h^0.* First, Proposition 4.1 implies directly estimates for M and J.

Lemma 4.2. *It holds for all $(\alpha,\alpha') \in (\mathbf{N}^d)^2$ that*

$$|\partial_z^\alpha \partial_{z'}^{\alpha'} M(z,z')| + |\partial_z^\alpha \partial_{z'}^{\alpha'} J(z,z')| \lesssim_{\alpha,\alpha'} \begin{cases} 1 & \text{if } |\alpha| + |\alpha'| = 0, \\ \widetilde{h}^{-2\delta(|\alpha|+|\alpha'|-r_0)} & \text{if } |\alpha|+|\alpha'| \geq 1. \end{cases}$$

On the other hand, by Bernstein's inequalities (see Lemma 2.6) and the fact that $|\xi| \sim \widetilde{h}^{-2}$ on the support of $\widetilde{\varphi}(h\xi)$, we can estimate the derivatives of q_h, given by (4.19), as follows.

Lemma 4.3. *We have for all $(\alpha,\beta) \in (\mathbf{N}^d)^2$,*

$$\left|\partial_x^\alpha \partial_\xi^\beta q_h(x,\xi)\right| \lesssim_{\alpha,\beta} \begin{cases} \widetilde{h}^{-1+2|\beta|}, & \text{if } |\alpha| = 0, \\ \widetilde{h}^{-1-2\delta(|\alpha|-(\frac{1}{2}+r_1))+2|\beta|}, & \text{if } |\alpha| \geq 1. \end{cases}$$

We now study the regularity of the symbol p_h^0.

Proposition 4.4. *Choosing* $r_0,\ r_1$ *satisfying*

$$2\delta(1-r_0)\le 1,\quad 2\delta(2-r_0)\le 2,\quad 2\delta(\frac{1}{2}-r_1)\le 1,\quad 2\delta(\frac{3}{2}-r_1)\le 2, \tag{4.27}$$

the symbol p_h^0 *then verifies*

(i) *for all* $(\alpha,\beta)\in\mathbf{N}^d,\ |\alpha|\le 2,$

$$\left|\partial_z^\alpha\partial_\xi^\beta p_h^0(z,\xi)\right|\lesssim_{\alpha,\beta}\widetilde{h}^{-1+|\beta|}\mathbb{1}_{\widetilde{h}|\xi|\sim 1}(\xi), \tag{4.28}$$

(ii) *for all* $(\alpha,\beta)\in\mathbf{N}^d,\ |\alpha|\ge 3,$

$$\left|\partial_z^\alpha\partial_\xi^\beta p_h^0(z,\xi)\right|\lesssim_{\alpha,\beta}\widetilde{h}^{-1-(2\delta-1)(|\alpha|-2)+|\beta|}\mathbb{1}_{\widetilde{h}|\xi|\sim 1}(\xi). \tag{4.29}$$

Proof. To simplify notation, we denote in this proof $q\equiv q_h$.

(i) (4.28) is trivial when $\alpha=0$. The argument below is independent of the dimension, so let us further simplify the notation by writing as if $d=1$. For $|\alpha|=1$, we compute

$$\begin{aligned}\partial_z^\alpha p^0(z,\xi)&=q_x\big(X(\widetilde{h}z),M_0(\widetilde{h}z)\widetilde{h}^{-1}\xi\big)\widetilde{h}X'(\widetilde{h}z)\\&\quad+q_\xi\big(X(\widetilde{h}z),M_0(\widetilde{h}z)\widetilde{h}^{-1}\xi\big)(\widetilde{h}^{-1}\xi)\widetilde{h}M_0'(\widetilde{h}z).\end{aligned} \tag{4.30}$$

For $|\alpha|=2$, we have

$$\begin{aligned}\partial_z^\alpha p_h^0(z,\xi)&=q_{xx}(\cdots)\widetilde{h}^2(X')^2+2q_{x\xi}(\cdots)X'M_0'\widetilde{h}\xi+q_{\xi\xi}(\cdots)(M_0')^2\xi^2\\&\quad+q_x(\cdots)\widetilde{h}^2X''+q_\xi(\cdots)(\widetilde{h}\xi)M_0''.\end{aligned} \tag{4.31}$$

Remark that $\widetilde{h}|\xi|\sim 1$ on the support of $p^0(z,\xi)$. Using Proposition 4.1 and Lemmas 4.2 and 4.3 one deduces easily that

$$\begin{aligned}&\left|\partial_z^\alpha p^0(z,\xi)\right|\lesssim_\alpha\widetilde{h}^{-1-2\delta(\frac{1}{2}-r_1)+1}+\widetilde{h}^{-1+2-1-2\delta(1-r_0)},\quad |\alpha|=1,\\&\left|\partial_z^\alpha p^0(z,\xi)\right|\lesssim_\alpha\widetilde{h}^{-1-2\delta(\frac{3}{2}-r_1)+2}+\widetilde{h}^{-1-2\delta(\frac{1}{2}-r_1)+2-2\delta(1-r_0)}+\widetilde{h}^{-1+4-4\delta(1-r_0)-2}\\&\qquad+\widetilde{h}^{-1-2\delta(\frac{1}{2}-r_1)+2-2\delta(1-r_0)}+\widetilde{h}^{-1+2-2\delta(2-r_0)},\quad |\alpha|=2.\end{aligned}$$

Under conditions (4.27), we get

$$\left|\partial_z^\alpha p^0(z,\xi)\right|\lesssim_\alpha\widetilde{h}^{-1}\mathbb{1}_{\widetilde{h}|\xi|\sim 1}(\xi),\quad |\alpha|\le 2.$$

To obtain (i) it remains to estimate $\partial_\xi^\beta(\partial_z^\alpha p_0^h)$ for $|\alpha|\le 2$ and $\beta\in\mathbf{N}^d$. From the explicit expressions (4.30), (4.31) of $\partial_z^\alpha p_h^0$, we see that there are two possibilities when differentiating once in ξ. One possibility is that the derivative falls down to q. This makes appear the factor $M_0(\widetilde{h}z)\widetilde{h}^{-1}$ while we gain $\widetilde{h}^2$ when differentiating q in ξ (by Lemma 4.3); we thus gain $\widetilde{h}$. Another possibility is that the derivative falls down to $\xi^\nu,\ \nu=1,2$, which results in $\nu\xi^{\nu-1}$. Since $\xi\sim\widetilde{h}^{-1}$ on the support of p_h^0 one deduces that $\xi^{\nu-1}\sim\xi^\nu\widetilde{h}$, which means that we still gain $\widetilde{h}$. Therefore, in both cases we gain $\widetilde{h}$ when differentiating once in ξ, and thus (4.28) follows.

(ii) As just explained above, it suffices to prove (4.29) with $\beta=0$. From the formula (4.31), the proof of (4.29) reduces to showing for $|\alpha|\ge 0$ that

$$|\partial_z^\alpha A_j(z,\xi,h)|\lesssim_{\alpha,\beta}\widetilde{h}^{-1-(2\delta-1)|\alpha|}\mathbb{1}_{\widetilde{h}|\xi|\sim 1}(\xi),\qquad j=\overline{1,5},$$

with

$$\begin{cases} A_1 = q_{x\xi}\big(X(\widetilde{h}z), M_0(\widetilde{h}z)\widetilde{h}^{-1}\xi\big)\widetilde{h}^{-1}, \\ A_2 = q_{xx}\big(X(\widetilde{h}z), M_0(\widetilde{h}z)\widetilde{h}^{-1}\xi\big)\widetilde{h}^{2}, \\ A_3 = q_{\xi\xi}\big(X(\widetilde{h}z), M_0(\widetilde{h}z)\widetilde{h}^{-1}\xi\big)\widetilde{h}^{-4}, \\ A_4 = q_{x}\big(X(\widetilde{h}z), M_0(\widetilde{h}z)\widetilde{h}^{-1}\xi\big)\widetilde{h}, \\ A_5 = q_{\xi}\big(X(\widetilde{h}z), M_0(\widetilde{h}z)\widetilde{h}^{-1}\xi\big)\widetilde{h}^{-2}, \end{cases}$$

and

$$|\partial_z^\alpha B_j(z,h)| \lesssim_{\alpha,\beta} \widetilde{h}^{-(2\delta-1)|\alpha|}, \qquad j = \overline{1,4},$$

with

$$\begin{cases} B_1 = X'(\widetilde{h}z), \\ B_2 = \widetilde{h}M_0'(\widetilde{h}z), \\ B_3 = \widetilde{h}X''(\widetilde{h}z), \\ B_4 = \widetilde{h}^2 M''(\widetilde{h}z). \end{cases}$$

(1) B_j. By Lemma 4.11,

$$|\partial_z^\alpha B_1| = |\partial_z^\alpha X'(\widetilde{h}z)| = \widetilde{h}^{|\alpha|}|(\partial_x^{\alpha+1}X)(\widetilde{h}z)| \lesssim \widetilde{h}^{|\alpha|-2\delta|\alpha|} \lesssim \widetilde{h}^{-1-(2\delta-1)|\alpha|}.$$

On the other hand, (4.12) and the condition $2\delta(1-r_0) \le 1$ imply that

$$|\partial_z^\alpha B_3| = \widetilde{h}|\partial_z^\alpha X''(\widetilde{h}z)| = \widetilde{h}^{1+|\alpha|}|(\partial_x^{\alpha+2}X)(\widetilde{h}z)| \lesssim \widetilde{h}^{1+|\alpha|-2\delta(|\alpha|+1-r_0)} \lesssim \widetilde{h}^{-(2\delta-1)|\alpha|}.$$

Observing that $M_0'(\widetilde{h}z)$ is as smooth as $X''(\widetilde{h}z)$, the preceding estimate also holds for B_2. Regarding B_4, we use Lemma 4.2 and the condition $2\delta(1-r_0) \le 1$ to estimate

$$\begin{aligned} |\partial_z^\alpha B_4| = \widetilde{h}^2|\partial_z^\alpha M_0''(\widetilde{h}z)| = \widetilde{h}^{2+|\alpha|}|(\partial_x^{\alpha+2}M_0)(\widetilde{h}z)| &\lesssim \widetilde{h}^{2+|\alpha|-2\delta(|\alpha|+1-r_0)} \\ &\lesssim \widetilde{h}^{-(2\delta-1)|\alpha|}. \end{aligned}$$

(2) A_1. For $\alpha = 0$, Lemma 4.3 gives

$$|A_1| \lesssim \widetilde{h}^{-1-2\delta(\frac{1}{2}-r_1)+2-1} \lesssim \widetilde{h}^{-1}$$

since $2\delta(\frac{1}{2}-r_1) \le 1$. Considering now $|\alpha| \ge 1$, using the Faà-di-Bruno formula we see that $\partial_z^\alpha A_1$ is a linear combination of terms of the form

$$C_1 = \widetilde{h}^{|\alpha|-1}\big(\partial_x^{a+1}\partial_\xi^{b+1}q\big)(\cdots)\prod_{j=1}^{r}\big((\partial_x^{L_j}X)(\widetilde{h}z)\big)^{P_j}\big((\partial_x^{L_j}M_0)(\widetilde{h}z)\widetilde{h}^{-1}\xi\big)^{Q_j},$$

where $1 \le |a|+|b| \le |\alpha|$, $|L_j| \ge 1$ $\forall j = \overline{1,r}$, and

$$\sum_{j=1}^{r} P_j = a, \quad \sum_{j=1}^{r} Q_j = b, \quad \sum_{j=1}^{r}(|P_j|+|Q_j|)L_j = \alpha.$$

According to Lemma 4.3,

$$|\big(\partial_x^{a+1}\partial_\xi^{b+1}q\big)(\cdots)| \lesssim \widetilde{h}^{-1-2\delta(|a|+\frac{1}{2}-r_1)+2(|b|+1)}.$$

On the other hand, since $|L_j| \geq 1$ Lemmas 4.3 and 4.2 imply that the product $\prod_{j=1}^r$ appearing in C_1 is bounded in absolute value by $\widetilde{h}^K$ with

$$K = \sum_{j=1}^r \Big(-2\delta(|L_j|-1)|P_j| - 2\delta(|L_j|-r_0)|Q_j|\Big) - 2\sum_{j=1}^r |Q_j|$$
$$= -2\delta|\alpha| + 2\delta|a| + 2\delta r_0|b| - 2|b|.$$

Therefore, $|C_1| \lesssim \widetilde{h}^L$ with

$$L = |\alpha| - 1 - 1 - 2\delta(|a| + \frac{1}{2} - r_1) + 2(|b|+1) - 2\delta|\alpha| + 2\delta|a| + 2\delta r_0|b| - 2|b|$$
$$\geq -1 - (2\delta-1)|\alpha| + 1 - 2\delta(\frac{1}{2} - r_1)$$
$$\geq -1 - (2\delta-1)|\alpha|,$$

where we have used again the condition $2\delta(\frac{1}{2} - r_1) \leq 1$. The proof for A_1 is complete.

(3) $A_2,\ A_3,\ A_4,\ A_5$. The estimate for these terms can be derived along the same lines as for A_3, where one needs to make use of the condition $2\delta(\frac{3}{2} - r_1) \leq 2$ for A_2 and the condition $2\delta(\frac{1}{2} - r_1) \leq 1$ for A_4. □

From now on, we always assume condition (4.27) for r_0 and r_1.

4.3.2. *The symbol* r_h. The next lemma provides the order of r_h and shows that it decays in ξ faster than in (z, z'), which shall be important in proving our "pseudo-dispersive estimates" in section 4.4.

Lemma 4.5. *For all* $(\alpha, \alpha', \xi) \in (\mathbf{N}^d)^3$, *we have*

$$\left|\partial_z^\alpha \partial_{z'}^{\alpha'} \partial_\xi^\beta r_h(z, z', \xi)\right| \lesssim_{\alpha,\alpha',\beta} \widetilde{h}^{1-2\delta(1-r_0)-(2\delta-1)(|\alpha|+|\alpha'|)+|\beta|} \mathbb{1}_{\{\widetilde{h}|\xi|\sim 1\}}(\xi).$$

Consequently, $r_h \in S^{-1+2\delta(1-r_0)}_{1,(2\delta-1),(2\delta-1)}$.

Proof. Recall the definition (4.26) of r_h. On the support of this symbol, $|\xi| \sim \widetilde{h}^{-1}$. In this proof, all the estimates are uniform in $s \in [0,1]$. It follows from Lemma 4.2 that

$$\forall(\alpha, \alpha') \in (\mathbf{N}^d)^2, \quad \left|\partial_z^\alpha \partial_{z'}^{\alpha'} J\big(\widetilde{h}z, \widetilde{h}[z,z']_s\big)\right| \lesssim_{\alpha,\alpha'} \widetilde{h}^{-(2\delta-1)(|\alpha|+|\alpha'|)}.$$

Next, setting

$$\widetilde{q}_h(x,\xi) = S_{(j-3)\delta}(\gamma)(x,\xi)\widetilde{\varphi}(\xi)$$

we see that

$$q_h\big(X(\widetilde{h}z), M\big(\widetilde{h}z, \widetilde{h}[z,z']_s\big)\widetilde{h}^{-1}\xi\big) = \widetilde{h}^{-1}\widetilde{q}_h\big(X(\widetilde{h}z), M\big(\widetilde{h}z, \widetilde{h}[z,z']_s\big)\widetilde{h}\xi\big).$$

The proof then boils down to showing, for all $(\alpha, \alpha', \beta) \in (\mathbf{N}^d)^3$, that

$$\left|\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} \partial_\xi \partial_{z'} \widetilde{q}_h(X(z), M(z,[z,z']_s)\xi)\right| \lesssim_{\alpha,\alpha',\beta} \widetilde{h}^{-2\delta(|\alpha|+|\alpha'|)-2\delta(1-r_0)}. \tag{4.32}$$

We compute

$$\Xi := \partial_\xi \partial_{z'} \widetilde{q}(X(z), M(z,[z,z']_s)\xi) = s\widetilde{q}_{\xi\xi}(\cdots)M_{z'}\xi M + s\widetilde{q}_\xi(\cdots)M_{z'},$$

which is bounded by $\widetilde{h}^{-2\delta(1-r_0)}$ in view of Lemma 4.2 and the fact that $|\xi| \sim 1$ on the support of $\widetilde{q}$. For the same reason we see that taking ξ-derivatives of Ξ is harmless (notice that M is bounded), so we only need to prove (4.32) for $|\beta| = 0$. Indeed, by Lemma 4.2

$$\left|\left(\partial_z^\alpha \partial_{z'}^{\alpha'} M\right)(\cdot)\right| + \left|\left(\partial_z^\alpha \partial_{z'}^{\alpha'} M_{z'}\right)(\cdot)\right| \lesssim \widetilde{h}^{-2\delta(|\alpha|+|\alpha'|)-2\delta(1-r_0)}. \tag{4.33}$$

On the other hand, using the Faà-di-Bruno formula (as in the proof of Proposition 4.4) we can prove that

$$\left|\partial_z^\alpha \partial_{z'}^{\alpha'} (\partial_\xi^\gamma \widetilde{q})\big(X(z), M(z,[z,z']_s)\xi\big)\right| \lesssim \widetilde{h}^{-2\delta(|\alpha|+|\alpha'|)},$$

from which we conclude the proof. □

In view of equation (4.24) we have proved that

$$\left(\mathcal{L}_j u_j\right)(t, X(t,\widetilde{h}z)) = \left(\partial_t + i\operatorname{Op}(p_h^0)\right) w_h(t,z) + \operatorname{Op}(r_h) w_h(t,z), \tag{4.34}$$

with $r_h \in S^{-1+2\delta(1-r_0)}_{1,(2\delta-1),(2\delta-1)}$ and $w_h(t,z) = u_j(t, X(t,\widetilde{h}z))$.

4.4. **A "pseudo-dispersive estimate" for $\mathcal{L}_j$.** In this step, we shall show that Proposition 2.9 can be applied to the evolution operator

$$L_h := D_t + \operatorname{Op}^w(p_h^0).$$

Henceforth, we set

$$\delta = \frac{3}{4}, \quad \lambda = \widetilde{h}^{-1}.$$

Proposition 4.4 shows that p_h^0 belongs to λS^2_λ. Using Lemma 2.11 to replace $\operatorname{Op}(p_h^0)$ in (4.34) with $\operatorname{Op}^w(p_h^0)$ we have

$$\operatorname{Op}^w(p_h^0) = \operatorname{Op}(p_h^0) + \operatorname{Op}(r'_h) \tag{4.35}$$

with $r'_h \in S^0_{1,\frac{1}{2},\frac{1}{2}}$. On the other hand, since $2\delta(1-r_0) \leq 1$, Lemma 4.5 combined with (4.34) and (4.35) leads to the following.

Proposition 4.6. *There exists a symbol $r_h^1 \in S^0_{1,\frac{1}{2},\frac{1}{2}}$ such that*

$$\frac{1}{i}\left(\mathcal{L}_j u_j\right)(t, X(t,\widetilde{h}z)) = \left(D_t + \operatorname{Op}^w(p_h^0)\right) w_h(t,z) + \operatorname{Op}(r_h^1) w_h(t,z). \tag{4.36}$$

Next, we recall the following proposition, which says that the characteristic set of p_h^0 has d nonvanishing principal curvatures.

Proposition 4.7 ([4, Proposition 4.16])**.** *Let $\mathcal{C}$ be a fixed annulus in $\mathbf{R}^d$. For any $0 < \delta < 1$ there exist $m_0 > 0,\ h_0 > 0$ such that*

$$\sup_{(t,x,\xi,h)\in I\times\mathbf{R}^d\times(\mathcal{C}\times(0,h_0])} \left|\det \partial_\xi^2 S_{\delta j}(\gamma)(t,x,\xi)\right| \geq m_0.$$

Let $\mathcal{S}_j$ and S_h denote the propagator of $\mathcal{L}_j$ and L_h, respectively. We are now in position to apply Theorem 2.9 to derive dispersive estimates for S_h.

Proposition 4.8. *For any symbol $\chi \in S^0_\lambda$ satisfying for all $z \in \mathbf{R}^d$ $\operatorname{supp}\chi(z,\cdot) \subset \lambda\mathcal{C}^2$, we have*

$$\left\|S_h(t,t_0)\big(\chi(z,D_z)f\big)\right\|_{L^\infty} \lesssim \widetilde{h}^{-\frac{d}{2}} \left|t-t_0\right|^{-\frac{d}{2}} \|f\|_{L^1} \tag{4.37}$$

for all $t,\ t_0 \in [0,1]$, and $0 < \widetilde{h} \leq \widetilde{h}_0$.

If, in addition, $\chi(z, D_z) : L^2 \to L^2$, *then for any* $r \in [2, \infty]$ *there holds by interpolation*

$$\left\|S_h(t,t_0)\big(\chi(z,D_z)f\big)\right\|_{L^r} \lesssim \widetilde{h}^{-d(\frac{1}{2}-\frac{1}{r})}\,|t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})}\,\|f\|_{L^{r'}}\,, \tag{4.38}$$

where r' *is the conjugate exponent of* r*; i.e.,* $\frac{1}{r}+\frac{1}{r'}=1$.

Proof. We have seen that $p_h^0 \in \lambda S^2_\lambda$. On the other hand, $\widetilde{\varphi} \equiv 1$ in $\mathcal{C}^4$. Proposition 4.7 then gives

$$\sup_{(t,x,\xi,h)\in I\times\mathbf{R}^d\times\mathcal{C}^4\times(0,h_0]} \left|\det\big(\partial_\xi^2 S_{\delta j}(\gamma)(t,x,\xi)\widetilde{\varphi}(\xi)\big)\right| \gtrsim 1.$$

Remark that (4.11) implies $|M_0(y)| \geq c_0$ for all $y \in \mathbf{R}^d$ (by choosing T small enough as explained after Proposition 4.1). Consequently,

$$\sup_{(t,x,\xi,h)\in I\times\mathbf{R}^d\times(\lambda\mathcal{C}^3)\times(0,h_0]} \left|\det\partial_\xi^2 p_h^0\right| \gtrsim \lambda^{-d}$$

if $c_3 < c_4$ is chosen appropriately. In other words, condition (**A**) in Theorem 2.9 is fulfilled with $c = c_3$, and thus the proposition follows. ☐

Let φ_1 be a smooth function verifying

$$\operatorname{supp}\varphi_1 \subset \{(2c_2)^{-1} \leq |\xi| \leq 2c_2\}, \quad \varphi_1 \equiv 1 \text{ in } \{(2c_1)^{-1} \leq |\xi| \leq 2c_1\}.$$

Lemma 4.9. *For* $f(t,z) = g(t, X(t,\widetilde{h}z))$ *we have*

$$\left(\varphi_1(hD_x)g\right)(t, X(t,\widetilde{h}z)) = \varphi_h^*(z,D_z)f(t,z) - i\operatorname{Op}(r_h^2)f(t,z), \tag{4.39}$$

with

$$\varphi_h^*(z,\xi) = \varphi_1\big(M_0(\widetilde{h}z)\widetilde{h}\xi\big), \tag{4.40}$$

$$r_h^2(z,z',\xi) = \int_0^1 \partial_\xi\partial_{z'}\varphi_1\big(M(\widetilde{h}z,\widetilde{h}[z,z']_s)\widetilde{h}\xi\big)J\big(\widetilde{h}z,\widetilde{h}[z,z']_s\big)ds. \tag{4.41}$$

Moreover, for every $(\alpha,\alpha',\xi) \in (\mathbf{N}^d)^3$ *there hold*

$$\left|\partial_z^\alpha\partial_\xi^\beta\varphi_h^*(z,\xi)\right| \lesssim_{\alpha,\beta} \widetilde{h}^{-(2\delta-1)|\alpha|+|\beta|}1_{\{\widetilde{h}|\xi|\sim 1\}}, \tag{4.42}$$

$$\left|\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_\xi^\beta r_h^2(z,z',\xi)\right| \lesssim_{\alpha,\alpha',\beta} \widetilde{h}^{2-2\delta(1-r_0)-(2\delta-1)(|\alpha|+|\alpha'|)+|\beta|}1_{\{\widetilde{h}|\xi|\sim 1\}}. \tag{4.43}$$

Proof. The formulas (4.39), (4.40), and (4.41) are derived along the same lines as in sections 4.2 and 4.3, where we performed the change of variables $x = X(t,\widetilde{h}z)$ to derive (4.34).

(1) Proof of (4.42).

Observe first that

$$\partial_\xi^\beta\varphi_h^*(z,\xi) = (\partial^\gamma\varphi_1)\big(M_0(\widetilde{h}z)\widetilde{h}\xi\big)\big(M_0(\widetilde{h}z)\big)^\gamma\widetilde{h}^{|\beta|}$$

where $|\gamma| = |\beta|$. Next, Lemma 4.2 implies that for all $\alpha \in \mathbf{N}^d$,

$$\left|\partial_z^\alpha(\partial^\gamma\varphi_1)\big(M_0(\widetilde{h}z)\widetilde{h}\xi\big)\right| + \left|\partial_z^\alpha\big(M_0(\widetilde{h}z)\big)^\gamma\right| \lesssim \widetilde{h}^{-(2\delta-1)|\alpha|},$$

and thus (4.42) follows.

(2) For (4.43) one proceeds exactly as in the proof of Lemma 4.5. ☐

Corollary 4.10. *If g is spectrally supported in the annulus $\frac{1}{h}\mathcal{C}^1$, then for all $r \in [2,\infty]$ we have*

$$\left\| S_h(t,t_0)\big(g \circ X(t_0,\widetilde{h}z)\big)\right\|_{L^r} \lesssim \widetilde{h}^{-d(\frac{2}{r'}-\frac{1}{2})}\, |t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})}\, \|g\|_{L^{r'}} + \left\| S(t,t_0)\,\mathrm{Op}(r_h^2)\big(g \circ X(t_0,\widetilde{h}z)\big)\right\|_{L^r} \tag{4.44}$$

for all $t,\ t_0 \in [0,1]$ and $0 < \widetilde{h} \leq \widetilde{h}_0$.

Proof. We first apply the identity (4.39) with $t = t_0$ and notice that $\varphi_1(h\xi) = 1$ if $\xi \in \frac{1}{h}\mathcal{C}^1$ to have

$$g \circ X(t_0,\widetilde{h}z) = \varphi_h^*(z,D_z)\big(g \circ X(t_0,\widetilde{h}z)\big)(z) - i\,\mathrm{Op}(r_h^2)\big(g \circ X(t_0,\widetilde{h}z)\big)(z).$$

The estimate (4.42) implies that $\varphi_h^* \in S^0_\lambda \cap S^0_{1,\frac{1}{2}}$ $(\lambda = \widetilde{h}^{-1})$, so the estimate (4.38) applied to $\chi = \varphi_h^*$ and $f(z) = g \circ X(t_0,\widetilde{h}z)$ gives for all $r \in [2,\infty]$,

$$\left\| S_h(t,t_0)\big(g \circ X(t_0,\widetilde{h}z)\big)\right\|_{L^r_z} \lesssim \widetilde{h}^{-d(\frac{1}{2}-\frac{1}{r})}\, |t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})} \left\| g \circ X(t_0,\widetilde{h}z)\right\|_{L^{r'}_z} + \left\| S(t,t_0)\,\mathrm{Op}(r_h^2)\big(g \circ X(t_0,\widetilde{h}z)\big)\right\|_{L^r_z}.$$

Finally, since X is Lipschitz we have

$$\left\| g \circ X(t_0,\widetilde{h}z)\right\|_{L^{r'}_z} \lesssim \widetilde{h}^{-\frac{d}{r'}}\, \|g\|_{L^{r'}_z},$$

from which we conclude the proof. □

To control the right-hand side of the estimate in the preceding corollary, we use the following lemma, whose proof is identical to that of Lemma 3.4.

Lemma 4.11. *For any $\mu \in \mathbf{R}$, the operators $S_h(t,s)$, $\mathcal{S}_j(t,s)$ are bounded on $H^\mu(\mathbf{R}^d)$ uniformly in $t,s \in I$.*

Henceforth, we choose $\varepsilon_0 > 0$ arbitrarily small and

$$r_1 = \frac{1}{6}; \quad r_0 = \frac{2}{3} + \varepsilon_0 \quad \text{when } d = 1, \quad r_0 = 1 \quad \text{when } d = 2 \tag{4.45}$$

so that

$$2\delta(1-r_0) = \frac{1}{2} - \frac{3}{2}\varepsilon_0, \quad 2\delta(2-r_0) = 2 - \frac{3}{2}\varepsilon_0 \quad \text{when } d = 1,$$
$$2\delta(1-r_0) = 0, \quad 2\delta(2-r_0) = \frac{3}{2} \quad \text{when } d = 2,$$

and (4.27) is fulfilled.

The next proposition shows what we called "pseudo-dispersive estimates" above.

Proposition 4.12. *If $\mathcal{L}_j u_j(t,x) = 0$ and $u_j(t)$ are spectrally supported in $\frac{1}{h}\mathcal{C}^1$ for all $t \in [0,T]$, then for any $t_0 \in [0,T]$ and $r \in [2,\infty]$ we have*

$$\|u_j(t)\|_{L^r} \lesssim \widetilde{h}^{-d(\frac{2}{r'}-\frac{1}{2})}\, |t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})}\, \|u_j(t_0)\|_{L^{r'}} + \widetilde{h}^{-\frac{d}{2}}\, \|u_j(t_0)\|_{L^2},$$

where $r = \infty$ when $d = 1$ and $r \in [2,\infty)$ when $d = 2$.

Proof. First, equation (4.36) implies that $w_h(t,z) := u_j(t, X(t, \widetilde{h}z))$ satisfies

$$w_h(t) = S_h(t,t_0)w_h(t_0) - \int_{t_0}^t S(t,s)\left(\mathrm{Op}_h(r_h^1)\right) w_h(s)ds =: (1) + (2).$$

Applying Corollary 4.10 to $g(x) = u_j(t_0, x)$ and then using the Sobolev embeddings

$$H^{\frac{d}{2}+\varepsilon} \hookrightarrow L^\infty, \quad H^{\frac{d}{2}} \hookrightarrow L^r \quad \forall r \in [2,\infty)$$

together with Lemma 4.11 one gets

$$\|(1)\|_{L^r} \lesssim \widetilde{h}^{-d(\frac{2}{r'}-\frac{1}{2})}\,|t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})}\,\|u_j(t_0)\|_{L^{r'}} + \left\|\mathrm{Op}(r_h^2)w_h(t_0)\right\|_{H^{\frac{d}{2}+\varepsilon_r}},$$

where

$$\varepsilon_r = 0 \quad \text{if } r \in [2,\infty), \quad \varepsilon_r = \frac{3}{2}\varepsilon_0 \quad \text{if } r = \infty.$$

By (4.43), $r_h^2 \in S^{-2+2\delta(1-r_0)}_{1,\frac{1}{2},\frac{1}{2}}$; hence

$$\left\|\mathrm{Op}(r_h^2)w_h(t_0)\right\|_{H^{\frac{d}{2}+\varepsilon_r}} \lesssim \|w_h(t_0)\|_{H^{\frac{d}{2}+\varepsilon_r-2+2\delta(1-r_0)}}.$$

Similarly, since $r_h^1 \in S^{-1+2\delta(1-r_0)}_{1,\frac{1}{2},\frac{1}{2}}$ one deduces with the aid of Lemma 4.11 that

$$\|(2)\|_{L^r} \lesssim \int_{t_0}^t \|w_h(s)\|_{H^{\frac{d}{2}+\varepsilon_r-1+2\delta(1-r_0)}}\,ds.$$

Putting together the above estimates leads to

$$\|w_h(t)\|_{L^r} \lesssim \widetilde{h}^{-d(\frac{2}{r'}-\frac{1}{2})}\,|t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})}\,\|u_j(t_0)\|_{L^{r'}} + \int_{t_0}^t \|w_h(s)\|_{H^{\frac{d}{2}+\varepsilon_r-1+2\delta(1-r_0)}}\,ds.$$

When $d = 1,\ r = \infty,\ \frac{d}{2} + \varepsilon_r - 1 + 2\delta(1-r_0) = 0$, and since $X(t,\cdot) \in W^{1,\infty}(\mathbf{R}^d)$ we have for all $s \in [t_0, t]$,

$$\|w_h(s)\|_{H^{\frac{d}{2}+\varepsilon_r-1+2\delta(1-r_0)}} \lesssim \widetilde{h}^{-\frac{d}{2}}\,\|u_j(s)\|_{L^2}.$$

When $d = 2,\ r \in [2,\infty),\ \frac{d}{2} + \varepsilon_r - 1 + 2\delta(1-r_0) = 0$, and thus

$$\|w_h(s)\|_{H^{\frac{d}{2}+\varepsilon_r-1+2\delta(1-r_0)}} \lesssim \widetilde{h}^{-\frac{d}{2}}\,\|u_j(s)\|_{L^2} \quad \forall s \in [t_0,t].$$

In addition, by Lemma 4.11, $\|u_j(s)\|_{L^2} \lesssim \|u_j(t_0)\|_{L^2}$ for all $s \in [t_0, t]$. Finally, noticing that

$$\|u_j(t)\|_{L^r} \lesssim \widetilde{h}^{\frac{1}{r}}\,\|w_h(t)\|_{L^r} \lesssim \|w_h(t)\|_{L^r},$$

we conclude the proof. □

Remark 4.13. Strictly speaking, the preceding estimate is not a standard dispersive estimate since its right-hand side does not decay in time. The appearance of the nondecaying term $\widetilde{h}^{-\frac{d}{2}}\,\|u_j(t_0)\|_{L^2}$ is however harmless for the purpose of proving Strichartz estimates in the next section.

4.5. **Strichartz estimates.**

Proposition 4.14. *Suppose that $\mathcal{L}_j u_j(t,x) = 0$ and $u_j(t)$, $t \in I := [0,T]$ is spectrally supported in $\frac{1}{h}\mathcal{C}^1$.*

(i) *When $d = 1$ we have*

$$\|u_j\|_{L^4(I;L^\infty)} \lesssim h^{-\frac{3}{8}} \|u_j|_{t=0}\|_{L^2}. \tag{4.46}$$

(ii) *When $d = 2$ we have with $q > 2$, $\frac{2}{q} + \frac{2}{r} = 1$,*

$$\|u_j\|_{L^q(I;L^r)} \lesssim h^{\frac{1}{4}-\frac{1}{r'}} \|u_j|_{t=0}\|_{L^2}. \tag{4.47}$$

Consequently, for any $s_0 \in \mathbf{R}$ and $\varepsilon > 0$,

$$\|u_j\|_{L^{2+\varepsilon}(I;W^{s_0-\frac{3}{4}-\varepsilon,\infty})} \lesssim \|u_j|_{t=0}\|_{H^{s_0}}. \tag{4.48}$$

Proof. For the two estimates (4.46) and (4.47), using the TT^* argument, one needs to show that

$$K := \int_I S(t,s)ds : L^{q'}(I;L^{r'}) \to L^q(I,L^r)$$

with norm bounded by h^M where $M = -\frac{3}{4}$ when $d = 1$ and $M = \frac{1}{2} - \frac{2}{r'}$ when $d = 2$. Moreover, since u_j is spectrally supported in $\frac{1}{h}\mathcal{C}^1$, it suffices to prove

$$\|Kf\|_{L^q(I,L^r)} \lesssim h^M \|f\|_{L^{q'}(I;L^{r'})} \tag{4.49}$$

for every f spectrally supported in $\frac{1}{h}\mathcal{C}^1$.

In view of the "pseudo-dispersive estimate" in Proposition 4.12,

$$\|K(t)f\|_{L^r} \lesssim (1) + (2), \quad \text{with}$$

$$(1) = h^{-d(\frac{1}{r'}-\frac{1}{4})} \int_I |t-s|^{-d(\frac{1}{2}-\frac{1}{r})} \|f(s)\|_{L^{r'}} \, ds,$$

$$(2) = h^{-\frac{d}{4}} \int_I \|f(s)\|_{L^2} \, ds.$$

(i) $d = 1$, $(q,r) = (4,\infty)$. By the Hardy-Littlewood-Sobolev inequality, $\|(1)\|_{L^q_t}$ is bounded by the right-hand side of (4.49). On the other hand, (2) can be estimated using Sobolev embedding as

$$(2) \lesssim h^{-\frac{d}{4}} h^{-\frac{d}{2}} \int_I \|f(s)\|_{L^1} \, ds \lesssim h^{-\frac{3d}{4}} \|f\|_{L^1L^1},$$

which concludes the proof of (4.46).

(ii) $d = 2$, $q > 2$, $\frac{2}{q} + \frac{2}{r} = 1$. Again, the Hardy-Littlewood-Sobolev inequality yields

$$\|(1)\|_{L^q} \lesssim h^{-d(\frac{1}{r'}-\frac{1}{4})} \|f\|_{L^{q'}(I;L^{r'})}.$$

For (2) one uses the embedding $L^{r'} \hookrightarrow L^2$, $r' \in [1,2)$,

$$\|f(s)\|_{L^2} \lesssim h^{\frac{d}{2}-\frac{d}{r'}} \|f(s)\|_{L^{r'}}$$

to get

$$(2) \lesssim h^{\frac{d}{4}-\frac{d}{r'}} \|f\|_{L^1L^{r'}}.$$

The estimate (4.47) then follows.

Now, for any $\varepsilon > 0$, let $q = 2 + \varepsilon$; then

$$\frac{2}{r} = \frac{\varepsilon}{\varepsilon + 2}, \quad \frac{1}{4} - \frac{1}{r'} = -\frac{3}{4} + \frac{\varepsilon}{2(\varepsilon + 2)}.$$

Multiplying both sides of (4.47) by h^{-s_0} yields

$$\|u_j\|_{L^q(I;W^{s_0-\frac{3}{4}+\frac{\varepsilon}{2(\varepsilon+2)},r})} \lesssim \|u_j|_{t=0}\|_{H^{s_0}}. \tag{4.50}$$

Writing $s_0 - \frac{3}{4} + \frac{\varepsilon}{2(\varepsilon+2)} = a + b,\ a = s_0 - \frac{3}{4} - \varepsilon,\ b = \varepsilon + \frac{\varepsilon}{2(\varepsilon+2)} > \frac{d}{r}$ we obtain (4.48) from (4.50) and the Sobolev embedding $W^{a+b,r}(\mathbf{R}^d) \hookrightarrow W^{a,\infty}$ with $b > \frac{d}{r}$. ☐

Recall from (4.1) and (4.45) that we have required that

$$V \in L^\infty(I; W^{\rho,\infty}(\mathbf{R}^d)),\ \gamma(\cdot,\xi) \in L^\infty(I; W^{\frac{2}{3},\infty}(\mathbf{R}^d)), \tag{4.51}$$

where $\rho = \frac{5}{3} + \varepsilon_0,\ \varepsilon_0 > 0$ when $d = 1$ and $\rho = 2$ when $d = 2$.

Theorem 4.15. *Let $s_0 \in \mathbf{R}$, $I = [0,T],\ T \in (0,+\infty)$, and $u \in L^\infty(I, H^{s_0}(\mathbf{R}^d))$ be a solution to the problem*

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = f, \quad u|_{t=0} = u^0,$$

where $\gamma = \sqrt{\lambda \mathfrak{a}}$ as defined in Theorem 2.2.

(1) *When $d = 1$, if for some $\varepsilon_0 > 0$, $V \in L^\infty(I, W^{\frac{5}{3}+\varepsilon_0,\infty}(\mathbf{R}^d))$, and $\eta \in L^\infty(I, W^{\frac{5}{3},\infty}(\mathbf{R}^d))$, then*

$$\|u\|_{L^4(I;W^{s_0-\frac{3}{8},\infty})} \lesssim \|u^0\|_{H^{s_0}} + \|f\|_{L^1(I,H^{s_0})}.$$

(2) *When $d = 2$, if $V \in L^\infty(I, W^{2,\infty}(\mathbf{R}^d))$ and $\eta \in L^\infty(I, W^{\frac{5}{3},\infty}(\mathbf{R}^d))$, then for every $\varepsilon > 0$,*

$$\|u\|_{L^{2+\varepsilon}(I;W^{s_0-\frac{3}{4}-\varepsilon,\infty})} \lesssim_\varepsilon \|u^0\|_{H^{s_0}} + \|f\|_{L^1(I,H^{s_0})}.$$

In the above estimates, the dependence constants depend on a finite number of semi-norms of the symbols V and γ.

Proof. If u is a solution to (2.3) with data u^0, then by (4.7), the dyadic piece $\Delta_j u$ is a solution to $\mathcal{L}_j \Delta_j u = F_j$ with F_j given by (4.8) spectrally supported in $\frac{1}{h}\mathcal{C}^1$ with c_1 sufficiently large. Under the regularity assumptions of V and γ in (1) and (2), condition (4.51) is fulfilled. Using Duhamel's formula and applying the Strichartz estimates in Proposition 4.14 we deduce that

$$\|\Delta_j u\|_{L^q(I;W^{s_0-\frac{d}{2}+\mu,\infty})} \lesssim \|\Delta_j u^0\|_{H^{s_0}} + \|F_j\|_{L^1(I;H^{s_0})}, \tag{4.52}$$

where

$$q = 4,\ \mu = \frac{1}{8} \quad \text{when } d = 1; \quad q = 2 + \varepsilon,\ \mu = \frac{1}{4} - \varepsilon \quad \text{when } d = 2.$$

We are left with the estimate for $F_j = F_j^1 + R_j + F_j^3$ where F_j^k are given by (4.3) and (4.8). Defining

$$\widetilde{\Delta}_j = \sum_{|k-j|\le 3} \Delta_k,$$

it follows from (4.5) that

$$\|R_j\|_{H^{s_0}} \lesssim \|\widetilde{\Delta}_j u\|_{H^{s_0}}.$$

Using the symbolic calculus Theorem A.5 one obtains without any difficulty that

$$\|F_j^1\|_{L^1(I;H^{s_0})} \lesssim \|\widetilde{\Delta}_j u\|_{L^1(I;H^{s_0})}.$$

For F_j^3 we use (3.23) to obtain that if

$$V \in L^\infty(I, W^{\frac{4}{3},\infty}(\mathbf{R}^d)), \quad \eta \in L^\infty(I, W^{\frac{5}{3},\infty}(\mathbf{R}^d)),$$

then

$$\left\|F_j^3\right\|_{L^1(I;H^{s_0})} \lesssim \|\Delta_j u\|_{L^1(I;H^{s_0})} + \|\Delta_j f\|_{L^1(I;H^{s_0})}. \tag{4.53}$$

Finally, putting together the above estimates we conclude from (4.52) that

$$\begin{aligned}\|u\|_{L^q(I;W^{s_0-\frac{d}{2}+\mu,\infty})} &\leq \sum_j \|\Delta_j u\|_{L^q(I;W^{s_0-\frac{d}{2}+\mu,\infty})} \\ &\lesssim \left\|u^0\right\|_{H^{s_0}} + \|u\|_{L^1(I;H^{s_0})} + \|f\|_{L^1(I;H^{s_0})} \\ &\lesssim \left\|u^0\right\|_{H^{s_0}} + \|f\|_{L^1(I;H^{s_0})},\end{aligned}$$

where in the last inequality we have used the energy estimate

$$\|u\|_{L^\infty(I;H^{s_0})} \lesssim \left\|u^0\right\|_{H^{s_0}} + \|f\|_{L^1(I;H^{s_0})}.$$

The proof is complete. □

According to Remark 2.3, after having the estimate for the u-solution to (2.3) one can use the symbolic calculus to obtain the desired estimates for the original solution (η, ψ, B, V) as stated in Theorem 1.2.

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Appendix A

Definition A.1.

(1) (Littlewood-Paley decomposition). Let $\psi \in C_0^\infty(\mathbf{R}^d)$ be such that

$$\psi(\theta) = 1 \quad \text{for } |\theta| \leq 1, \qquad \psi(\theta) = 0 \quad \text{for } |\theta| > 2. \tag{A.1}$$

Define

$$\psi_k(\theta) = \kappa(2^{-k}\theta) \quad \text{for } k \in \mathbf{Z}, \qquad \varphi_0 = \kappa_0, \qquad \text{and} \qquad \varphi_k = \psi_k - \psi_{k-1} \quad \text{for } k \geq 1.$$

Given a temperate distribution u and an integer k, we introduce $S_k u = \psi_k(D_x)u$, $\Delta_k u = S_k u - S_{k-1}u$ for $k \geq 1$, and $\Delta_0 u = S_0 u$. Then we have the formal decomposition

$$u = \sum_{k=0}^{\infty} \Delta_k u. \tag{A.2}$$

(2) (Hölder spaces). For $k \in \mathbf{N}$, we denote by $W^{k,\infty}(\mathbf{R}^d)$ the usual Sobolev spaces. For $\rho = k+\sigma$, $k \in \mathbf{N}$, $\sigma \in (0,1)$ denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the space of functions whose derivatives up to order k are bounded and uniformly Hölder continuous with exponent σ.

Definition A.2. Let u be a tempered distribution in $\mathbf{R}^d$. We define the spectrum of u to be the support of the Fourier transform of u. Then u is said to be spectrally supported in a set $A \subset \mathbf{R}^d$ if the spectrum of u is contained in A.

Let us review notation and results about Bony's paradifferential calculus (cf. [7], [22]).

Definition A.3.

(1) (Symbols). Given $\rho \in [0,\infty)$ and $m \in \mathbf{R}$, $\Gamma^m_\rho(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbf{R}^d)$ and there exists a constant C_α such that

$$\forall |\xi| \geq \frac{1}{2}, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}. \tag{A.3}$$

Letting $a \in \Gamma^m_\rho(\mathbf{R}^d)$, we define the semi-norm

$$M^m_\rho(a) = \sup_{|\alpha| \leq 2(d+2)+\rho} \sup_{|\xi| \geq 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)}. \tag{A.4}$$

(2) (Paradifferential operators). Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi-\eta,\eta) \widehat{a}(\xi-\eta,\eta) \rho(\eta) \widehat{u}(\eta)\, d\eta, \tag{A.5}$$

where $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta} a(x,\xi)\, dx$ is the Fourier transform of a with respect to the first variable, χ and ρ are two fixed C^∞ functions such that

$$\rho(\eta) = 0 \quad \text{for } |\eta| \leq \frac{1}{5}, \qquad \rho(\eta) = 1 \quad \text{for } |\eta| \geq \frac{1}{4}, \tag{A.6}$$

and $\chi(\theta,\eta)$ is defined by $\chi(\theta,\eta) = \sum_{k=0}^{+\infty} \kappa_{k-3}(\theta) \varphi_k(\eta)$.

We remark that the cut-off function χ in the preceding definition has the following properties for some $0 < \varepsilon_1 < \varepsilon_2 < 1$:

$$\begin{cases} \chi(\eta,\xi) = 1 & \text{for } |\eta| \leq \varepsilon_1(1+|\xi|), \\ \chi(\eta,\xi) = 0 & \text{for } |\eta| \geq \varepsilon_2(1+|\xi|). \end{cases} \tag{A.7}$$

Definition A.4. Let $m \in \mathbf{R}$. An operator T is said to be of order m if, for all $\mu \in \mathbf{R}$, it is bounded from H^μ to $H^{\mu-m}$.

Symbolic calculus for paradifferential operators is summarized in the following theorem.

Theorem A.5 (Symbolic calculus)**.** *Let $m \in \mathbf{R}$ and $\rho \in [0,\infty)$.*

(i) *If $a \in \Gamma^m_0(\mathbf{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \leq K M^m_0(a). \tag{A.8}$$

(ii) *If $a \in \Gamma^m_\rho(\mathbf{R}^d), b \in \Gamma^{m'}_\rho(\mathbf{R}^d)$, then $T_a T_b - T_{a\sharp b}$ is of order $m+m'-\rho$ with*

$$a\sharp b := \sum_{|\alpha|<\rho} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi).$$

Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_a T_b - T_{a\sharp b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K M^m_\rho(a) M^{m'}_0(b) + K M^m_0(a) M^{m'}_\rho(b). \tag{A.9}$$

(iii) *Let* $a \in \Gamma^m_\rho(\mathbf{R}^d)$. *Denote by* $(T_a)^*$ *the adjoint operator of* T_a *and by* $\overline{a}$ *the complex conjugate of* a. *Then* $(T_a)^* - T_b$ *is of order* $m - \rho$ *with*

$$b := \sum_{|\alpha|<\rho} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \bar{a}(x, \xi).$$

Moreover, for all μ *there exists a constant* K *such that*

$$\|(T_a)^* - T_b\|_{H^\mu \to H^{\mu - m + \rho}} \leq K M^m_\rho(a). \tag{A.10}$$

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