TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 12, December 2018, Pages 8999–9023
https://doi.org/10.1090/tran/7468
Article electronically published on August 17, 2018

SL(n) COVARIANT VECTOR VALUATIONS ON POLYTOPES

CHUNNA ZENG AND DAN MA

Abstract. All SL(n) covariant vector valuations on convex polytopes in $\mathbb{R}^n$ are completely classified without any continuity assumptions. The moment vector turns out to be the only such valuation if $n \geq 3$, while two new functionals show up in dimension two.

1. Introduction

The study and classification of geometric notions which are compatible with transformation groups are important tasks in geometry as proposed in Felix Klein's Erlangen program in 1872. As many functions defined on geometric objects satisfy the inclusion-exclusion principle, the property of being a valuation is natural to consider in the classification. Here, a map $\mu : \mathcal{S} \to \langle \mathcal{A}, + \rangle$ is called a *valuation* on a collection $\mathcal{S}$ of sets with values in an abelian semigroup $\langle \mathcal{A}, + \rangle$ if

$$\mu(P) + \mu(Q) = \mu(P \cup Q) + \mu(P \cap Q)$$

whenever P, Q, $P \cap Q$, and $P \cup Q$ are contained in $\mathcal{S}$.

At the beginning of the twentieth century, valuations were first constructed by Dehn in his solution of Hilbert's third problem. Nearly 50 years later, Hadwiger initiated a systematic study of valuations by his celebrated characterization theorem. He showed that all continuous and rigid motion invariant valuations on the space of convex bodies (i.e., compact convex sets) in $\mathbb{R}^n$ are linear combinations of intrinsic volumes.

The classification of valuations using compatibility with certain linear maps and the topology induced by the Hausdorff metric is a classical part of geometry with important applications in integral geometry (see [10], [26, Chap. 6]). Such results turned out to be extremely fruitful and useful, especially in the affine geometry of convex bodies. Examples include intrinsic volumes, affine surface areas, the projection body operator, and the intersection body operator (see [1–6, 8, 9, 11–13, 15–22, 24, 25]).

Recently, Ludwig and Reitzner [23] established a characterization of SL(n) invariant valuation on $\mathcal{P}^n$, the space of convex polytopes in $\mathbb{R}^n$, without any continuity assumptions.

Received by the editors April 25, 2017, and, in revised form, November 20, 2017.

2010 *Mathematics Subject Classification.* Primary 52B45, 52A20.

Key words and phrases. Moment vector, valuation, convex polytope, SL(n) covariance.

The first author was supported in part by the National Natural Science Foundation of China (Project No. 11801048), by the Chinese Scholarship Council and by the Natural Science Foundation Project of CSTC (Grant No. cstc2017jcyjAX0022).

The second author was supported in part by Shanghai Sailing Program 17YF1413800 and by the National Natural Science Foundation of China (Project No. 11701373). The second author is the corresponding author.

Theorem 1.1. *A functional $z : \mathcal{P}^n \to \mathbb{R}$ is an* $\mathrm{SL}(n)$ *invariant valuation if and only if there exist constants $c_0, c_0', d_0 \in \mathbb{R}$ and solutions $\alpha, \beta : [0,\infty) \to \mathbb{R}$ of Cauchy's functional equation such that*

$$z(P) = c_0 V_0(P) + c_0'(-1)^{\dim P}\chi_{\mathrm{relint}\, P}(0) + \alpha(V_n(P)) + d_0\chi_P(0) + \beta(V_n([0,P]))$$

for every $P \in \mathcal{P}^n$, where V_0 and V_n denote the Euler characteristic and the volume, respectively, $[0,P]$ denotes the convex hull of P and the origin, and χ denotes the indicator function.

The aim of this paper is to obtain a complete classification of $\mathrm{SL}(n)$ covariant vector valuations on $\mathcal{P}^n$. This also corresponds to the following classification results on $\mathcal{P}^n_{(0)}$, the space of convex polytopes containing the origin in their interiors, due to Haberl and Parapatits [7].

Theorem 1.2. *Let $n \geq 3$. A functional $\mu : \mathcal{P}^n_{(0)} \to \mathbb{R}^n$ is a measurable and* $\mathrm{SL}(n)$ *covariant valuation if and only if there exists a constant $c \in \mathbb{R}$ such that*

$$\mu(P) = cm(P)$$

for every $P \in \mathcal{P}^n_{(0)}$.

Theorem 1.3. *A functional $\mu : \mathcal{P}^2_{(0)} \to \mathbb{R}^2$ is a measurable and* $\mathrm{SL}(2)$ *covariant valuation if and only if there exist constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\mu(P) = c_1 m(P) + c_2 \rho_{\frac{\pi}{2}} m(P^*)$$

for every $P \in \mathcal{P}^2_{(0)}$, where $\rho_{\frac{\pi}{2}}$ denotes the counterclockwise rotation in $\mathbb{R}^2$ by the angle $\pi/2$ and P^* denotes the polar body of P.

Here, a functional $\mu : \mathcal{P}^n \to \mathbb{R}^n$ is called $\mathrm{SL}(n)$ *covariant* if $\mu(\phi P) = \phi\mu(P)$ for all $P \in \mathcal{P}^n$ and $\phi \in \mathrm{SL}(n)$. The vector $m(P)$ is the *moment vector* of P, which is defined as

$$m(P) = \int_P x dx$$

for every $P \in \mathcal{P}^n$. It coincides with the centroid of P multiplied by the volume of P, which makes it a basic notion in mechanics, engineering, physics, and geometry. Earlier results on characterizations of moment vectors can be found in [14, 26]. Throughout this paper, a functional with values in a Euclidean space is called *measurable* if the preimage of every open set is a Borel set with respect to the corresponding topology.

Denote by $\mathcal{P}^n_0$ the subspace of convex polytopes containing the origin. First, we consider valuations defined on $\mathcal{P}^n_0$ and obtain the following result.

Theorem 1.4. *Let $n \geq 3$. A functional $\mu : \mathcal{P}^n_0 \to \mathbb{R}^n$ is an* $\mathrm{SL}(n)$ *covariant valuation if and only if there exists a constant $c \in \mathbb{R}$ such that*

$$\mu(P) = cm(P)$$

for every $P \in \mathcal{P}^n_0$.

Solutions of Cauchy's functional equation show up only in dimension two.

Theorem 1.5. *A functional $\mu : \mathcal{P}^2_0 \to \mathbb{R}^2$ is an* $\mathrm{SL}(2)$ *covariant valuation if and only if there exist constants $c_1, c_2 \in \mathbb{R}$ and a solution of Cauchy's functional equation $\alpha : [0,\infty) \to \mathbb{R}$ such that*

$$\mu(P) = c_1 m(P) + c_2 e(P) + h_\alpha(P)$$

for every $P \in \mathcal{P}_0^n$*, where the functionals* $e, h_\alpha : \mathcal{P}_0^2 \to \mathbb{R}^2$ *are defined in section* 2.

Next, we consider the classification of measurable SL(2) covariant valuations. It is well known that all measurable solutions of Cauchy's functional equation are linear. This immediately leads to the following corollary.

Corollary 1.1. *A functional* $\mu : \mathcal{P}_0^2 \to \mathbb{R}^2$ *is a measurable and* SL(2) *covariant valuation if and only if there exist constants* $c_1, c_2, c_3 \in \mathbb{R}$ *such that*

$$\mu(P) = c_1 m(P) + c_2 e(P) + c_3 h(P)$$

for every $P \in \mathcal{P}_0^2$*, where the functional* $h : \mathcal{P}_0^2 \to \mathbb{R}^2$ *is defined in section* 3.

Next, we consider the space of all convex polytopes $\mathcal{P}^n$. This step is as in the classification of convex body valued valuations by Schuster and Wannerer [27] and Wannerer [28].

Theorem 1.6. *Let* $n \geq 3$*. A functional* $\mu : \mathcal{P}^n \to \mathbb{R}^n$ *is an* SL(n) *covariant valuation if and only if there exist constants* $c_1, c_2 \in \mathbb{R}$ *such that*

$$\mu(P) = c_1 m(P) + c_2 m([0, P]) \tag{1.1}$$

for every $P \in \mathcal{P}^n$.

Again, the case of dimension two is different. We prove the following result.

Theorem 1.7. *A functional* $\mu : \mathcal{P}^2 \to \mathbb{R}^2$ *is an* SL(2) *covariant valuation if and only if there exist constants* $c_1, c_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ *and solutions of Cauchy's functional equation* $\alpha, \gamma : [0, \infty) \to \mathbb{R}$ *such that*

$$\begin{aligned} \mu(P) = c_1 m(P) + \tilde{c}_1 m([0, P]) + c_2 e(P) + \tilde{c}_2 e([0, v_1, \dots, v_r]) + h_\alpha([0, P]) \\ + \sum_{i=2}^{r} h_\gamma([0, v_{i-1}, v_i]) \end{aligned}$$

for every polytope $P \in \mathcal{P}^2$ *with vertices* $v_1, \dots, v_r$ *visible from the origin and labeled counterclockwise, where a vertex* v *of* P *is called visible from the origin if* $P \cap \operatorname{relint}[0, v] = \varnothing$.

Similarly, we have the following corollary.

Corollary 1.2. *A functional* $\mu : \mathcal{P}^2 \to \mathbb{R}^2$ *is a measurable and* SL(2) *covariant valuation if and only if there exist constants* $c_1, c_2, c_3, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \in \mathbb{R}$ *such that*

$$\begin{aligned} \mu(P) = c_1 m(P) + \tilde{c}_1 m([0, P]) + c_2 e([0, P]) + c_3 h([0, P]) + \tilde{c}_2 e([0, v_1, \dots, v_r]) \\ + \tilde{c}_3 h([0, v_1, \dots, v_r]) \end{aligned}$$

for every polytope $P \in \mathcal{P}^2$*, with vertices* $v_1, \dots, v_r$ *visible from the origin and labeled counterclockwise.*

2. Notation and preliminary results

We work in n-dimensional Euclidean space $\mathbb{R}^n$. The standard basis of $\mathbb{R}^n$ consists of $e_1, e_2, \dots, e_n$. The coordinates of a vector $x \in \mathbb{R}^n$ with respect to the standard basis are denoted by $x_1, x_2, \dots, x_n$. Denote the vector with all coordinates 1 by $\mathbf{1}$, the $n \times n$ identity matrix by $I_n = (e_1, \dots, e_n)$, and the determinant of a matrix A by $\det A$. The affine hull, the dimension, the interior, the relative interior, and the boundary of a given set in $\mathbb{R}^n$ are denoted by dim, aff, int, relint, and bd, respectively.

The convex hull of $k+1$ affinely independent points is called a k-dimensional simplex for all natural number k's. Generally, we denote by $[v_1, v_2, \dots, v_k]$ the convex hull of $v_1, v_2, \dots, v_k \in \mathbb{R}^n$. Two special simplices are the k-dimensional standard simplex $T^k = [0, e_1, e_2, \dots, e_k]$ and $\tilde{T}^{k-1} = [e_1, e_2, \dots, e_k]$, which is a $(k-1)$-dimensional simplex. For $i = 1, \dots, n$, let $\mathcal{T}^i$ be the set of i-dimensional simplices with one vertex at the origin and, let $\tilde{\mathcal{T}}^{i-1}$ be the set of $(i-1)$-dimensional simplices $T \subset \mathbb{R}^n$ with $0 \notin \operatorname{aff} T$.

We now recall some basic results on valuations (see [10, 24]). Let $\mathcal{Q}^n$ be either $\mathcal{P}^n$ or $\mathcal{P}^n_0$. The first lemma is the inclusion-exclusion principle.

Lemma 2.1. *Let $\mathcal{A}$ be an abelian group, and let $\mu : \mathcal{Q}^n \to \mathcal{A}$ be a valuation. Then,*

$$\mu(P_1 \cup \dots \cup P_k) = \sum_{\varnothing \neq S \subseteq \{1,2,\dots,k\}} (-1)^{|S|-1} \mu\Big(\bigcap_{i \in S} P_i\Big)$$

for all $k \in \mathbb{N}$ and $P_1, P_2, \dots, P_k \in \mathcal{Q}^n$, with $P_1 \cup \dots \cup P_k \in \mathcal{Q}^n$.

We define a *triangulation* of a k-dimensional polytope P into simplices as a set of k-dimensional simplices $\{T_1, \dots, T_r\}$ which have pairwise disjoint interiors, with $P = \bigcup T_i$ and with the property that, for an arbitrary $1 \le i_1 < \dots < i_j \le r$, the intersections $T_{i_1} \cap \dots \cap T_{i_j}$ are again simplices. Therefore, we can make full use of the inclusion-exclusion principle (see [24]).

Lemma 2.2. *Let $\mathcal{A}$ be an abelian group, and let $\mu : \mathcal{P}^n_0 \to \mathcal{A}$ be a valuation. Then, μ is determined by its values on n-dimensional simplices with one vertex at the origin and its value on $\{0\}$.*

A valuation on $\mathcal{Q}^n$ is called *simple* if $\mu(P) = 0$ for all $P \in \mathcal{Q}^n$ with $\dim P < n$.

Denote by $\mathrm{SL}^{\pm}(n)$ the group of volume-preserving linear maps, i.e., those with determinant 1 or -1. A functional $\mu : \mathcal{Q}^n \to \mathbb{R}^n$ is called $\mathrm{SL}^{\pm}(n)$ *covariant* if $\mu(\phi P) = \phi\mu(P)$ for all $P \in \mathcal{Q}^n$ and $\phi \in \mathrm{SL}^{\pm}(n)$ and, following [7], it is called $\mathrm{SL}^{\pm}(n)$ *signum covariant* if $\mu(\phi P) = (\det \phi)\phi\mu(P)$ for all $P \in \mathcal{Q}^n$ and $\phi \in \mathrm{SL}^{\pm}(n)$. Let $\mu : \mathcal{Q}^n \to \mathbb{R}^n$ be an $\mathrm{SL}(n)$ covariant valuation. We have $\mu = \mu^+ + \mu^-$, where

$$\mu^+(P) = \frac{1}{2}\big(\mu(P) + \theta\mu(\theta^{-1}P)\big) \quad \text{and} \quad \mu^-(P) = \frac{1}{2}\big(\mu(P) - \theta\mu(\theta^{-1}P)\big)$$

for some fixed $\theta \in \mathrm{SL}^{\pm}(n) \setminus \mathrm{SL}(n)$. Clearly, μ^+ and μ^- are valuations. Moreover, it is not hard to see that μ^+ is $\mathrm{SL}^{\pm}(n)$ covariant and μ^- is $\mathrm{SL}^{\pm}(n)$ signum covariant.

The solution of Cauchy's functional equation is one of the main ingredients in our proof. Since we do not assume continuity, functionals also depend on solutions $\alpha : [0, \infty) \to \mathbb{R}$ of *Cauchy's functional equation*, that is,

$$\alpha(s+t) = \alpha(s) + \alpha(t)$$

for all $s, t \in [0, \infty)$. If we add the condition that α is measurable, then α has to be linear.

Let $\lambda \in (0,1)$ and denote by H the hyperplane through the origin with the normal vector $(1-\lambda)e_1 - \lambda e_2$. Write H^+ and H^- as the two half-spaces bounded by H. This hyperplane induces a series of dissections of T^i as well as $\tilde{T}^{i-1}$ for $i = 2, \dots, n$. Let $\mu : \mathcal{Q}^n \to \mathbb{R}^n$ be an $\mathrm{SL}(n)$ covariant valuation. There are two interpolations corresponding to these dissections. First, assume that $i < n$. By the inclusion-exclusion principle we get

$$\mu(T^i) + \mu(T^i \cap H) = \mu(T^i \cap H^+) + \mu(T^i \cap H^-). \tag{2.1}$$

Definition 2.1. Let $\lambda \in (0,1)$. The linear transform $\phi_1 \in \mathrm{SL}(n)$ is given by

$$\phi_1 e_1 = \lambda e_1 + (1-\lambda)e_2, \phi_1 e_2 = e_2, \phi_1 e_n = e_n/\lambda, \phi_1 e_j = e_j \text{ for } 3 \le j \le n-1,$$

and $\psi_1 \in \mathrm{SL}(n)$ is given by

$$\psi_1 e_1 = e_1, \psi_1 e_2 = \lambda e_1 + (1-\lambda)e_2, \psi_1 e_n = e_n/(1-\lambda), \psi_1 e_j = e_j \text{ for } 3 \le j \le n-1.$$

It is clear that $T^i \cap H^+ = \psi_1 T^i$, $T^i \cap H^- = \phi_1 T^i$, and $T^i \cap H = \phi_1 T^{i-1}$. Then, equation (2.1) becomes

$$\mu(T^i) + \mu(\phi_1 T^{i-1}) = \mu(\phi_1 T^i) + \mu(\psi_1 T^i).$$

Since μ is $\mathrm{SL}(n)$ covariant, we derive

$$(\phi_1 + \psi_1 - I_n)\,\mu(T^i) = \phi_1 \mu(T^{i-1}). \tag{2.2}$$

Second, we consider the dissection of sT^n for $s > 0$. Again, by the inclusion-exclusion principle, we have

$$\mu(sT^n) + \mu(sT^n \cap H) = \mu(sT^n \cap H^+) + \mu(sT^n \cap H^-). \tag{2.3}$$

Definition 2.2. Let $\lambda \in (0,1)$. The linear transform $\phi_2 \in \mathrm{GL}(n)$ is given by

$$\phi_2 e_1 = \lambda e_1 + (1-\lambda)e_2, \phi_2 e_2 = e_2, \phi_2 e_j = e_j \text{ for } 3 \le j \le n,$$

and $\psi_2 \in \mathrm{GL}(n)$ is given by

$$\psi_2 e_1 = e_1, \psi_2 e_2 = \lambda e_1 + (1-\lambda)e_2, \psi_2 e_j = e_j \text{ for } 3 \le j \le n.$$

It is clear that $sT^n \cap H^+ = \psi_2 sT^n$, $sT^n \cap H^- = \phi_2 sT^n$, and $sT^n \cap H = \phi_2 sT^{n-1}$. Then, equation (2.3) becomes

$$\mu(sT^n) + \mu(\phi_2 sT^{n-1}) = \mu(\phi_2 sT^n) + \mu(\psi_2 sT^n).$$

Since $\phi_2/\sqrt[n]{\lambda}$ and $\psi_2/\sqrt[n]{1-\lambda}$ belong to $\mathrm{SL}(n)$, we obtain

$$\mu(sT^n) + \lambda^{-1/n}\phi_2\mu(\sqrt[n]{\lambda} sT^{n-1}) = \lambda^{-1/n}\phi_2\mu(\sqrt[n]{\lambda} sT^n) + (1-\lambda)^{-1/n}\psi_2\mu(\sqrt[n]{1-\lambda} sT^n).$$

Replacing s by $\sqrt[n]{s}$ in the equation above yields

$$\begin{aligned}\mu(\sqrt[n]{s}T^n) + \lambda^{-1/n}\phi_2\mu(\sqrt[n]{\lambda s}T^{n-1}) &= \lambda^{-1/n}\phi_2\mu(\sqrt[n]{\lambda s}T^n) \\ &\quad + (1-\lambda)^{-1/n}\psi_2\mu(\sqrt[n]{(1-\lambda)s}T^n).\end{aligned} \tag{2.4}$$

On $\mathcal{P}_0^2$, two new functionals appear in the classification results. Define $e : \mathcal{P}_0^2 \to \mathbb{R}^2$ as

$$e(P) = v + w$$

if $\dim P = 2$ and P has two edges $[0,v]$ and $[0,w]$, or $\dim P = 2$ and P has an edge $[v,w]$ that contains the origin in its relative interior;

$$e(P) = 2(v+w)$$

if $\dim P = 1$ and $P = [v,w]$ contains the origin; or

$$e(P) = 0$$

otherwise.

In order to prove that e is a valuation on $\mathcal{P}_0^2$, we use the following terminology. We say μ defined on $\mathcal{P}_0^2$ is a *weak valuation* if

$$\mu(P \cap L^+) + \mu(P \cap L^-) = \mu(P) + \mu(P \cap L) \tag{2.5}$$

for every $P \in \mathcal{P}_0^2$ and line L through the origin in the plane, where L^+ and L^- are two half-planes bounded by L. Indeed, we have the following implication (see [26, Theorem 6.2.3] for a version on $\mathcal{P}^2$).

Lemma 2.3. *Every weak valuation is a valuation on* $\mathcal{P}_0^2$.

Proof. Let μ be a weak valuation on $\mathcal{P}_0^2$. Write S_0^2 as the space of triangles in $\mathbb{R}^2$ with one vertex at the origin. Note that S_0^2 is a *generating set* of $\mathcal{P}_0^2$, i.e., a subset of $\mathcal{P}_0^2$ that is closed under finite intersections and such that every element of $\mathcal{P}_0^2$ is a finite union of elements therein. Due to Groemer's integral theorem (see [10, Theorem 2.2.1]), it suffices to show that μ is a valuation on S_0^2.

Let $S_1, S_2 \in S_0^2$, with $S = S_1 \cup S_2 \in S_0^2$ as well. The statement is trivial if one of them includes the other. Otherwise, write $S_3 = S_1 \cap S_2$. There are two cases.

First, if S_3 is a line segment, write $L = \operatorname{span} S_3$. Without loss of generality, assume $S_1 = S \cap L^+$ and $S_2 = S \cap L^-$. Since μ is a weak valuation, we have

$$\begin{aligned}\mu(S_1) + \mu(S_2) &= \mu(S \cap L^+) + \mu(S \cap L^-) \\ &= \mu(S) + \mu(S \cap L) = \mu(S_1 \cup S_2) + \mu(S_1 \cap S_2).\end{aligned}$$

Next, if $\dim S_3 = 2$, write $S_4 = \operatorname{cl}(S_1 \setminus S_3)$, $S_5 = \operatorname{cl}(S_2 \setminus S_3)$, $L_1 = \operatorname{span}(S_3 \cap S_4)$, and $L_2 = \operatorname{span}(S_3 \cap S_5)$. Without loss of generality, assume $S_4 = S_1 \cap L_1^+$, $S_3 = S_1 \cap L_1^- = S_2 \cap L_2^+$, and $S_5 = S_2 \cap L_2^-$. Since μ is a weak valuation, we have

$$\mu(S_3) + \mu(S_4) = \mu(S_1 \cap L_1^-) + \mu(S_1 \cap L_1^+) = \mu(S_1) + \mu(S_3 \cap S_4)$$

and

$$\mu(S_3) + \mu(S_5) = \mu(S_2 \cap L_2^+) + \mu(S_2 \cap L_2^-) = \mu(S_2) + \mu(S_3 \cap S_5).$$

Summing the two equations above gives

$$\mu(S_1 \cup S_2) + \mu(S_1 \cap S_2) = \mu(S) + \mu(S_3) = \mu(S_1) + \mu(S_2).$$

Therefore, μ is a valuation on $\mathcal{P}_0^2$. □

Lemma 2.4. *The functional e is an* $\mathrm{SL}(2)$ *covariant valuation on* $\mathcal{P}_0^2$.

Proof. By the definition it is clear that e is $\mathrm{SL}(2)$ covariant.

Next, we are going to prove that e is a valuation on $\mathcal{P}_0^2$. Due to Lemma 2.3, it suffices to show that e is a weak valuation via the following four cases.

First, let $\dim P = 2$, and let P have two edges, $[0, v]$ and $[0, w]$. Then, we have $e(P) = v + w$. Assume that a line L through the origin intersects an edge of P at u. It follows that $e(P \cap L^+) = w + u$, $e(P \cap L^-) = u + v$ and $e(P \cap L) = 2u$.

Second, let $\dim P = 2$, and let P have an edge $[v, w]$ that contains the origin in its relative interior. Then, we have $e(P) = v + w$. Assume that a line L through the origin intersects an edge of P at u. It follows that $e(P \cap L^+) = w + u$, $e(P \cap L^-) = u + v$ and $e(P \cap L) = 2u$.

Third, let $\dim P = 2$, and let P contain the origin in its interior. Then, we have $e(P) = 0$. Assume that a line L through the origin intersects two edges of P at v and w, respectively. It follows that $e(P \cap L^+) = v + w$, $e(P \cap L^-) = v + w$, and $e(P \cap L) = 2(v + w)$.

Finally, let $\dim P = 1$, and let $P = [v, w]$ contain the origin. Then, we have $e(P) = 2(v + w)$. For every line L through the origin, we get $e(P \cap L^+) = 2w$, $e(P \cap L^-) = 2v$, and $e(P \cap L) = 0$. $\square$

Let $\alpha : [0, \infty) \to \mathbb{R}$ be a solution of Cauchy's functional equation. Define $h_\alpha : \mathcal{P}_0^2 \to \mathbb{R}^2$ as

$$h_\alpha(P) = \sum_{i=2}^{r} \frac{\alpha\left(\det(v_{i-1}, v_i)\right)}{\det(v_{i-1}, v_i)}(v_{i-1} - v_i)$$

if $\dim P = 2$ and $P = [0, v_1, \dots, v_r]$, with $0 \in \operatorname{bd} P$ and the vertices $\{0, v_1, \dots, v_r\}$ labeled counterclockwise;

$$h_\alpha(P) = \frac{\alpha\left(\det(v_r, v_1)\right)}{\det(v_r, v_1)}(v_r - v_1) + \sum_{i=2}^{r} \frac{\alpha\left(\det(v_{i-1}, v_i)\right)}{\det(v_{i-1}, v_i)}(v_{i-1} - v_i)$$

if $0 \in \operatorname{int} P$ and $P = [v_1, \dots, v_r]$, with the vertices $\{v_1, \dots, v_r\}$ labeled counterclockwise; or

$$h_\alpha(P) = 0$$

if $P = \{0\}$ or P is a line segment.

Lemma 2.5. *If $\alpha : [0, \infty) \to \mathbb{R}$ is a solution of Cauchy's functional equation, then the functional h_α is an* SL(2) *covariant valuation on $\mathcal{P}_0^2$.*

Proof. Let $\alpha : [0, \infty) \to \mathbb{R}$ be a solution of Cauchy's functional equation. We write $\alpha^* = \alpha(s)/s$ for $s > 0$. As a first step, we show that h_α is SL(2) covariant. First, let $P \in \mathcal{P}_0^2$ and $\dim P = 2$. If $P = [0, v_1, \dots, v_r]$ or $P = [v_1, \dots, v_r]$, with $0 \in [v_1, v_r]$, then

$$\begin{aligned} h_\alpha(\phi P) &= \sum_{i=2}^{r} \alpha^*\left(\det(\phi v_{i-1}, \phi v_i)\right)(\phi v_{i-1} - \phi v_i) \\ &= \phi \sum_{i=2}^{r} \alpha^*\left(\det(v_{i-1}, v_i)\right)(v_{i-1} - v_i) \\ &= \phi h_\alpha(P) \end{aligned}$$

for every $\phi \in \mathrm{SL}(2)$. Similarly, if $0 \in \operatorname{int} P$, we also have $h_\alpha(\phi P) = \phi h_\alpha(P)$ for every $\phi \in \mathrm{SL}(2)$. If $P = \{0\}$ or $\dim P = 1$, then $h_\alpha(\phi P) = \phi h_\alpha(P) = 0$ for every $\phi \in \mathrm{SL}(2)$.

As a second step, we are going to show that h_α is a valuation on $\mathcal{P}_0^2$. Due to Lemma 2.3, it suffices to show that h_α is a weak valuation via the following two cases.

First, let $\dim P = 2$ and $P = [0, v_1, \dots, v_r]$, with $0 \in \operatorname{bd} P$ and the vertices $\{0, v_1, \dots, v_r\}$ labeled counterclockwise. Then, we have

$$h_\alpha(P) = \sum_{i=2}^{r} \alpha^*\left(\det(v_{i-1}, v_i)\right)(v_{i-1} - v_i).$$

(i) Assume L passes through a vertex of P; say, v_j. Without loss of generality, we have $P \cap L^+ = [0, v_1, \dots, v_j]$ and $P \cap L^- = [0, v_j, \dots, v_r]$. Thus,

$$h_\alpha(P\cap L^+)=\sum_{i=2}^{j}\alpha^*\left(\det(v_{i-1},v_i)\right)(v_{i-1}-v_i)$$

and

$$h_\alpha(P\cap L^-)=\sum_{i=j+1}^{r}\alpha^*\left(\det(v_{i-1},v_i)\right)(v_{i-1}-v_i).$$

(ii) Assume L intersects the edge $[v_j,v_{j+1}]$ at u. Without loss of generality, we have $P\cap L^+=[0,v_1,\dots,v_j,u]$ and $P\cap L^-=[0,u,v_{j+1},\dots,v_r]$. Thus,

$$h_\alpha(P\cap L^+)=\alpha^*\left(\det(v_j,u)\right)(v_j-u)+\sum_{i=2}^{j}\alpha^*\left(\det(v_{i-1},v_i)\right)(v_{i-1}-v_i)$$

and

$$h_\alpha(P\cap L^-)=\alpha^*\left(\det(u,v_{j+1})\right)(u-v_{j+1})+\sum_{i=j+2}^{r}\alpha^*\left(\det(v_{i-1},v_i)\right)(v_{i-1}-v_i).$$

Equation (2.5) follows from the fact that

$$\alpha^*\left(\det(v_j,v_{j+1})\right)(v_j-v_{j+1})=\alpha^*\left(\det(v_j,u)\right)(v_j-u)+\alpha^*\left(\det(u,v_{j+1})\right)(u-v_{j+1}). \tag{2.6}$$

Indeed, let $s=\sqrt{\det(v_j,v_{j+1})}$ and $\phi=(v_j,v_{j+1})/s\in \mathrm{SL}(2)$. Then,

$$v_j=\phi(se_1)\text{ and }v_{j+1}=\phi(se_2). \tag{2.7}$$

Since $u\in \operatorname{relint}[v_j,v_{j+1}]$, there exists $\lambda\in(0,1)$ such that $u=\lambda v_j+(1-\lambda)v_{j+1}$. Setting $v=\lambda e_1+(1-\lambda)e_2$, we obtain

$$u=\phi(sv). \tag{2.8}$$

Because of (2.7) and (2.8), the right-hand side of (2.6) equals

$$\begin{aligned}&\phi\left(s\alpha^*\left(s^2(1-\lambda)\right)(e_1-v)+s\alpha^*\left(s^2\lambda\right)(v-e_2)\right)\\&=s\alpha^*(s^2)\phi(e_1-e_2)=\alpha^*\left(\det(v_j,v_{j+1})\right)(v_j-v_{j+1}),\end{aligned}$$

as $v=\lambda e_1+(1-\lambda)e_2$ and by the additivity property of α.

Second, let $0\in \operatorname{int}P$ and $P=[v_1,\dots,v_r]$, with vertices $\{v_1,\dots,v_r\}$ labeled counterclockwise. Then, we have

$$h_\alpha(P)=\alpha^*\left(\det(v_r,v_1)\right)(v_r-v_1)+\sum_{i=2}^{r}\alpha^*\left(\det(v_{i-1},v_i)\right)(v_{i-1}-v_i).$$

(i) Assume L passes through v_1 and v_j. Without loss of generality, we have $P\cap L^+=[0,v_1,\dots,v_j]$ and $P\cap L^-=[0,v_j,\dots,v_r,v_1]$. Thus,

$$h_\alpha(P\cap L^+)=\sum_{i=2}^{j}\alpha^*\left(\det(v_{i-1},v_i)\right)(v_{i-1}-v_i)$$

and

$$h_\alpha(P\cap L^-)=\alpha^*\left(\det(v_r,v_1)\right)(v_r-v_1)+\sum_{i=j+1}^{r}\alpha^*\left(\det(v_{i-1},v_i)\right)(v_{i-1}-v_i).$$

(ii) Assume L passes through v_1 and intersects the edge $[v_j, v_{j+1}]$. Without loss of generality, we have $P \cap L^+ = [0, v_1, \ldots, v_j, u]$ and $P \cap L^- = [0, u, v_{j+1}, \ldots, v_r, v_1]$. Thus,

$$h_\alpha(P \cap L^+) = \alpha^* (\det(v_j, u)) (v_j - u) + \sum_{i=2}^{j} \alpha^* (\det(v_{i-1}, v_i)) (v_{i-1} - v_i)$$

and

$$\begin{aligned} h_\alpha(P \cap L^-) = \alpha^* (\det(v_r, v_1)) (v_r - v_1) + \alpha^* (\det(u, v_{j+1})) (u - v_{j+1}) \\ + \sum_{i=j+2}^{r} \alpha^* (\det(v_{i-1}, v_i)) (v_{i-1} - v_i). \end{aligned}$$

Equation (2.5) follows from (2.6).

(iii) Assume L intersects the edge $[v_r, v_1]$ at u_1 and the edge $[v_j, v_{j+1}]$ at u_2. Without loss of generality, we have $P \cap L^+ = [0, u_1, v_1, \ldots, v_j, u_2]$ and $P \cap L^- = [0, u_2, v_{j+1}, \ldots, v_r, u_1]$. Thus,

$$\begin{aligned} h_\alpha(P \cap L^+) = \alpha^* (\det(u_1, v_1)) (u_1 - v_1) + \alpha^* (\det(v_j, u_2)) (v_j - u_2) \\ + \sum_{i=2}^{j} \alpha^* (\det(v_{i-1}, v_i)) (v_{i-1} - v_i) \end{aligned}$$

and

$$\begin{aligned} h_\alpha(P \cap L^-) = \alpha^* (\det(v_r, u_1)) (v_r - u_1) + \alpha^* (\det(u_2, v_{j+1})) (u_2 - v_{j+1}) \\ + \sum_{i=j+2}^{r} \alpha^* (\det(v_{i-1}, v_i)) (v_{i-1} - v_i). \end{aligned}$$

Equation (2.5) follows from an analogue of (2.6). □

3. SL(n) COVARIANT VALUATIONS ON $\mathcal{P}_0^n$

3.1. **The two-dimensional case.** First, we give the representation of such valuations on sT^2 for $s > 0$.

Lemma 3.1. *If $\mu : \mathcal{P}_0^2 \to \mathbb{R}^2$ is an* SL(2) *covariant valuation, then there exist constants $c_1, c_2 \in \mathbb{R}$ and a solution of Cauchy's functional equation $\alpha : [0, \infty) \to \mathbb{R}$ such that*

$$\mu(sT^2) = c_1 m(sT^2) + c_2 s(e_1 + e_2) + \frac{\alpha(s^2)}{s}(e_1 - e_2)$$

for $s > 0$.

Proof. First, we decompose μ as $\mu = \mu^+ + \mu^-$, where μ^+ is an $\mathrm{SL}^{\pm}(2)$ covariant valuation and μ^- is an $\mathrm{SL}^{\pm}(2)$ signum covariant one.

Next, let $v = (v_1, v_2)^t \in \mathbb{R}^2$, with $v_1 v_2 \neq 0$,

$$\rho_1 = \begin{pmatrix} v_1 & 0 \\ v_2 & 1/v_1 \end{pmatrix}, \rho_2 = \begin{pmatrix} v_1 & 0 \\ v_2 & -1/v_1 \end{pmatrix}, \text{ and } \rho_3 = \begin{pmatrix} v_1 & -1/v_2 \\ v_2 & 0 \end{pmatrix}.$$

Then, we have $v = \rho_1 e_1 = \rho_2 e_1$. The $\mathrm{SL}^{\pm}(2)$ covariance of μ^+ implies

$$\begin{aligned} \mu^+([0, v]) &= \mu^+(\rho_1 T^1) = \rho_1 \mu^+(T^1) \\ &= \mu^+(\rho_2 T^1) = \rho_2 \mu^+(T^1). \end{aligned}$$

Setting $\mu^+(T^1) = (x_1^+, x_2^+)^t$, we obtain

$$v_1 x_1^+ = v_1 x_1^+,$$
$$v_2 x_1^+ + x_2^+/v_1 = v_2 x_1^+ - x_2^+/v_1.$$

Thus, $x_2^+ = 0$, and there exists a constant $c \in \mathbb{R}$ such that $\mu^+(T^1) = ce_1$. For $s > 0$, we apply

$$\rho_0 = \begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix}$$

and get

$$\mu^+(sT^1) = \mu^+(\rho_0 T^1) = \rho_0 \mu^+(T^1) = cse_1. \tag{3.1}$$

On the other hand, the $\mathrm{SL}^{\pm}(2)$ signum covariance of μ^- implies

$$\begin{aligned}\mu^-([0, v]) &= \mu^-(\rho_1 T^1) = \rho_1 \mu^-(T^1) \\ &= \mu^-(\rho_2 T^1) = -\rho_2 \mu^-(T^1) \\ &= \mu^-(\rho_3 T^1) = \rho_3 \mu^-(T^1).\end{aligned}$$

Setting $\mu^-(T^1) = (x_1^-, x_2^-)^t$, we obtain

$$v_1 x_1^- = -v_1 x_1^- = v_1 x_1^- - x_2^-/v_2,$$
$$v_2 x_1^- + x_2^-/v_1 = -v_2 x_1^- + x_2^-/v_1 = v_2 x_1^-.$$

Thus, $x_1^- = x_2^- = 0$, which implies $\mu^-(T^1) = 0$. Similarly, we get

$$\mu^-(sT^1) = 0 \tag{3.2}$$

for $s > 0$ and

$$\mu([0, v]) = \rho_1(\mu^+(T^1) + \mu^-(T^1)) = cv. \tag{3.3}$$

Finally, we use the dissection in Definition 2.2. It follows from (2.4) and (3.1) that, for $s > 0$,

$$\mu^+(\sqrt{s}T^2) + c\sqrt{s}(\lambda, 1-\lambda)^t = \sqrt{\lambda}^{-1} \phi_2 \mu^+(\sqrt{\lambda s}T^2) + \sqrt{1-\lambda}^{-1} \psi_2 \mu^+(\sqrt{(1-\lambda)s}T^2).$$

Setting $\lambda = a/(a+b)$ and $s = a+b$ for $a, b > 0$, we have

$$\frac{1}{\sqrt{a+b}} \mu^+(\sqrt{a+b}T^2) + \frac{c}{a+b}(a, b)^t = \frac{1}{\sqrt{a}} \phi_2 \mu^+(\sqrt{a}T^2) + \frac{1}{\sqrt{b}} \psi_2 \mu^+(\sqrt{b}T^2).$$

Write $g^+(x) = \mu^+(\sqrt{x}T^2)/\sqrt{x} = (g_1^+(x), g_2^+(x))^t$ for $x > 0$. Then, the equation above becomes

$$\begin{aligned} g_1^+(a+b) + \frac{ca}{a+b} &= \frac{a}{a+b} g_1^+(a) + g_1^+(b) + \frac{a}{a+b} g_2^+(b), \\ g_2^+(a+b) + \frac{cb}{a+b} &= \frac{b}{a+b} g_1^+(a) + g_2^+(a) + \frac{b}{a+b} g_2^+(b) \end{aligned} \tag{3.4}$$

and, equivalently,

$$g_1^+(a+b) + g_2^+(a+b) + c = g_1^+(a) + g_2^+(a) + g_1^+(b) + g_2^+(b),$$
$$b(g_1^+(a+b) - g_1^+(b)) = a(g_2^+(a+b) - g_2^+(a)).$$

Moreover, applying

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we have $\mu^+(sT^2) = \mu^+(\sigma sT^2) = \sigma\mu^+(sT^2)$. Hence, $\mu_1^+(sT^2) = \mu_2^+(sT^2)$, which implies $g_1^+ = g_2^+$. Consequently,

$$g_1^+(a+b) + c/2 = g_1^+(a) + g_1^+(b),$$
$$b(g_1^+(a+b) - g_1^+(b)) = a(g_1^+(a+b) - g_1^+(a)).$$

It follows that

$$g_1^+(x) = \gamma(x) + c/2 \quad \text{for } x > 0, \tag{3.5}$$

where $\gamma : [0,\infty] \to \mathbb{R}$ is a solution of Cauchy's functional equation. Inserting (3.5) into (3.4), we see that γ is linear, i.e., there exist constants $c_1', c_2 \in \mathbb{R}$ such that $g_1^+(x) = g_2^+(x) = c_1'x + c_2$, where $c_2 = c/2$. Therefore,

$$\mu^+(sT^2) = c_1's^3(e_1+e_2) + c_2s(e_1+e_2) = c_1m(sT^2) + c_2s(e_1+e_2), \tag{3.6}$$

where $c_1 = 6c_1'$, and in the second step we use $m(sT^2) = s^3(e_1+e_2)/3!$.

On the other hand, by (2.4) and (3.2), we obtain

$$\mu^-(\sqrt{s}T^2) = \sqrt{\lambda}^{-1}\phi_2\mu^-(\sqrt{\lambda s}T^2) + \sqrt{1-\lambda}^{-1}\psi_2\mu^-(\sqrt{(1-\lambda)s}T^2).$$

By putting $\lambda = a/(a+b)$ and $s = a+b$ for $a, b > 0$ we obtain

$$\frac{1}{\sqrt{a+b}}\mu^-(\sqrt{a+b}T^2) = \frac{1}{\sqrt{a}}\phi_2\mu^-(\sqrt{a}T^2) + \frac{1}{\sqrt{b}}\psi_2\mu^-(\sqrt{b}T^2).$$

Write $g^-(x) = \mu^-(\sqrt{x}T^2)/\sqrt{x} = (g_1^-(x), g_2^-(x))^t$ for $x > 0$. Then, the equation above becomes

$$g_1^-(a+b) + g_2^-(a+b) = g_1^-(a) + g_2^-(a) + g_1^-(b) + g_2^-(b),$$
$$b(g_1^-(a+b) - g_1^-(b)) = a(g_2^-(a+b) - g_2^-(a)).$$

Moreover, applying σ again, we have $\mu^-(sT^2) = \mu^-(\sigma sT^2) = -\sigma\mu^-(sT^2)$. Then $\mu_1^-(sT^2) + \mu_2^-(sT^2) = 0$, which implies $g_1^-(s) + g_2^-(s) = 0$. This implies

$$(a+b)g_1^-(a+b) = ag_1^-(a) + bg_1^-(b).$$

Therefore, $g_1^-(x) = -g_2^-(x) = \alpha(x)/x$, where $\alpha : [0,\infty) \to \mathbb{R}$ is a solution of Cauchy's functional equation. It follows that

$$\mu^-(sT^2) = \frac{\alpha(s^2)}{s}(e_1 - e_2). \tag{3.7}$$

Combining (3.6) and (3.7) completes the proof. □

Next, we consider the valuation on triangles with one vertex at the origin. Let $P = [0, v, w]$ with determinant $\det(v,w) > 0$. Set $\phi = (v,w) \in \mathrm{GL}(2)$ such that $\phi e_1 = v$ and $\phi e_2 = w$. By Lemma 3.1 there exist constants $c_1, c_2 \in \mathbb{R}$ and a solution of Cauchy's functional equation $\alpha : [0,\infty) \to \mathbb{R}$ such that

$$\begin{aligned}\mu(P) = \mu(\phi T^2) &= \sqrt{\det(v,w)}^{-1}\phi\mu\left(\sqrt{\det(v,w)}T^2\right)\\ &= c_1m(P) + c_2(v+w) + \frac{\alpha(\det(v,w))}{\det(v,w)}(v-w),\end{aligned} \tag{3.8}$$

where in the last step we use $m(\phi P) = |\det\phi|\,\phi m(P)$ for $\phi \in \mathrm{GL}(2)$.

Lemma 3.2. *If $\mu : \mathcal{P}_0^2 \to \mathbb{R}^2$ is an* SL(2) *covariant valuation, then there exist constants $c_1, c_2 \in \mathbb{R}$ and a solution of Cauchy's functional equation $\alpha : [0,\infty) \to \mathbb{R}$ such that*

$$\mu(P) = c_1 m(P) + c_2 e(P) + h_\alpha(P)$$

for every $P \in \mathcal{P}_0^2$ with $\dim P = 2$.

Proof. First, assume that the origin is a vertex of P. Let $P = [0, v_1, v_2, \ldots, v_r]$ be a polygon which has edges $[0, v_1], [v_1, v_2], \ldots, [v_{r-1}, v_r], [v_r, 0]$ labeled counterclockwise. Triangulate P into the simplices $[0, v_1, v_2], [0, v_2, v_3], \ldots, [0, v_{r-1}, v_r]$. By the inclusion-exclusion principle, (3.3), and (3.8) there exist constants $c_1, c_2 \in \mathbb{R}$ and a solution of Cauchy's functional equation $\alpha : [0,\infty) \to \mathbb{R}$ such that

$$\begin{aligned}\mu(P) =&\mu([0,v_1,v_2]) + \cdots + \mu([0,v_{r-1},v_r]) - \mu([0,v_2]) - \cdots - \mu([0,v_{r-1}])\\ =&c_1 m(P) + c_2(v_1+v_r) + \sum_{i=2}^{r} \frac{\alpha(\det(v_{i-1},v_i))}{\det(v_{i-1},v_i)}(v_{i-1}-v_i).\end{aligned} \tag{3.9}$$

Second, assume that the origin is contained in the relative interior of an edge of P. Let $P = [v_1, \ldots, v_r]$, with $0 \in \operatorname{relint}[v_1, v_r]$ and $[v_1, v_2], \ldots, [v_{r-1}, v_r], [v_r, v_1]$ labeled counterclockwise. Triangulate P into simplices $[0, v_1, v_2], [0, v_2, v_3], \ldots, [0, v_{r-1}, v_r]$. By the inclusion-exclusion principle, (3.3) and (3.8), we obtain

$$\begin{aligned}\mu(P) =&\mu([0,v_1,v_2]) + \cdots + \mu([0,v_{r-1},v_r]) - \mu([0,v_2]) - \cdots - \mu([0,v_{r-1}])\\ =&c_1 m(P) + c_2(v_1+v_r) + \sum_{i=2}^{r} \frac{\alpha(\det(v_{i-1},v_i))}{\det(v_{i-1},v_i)}(v_{i-1}-v_i).\end{aligned} \tag{3.10}$$

Third, assume that $0 \in \operatorname{int} P$. Let $P = [v_1, v_2, \ldots, v_r]$ be a polygon which has edges $[v_1, v_2], \ldots, [v_{r-1}, v_r]$ labeled counterclockwise. Triangulate P into simplices $[0, v_1, v_2]$, $[0, v_2, v_3], \ldots, [0, v_{r-1}, v_r]$, $[0, v_r, v_1]$. By the inclusion-exclusion principle, (3.3) and (3.8), we have

$$\begin{aligned}\mu(P) =&\mu([0,v_1,v_2]) + \cdots + \mu([0,v_{r-1},v_r]) + \mu([0,v_r,v_1])\\ &- \mu([0,v_1]) - \mu([0,v_2]) - \cdots - \mu([0,v_r])\\ =&c_1 m(P) + \frac{\alpha(\det(v_r,v_1))}{\det(v_r,v_1)}(v_r - v_1) + \sum_{i=2}^{r} \frac{\alpha(\det(v_{i-1},v_i))}{\det(v_{i-1},v_i)}(v_{i-1}-v_i).\end{aligned} \tag{3.11}$$

Combining (3.9), (3.10), and (3.11) and the definitions of e and h_α on $\mathcal{P}_0^2$ completes the proof. ☐

Using $\mu(\{0\}) = 0$, (3.3), and Lemmas 2.4, 2.5, and 3.2, we complete the proof of Theorem 1.5.

Finally, we consider measurable SL(2) covariant valuations. Define the functional $h : \mathcal{P}_0^2 \to \mathbb{R}^2$ by

$$h(P) = v_1 - v_r$$

if $\dim P = 2$ and $P = [0, v_1, \ldots, v_r]$, with $0 \in \operatorname{bd} P$ and the vertices $\{0, v_1, \ldots, v_r\}$ labeled counterclockwise;

$$h(P) = 0$$

if $0 \in \operatorname{int} P$ or P is a line segment or $P = \{0\}$.

If we assume that h_α is a measurable and SL(2) covariant valuation, then α is linear. There exists a constant c_3 such that $h_\alpha(P) = c_3 h(P)$. Because h_α is a simple valuation, we know that h is also a simple valuation on $\mathcal{P}_0^2$. Using Theorem 1.5, we obtain Corollary 1.1.

3.2. **The higher-dimensional case.** In this section, we first give the following propositions about simplices containing the origin.

Proposition 3.1. *Let $n \geq 3$. If $\mu : \mathcal{P}_0^n \to \mathbb{R}^n$ is an* SL(n) *covariant valuation, then there exists a constant $a \in \mathbb{R}$ such that $\mu(T^n) = a\mathbf{1}$.*

Proof. We first consider $\mu(T^3)$. Write $\mu(T^3) = (x_1, x_2, x_3)^t$ and

$$\sigma_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The SL(3) covariance of μ implies

$$\mu(T^3) = \mu(\sigma_0 T^3) = \sigma_0 \mu(T^3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}.$$

Thus, $x_1 = x_2 = x_3$.

Next, we consider $\mu(T^n)$ for $n \geq 4$ using a similar argument. Write $\mu(T^n) = (x_1, \ldots, x_n)^t$ and

$$\sigma = \begin{pmatrix} I_r & & \\ & \sigma_0 & \\ & & I_{n-r-3} \end{pmatrix} \in \mathrm{SL}(n),$$

where $r = 0, 1, \ldots, n-3$ and σ_0 moves along the main diagonal of σ. Using the SL(n) covariance of μ, we have $\mu(T^n) = \mu(\sigma T^n) = \sigma\mu(T^n)$. This yields $x_1 = \cdots = x_n$. Thus, $\mu(T^n) = a\mathbf{1}$, with $a = x_1$. □

Proposition 3.2. *If $\mu : \mathcal{P}_0^3 \to \mathbb{R}^3$ is an* SL(3) *covariant valuation, then there exists a constant $c \in \mathbb{R}$ such that $\mu(T^1) = 2ce_1$ and $\mu(T^2) = c(e_1 + e_2)$.*

Proof. Write $\mu(T^1) = (x_1, x_2, x_3)^t$ and $\mu(T^2) = (y_1, y_2, y_3)^t$. Set

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The SL(n) covariance of μ implies that $\mu(T^1) = \mu(\sigma_1 T^1) = \sigma_1 \mu(T^1)$ and $\mu(T^2) = \mu(\sigma_2 T^2) = \sigma_2 \mu(T^2)$. Thus, we have $\mu(T^1) = (x_1, 0, 0)^t$ and $\mu(T^2) = (y_1, y_1, 0)^t$.

Now, we use the dissection in Definition 2.1. Then, equation (2.2) is equivalent to

$$\begin{pmatrix} \lambda & \lambda & 0 \\ 1-\lambda & 1-\lambda & 0 \\ 0 & 0 & \frac{1}{\lambda} + \frac{1}{1-\lambda} - 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 1-\lambda & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}.$$

This yields $x_1 = 2y_1$. Therefore, there exists a constant c such that $\mu(T^1) = 2ce_1$ and $\mu(T^2) = c(e_1 + e_2)$. □

We now investigate SL(n) covariant valuations on $\mathcal{T}^k$ for the three-dimensional case and the n-dimensional case for $n \geq 4$, respectively.

Lemma 3.3. *If $\mu : \mathcal{P}_0^3 \to \mathbb{R}^3$ is an* SL(3) *covariant valuation, then μ is simple.*

Proof. Note that for $k \leq 2$, every simplex $T \in \mathcal{T}^k$ is an SL(3) image of T^k. Thus, it suffices to prove that μ vanishes on the standard simplices $\{0\}, T^1$, and T^2.

First, let $\mu(\{0\}) = (x_1, x_2, x_3)^t$, and let σ_1 be the same as in the proof of Proposition 3.2, while

$$\sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the SL(3) covariance of μ, we have

$$\begin{aligned} \mu(\{0\}) &= \mu(\sigma\{0\}) = \sigma\mu(\{0\}) \\ &= \mu(\sigma_1\{0\}) = \sigma_1\mu(\{0\}). \end{aligned}$$

This yields $x_1 = x_2 = x_3 = 0$. Therefore, $\mu(\{0\}) = 0$.

Next, let $T_{23} = [0, e_2, e_3]$ and σ_0 be the same as in the proof of Proposition 3.1. It follows from $T_{23} = \sigma_0 T^2$ and Proposition 3.2 that

$$\mu(T_{23}) = \mu(\sigma_0 T^2) = \sigma_0\mu(T^2) = c(e_2 + e_3).$$

Setting

$$\rho = \begin{pmatrix} s^{-2} & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix},$$

we obtain

$$\mu(sT_{23}) = \mu(\rho T_{23}) = \rho\mu(T_{23}) = cs(e_2 + e_3) \tag{3.12}$$

for every $s > 0$.

Finally, we use the dissection in Definition 2.2. By (2.4) and (3.12) it follows that

$$\mu(\sqrt[3]{s}T^3) + c\sqrt[3]{s}(\lambda, 1-\lambda, 1)^t = \lambda^{-1/3}\phi_2\mu(\sqrt[3]{\lambda s}T^3) + (1-\lambda)^{-1/3}\psi_2\mu(\sqrt[3]{(1-\lambda)s}T^3).$$

We set $\lambda = a/(a+b)$ and $s = a+b$ for $a, b > 0$ to get

$$\frac{1}{\sqrt[3]{a+b}}\mu(\sqrt[3]{a+b}T^3) + \frac{c}{a+b}(a, b, a+b)^t = \frac{1}{\sqrt[3]{a}}\phi_2\mu(\sqrt[3]{a}T^3) + \frac{1}{\sqrt[3]{b}}\psi_2\mu(\sqrt[3]{b}T^3).$$

Write $g(x) = \mu(\sqrt[3]{x}T^3)/\sqrt[3]{x} = (g_1(x), g_2(x), g_3(x))^t$ for $x > 0$. Now, the equation above is equivalent to

$$\begin{aligned} g_1(a+b) + g_2(a+b) + c &= g_1(a) + g_2(a) + g_1(b) + g_2(b), \\ g_3(a+b) + c &= g_3(a) + g_3(b). \end{aligned} \tag{3.13}$$

By Proposition 3.1 we obtain $g_1(x) = g_2(x) = g_3(x)$. Thus, (3.13) yields

$$\begin{aligned} g_1(a+b) + c/2 &= g_1(a) + g_1(b), \\ g_1(a+b) + c &= g_1(a) + g_1(b). \end{aligned}$$

Therefore, $c = 0$. □

Lemma 3.4. *Let $n \geq 4$. If $\mu : \mathcal{P}_0^n \to \mathbb{R}^n$ is an* SL(n) *covariant valuation, then μ is simple.*

Proof. Notice that for $k \le n-1$, every simplex $T \in \mathcal{T}^k$ is an SL(n) image of T^n. It suffices to prove that μ vanishes on the standard simplex T^k. We prove the statement by induction on $k = \dim T$.

For $k = 0$, let $\mu(\{0\}) = (w_1, \dots, w_n)^t$,

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \sigma_1 = \begin{pmatrix} I_r & & \\ & \sigma & \\ & & I_{n-r-2} \end{pmatrix} \in \mathrm{SL}(n),$$

where $r = 0, 1, \dots, n-2$ and σ moves along the main diagonal of σ_1. Using the SL(n) covariance of μ, we have $\mu(\{0\}) = \mu(\sigma_1 \{0\}) = \sigma_1 \mu(\{0\})$. Therefore, $w_1 = \cdots = w_n = 0$.

For $k = 1$, let $\mu(T^1) = (v_1, \dots, v_n)^t$ and

$$\sigma_3 = \begin{pmatrix} I_l & & \\ & \sigma & \\ & & I_{n-l-2} \end{pmatrix} \in \mathrm{SL}(n),$$

where $l = 1, \dots, n-2$ and σ moves along the main diagonal of σ_3. Using the SL(n) covariance of μ, we obtain $\mu(T^1) = \mu(\sigma_3 T^1) = \sigma_3 \mu(T^1)$. Therefore, $v_2 = \cdots = v_n = 0$ and there exists a constant c such that $\mu(T^1) = 2ce_1$.

For $k = 2$, let $\mu(T^2) = (x_1, \dots, x_n)^t$,

$$\sigma_4 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & I_{n-3} \end{pmatrix} \text{ and } \sigma_5 = \begin{pmatrix} I_r & & \\ & \sigma & \\ & & I_{n-r-2} \end{pmatrix} \in \mathrm{SL}(n),$$

where $r = 2, \dots, n-2$, σ_2 is the same as in the proof of Proposition 3.2, and σ moves along the main diagonal of σ_5. By the SL(n) covariance of μ we have $\mu(T^2) = \mu(\sigma_4 T^2) = \sigma_4 \mu(T^2)$ and $\mu(T^2) = \mu(\sigma_5 T^2) = \sigma_5 \mu(T^2)$. Therefore, $x_1 = x_2$ and $x_3 = \cdots = x_n = 0$. We use the dissection in Definition 2.1. Then (2.2) is equivalent to

$$\begin{pmatrix} \lambda & \lambda & 0 & \cdots & 0 \\ 1-\lambda & 1-\lambda & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\lambda} + \frac{1}{1-\lambda} - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 1-\lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} 2c \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}.$$

This yields $x_1 = c$. Moreover, we know that $\mu(T^2) = c(e_1 + e_2)$ and $\mu([0, e_2, e_3]) = c(e_2 + e_3)$.

For $k = 3$, let $\mu(T^3) = (y_1, \dots, y_n)^t$,

$$\sigma_6 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & I_{n-3} \end{pmatrix}, \text{ and } \sigma_7 = \begin{pmatrix} I_q & & \\ & \sigma & \\ & & I_{n-q-2} \end{pmatrix} \in \mathrm{SL}(n),$$

where $q = 3, \dots, n-2$, σ_0 is the same as in the proof of Proposition 3.1 and σ moves along the main diagonal of σ_7. By the SL(n) covariance of μ we have

$\mu(T^3) = \mu(\sigma_6 T^3) = \sigma_6\mu(T^3)$ and $\mu(T^3) = \mu(\sigma_7 T^3) = \sigma_7\mu(T^3)$. This yields $y_1 = y_2 = y_3$ and $y_4 = \cdots = y_n = 0$.

For T^3, we take the same dissection as above and similarly obtain $y_1 = c = 0$. Therefore, $\mu(T^1) = \mu(T^2) = \mu(T^3) = 0$.

Next, assume that $\mu(T) = 0$ for all T's with $\dim T \le k-1$ and $k \ge 4$. We are going to prove the statement for $\dim T = k \le n-1$. By the induction hypothesis we know that $\mu(T^{k-1}) = 0$. Let $\mu(T^k) = (z_1, \dots, z_n)^t$,

$$\sigma_8 = \begin{pmatrix} I_r & & & \\ & \sigma_0 & & \\ & & I_{k-r-3} & \\ & & & I_{n-k} \end{pmatrix}, \text{ and } \sigma_9 = \begin{pmatrix} I_k & & & \\ & I_l & & \\ & & \sigma & \\ & & & I_{k-l-2} \end{pmatrix},$$

where $r = 0, 1, \dots, k-3, l = 0, \dots, n-k-2$, and σ, σ_0 moves along the main diagonal of σ_8 and σ_9, respectively. By the $\mathrm{SL}(n)$ covariance we have $\mu(T^k) = \mu(\sigma_8 T^k) = \sigma_8\mu(T^k)$ and $\mu(T^k) = \mu(\sigma_9 T^k) = \sigma_9\mu(T^k)$. Therefore, $z_1 = \cdots = z_k$ and $z_{k+1} = \cdots = z_n = 0$. Now, we use a dissection which is slightly different from Definition 2.1. Denote by H_λ the hyperplane through the origin with the normal vector $(1-\lambda)e_{k-1} - \lambda e_k$. Define $\phi \in \mathrm{SL}(n)$ by

$$\phi e_{k-1} = e_{k-1}, \quad \phi e_k = \lambda e_{k-1} + (1-\lambda)e_k, \quad \phi e_n = e_n/(1-\lambda),$$
$$\phi e_j = e_j \quad \text{for} \quad j \ne k-1, k, n,$$

and $\psi \in \mathrm{SL}(n)$ by

$$\psi e_{k-1} = \lambda e_{k-1} + (1-\lambda)e_k, \quad \psi e_k = e_k, \quad \psi e_n = e_n/\lambda, \quad \psi e_j = e_j \quad \text{for } j \ne k-1, k, n.$$

By the $\mathrm{SL}(n)$ covariance and since $\mu(T^{k-1}) = 0$, as in (2.2), we have $\big(\phi + \psi - I_n)\mu(T^k\big) = 0$. This implies $z_1 = \cdots = z_k = 0$. Therefore, the proof of Lemma 3.4 is complete. □

Finally, we obtain the following classification.

Proof of Theorem 1.4. It is clear that the moment vector is an $\mathrm{SL}(n)$ covariant valuation on $\mathcal{P}_0^n$. It remains to show the reverse statement.

We use the dissection in Definition 2.2. By (2.4) and Lemmas 3.3 and 3.4 we obtain for $s > 0$

$$\mu(\sqrt[n]{s}T^n) = \lambda^{-1/n}\phi_2\mu(\sqrt[n]{\lambda s}T^n) + (1-\lambda)^{-1/n}\psi_2\mu(\sqrt[n]{(1-\lambda)s}T^n).$$

By Proposition 3.1 there exists a function $f : [0, \infty) \to \mathbb{R}$ such that $\mu(T^n) = f(1)\mathbf{1}$ and

$$\mathbf{1}f(s^{\frac{1}{n}}) = \lambda^{-\frac{1}{n}}\phi_2\mathbf{1}f\big((s\lambda)^{\frac{1}{n}}\big) + (1-\lambda)^{-\frac{1}{n}}\psi_2\mathbf{1}f\left(\big(s(1-\lambda)\big)^{\frac{1}{n}}\right).$$

In other words,

$$f(s^{\frac{1}{n}}) = \lambda^{\frac{n-1}{n}}f\big((s\lambda)^{\frac{1}{n}}\big) + (1-\lambda)^{-\frac{1}{n}}(1+\lambda)f\left(\big(s(1-\lambda)\big)^{\frac{1}{n}}\right),$$
$$f(s^{\frac{1}{n}}) = (2-\lambda)\lambda^{-\frac{1}{n}}f\big((s\lambda)^{\frac{1}{n}}\big) + (1-\lambda)^{\frac{n-1}{n}}f\left(\big(s(1-\lambda)\big)^{\frac{1}{n}}\right),$$
$$f(s^{\frac{1}{n}}) = \lambda^{-\frac{1}{n}}f\big((s\lambda)^{\frac{1}{n}}\big) + (1-\lambda)^{-\frac{1}{n}}f\left(\big(s(1-\lambda)\big)^{\frac{1}{n}}\right).$$

We set $s = a+b, \lambda = a/(a+b)$ for $a, b > 0$ and $g(x) = x^{-\frac{1}{n}}f(x^{\frac{1}{n}})$ for $x > 0$ to get

$$g(a+b) = g(a) + g(b),$$
$$g(a)/g(b) = a/b.$$

Hence, $f(x) = ax^{n+1}$. By Proposition 3.1 and $m(sT^n) = s^{n+1}\mathbf{1}/(n+1)!$ we know that $\mu(sT^n) = as^{n+1}\mathbf{1} = a(n+1)!m(sT^n)$. In other words, there exists a constant $c \in \mathbb{R}$ such that $\mu(sT^n) = cm(sT^n)$. Therefore, $\mu(T) = cm(T)$ for each $T \in \mathcal{T}^n$. Next, we dissect $P \in \mathcal{P}_0^n$ into simplices with one vertex at the origin. Since μ is simple and by the inclusion-exclusion principle, we obtain $\mu(P) = cm(P)$. □

4. SL(n) covariant valuations on $\mathcal{P}^n$

4.1. **The two-dimensional case.** First, we consider $s\tilde{T}^1$ for $s > 0$.

Lemma 4.1. *If $\mu : \mathcal{P}^2 \to \mathbb{R}^2$ is an* SL(2) *covariant valuation, then there exist constants $c_1, c_2 \in \mathbb{R}$ and a solution of Cauchy's functional equation $\beta : [0, \infty) \to \mathbb{R}$ such that*

$$\mu(s\tilde{T}^1) = \tilde{c}_1 m([0, s\tilde{T}^1]) + \tilde{c}_2 s(e_1 + e_2) + \frac{\beta(s^2)}{s}(e_1 - e_2)$$

for $s > 0$.

Proof. First, we decompose μ as $\mu = \mu^+ + \mu^-$, where μ^+ is an SL$^\pm$(2) covariant valuation and μ^- is an SL$^\pm$(2) signum covariant one.

Next, let $v = (v_1, v_2)^t \in \mathbb{R}^2$, with $v_1 v_2 \neq 0$. We have $v = \rho_1 e_1 = \rho_2 e_1$ for the same ρ_1 and ρ_2 as in the proof of Lemma 3.1. The SL$^\pm$(2) covariance of μ^+ implies

$$\begin{aligned} \mu^+(\{v\}) &= \mu^+(\rho_1 \{e_1\}) = \rho_1 \mu^+ \{e_1\} \\ &= \mu^+(\rho_2 \{e_1\}) = \rho_2 \mu^+ \{e_1\}. \end{aligned}$$

Setting $\mu^+(\{e_1\}) = (\tilde{x}_1^+, \tilde{x}_2^+)^t$, we obtain

$$\begin{aligned} v_1 \tilde{x}_1^+ &= v_1 \tilde{x}_1^+, \\ v_2 \tilde{x}_1^+ + \tilde{x}_2^+/v_1 &= v_2 \tilde{x}_1^+ - \tilde{x}_2^+/v_1. \end{aligned}$$

Thus, $\tilde{x}_2^+ = 0$ and there exists a constant $\tilde{c} \in \mathbb{R}$ such that $\mu^+(\{e_1\}) = \tilde{c}e_1$. For $s > 0$, applying the same ρ_0 as in the proof of Lemma 3.1, we obtain

$$\mu^+(\{se_1\}) = \mu^+(\rho_0 \{e_1\}) = \rho_0 \mu^+(\{e_1\}) = \tilde{c}se_1. \tag{4.1}$$

On the other hand, the SL$^\pm$(2) signum covariance of μ^- implies

$$\begin{aligned} \mu^-(\{v\}) &= \mu^-(\rho_1 \{e_1\}) = \rho_1 \mu^-(\{e_1\}) \\ &= \mu^-(\rho_2 \{e_1\}) = -\rho_2 \mu^-(\{e_1\}) \\ &= \mu^-(\rho_3 \{e_1\}) = \rho_3 \mu^-(\{e_1\}), \end{aligned}$$

where ρ_3 is the same as in the proof of Lemma 3.1. Setting $\mu^-(\{e_1\}) = (\tilde{x}_1^-, \tilde{x}_2^-)^t$, we obtain

$$\begin{aligned} v_1 \tilde{x}_1^- &= -v_1 \tilde{x}_1^- = v_1 \tilde{x}_1^- - \tilde{x}_2^-/v_2, \\ v_2 \tilde{x}_1^- + \tilde{x}_2^-/v_1 &= -v_2 \tilde{x}_1^- + \tilde{x}_2^-/v_1 = v_2 \tilde{x}_1^-. \end{aligned}$$

Thus, $\tilde{x}_1^- = \tilde{x}_2^- = 0$, which implies $\mu^-(\{e_1\}) = 0$. Similarly, we have

$$\mu^-(\{se_1\}) = 0 \tag{4.2}$$

for $s > 0$ and $\mu(\{v\}) = \mu(\rho_1 \{e_1\}) = \rho_1(\mu^+ \{e_1\} + \mu^- \{e_1\}) = \tilde{c}v$.

Second, we use the dissection in Definition 2.2. By the valuation property of μ^+, (2.4), and (4.1), we obtain

$$\mu^+(\sqrt{s}\tilde{T}^1)+\tilde{c}\sqrt{s}(\lambda,1-\lambda)^t = \sqrt{\lambda}^{-1}\phi_2\mu^+(\sqrt{\lambda s}\tilde{T}^1)+\sqrt{1-\lambda}^{-1}\psi_2\mu^+(\sqrt{(1-\lambda)s}\tilde{T}^1).$$

Setting $\lambda = a/(a+b)$ and $s = a+b$ for $a, b > 0$, we have

$$\frac{1}{\sqrt{a+b}}\mu^+(\sqrt{a+b}\tilde{T}^1)+\frac{c}{a+b}(a,b)^t = \frac{1}{\sqrt{a}}\phi_2\mu^+(\sqrt{a}\tilde{T}^1)+\frac{1}{\sqrt{b}}\psi_2\mu^+(\sqrt{b}\tilde{T}^1).$$

Write $g^+(x) = \mu^+(\sqrt{x}\tilde{T}^1)/\sqrt{x} = (g_1^+(x), g_2^+(x))^t$ for $x > 0$. Then, the equation above becomes

$$\begin{aligned} g_1^+(a+b)+\frac{\tilde{c}a}{a+b} &= \frac{a}{a+b}g_1^+(a)+g_1^+(b)+\frac{a}{a+b}g_2^+(b), \\ g_2^+(a+b)+\frac{\tilde{c}b}{a+b} &= \frac{b}{a+b}g_1^+(a)+g_2^+(a)+\frac{b}{a+b}g_2^+(b). \end{aligned} \tag{4.3}$$

As in the proof of Lemma 3.1, we obtain $g_1^+ = g_2^+$. Combined with (4.3), it follows that there exist constants $\tilde{c}_1', \tilde{c}_2$ such that $g_1^+(x) = g_2^+(x) = \tilde{c}_1'x+\tilde{c}_2$, where $\tilde{c}_2 = \tilde{c}/2$. Therefore,

$$\mu^+(s\tilde{T}^1) = \tilde{c}_1's^3(e_1+e_2)+\tilde{c}_2s(e_1+e_2) = \tilde{c}_1m([0,s\tilde{T}^1])+\tilde{c}_2s(e_1+e_2), \tag{4.4}$$

where $\tilde{c}_1 = 6\tilde{c}_1'$ and in the second step we use $m([0,s\tilde{T}^1]) = s^3(e_1+e_2)/3!$.

On the other hand, by the valuation property of μ^-, (2.4) and (4.2), we obtain

$$\mu^-(\sqrt{s}\tilde{T}^1) = \sqrt{\lambda}^{-1}\phi_2\mu^-(\sqrt{\lambda s}\tilde{T}^1)+\sqrt{1-\lambda}^{-1}\psi_2\mu^-(\sqrt{(1-\lambda)s}\tilde{T}^1).$$

Putting $\lambda = a/(a+b)$ and $s = a+b$ for $a, b > 0$, we obtain

$$\frac{1}{\sqrt{a+b}}\mu^-(\sqrt{a+b}\tilde{T}^1) = \frac{1}{\sqrt{a}}\phi_2\mu^-(\sqrt{a}\tilde{T}^1)+\frac{1}{\sqrt{b}}\psi_2\mu^-(\sqrt{b}\tilde{T}^1).$$

Write $g^-(x) = \mu^-(\sqrt{x}\tilde{T}^1)/\sqrt{x} = (g_1^-(x), g_2^-(x))^t$ for $x > 0$. Then, the equation above becomes

$$\begin{aligned} g_1^-(a+b) &= \frac{a}{a+b}g_1^-(a)+g_1^-(b)+\frac{a}{a+b}g_2^-(b), \\ g_2^-(a+b) &= \frac{b}{a+b}g_1^-(a)+g_2^-(a)+\frac{b}{a+b}g_2^-(b). \end{aligned}$$

Moreover, applying the same σ as in the proof of Lemma 3.1, we have $\mu^-(s\tilde{T}^1) = \mu^-(\sigma s\tilde{T}^1) = -\sigma\mu^-(s\tilde{T}^1)$. Then, $\mu_1^-(s\tilde{T}^1)+\mu_2^-(s\tilde{T}^1) = 0$, which implies $g_1^-+g_2^- = 0$. Hence,

$$(a+b)g_1^-(a+b) = ag_1^-(a)+bg_1^-(b).$$

Therefore, $g_1^-(x) = -g_2^-(x) = \beta(x)/x$, where $\beta : [0,\infty) \to \mathbb{R}$ is a solution of Cauchy's functional equation. It follows that

$$\mu^-(s\tilde{T}^1) = \frac{\beta(s^2)}{s}(e_1-e_2). \tag{4.5}$$

Combining (4.4) and (4.5) completes the proof. □

Next, we derive the representation for one-dimensional convex polygons.

Lemma 4.2. *If $\mu : \mathcal{P}^2 \to \mathbb{R}^2$ is an* SL(2) *covariant valuation, then there exist constants $c_2, \tilde{c}_1, \tilde{c}_2$ and a solution of Cauchy's functional equation $\beta : [0,\infty) \to \mathbb{R}$ such that*

$$\mu(P) = \begin{cases} \tilde{c}_1 m([0,P]) + \tilde{c}_2(v_1+v_2) \\ \quad + \frac{\beta(\det(v_1,v_2))}{\det(v_1,v_2)}(v_1-v_2) & \text{if } 0 \notin \operatorname{aff} P \text{ and} \det(v_1,v_2) > 0; \\ 2(\tilde{c}_2 - c_2)v_1 + 2c_2 v_2 & \text{if } 0 \in \operatorname{aff} P \setminus P, \end{cases}$$

for every line segment $P = [v_1, v_2]$ in $\mathcal{P}^2$.

Proof. First, assume that $0 \notin \operatorname{aff} P$ and $\phi = (v_1, v_2) \in \mathrm{GL}(2)$ such that $\phi e_1 = v_1$ and $\phi e_2 = v_2$. By Lemma 4.1 there exist constants $\tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ and a solution of Cauchy's functional equation $\beta : [0,\infty) \to \mathbb{R}$ such that

$$\begin{aligned} \mu(P) = \mu(\phi \tilde{T}^1) =& \sqrt{\det(v_1,v_2)}^{-1} \phi\mu\left(\sqrt{\det(v_1,v_2)}\tilde{T}^1\right) \\ =& \tilde{c}_1 m([0,P]) + \tilde{c}_2(v_1+v_2) + \frac{\beta(\det(v_1,v_2))}{\det(v_1,v_2)}(v_1 - v_2). \end{aligned}$$

Second, assume that $0 \in \operatorname{aff} P \setminus P$. Then $0, v_1$, and v_2 are on the same line. Since μ is a valuation, we obtain $\mu([0,v_1]) + \mu([v_1,v_2]) = \mu([0,v_2]) + \mu(\{v_1\})$. Since there exists a constant $c_2 \in \mathbb{R}$ such that $\mu([0,v]) = 2c_2 v$ and $\mu(\{v_1\}) = 2\tilde{c}_2 v_1$, we have $\mu(P) = 2(\tilde{c}_2 - c_2)v_1 + 2c_2 v_2$. □

Finally, we treat convex polygons of dimension two.

Lemma 4.3. *If $\mu : \mathcal{P}^2 \to \mathbb{R}^2$ is an* SL(2) *covariant valuation, then there exist constants $c_1, c_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ and solutions of Cauchy's functional equation $\alpha, \gamma : [0,\infty) \to \mathbb{R}$ such that*

$$\begin{aligned} \mu(P) = (c_1 - \tilde{c}_1)m(P) + \tilde{c}_1 m([0,P]) + c_2 e([0,P]) + \tilde{c}_2 e([0,v_1,\dots,v_r]) + h_\alpha([0,P]) \\ + \sum_{i=2}^{r} h_\gamma([0,v_{i-1},v_i]) \end{aligned}$$

for every $P \in \mathcal{P}^2 \setminus \mathcal{P}^2_0$ with $\dim P = 2$, with vertices $v_1, \dots, v_r$ visible from the origin and labeled counterclockwise.

Proof. Let $P \in \mathcal{P}^2 \setminus \mathcal{P}^2_0$. Let $E_i = [v_i, v_{i+1}]$ be the edges of P visible from the origin for $i = 1, \dots, r$. Assume that the edges $E_1, E_2, \dots, E_r$ are labeled counterclockwise. Clearly, $[0,P] = P \cup [0,E_1] \cup \dots \cup [0,E_r]$. Note that $[0,P], [0,E_1], \dots, [0,E_r] \in \mathcal{P}^2_0$. By the inclusion-exclusion principle, Theorem 1.5, and (4.1), we have

$$\begin{aligned} \mu([0,P]) =& \mu(P) + \sum_{i=1}^{r} \mu[0,E_i] - \sum_{i=1}^{r} \mu(\underbrace{[0,E_i] \cap P}_{=E_i}) - \sum_{1 \le j < k \le r} \mu(\underbrace{[0,E_j] \cap [0,E_k]}_{\in \mathcal{P}^2_0}) \\ &+ \sum_{1 \le j < k \le r} \mu([0,E_j] \cap [0,E_k] \cap P). \end{aligned}$$

Thus, there exist solutions of Cauchy's functional equation $\alpha, \beta, \gamma : [0, \infty) \to \mathbb{R}$ such that

$$\begin{aligned}
\mu(P) =&\mu([0,P]) - \sum_{i=1}^{r} \mu[0,E_i] + \sum_{i=1}^{r} \mu(E_i) + \sum_{i=2}^{r-1} \mu([0,v_i]) - \sum_{i=2}^{r-1} \mu(\{v_i\}) \\
=&c_1 m([0,P]) + c_2 e([0,P]) + h_\alpha([0,P]) - c_1 m\big(\mathrm{cl}([0,P] \setminus P)\big) \\
&- c_2\big(v_1 + 2\sum_{i=1}^{r-1} v_i + v_r\big) \\
&- \sum_{i=2}^{r} \frac{\alpha(\det(v_{i-1},v_i))}{\det(v_{i-1},v_i)}(v_{i-1}-v_i) + \tilde{c}_1 m\big(\mathrm{cl}([0,P] \setminus P)\big) \\
&+ \tilde{c}_2\big(v_1 + \sum_{i=2}^{r-1} v_i + v_r\big) \\
&+ \sum_{i=2}^{r} \frac{\beta(\det(v_{i-1},v_i))}{\det(v_{i-1},v_i)}(v_{i-1}-v_i) + 2c_2 \sum_{i=2}^{r-1} v_i - 2\tilde{c}_2 \sum_{i=2}^{r-1} v_i \\
=&(c_1 - \tilde{c}_1) m(P) + \tilde{c}_1 m([0,P]) + h_\alpha([0,P]) + c_2 e([0,P]) + \tilde{c}_2(v_1 + v_r) \\
&- \sum_{i=2}^{r} \frac{\alpha(\det(v_{i-1},v_i))}{\det(v_{i-1},v_i)}(v_{i-1}-v_i) + \sum_{i=2}^{r} \frac{\beta(\det(v_{i-1},v_i))}{\det(v_{i-1},v_i)}(v_{i-1}-v_i) \\
=&(c_1 - \tilde{c}_1) m(P) + \tilde{c}_1 m([0,P]) + h_\alpha([0,P]) + c_2 e([0,P]) + \tilde{c}_2(v_1 + v_r) \\
&+ \sum_{i=2}^{r} \gamma \frac{\alpha(\det(v_{i-1},v_i))}{\det(v_{i-1},v_i)}(v_{i-1}-v_i) \\
=&(c_1 - \tilde{c}_1) m(P) + \tilde{c}_1 m([0,P]) + \tilde{c}_2 e([0,P]) + \tilde{c}_2 e([0,v_1,\ldots,v_r]) + h_\alpha([0,P]) \\
&+ \sum_{i=2}^{r} h_\gamma([0,v_{i-1},v_i]).
\end{aligned}$$

□

Using Theorem 1.5 and Lemmas 4.2 and 4.3, we complete the proof of Theorem 1.7. Similarly, we obtain Corollary 1.2.

4.2. **The higher-dimensional case.** We consider $\mathrm{SL}(n)$ covariant valuations on $\tilde{\mathcal{T}}^k$ for the three-dimensional case and the n-dimensional case for $n \geq 4$, respectively.

Lemma 4.4. *If $\mu : \mathcal{P}^3 \to \mathbb{R}^3$ is an $\mathrm{SL}(3)$ covariant valuation, then $\mu(T) = 0$ for every $T \in \tilde{\mathcal{T}}^k$ with $0 \leq k \leq 1$.*

Proof. It suffices to consider the valuation on $\{e_1\}, \tilde{T}^1$, and $\tilde{T}^2$. First, applying the same σ_1 as in the proof of Proposition 3.2 shows that there exists a constant $c \in \mathbb{R}$ such that $\mu(\{e_1\}) = \mu(\sigma_1\{e_1\}) = \sigma_1 \mu(\{e_1\}) = 2ce_1$.

Let $\mu(\tilde{T}^1) = (x_1, x_2, x_3)^t$, and let σ_2 be the same as in the proof of Proposition 3.2. The $\mathrm{SL}(3)$ covariance of μ implies that $\mu(\tilde{T}^1) = \mu(\sigma_2 \tilde{T}^1) = \sigma_2 \mu(\tilde{T}^1)$. Then $\mu(\tilde{T}^1) = (x_1, x_1, 0)^t$. Let $v = \lambda e_1 + (1-\lambda)e_2$ where $\lambda \in (0,1)$. We use the dissection in Definition 2.1. By the valuation property of μ we have

$$\mu(\tilde{T}^1) + \mu(\{v\}) = \mu(\phi_1 \tilde{T}^1) + \mu(\psi_1 \tilde{T}^1).$$

Using the SL(3) covariance of μ, we obtain $\mu(\tilde{T}^1) = c(e_1 + e_2)$. Let $\tilde{T}_{23} = [e_2, e_3]$. Since $\mu(\tilde{T}_{23}) = \mu(\sigma_0 \tilde{T}^1) = \sigma_0 \mu(\tilde{T}^1)$ for the same σ_0 as in the proof of Proposition 3.1, we have $\mu(\tilde{T}_{23}) = c(e_2 + e_3)$. Note that

$$\mu(s\tilde{T}_{23}) = \mu(\rho\tilde{T}_{23}) = \rho\mu(\tilde{T}_{23}) = cs(e_2 + e_3) \tag{4.6}$$

for the same ρ as in the proof of Lemma 3.3 and every $s > 0$.

Next, we use the dissection in Definition 2.2. By (2.4), (3.3), and (4.6) it follows that

$$\mu(\sqrt[3]{s}\tilde{T}^2) + c\sqrt[3]{s}(\lambda, 1-\lambda, 1)^t = \lambda^{-1/3}\phi_2\mu(\sqrt[3]{\lambda s}\tilde{T}^2) + (1-\lambda)^{-1/3}\psi_2\mu(\sqrt[3]{(1-\lambda)s}\tilde{T}^2).$$

Setting $\lambda = a/(a+b), s = a+b$ for $a, b > 0$ and $g(x) = \mu(\sqrt[3]{x}\tilde{T}^2)/\sqrt[3]{x} = (g_1(x), g_2(x), g_3(x))^t$ for $x > 0$, we obtain

$$\begin{aligned} g_1(a+b) + \frac{ca}{a+b} &= \frac{a}{a+b}g_1(a) + g_1(b) + \frac{a}{a+b}g_2(b), \\ g_2(a+b) + \frac{cb}{a+b} &= \frac{b}{a+b}g_1(a) + g_2(a) + \frac{b}{a+b}g_2(b), \\ g_3(a+b) + c &= g_3(a) + g_3(b). \end{aligned}$$

Due to Proposition 3.1, we have $g_1(x) = g_2(x) = g_3(x)$. It follows that $\mu(\{e_1\}) = \mu(\tilde{T}^1) = 0$. □

Lemma 4.5. *Let $n \geq 4$. If $\mu : \mathcal{P}^n \to \mathbb{R}^n$ is an* SL(n) *covariant valuation, then $\mu(T) = 0$ for every $T \in \tilde{\mathcal{T}}^k$ with $0 \leq k \leq n-2$.*

Proof. It suffices to prove that μ vanishes on $\tilde{\mathcal{T}}^k$ for $0 \leq k \leq n-2$. We prove the statement by induction on $k = \dim T$. For $k = 0$, write $\mu(\{e_1\}) = x = (x_1 \ldots, x_n)^t$. By the SL($n$) covariance of μ we have $\mu(\{e_1\}) = \mu(\sigma_3 \{e_1\}) = \sigma_3\mu(\{e_1\})$. Hence, $x_2 = \cdots = x_n = 0$, and there exists a constant c such that $\mu(\{e_1\}) = 2ce_1$.

For $k = 1$, write $\mu(\tilde{T^1}) = (x_1, \ldots, x_n)^t$. Using the SL($n$) covariance of μ, we have $\mu(\tilde{T^1}) = \mu(\sigma_4\tilde{T^1}) = \sigma_4\mu(\tilde{T^1})$ and $\mu(\tilde{T^1}) = \mu(\sigma_5\tilde{T^1}) = \sigma_5\mu(\tilde{T^1})$ for the same σ_4 and σ_5 as in the proof of Lemma 3.4. Therefore, $x_1 = x_2$ and $x_3 = x_4 = \cdots = x_n = 0$. Moreover, we know that $\mu(\tilde{T^1}) = c(e_1 + e_2)$ and $\mu([e_2, e_3]) = c(e_2 + e_3)$.

For $k = 2$, write $\mu(\tilde{T^2}) = (y_1, \ldots, y_n)^t$. By the SL($n$) covariance of μ we have $\mu(\tilde{T^2}) = \mu(\sigma_6 T^2) = \sigma_6\mu(\tilde{T^2})$ and $\mu(\tilde{T^2}) = \mu(\sigma_7\tilde{T^2}) = \sigma_7\mu(\tilde{T^2})$ for the same σ_6 and σ_7 as in the proof of Lemma 3.4. This yields $y_1 = y_2 = y_3$ and $y_4 = \cdots = y_n = 0$. We use the dissection in Definition 2.1. Since μ is an SL(n) covariant valuation, we have $\big(\phi_1 + \psi_1 - I_n\big)\mu(\tilde{T}^2) = \psi_1\mu([e_2, e_3])$. Thus, the equation above is equivalent to $y_1 = c = 0$. Therefore, we obtain $\mu(\{e_1\}) = \mu(\tilde{T}^1) = \mu(\tilde{T}^2) = 0$.

Next assume that $\mu(\tilde{T}) = 0$ for all $\tilde{T}$'s with $\dim\tilde{T} \leq k-1$. We prove the statement for $\dim\tilde{T} = k \leq n-2$. By the induction hypothesis we know that $\mu(\tilde{T}^{k-1}) = 0$. Let $\mu(\tilde{T}^k) = (z_1, \ldots, z_n)^t$. By the SL($n$) covariance we have $\mu(\tilde{T}^k) = \mu(\sigma_8\tilde{T}^k) = \sigma_8\mu(\tilde{T}^k)$ and $\mu(\tilde{T}^k) = \mu(\sigma_9\tilde{T}^k) = \sigma_9\mu(\tilde{T}^k)$ for the same σ_8 and σ_9 as in the proof of Lemma 3.4. Therefore, $z_1 = \cdots = z_k$, and $z_{k+1} = \cdots = z_n = 0$.

Denote by H_λ the hyperplane through $\lambda e_{k-1} + (1-\lambda)e_k$ and e_i for $i \neq k-1, k$. Then H_λ dissects $\tilde{T}^k$ into $\phi_2\tilde{T}^k$ and $\psi_2\tilde{T}^k$ in a way that is as in the dissection in Definition 2.1. Since μ is a valuation, we have

$$\mu(\tilde{T}^k) + \mu(\psi_2\tilde{T}^{k-1}) = \mu(\phi_2\tilde{T}^k) + \mu(\psi_2\tilde{T}^k).$$

By the $\mathrm{SL}(n)$ covariance and since $\mu(\tilde{T}^{k-1})=0$ the equation above can be rewritten as $\big(\phi_2+\psi_2-I_n\big)\mu(\tilde{T}^k\big)=0$. This yields $z_1=\cdots=z_k=0$, which completes the proof. □

Lemma 4.6. *Let $n\geq 3$. If $\mu:\mathcal{P}^n\to\mathbb{R}^n$ is an $\mathrm{SL}(n)$ covariant valuation, then μ vanishes on every polytope $P\in\mathcal{P}^n$ with $\dim P\leq n-2$.*

Proof. Note that μ vanishes on at most $(n-1)$-dimensional polytopes in $\mathcal{P}^n_0$, and thus we just need to take care of polytopes in $\mathcal{P}^n\setminus\mathcal{P}^n_0$. We assume that $P\in\mathcal{P}^n\setminus\mathcal{P}^n_0$ and prove the statement by induction on $k=\dim P$. For $k=0$, by Lemmas 4.4 and 4.5, we have $\mu(\{x\})=\mu(\{e_1\})=0$. Assume $\mu(P)=0$ for all $P\in\mathcal{P}^n\setminus\mathcal{P}^n_0$ with $\dim P\leq k-1$. We prove the statement for $\dim P=k\leq n-2$.

First, let P be a k-dimensional polytope with $0\notin\operatorname{aff}P$. Triangulate P into k-dimensional simplices $T_1,\ldots,T_r$. By the inclusion-exclusion principle, the induction assumption, and Lemmas 4.4 and 4.5, we have $\mu(P)=0$.

Second, let P be a k-dimensional polytope with $0\in\operatorname{aff}P$. Let $F_1,\ldots,F_r$ be the facets of P visible from the origin. Triangulate the facets F_i into $(k-1)$-dimensional simplices $T_1',\ldots,T_l'$ and thus the closure of $[0,P]\backslash P$ into simplices $T_1=[0,T_1'],\ldots,T_l=[0,T_l']$, with a vertex at the origin. Using the inclusion-exclusion principle, that μ vanishes on $\mathcal{P}^n_0$ and the induction assumption, we have

$$\begin{aligned}0=\mu(\underbrace{[0,P]}_{\in\mathcal{P}^n_0})=&\sum_{j=1}^{r}(-1)^{j-1}\sum_{1\leq i_1\leq\cdots\leq i_j\leq r}\mu(\underbrace{T_{i_1}\cap\cdots\cap T_{i_j}}_{\in\mathcal{P}^n_0})\\&+\sum_{j=1}^{r}(-1)^{j}\sum_{1\leq i_1\leq\cdots\leq i_j\leq r}\mu(\underbrace{T_{i_1}\cap\cdots\cap T_{i_j}\cap P}_{\dim\leq k-1})+\mu(P)\\=&\mu(P).\end{aligned}$$

This completes the proof. □

Next, we establish the classification on all convex polytopes of dimension $n-1$.

Lemma 4.7. *Let $n\geq 3$. If $\mu:\mathcal{P}^n\to\mathbb{R}^n$ is an $\mathrm{SL}(n)$ covariant valuation, then there exists a constant $\tilde{c}\in\mathbb{R}$ such that*

$$\mu(P)=\tilde{c}m([0,P])$$

for every $(n-1)$-dimensional polytope $P\in\mathcal{P}^n$.

Proof. First, it suffices to consider $s\tilde{T}^{n-1}$ for $s>0$. We use the dissection in Definition 2.2. By (2.4), (3.3), and Lemma 4.6 we have

$$\mu(\sqrt[n]{s}\tilde{T}^{n-1})=\lambda^{-1/n}\phi_2\mu(\sqrt[n]{\lambda s}\tilde{T}^{n-1})+(1-\lambda)^{-1/n}\psi_2\mu(\sqrt[n]{(1-\lambda)s}\tilde{T}^{n-1}).$$

As in Proposition 3.1, there exists a function f on $\mathbb{R}$ such that $\mu(\tilde{T}^{n-1})=f(1)\mathbf{1}$ and

$$\mathbf{1}f(s^{\frac{1}{n}})=\lambda^{-\frac{1}{n}}\phi_2\mathbf{1}f\big((s\lambda)^{\frac{1}{n}}\big)+(1-\lambda)^{-\frac{1}{n}}\psi_2\mathbf{1}f\left(\big(s(1-\lambda)\big)^{\frac{1}{n}}\right).$$

Furthermore, using a similar argument as in the proof of Theorem 1.4, we obtain that there exists a constant $c_2\in\mathbb{R}$ such that

$$\mu(s\tilde{T}^{n-1})=c_2m([0,s\tilde{T}^{n-1}]).\tag{4.7}$$

Second, let P be an $(n-1)$-dimensional polytope with $0 \notin \operatorname{aff} P$. Triangulate P into simplices $T_1, \dots, T_r$. Using the inclusion-exclusion principle, (4.7), and Lemma 4.6 we have

$$\mu(P) = \sum_{j=1}^{r} \mu(T_j) = c_2 m([0, P]).$$

Finally, let P be an $(n-1)$-dimensional polytope with $0 \in \operatorname{aff} P$. Then the polytope $[0, P]$ is $(n-1)$ dimensional and $m([0, P]) = 0$. Thus, for $P \in \mathcal{P}_0^n$ the assertion is trivial. Assume that $0 \notin P$ and triangulate the facets of P visible from the origin as in the proof of Lemma 4.6. Dissect the closure of $[0, P] \setminus P$ into simplices $T_1, \dots, T_r$ with a vertex at the origin. From Lemmas 3.3, 3.4, and 4.6, and the inclusion-exclusion principle, we obtain

$$0 = \mu([0, P]) = \sum_{j=1}^{r} \mu(T_j) + \mu(P) = \mu(P),$$

which completes the proof of the lemma. □

Finally, we establish the classification in Theorem 1.6.

Proof of Theorem 1.6. It is clear that the expression in (1.1) is an SL(n) covariant valuation. It remains to show the reverse statement.

For $P \in \mathcal{P}_0^n$, by $m\big(\mathrm{cl}([0, P] \setminus P)\big) = 0$ and Theorem 1.4, the assertion holds. So we focus on the polytopes in $\mathcal{P}^n \setminus \mathcal{P}_0^n$. Assume that $P \in \mathcal{P}^n \setminus \mathcal{P}_0^n$ with dimension n. Let $F_1, \dots, F_r$ be the facets of P visible from the origin. By Theorem 1.4, Lemmas 4.6 and 4.7, and the inclusion-exclusion principle there exist constants $c, \tilde{c} \in \mathbb{R}$ such that

$$\begin{aligned}
cm([0, P]) =& \mu([0, P]) \\
=& \sum_{j=1}^{r} (-1)^{j-1} \sum_{1 \le i_1 \le \cdots \le i_j \le r} \mu(\underbrace{[0, F_{i_1}] \cap \cdots \cap [0, F_{i_j}]}_{\in \mathcal{P}_0^n}) \\
&+ \sum_{j=2}^{r} (-1)^{j} \sum_{1 \le i_1 \le \cdots \le i_j \le r} \mu(\underbrace{[0, F_{i_1}] \cap \cdots \cap [0, F_{i_j}] \cap P}_{\dim \le n-2}) \\
&- \sum_{i=1}^{r} \mu(\underbrace{[0, F_i] \cap P}_{= F_i}) + \sum_{i=1}^{r} \mu\left([0, F_i]\right) + \mu(P) \\
=& \sum_{i=1}^{r} \mu[0, F_i] + \mu(P) - \sum_{i=1}^{r} \mu(F_i) \\
=& c \sum_{i=1}^{r} m([0, F_i]) + \mu(P) - \tilde{c} \sum_{i=1}^{r} m([0, F_i]).
\end{aligned}$$

Since the moment vector is a simple valuation on $\mathcal{P}^n$, we have $\mu(P) = (c - \tilde{c}) m(P) + \tilde{c} m([0, P])$. □

Acknowledgment

The authors wish to thank the referee for the valuable suggestions and the careful reading of the original manuscript.

References

[1] S. Alesker, *Continuous rotation invariant valuations on convex sets*, Ann. of Math. (2) **149** (1999), no. 3, 977–1005, DOI 10.2307/121078. MR1709308

[2] S. Alesker, *Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture*, Geom. Funct. Anal. **11** (2001), no. 2, 244–272, DOI 10.1007/PL00001675. MR1837364

[3] Christoph Haberl, *Blaschke valuations*, Amer. J. Math. **133** (2011), no. 3, 717–751, DOI 10.1353/ajm.2011.0019. MR2808330

[4] Christoph Haberl, *Minkowski valuations intertwining with the special linear group*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 5, 1565–1597, DOI 10.4171/JEMS/341. MR2966660

[5] Christoph Haberl and Lukas Parapatits, *The centro-affine Hadwiger theorem*, J. Amer. Math. Soc. **27** (2014), no. 3, 685–705, DOI 10.1090/S0894-0347-2014-00781-5. MR3194492

[6] Christoph Haberl and Lukas Parapatits, *Valuations and surface area measures*, J. Reine Angew. Math. **687** (2014), 225–245, DOI 10.1515/crelle-2012-0044. MR3176613

[7] Christoph Haberl and Lukas Parapatits, *Moments and valuations*, Amer. J. Math. **138** (2016), no. 6, 1575–1603, DOI 10.1353/ajm.2016.0047. MR3595495

[8] Christoph Haberl and Lukas Parapatits, *Centro-affine tensor valuations*, Adv. Math. **316** (2017), 806–865, DOI 10.1016/j.aim.2017.06.030. MR3672921

[9] Daniel A. Klain, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), no. 1, 80–101, DOI 10.1006/aima.1996.0048. MR1399604

[10] Daniel A. Klain and Gian-Carlo Rota, *Introduction to geometric probability*, Lezioni Lincee. [Lincei Lectures], Cambridge University Press, Cambridge, 1997. MR1608265

[11] Jin Li and Gangsong Leng, *L_p Minkowski valuations on polytopes*, Adv. Math. **299** (2016), 139–173, DOI 10.1016/j.aim.2016.05.009. MR3519466

[12] Jin Li and Dan Ma, *Laplace transforms and valuations*, J. Funct. Anal. **272** (2017), no. 2, 738–758, DOI 10.1016/j.jfa.2016.09.011. MR3571907

[13] Jin Li, Shufeng Yuan, and Gangsong Leng, *L_p-Blaschke valuations*, Trans. Amer. Math. Soc. **367** (2015), no. 5, 3161–3187, DOI 10.1090/S0002-9947-2015-06047-4. MR3314805

[14] Monika Ludwig, *Moment vectors of polytopes*, Rend. Circ. Mat. Palermo (2) Suppl. **70** (2002), 123–138. IV International Conference in "Stochastic Geometry, Convex Bodies, Empirical Measures & Applications to Engineering Science", Vol. II (Tropea, 2001). MR1962589

[15] Monika Ludwig, *Projection bodies and valuations*, Adv. Math. **172** (2002), no. 2, 158–168, DOI 10.1016/S0001-8708(02)00021-X. MR1942402

[16] Monika Ludwig, *Valuations of polytopes containing the origin in their interiors*, Adv. Math. **170** (2002), no. 2, 239–256, DOI 10.1006/aima.2002.2077. MR1932331

[17] Monika Ludwig, *Ellipsoids and matrix-valued valuations*, Duke Math. J. **119** (2003), no. 1, 159–188, DOI 10.1215/S0012-7094-03-11915-8. MR1991649

[18] Monika Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4191–4213, DOI 10.1090/S0002-9947-04-03666-9. MR2159706

[19] Monika Ludwig, *Intersection bodies and valuations*, Amer. J. Math. **128** (2006), no. 6, 1409–1428. MR2275906

[20] Monika Ludwig, *Minkowski areas and valuations*, J. Differential Geom. **86** (2010), no. 1, 133–161. MR2772547

[21] Monika Ludwig, *Fisher information and matrix-valued valuations*, Adv. Math. **226** (2011), no. 3, 2700–2711, DOI 10.1016/j.aim.2010.08.021. MR2739790

[22] Monika Ludwig and Matthias Reitzner, *A classification of* $\mathrm{SL}(n)$ *invariant valuations*, Ann. of Math. (2) **172** (2010), no. 2, 1219–1267, DOI 10.4007/annals.2010.172.1223. MR2680490

[23] Monika Ludwig and Matthias Reitzner, $\mathrm{SL}(n)$ *invariant valuations on polytopes*, Discrete Comput. Geom. **57** (2017), no. 3, 571–581, DOI 10.1007/s00454-016-9838-7. MR3614772

[24] Lukas Parapatits, $\mathrm{SL}(n)$*-contravariant L_p-Minkowski valuations*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1195–1211, DOI 10.1090/S0002-9947-2013-05750-9. MR3145728

[25] Lukas Parapatits, $\mathrm{SL}(n)$*-covariant L_p-Minkowski valuations*, J. Lond. Math. Soc. (2) **89** (2014), no. 2, 397–414, DOI 10.1112/jlms/jdt068. MR3188625

[26] Rolf Schneider, *Convex bodies: The Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. MR3155183

[27] Franz E. Schuster and Thomas Wannerer, GL(n) *contravariant Minkowski valuations*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 815–826, DOI 10.1090/S0002-9947-2011-05364-X. MR2846354

[28] Thomas Wannerer, GL(n) *equivariant Minkowski valuations*, Indiana Univ. Math. J. **60** (2011), no. 5, 1655–1672, DOI 10.1512/iumj.2011.60.4425. MR2997003

School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, People's Republic of China; and Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8–10/1046, 1040 Wien, Austria

Email address: zengchn@163.com

Department of Mathematics, Shanghai Normal University, Shanghai 200234, People's Republic of China

Email address: madan@shnu.edu.cn