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INTERPOLATION OF THE MEASURE OF NONCOMPACTNESS OF BILINEAR OPERATORS

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Abstract. We study interpolation of the measure of noncompactness of bilinear operators. We prove a result of a general nature which states that for a large class of interpolation functors preserving bilinear interpolation estimates of measures of noncompactness of interpolated linear operators between Banach couples can be lifted to bilinear operators. As an application, we show that the measure of noncompactness of bilinear operators behave well under the real method of interpolation.

1. Introduction

An important topic in the theory of interpolation is the study of which properties of operators are inherited to Banach spaces obtained by classical interpolation methods. The well known methods of interpolation give estimates of norms of interpolated operators. These estimates are used to study several important properties of operators. It should be mentioned that the compactness property has been studied intensively in recent years. An important question related to the behavior of interpolation of compact operators is whether an operator acting between Banach couples and which acts compactly on one or both of the "endpoint" spaces also acts compactly on the interpolation spaces generated by the couples. This problem was first studied by Lions and Peetre [16] in the case where the couple in the domain or in the target is generated by a single Banach space. Persson [19] extended this result for a large class of interpolation methods under the assumption that the target couple satisfies a certain approximation condition. Hayakawa [15] established the so-called "two-sided" interpolation for compact operators for the real method without any approximation condition. Cwikel [9] showed, solving a long-standing problem, that "one-sided" variant of Hayakava's result is true. One of the open problems in the field concerns whether or not a similar result is true for the complex interpolation spaces. Calderón [4, 10.4] proved an one-sided interpolation for compact operators for the complex method under certain approximation condition. We refer to the Cwikel and Kalton paper [10] about interpolation of compact operators by the complex method. A novel approach to the interpolation of compact operators has been developed in papers [8] and [7].

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We refer also to the paper by Cobos, Cwikel, and Matos [5] and references therein, in which the authors proved almost optimal conditions for compactness-type interpolation results in the sense of Lions–Peetre. In this paper the authors consider the quantities called measures of noncompactness and prove interpolation variants of Lions–Peetre results for linear operators. The interpolation estimates for the measure of noncompactness of linear operators between real method spaces were first proved by Teixeira and Edmunds [22] under certain interpolation approximation condition on the target space. Cobos, Fernández-Martínez and Martínez [6] established a quantitative strengthening of this result. They showed that measure of noncompactness of interpolated operators between real classical interpolation spaces is logarithmically convex up to a multiplicative constant. This result was extended by Szwedek [21] for the class the abstract real method spaces generated by the parameters of the real method of interpolation.

A natural question to ask is: Does there exist an abstract approach to prove variants of the mentioned results for bilinear operators? In the present paper, we develop our main tool for deriving a positive answer to this question, and we study interpolation of the measure of noncompactness of bilinear operators. In Section 2 we prove some preliminary results. Combining these results we prove in Section 3 theorems of general nature. The main result states that for a large class of interpolation functors preserving bilinear interpolation estimates of measures of noncompactness of interpolated linear operators between Banach couples can be lifted to bilinear operators. As a byproduct, this leads to interpolation theorems on compactness of one-sided compact bilinear operators. As an application, we show that the measure of noncompactness behaves well under the real method of interpolation. This yields a variant of the compactness interpolation theorem for bilinear operators between real method spaces. In Section 4 one-sided interpolation of measure of noncompactness of bilinear operators is studied. The results we prove in this section correspond to the classical theorems of Lions–Peetre, Persson, and Teixeira-Edmunds for linear operators.

2. Notation and preliminary results

We shall require considerable notation in what follows. An operator $S\colon X \to Y$ between normed spaces is a continuous linear mapping. If X, Y are two normed spaces, then the product $X \times Y$ is equipped with the standard norm $\|(x,y)\| := \max\{\|x\|_X, \|y\|_Y\}$, for all $(x,y) \in X \times Y$. A 2-linear mapping $T\colon X \times Y \to Z$ between normed spaces is called a bilinear operator if it is bounded. As usual, the norm of T is defined by

$$\|T\| = \inf\{C > 0;\ \|T(x,y)\|_Z \le C\|x\|_X\,\|y\|_Y \text{ for all } (x,y) \in X \times Y\}.$$

The space of all bilinear operators from $X \times Y$ into Z is denoted by $\mathcal{L}_2(X,Y;Z)$. If $Z = \mathbb{K}$ is a scalar field we write $\mathcal{L}_2(X,Y)$ instead of $\mathcal{L}_2(X,Y;\mathbb{K})$. A bilinear operator $T\colon X \times Y \to Z$ is said to be compact if $T(B_X \times B_Y)$ is precompact, where B_X is the closed unit ball in the normed space X. For examples of bilinear compact operators, we refer to article [1].

For basic notation for interpolation theory, we refer to [2] and [3]. We shall recall that a mapping F from the category $\vec{\mathbf{B}}$ of all couples of Banach spaces into the category of all Banach spaces is said to be an *interpolation functor* if, for any couple $\vec{X} := (X_0, X_1)$, the Banach space $F(X_0, X_1)$ is intermediate with respect to $\vec{X}$ (i.e.,

$\Delta(\vec{X}) := X_0 \cap X_1 \hookrightarrow F(\vec{X}) \hookrightarrow \Sigma(\vec{X}) := X_0 + X_1$), and $T\colon F(X_0, X_1) \to F(Y_0, Y_1)$ for all $T\colon (X_0, X_1) \to (Y_0, Y_1)$; here as usual the notation $T\colon (X_0, X_1) \to (Y_0, Y_1)$ means that $T\colon X_0 + X_1 \to Y_0 + Y_1$ is a linear operator such that the restrictions of T to the space X_j is a bounded operator from X_j to Y_j, for both $j = 0$ and $j = 1$.

The real and the complex methods of interpolation are of powerful interest in applications in various areas of modern analysis. We recall here that the real interpolation functor $K_E(\vec{X})$, generated by any Banach sequence lattice E modeled on $\mathbb{Z}$, such that $\{\min\{1, 2^n\}\}_{n\in\mathbb{Z}} \in E$, is defined for any Banach couple $\vec{X} = (X_0, X_1)$ to be the Banach space of all $x \in X_0 + X_1$ equipped with the norm

$$\|x\| = \|\{K(2^n, x; \vec{X})\}\|_E.$$

In particular, the choice $E = \ell_q(2^{-n\theta})$ yields the usual real interpolation space denoted by $\vec{X}_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$.

If X is an intermediate Banach space with respect to a couple $\vec{X} = (X_0, X_1)$, we let X° be the closed hull of $\Delta(\vec{X})$ in X. A Banach couple $\vec{X}$ is called *regular* if $X_j^\circ = X_j$ for $j = 0, 1$. We also define its *relative completion* X^c to be the Banach space of all limits in $X_0 + X_1$ of sequences that are bounded in X and equipped with the natural norm. If $X^c = X$, then X is said to be relatively complete.

If F is an interpolation functor, F° (resp., F^c) is the functor defined by $F^\circ(\vec{X}) = F(\vec{X})^\circ$ (resp., $F^c(\vec{X}) = F(\vec{X})^c$) for all $\vec{X} \in \vec{\mathbf{B}}$. We say that F is regular (resp., relatively complete) on $\vec{X}$ if $F(\vec{X}) = F(\vec{X})^\circ$ (resp., $F(\vec{X}) = F(\vec{X})^c$). If $F = F^\circ$ (resp., $F = F^c$), then it is called a regular (resp., relatively complete) interpolation functor. We will use the well known facts that $(\cdot)_{\theta,q}$ is both regular and relatively complete functor for any $0 < \theta < 1$ and $1 \le q < \infty$.

The space X', dual to the intermediate space X of a Banach couple $\vec{X} = (X_0, X_1)$, is defined by $X' = (X_0 \cap X_1, \|\cdot\|_X)^*$. In what follows, the isometric isomorphism ε_X of X' onto $(X^\circ)^*$ is defined by $\varepsilon_X(x') = \bar{x}'$, where $\bar{x}'$ is the unique continuous extension of x' onto X°. If $X = \Sigma(\vec{X})$, then the map ε_X will be denoted by $\varepsilon_{\vec{X}}$. The Banach couple (X_0', X_1') is denoted by $\vec{X}'$, and the category of all dual couples $\vec{X}'$ is denoted by $\vec{\mathbf{B}}'$.

We will use the well known duality formulas which state that, for any Banach couple (X_0, X_1), we have $(X_0 \cap X_1)' = X_0' + X_1'$ and $(X_0 + X_1)' = X_0' \cap X_1'$ with equality of norms (see, e.g., [3, Prop. 2.4.6]). For the sake of completeness we recall here an important concept of a dual operator (see [3, Def. 2.4.18]). The operator

$$T' := (T|_{\Delta(\vec{X})})^*$$

is called a *dual operator* with respect to an operator $T\colon \vec{X} \to \vec{Y}$ between Banach couples. We note that (see [3, Prop. 2.4.19])

$$T'\colon \vec{Y}' \to \vec{X}' \quad \text{and} \quad \|T'\|_{\vec{Y}' \to \vec{X}'} \le \|T\|_{\vec{X} \to \vec{Y}},$$

where strict equality takes place for regular couple $\vec{X}$.

Let $\vec{X}$ be a regular couple; i.e., $\Delta(\vec{X})$ is dense in both X_0 and X_1. Following [3, Def. 2.4.10], we define the map $\kappa_{\vec{X}}$ by

$$\kappa_{\vec{X}} = (\varepsilon_{\vec{X}})^* \, \kappa_{\Sigma(\vec{X})},$$

where, as usual for a given Banach space Y, we denote by $\kappa_Y\colon Y \to Y^{**}$ the canonical isometric embedding defined by $\kappa_Y y(y^*) = y^*(y)$, for all $(y, y^*) \in Y \times Y^*$.

We note that by the abovementioned duality formulas, we get that

$$((X_0 + X_1)')^* = (X_0' \cap X_1')^* = X_0'' + X_1''$$

isometrically. This implies that

$$\kappa_{\vec{X}} \colon (X_0, X_1) \to (X_0'', X_1'')$$

with norm less or equal than 1.

In what follows, for a given Banach space A, $\langle \cdot, \cdot \rangle_A$ will stand for a canonical bilinear form on $A^* \times A$, and the corresponding subscript will be omitted if there is no confusion.

We will use the following easily verified formula (see [3, Prop. 2.4.11]):

$$\langle \kappa_{\vec{X}}(x), x' \rangle_{\Delta(\vec{X}')} = \langle \varepsilon_{\vec{X}}(x'), x \rangle_{\Sigma(\vec{X})},$$

which is true for all $x \in X_0 + X_1$, $x' \in X_0' \cap X_1'$.

For a given interpolation functor, F, we define a mapping F' from the category $\vec{\mathbf{B}}'$ into the category $\mathbf{B}$ of all Banach spaces by

$$F'(\vec{X}') := F(\vec{X})', \quad \vec{X}' \in \vec{\mathbf{B}}'.$$

Following [3, Def. 2.4.29] the functor *DF* is called a *dual* for a given functor F if it is maximal among those functors G for which

$$G(\vec{X}') \hookrightarrow F(\vec{X})'$$

with norm of the inclusion map less or equal than 1 for every regular Banach couple $\vec{X}$. We refer to [3, pp. 201–209] for descriptions of the dual functors *DF* for the functors of an orbit and co-orbit in the sense of Aronszajn and Gagliardo.

Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$, and $\vec{Z} = (Z_0, Z_1)$ be fixed Banach couples, and let X, Y, and Z be Banach spaces such that $X_0 \cap X_1 \hookrightarrow X$, $Y_0 \cap Y_1 \hookrightarrow Y$, and $Z_0 \cap Z_1 \hookrightarrow Z$. For typographical convenience, we denote the space $\mathcal{L}_2\big(\widetilde{X}, \widetilde{Y}; \widetilde{Z}\big)$ by $\mathcal{B}(X, Y; Z)$, and in what follows, $\widetilde{X}$ denotes $X_0 \cap X_1$ endowed with the induced norm $\|\cdot\|_X$; i.e., $\widetilde{X} := (X_0 \cap X_1, \|\cdot\|_X)$. In the case when Z is a scalar field, we write $\mathcal{B}(X, Y)$ for short. We notice that for every $T \in \mathcal{B}(X, Y)$ we have

$$|T(x, y)| \le C\|x\|_X \|y\|_Y, \quad (x, y) \in \Delta(\vec{X}) \times \Delta(\vec{Y}),$$

where $C = \|T\|_{\widetilde{X} \times \widetilde{Y}}$. This implies that

$$|T(x, y)| \le C\|x\|_{\Delta(\vec{X})} \|y\|_{\Delta(\vec{Y})}, \quad (x, y) \in \Delta(\vec{X}) \times \Delta(\vec{Y}).$$

Hence $T \in \mathcal{B}\big(\Delta(\vec{X}), \Delta(\vec{Y})\big)$ with $\|T\|_{\Delta(\vec{X}) \times \Delta(\vec{Y})} \le C$, and so

$$\mathcal{B}(X, Y) \hookrightarrow \mathcal{B}\big(\Delta(\vec{X}), \Delta(\vec{Y})\big)$$

with the norm of the inclusion map less than or equal to 1. In particular, we conclude that, for any Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$,

$$\mathcal{B}(X_j, Y_j) \hookrightarrow \mathcal{B}\big(\Delta(\vec{X}), \Delta(\vec{Y})\big), \quad j \in \{0, 1\}.$$

Thus, $(\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1))$ forms a Banach couple. In the case when Banach couples $\vec{X}$ and $\vec{Y}$ are fixed and there is no confusion, we write $\vec{\mathcal{B}}$ instead of $\big(\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1)\big)$.

For given Banach couples $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$, and $\vec{Z} = (Z_0, Z_1)$, we denote by $\mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$ the space of all 2-linear mappings $T \colon \Delta(\vec{X}) \times \Delta(\vec{Y}) \to \Delta(\vec{Z})$, such that $T\colon \widetilde{X}_j \times \widetilde{Y}_j \to \widetilde{Z}_j$ are bounded bilinear maps for both $j = 0$ and $j = 1$ $\big($i.e., $T \in \mathcal{B}(X_0, Y_0; Z_0) \cap \mathcal{B}(X_1, Y_1; Z_1)\big)$.

If X, Y, and Z are intermediate Banach spaces with respect to $\vec{X}$, $\vec{Y}$, and $\vec{Z}$, respectively, then X, Y, and Z are said to be *bilinear interpolation* with respect to $\vec{X}$, $\vec{Y}$, and $\vec{Z}$ $\big((X, Y; Z) \in \mathcal{B}int(\vec{X}, \vec{Y}; \vec{Z})$ for short$\big)$ if, for every $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$, we have $T \in \mathcal{B}(X, Y; Z)$.

Throughout the paper we write $(X, Y; Z) \in \mathcal{B}int_C(\vec{X}, \vec{Y}; \vec{Z})$ whenever there exists a constant $C > 0$ such that, for every operator $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$, we have

$$\|T(x, y)\|_Z \le C\|x\|_X\, \|y\|_Y, \quad (x, y) \in \Delta(\vec{X}) \times \Delta(\vec{Y})\,.$$

If F_0, F_1, and F_2 are interpolation functors such that

$$\big(F_0(\vec{X}), F_1(\vec{Y}); F_2(\vec{Z})\big) \in \mathcal{B}int(\vec{X}, \vec{Y}; \vec{Z})$$

for all Banach couples $\vec{X}$, $\vec{Y}$, and $\vec{Z}$, then we say that a bilinear interpolation theorem $F_0 \times F_1 \to F_2$ holds. If there exists a constant $C > 0$ such that

$$\big(F_0(\vec{X}), F_1(\vec{Y}); F_2(\vec{Z})\big) \in \mathcal{B}int_C(\vec{X}, \vec{Y}; \vec{Z})$$

for all Banach couples $\vec{X}$, $\vec{Y}$, and $\vec{Z}$, then we say that a bilinear interpolation theorem $F_0 \times F_1 \to F_2$ holds with a constant C. We refer to general abstract results on interpolation of bilinear operators to [17].

The following theorems will be useful later.

Theorem 2.1. *Let F_0, F_1, F be interpolation functors, and let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ be Banach couples. Assume that $\big(F_0(\vec{X}), F_1(\vec{B}); F(\vec{Y})'\big) \in \mathcal{B}int_C(\vec{X}, \vec{B}; \vec{Y}')$ for any Banach couple $\vec{B}$. Then, the following statements are true*:

(i) *For every $T \in \mathcal{B}(X_0, Y_0) \cap \mathcal{B}(X_1, Y_1)$, we have $T \in \mathcal{B}\big(F_0(\vec{X}), F(\vec{Y})\big)$ with*

$$\|T\|_{\mathcal{B}(F_0(\vec{X}),\, F(\vec{Y}))} \le C\|T\|_{F_1(\mathcal{B}(X_0, Y_0),\, \mathcal{B}(X_1, Y_1))}.$$

(ii) *If F_1 is a regular functor, then*

$$F_1\big(\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1)\big) \hookrightarrow \mathcal{B}\big(F_0(\vec{X}), F(\vec{Y})\big)$$

with norm of the inclusion map less than or equal to C.

Proof. (i). We let $\vec{B} = (B_0, B_1)$, where $B_j := \mathcal{B}(X_j, Y_j)$ for $j \in \{0, 1\}$. Define a map Φ on $\Delta(\vec{X}) \times \Delta(\vec{B})$ by

$$\Phi(x, T) = T(x, \cdot), \quad (x, T) \in \Delta(\vec{X}) \times \Delta(\vec{B}).$$

Clearly $\Phi\colon \Delta(\vec{X}) \times \Delta(\vec{B}) \to \Delta(\vec{Y}')$ is a 2-linear map and

$$\Phi\colon \widetilde{X}_j \times \widetilde{B}_j \to Y_j'$$

with $\|\Phi\|_{\widetilde{X}_j \times \widetilde{B}_j \to Y_j'} \le 1$ for $j \in \{0, 1\}$. Thus, $\Phi \in \mathcal{B}(\vec{X}, \vec{B}; \vec{Y}')$, and our hypothesis yields that

$$\|\Phi(x, T)\|_{F(\vec{Y})'} \le C\|x\|_{F_0(\vec{X})}\, \|T\|_{F_1(\vec{B})}, \quad (x, T) \in \Delta(\vec{X}) \times \Delta(\vec{B}).$$

Since

$$\|\Phi(x, T)\|_{F(\vec{Y})'} = \sup\{|T(x, y)|;\ \|y\|_{F(\vec{Y})} \le 1,\ y \in \Delta(\vec{Y})\},$$

the desired estimate follows.

(ii). We use the following result from [4] which states that if (A, B) is a Banach couple and $M \subset A \cap B$ is a dense subset in A such that

$$\|x\|_B \leq C\|x\|_A, \quad x \in M,$$

then $A \hookrightarrow B$, with the norm of inclusion map less than or equal to C.

Let $A := F_1\big(\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1)\big)$ and $B := \mathcal{B}\big(F_0(\vec{X}), F(\vec{Y})\big)$. Since we have $A \hookrightarrow \mathcal{B}(\Delta(\vec{X}), \Delta(\vec{Y}))$ and $B \hookrightarrow \mathcal{B}(\Delta(\vec{X}), \Delta(\vec{Y}))$, (A, B) forms a Banach couple. We notice that, from our hypothesis, F_1 is a regular functor; then it follows that $M := \mathcal{B}(X_0, Y_0) \cap \mathcal{B}(X_1, Y_1)$ is a dense subspace in A. To conclude, it is enough to apply the mentioned result and the statement(i) to the Banach spaces A and B. □

In a similar fashion we prove the following result.

Theorem 2.2. *Let F_0, F_1, F be interpolation functors, and let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ be Banach couples. Assume that $\big(F_0(\vec{B}), F_1(\vec{X}); F(\vec{Y})'\big) \in \mathcal{B}int_C(\vec{B}, \vec{X}; \vec{Y}')$ for any Banach couple $\vec{B}$. Then the following statements are true*:

(i) *For every $T \in \mathcal{B}(X_0, Y_0) \cap \mathcal{B}(X_1, Y_1)$, we have $T \in \mathcal{B}\big(F_1(\vec{X}), F(\vec{Y})\big)$ with*

$$\|T\|_{\mathcal{B}(F_1(\vec{X}),\, F(\vec{Y}))} \leq C\|T\|_{F_0(\mathcal{B}(X_0,Y_0),\, \mathcal{B}(X_1,Y_1))}.$$

(ii) *If F_0 is a regular functor, then*

$$F_0(\mathcal{B}\big(X_0, Y_0), \mathcal{B}(X_1, Y_1)\big) \hookrightarrow \mathcal{B}\big(F_1(\vec{X}), F(\vec{Y})\big)$$

with norm of the inclusion map less than or equal to C.

As an application of Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Assume that F_0, F_1, and F are interpolation functors such that a bilinear interpolation theorem $F_0 \times F_1 \to DF$ holds with a constant $C > 0$. If F_1 is a regular functor, then for all Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$, one has*

$$F_1\big(\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1)\big) \hookrightarrow \mathcal{B}\big(F_0(\vec{X}), F(\vec{Y}^{\circ})\big)$$

with the norm of the inclusion map less than or equal to C.

Proof. Since DF is a dual functor to F, $DF(\vec{Y}') \hookrightarrow F(\vec{Y}^{\circ})'$ with the norm of the inclusion map less than or equal to 1 for any Banach couple $\vec{Y}$, then the required continuous inclusion follows from Theorem 2.1(ii). □

We introduce an interpolation variant of a general adjoint operator. This is motivated by work done by Ramanujan and Schock [20], where they studied ideals of bilinear operators between Banach spaces. Let X, Y, and Z be normed spaces. Following Ramanujan and Schock [20], for a given bilinear operator $T\colon X \times Y \to Z$ we define the generalized linear map $T^{\times}\colon Z^* \to \mathcal{L}_2(X, Y)$ by

$$T^{\times} z^*(x, y) = z^*(T(x, y)), \quad z^* \in Z^*, \quad (x, y) \in X \times Y.$$

Clearly $T^{\times}$ is a linear operator, and $\|T\| = \|T^{\times}\|$. In [20, Theorem 2.6] the analogous of Schauder's theorem was proved, which states that if $T\colon X \times Y \to Z$ is a bilinear operator between Banach spaces, then T is compact if, and only if, $T^{\times}$ is compact. Here, T is a compact bilinear operator, which means that $T(B_X \times B_Y)$ is a relatively compact subset in Z. We point out that the proof works for normed spaces.

Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$, and $\vec{Z} = (Z_0, Z_1)$ be Banach couples, and let $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$. Clearly $T_0^\times = T_1^\times$ on $Z_0' \cap Z_1'$, where we put $T_j := T$ for a bilinear operator $T\colon \widetilde{X}_j \times \widetilde{Y_j} \to \widetilde{Z}_j$ for each $j \in \{0, 1\}$. This implies that the formula

$$T^\otimes(z') := T_0^\times(z_0') + T_1^\times(z_1'), \quad z' = z_0' + z_1' \in Z_0' + Z_1'$$

with $z_0' \in Z_0'$ and $z_1' \in Z_1'$ defines a linear mapping from $Z_0' + Z_1'$ into $\mathcal{B}(X_0, Y_0) + \mathcal{B}(X_1, Y_1)$.

Since $T^\otimes|_{Z_j'} = T_j^\times\colon Z_j' \to \mathcal{B}(X_j, Y_j)$ is a bounded operator for each $j \in \{0, 1\}$, we conclude that

$$T^\otimes\colon (Z_0', Z_1') \to \big(\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1)\big).$$

For given Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ we define the 2-linear mapping

$$J\colon \Delta(\vec{X}) \times \Delta(\vec{Y}) \to \big(\mathcal{B}(X_0, Y_0) \cap \mathcal{B}(X_1, Y_1)\big)^* = \mathcal{B}(X_0, Y_0)' + \mathcal{B}(X_1, Y_1)'$$

by the formula $\langle J(x, y), B\rangle := B(x, y)$ for all $(x, y) \in \Delta(\vec{X}) \times \Delta(\vec{Y})$, $B \in \mathcal{B}(X_0, Y_0) \cap \mathcal{B}(X_1, Y_1)$. Clearly

$$J\colon \big(\widetilde{X}_0 \times \widetilde{Y}_0, \widetilde{X}_1 \times \widetilde{Y}_1\big) \to \big(\mathcal{B}(X_0, Y_0)',\, \mathcal{B}(X_1, Y_1)'\big)$$

with $\|J\|_{\widetilde{X}_j \times \widetilde{Y}_j \to \mathcal{B}(X_j, Y_j)'} = 1$ for each $j \in \{0, 1\}$.

Proposition 2.1. *Let X, Y, and Z be Banach spaces intermediate with respect to Banach couples $\vec{X}$, $\vec{Y}$, and $\vec{Z}$, respectively. Then for every bilinear operator $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$,*

(i) $(T^\otimes)' J = \kappa_{\vec{Z}^\circ} T$.

(ii) *If, in addition, $T \in \mathcal{B}(X, Y; Z)$, then the operator $T^\times\colon Z' \to \mathcal{B}(X, Y)$ satisfies $T^\times = T^\otimes|_{Z'}$.*

Proof. (i). For all $(x, y) \in \Delta(\vec{X}) \times \Delta(\vec{Y})$ and $z' \in Z_0' + Z_1'$, one has

$$\begin{aligned}\big\langle (T^\otimes)' J(x, y), z'\big\rangle &= \big\langle (T^\otimes)'(J(x, y)), z'\big\rangle = \big\langle J(x, y), T^\otimes z'\big\rangle \\ &= T^\otimes z'(x, y) = z'(T(x, y)) = \big\langle \kappa_{\vec{Z}^\circ} T(x, y), z'\big\rangle.\end{aligned}$$

Thus, we get that

$$(T^\otimes)' J(x, y) = \kappa_{\vec{Z}^\circ} T(x, y),$$

and so the required formula holds.

(ii). Since $Z_0 \cap Z_1 \hookrightarrow Z$, we obtain $Z' \hookrightarrow Z_0' + Z_1'$. If $z' \in Z'$ and $z' = z_0' + z_1'$ with $z_0' \in Z_0'$, $z_1' \in Z_1'$, then we have for all $(x, y) \in \Delta(\vec{X}) \cap \Delta(\vec{Y})$

$$T^\times z'(x, y) = z'(T(x, y)) = z_0'(T(x, y)) + z_1'(T(x, y)) = T^\otimes z'(x, y).$$

Thus, $T^\times z' = T^\otimes z'$ for all $z' \in Z'$, and this completes the proof. □

3. Main results

Let us recall that if A is a bounded subset of a metric space $\mathcal{X}$, the *Kuratowski measure of noncompactness* of A is defined by

$$\psi_{\mathcal{X}}(A) = \inf\{\varepsilon > 0;\ A \text{ may be covered by finitely many sets of diameter} \le \varepsilon\};$$

the *ball measure of noncompactness* of A is defined by

$$\widetilde{\psi}_{\mathcal{X}}(A) = \inf\{\varepsilon > 0;\ A \text{ may be covered by finitely many balls of radius} \le \varepsilon\}.$$

It is easy to check that $\widetilde{\psi}_{\mathcal{X}}(A) \le \psi_{\mathcal{X}}(A) \le 2\widetilde{\psi}_{\mathcal{X}}(A)$ for every bounded set A in $\mathcal{X}$.

Recall that in a Banach space X, a set S is called an ε-net of A if $A \subset S + \varepsilon B_X$. Thus, the definition of $\widetilde{\psi}_X$ measure in a Banach space is equivalent to the following:

$$\widetilde{\psi}_X(A) = \inf\{\varepsilon > 0;\ A \text{ has a finite } \varepsilon\text{-net}\}.$$

The Kuratowski and the ball measure of noncompactness of a linear operator $S\colon X \to Y$ between normed spaces are defined by

$$\beta(S\colon X \to Y) := \psi_Y(S(B_X))$$

and

$$\widetilde{\beta}(S\colon X \to Y) := \widetilde{\psi}_Y(S(B_X)),$$

respectively. An operator $S\colon X \to Y$ between normed spaces is called a *k-ball contraction* if $\widetilde{\psi}_Y(S(B)) \le k\widetilde{\psi}_X(B)$ for every bounded set $B \subset X$. We will use the following easily verified fact later on without any references (cf. [12, Lemma 2.8(ii)]):

$$\widetilde{\beta}(S\colon X \to Y) = \inf\{k > 0;\ S \text{ is a } k\text{-ball contraction}\}.$$

Let X, Y, and Z be normed spaces. Following the linear case, the Kuratowski, and the ball measure of noncompactness of a bilinear operator $T\colon X \times Y \to Z$ are defined by

$$\beta(T\colon X \times Y \to Z) := \psi_Z(T(B_X \times B_Y))$$

and

$$\widetilde{\beta}(T\colon X \times Y \to Z) = \widetilde{\psi}_Z(T(B_X \times B_Y)),$$

respectively. Clearly that $T\colon X \times Y \to Z$ is a compact operator if, and only if, $\beta(T\colon X \times Y \to Z) = 0$.

In what follows we will use the following easily verified properties:

(i) $\beta(T\colon X \times Y \to Z) \le \|T\|$ since $T(B_X \times B_Y) \subset \|T\|B_Z$.

(ii) $\widetilde{\beta}(T) \le \beta(T) \le 2\widetilde{\beta}(T)$.

(iii) $\widetilde{\beta}(T) \le 2\tilde{\beta}(IT)$ for any metric injection $I\colon Z \to W$; that is, $\|Iz\|_W = \|z\|_Z$ for all $z \in Z$, where W is a normed space.

The following relationships between measures of noncompactness of a linear operator $S\colon X \to Y$ acting between Banach spaces and its dual operator $S^*\colon Y^* \to X^*$ hold (see [12, Theorem 2.9])

(iv) $\beta(S) \le \widetilde{\beta}(S^*)$ and $\beta(S^*) \le \widetilde{\beta}(S)$.

We notice that the proof of these estimates works for operators between normed spaces.

We now state and prove the following lemma, which will be the focal point for our later arguments.

Lemma 3.1. *Let X, Y, and Z be Banach spaces intermediate with respect to Banach couples $\vec{X}$, $\vec{Y}$, and $\vec{Z}$, respectively, and suppose that $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$.*

(i) *If $T \in \mathcal{B}(X, Y; Z)$, then*

$$\beta\big(T^\times\colon Z' \to \mathcal{B}(X, Y)\big) \le \widetilde{\beta}\big(T\colon \widetilde{X} \times \widetilde{Y} \to \widetilde{Z}\big).$$

In particular we have for both $j = 0$ and $j = 1$,

$$\beta(T^\times\colon Z_j' \to \mathcal{B}(X_j, Y_j)) \le \widetilde{\beta}\big(T\colon \widetilde{X}_j \times \widetilde{Y}_j \to \widetilde{Z}_j\big).$$

(ii) *If* $(X,Y;Z) \in \mathcal{B}int(\vec{X},\vec{Y};\vec{Z})$, *where* Z *is a regular and relatively complete Banach space, then*

$$\beta\big(T\colon \widetilde{X}\times\widetilde{Y}\to\widetilde{Z}\big) \le 2\,\widetilde{\beta}(T^{\times}\colon Z'\to\mathcal{B}(X,Y)).$$

Proof. (i) Put $\widetilde{\beta}(T) := \widetilde{\beta}\big(T\colon \widetilde{X}\times\widetilde{Y}\to\widetilde{Z}\big)$. Let $S\subset Z'$ be a bounded set with $\operatorname{diam}(S)\le d$, $d>0$. Given $\varepsilon>0$, there exists a finite subset $\{z_1,\dots,z_n\}$ of $\Delta(\vec{Z})$ such that

$$T(B_{\widetilde{X}}\times B_{\widetilde{Y}})\subset\bigcup_{j=1}^{n}\big(z_i+rB_{\widetilde{Z}}\big),$$

where $r=\widetilde{\beta}(T)+\frac{\varepsilon}{2d}$.

For each $1\le j\le n$, we define the set $\{z'(z_j);\, z'\in S\}\subset\mathbb{K}$. Since S is bounded, all sets are relatively compact. Without loss of generality, we may assume that $\mathbb{K}=\mathbb{R}$. Thus, for each $1\le j\le n$, each above set can be covered by closed intervals $I_{j,1},\dots,I_{j,k(j)}$ with length less than or equal to $\varepsilon/2$.

Let $p=(p_1,\dots,p_n)$, where $p_j\in\{1,2,\dots,k(j)\}$, and let us set

$$E_p := \{z'\in S;\ \langle z_j,z'\rangle\in I_{j,p_j},\ 1\le j\le n\}.$$

We have

$$T^{\times}(S)\subset\bigcup_{p}T^{\times}(E_p).$$

We claim that $\operatorname{diam}(T^{\times}(E_p))<d\widetilde{\beta}(T)+\varepsilon$. To prove it, fix p and take $z_1',\,z_2'\in E_p$. Then

$$\begin{aligned}\|T^{\times}z_1'-T^{\times}z_2'\|_{\mathcal{B}(X,Y)} &= \sup\big\{|\langle T(x,y),z_1'-z_2'\rangle|;\ (x,y)\in B_{\widetilde{X}}\times B_{\widetilde{Y}}\big\}\\ &= \sup\big\{|\langle z,z_1'-z_2'\rangle|;\ z\in T(B_{\widetilde{X}}\times B_{\widetilde{Y}})\big\}.\end{aligned}$$

Now, we observe that, for every $z\in T(B_{\widetilde{X}}\times B_{\widetilde{Y}})$, there exists $1\le j\le n$ such that $z\in z_j+rB_{\widetilde{Z}}$. Since $z_1',\,z_2'\in E_p$, $|\langle z_j,z_1'-z_2'\rangle|\le\varepsilon/2$. This implies, by $\|z_1'-z_2'\|_{Z'}\le d$ and $\|z-z_j\|_Z\le\widetilde{\beta}(T)+\varepsilon/2d$,

$$|\langle z-z_j,z_1'-z_2'\rangle|\le\|z_1'-z_2'\|_{Z'}\,\|z-z_j\|_Z\le d\widetilde{\beta}(T)+\varepsilon/2.$$

Hence, for $z\in T(B_{\widetilde{X}}\times B_{\widetilde{Y}})$,

$$|\langle z,z_1'-z_2'\rangle|\le|\langle z-z_j,z_1'-z_2'\rangle|+|\langle z_j,z_1'-z_2'\rangle|\le d\widetilde{\beta}(T)+\varepsilon.$$

Combining the above estimates, yields

$$\big|(T^{\times}z_1'-T^{\times}z_2')(x,y)\big|\le d\widetilde{\beta}(T)+\varepsilon,\quad (x,y)\in B_{\widetilde{X}}\times B_{\widetilde{Y}}.$$

Since $\varepsilon>0$ is arbitrary, we get that

$$\big\|T^{\times}z_1'-T^{\times}z_2'\big\|_{\mathcal{B}(X,Y)}\le d\widetilde{\beta}(T),$$

and this completes the proof.

(ii) We put $\vec{\mathcal{B}} := (\mathcal{B}(X_0,Y_0),\mathcal{B}(X_1,Y_1))$. Our hypothesis means that $\Delta(\vec{\mathcal{B}})\subset\mathcal{B}(X,Y)$. Since both spaces $\Delta(\vec{\mathcal{B}})$ and $\mathcal{B}(X,Y)$ are continuously embedded into $\mathcal{B}(\Delta(\vec{X}),\Delta(\vec{Y}))$, it follows from the closed graph theorem that $\Delta(\vec{\mathcal{B}})\hookrightarrow\mathcal{B}(X,Y)$. This implies

$$\mathcal{B}(X,Y)'\hookrightarrow\big(\mathcal{B}(X_0,Y_0)\cap\mathcal{B}(X_1,Y_1)\big)^{*}=\mathcal{B}(X_0,Y_0)'+\mathcal{B}(X_1,Y_1)'.$$

Now, observe that the 2-linear mapping $J\colon \Delta(\vec{X}) \times \Delta(\vec{Y}) \to \Delta(\vec{\mathcal{B}})^*$ satisfies (by $\Delta(\vec{\mathcal{B}}) \subset \mathcal{B}(X,Y)$)

$$|\langle J(x,y), S\rangle| = |S(x,y)| \leq \|S\|_{\mathcal{B}(X,Y)} \|x\|_{\widetilde{X}} \|y\|_{\widetilde{Y}},$$

for all $S \in \Delta(\vec{\mathcal{B}})$ and $(x,y) \in \widetilde{X} \times \widetilde{Y}$. This yields that $J\colon \widetilde{X} \times \widetilde{Y} \to \mathcal{B}(X,Y)'$ is a bilinear operator, with $\|J\| \leq 1$, and so

$$J(B_{\widetilde{X}} \times B_{\widetilde{Y}}) = J(B_{\widetilde{X}\times\widetilde{Y}}) \subset B_{\mathcal{B}(X,Y)'}\,.$$

The hypothesis that Z is a regular and relatively complete Banach space yields that $\kappa_{\vec{Z}^\circ}\colon Z \to Z''$ is an isometric injection. In particular, we have that

$$\|\kappa_{\vec{Z}^\circ}(T(x,y))\|_{Z''} = \|T(x,y)\|_Z, \quad (x,y) \in \widetilde{X} \times \widetilde{Y}.$$

Combining all together, with $(T^{\otimes})'J = \kappa_{\vec{Z}^\circ}T$ and $T^{\otimes}|_{Z'} = T^{\times}\colon Z' \to \mathcal{B}(X,Y)$ (by Proposition 2.1), yields

$$\begin{aligned}\beta\big(T\colon \widetilde{X} \times \widetilde{Y} \to \widetilde{Z}\big) &= \beta\big(\kappa_{\vec{Z}^\circ}T\colon \widetilde{X} \times \widetilde{Y} \to Z''\big) = \beta\big((T^{\otimes})'J\colon \widetilde{X} \times \widetilde{Y} \to Z''\big)\\ &= \psi_{Z''}\big((T^{\times})'J(B_{\widetilde{X}} \times B_{\widetilde{Y}})\big) \leq \psi_{Z''}\big((T^{\times})'(B_{\mathcal{B}(X,Y)'})\big)\\ &= \beta\big((T^{\times})'\colon \mathcal{B}(X,Y)' \to Z''\big).\end{aligned}$$

Now we apply the mentioned properties (i)–(iv) on the measure of noncompactness for linear operators. At first, by property (iv), we get the following estimate

$$\begin{aligned}\beta\big((T^{\times})'\colon \mathcal{B}(X,Y)' \to Z''\big) &= \beta\big((T^{\times}|_W)^*\colon \big(\Delta(\vec{\mathcal{B}}), \|\cdot\|_{\mathcal{B}(X,Y)}\big)^* \to \big(\Delta(\vec{Z}'), \|\cdot\|_{Z'}\big)^*\big)\\ &\leq \widetilde{\beta}\big(T^{\times}|_W\colon W \to \big(\Delta(\vec{\mathcal{B}}), \|\cdot\|_{\mathcal{B}(X,Y)}\big)\big),\end{aligned}$$

where $W := (\Delta(\vec{Z}'), \|\cdot\|_{Z'})$. Since $B_W \subset B_{Z'}$, it follows by property (ii) that

$$\begin{aligned}\widetilde{\beta}\big(T^{\times}|_W\colon W \to \big(\Delta(\vec{\mathcal{B}}), \|\cdot\|_{\mathcal{B}(X,Y)}\big)\big) &\leq 2\,\widetilde{\beta}\big(T^{\times}|_W\colon W \to \mathcal{B}(X,Y)\big)\\ &\leq 2\,\widetilde{\beta}\big(T^{\times}\colon Z' \to \mathcal{B}(X,Y)\big).\end{aligned}$$

In consequence, we conclude that

$$\beta(T) \leq 2\widetilde{\beta}\big(T^{\times}\colon Z' \to \mathcal{B}(X,Y)\big),$$

and this completes the proof. □

Before we state and prove the first result on interpolation of bilinear compact operators, we introduce the following definition. An interpolation functor F has the left (resp., right) side compactness property if, for all Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$, and every operator $T\colon \vec{X} \to \vec{Y}$ such that $T\colon X_0 \to Y_0$ (resp., $T\colon \widetilde{X_1 \to Y_1}$) is compact, then $T\colon \widetilde{F}(\vec{X}) \to \widetilde{F}(\vec{Y})$ is also compact, where $\widetilde{F}(\vec{A}) := \widetilde{F(\vec{A})}$ for any Banach couple $\vec{A}$.

Theorem 3.1. *Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$, and $\vec{Z} = (Z_0, Z_1)$ be Banach couples, and let F, G, and F_j, for $j \in \{0,1\}$, be interpolation functors such that $\big(F_0(\vec{X}), G(\vec{B}); F_1'(\vec{Y}')\big) \in \mathcal{B}int_C(\vec{X}, \vec{B}; \vec{Y}')$ for all Banach couples $\vec{B}$, for some $C > 0$. Assume that G is regular and that it has the left (resp., the right) side compactness property, $F(\vec{Z})' \hookrightarrow G(\vec{Z}')$, and $F(\vec{Z})$ is regular and relatively complete. Then, for any $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$ such that $T \in \mathcal{B}(F_0(\vec{X}), F_1(\vec{Y}); F(\vec{Z}))$, we have that $T\colon \widetilde{F}_0(\vec{X}) \times \widetilde{F}_1(\vec{Y}) \to \widetilde{F}(\vec{Z})$ is a compact operator whenever $T\colon \widetilde{X}_0 \times \widetilde{Y}_0 \to \widetilde{Z}_0$ (resp., $T\colon \widetilde{X}_1 \times \widetilde{Y}_1 \to \widetilde{Z}_1$) is compact.*

Proof. Assume, without loss of generality, that the functor G has the left side compactness property. Fix $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$ and put $X := F_0(\vec{X})$, $Y := F_1(\vec{Y})$, and $Z := F(\vec{Z})$. It follows by the discussion after Corollary 2.1 that

$$T^{\otimes}\colon (Z_0', Z_1') \to (\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1)).$$

Suppose that the operator $T\colon \widetilde{X}_0 \times \widetilde{Y}_0 \to \widetilde{Z}_0$ is compact (i.e., $\beta\big(T\colon \widetilde{X}_0 \times \widetilde{Y}_0 \to \widetilde{Z}_0\big) = 0$ and that $T \in \mathcal{B}(X, Y; Z)$. As a consequence of Lemma 3.1(i) (by $T^{\otimes}|_{Z_0'} = T^{\times}$), $T^{\times}\colon Z_0' \to \mathcal{B}(X_0, Y_0)$ is compact, so

$$T^{\otimes}\colon G(Z_0', Z_1') \to G(\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1))$$

is also a compact operator. From Theorem 2.1 we get that

$$G(\mathcal{B}(X_0, Y_0), \mathcal{B}(X_1, Y_1)) \hookrightarrow \mathcal{B}(X, Y).$$

Combining Proposition 2.1(ii) with our assumption that $Z' = F(\vec{Z})' \hookrightarrow G(\vec{Z}')$, we conclude that

$$T^{\times}\colon Z' \to \mathcal{B}(X, Y)$$

is a compact operator, so $\widetilde{\beta}(T^{\times}\colon Z' \to \mathcal{B}(X, Y)) = 0$. This yields, by Lemma 3.1(ii), that

$$\beta\big(T\colon \widetilde{X} \times \widetilde{Y} \to \widetilde{Z}\big) = 0,$$

and this completes the proof. □

For the case of the real method we have the following result that seems interesting on its own.

Theorem 3.2. *Let $\theta \in (0,1)$, $p, q \in [1, \infty)$, and $r \in (1, \infty)$ satisfy $1/p + 1/q = 1 + 1/r$, and let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$, and $\vec{Z} = (Z_0, Z_1)$ be Banach couples. Then, there exists a constant $C = C(\theta) > 0$ such that for any bilinear operator $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$,*

$$\widetilde{\beta}\big(T\colon \widetilde{\vec{X}}_{\theta,p} \times \widetilde{\vec{Y}}_{\theta,q} \to \widetilde{\vec{Z}}_{\theta,r}\big) \le C\widetilde{\beta}\big(T\colon \widetilde{X}_0 \times \widetilde{Y}_0 \to \widetilde{Z}_0\big)^{1-\theta}\widetilde{\beta}\big(T\colon \widetilde{X}_1 \times \widetilde{Y}_1 \to \widetilde{Z}_1\big)^{\theta}.$$

Proof. We consider the following interpolation functors, $F_0 = (\cdot)_{\theta,p}$, $F_1 = (\cdot)_{\theta,q}$, $F = (\cdot)_{\theta,r}$, and $G = (\cdot)_{\theta,r'}$, where as usual $1/r' := 1 - 1/r$. By the well known duality formula $((Y_0, Y_1)_{\theta,q})' = (Y_0', Y_1')_{\theta,q'}$, up to equivalence of norms (see, e.g., [2] or [3]), it follows that

$$F_1(Y_0, Y_1)' = (Y_0', Y_1')_{\theta,q'}.$$

Our hypothesis $1/p + 1/q = 1 + 1/r$ implies $1/p + 1/r' = 1 + 1/q'$. Thus, the Lions–Peetre theorem [16] on interpolation of bilinear operators by the real method is applicable, and we can deduce the existence of $C > 0$ such that

$$\big(F_0(\vec{X}), F_1(\vec{Y}); F(\vec{Z})\big) \in \mathcal{B}int_C(\vec{X}, \vec{Y}; \vec{Z})$$

and

$$(*) \qquad \big(F_0(\vec{X}), G(\vec{B}); F_1(\vec{Y})'\big) \in \mathcal{B}int_C(\vec{X}, \vec{B}; \vec{Y}')$$

for all Banach couples $\vec{B}$.

We put $\mathcal{B}_j := \mathcal{B}_j(X_j, Y_j)$ for $j = 0, 1$. Then, from Lemma 3.1, we get that

$$T^{\otimes}\colon (Z_0', Z_1') \to (\mathcal{B}_0, \mathcal{B}_1).$$

Applying [6, Theorem 1.2], we deduce that there exists a constant $C_1 = C_1(\theta) > 0$ such that

$$\widetilde{\beta}(T^{\otimes}\colon G(Z_0', Z_1') \to G(\mathcal{B}_0, \mathcal{B}_1)) \le C_1 \widetilde{\beta}(T^{\times}\colon Z_0' \to \mathcal{B}_0)^{1-\theta} \widetilde{\beta}(T^{\times}\colon Z_1' \to \mathcal{B}_1)^{\theta}.$$

Since $r' \in [1, \infty)$, G is a regular functor. Thus Theorem 2.1 is applicable (by $(*)$), and we can deduce that

$$G(\mathcal{B}_0, \mathcal{B}_1) \hookrightarrow \mathcal{B}(F_0(\vec{X}), F_1(\vec{Y})).$$

Combining $T^{\otimes}|_{F(\vec{Z})'} = T^{\times}$ (by Proposition 2.1) with the above estimate and the duality formula (up to equivalence of norms), one has

$$F(\vec{Z})' = ((Z_0, Z_1)_{\theta,r})' = (Z_0', Z_1')_{\theta,r'} = G(\vec{Z}').$$

Thus, we get that there exists a constant $C_2 = C_2(\theta) > 0$ such that

$$\widetilde{\beta}\big(T^{\times}\colon F(\vec{Z})' \to \mathcal{B}(F_0(\vec{X}), F_1(\vec{Y})\big) \le C_2 \widetilde{\beta}(T^{\times}\colon Z_0' \to \mathcal{B}_0)^{1-\theta} \widetilde{\beta}(T^{\times}\colon Z_1' \to \mathcal{B}_1)^{\theta}.$$

To finish we recall that $F = (\cdot)_{\theta,r}$ is both a regular and relatively complete functor. Thus Lemma 3.1(ii) applies and the required estimate follows. □

It is worth pointing out that Theorem 3.2 is a quantitative version of the qualitative result given in Theorem 3.3 of the recent paper [14] by Fernández-Cabrera and Martínez.

4. One-sided interpolation of the measure of noncompactness

In the theory of interpolation of operators the set of *interpolation functions* (denoted by Φ) plays a key role. We recall that $\varphi \in \Phi$ if $\varphi\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ and φ is nondecreasing in each variable and positively homogeneous (i.e., $\varphi(\lambda s, \lambda t) = \lambda \varphi(s,t)$ for all $\lambda, s, t > 0$). Note that interpolation functions are continuous by the monotonicity. Clearly every $\varphi \in \Phi$ can be extended by continuity to $[0,\infty) \times [0,\infty)$. In what follows, this extension will be denoted by the same symbol φ. We denote by Φ_0 the set of interpolation functions such that $\varphi(s,0) = 0$ and $\varphi(0,t) = 0$, for all $s, t > 0$. The simplest examples of interpolation functions are $as + bt$, $\max\{as, bt\}$, and $\min\{as, bt\}$, where $a, b > 0$ and the power functions $s^{1-\theta} t^{\theta}$, where $0 \le \theta \le 1$.

We will need to introduce a function which measures the position of a given intermediate space within a given Banach couple. Let $\vec{X} = (X_0, X_1)$ be a Banach couple, and let X be a Banach space such that $X_0 \cap X_1 \hookrightarrow X$. The *characteristic function of* X with respect to (X_0, X_1) is defined by (see [11])

$$\varphi_X(s,t) = \sup\big\{\|x\|_X;\ x \in X_0 \cap X_1, \|x\|_{X_0} \le s, \|x\|_{X_1} \le t\big\}, \quad s, t > 0.$$

We prove a bilinear variant of Lions–Peetre result on one-sided interpolation of compact operators.

Theorem 4.1. *Let $\vec{Z} = (Z_0, Z_1)$ be a Banach couple, let Z be an intermediate Banach space with respect to $\vec{Z}$, and let X and Y be any Banach spaces. If $T\colon X \times Y \to Z_0 \cap Z_1$ is a bounded bilinear operator, then*

$$\widetilde{\beta}(T\colon X \times Y \to Z) \le 2\varphi_Z\big(\widetilde{\beta}(T\colon X \times Y \to Z_0), \widetilde{\beta}(T\colon X \times Y \to Z_1)\big),$$

where φ_Z is the characteristic function of Z with respect to (Z_0, Z_1). In particular this implies that, if the restrictions $T\colon X \times Y \to Z_i$ is compact for $i = 0$ or $i = 1$, then $T\colon X \times Y \to Z$ is also compact whenever $\varphi_Z \in \Phi_0$.

Proof. Let $\widetilde{\beta}(T_i) := \widetilde{\beta}(T\colon X \times Y \to Z_i)$ for $i \in \{0,1\}$. For every $\varepsilon > 0$ and each $i \in \{0,1\}$ there exists a finite subset $S(\varepsilon, i)$ of Z such that

$$(*) \qquad T(B_X \times B_Y) \subset \bigcup_{z \in S(\varepsilon,i)} \big(z + (\varepsilon + \widetilde{\beta}(T_i))B_{Z_i}\big).$$

Similarly as in the linear case (see [5, Theorem 3.2]), we construct a subset $\widetilde{Z}(\varepsilon)$ of $Z_0 \cap Z_1$ as follows: for each choice $z_0 \in S(\varepsilon,0)$ and $z_1 \in S(\varepsilon,1)$, if the set

$$E(z_0,z_1) = \big(z_0 + (\varepsilon + \widetilde{\beta}(T_0))B_{Z_0}\big) \cap \big(z_1 + (\varepsilon + \widetilde{\beta}(T_1))B_{Z_1}\big)$$

is nonempty, we choose exactly one element $w = w(z_0,z_1)$ from this set; i.e.,

$$\widetilde{Z}(\varepsilon) = \{w = w(z_0,z_1);\, z_0 \in S(\varepsilon,0),\, z_1 \in S(\varepsilon,1),\, E(z_0,z_1) \neq \emptyset\}.$$

Fix $(x,y) \in B_X \times B_Y$. Then, it follows from $(*)$ that there exist $z_0 \in S(\varepsilon,0)$ and $z_1 \in S(\varepsilon,1)$ such that

$$T(x,y) \in \big(z_0 + (\varepsilon + \widetilde{\beta}(T_0))B_{Z_0}\big) \cap \big(z_1 + (\varepsilon + \widetilde{\beta}(T_1))B_{Z_1}\big),$$

so $E(z_0,z_1) \neq \emptyset$. Thus, for any $w = w(z_0,z_1) \in \widetilde{Z}(\varepsilon)$ one has

$$\|T(x,y) - w\|_{Z_0} \le \|T(x,y) - z_0\|_{Z_0} + \|z_0 - w\|_{Z_1} \le 2(\varepsilon + \widetilde{\beta}(T_0))$$

and

$$\|T(x,y) - w\|_{Z_1} \le \|T(x,y) - z_1\|_{Z_1} + \|z_1 - w\|_{Z_1} \le 2(\varepsilon + \widetilde{\beta}(T_1)).$$

This implies that

$$\|T(x,y) - w\|_Z \le 2\varphi_Z\big(\varepsilon + \widetilde{\beta}(T_0),\, \varepsilon + \widetilde{\beta}(T_1)\big).$$

Hence

$$\widetilde{\beta}(T\colon X \times Y \to Z) \le 2\varphi_Z\big(\varepsilon + \widetilde{\beta}(T_0),\, \varepsilon + \widetilde{\beta}(T_1)\big).$$

Since ε is arbitrary, the required estimate follows. □

Before we state and prove a second variant of the Lions–Petree result, we need to recall the definition of the next important interpolation function in the theory of interpolation. Let $\vec{X} = (X_0, X_1)$ be a Banach couple, and let X be a Banach space such that $X \hookrightarrow X_0 + X_1$. Following [11], we define a function $\psi_X\colon (0,\infty)\times(0,\infty) \to (0,\infty)$ by

$$\psi_X(s,t) = \sup\{K(s,t;x;\vec{X});\, x \in B_X\},$$

where for every $s,t > 0$ and $x \in X_0 + X_1$,

$$K(s,t,x;\vec{X}) = \inf\{s\|x_0\|_{X_0} + t\|x_1\|_{X_1};\, x = x_0 + x_1,\, x_j \in X_j,\, j = 0,1\}.$$

Clearly, ψ_X is an interpolation function whenever X is a nontrivial space.

In the sequel we put, as usual for every $x \in X_0 + X_1$,

$$K(t,x;\vec{X}) := K(1,t,x;\vec{X}), \quad t > 0.$$

Theorem 4.2. *Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ be Banach couples and let Z be a Banach space. Assume that X and Y are Banach spaces such that $X \hookrightarrow X_0 + X_1$, $Y \hookrightarrow Y_0 + Y_1$ and let $T\colon (X_0+X_1)\times(Y_0+Y_1) \to Z$ be a bounded bilinear operator. Then, there exists a constant C depending on T such that the following estimate holds for all $t > 0$:*

$$\begin{aligned}\widetilde{\beta}(T\colon X \times Y \to Z) \le \psi_X(1,t)\,\psi_Y(1,t)\big(&\widetilde{\beta}(T\colon X_0 \times Y_0 \to Z)\\ &+ t^{-2}\widetilde{\beta}(T\colon X_1 \times Y_1 \to Z)\big) + Ct^{-1}(\psi_X(1,t) + \psi_Y(1,t)).\end{aligned}$$

If, in addition $\psi_X(1,t)/t \to 0$ *and* $\psi_Y(1,t)/t \to 0$ *as* $t \to \infty$*, and the restriction* $T\colon X_0 \times Y_0 \to Z$ *is compact, then* $T\colon X \times Y \to Z$ *is also a compact bilinear operator.*

Proof. For simplicity of notation we put $\rho_X(t) = \psi_X(1,t)$ and $\rho_Y(t) = \psi_Y(1,t)$, for all $t > 0$ and $\widetilde{\beta}(T_i) := \widetilde{\beta}(T\colon X_i \times Y_i \to Z)$ for $i \in \{0,1\}$. For every $\varepsilon > 0$ and each $i \in \{0,1\}$ there exists a finite subset $Z(\varepsilon, i)$ of Z such that

$$(**) \qquad T(B_{X_i} \times B_{Y_i}) \subset \sum_{z \in Z(\varepsilon,i)} \big(z + (\varepsilon + \widetilde{\beta}(T_i))B_Z\big).$$

Let $(x,y) \in B_X \times B_Y$, and let $t, \delta > 0$ be fixed. Since $K(t,x;\vec{X}) < \delta + \rho_X(t)$ and $K(t,y;\vec{Y}) < \delta + \rho_Y(t)$, there exist decompositions $x = x_0 + x_1$ with $x_j \in X_i$ and $y = y_0 + y_1$ with $y_i \in Y_i$, for $i \in \{0,1\}$, such that $\|x_0\|_{X_0} + t\|x_1\|_{X_1} < \delta + \rho_X(t)$ and $\|y_0\|_{Y_0} + t\|y_1\|_{Y_1} < \delta + \rho_Y(t)$. Hence

$$\pm x_i \in t^{-i}(\delta + \rho_X(t))B_{X_i}, \quad y_i \in t^{-i}(\delta + \rho_Y(t))B_{Y_i}, \quad i \in \{0,1\}.$$

Combining with $(**)$, we obtain that for some $z_0 \in Z(\varepsilon, 0)$ and $z_1 \in Z(\varepsilon, 1)$ we have

$$\begin{aligned} T(x_0, y_0) \in & (\delta + \rho_X(t))(\delta + \rho_Y(t))z_0 \\ & + (\delta + \rho_X(t))(\delta + \rho_Y(t))\big(\varepsilon + \widetilde{\beta}(T_0)\big)B_Z, \end{aligned}$$

and

$$\begin{aligned} T(-x_1, y_1) \in & t^{-2}(\delta + \rho_X(t))(\delta + \rho_Y(t))z_1 \\ & + t^{-2}(\delta + \rho_X(t))(\delta + \rho_Y(t))\big(\varepsilon + \widetilde{\beta}(T_1)\big)B_Z. \end{aligned}$$

Since the bilinear map $T\colon (X_0 + X_1) \times (Y_0 + Y_1) \to Z$ is bounded, $X \hookrightarrow X_0 + X_1$ and $Y \hookrightarrow Y_0 + Y_1$, then there exists a constant $C > 0$ such that, $T\colon X \times Y_1 \to Z$ and $T\colon (X_0 + X_1) \times Y \to Z$ with norms less than or equal to C.

Combining, we get from the formula $T(x,y) = T(x_0 + x_1, y_0 + y_1) = T(x_0, y_0) + T(-x_1, y_1) + T(x, y_1) + T(x_1, y)$ that

$$\begin{aligned} T(x,y) \in & (\delta + \rho_X(t))(\delta + \rho_Y(t))(z_0 + t^{-2}z_1) + (\delta + \rho_X(t))(\delta + \rho_Y(t))\big(\varepsilon + \widetilde{\beta}(T_0)\big)B_Z \\ & + Ct^{-1}(2\delta + \rho_X(t) + \rho_Y(t))B_Z + t^{-2}(\delta + \rho_X(t))(\delta + \rho_Y(t))\big(\varepsilon + \widetilde{\beta}(T_1)\big)B_Z. \end{aligned}$$

Observe that the set $\{(\delta + \rho_X(t))(\delta + \rho_Y(t))(z_0 + t^{-2}z_1);\, z_0 \in Z(\varepsilon,0), z_1 \in Z(\varepsilon,1)\}$ is finite, and so we conclude that

$$\begin{aligned} \widetilde{\beta}(T\colon X \times Y \to Z) \le & (\delta + \rho_X(t))\big(\delta + \rho_Y(t)\big)\big(\varepsilon + \widetilde{\beta}(T_0) + t^{-2}(\varepsilon + \widetilde{\beta}(T_1))\big) \\ & + Ct^{-1}(2\delta + \rho_X(t) + \rho_Y(t)). \end{aligned}$$

Since ε and δ may be chosen arbitrarily small,

$$\widetilde{\beta}(T\colon X \times Y \to Z) \le \rho_X(t)\rho_Y(t)\big(\widetilde{\beta}(T_0) + t^{-2}\widetilde{\beta}(T_1)\big) + Ct^{-1}(\rho_X(t) + \rho_Y(t)).$$

If we assume that the restriction $T\colon X_0 \times Y_0 \to Z$ is a compact bilinear operator, then $\widetilde{\beta}(T_0) = 0$. Thus, the above estimate yields

$$\begin{aligned} \widetilde{\beta}(T\colon X \times Y \to Z) \le & \lim_{t\to\infty} \frac{\rho_X(t)}{t}\,\frac{\rho_Y(t)}{t}\,\widetilde{\beta}(T_1) \\ & + C \lim_{t\to\infty} \Big(\frac{\rho_X(t)}{t} + \frac{\rho_Y(t)}{t}\Big) = 0, \end{aligned}$$

and this completes the proof. ∎

To show general examples we recall, following [18], that the function φ, which corresponds to an exact interpolation functor F by the equality

$$F(s\mathbb{R}, t\mathbb{R}) = \varphi(s,t)\mathbb{R}, \quad s,t > 0$$

is called the *characteristic function* of the functor $\mathcal{F}$. Here $\alpha\mathbb{R}$ denotes $\mathbb{R}$ equipped with the norm $\|\cdot\|_{\alpha\mathbb{R}} = \alpha|\cdot|$ for $\alpha > 0$.

We note that, for any exact interpolation functor F (see [18, p. 372]) and for any Banach couple $\vec{X} = (X_0, X_1)$, we have

$$\|x\|_{\mathcal{F}(\vec{X})} \le \varphi(\|x\|_{X_0}, \|x\|_{X_1}), \quad x \in X_0 \cap X_1$$

and moreover, by [18, Lemma 7.7.1], for all $s,t > 0$,

$$K(s,t,x;\vec{X}) \le \varphi_*(s,t)\|x\|_{F(\vec{X})}, \quad x \in F(\vec{X}),$$

where $\varphi_*(s,t) := 1/\varphi(s^{-1}, t^{-1})$. Hence, for a Banach space $X := F(\vec{X})$,

$$\varphi_X(s,t) \le \varphi(s,t), \quad \psi_X(s,t) \le \varphi_*(s,t), \quad s,t > 0.$$

In particular, this implies that $\varphi_X \in \Phi_0$ and $\psi_X \in \Phi_0$ whenever $\varphi \in \Phi_0$. Combining these observations with the above results, we obtain variants of one-sided compactness results for the real interpolation functor K_E generated by any Banach sequence lattice E modelled on $\mathbb{Z}$, such that $\{\min\{1, 2^n\}\}_n \in E$. It is obvious that the characteristic function of the exact interpolation functor K_E is given by $\varphi(s,t) = \|\{\min\{s, t2^n\}\}_n\|_E$ for all $s,t > 0$. In particular, applying this observation to special type of spaces E, we recover the results of Lions–Peetre type from [13].

We conclude this section by proving a bilinear variant of Teixeira and Edmunds' result proved for linear operators [22], under the condition that the couple in the target of the operator satisfies the approximation property (H). At first, we introduce a minor modification of the previously mentioned approximation property.

A Banach couple $\vec{X} = (X_0, X_1)$ is said to have the approximation property $(\widetilde{H})$ if there exists a constant $c > 0$ such that, for any given $\varepsilon > 0$ and any finite subsets $F_0 \subset X_0$ and $F_1 \subset X_1$, there is an operator $P\colon \vec{X} \to \vec{X}$ with $\|P\|_{\vec{X}\to\vec{X}} \le c$, $P\colon X_0 + X_1 \to X_0 \cap X_1$, $P\colon X_0 \to X_0$ is compact, and $\|x - Px\|_{X_i} \le \varepsilon$ for each $x \in F_i$, $i \in \{0,1\}$.

We note that it follows from [19] that a Banach couple (L_{p_0}, L_{p_1}) of L_p spaces on a locally compact measure space with Radon measure, with $1 \le p_0 < \infty$ and $1 \le p_1 < \infty$ satisfies the approximation property with $c = 1$. In fact, a minor modification of the original proof shows that this is true for any measure space (see [13]).

We need the following result which is a bilinear variant of a lemma [22, p. 37] for linear operators.

Lemma 4.1. *Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ and $\vec{Z} = (Z_0, Z_1)$ be Banach couples, and let $T \in \mathcal{B}(\vec{X}, \vec{Y}; \vec{Z})$. Suppose that $\vec{Z}$ has the approximation property $(\widetilde{H})$. Then there exists a constant C such that for every $\varepsilon > 0$ there exists $P\colon \vec{Z} \to \vec{Z}$ satisfying*

$$\|T - PT\|_{\widetilde{X}_i \times \widetilde{Y}_i \to \widetilde{Z}_i} \le C\widetilde{\beta}\big(T\colon \widetilde{X}_i \times \widetilde{Y}_i \to \widetilde{Z}_i\big) + \varepsilon, \quad i \in \{0,1\}.$$

Proof. For typographical convenience we put $\widetilde{\beta}(T_i) := \widetilde{\beta}\big(T_i\colon \widetilde{X}_i \times \widetilde{Y}_i \to \widetilde{Z}_i\big)$, for $i \in \{0,1\}$. Let c be a positive constant in the definition of the approximation property $(\widetilde{H})$ of $\vec{Z}$. Fix $\varepsilon > 0$. There are finite sets $F_0 \subset \widetilde{Z}_0$ and $F_1 \subset \widetilde{Z}_1$ such that, for each $i \in \{0,1\}$, we have

$$\sup_{(x,y)\in B_{\widetilde{X}_i}\times B_{\widetilde{Y}_i}} \|T(x,y) - z\|_{Z_i} \le \widetilde{\beta}(T_i) + \frac{\varepsilon}{2(1+c)}, \quad z \in F_i.$$

Our hypothesis implies that there exists $P\colon \vec{Z} \to \vec{Z}$ such that, for each $i \in \{0,1\}$, we have with $C = 1 + c$,

$$\|Pz - z\|_{Z_i} < \frac{\varepsilon}{2C}, \quad z \in F_i.$$

Combining the above estimates for all $(x,y) \in B_{\widetilde{X}_i} \times B_{\widetilde{Y}_i}$ and each $i \in \{0,1\}$, we get that

$$\begin{aligned}\|(I-P)T(x,y)\|_{Z_i} &= \|(I-P)(T(x,y)-z) + (I-P)z\|_{Z_i}\\ &\le (1+c)\|T(x,y)-z\|_{Z_i} + \|z - Pz\|_{Z_i}\\ &\le C\Big(\|T(x,y)-z\|_{Z_i} + \frac{\varepsilon}{2C}\Big)\\ &\le C\widetilde{\beta}(T_i) + \varepsilon.\end{aligned}$$

and this completes the proof. □

In what follows, we write $(X,Y;Z) \in \mathcal{B}int(\vec{X},\vec{Y};\vec{Z})$ is a φ-bilinear interpolation for some $\varphi \in \Phi$ whenever we have

$$\|T\|_{\widetilde{X}\times\widetilde{Y}\to\widetilde{Z}} \le \varphi\big(\|T\|_{\widetilde{X}_0\times\widetilde{Y}_0\to\widetilde{Z}_0}, \|T\|_{\widetilde{X}_1\times\widetilde{Y}_1\to\widetilde{Z}_1}\big).$$

We note that $\big([\vec{X}]_\theta, [\vec{Y}]_\theta; [\vec{Z}]_\theta\big) \in \mathcal{B}int\big(\vec{X},\vec{Y};\vec{Z}\big)$ and $\big(\vec{X}_{\theta,p}, \vec{Y}_{\theta,q}; \vec{Z}_{\theta,r}\big) \in \mathcal{B}int\big(\vec{X},\vec{Y};\vec{Z}\big)$ are φ-bilinear interpolation with $\varphi(s,t) = s^{1-\theta}t^\theta$ for all $s,t > 0$, and $p,q,r \in [1,\infty)$ with $1/p + 1/q = 1 + 1/r$, where $[\,\cdot\,]_\theta$ and $(\cdot)_{\theta,q}$, with $0 < \theta < 1$, $1 \le q < \infty$, are the complex and the real method of interpolation, respectively (see [4], [16]).

We are ready to state a general bilinear version of the mentioned result.

Theorem 4.3. *Let X, Y, and Z be intermediate Banach spaces with respect to $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$, and $\vec{Z} = (Z_0, Z_1)$, respectively, such that $(X,Y;Z) \in \mathcal{B}int(\vec{X},\vec{Y};\vec{Z})$ is a φ-bilinear interpolation for some $\varphi \in \Phi_0$. Suppose that $\vec{Z}$ has the approximation property $(\widetilde{H})$. Then, there exists a constant $C > 0$ such that for every bilinear operator $T \in \mathcal{B}(\vec{X},\vec{Y};\vec{Z})$,*

$$\widetilde{\beta}(T\colon \widetilde{X}\times\widetilde{Y} \to \widetilde{Z}) \le \varphi\big(\widetilde{\beta}\big(T\colon \widetilde{X}_0\times\widetilde{Y}_0 \to \widetilde{Z}_0\big), \widetilde{\beta}\big(T\colon \widetilde{X}_1\times\widetilde{Y}_1 \to \widetilde{Z}_1\big)\big).$$

In particular this implies that $T\colon \widetilde{X}\times\widetilde{Y} \to \widetilde{Z}$ is a compact operator if $T\colon \widetilde{X}_i\times\widetilde{Y}_i \to \widetilde{Z}_i$ is compact for $i = 0$ or $i = 1$.

Proof. Fix $\varepsilon > 0$. Then it follows by Lemma 4.1 that there exists $P\colon \vec{Z} \to \vec{Z}$, such that $P\colon Z_0 + Z_1 \to Z_0 \cap Z_1$, $P\colon Z_0 \to Z_0$ is compact and

$$\|T - PT\|_{\widetilde{X}_i\times\widetilde{Y}_i\to\widetilde{Z}_i} \le C\widetilde{\beta}\big(T\colon \widetilde{X}_i\times\widetilde{Y}_i \to \widetilde{Z}_i\big) + \varepsilon, \quad i \in \{0,1\}.$$

Since $P\colon Z_0 \to Z_0$ is compact and

$$PT\colon \widetilde{X}\times\widetilde{Y} \to \widetilde{Z}_0 \cap \widetilde{Z}_1 \hookrightarrow Z,$$

$\widetilde{\beta}\big(PT\colon \widetilde{X}\times\widetilde{Y}\to\widetilde{Z}\big)=0$. Combining the above with $T=PT+(T-PT)$ yields

$$\begin{aligned}\widetilde{\beta}\big(T\colon \widetilde{X}\times\widetilde{Y}\to\widetilde{Z}\big) &\le \widetilde{\beta}\big(PT\colon \widetilde{X}\times\widetilde{Y}\to\widetilde{Z}\big)\\ &\quad+\widetilde{\beta}\big(T-PT\colon \widetilde{X}\times\widetilde{Y}\to\widetilde{Z}\big)\le \|T-PT\|_{\widetilde{X}\times\widetilde{Y}\to\widetilde{Z}}\\ &\quad\le C\varphi\big(\widetilde{\beta}\big(T\colon \widetilde{X}_0\times\widetilde{Y}_i\to\widetilde{Z}_i\big)+\varepsilon,\,\widetilde{\beta}\big(T\colon \widetilde{X}_1\times\widetilde{Y}_1\to\widetilde{Z}_1\big)+\varepsilon\big).\end{aligned}$$

Since ε may be chosen arbitrarily small, the desired statement follows. □

We close this section with the general observation that the above theorem may be applied for bilinear operators between a large class of interpolation spaces generated by interpolation functors. We begin with the following definition. Let F be an interpolation functor. If there is a constant $C>0$ such that for every $T\colon \vec{X}\to\vec{Y}$

$$\|T\|_{F(\vec{X})\to F(\vec{Y})}\le C\max\big\{\|T\|_{X_0\to Y_0},\|T\|_{X_1\to Y_1}\big\},$$

then F is called *bounded*. Clearly we always have $C\ge 1$, and if $C=1$ then F is called *exact*. For a bounded interpolation functor $\mathcal{F}$, we define the *fundamental function* $\phi_F\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ of F by

$$\phi_F(s,t):=\sup\big\{\|T\|_{F(\vec{X})\to F(\vec{Y})}\big\},\quad s,t>0$$

where the supremum is taken over all Banach couples $\vec{X}$, $\vec{Y}$, and all operators $T\colon \vec{X}\to\vec{Y}$ such that $\|T\|_{X_0\to Y_0}\le s$ and $\|T\|_{X_1\to Y_1}\le t$. We note that from the definition, it follows for all couples $\vec{X}$, $\vec{Y}$ and all $T\colon \vec{X}\to\vec{Y}$, that

$$\|T\|_{F(\vec{X})\to F(\vec{Y})}\le \phi_F\big(\|T\|_{X_0\to Y_0},\|T\|_{X_1\to Y_1}\big).$$

We also remark that it is well known that the fundamental function of the classical real interpolation functor $F=(\cdot)_{\theta,q}$ with $0<\theta<1$, $1\le q\le\infty$, is given by $\phi_F(s,t)=s^{1-\theta}t^{\theta}$ for all $s,t>0$.

Lemma 4.2. *Let F, G, and F_j for $j\in\{0,1\}$ be interpolation functors such that there exists $C_1>0$ with $\big(F_0(\vec{X}),G(\vec{Y});F_1(\vec{Z})'\big)\in \mathcal{B}int_{C_1}(\vec{X},\vec{Y};\vec{Z}')$ for all Banach couples $\vec{X}$, $\vec{Y}$, and $\vec{Z}$. Assume that G is a bounded regular functor such that $\|\mathrm{id}\colon F(\vec{Z})'\hookrightarrow G(\vec{Z}')\|\le C_2$ and $F(\vec{Z})$ is regular. If a bilinear interpolation theorem $F_0\times F_1\to F$ holds, then $\big(F_0(\vec{X}),F_1(\vec{Y});F(\vec{Z})\big)\in\mathcal{B}int(\vec{X},\vec{Y};\vec{Z})$ is a φ-bilinear interpolation for all Banach couples $\vec{X}$, $\vec{Y}$, and $\vec{Z}$ with $\varphi(s,t)\le C\phi_G(s,t)$ for all $s,t>0$, where ϕ_G is the fundamental function of G and $C=C_1C_2$.*

Proof. Fix $T\in\mathcal{B}(\vec{X},\vec{Y};\vec{Z})$. For simplicity of notation, we put $X:=F_0(\vec{X})$, $Y:=F_1(\vec{Y})$, and $Z:=F(\vec{Z})$. Since

$$T^{\otimes}\colon (Z_0',Z_1')\to(\mathcal{B}(X_0,Y_0),\mathcal{B}(X_1,Y_1))$$

and then, by interpolation property

$$(\diamond)\qquad T^{\otimes}\colon G(Z_0',Z_1')\to G(\mathcal{B}(X_0,Y_0),\mathcal{B}(X_1,Y_1)).$$

Applying Theorem 2.1(ii), we get that

$$G\big(\mathcal{B}(X_0,Y_0),\mathcal{B}(X_1,Y_1)\big)\hookrightarrow\mathcal{B}\big(F_0(\vec{X}),F_1(\vec{Y})\big)=\mathcal{B}(X,Y)$$

with the norm of the inclusion map less than or equal to C_1. Since $\|\mathrm{id}\colon F(\vec{Z})'\hookrightarrow G(\vec{Z}')\|\le C_2$, we conclude that

$$T^{\times}\colon F(\vec{Z})'\to\mathcal{B}(X,Y)$$

with norm less than or equal to $C := C_1C_2$. Combining ($\diamond$) with the definition of the fundamental function of G, one has

$$\|T^\times\|_{Z'\to\mathcal{B}(X,Y)} \le C\phi_G\big(\|T^\times\|_{Z_0'\to\mathcal{B}(X_0,Y_0)}, \|T^\times\|_{Z_1'\to\mathcal{B}(X_1,Y_1)}\big).$$

Our hypothesis is that a bilinear interpolation theorem $F_0 \times F_1 \to F$ holds. In particular this implies that $T\colon \widetilde{X}\times\widetilde{Y}\to\widetilde{Z}$ is a bilinear operator, and so we have

$$\|T\|_{\widetilde{X}\times\widetilde{Y}\to\widetilde{Z}} = \|T^\times\|_{Z'\to\mathcal{B}(X,Y)}.$$

Hence the above inequality yields

$$\|T\|_{\widetilde{X}\times\widetilde{Y}\to\widetilde{Z}} \le C\phi_G\big(\|T\|_{\widetilde{X}_0\times\widetilde{Y}_0\to Z_0}, \|T\|_{\widetilde{X}_1\times\widetilde{Y}_1\to Z_1}\big),$$

and this completes the proof. □

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References

[1] Árpád Bényi and Rodolfo H. Torres, *Compact bilinear operators and commutators*, Proc. Amer. Math. Soc. **141** (2013), no. 10, 3609–3621, DOI 10.1090/S0002-9939-2013-11689-8. MR3080183

[2] Jöran Bergh and Jörgen Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223. MR0482275

[3] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation functors and interpolation spaces. Vol. I*, North-Holland Mathematical Library, vol. 47, North-Holland Publishing Co., Amsterdam, 1991. Translated from the Russian by Natalie Wadhwa; With a preface by Jaak Peetre. MR1107298

[4] A.-P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190, DOI 10.4064/sm-24-2-113-190. MR0167830

[5] Fernando Cobos, Michael Cwikel, and Pedro Matos, *Best possible compactness results of Lions-Peetre type*, Proc. Edinburgh Math. Soc. (2) **44** (2001), no. 1, 153–172, DOI 10.1017/S0013091598001163. MR1879216

[6] Fernando Cobos, Pedro Fernández-Martínez, and Antón Martínez, *Interpolation of the measure of non-compactness by the real method*, Studia Math. **135** (1999), no. 1, 25–38. MR1686369

[7] Fernando Cobos, Thomas Kühn, and Tomas Schonbek, *One-sided compactness results for Aronszajn-Gagliardo functors*, J. Funct. Anal. **106** (1992), no. 2, 274–313, DOI 10.1016/0022-1236(92)90049-O. MR1165856

[8] Fernando Cobos and Jaak Peetre, *Interpolation of compactness using Aronszajn-Gagliardo functors*, Israel J. Math. **68** (1989), no. 2, 220–240, DOI 10.1007/BF02772662. MR1035891

[9] Michael Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. **65** (1992), no. 2, 333–343, DOI 10.1215/S0012-7094-92-06514-8. MR1150590

[10] M. Cwikel and N. J. Kalton, *Interpolation of compact operators by the methods of Calderón and Gustavsson-Peetre*, Proc. Edinburgh Math. Soc. **38** (1995), no. 2, 261–276, DOI 10.1017/S0013091500019076. MR1335873

[11] V. I. Dmitriev, S. G. Kreĭn, and V. I. Ovčinnikov, *Fundamentals of the theory of interpolation of linear operators* (Russian), Geometry of linear spaces and operator theory (Russian), Jaroslav. Gos. Univ., Yaroslavl, 1977, pp. 31–74. MR0634076

[12] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1987. Oxford Science Publications. MR929030

[13] L. M. Fernández-Cabrera and A. Martínez, *On interpolation properties of compact bilinear operators*, Math. Nachr. **290** (2017), no. 11-12, 1663–1677, DOI 10.1002/mana.201600203. MR3683453

[14] L. M. Fernández-Cabrera and A. Martínez, *Real interpolation of compact bilinear operators*, J. Fourier Anal. Appl. (2017). https://doi.org/10.1007/s00041-017-9561-7.
[15] Kantaro Hayakawa, *Interpolation by the real method preserves compactness of operators.*, J. Math. Soc. Japan **21** (1969), 189–199, DOI 10.2969/jmsj/02120189. MR0250057
[16] J.-L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation* (French), Inst. Hautes Études Sci. Publ. Math. **19** (1964), 5–68. MR0165343
[17] Mieczysław Mastyło, *Bilinear interpolation theorems and applications*, J. Funct. Anal. **265** (2013), no. 2, 185–207, DOI 10.1016/j.jfa.2013.05.001. MR3056700
[18] V. I. Ovchinnikov, *The method of orbits in interpolation theory*, Math. Rep. **1** (1984), no. 2, i–x and 349–515. MR877877
[19] Arne Persson, *Compact linear mappings between interpolation spaces*, Ark. Mat. **5** (1964), 215–219 (1964), DOI 10.1007/BF02591123. MR0166598
[20] M. S. Ramanujan and E. Schock, *Operator ideals and spaces of bilinear operators*, Linear and Multilinear Algebra **18** (1985), no. 4, 307–318, DOI 10.1080/03081088508817695. MR826681
[21] Radosław Szwedek, *Measure of non-compactness of operators interpolated by the real method*, Studia Math. **175** (2006), no. 2, 157–174, DOI 10.4064/sm175-2-4. MR2261731
[22] M. F. Teixeira and D. E. Edmunds, *Interpolation theory and measures of noncompactness*, Math. Nachr. **104** (1981), 129–135, DOI 10.1002/mana.19811040110. MR657887

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