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THE UNIFORM MARTIN'S CONJECTURE FOR MANY-ONE DEGREES

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Abstract. We study functions from reals to reals which are uniformly degree invariant from Turing equivalence to many-one equivalence, and we compare them "on a cone". We prove that they are in one-to-one correspondence with the Wadge degrees, which can be viewed as a refinement of the uniform Martin's conjecture for uniformly invariant functions from Turing equivalence to Turing equivalence.

Our proof works in the general case of many-one degrees on $\mathcal{Q}^\omega$ and Wadge degrees of functions $\omega^\omega \to \mathcal{Q}$ for any better-quasi-ordering $\mathcal{Q}$.

1. Introduction

The uniform version of Martin's conjecture for functions from Turing equivalence to Turing equivalence was proved by Slaman and Steel in [Ste82, SS88]. We prove it for functions that are uniformly degree invariant from Turing to many-one equivalence, getting a finer and richer structure. Let us explain this in more detail.

Often in mathematics, we consider a class of objects, some of which show up more often than others. It is often the case that those objects that occur naturally behave better than the rest. The contrast between the general behavior and the behavior of naturally occurring objects can be quite interesting and intriguing. For instance, not all continuous functions are differentiable, although most naturally occurring ones are; not all sets of reals are measurable, although most naturally occurring ones are. In this paper, we consider the class of many-one degrees, which has been widely studied in computability theory since its beginnings (see [Odi89, Chapters III and VI]).

Definition 1. For sets $A, B \subseteq \mathbb{N}$, we say that A is *many-one reducible* to B (sometimes referred to as m reducible and written $A \leq_m B$) if there is a computable function $\psi\colon \mathbb{N} \to \mathbb{N}$ such that

$$n \in A \iff \psi(n) \in B \quad \text{for all } n\text{'s} \in \mathbb{N}.$$

As usual, from this preordering we define an equivalence relation $\equiv_m$ by

$$A \equiv_m B \iff A \leq_m B \text{ and } B \leq_m A,$$

and we call the equivalence classes *m-degrees*, which are partially ordered by $\leq_m$.

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This reducibility tries to measure the information content within a set: if $A \leq_m B$, then B contains all the information encoded in A. It is one of the most natural reducibilities in computability theory. Let us look at some examples. The following three sets are m equivalent:

- K, the *Halting problem*, namely the set of all programs that halt and do not run forever;
- the *word problem*, namely the set of pairs of a finite set of generators and a finite set of relations generating a group that is trivial;
- *Hilbert's 10th problem*, namely the set of polynomials in $\mathbb{Z}[X_1, X_2, \ldots]$ which have integer solutions.

This is actually how one proves that the last two sets are noncomputable: one shows that the Halting problem, which can be easily shown to be noncomputable with a diagonalization argument, m-reduces to them. Continuing with more examples, if we consider that the set $P = \{p(Y, X_1, \ldots, X_k) \in \mathbb{Z}[Y, X_1, X_2, \ldots] : p$ has integer solutions for exactly one value of $Y\}$, then P is strictly more complicated than the Halting problem K. That is, $K \leq_m P$ but $P \not\leq_m K$. Computability theories would recognize P as the complete d.c.e. m-degree. Then the set T, of finite presentations of torsion-free groups, is strictly higher up [Lem97], and the set L of computable linear orderings which do not contain a copy of $\mathbb{Q}$ is even higher. (For computability theorists: these are the complete Π^0_2 and complete Π^1_1 m-degrees.)

There are various other natural m-degrees that computability theorists know about. But the natural examples are still few and far between, and except for taking complements, they seem to be linearly ordered—even well-ordered. For instance, we know of no natural m-degree of a c.e. set that is neither complete nor computable, despite there being infinitely many such degrees. We know of no natural m-degree of a Σ^1_1 set that is neither Σ^1_1 complete nor hyperarithmetic—again, despite there being lots of them. The general structure of the m-degrees is quite complex: there are continuum-size antichains; every countable poset embeds in it, even below K [KP54]; its first-order theory is extremely complicated—it is computably isomorphic to true second-order arithmetic, etc. [NS80] (see [Odi89, Chapter VI].)

In this paper, we give a complete characterization of the *natural* many-one degrees. In the same sense, a characterization of the natural Turing degrees is already well known and follows from the uniform Martin's conjecture, which was proved by Slaman and Steel [Ste82, SS88]: the natural Turing degrees are, essentially, the iterates of the Turing jump through the transfinite. Becker [Bec88] gave a detailed analysis of the well order of natural nonzero Turing degrees by relating them to the universal sets of reasonable point classes. It turns out that the answer for the many-one degrees is richer: the natural many-one degrees are in one-to-one correspondence with the Wadge degrees. Except for a few ideas that we borrowed from the proof of the uniform Martin's conjecture, most of our argument is completely different. Our results can be viewed as a refinement of the uniform Martin's conjecture, since the jump of a natural Turing degree is a natural m-degree. However, there are many natural m-degrees that are not differentiated by Turing equivalence. Indeed, every natural Turing degree contains a lot of m-inequivalent natural m-degrees; for instance, the complete c.e. set is Turing equivalent to the complete d.c.e. set, though they are not m equivalent.

We do not have a formal mathematical definition for what it means to be a natural m-degree. Thus, there will have to be an empirical, nonmathematical claim in our argument:

> Natural m-degrees induce Turing to many-one, uniformly degree-invariant functions, as in Definition 2.

This claim comes from the observation that, in computability, all proofs *relativize*, which is also empirically observed. That is, for any given theorem, if we change the notion of computability to that of computability relative to an oracle X, the resulting theorem can then still be proved using the same proof. Furthermore, the notions we deal with in computability theory also relativize, and so do their properties. Thus, if we have a natural m-degree $\mathbf{s}$, we can associate with it a function that, given an oracle X, returns the relativization of $\mathbf{s}$ to X, denoted $\mathbf{s}^X$. Furthermore, if we relativize to an oracle $Y \equiv_T X$, the classes of partial X-computable functions and of partial Y-computable functions are the same, so we should obtain the same m-degrees. We let the interested reader contemplate this fact further; see also [Ste82, Bec88, DS97] for criteria on natural degrees. We will now move on to the purely mathematical results.

Here is the definition of the uniformly degree-invariant functions we mentioned above.

Definition 2. We say that a function $f\colon \omega^\omega \to 2^\omega$ is *uniformly* $(\leq_T, \leq_m)$*-order preserving* (abbreviated $(\leq_T, \leq_m)$-UOP) if, for every $X, Y \in \omega^\omega$,

$$X \leq_T Y \implies f(X) \leq_m f(Y),$$

and furthermore, there is a computable function $u\colon \omega \to \omega$ such that, for all cases in which $X, Y \in \omega^\omega$,

$$X \leq_T Y \text{ via } e \implies f(X) \leq_m f(Y) \text{ via } u(e).$$

(By $X \leq_T Y$ via e, we mean that it is the eth Turing functional Φ_e that Turing reduces X to Y, and analogously with m-reducibility.)

We say that f is *uniformly* $(\equiv_T, \equiv_m)$ *invariant* (abbreviated $(\equiv_T, \equiv_m)$-UI) if there is a computable function $u\colon \omega^2 \to \omega^2$ such that, for all cases in which $X, Y \in \omega^\omega$,

$$X \equiv_T Y \text{ via } (i, j) \implies f(X) \equiv_m f(Y) \text{ via } u(i, j).$$

There is a natural notion of *largeness* for sets of Turing degrees given by Martin's measure: a Turing-degree-invariant set $\mathcal{A} \subseteq \omega^\omega$ has the Martin measure 1 if it contains a *Turing cone*, i.e., a set of the form $\{X \in \omega^\omega : Y \geq_T X\}$ for some $X \in \omega^\omega$, and has the Martin measure 0 otherwise. Martin proved that if determinacy holds for all sets in a point class Γ, then this is a σ-additive measure on the degree-invariant sets in Γ [Mar68]. He used this notion of largeness to compare $\equiv_T$-to-$\equiv_T$–invariant functions: Given two such functions, we say that f is *Turing reducible to g on a cone* if $f(X) \leq_T g(X)$ for all X's on a set of Martin's measure 1.

> Martin's conjecture: The nonconstant $\equiv_T$-to-$\equiv_T$–invariant functions are well-ordered by Turing reducibility on a cone.

Martin's conjecture is one of the most important open questions in computability theory. As we mentioned above, the case of *uniformly* $\equiv_T$-to-$\equiv_T$–invariant functions was proved by Slaman and Steel [Ste82, SS88], and what we are reminded of is Steel's conjecture that claims that every $\equiv_T$-to-$\equiv_T$–invariant function is Turing equivalent on a cone to a uniformly invariant one. Andrew Marks has been making quite a bit of progress on this.

In this paper, we instead use Martin's notion of largeness to extend the *many-one ordering* from sets to $(\equiv_T, \equiv_m)$-UI functions. As it turns out, we obtain a much richer structure than for the uniformly $\equiv_T$-to-$\equiv_T$–invariant functions, as Turing equivalence is a much coarser equivalence relation.

Definition 3. For $A, B \subseteq \omega$ and an oracle $C \in \omega^\omega$, we say that A is many-one reducible to B relative to C (and write $A \leq_m^C B$) if there is a C-computable function Φ_e^C such that

$$(\forall n \in \omega)\; n \in A \iff \Phi_e^C(n) \in B.$$

Given $f, g \colon \omega^\omega \to 2^\omega$, we say that f is *many-one reducible to g on a cone* (and write $f \leq_{\mathbf{m}}^\triangledown g$) if

$$(\exists C \in \omega^\omega)(\forall X \geq_T C)\; f(X) \leq_m^C g(X).$$

It is clear that $\leq_{\mathbf{m}}^\triangledown$ is a preordering and hence induces an equivalence on functions we denote by $\equiv_{\mathbf{m}}^\triangledown$. Our objective is to compare $\equiv_{\mathbf{m}}^\triangledown$-degrees of $(\equiv_T, \equiv_m)$-UI functions with the Wadge degrees.

Definition 4 (Wadge [Wad83])**.** Given $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$, we say that $\mathcal{A}$ is *Wadge reducible* to $\mathcal{B}$ (and write $\mathcal{A} \leq_w \mathcal{B}$) if there is a continuous function $f \colon \omega^\omega \to \omega^\omega$ such that $X \in \mathcal{A} \iff f(X) \in \mathcal{B}$ for all X's $\in \omega^\omega$.

Again, $\leq_w$ is a preordering which induces an equivalence $\equiv_w$ and a degree structure. The Wadge degrees are rather well-behaved, at least under enough determinacy. If we assume Γ-determinacy, then the Wadge degrees of sets in Γ are semi–well-ordered in the sense that they are well founded and that all antichains have a size of at most 2 (as proved by Wadge [Wad83], and Martin and Monk [KLS12]). Furthermore, they are all natural, and we can assign *names* to each of them using an ordinal and a symbol from $\{\Sigma, \Pi\}$ (see [VW78]), a name from which we can understand the nature of that Wadge degree.

Here is our main theorem for the case of sets.

Theorem 5 (axiom of determinacy and dependent choice (AD+DC))**.** *There is an isomorphism between the partial ordering of $\equiv_{\mathbf{m}}^\triangledown$-degrees of $(\equiv_T, \equiv_m)$-UI functions ordered by $\leq_{\mathbf{m}}^\triangledown$ and the partial ordering of Wadge degrees of subsets of ω^ω ordered by Wadge reducibility.*

The definition of the isomorphism is merely an *uncurrying* function, which is not complicated (see section 2). It is the proof that it is a correspondence that requires work. We get the following simple corollaries. The clopen Wadge degrees correspond to the constant functions. Then the open nonclopen Wadge degree corresponds to the $(\equiv_T, \equiv_m)$-UI function that gives the complete c.e. set. Thus, there are no $(\equiv_T, \equiv_m)$-UI functions strictly in between the constant functions and the complete ones. The Hausdorf–Kuratowski difference hierarchy of $\mathbf{\Delta}_2^0$ sets of reals corresponds to the Ershov hierarchy of Δ_2^0 sets of natural numbers. Thus, up to $\equiv_{\mathbf{m}}^\triangledown$ equivalence, the only $\mathbf{\Delta}_2^0$ $(\equiv_T, \equiv_m)$-UI functions are the ones corresponding to the Ershov hierarchy. The complete Wadge degree of the Σ_1^1 set of reals corresponds to the $(\equiv_T, \equiv_m)$-UI function given by the complement of the hyperjump. Since every Wadge degree of a $\mathbf{\Sigma}_1^1$ set must be either $\mathbf{\Sigma}_1^1$ complete or Borel, we get that, up to $\equiv_{\mathbf{m}}^\triangledown$ equivalence, a $\mathbf{\Sigma}_1^1$ $(\equiv_T, \equiv_m)$-UI function must be either complete or hyperarithmetic.

In our construction of section 3, we actually assign a $(\leq_T, \leq_m)$-UOP function to each Wadge degree. Thus, our proof also gives the following theorem.

Theorem 6 (AD+DC)**.** *Every $(\equiv_T, \equiv_m)$-UI function $\omega^\omega \to 2^\omega$ is $\equiv^\nabla_{\mathbf{m}}$ equivalent to a $(\leq_T, \leq_m)$-UOP one.*

1.1. **The extension to better-quasi-orderings.** Our main theorem will actually be more general than Theorem 5. A subset of ω can be viewed as a function $\omega \to 2$, and a subset of ω^ω as a function $\omega^\omega \to 2$. Instead, we will consider functions $\omega \to \mathcal{Q}$ and $\omega^\omega \to \mathcal{Q}$, where $\mathcal{Q}$ is a better-quasi-ordering (bqo). The definition of better-quasi-ordering is complicated (Definition 10), so for now, let us just say that better-quasi-orderings are well founded, have no infinite antichains, and have nice closure properties.

The generalizations of all of the notions defined above are straightforward. We include them for completeness.

Definition 7. Let $(\mathcal{Q}; \leq_\mathcal{Q})$ be a quasi-ordered set. For $A, B \in \mathcal{Q}^\omega$ and an oracle $C \in \omega^\omega$, we say that A is *$\mathcal{Q}$-many-one reducible to B relative to C* (written $A \leq_m^C B$) if there is a C-computable function $\Phi_e^C\colon \omega \to \omega$ such that

$$(\forall n \in \omega)\ A(n) \leq_\mathcal{Q} B(\Phi_e^C(n)).$$

For functions $\omega^\omega \to \mathcal{Q}^\omega$, the definitions of $(\leq_T, \leq_m)$-UOP, $(\leq_T, \leq_m)$-UI, and $\leq^\nabla_{\mathbf{m}}$ are then exactly as before, using the new notion of $\mathcal{Q}$-many-one reducibility.

For $\mathcal{Q}$-valued functions $\mathcal{A}, \mathcal{B}\colon \omega^\omega \to \mathcal{Q}$, we say that $\mathcal{A}$ is *$\mathcal{Q}$-Wadge reducible to $\mathcal{B}$* (written $\mathcal{A} \leq_w \mathcal{B}$) if there is a continuous function $\theta\colon \omega^\omega \to \omega^\omega$ such that

$$(\forall X \in \omega^\omega)\ \mathcal{A}(X) \leq_\mathcal{Q} \mathcal{B}(\theta(X)).$$

On the one hand, considering the general case does not add to the complexity of the proof—the proofs for 2 and for a general $\mathcal{Q}$ are essentially the same. There are bqos $\mathcal{Q}$ other than 2 for which the $\mathcal{Q}$-many-one degrees are interesting too. For $\mathcal{Q} = 3$, the poset with three incomparable elements, Marks [Mar17] proved that many-one equivalence on 3^ω is a uniformly universal countable Borel equivalence relation, while this is not the case for 2^ω. Since $(\equiv_T, \equiv_m)$-UI functions are nothing more than uniform reductions from Turing to many-one equivalence, understanding such functions can shed light on the structure of countable, degree-invariant Borel equivalence relations. For $\mathcal{Q} = (\omega; \leq)$, we maintain that, for $f, g\colon \omega \to \omega$, $f \leq_m g$ if and only if there is a computable speed up of g that grows faster than f, that is, if there is a computable $h\colon \omega \to \omega$ such that $g \circ h(n) \geq f(n)$ for all n's $\in \omega$. On the side of the Wadge degrees, Steel showed that when $\mathcal{Q}$ is the class of ordinals, the Wadge degrees are well founded. In [KM], Kihara and Montalbán provide a full description of the Wadge degrees of $\mathcal{Q}$-valued Borel functions for each bqo $\mathcal{Q}$, extending the work of Duparc [Dup01, Dup03], Selivanov [Sel07], and others.

Here is our main theorem.

Theorem 8 (AD$^+$)**.** *There is an isomorphism between the partial ordering of $\equiv^\nabla_{\mathbf{m}}$-degrees of $(\equiv_T, \equiv_m)$-UI functions $\omega^\omega \to \mathcal{Q}^\omega$ ordered by $\leq^\nabla_{\mathbf{m}}$ and the partial ordering of $\mathcal{Q}$-Wadge degrees of functions $\omega^\omega \to \mathcal{Q}$ ordered by $\mathcal{Q}$-Wadge reducibility.*

Theorem 9 (AD$^+$)**.** *Every $(\equiv_T, \equiv_m)$-UI function $\omega^\omega \to Q^\omega$ is $\equiv^\nabla_{\mathbf{m}}$ equivalent to a $(\leq_T, \leq_m)$-UOP one.*

1.2. **Background facts on $\mathcal{Q}$-Wadge degrees.** In the 1970s, Martin and Monk [KLS12] showed that the Wadge degrees of subsets of ω^ω are well founded, and hence semi–well-ordered by Wadge's lemma [Wad83]. Steel then showed that the Wadge degrees of ordinal-valued functions with domain ω^ω are well-ordered (see [Dup03, Theorem 1]). Later, van Engelen, Miller, and Steel [vEMS87] employed bqo theory to unify these results, and they showed that if $\mathcal{Q}$ is bqo, then so are the Wadge degrees of $\mathcal{Q}$-valued Borel functions. More recently, Block [Blo14] introduced the notion of a *very strong better-quasi-order* to remove the Borel-ness assumption from van Engelen–Miller–Steel's theorem. (We show in subsection 1.3 that under AD^+, bqos and very strong bqos are the same thing.)

To define bqos, we need to introduce some notation. Let $[\omega]^\omega$ be the set of all strictly increasing sequences on ω, whose topology is inherited from ω^ω. We also assume that a quasi-order $\mathcal{Q}$ is equipped with the discrete topology. Given $X \in [\omega]^\omega$, by X^- we denote the result of dropping the first entry from X (or, equivalently, $X^- = X \setminus \{\min X\}$ if we think of $X \in [\omega]^\omega$ as an infinite subset of ω).

Definition 10 (Nash-Williams [NW65])**.** A quasi-order $\mathcal{Q}$ is called a *bqo* if, for any continuous function $f\colon [\omega]^\omega \to \mathcal{Q}$, there is an $X \in [\omega]^\omega$ such that $f(X) \leq_{\mathcal{Q}} f(X^-)$.

The formulation of the definition above is due to Simpson [Sim85]. It is not hard to prove that every bqo is also a *well-quasi-order*, that is, that it is well founded and that it has no infinite antichain.

Example 11. For a natural number k, the discrete order $\mathcal{Q} = (k; =)$, which we will denote by k, is a bqo. More generally, every finite partial ordering is a bqo. For $\mathcal{Q} = k$, the $\mathcal{Q}$-valued functions are called *k-partitions.*

Let us now state the key facts that we will be using for the $\mathcal{Q}$-Wadge degrees. Special cases of the following facts were proved by van Engelen, Miller, and Steel [vEMS87, Theorem 3.2] for Borel functions, and by Block [Blo14, Theorem 3.3.10] for very strong bqos $\mathcal{Q}$ under AD. AD^+ proves the general result.

Fact 12 (AD^+)**.** *If $\mathcal{Q}$ is a bqo, then the Wadge degrees of $\mathcal{Q}$-valued functions on ω^ω form a bqo too.*

There are two more facts about $\mathcal{Q}$-Wadge degrees that we will use throughout the paper.

Definition 13. We say that a $\mathcal{Q}$-Wadge degree $\mathbf{a}$ is *σ-join-reducible* if $\mathbf{a}$ is the least upper bound of a countable collection $(\mathbf{b}_i)_{i\in\omega}$ of $\mathcal{Q}$-Wadge degrees such that $\mathbf{b}_i <_w \mathbf{a}$. Otherwise, we say that $\mathbf{a}$ is *σ join irreducible.*

The following fact gives a better way to characterize σ join reducibility. Its proof uses the well-foundedness of the $\mathcal{Q}$-Wadge degrees, which is an immediate consequence of Fact 12. For $X \in \omega^\omega$, we use the symbol $X \upharpoonright n$ to denote the unique initial segment of X of length n, and, for a finite string $\sigma \in \omega^{<\omega}$, $[\sigma]$ denotes the set of all reals extending σ.

Fact 14 (AD^+)**.** *Let $\mathcal{Q}$ be a bqo. A function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is σ join irreducible if and only if there is an $X \in \omega^\omega$ such that $\mathcal{A} \leq_w \mathcal{A} \upharpoonright [X \upharpoonright n]$ for every $n \in \omega$.*

A function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is σ join reducible if and only if it is Wadge equivalent to a function of the form $\bigoplus_{n\in\omega} \mathcal{A}_n$, where each $\mathcal{A}_n$ is σ join irreducible and $\mathcal{A}_n <_w \mathcal{A}$, and where $\bigoplus_{n\in\omega} \mathcal{A}_n$ is defined by $(\bigoplus_{n\in\omega} \mathcal{A}_n)(n^\frown X) = \mathcal{A}_n(X)$.

The third fact that we need is a generalization of Steel–van Wesep's theorem [VW78] from $\mathcal{Q} = 2$ to a general $\mathcal{Q}$, proved by Block [Blo14]. The following generalization of self-duality is due to Louveau and Saint-Raymond [LSR90].

Definition 15. We say that a function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is *self-dual* if there is a continuous function $\theta\colon \omega^\omega \to \omega^\omega$ such that $\mathcal{A}(\theta(X)) \not\leq_{\mathcal{Q}} \mathcal{A}(X)$ for all X's $\in \omega^\omega$.

Assuming AD, Block [Blo14, Proposition 3.5.4] showed the following fact for very strong bqos. We get it for all bqos under AD^+.

Fact 16 (AD^+). *Let $\mathcal{Q}$ be a bqo. Then a $\mathcal{Q}$-valued function on ω^ω is self-dual if and only if it is σ join reducible.*

1.3. **The set-theoretic assumptions.** Our main theorems are stated under the assumption of AD^+, which is an extension of the AD introduced by Woodin [Woo99]. If we want to assume less than AD^+, our results will still be true for restricted classes of functions. For instance, they will still be true for Borel functions just in ZFC, and they will still be true for projective functions if we assume DC+PD.

Let Γ be a point class of sets of reals containing all Borel sets closed under countable unions, finite intersections, and continuous substitutions. We concentrate on Γ functions $f : \omega^\omega \to \mathcal{Q}$ whose range is countable, where a function $g : \omega^\omega \to \mathcal{Q}^\omega$ can also be thought of as a function from $\omega^\omega \times \omega$ ($\simeq \omega^\omega$) to $\mathcal{Q}$ in an obvious way (see also Definition 18).

For our results to hold for functions in Γ, we need to assume, first, that all Wadge-like games (introduced in section 4) for Γ functions are determined, and, second, that Facts 12, 14, and 16 hold for functions in Γ. The first assertion is ensured by assuming that *all sets in* Γ *are determined* whenever the ranges of functions are countable. Our assumption of countability of the range is used only to ensure this part (and thus this restriction can be removed under AD).

We will now argue that assuming that *all sets in* Γ *are Ramsey* gives us these three facts for any bqo $\mathcal{Q}$. Note that this Γ-Ramsey hypothesis actually implies that all sets in Γ are completely Ramsey (that is, all sets in Γ have the Baire property with respect to the Ellentuck topology) under our assumption on Γ (see Brendle and Löwe [BL99, Lemma 2.1]). Fact 14 uses only the well-foundedness of $\mathcal{Q}$-Wadge degrees of Γ functions, which clearly follows from Fact 12, on top of ZFC. For Facts 12 and 16, we need the following observation.

Observation 17. Suppose that all sets in Γ are determined and Ramsey, and let $\mathcal{Q}$ be a bqo. We say that $\mathcal{Q}$ is a Γ-*bqo* if, for every Γ function $f\colon [\omega]^\omega \to \mathcal{Q}$, there is an $X \in [\omega]^\omega$ such that $f(X) \leq_{\mathcal{Q}} f(X^-)$.

Our assumption on Γ implies that if $\mathcal{Q}$ is a bqo, it is also a Γ-bqo: this is because every such f in Γ has the Baire property with respect to the Ellentuck topology by our assumption that all sets in Γ are completely Ramsey. Louveau and Simpson [LS82] showed that, for every Ellentuck–Baire function $f : [\omega]^\omega \to \mathcal{Y}$, where $\mathcal{Y}$ is a metric space, not necessarily separable, there exists an infinite set $X \subseteq \omega$ such that f is continuous when restricted to $[X]^\omega$. By applying this to the discrete metric space $\mathcal{Y} := \mathcal{Q}$, the above argument verifies our claim.

One can then carry out the van Engelen–Miller–Steel proof [vEMS87, Theorem 3.2] for Γ functions exactly as Block did in [Blo14, Theorem 3.3.10] to determine that the $\mathcal{Q}$-Wadge degrees of functions in Γ are bqo. The argument requires only that Γ is closed under a countable (separated) union and a continuous substitution.

Similarly, we can use Block's argument [Blo14, Theorem 3.4.4] to show the Steel–van Wesep theorem [VW78] for $\mathcal{Q}$-valued Γ functions, that is, that Wadge self-duality and Lipschitz self-duality are equivalent for Γ functions. Fact 16 then follows from the standard argument (see [VW78, section 3] or [Blo14, Proposition 3.5.4]) and Fact 14.

Therefore, what we actually prove in this paper is the following:

> If we assume that all sets in Γ are determined and Ramsey, then our main Theorems 5 and 8 hold when restricted to Γ functions whose range is countable.

In particular, Theorems 5, 6, 8, and 9 for Borel functions can be proved in ZFC (since all Borel sets are determined and Ramsey under ZFC [Mar75, GP73]), and for projective functions can be proved under PD (since all projective sets are Ramsey under PD [HK81]; indeed, $\mathbf{\Delta}^1_n$-determinacy implies that all $\mathbf{\Pi}^1_n$ sets are Ramsey for any positive, even number n). Our assumption AD^+ implies that all sets of reals are determined and Ramsey.

We also notice that our hypothesis that all Γ sets are Ramsey is used only to ensure that every bqo is Γ-bqo. For $\mathcal{Q} = 2$, we can prove our main theorem without assuming the Γ-Ramsey hypothesis. This is because the discrete ordered set $2 = \{0, 1\}$ is a very strong bqo (i.e., Γ-bqo for any Γ) within AD+DC (see Block [Blo14, Corollary 3.3.9]). Indeed, Wadge's lemma, Martin–Monk's lemma, and Steel–van Wesep's theorem are all provable in AD+DC, and these are all that we need to prove our main theorem. This is the reason why we can state Theorems 5 and 6 assuming only AD+DC.

We will not mention these assumptions anymore through the rest of the paper. The reader may either assume AD^+ or assume that we are working only with functions in a point class Γ, all of whose sets are determined and Ramsey.

2. The Plan

The mapping $\mathfrak{A}$ that we will use to embed the $(\equiv_T, \equiv_m)$-UI functions onto the $\mathcal{Q}$-Wadge degrees is quite simple: it is an uncurrying function. The difficult part will be to prove that it actually gives a one-to-one correspondence.

Definition 18. Given $f\colon \omega^\omega \to \mathcal{Q}^\omega$, we define a function $\mathfrak{A}(f)\colon \omega^\omega \to \mathcal{Q}$ as follows:

$$\mathfrak{A}(f)(n^\frown X) = f(X)(n)$$

for $n \in \omega$ and $X \in \omega^\omega$. Here, $n^\frown X$ is the concatenation of n and X.

This function will only work well on a subset of the $(\equiv_T, \equiv_m)$-UI functions, the $\mathfrak{A}$-minimal functions, which we define below. Before doing so, we need to introduce the following notion.

Definition 19. By *perfect tree*, we mean a map $\psi_T\colon \omega^{<\omega} \to \omega^{<\omega}$ together with its image $S_T = \{\sigma : (\exists \tau)\ \sigma \subseteq \psi_T(\tau)\}$, satisfying $\sigma \subseteq \tau \iff \psi_T(\sigma) \subseteq \psi_T(\tau)$ for all cases in which $\sigma, \tau \in \omega^{<\omega}$. In other words, a perfect tree is a pair $T = (\psi_T, S_T)$. Abusing notation, we simply write $T(\cdot)$ and T for $\psi_T(\cdot)$ and S_T, respectively. For each $X \in \omega^\omega$, we can define $T[X] \in \omega^\omega$ in a obvious way; we often think of ψ_T directly as a continuous map $T[\cdot]\colon \omega^\omega \to \omega^\omega$. We use $[T]$ to denote the image $\{T[X] : X \in \omega^\omega\}$ of the map $T[\cdot]$, whose element is called a *path* through T.

A *pointed perfect tree* is a perfect tree which is computable from each of its paths. In other words, it is a perfect tree $T = (\psi_T, S_T)$ such that $T \leq_T Y$ for any $Y \in [T]$, where $T \leq_T Y$ means that $\psi_T \oplus S_T \leq_T Y$. By a *uniformly pointed perfect tree* (abbreviated as *u.p.p. tree*), we mean a perfect tree whose pointedness is witnessed by a Turing reduction independent of Y. In other words, it is a perfect tree $T = (\psi_T, S_T)$ such that there is an index e such that $\Phi_e(Y) = \psi_T \oplus S_T$ for any $Y \in [T]$.

The main property of u.p.p. trees is that, for every $X \geq_T T$, we have $X \equiv_T T[X]$,[1] and we can compute the indices for this Turing equivalence given the index for $X \geq_T T$. Here is how u.p.p. trees interact with a $(\equiv_T, \equiv_m)$-UI function. In the statement of the lemma, we view the trees as maps $\omega^\omega \to \omega^\omega$.

Lemma 20. *Let $f\colon \omega^\omega \to \mathcal{Q}^\omega$ be a $(\equiv_T, \equiv_m)$-UI function, and let S and T be u.p.p. trees.*

(1) *If $S \leq_T T$, then $\mathfrak{A}(f \circ T) \leq_w \mathfrak{A}(f \circ S)$.*
(2) *If f is $(\leq_T, \leq_m)$-UOP, then $\mathfrak{A}(f \circ T) \equiv_w \mathfrak{A}(f)$.*
(3) *$f \circ T \equiv^\nabla_{\mathbf{m}} f$.*

Proof. For (1), it is not hard to see that, since S and T are uniformly pointed and $S \leq_T T$, one can computably extract the triple (S, T, X) from $T[X]$ and the pair (T, X) from $S[T \oplus X]$ in a uniform manner. Therefore, there is a pair of Turing reductions witnessing $T[X] \equiv_T S[T \oplus X]$ which does not depend on X. Thus, since f is $(\equiv_T, \equiv_m)$-UI, there is a computable function Ψ such that

$$f(T[X])(n) \leq_{\mathcal{Q}} f(S[T \oplus X])(\Psi(n))$$

for any $n \in \omega$. Consequently, we have

$$\mathfrak{A}(f \circ T)(n^\frown X) = \mathfrak{A}(f)(n^\frown T[X]) \leq_{\mathcal{Q}} \mathfrak{A}(f)(\Psi(n)^\frown S[T \oplus X]) = \mathfrak{A}(f \circ S)(\Psi(n)^\frown T \oplus X).$$

For (2), we need to show only that $\mathfrak{A}(f \circ T) \geq_w \mathfrak{A}(f)$, as the other reduction follows from (1). There is an index that we can use to compute X from $T[X]$ for all X's, and hence there is a computable function ψ witnessing $f(X) \leq_m f(T[X])$ for all X's. We then have

$$\mathfrak{A}(f)(n^\frown X) = f(X)(n) \leq_{\mathcal{Q}} f(T[X])(\psi(n)) = \mathfrak{A}(f \circ T)(\psi(n)^\frown X).$$

For (3), assume that $X \geq_T T$. Then $X \equiv_T T[X]$, so let (i, j) be a pair of indices witnessing this. Let $u = (u_0, u_1)$, and witness that f is $(\equiv_T, \equiv_m)$-UI, where $u_i = \pi_i \circ u$ for the ith projection π_i. Then we have $f(X)(n) \leq_{\mathcal{Q}} f(T[X])(\Phi_{u_0(i,j)}(n))$ and $f(T[X])(n) \leq_{\mathcal{Q}} f(X)(\Phi_{u_1(j,i)}(n))$ for any $n \in \omega$. This clearly implies that $f \circ T \equiv^\nabla_{\mathbf{m}} f$. $\square$

Since the $\mathcal{Q}$-Wadge degrees are well founded (actually better-quasi-ordered by Fact 12), by Lemma 20 (1), we determine that there is a C such that the $\mathcal{Q}$-Wadge degree of $\mathfrak{A}(f \circ T)$ is the same for all u.p.p. trees $T \geq_T C$.

Definition 21. We say that $f\colon \omega^\omega \to \mathcal{Q}^\omega$ is $\mathfrak{A}$-*minimal* if for all u.p.p. trees T, $\mathfrak{A}(f \circ T) \equiv_w \mathfrak{A}(f)$.

[1] In fact, for any pointed perfect tree T, we always have $X \leq_T \psi_T \oplus S_T \oplus T[X] \leq_T T[X] \leq_T \psi_T \oplus X$ (the first inequality is proven by searching for σ such that $\psi_T(\sigma) \subseteq T[X]$), and if moreover $X \geq_T \psi_T$, then $\psi_T \oplus X \leq_T X$ and hence $X \equiv_T T[X]$.

It follows from the lemma above that every $(\equiv_T, \equiv_m)$-UI function is $\equiv^{\triangledown}_{\mathbf{m}}$ equivalent to a $\mathfrak{A}$-minimal one, and that if f is $(\leq_T, \leq_m)$-UOP, it is $\mathfrak{A}$ minimal already. We can thus concentrate only on the $\mathfrak{A}$-minimal $(\equiv_T, \equiv_m)$-UI functions.

Lemma 22. *Let $f, g\colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\equiv_T, \equiv_m)$-UI, $\mathfrak{A}$-minimal functions. Then $f \leq^{\triangledown}_{\mathbf{m}} g$ implies $\mathfrak{A}(f) \leq_w \mathfrak{A}(g)$.*

Proof. There is a $C \in \omega^\omega$ such that, for each $X \geq_T C$, there is some e such that Φ^C_e is a many-one reduction witnessing $f(X) \leq^C_m g(X)$. We then use Martin's lemma (see [MSS16, Lemma 3.5]), saying that if ω^ω is partitioned into countably many subsets, then one of them contains all infinite paths through a u.p.p. tree, to obtain an index e and a u.p.p. tree T such that, for all Y's $\in [T]$, $f(Y) \leq^C_m g(Y)$ via Φ^C_e. We thus get that, for all X's $\in \omega^\omega$ and $n \in \omega$, $f(T[X])(n) \leq_{\mathcal{Q}} g(T[X])(\Phi^C_e(n))$, and hence that $\mathfrak{A}(f \circ T) \leq_w \mathfrak{A}(g \circ T)$. Since both f and g are $\mathfrak{A}$ minimal, this implies that $\mathfrak{A}(f) \leq_w \mathfrak{A}(g)$. $\square$

We now have a well-defined map from the $\leq^{\triangledown}_{\mathbf{m}}$-degrees of $(\equiv_T, \equiv_m)$-UI functions to the $\mathcal{Q}$-Wadge degrees: given a $(\equiv_T, \equiv_m)$-UI function f, let g be a $(\equiv_T, \equiv_m)$-UI function that is $\mathfrak{A}$ minimal and $\equiv^{\triangledown}_{\mathbf{m}}$ equivalent to f, and let the image of the $\equiv^{\triangledown}_{\mathbf{m}}$-degree of f be the $\mathcal{Q}$-Wadge degree of $\mathfrak{A}(g)$. To show that this map is an isomorphism, i.e., Theorem 8, and to also get Theorem 9, we will show the following two propositions.

Proposition 23. *For every $\mathcal{Q}$-Wadge degree $\mathcal{A}$, there is a $(\leq_T, \leq_m)$-UOP function g such that $\mathfrak{A}(g) \equiv_w \mathcal{A}$.*

Remark 24. Let us say that g is *in standard form* if either $\mathfrak{A}(g)$ is non–self-dual or it is of the form $\bigoplus g_n$, where $\mathfrak{A}(g_n)$ is non–self-dual for each n, where we define $\bigoplus_n g_n\colon \omega^\omega \to \mathcal{Q}^\omega$ by $(\bigoplus_n g_n)(X)(\langle m, k\rangle) = g_m(X)(k)$. It will follow from the proof of Proposition 23 in the next section that we can assume g is of the form $\bigoplus g_n$ and hence is in standard form. We can then use Lemma 20 to find an oracle C such that, for all u.p.p. trees S, $\mathfrak{A}(g_n \circ S)$ has a minimal Wadge degree, and hence each of the g_n's is $\mathfrak{A}$ minimal.

Proposition 25. *Let $f, g\colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\equiv_T, \equiv_m)$-UI, $\mathfrak{A}$-minimal functions. Then $f \leq^{\triangledown}_{\mathbf{m}} g$ if and only if $\mathfrak{A}(f) \leq_w \mathfrak{A}(g)$.*

We will prove Proposition 23 and Remark 24 in section 3. We will prove Proposition 25 in sections 4 and 5.2.

3. Surjectivity

The next step is to show that $\mathfrak{A}$ is onto. We devote this subsection to proving Proposition 23.

Given an oracle $C \in \omega^\omega$, a function $p\colon \omega \to \omega$ is said to be *C-primitive recursive* if it can be obtained by using the usual axioms of primitive recursive functions, including the function $n \mapsto C(n)$ in the list of initial functions. A *primitive recursive functional* is a function $P\colon \omega^\omega \to \omega^\omega$ such that $P(C)$ is C-primitive recursive uniformly in C. Let $(\mathrm{PRec}_e)_{e\in\omega}$ be an effective list of all primitive recursive functionals from ω^ω into ω^ω, so $\mathrm{PRec}\colon (e, X) \mapsto \mathrm{PRec}_e(X)$ is computable. We now introduce the following operation $\mathfrak{B}$ that will almost work as an inverse of $\mathfrak{A}$.

Definition 26. Given $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ and $C \in \omega^\omega$, let $\mathfrak{B}^C(\mathcal{A})\colon \omega^\omega \to \mathcal{Q}^\omega$ be defined by

$$\mathfrak{B}^C(\mathcal{A})(X)(e) = \mathcal{A}(\mathrm{PRec}_e(C \oplus X)).$$

We will show that, for some large enough C, $\mathfrak{B}^C(\mathcal{A})$ is $(\leq_T, \leq_m)$-UOP, and that the $\equiv^\triangledown_{\mathbf{m}}$-degree of $\mathfrak{B}^C(\mathcal{A})$ is independent of C. We start by showing that $\mathfrak{B}^C(\mathcal{A})$ is always an inverse of $\mathcal{A}$, even if $\mathfrak{B}^C(\mathcal{A})$ is not $(\equiv_T, \equiv_m)$-UI.

Lemma 27. *For any $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ and $C \in \omega^\omega$, we have $\mathfrak{A}(\mathfrak{B}^C(\mathcal{A})) \equiv_w \mathcal{A}$.*

Proof. Note that $\mathfrak{A}(\mathfrak{B}^C(\mathcal{A}))(e^\frown X) = \mathcal{A}(\mathrm{PRec}_e(C \oplus X))$. Let i be an index of the function $C \oplus X \mapsto X$; that is, $X = \mathrm{PRec}_i(C \oplus X)$. Then, given X, one can easily see that $\mathcal{A}(X) = \mathcal{A}(\mathrm{PRec}_i(C \oplus X)) = \mathfrak{A}(\mathfrak{B}^C(\mathcal{A}))(i^\frown X)$. Thus, $\mathcal{A} \leq_w \mathfrak{A}(\mathfrak{B}^C(\mathcal{A}))$. For the other reduction, notice that the map $(e, X) \mapsto \mathrm{PRec}_e(C \oplus X)$ is continuous, which indicates that $\mathfrak{A}(\mathfrak{B}^C(\mathcal{A})) \leq_w \mathcal{A}$. $\square$

The following lemma shows that, when $\mathfrak{B}^C(\mathcal{A})$ is $(\leq_T, \leq_m)$-UOP, $\mathfrak{B}^C(\mathcal{A})$ always gives us the same function up to $\equiv^\triangledown_{\mathbf{m}}$, independently of the oracle C.

Lemma 28. *Let $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$, and let $C, D \in \omega^\omega$. If $\mathfrak{B}^C(\mathcal{A})$ and $\mathfrak{B}^D(\mathcal{A})$ are $(\leq_T, \leq_m)$-UOP, then $\mathfrak{B}^C(\mathcal{A}) \equiv^\triangledown_{\mathbf{m}} \mathfrak{B}^D(\mathcal{A})$.*

Proof. It suffices to show that, for any $X \geq_T C \oplus D$, $\mathfrak{B}^C(\mathcal{A})(X) \leq_m \mathfrak{B}^D(\mathcal{A})(X)$ holds. Let v be such that $\mathrm{PRec}_e(C \oplus X) = \mathrm{PRec}_{v(e)}(D \oplus C \oplus X)$. Note that v gives us a many-one reduction

$$\mathfrak{B}^C(\mathcal{A})(X) \leq_m \mathfrak{B}^D(\mathcal{A})(C \oplus X).$$

For $X \geq_T C$, since $C \oplus X \leq_T X$ and $\mathfrak{B}^D(\mathcal{A})$ is $(\leq_T, \leq_m)$-UOP, we get

$$\mathfrak{B}^D(\mathcal{A})(C \oplus X) \leq_m \mathfrak{B}^D(\mathcal{A})(X).$$

We thus get $\mathfrak{B}^C(\mathcal{A})(X) \leq_m \mathfrak{B}^D(\mathcal{A})(X)$, as needed. The other inequality is analogous. $\square$

What is left to show that is that $\mathfrak{B}^C(\mathcal{A})$ is $(\leq_T, \leq_m)$-UOP for some C. We will not get exactly this—but close enough. We start with the case when $\mathcal{A}$ is not self-dual, for which we first need to prove a quick lemma. We say that a function $\theta\colon \omega^\omega \to \omega^\omega$ is *Lipschitz* if $\theta(X) \upharpoonright n$ depends only on $X \upharpoonright n$ for every $X \in \omega^\omega$, $n \in \omega$, or in other words, if $X \upharpoonright n = Y \upharpoonright n \implies \theta(X) \upharpoonright n = \theta(Y) \upharpoonright n$. Note that this property is stronger than being Lipschitz with respect to the standard ultrametric on ω^ω. In the metric context, such a function is called a metric map, or a nonexpansive map. However, we simply call it a Lipschitz function to make the connection with a Lipschitz game (see section 4) apparent.

Lemma 29. *Let $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ be not self-dual, $\mathcal{B}\colon \omega^\omega \to \mathcal{Q}$, and $\mathcal{D} \subseteq \omega^\omega$. If there is a continuous function $\theta\colon \mathcal{D} \to \omega^\omega$ such that $\mathcal{B}(X) \leq_{\mathcal{Q}} \mathcal{A}(\theta(X))$ for all X's $\in \mathcal{D}$, then there is a Lipchitz $\hat{\theta}\colon \omega^\omega \to \omega^\omega$ such that $\mathcal{B}(X) \leq_{\mathcal{Q}} \mathcal{A}(\hat{\theta}(X))$ for all X's $\in \mathcal{D}$.*

Proof. Consider the following variation of the Wadge game, which we denote by $G_{\mathrm{diag}}(\mathcal{A}, \mathcal{B} \upharpoonright \mathcal{D})$: Players I and II choose $x_n, y_n \in \omega$ alternately and produce $X = (x_n)_{n\in\omega}$ and $Y = (y_n)_{n\in\omega}$, respectively. Player II wins if $Y \in \mathcal{D}$ and $\mathcal{A}(X) \not\geq_{\mathcal{Q}} \mathcal{B}(Y)$. A winning strategy for II would give us a Lipchitz function Ψ such that $\mathcal{A}(X) \not\geq_{\mathcal{Q}} \mathcal{B}(\Psi(X))$ for all X's $\in \omega^\omega$. Composing with θ, we would then have that $\mathcal{A}(X) \not\geq_{\mathcal{Q}} \mathcal{A}(\theta \circ \Psi(X))$, contradicting the belief that $\mathcal{A}$ is not self-dual. Thus, Player

I must have a winning strategy, which gives us a Lipchitz function $\hat{\theta}\colon \omega^\omega \to \omega^\omega$. $\hat{\theta}$ must satisfy that, for all X's $\in \mathcal{D}$, $\mathcal{A}(\hat{\theta}(X)) \geq_{\mathcal{Q}} \mathcal{B}(X)$, as wanted. □

As in the previous proof, we can always identify a winning strategy τ with a Lipchitz function θ_τ. Moreover, $n \mapsto \tau(X \upharpoonright n)$ is $(\tau \oplus X)$-primitive recursive uniformly in $\tau \oplus X$. In other words, there is a primitive recursive code e such that, if τ defines a Lipschitz function θ_τ, then we have $\theta_\tau(X) = \mathrm{PRec}_e(\tau \oplus X)$.

Lemma 30. *If $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is not self-dual, there exists a C such that $\mathfrak{B}^C(\mathcal{A})$ is $(\leq_T, \leq_m)$-UOP.*

Proof. We will construct an oracle $C \in \omega^\omega$ and a computable function $q\colon \omega \to \omega$ such that, if $X \leq_T Y$ via Φ_d, then $\mathfrak{B}^C(\mathcal{A})(X) \leq_m \mathfrak{B}^C(\mathcal{A})(Y)$ via $q(d)$. Fix $p \in \mathcal{Q}$ and, for each $d \in \omega$, consider the following function $\mathcal{B}_d\colon \omega^\omega \to \mathcal{Q}$:

$$\mathcal{B}_d(e, C, Y) = \begin{cases} \mathcal{A}(\mathrm{PRec}_e(C \oplus \Phi_d(Y))) & \text{if } \Phi_d(Y) \text{ is total,} \\ p & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}_d$ be the set of all (e, C, Y)'s such that $\Phi_d(Y)$ is total. The continuous function $(e, C, Y) \mapsto \mathrm{PRec}_e(C \oplus \Phi_d(Y))$ reduces $\mathcal{B}_d$ to $\mathcal{A}$ on the domain $\mathcal{D}_d$. Therefore, by the previous lemma, there is a total Lipschitz function $\hat{\theta}_d$ such that, for all cases, $(e, C, Y) \in \mathcal{D}_d$, $\mathcal{B}_d(e, C, Y) \leq_{\mathcal{Q}} \mathcal{A}(\hat{\theta}_d(e, C, Y))$. Let

$$C = \bigoplus_{d \in \omega} \hat{\theta}_d.$$

We claim that $\mathfrak{B}^C$ is $(\leq_T, \leq_m)$-UOP. Given d and e, one can effectively find $q(d, e)$ such that

$$\hat{\theta}_d(e, C, Y) = \mathrm{PRec}_{q(d,e)}(C \oplus Y) \quad (\forall Y \in \omega^\omega).$$

Let $X \leq_T Y$ and suppose that $X = \Phi_d(Y)$ for some Turing functional Φ_d. Since $\Phi_d(Y)$ is total, we then have

$$\begin{aligned} \mathcal{A}(\mathrm{PRec}_e(C \oplus X)) &= \mathcal{A}(\mathrm{PRec}_e(C \oplus \Phi_d(Y))) \\ &= \mathcal{B}_d(e, C, Y) \leq_{\mathcal{Q}} \mathcal{A}(\hat{\theta}_d(e, C, Y)) = \mathcal{A}(\mathrm{PRec}_{q(d,e)}(C \oplus Y)). \end{aligned}$$

Consequently, whenever $X \leq_T Y$ via Φ_d, we have $\mathfrak{B}^C(\mathcal{A})(X)(e) \leq_{\mathcal{Q}} \mathfrak{B}^C(\mathcal{A})(Y)$ $(q(d, e))$. In other words, $\mathfrak{B}^C(\mathcal{A})$ is $(\leq_T, \leq_m)$-UOP, as desired. □

We are now ready to show that $\mathfrak{A}$ is onto.

Proof of Proposition 23. If $\mathcal{A}$ is non–self-dual, let C be as in Lemma 30, and then we have $\mathfrak{B}^C(\mathcal{A})$ being $(\leq_T, \leq_m)$-UOP and, by Lemma 27, we have $\mathfrak{A}(\mathfrak{B}^C(\mathcal{A})) \equiv_w \mathcal{A}$.

Suppose now that $\mathcal{A}$ is self-dual. By Fact 16, $\mathcal{A}$ is σ join reducible; that is, there exists a sequence $\mathcal{A}_0$, $\mathcal{A}_1, \dots$ of non–self-dual functions from ω^ω to $\mathcal{Q}$ such that $\mathcal{A} \equiv_w \bigoplus_n \mathcal{A}_n$. By Lemma 30, for each n, there is a $C_n \in \omega^\omega$ such that $\mathfrak{B}^{C_n}(\mathcal{A}_n)$ is $(\leq_T, \leq_m)$-UOP, and moreover, the proof of Lemma 30 provides an effective way of computing the witness of the fact that $\mathfrak{B}^{C_n}(\mathcal{A}_n)$ is $(\leq_T, \leq_m)$-UOP from a given n. Put $C = \bigoplus C_n$, and then $\mathfrak{B}^C(\mathcal{A}_n)$ is also $(\leq_T, \leq_m)$-UOP. We claim that

$$\mathfrak{A}(\bigoplus_n \mathfrak{B}^C(\mathcal{A}_n)) \equiv_w \mathcal{A}.$$

On the one hand, we have that

$$\mathfrak{A}(\bigoplus_n \mathfrak{B}^C(\mathcal{A}_n))(\langle m, e\rangle^\frown X) = \mathcal{A}(m^\frown \mathrm{PRec}_e(C \oplus X)),$$

and on the other hand that $\mathcal{A}(m^\frown X) = \mathfrak{A}(\bigoplus_n \mathfrak{B}^C(\mathcal{A}_n))(\langle m, e\rangle^\frown X)$, where e is such that $\mathrm{PRec}_e(C \oplus X) = X$.

Notice that $\bigoplus_n \mathfrak{B}^C(\mathcal{A}_n)$ not only is $(\leq_T, \leq_m)$-UOP but is also in standard form, as needed for Remark 24. □

4. The games and the embedding lemma

4.1. **The Wadge game G_w.** Wadge [Wad83, section 1 B] introduced a perfect-information, infinite, two-player game, known as the *Wadge game*, which can be used to define Wadge reducibility. For $\mathcal{Q}$-valued functions $\mathcal{A}, \mathcal{B} \colon \omega^\omega \to \mathcal{Q}$, here is the $\mathcal{Q}$-valued version $G_{\mathrm{Wadge}}(\mathcal{A}, \mathcal{B})$ of the Wadge game: in the nth round of the game, Player I chooses $x_n \in \omega$ and II chooses $y_n \in \omega \cup \{\mathsf{pass}\}$ alternately (where $\mathsf{pass} \notin \omega$), and eventually Players I and II produce infinite sequences $X = (x_n)_{n\in\omega}$ and $Y = (y_n)_{n\in\omega}$, respectively. We write Y^{p} for the result dropping all passes from Y. We say that *Player II wins the game* $G_{\mathrm{Wadge}}(\mathcal{A}, \mathcal{B})$ if

$$Y^{\mathsf{p}} \text{ is an infinite sequence and } \mathcal{A}(X) \leq_{\mathcal{Q}} \mathcal{B}(Y^{\mathsf{p}}).$$

One can play the same game for functions $f, g : \omega^\omega \to \mathcal{Q}^\omega$ by identifying them with their uncurrying $\mathfrak{A}(f), \mathfrak{A}(g)$: given $\mathcal{Q}^\omega$-valued functions f, g, we use the symbol $G_w(f, g)$ to denote $G_{\mathrm{Wadge}}(\mathfrak{A}(f), \mathfrak{A}(g))$.

In other words, Player I plays natural numbers $m, x_0, x_1, \dots \in \omega$, and Player II plays $y_0, y_1, y_2, \dots \in \omega \cup \{\mathsf{pass}\}$ alternately. Player II wins the game $G_w(f, g)$ if Y^{p} is infinite, and $f(X)(m) \leq_{\mathcal{Q}} g((Y_{>i})^{\mathsf{p}})(y_i)$, where i is the least number such that $y_i \neq \mathsf{pass}$, and $Y_{>i} = (y_n)_{n>i}$. As in Wadge [Wad83, Theorem B8], one can easily check that $\mathfrak{A}(f) \leq_w \mathfrak{A}(g)$ holds if and only if Player II wins the game $G_w(f, g)$.

4.2. **The m-game $G_{\mathbf{m}}$.** A second version of the Wadge game that will be useful to us is the game we call $G_{\mathbf{m}}(f, g)$, where Player II is not allowed to pass in his first move, but he can pass in subsequent moves. In other words, in the game $G_{\mathbf{m}}(f, g)$, Player I plays natural numbers $m, x_0, x_1, \dots$, and Player II plays $n, y_0, y_1, \dots$ alternately, where $n, m, x_0, x_1, \dots \in \omega$ and $y_0, y_1, \dots \in \omega \cup \{\mathsf{pass}\}$. Player II wins the game $G_{\mathbf{m}}(f, g)$ if Y^{p} is infinite and $f(X)(m) \leq_{\mathcal{Q}} g(Y^{\mathsf{p}})(n)$.

4.3. **The Lipchitz-game $G_{\mathbf{lip}}$.** A third version of the Wadge game that will also be useful to us is the game we call $G_{\mathbf{lip}}(f, g)$, where Player II is not allowed to pass at any time. The rest is the same. This game (for sets) was introduced by Wadge [Wad83, section 1 B].

4.4. **The modified m-game $\tilde{G}_{\mathbf{m}}$.** Steel [Ste82, Lemma 1] introduced a perfect-information, infinite, two-player game $\tilde{G}_{\mathbf{m}}(f, g)$ to study uniformly Turing degree-invariant functions. Here is a small variation of its $\mathcal{Q}$-valued version: alternately, Player I plays natural numbers $m, x_0, x_1, \dots$, and Player II plays $\langle n, j\rangle, y_0, y_1, \dots$, with $\langle n, j\rangle \in \omega^2$ and $y_0, y_1, \dots \in \omega \cup \{\mathsf{pass}\}$. Player II wins the game $\tilde{G}_{\mathbf{m}}(f, g)$ if Y^{p} is infinite,

$$\Phi_j^{Y^{\mathsf{p}}} = X, \text{ and } f(X)(m) \leq_{\mathcal{Q}} g(Y^{\mathsf{p}})(n),$$

where $X = (x_n)_{n\in\omega}$ and $Y = (y_n)_{n\in\omega}$.

4.5. **The plan for embeddability.** The following lemmas lay out the plan to prove the right-to-left direction of Proposition 25, which states that $\mathfrak{A}$ is an order-preserving embedding when restricted to $\mathfrak{A}$-minimal functions. Recall that the left-to-right direction of Proposition 25 was already proved in Lemma 22. The lemmas are quite similar in form, except that one assumes that f is $(\leq_T, \leq_m)$-UOP, and the other that g is $(\leq_T, \leq_m)$-UOP.

Lemma 31. *Let $f, g\colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\equiv_T, \equiv_m)$-UI, $\mathfrak{A}$-minimal functions. Suppose also that f is $(\leq_T, \leq_m)$-UOP. Each of the following statements implies the next one:*

(1) $\mathfrak{A}(f) \leq_w \mathfrak{A}(g)$.
(2) *For every u.p.p. tree S, II wins $G_w(f, g \circ S)$.*
(3) *For every u.p.p. tree S, II wins $G_{\mathbf{lip}}(f, g \circ S)$.*
(4) *II wins $\tilde{G}_{\mathbf{m}}(f, g)$.*
(5) $f \leq^{\triangledown}_{\mathbf{m}} g$.

Lemma 32. *Let $f, g\colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\equiv_T, \equiv_m)$-UI, $\mathfrak{A}$-minimal functions. Suppose also that g is $(\leq_T, \leq_m)$-UOP and in standard form (as in Remark 24). Each of the following statements implies the next one:*

(1) $\mathfrak{A}(f) \leq_w \mathfrak{A}(g)$.
(2) *II wins $G_w(f, g)$.*
(3) *There is a u.p.p. tree T such that II wins $G_{\mathbf{m}}(f \circ T, g)$.*
(4) $f \leq^{\triangledown}_{\mathbf{m}} g$.

First, let us see how the lemmas imply the right-to-left direction of Proposition 25.

Proof of Proposition 25. Consider $(\equiv_T, \equiv_m)$-UI, $\mathfrak{A}$-*minimal* functions $f, g\colon \omega^\omega \to \mathcal{Q}^\omega$. The problem is that maybe neither of them is $(\leq_T, \leq_m)$-UOP. By Proposition 23, there is a $(\leq_T, \leq_m)$-UOP function h such that $\mathfrak{A}(g) \equiv_w \mathfrak{A}(h)$. Furthermore, as noted in Remark 24, we can assume that h is in standard form. We apply Lemma 32 to f and h and Lemma 31 to h and g, and we then apply the transitivity of $\leq^{\triangledown}_{\mathbf{m}}$. □

Let us start by proving the easiest implication in Lemmas 31 and 32. Since f and g are $\mathfrak{A}$ minimal, we have $\mathfrak{A}(f) \leq_w \mathfrak{A}(g)$ if and only if, for every u.p.p. tree S, $\mathfrak{A}(f) \leq_w \mathfrak{A}(g \circ S)$. The equivalences between (1) and (2) in both lemmas then follow from the equivalence between $\mathcal{Q}$-Wadge reducibility and the Wadge game.

The implication from (4) to (5) follows from the equivalence between $\leq^{\triangledown}_{\mathbf{m}}$ reducibility and the modified $\mathbf{m}$-game $\tilde{G}_{\mathbf{m}}$ (Lemma 35).

5. The proof of the embeddability lemmas

This section is dedicated to proving the rest of Lemmas 31 and 32.

5.1. **The case when f is $(\leq_T, \leq_m)$-UOP.** We start with the proof of Lemma 31. The implication from (2) to (3) in Lemma 31 follows from the next lemma and an application of determinacy.

Lemma 33. *Let $f\colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\leq_T, \leq_m)$-UOP, and let $g\colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\equiv_T, \equiv_m)$-UI. If Player* I *has a winning strategy for $G_{\mathbf{lip}}(f, g)$, then Player* I *has a winning strategy for $G_w(f, g)$.*

Proof. Let τ be Player I's strategy in $G_{\mathbf{lip}}(f,g)$. The difficulty in defining a strategy in $G_w(f,g)$ is that now Player II is allowed to pass.

Let Φ_i be a computable operator that removes the 0s from the input and reduces the rest of the entries by 1. That is, $\Phi_i(\sigma^\frown 0) = \Phi_i(\sigma)$ and $\Phi_i(\sigma^\frown(n+1)) = \Phi_i(\sigma)^\frown n$. Since f is $(\leq_T, \leq_m)$-UOP, there is a computable function p such that $f(\Phi_i(X))(n) \leq_{\mathcal{Q}} f(X)(p(n))$ for all X's $\in \omega^\omega$.

We are now ready to describe a winning strategy for Player I in the Wadge game $G_w(f,g)$. Let $Y = (y_s)_{s\in\omega}$ be a sequence produced by Player II in the Wadge game $G_w(f,g)$. We will play a run of $G_{\mathbf{lip}}(f,g)$ at the same time, where Player II plays Y^{p}. Let Player I's first move in $G_w(f,g)$ be $x_0 = p(n)$, where n is Player I's move in $G_{\mathbf{lip}}(f,g)$. At any round s, if Player II's move y_s is pass, then let Player I's next move be $x_{s+1} = 0$. If Player II's move is $y_s \neq \mathsf{pass}$, then let Player I follow the winning strategy τ in the game $G_{\mathbf{lip}}(f,g)$ and then add 1; that is, let Player I's next move be $x_{s+1} = \tau(\langle y_0, \dots, y_s\rangle^{\mathsf{p}}) + 1$.

Assume that $(y_s)_{s\in\omega}$ contains infinitely many natural numbers; otherwise, Player I wins. If Player I follows the above strategy as we described and plays a sequence $p(n)^\frown X$, where $X = \langle x_1, x_2, \dots\rangle$, we have $\Phi_i(X) = \tau(Y^{\mathsf{p}})^-$ and then get

$$\mathfrak{A}(f)(p(n)^\frown X) = f(X)(p(n)) \geq_{\mathcal{Q}} f(\Phi_i(X))(n) = f(\tau(Y^{\mathsf{p}})^-)(n) \not\leq_{\mathcal{Q}} \mathfrak{A}(g)(Y^{\mathsf{p}}).$$

Consequently, Player I wins the Wadge game $G_w(f,g)$. $\square$

The implication from (3) to (4) in Lemma 31 follows from the next lemma and an application of determinacy.

Lemma 34. *Let $f,g\colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\equiv_T, \equiv_m)$-UI functions. If Player* I *has a winning strategy for $\tilde{G}_{\mathbf{m}}(f,g)$, then Player* I *has a winning strategy for $G_{\mathbf{lip}}(f, g\circ S)$ for some u.p.p. tree S.*

Proof. Let τ be Player I's strategy in $\tilde{G}_{\mathbf{m}}(f,g)$. The difficulty in defining a strategy in $G_{\mathbf{lip}}(f, g\circ S)$ is that now Player II does not need to play a correct index e to compute Player I's moves.

For each m, e, Z, let n and $\theta(m,e,Z)$ be such that $(n, \theta(m,e,Z))$ is Player I's answer to II playing $(\langle m, e\rangle, Z)$ in $\tilde{G}_{\mathbf{m}}(f,g)$. Let $S \geq_T \tau$ be a u.p.p. tree. Then there is a computable operator Ψ such that, for every $Z\in\omega^\omega$ with $Z \in [S]$, we have $\Psi^Z(m,e) = \theta(m,e,Z)$. By the recursion theorem, there is a computable function $e(m)$ such that $\Phi^Z_{e(m)} = \Psi^Z(m, e(m))$.

To define Player I's strategy in $G_{\mathbf{lip}}(f, g\circ S)$ in answer to Player II moving (m, Y), all that we have to do is imitate Player I's strategy in $\tilde{G}_{\mathbf{m}}(f,g)$ in answer to Player II moving $(\langle m, e(m)\rangle, S[Y])$. Notice that since Player II is not allowed to pass, $Y = Y^{\mathsf{p}}$, and hence $S[Y]$ computes Player I's moves using $\Phi_{e(m)}$. $\square$

The implication from (4) to (5) follows from the next lemma.

Lemma 35. *Let $f,g\colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\equiv_T, \equiv_m)$-UI functions. If Player* II *has a winning strategy for $\tilde{G}_{\mathbf{m}}(f,g)$, then $f \leq^{\triangledown}_{\mathbf{m}} g$.*

Proof. Consider a winning strategy σ for Player II in $\tilde{G}_{\mathbf{m}}(f,g)$. Suppose that the answer to Player I playing $n^\frown X$ is Player II playing $\langle \psi(n), j_n\rangle^\frown Y_{n,X}$. Since σ is winning, we get $f(X)(n) \leq_{\mathcal{Q}} g(Y^{\mathsf{p}}_{n,X})(\psi(n))$ for all n's $\in\omega$ and X's $\in\omega^\omega$. Also, if we take an X that can compute the strategy, we get, for each n, a pair (i_n, j_n) of indices for the Turing equivalence between X and $Y^{\mathsf{p}}_{n,X}$: X computes $Y^{\mathsf{p}}_{n,X}$ using n

and the strategy, and $Y^{\mathsf{p}}_{n,X}$ computes X using Φ_{j_n}. Let $u = (u_0, u_1)$ witness that g is $(\equiv_T, \equiv_m)$-UI. Then $u_0(i_n, j_n)$ is a witness of $g(Y^{\mathsf{p}}_{n,X}) \leq_m g(X)$ since (i_n, j_n) witnesses $X \equiv_T Y^{\mathsf{p}}_{n,X}$. Thus,

$$f(X)(n) \leq_{\mathcal{Q}} g(Y^{\mathsf{p}}_{n,X})(\psi(n)) \leq_{\mathcal{Q}} g(X)(\Phi_{u(i_n,j_n)} \circ \psi(n)).$$

This implies that $f(X) \leq_m g(X)$ whenever $X \in \omega^\omega$ computes Player II's strategy. $\square$

This finishes the proof of Lemma 31.

5.2. **The case in which g is $(\leq_T, \leq_m)$-UOP.** We now concentrate on the proof of Lemma 32. The implication from (3) to (4) follows from the next lemma.

Lemma 36. *Let $f \colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\equiv_T, \equiv_m)$-UI, and let $g \colon \omega^\omega \to \mathcal{Q}^\omega$ be $(\leq_T, \leq_m)$-UOP. If there is a u.p.p. tree T such that Player* II *has a winning strategy for $G_{\mathbf{m}}(f \circ T, g)$, then $f \leq^{\nabla}_{\mathbf{m}} g$.*

Proof. The proof is very similar to that of Lemma 35, with the exceptions that now we do not need to have Y^{p} compute X, and that we need to consider the tree T.

Consider a winning strategy for Player II in $G_{\mathbf{m}}(f \circ T, g)$. Suppose that the answer to (n, X) is (m, Y). From the strategy, we get a function ψ that outputs m given n and satisfies $f(X)(n) \leq_{\mathcal{Q}} g(Y)(\psi(n))$ for $n \in \omega$ and $X \in [T]$. If we take an $X \in [T]$ that can compute the strategy, then X can compute Y^{p} uniformly using n. Let $i(n)$ be an index for the Turing reduction from Y^{p} to X. Thus,

$$f(X)(n) \leq_{\mathcal{Q}} g(Y^{\mathsf{p}})(\Psi(n)) \leq_{\mathcal{Q}} g(X)(\Phi_{u(i(n))} \circ \Psi(n)),$$

where u witnesses that g is $(\leq_T, \leq_m)$-UOP, and hence $f(X) \leq_m g(X)$ for all X's $\in [T]$ that compute the strategy. Now, if we take any $X \geq_T T$, we have $X \equiv_T T[X]$, and hence $f(X) \leq_m f(T[X])$ and $g(T[X]) \leq_m g(X)$, since f and g are $(\equiv_T, \equiv_m)$-UI. Putting all of this together, we get $f(X) \leq_m g(X)$ for all X's that compute T and the strategy. This shows that $f \leq^{\nabla}_{\mathbf{m}} g$. $\square$

All that is left to finish the proof of Lemma 32 is to prove that (2) implies (3), connecting the Wadge game and the **m**-game. This will then finish the proofs of Proposition 25 and our main theorems. The proof is divided into two cases: the case when $\mathfrak{A}(g)$ is σ join irreducible, and the case in which $\mathfrak{A}(g)$ is σ join reducible and g is in standard form (by Fact 16 and Remark 24). The existence of the u.p.p. tree T mentioned in (3) is needed only in the latter case.

Lemma 37. *Let f, g be functions $\omega^\omega \to \mathcal{Q}^\omega$, and assume that $\mathfrak{A}(g)$ is σ join irreducible. If Player* II *has a winning strategy for $G_w(f, g)$, then Player* II *has a winning strategy for $G_{\mathbf{m}}(f, g)$.*

Proof. By Fact 14, if $\mathfrak{A}(g)$ is σ join irreducible, there is a $Z \in \omega^\omega$ such that $\mathfrak{A}(g) \leq_w \mathfrak{A}(g) \restriction [Z \restriction n]$ for any $n \in \omega$. In particular, Player II has a winning strategy τ for $G_{\mathrm{Wadge}}(\mathfrak{A}(f), \mathfrak{A}(g) \restriction [Z(0)])$. In the game $G_{\mathbf{m}}(f, g)$, Player II plays $Z(0)$ and then follows τ. This clearly gives II's winning strategy for $G_{\mathbf{m}}(f, g)$. $\square$

We now move to the last case of $\mathfrak{A}(g)$ being σ join reducible. We say that a closed set $P \subseteq 2^\omega$ is *thin* if, for every Π^0_1 set $Q \subseteq 2^\omega$, the intersection $P \cap Q$ is clopen in P. We also say that a closed set $P \subseteq 2^\omega$ is *almost thin* if there are at most finitely many X's $\in 2^\omega$ such that $P \cap [X \restriction n]$ is not thin for any $n \in \omega$. Recall that $[\sigma]$ denotes the set of all reals extending σ. For a number $k \in \omega$, we also use $[k]$ to denote $[\langle k\rangle]$.

Cenzer et al. [CDJS93, Theorem 2.10] showed that an element X of a thin Π^0_1 class satisfies that $X' \leq_T X \oplus \emptyset''$. We extend their result as follows.

Lemma 38. *Let $T \subseteq 2^{<\omega}$ be a tree such that $[T]$ is almost thin. Then, for every $X \in [T]$, either $X' \leq_T X \oplus T''$ or $X \leq_T T''$ holds.*

Proof. We first claim that if $[T]$ is thin, then $X' \leq_T X \oplus T''$ for any $X \in [T]$. Given e, let Q_e be the Π^0_1 set consisting of oracles $X \in 2^\omega$ such that $\Phi^X_e(e)$ diverges. Since $[T]$ is thin, $Q_e \cap [T]$ is clopen in $[T]$. Therefore, there is a height $h(e)$ such that $\Phi^X_e(e)$ converges if and only if $\Phi^{X\restriction h(e)}_e(e)$ converges for every $X \in [T]$. Note that such an h can be computed from T'' by searching for the smallest $h(e)$ such that, if σ is an extendible node of T of length $h(e)$ and $\Phi^\tau_e(e)$ converges for some node $\tau \succeq \sigma$ in T, then $\Phi^\sigma_e(e)$ already converges. This shows that $X' \leq_T X \oplus T''$ for every $X \in [T]$.

Now, let us assume that T is almost thin. If $X \in [T]$ satisfies the requirement that $[T] \cap [X\restriction n]$ is thin for some n, then we can apply the previous argument to the closed set $[T] \cap [X\restriction n]$ and obtain that $X' \leq_T X \oplus T''$. There are finitely many X's for which $[T] \cap [X\restriction n]$ is not thin for any n. Again, by restricting ourselves to a tree of the form $[T] \cap [X\restriction n]$, let us assume that X is the only path in $[T]$ for which $[T] \cap [X\restriction n]$ is not thin for any n. We will show that $X \leq_T T''$.

Let $Q \subseteq 2^{<\omega}$ be a computable tree witnessing that T is not thin, i.e., such that $[Q] \cap [T]$ is not clopen in $[T]$. Let $S \subseteq 2^{<\omega}$ be the set of strings $\sigma \in T$ such that $[Q] \cap [T] \cap [\sigma]$ is not clopen in $[T] \cap [\sigma]$. First, let us observe that X is the only path through S: S must have some path, as otherwise there is some ℓ such that, for all σ's $\in 2^\ell$, $[Q] \cap [T] \cap [\sigma]$ is clopen in $[T] \cap [\sigma]$, and hence $[Q] \cap [T]$ would be clopen in $[T]$. Suppose that $Y \in [T]$, but $Y \neq X$. Then there is some n such that $[T] \cap [Y\restriction n]$ is thin, and hence $[Q] \cap [T] \cap [Y\restriction n]$ is clopen in $[T] \cap [Y\restriction n]$. Thus, $Y\restriction n \notin S$. It follows that X is the only path through S.

Second, let us observe that S is Π^0_1 relative to T''. A string σ is not in S if and only if there exists an $\ell \geq |\sigma|$ such that, for every $\tau \in 2^\ell$ extending σ, either $\tau \notin Q$ (and hence $[Q] \cap [T] \cap [\tau] = \emptyset$) or every $\gamma \in T$ which extends τ and is extendible in T belongs to Q too (and hence $[Q] \cap [T] \cap [\tau] = [T] \cap [\tau]$).

Since X is the only path on a Π^0_1 class relative to T'', we find that $X \leq_T T''$. $\square$

Lemma 39. *Let $f\colon \omega^\omega \to \mathcal{Q}^\omega$ be a $(\equiv_T, \equiv_m)$-UI function, and let $g\colon \omega^\omega \to \mathcal{Q}^\omega$ be a $(\leq_T, \leq_m)$-UOP function such that $\mathfrak{A}(g)$ is σ join reducible and g is in standard form. If Player* II *wins $G_w(f, g)$, then, for some u.p.p. tree T, Player* II *has a winning strategy for $G_{\mathbf{m}}(f \circ T, g)$.*

Proof. Since g is in standard form, we have g being of the form $\bigoplus_{n\in\omega} g_n$, where $\mathfrak{A}(g_n)$ is σ join irreducible. By Fact 14, there are z_n's $\in \omega$ such that $\mathfrak{A}(g_n) \leq_w \mathfrak{A}(g_n)\restriction[z_n]$ since $\mathfrak{A}(g_n)$ is σ join irreducible.

We say that a subset D of a quasi-order $\mathcal{P}$ is *directed* if for any $p, q \in D$ there is an $r \in D$ such that $p, q \leq_{\mathcal{P}} r$. By the Erdös–Tarski theorem [ET43], if $\mathcal{P}$ has no infinite antichains, then $\mathcal{P}$ is covered by a finite collection $(D_m)_{m<l}$ of directed sets. We now consider the quasi-order $\leq_\omega$ on ω defined by $m \leq_\omega n$ if and only if $\mathfrak{A}(g_m) \leq_w \mathfrak{A}(g_n)$. Since $(\omega; \leq_\omega)$ is bqo, it is covered by finitely many directed sets $(D_m)_{m<\ell}$.

Given numbers m and n, consider the following closed set:

$$\mathcal{F}_{m,n} = \{X \in 2^\omega : (\forall i \in D_m)(\forall k \in \omega)[\mathfrak{A}(f) \restriction [n^\frown X \restriction k] \not\leq_w \mathfrak{A}(g_i)]\}.$$

Let $C \geq_T \bigoplus_m D_m$ be a sufficiently powerful oracle deciding whether $\mathfrak{A}(f) \restriction [n^\frown \tau] \not\leq_w \mathfrak{A}(g_i)$ given that $n, i \in \omega$ and $\tau \in 2^{<\omega}$. In particular, we find that

$$\mathcal{F}_{m,n} \text{ is } \Pi^0_1(C).$$

Case 1. For all n's $\in \omega$, there is an $m < \ell$ such that $\mathcal{F}_{m,n}$ is almost thin.

In this case, by Lemma 38, every element $X \in \mathcal{F}_{m,n}$ satisfies $X' \leq_T X \oplus C''$ or $X \leq_T C''$. Thus, no X with $X >_T C''$ belongs to $\mathcal{F}_{m,n}$. Let K be the compact set $\{X \oplus C''' : X \in 2^\omega\}$. Since K is disjoint from $\mathcal{F}_{m,n}$, for every $X \in K$, there in an $i \in D_m$ and a $k \in \omega$ such that $\mathfrak{A}(f) \restriction [n^\frown X \restriction k] \leq_w \mathfrak{A}(g_i)$. By the compactness of K, such an i can be chosen from a finite set $E \subseteq D_m$. Since D_m is directed, there is an $i(n) \in D_m$ such that $e \leq_\omega i(n)$ for any $e \in E$. Let T be a u.p.p. tree whose image is inside K.

We now claim that Player II has a winning strategy for the game $G_{\mathbf{m}}(f \circ T, g)$. If Player I's first move is n, Player II chooses a pair $\langle i(n), z_{i(n)} \rangle$. Given Player I's move X, Player II waits for a round s such that $\mathfrak{A}(f) \restriction [n^\frown T[X] \restriction s] \leq_w \mathfrak{A}(g_{i(n)})$. Such an s exists by our choice of $i(n)$. By the definition of $z_{i(n)}$, we have $\mathfrak{A}(f) \restriction [n^\frown T[X] \restriction s] \leq_w \mathfrak{A}(g_{i(n)}) \restriction [z_{i(n)}]$, and then Player II follows a winning strategy witnessing this. This procedure gives a desired winning strategy for Player II.

Case 2. Otherwise, there is an $n \in \omega$ such that $\mathcal{F}_{m,n}$ is not almost thin for any $m < \ell$.

In this case, there is a sequence of different reals $(X_m)_{m<\ell}$ such that $\mathcal{F}_{m,n} \cap [X_m \restriction k]$ is not thin for any k. Therefore, there is a sequence $(\sigma_m)_{m<\ell}$ of pairwise incomparable strings such that $\mathcal{F}_{m,n} \cap [\sigma_m]$ is not thin for any $m < \ell$.

For each $m < \ell$, let Q_m be a computable tree witnessing that $\mathcal{F}_{m,n} \cap [\sigma_m]$ is not thin. Let $(\tau^m_k)_{k \in \omega}$ be the set of minimal strings extending σ_m, not in Q_m. Thus, for each $m < \ell$, $(\tau^m_k)_{k\in\omega}$ is a computable sequence of pairwise incomparable strings extending σ_m such that τ^m_k is extendible in $\mathcal{F}_{m,n}$ for infinitely many k's $\in \omega$. Since τ^m_k is not comparable with τ^i_j whenever $(i,j) \neq (m,k)$, there is a fixed pair (d,e) of indices of computable functions witnessing $0^{\ell k+m}1^\frown X \equiv_T \tau^{m\frown}_k X$. Let u show that f is $(\equiv_T, \equiv_m)$-UI, and then we have

$$f(\tau^{m\frown}_k X)(n) \leq_{\mathcal{Q}} f(0^{\ell k+m}1^\frown X)(\Phi_{u(d,e)}(n)).$$

We claim that $\mathfrak{A}(f) \not\leq_w \mathfrak{A}(g)$ (i.e., that I wins $G_w(f,g)$), showing that Case 2 was not possible to begin with. Player I first chooses $\Phi_{u(d,e)}(n)$. Then Player I plays along 0^ω until Player II moves to some $\langle i, y_0 \rangle \neq \texttt{pass}$ at some round s. Let m be such that $i \in D_m$. Player I searches for a large k so that $s \leq lk + m$ and that τ^m_k is extendible in $\mathcal{F}_{m,n}$. Then, $\mathfrak{A}(f) \restriction [n^\frown \tau^m_k] \not\leq_w \mathfrak{A}(g_i)$, since $i \in D_m$, and therefore Player I has a winning strategy for the game $G_w(\mathfrak{A}(f) \restriction [n^\frown \tau^m_k], \mathfrak{A}(g_i))$. In this game, given Player II's play $Y = (y_n)_{n\in\omega}$, Player I's winning strategy yields a play of the form $(n, \tau^{m\frown}_k \theta(Y))$. Then I's play $\Phi_{u(d,e)}(n)^\frown 0^{\ell k+m}1^\frown\theta(Y)$ in the original game clearly gives a winning strategy. □

Remark 40. One might think that the proof is based on the assumption that the domain of f is 2^ω, or at least compact. The trick here is the use of a u.p.p. tree. Indeed, for any $(\equiv_T, \equiv_m)$-UI function f, its restriction $f \restriction 2^\omega$ already contains full information of f, since every $X \in \omega^\omega$ is Turing equivalent to some $X^* \in 2^\omega$ in a uniform manner.

We again emphasize that considering the general case does not add to the complexity of the proof. For $\mathcal{Q} = 2$, we can remove the second paragraph (that is, the use of the Erdös–Tarski theorem) in the proof of Lemma 39 and replace $(D_m)_{m<\ell}$ with $(\{\omega\})$; however, all of the other arguments are still required.

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