

THE GENERATING FUNCTION OF THE M_2 -RANK OF PARTITIONS WITHOUT REPEATED ODD PARTS AS A MOCK MODULAR FORM

CHRIS JENNINGS-SHAFFER

ABSTRACT. By work of Bringmann, Ono, and Rhoades it is known that the generating function of the M_2 -rank of partitions without repeated odd parts is the so-called holomorphic part of a certain harmonic Maass form. Here we improve the standing of this function as a harmonic Maass form and show more can be done with this function. In particular, we show the related harmonic Maass form transforms like the generating function for partitions without repeated odd parts (which is a modular form). We then use these improvements to determine formulas for the rank differences modulo 7. Additionally, we give identities and formulas that allow one to determine formulas for the rank differences modulo c , for any $c > 2$.

1. INTRODUCTION

To begin, we recall that a partition of an integer is a nonincreasing sequence of positive integers that sum to n . For example the partitions of 5 are 5 , $4 + 1$, $3 + 2$, $3 + 1 + 1$, $2 + 2 + 1$, $2 + 1 + 1 + 1$, and $1 + 1 + 1 + 1 + 1$. It is standard to let $p(n)$ denote the number of partitions of n . From our example we see that $p(5) = 7$. Partitions have a rich history in number theory and combinatorics, going as far back as Euler. Today much emphasis is put on Ramanujan's work with partitions and his related work with q -series.

One point of interest is the congruences of Ramanujan,

$$\begin{aligned}p(5n + 4) &\equiv 0 \pmod{5}, \\p(7n + 5) &\equiv 0 \pmod{7}, \\p(11n + 6) &\equiv 0 \pmod{11}.\end{aligned}$$

Dyson in [8] proposed the following combinatorial explanation of the first two congruences. Given a partition of n , we define the rank of the partition to be the largest part minus the number of parts. If we group the partitions of $5n + 4$ according to the modulo 5 value of their rank, then it turns out we have five sets of equal size. If we group the partitions of $7n + 5$ according to the modulo 7 value of their rank, then it turns out we have seven sets of equal size. These two statements were later proved by Atkin and Swinnerton-Dyer in [2].

Another point of interest are the mock theta functions, which Ramanujan introduced in his famous last letter to Hardy. Mock theta functions are the topic of Watson's final address as president of the London Mathematical Society [21].

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We save the more technical discussion for the next section, but we give two of Ramanujan’s examples:

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots,$$

$$\phi(q) = 1 + \frac{q}{(1+q^2)} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots.$$

One of Ramanujan’s mock theta conjectures (which was proved by Watson in [21]) is that $2\phi(-q) - f(q) = \vartheta_4(0, q) \prod_{n=1}^{\infty} (1+q^n)^{-1}$, where $\vartheta_4(0, q)$ is a Jacobi theta function.

These two areas are actually strongly connected. In particular, we let $N(m, n)$ denote the number of partitions of n with rank m and write the generating function as

$$R(\zeta; q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) \zeta^m q^n.$$

Of the seven functions commonly referred to as the third-order mock theta functions, all seven can be expressed in terms of $R(\zeta; q)$ by replacing q by a power of q , letting ζ be a power of q times a root of unity, and possibly adjusting by a constant and a power of q . It is worth pointing out that it was Watson [21] who observed the third-order functions could be written in terms of the rank, and he did so nearly a decade before the rank function was formally defined. The fifth- and seventh-order mock theta functions can also be obtained in such a manner, if one also allows adding on certain infinite products. Explicit versions of these statements can be found in Section 5 of [12]. For this reason $R(\zeta; q)$ is called a universal mock theta function. With this we see that a strong understanding of $R(\zeta; q)$ leads to a better understanding of the mock theta functions.

Another universal mock theta function is $R_2(\zeta; q)$, the generating function of the M_2 -rank of partitions without repeated odd parts, which appears among the tenth-order mock theta functions. This also has an elegant combinatorial statement. The M_2 -rank of a partition without repeated odd parts is given by taking the ceiling of the largest part divided by 2, and then subtracting the number of parts. This rank was introduced by Berkovich and Garvan in [3] and further studied by Lovejoy and Osburn in [16]. We let $N_2(m, n)$ denote the number of partitions of n without repeated odd parts and M_2 -rank m . We denote the generating function by

$$R_2(\zeta; q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) \zeta^m q^n.$$

In this article we improve the understanding of the M_2 -rank as the holomorphic part of a harmonic Maass form. We determine a more precise formula for the transformation of the associated harmonic Maass form and do so on a larger group than previously known. To aid in determining dissection formulas $R_2(e^{\frac{2\pi i}{\ell}}; q) = \sum_{r=0}^{\ell-1} q^r A_r(q^\ell)$, we find additional harmonic Maass forms with holomorphic parts corresponding to the mock modular parts of the $A_r(q^\ell)$. Additionally, we give lower bounds on the orders of the cusps for the holomorphic parts of the harmonic Maass forms.

Partition ranks have been studied in terms of harmonic Maass forms in a number of recent works; perhaps the most influential is [5], and other important works are

[1, 6, 7, 17], of which the second and third deal with the M_2 -rank. Recently, Garvan [9] has made remarkable improvements to the transformations and levels from [5] for the harmonic Maass form related to $R(\zeta; q)$. In Garvan's work, one sees that on a rather large subgroup, the harmonic Maass form related to $R(\zeta; q)$ has the same multiplier as $\eta(\tau)^{-1}$, which is the generating function for partitions. The same phenomenon will occur here. That is to say the harmonic Maass form related to $R2(\zeta; q)$ will transform like $\frac{\eta(2\tau)}{\eta(4\tau)\eta(\tau)}$, which is the generating function for partitions without repeated odd parts. This also occurs with the Dyson rank of overpartitions, $\overline{R}(\zeta; q)$, which the author studied in [13]. Although not stated explicitly in that article, one can easily verify that on a certain subgroup the transformation formula for the Dyson rank of overpartitions agrees with $\frac{\eta(2\tau)}{\eta(\tau)^2}$, the generating function for overpartitions. The general explanation for this is that usually one can write the generating function of a rank function as the product of the generating function for the underlying partition function and a higher level Appell function. By the work of Zwegers in [22], Appell functions have modular completions and these modular completions have trivial multipliers on large subgroups of $SL_2(\mathbb{Z})$. With an improved understanding of the M_2 -rank, we give new identities for $R2(\zeta; q)$ when ζ is set equal to certain roots of unity. In the next section we formally state our definitions and results and then give an outline of the rest of the article.

For the reader who wishes to compare the various appearances of the the three rank functions mentioned above, we note the following. In the notation of Gordon and McIntosh [10], we have $R(\zeta; q) = h_3(\zeta, q) = (1 - \zeta)(1 + \zeta g_3(\zeta, q))$, $\overline{R}(\zeta; q) = (1 - \zeta)(1 - \zeta^{-1})g_2(\zeta, q)$, and $R2(\zeta; q) = h_2(\zeta, -q)$. In the notation of Hickerson and Mortenson [12], we have $R(\zeta; q) = (1 - \zeta)(1 + \zeta g(\zeta, q))$, $\overline{R}(\zeta; q) = (1 - \zeta)(1 - \zeta^{-1})h(\zeta, q)$, and $R2(\zeta; q) = (1 - \zeta)\zeta^{\frac{1}{2}}k(\zeta^{\frac{1}{2}}, -q)$.

2. STATEMENT OF RESULTS

As in the introduction we let $N_2(m, n)$ denote the number of partitions of n without repeated odd parts with M_2 -rank m . Furthermore, we let $N_2(k, c, n)$ denote the number of partitions of n without repeated odd parts with M_2 -rank congruent to k modulo c . To state our definitions and results, we use the standard product notation

$$\begin{aligned}
 (\zeta; q)_n &= \prod_{j=0}^{n-1} (1 - \zeta q^j), & (\zeta; q)_\infty &= \prod_{j=0}^{\infty} (1 - \zeta q^j), \\
 (\zeta_1, \dots, \zeta_k; q)_n &= (\zeta_1; q)_n \cdots (\zeta_k; q)_n, & (\zeta_1, \dots, \zeta_k; q)_\infty &= (\zeta_1; q)_\infty \cdots (\zeta_k; q)_\infty.
 \end{aligned}$$

Also we let $q = \exp(2\pi i\tau)$ for $\tau \in \mathcal{H}$, that is $\text{Im}(\tau) > 0$. For c a positive integer and a an integer, we let $\zeta_c^a = \exp(2\pi ia/c)$.

As noted in [16], one may use Theorem 1.2 of [15] to find that the generating function for $N_2(m, n)$ is given by

$$\begin{aligned}
 R2(\zeta; q) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) \zeta^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(\zeta q^2, \zeta^{-1} q^2; q^2)_n} \\
 &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1 - \zeta)(1 - \zeta^{-1})(-1)^n q^{2n^2+n} (1 + q^{2n})}{(1 - \zeta q^{2n})(1 - \zeta^{-1} q^{2n})} \right).
 \end{aligned}$$

Next for a and c integers, $c > 0$, and $c \nmid 2a$, we define

$$\begin{aligned} \mathcal{R}2(a, c; \tau) &= q^{-\frac{1}{8}} R2(\zeta_c^a; q), \\ \widetilde{\mathcal{R}}2(a, c; \tau) &= \mathcal{R}2(a, c; \tau) - i(1 - \zeta_c^a)\zeta_c^{-a} \left(e^{\frac{\pi i}{4}} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)dw}{\sqrt{-i(w + \tau)}} \right. \\ &\quad \left. + e^{-\frac{\pi i}{4}} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)dw}{\sqrt{-i(w + \tau)}} \right), \end{aligned}$$

where $g_{a,b}(\tau)$ is a theta function of Zwegers [23], the definition of which we give in the next section. The factor of $q^{-\frac{1}{8}}$ is necessary for certain transformations to work correctly. We notice that it is immediately clear that $\mathcal{R}2(a + c, c; \tau) = \mathcal{R}2(a, c; \tau)$; it is also true that $\widetilde{\mathcal{R}}2(a + c, c; \tau) = \widetilde{\mathcal{R}}2(a, c; \tau)$, and this follows immediately from properties of $g_{a,b}(\tau)$.

We will find $R2$, $\mathcal{R}2$, and $\widetilde{\mathcal{R}}2$ are related to the following functions. For k and c integers, $0 \leq k < c$, we define

$$\begin{aligned} S(k, c; \tau) &= \frac{q^{-c^2+4kc-2k^2+c-k}}{\left((-1)^{c-1}q^{c^2}, (-1)^{c-1}q^{3c^2}, q^{4c^2}; q^{4c^2}\right)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^{nc}q^{2c^2n^2+3c^2n}}{1 - q^{4c^2n+(1+4k-c)c}}, \\ \mathcal{S}(k, c; \tau) &= i^c q^{-\frac{1}{8}} S(k, c; \tau), \\ \widetilde{\mathcal{S}}(k, c; \tau) &= \mathcal{S}(k, c; \tau) - (-1)^k i^{1+c} e^{-\frac{\pi i}{4}} c \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(4c^2w)}{\sqrt{-i(w + \tau)}} dw. \end{aligned}$$

We note that $S(k, c; \tau)$ has a pole when $1 + 4k - c \equiv 0 \pmod{4c}$, however this case does not occur in our usage of $S(k, c; \tau)$. We will see in section 3 that we can define the functions $\mathcal{S}(k, c; \tau)$ and $\widetilde{\mathcal{S}}(k, c; \tau)$ for all integer values of k , however some care must be taken in doing so as the above definitions would not be the proper choice as it is not true that $\mathcal{S}(k + c, c; \tau) = \mathcal{S}(k, c; \tau)$.

Our main theorem is the following, which relates the M_2 -rank generating function to harmonic Maass forms and modular forms.

Theorem 2.1. *Suppose a and c are integers, $c > 0$, and $c \nmid 2a$. Let $t = \frac{c}{\gcd(4,c)}$.*

- (1) $\widetilde{\mathcal{R}}2(a, c; 8\tau)$ is a harmonic weak Maass form of weight $1/2$ on $\Gamma_0(256t^2) \cap \Gamma_1(8t)$, and $\mathcal{R}2(a, c; 8\tau)$ is a mock modular form.
- (2) The function

$$\widetilde{\mathcal{R}}2(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \widetilde{\mathcal{S}}(k, c; \tau)$$

is holomorphic in τ and has at worst poles at the cusps. Furthermore,

$$\begin{aligned} \widetilde{\mathcal{R}}2(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \widetilde{\mathcal{S}}(k, c; \tau) \\ = \mathcal{R}2(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \mathcal{S}(k, c; \tau). \end{aligned}$$

(3) *The function*

$$\frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)} \left(\mathcal{R}2(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \mathcal{S}(k, c; \tau) \right)$$

is a weakly holomorphic modular form of weight 1 on $\Gamma_0(4c^2 \cdot \gcd(c, 2)) \cap \Gamma_1(4c)$.

(4) *If c is odd, then the function*

$$\frac{\eta(4c^2\tau)\eta(c^2\tau)}{\eta(2c^2\tau)} \left(\mathcal{R}2(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \mathcal{S}(k, c; \tau) \right)$$

is a weakly holomorphic modular form of weight 1 on $\Gamma_0(4c^2) \cap \Gamma_1(4c)$.

In the sums above, when $1 + 4k - c \equiv 0 \pmod{4c}$, the summands

$$(-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \mathcal{S}(k, c; \tau) \quad \text{and} \quad (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \tilde{\mathcal{S}}(k, c; \tau)$$

are taken to be zero.

Part (1) of Theorem 2.1 is not entirely new. In [6] Bringmann, Ono, and Rhoades studied various general harmonic Maass forms, one of which has essentially $R2(\zeta_c^a, 8t^2\tau)$ as the holomorphic part, and the subgroup is $\Gamma_1(2^{10}t^4)$. However, the purpose of that article was to establish, for the first time, that various families of functions are mock modular forms, rather than to obtain a precise formula for the multipliers on large subgroups. Our construction of the harmonic Maass forms greatly differs from that of [6]. Additionally, Hickerson and Mortenson in [11] considered the function

$$D_2(a; M) = \sum_{n=0}^{\infty} \left(N_2(a, M, n) - \frac{p_o(n)}{M} \right) q^n,$$

where $p_o(n)$ is the number of partitions without repeated odd parts. There they showed that $D_2(a; M)$ can be expressed as the sum of two Appell–Lerch sums, of a form similar to $S(k, c; \tau)$, and a theta function.

In section 5 we give various formulas so that we may determine the orders of these modular forms at cusps. With this we can then verify various identities for the M_2 -rank function by passing to modular forms and using the valence formula. In particular, one can use this to give an exact description of the c -dissection of $R2(\zeta_c^a; q)$ in terms of generalized Lambert series and modular forms. We give these identities for $c = 7$. Similar formulas were determined by Lovejoy and Osburn for $c = 3$ and $c = 5$ in [16] and for $c = 6$ and $c = 10$ by Mao in [18]. Rather than using harmonic Maass forms, those formulas used the q -series techniques developed by Atkin and Swinnerton-Dyer [2] to determine rank difference formulas for the rank of partitions. Furthermore, in [11] Hickerson and Mortenson demonstrated how their identities for $D_2(a; M)$ can be used to prove the formulas for $c = 3$ and $c = 5$. With this in mind, we see that there are at least three different proof techniques for determining the c -dissection of $R2(\zeta_c^a; q)$. One might ask for the pros and cons of the approach we use in this article. Our approach with harmonic Maass forms shares a difficulty with the other two methods, which is that we must correctly guess the identity before we can prove it. This approach with harmonic Maass forms helps us to guess the identity, in that we know the weight and level of the modular forms.

Theorem 2.2. Let ζ_7 be a primitive seventh root of unity, $J_a = (q^a, q^{28-a}; q^{28})_\infty$ for $1 \leq a \leq 14$, $J_0 = (q^{28}; q^{28})_\infty$, and $A(m, n, r) = m\zeta_7 + n\zeta_7^2 + r\zeta_7^3 + r\zeta_7^4 + n\zeta_7^5 + m\zeta_7^6$. Then

$$R2(\zeta_7; q) = R2_0(q^7) + qR2_1(q^7) + q^2R2_2(q^7) + q^3R2_3(q^7) + q^4R2_4(q^7) + q^5R2_5(q^7) + q^6R2_6(q^7),$$

where each $R2_i(q)$ can be written as a sum of terms with $S(k, 7; \tau/7)$ and quotients of J_a . Due to the complexity of the expressions, we only state the definition of $R2_0(q)$ here and give the definitions of the other $R2_i(q)$ in section 6, equations (6.1)-(6.6). We have

$$\begin{aligned} R2_0(q) = & A(3, 2, 2)S(0, 7, \tau/7) + A(3, 2, 2)S(3, 7, \tau/7) + \frac{A(-51, -25, -35)q^{-1}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2} \\ & + \frac{A(69, 34, 44)q^{-1}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2} + \frac{A(27, 19, 20)q^{-1}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2} + \frac{A(28, 14, 22)q^{-1}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_{10}} \\ & + \frac{A(-45, -29, -27)q^{-1}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} + \frac{A(-39, -22, -24)q^{-1}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(48, 21, 33)q^{-1}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} \\ & + \frac{A(15, 8, 6)q^{-1}J_0J_6J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} + \frac{A(-25, -10, -20)q^{-1}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(-90, -43, -61)q^{-1}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} \\ & + \frac{A(14, 1, 9)q^{-1}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} + \frac{A(39, 23, 29)q^{-1}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} + \frac{A(26, 17, 20)q^{-1}J_0J_6J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} \\ & + \frac{A(3, 4, 0)J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(3, 3, 1)q^{-1}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} + \frac{A(-3, 0, -4)q^{-1}J_0J_6J_7J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_{12}} \\ & + \frac{A(3, 2, 2)q^{-1}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(27, 14, 20)q^{-1}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(-31, -13, -21)q^{-1}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} \\ & + \frac{A(36, 24, 20)q^{-1}J_0J_6J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(-8, -3, -7)q^{-1}J_0J_6J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(-52, -32, -33)q^{-1}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} \\ & + \frac{A(39, 22, 22)q^{-1}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(0, -1, 1)q^{-1}J_0J_6J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9J_{10}} + \frac{A(-1, -3, 1)J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8} \\ & + \frac{A(0, -1, 1)q^{-1}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(-24, -16, -16)q^{-1}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(24, 16, 16)q^{-1}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} \\ & + \frac{A(0, 1, -1)q^{-1}J_0J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9J_{10}} + \frac{A(-1, 1, 0)q^{-1}J_0J_6^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9J_{10}} + \frac{A(-3, -5, 0)q^{-1}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} \\ & + \frac{A(-3, -3, -1)q^{-1}J_0J_6^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9J_{10}} + \frac{A(-3, -3, -2)qJ_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(-24, -15, -17)q^{-1}J_0J_5J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} \\ & + \frac{A(0, -1, 1)q^{-1}J_0J_5J_8^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9J_{12}} + \frac{A(0, 1, -1)q^{-1}J_0J_5J_6J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9J_{10}} + \frac{A(-8, -4, -6)q^{-1}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} \\ & + \frac{A(-4, 0, -3)q^2J_0J_5^2J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(-54, -27, -37)q^{-1}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(38, 18, 26)q^{-1}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} \\ & + \frac{A(51, 25, 35)q^{-1}J_0J_6J_8^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9J_{10}} + \frac{A(0, -1, 1)q^{-1}J_0J_5J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} + \frac{A(-26, -13, -18)q^{-1}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_7J_8J_9} \end{aligned}$$

It is trivial to verify that the $q^\alpha S(k, 7; \tau/7)$ appearing in the definitions of the $R2_i(q)$ are all series with integral powers of q . One advantage to writing the identity in this form is that we can also read off formulas involving the rank differences,

$$R2_{r,s,c}(d; q) = \sum_{n=0}^{\infty} (N_2(r, c, cn + d) - N_2(s, c, cn + d)) q^n.$$

To explain this, we let $\zeta_c = \zeta_c^1$ and use that $1 + \zeta_c + \zeta_c^2 + \dots + \zeta_c^{c-1} = 0$ to find that

$$R2(\zeta_c; q) = \sum_{n=0}^{\infty} \sum_{r=0}^{c-1} N_2(r, c, n) \zeta_c^r q^n = \sum_{r=1}^{c-1} \zeta_c^r \sum_{d=0}^{c-1} q^d R2_{r,0,c}(d, q^c).$$

Since $N_2(m, n) = N_2(-m, n)$, we know $R2_{r,0,c}(d, q^c) = R2_{c-r,0,c}(d, q^c)$. When c is odd, we have

$$R2(\zeta_c; q) = \sum_{r=1}^{\frac{c-1}{2}} (\zeta_c^r + \zeta_c^{c-r}) \sum_{d=0}^{c-1} q^d R2_{r,0,c}(d; q^c),$$

whereas for even c we have

$$R2(\zeta_c; q) = \zeta_c^{\frac{c}{2}} \sum_{d=0}^{c-1} q^d R2_{\frac{c}{2},0,c}(d; q^c) + \sum_{r=1}^{\frac{c-2}{2}} (\zeta_c^r + \zeta_c^{c-r}) \sum_{d=0}^{c-1} q^d R2_{r,0,c}(d; q^c).$$

In the case of c being an odd prime, we have that $\zeta_c, \zeta_c^2, \dots, \zeta_c^{c-1}$ are linearly independent over \mathbb{Q} , so if

$$R2(\zeta_c; q) = \sum_{r=1}^{\frac{c-1}{2}} (\zeta_c^r + \zeta_c^{c-r}) \sum_{d=0}^{c-1} q^d S_r(d; q^c)$$

and each $S_r(d; q)$ is a series in q with rational coefficients, then $S_r(d; q) = R2_{r,0,c}(d; q)$. As an example, from the formula for $R2_0(q)$, we have that

$$\begin{aligned} R2_{1,0,7}(0; q) &= 3S(0, 7, \tau/7) + 3S(3, 7, \tau/7) - \frac{51q^{-1}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}^2} \\ &+ \frac{69q^{-1}J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}^2J_{13}^2J_{14}^2} + \frac{27q^{-1}J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}^2J_{13}J_{14}} \\ &+ \frac{28q^{-1}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} - \frac{45q^{-1}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{13}^2J_{14}^2} + \dots \end{aligned}$$

We note it is in fact more convenient to verify the identities for the $R2_{r,0,7}(d; q)$, as then we are working with integer coefficients rather than coefficients from $\mathbb{Z}[\zeta_7]$. When c is composite, we can also deduce an identity based on the minimal polynomial for ζ_c .

The rest of the article is organized as follows. In section 3 we recall the basics of modular and harmonic Maass forms and introduce the functions studied by Zwegers in [22, 23]; these functions allow us to relate our functions to harmonic Maass forms. In section 4 we work out the transformation formulas for our functions and prove Theorem 2.1. In section 5 we give formulas for the orders at cusps. In section 6 we use the results of sections 4 and 5 to prove Theorem 2.2.

3. MODULAR FORMS, HARMONIC MAASS FORMS, AND ZWEGERS' FUNCTIONS

We first recall some basic terminology and results for modular and Maass forms. For further details one may consult [5, 19, 20, 23]. We have that $SL_2(\mathbb{Z})$ is the multiplicative group of 2×2 integer matrices with determinant 1. The principal congruence subgroup of level N is

$$\Gamma(N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) : \alpha \equiv \delta \equiv 1 \pmod{N}, \beta \equiv \gamma \equiv 0 \pmod{N} \right\}.$$

A subgroup Γ of $SL_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma \supseteq \Gamma(N)$ for some N . Two congruence subgroups we use are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) : \gamma \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) : \alpha \equiv \delta \equiv 1 \pmod{N}, \gamma \equiv 0 \pmod{N} \right\}.$$

We recall $SL_2(\mathbb{Z})$ acts on \mathcal{H} via Möbius transformations, that is if $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then $B\tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$. Additionally, we let $j(B, \tau) = \gamma\tau + \delta$. One can verify that $j(BC, \tau) = j(B, C\tau) \cdot j(C, \tau)$.

We recall a weakly holomorphic modular form of integral weight k on a congruence subgroup Γ of $SL_2(\mathbb{Z})$ is a holomorphic function f on \mathcal{H} such that

- (1) if $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, then $f(B\tau) = j(B, \tau)^k f(\tau)$;
- (2) if $B \in SL_2(\mathbb{Z})$, then $j(B, \tau)^{-k} f(B\tau)$ has an expansion of the form $\sum_{n=n_0}^{\infty} a_n e^{\frac{2\pi i n \tau}{N}}$, where $n_0 \in \mathbb{Z}$ and N is a positive integer.

When k is a half integer, we require $\Gamma \subset \Gamma_0(4)$ and replace (1) with $f(A\tau) = \left(\frac{\gamma}{\delta}\right)^{2k} \epsilon(\delta)^{-2k} (\gamma\tau + \delta)^k f(\tau)$. Here $\left(\frac{m}{n}\right)$ is the Jacobi symbol extended to all integers n by

$$\left(\frac{0}{\pm 1}\right) = 1, \quad \left(\frac{m}{n}\right) = \begin{cases} \left(\frac{m}{|n|}\right) & \text{if } m > 0 \text{ or } n > 0, \\ -\left(\frac{m}{|n|}\right) & \text{if } m < 0 \text{ and } n < 0, \end{cases}$$

and $\epsilon(\delta)$ is 1 when $\delta \equiv 1 \pmod{4}$ and is i otherwise. For a modular form with multiplier, we replace (1) with $f(B\tau) = \psi(B)j(B, \tau)^k f(\tau)$ where $|\psi(B)| = 1$.

A harmonic weak Maass form satisfies the transformation law in (1). The condition of being holomorphic is replaced with being smooth and annihilated by the weight k hyperbolic Laplacian operator

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

where here and throughout the article $\tau = x + iy$. We note that with $\frac{\partial}{\partial \bar{\tau}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ and $\frac{\partial}{\partial \tau} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$, we have $\Delta_k = -4y^{2-k} \frac{\partial}{\partial \tau} y^k \frac{\partial}{\partial \bar{\tau}}$. Condition (2) is replaced with $j(B, \tau)^{-k} f(B\tau)$ having at most linear exponential growth as $y \rightarrow \infty$ in the form that there exists a polynomial $P_B(\tau) \in \mathbb{C}[q^{-\frac{1}{N}}]$ such that $j(B, \tau)^{-k} f(B\tau) - P_B(\tau) = O(e^{-\epsilon y})$, with $\epsilon > 0$, as $y \rightarrow \infty$.

If f is a harmonic weak Maass form of weight $2 - k$, with $k > 1$, on $\Gamma_1(N)$, then f can be written as $f = f^+ + f^-$, where f^+ and f^- have expansions of the form

$$f^+(\tau) = \sum_{n=n_0}^{\infty} a(n)q^n, \quad f^-(\tau) = \sum_{n=1}^{\infty} b(n)\Gamma(k-1, 4\pi n y)q^{-n}.$$

Here Γ is the incomplete Gamma function given by $\Gamma(\alpha, \beta) = \int_{\beta}^{\infty} e^{-t} t^{\alpha-1} dt$. We call f^+ the holomorphic part and f^- the nonholomorphic part. The non-holomorphic part is often written instead as an integral of the form

$$f^-(\tau) = \int_{-\bar{\tau}}^{\infty} g(w)(-i(w + \tau))^{k-2} dw,$$

where $g = \xi_{2-k}(f)$ and $\xi_k = 2iy^k \frac{\partial}{\partial \bar{\tau}}$. When f is a harmonic Maass form on some other congruence subgroup, similar expansions exist, but with q replaced by a fractional power of q .

We now recall the various functions needed in working with harmonic Maass forms. For $u, v, z \in \mathbb{C}$, $\tau \in \mathcal{H}$, $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$, and ℓ a positive integer, we let

$$\begin{aligned} \vartheta(z; \tau) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{\pi i n^2 \tau + 2\pi i n(z + \frac{1}{2})} = -iq^{\frac{1}{8}} e^{-\pi iz} (e^{2\pi iz}, e^{-2\pi iz} q, q; q)_{\infty}, \\ \mu(u, v; \tau) &= \frac{e^{\pi i u}}{\vartheta(v; \tau)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{\pi i n(n+1)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}}, \\ A_{\ell}(u, v; \tau) &= e^{\pi i \ell u} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\ell n} e^{\pi i \ell n(n+1)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}}. \end{aligned}$$

With $a = \text{Im}(z)/y$, we define

$$\begin{aligned} E(z) &= 2 \int_0^z e^{-\pi w^2} dw, \\ R(z; \tau) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} \left(\text{sgn}(n) - E((n+a)\sqrt{2y}) \right) (-1)^{n-\frac{1}{2}} e^{-\pi i n^2 \tau - 2\pi i n z}. \end{aligned}$$

We remark that $R(z; \tau)$ is not related to the rank generating function $R(\zeta; q)$ discussed in the introduction. For $a, b \in \mathbb{R}$, we set

$$g_{a,b}(\tau) = \sum_{n \in \mathbb{Z} + a} n e^{\pi i n^2 \tau + 2\pi i n b}.$$

Finally for $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$, we set

$$\begin{aligned} \tilde{\mu}(u, v; \tau) &= \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau), \\ \widehat{A}_{\ell}(u, v; \tau) &= A_{\ell}(u, v; \tau) \\ &\quad + \frac{i}{2} \sum_{k=0}^{\ell-1} e^{2\pi i k u} \vartheta\left(v + k\tau + \frac{(\ell-1)}{2}; \ell\tau\right) R\left(\ell u - v - k\tau - \frac{(\ell-1)}{2}; \ell\tau\right). \end{aligned}$$

Zwegers studied most of these functions in his revolutionary PhD thesis [23], giving their transformation formulas, which are essential to our results. The functions $A_{\ell}(u, v; \tau)$ and $\widehat{A}_{\ell}(u, v; \tau)$ are the level ℓ Appell functions from [22].

Proposition 3.1. *Suppose $\zeta = e^{2\pi iz}$, then*

$$R2(\zeta; q) = q^{\frac{1}{8}} (\zeta^{-1} - 1) \frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)} A_2\left(z, -\tau - \frac{1}{2}; 2\tau\right).$$

Proof. We have that

$$\begin{aligned} R2(\zeta; q) &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 + \sum_{n=1}^\infty \frac{(1-\zeta)(1-\zeta^{-1})(-1)^n q^{2n^2+n}(1+q^{2n})}{(1-\zeta q^{2n})(1-\zeta^{-1}q^{2n})} \right) \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 + \sum_{n=1}^\infty (-1)^n q^{2n^2+n} \left(\frac{(1-\zeta)}{(1-\zeta q^{2n})} + \frac{(1-\zeta^{-1})}{(1-\zeta^{-1}q^{2n})} \right) \right) \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} (1-\zeta) \sum_{n=-\infty}^\infty \frac{(-1)^n q^{2n^2+n}}{(1-\zeta q^{2n})} \\ &= (\zeta^{-1} - 1) \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} A_2 \left(z, -\tau - \frac{1}{2}; 2\tau \right). \end{aligned}$$

To finish the proof, we note that

$$\frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} = \frac{(q^2; q^2)_\infty}{(q; q)_\infty (q^4; q^4)_\infty} = q^{\frac{1}{8}} \frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)}. \quad \square$$

We note that we could instead only use μ . In particular one can easily verify that

$$R2(\zeta; q) = i(1-\zeta)(\zeta^{-1}\mu(2z, -\tau; 4\tau) - \mu(2z, \tau; 4\tau)),$$

and more generally that

$$\begin{aligned} \widehat{A}_2(u, v; \tau) &= \vartheta \left(v + \frac{1}{2}; 2\tau \right) \widetilde{\mu} \left(2u, v + \frac{1}{2}; 2\tau \right) \\ &\quad + e^{2\pi i u} \vartheta \left(v + \tau + \frac{1}{2}; 2\tau \right) \widetilde{\mu} \left(2u, v + \tau + \frac{1}{2}; 2\tau \right). \end{aligned}$$

We choose to use \widehat{A}_2 as this form makes it immediately apparent that we should expect $\widetilde{\mathcal{R}}2(a, c; \tau)$ to have the same multiplier as $\frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)}$ on a certain subgroup.

Proposition 3.2. *For a and c integers, $c > 0$, and $c \nmid 2a$, we have that*

$$\begin{aligned} \mathcal{R}2(a, c; \tau) &= (\zeta_c^{-a} - 1) \frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)} A_2 \left(\frac{a}{c}, -\tau - \frac{1}{2}; 2\tau \right), \\ \widetilde{\mathcal{R}}2(a, c; \tau) &= (\zeta_c^{-a} - 1) \frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)} \widehat{A}_2 \left(\frac{a}{c}, -\tau - \frac{1}{2}; 2\tau \right). \end{aligned}$$

Proof. We see that the first identity is a trivial corollary of the previous proposition. For the second identity, given the definitions of $\widetilde{\mathcal{R}}2$ and \widehat{A}_2 , we see we must show that

$$\begin{aligned} &\frac{(\zeta_c^{-a} - 1)\eta(2\tau)}{\eta(\tau)\eta(4\tau)} \left(\frac{i}{2} \vartheta(-\tau; 4\tau) R \left(\frac{2a}{c} + \tau; 4\tau \right) + \frac{i}{2} \zeta_c^a \vartheta(\tau; 4\tau) R \left(\frac{2a}{c} - \tau; 4\tau \right) \right) \\ &= -i(1 - \zeta_c^a) \zeta_{2c}^{-a} \left(e^{\frac{\pi i}{4}} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)dw}{\sqrt{-i(w+\tau)}} + e^{-\frac{\pi i}{4}} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)dw}{\sqrt{-i(w+\tau)}} \right). \end{aligned}$$

Using that

$$\begin{aligned} \frac{\eta(\tau)\eta(4\tau)}{\eta(2\tau)} &= q^{\frac{1}{8}} (q, q^3, q^4; q^4)_\infty, & \vartheta(\tau; 4\tau) &= -i (q, q^3, q^4; q^4)_\infty, \\ \vartheta(-\tau; 4\tau) &= i (q, q^3, q^4; q^4)_\infty, \end{aligned}$$

we find that

$$\begin{aligned} & \frac{(\zeta_c^{-a} - 1)\eta(2\tau)}{\eta(\tau)\eta(4\tau)} \left(\frac{i}{2} \vartheta(-\tau; 4\tau) R\left(\frac{2a}{c} + \tau; 4\tau\right) + \frac{i}{2} \zeta_c^a \vartheta(\tau; 4\tau) R\left(\frac{2a}{c} - \tau; 4\tau\right) \right) \\ &= \frac{(\zeta_c^{-a} - 1)q^{-\frac{1}{8}}}{2} \left(-R\left(\frac{2a}{c} + \tau; 4\tau\right) + \zeta_c^a R\left(\frac{2a}{c} - \tau; 4\tau\right) \right). \end{aligned}$$

Now, by Theorem 1.16 of [23], we have that

$$\begin{aligned} R\left(\frac{2a}{c} - \tau; 4\tau\right) &= -\exp\left(\frac{4\pi i\tau}{16} + \frac{2\pi i}{4}\left(-\frac{2a}{c} + \frac{1}{2}\right)\right) \int_{-4\tau}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(z) dz}{\sqrt{-i(z + 4\tau)}} \\ &= -2q^{\frac{1}{8}} e^{\frac{\pi i}{4}} \zeta_{2c}^{-a} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w) dw}{\sqrt{-i(w + \tau)}}, \end{aligned}$$

and similarly

$$\begin{aligned} R\left(\frac{2a}{c} + \tau; 4\tau\right) &= -\exp\left(\frac{4\pi i\tau}{16} - \frac{2\pi i}{4}\left(-\frac{2a}{c} + \frac{1}{2}\right)\right) \int_{-4\tau}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(z) dz}{\sqrt{-i(z + 4\tau)}} \\ &= -2q^{\frac{1}{8}} e^{-\frac{\pi i}{4}} \zeta_{2c}^a \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w) dw}{\sqrt{-i(w + \tau)}}. \end{aligned}$$

The proposition then follows. □

Proposition 3.3. *Suppose k and c are integers, $0 \leq k < c$, and $1 + 4k - c \not\equiv 0 \pmod{4c}$. Then*

$$\begin{aligned} \mathcal{S}(k; c; \tau) &= q^{-\frac{(1+4k-2c)^2}{8}} \mu \left((1 + 4k - c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right) \\ &= \begin{cases} \frac{i^c \eta(2c^2\tau) q^{-\frac{(3c-4k-1)(c-4k-1)}{8}}}{\eta(c^2\tau)\eta(4c^2\tau)} A_1 \left((1 + 4k - c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right) & \text{if } c \text{ is odd,} \\ \frac{i^c \eta(c^2\tau) q^{-\frac{(3c-4k-1)(c-4k-1)}{8}}}{\eta(2c^2\tau)^2} A_1 \left((1 + 4k - c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right) & \text{if } c \text{ is even,} \end{cases} \\ \tilde{\mathcal{S}}(k; c; \tau) &= q^{-\frac{(1+4k-2c)^2}{8}} \tilde{\mu} \left((1 + 4k - c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right) \\ &= \begin{cases} \frac{i^c \eta(2c^2\tau) q^{-\frac{(3c-4k-1)(c-4k-1)}{8}}}{\eta(c^2\tau)\eta(4c^2\tau)} \widehat{A}_1 \left((1 + 4k - c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right) & \text{if } c \text{ is odd,} \\ \frac{i^c \eta(c^2\tau) q^{-\frac{(3c-4k-1)(c-4k-1)}{8}}}{\eta(2c^2\tau)^2} \widehat{A}_1 \left((1 + 4k - c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right) & \text{if } c \text{ is even.} \end{cases} \end{aligned}$$

Proof. We find that

$$\mathcal{S}(k; c; \tau) = q^{-\frac{(1+4k-2c)^2}{8}} \mu \left((1 + 4k - c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right)$$

follows directly from definitions. Next, we note that when c is odd

$$\begin{aligned} & \frac{1}{((-1)^{c-1}q^{c^2}, (-1)^{c-1}q^{3c^2}, q^{4c^2}; q^{4c^2})_{\infty}} \\ &= \frac{1}{(q^{c^2}, q^{3c^2}, q^{4c^2}; q^{4c^2})_{\infty}} = \frac{(q^{2c^2}; q^{2c^2})_{\infty}}{(q^{c^2}; q^{c^2})_{\infty} (q^{4c^2}; q^{4c^2})_{\infty}} = \frac{q^{\frac{c^2}{8}} \eta(2c^2\tau)}{\eta(c^2\tau)\eta(4c^2\tau)}, \end{aligned}$$

and when c is even

$$\begin{aligned} \frac{1}{((-1)^{c-1}q^{c^2}, (-1)^{c-1}q^{3c^2}, q^{4c^2}; q^{4c^2})_\infty} &= \frac{1}{(-q^{c^2}, -q^{3c^2}, q^{4c^2}; q^{4c^2})_\infty} \\ &= \frac{(q^{c^2}, q^{3c^2}; q^{4c^2})_\infty}{(q^{4c^2}; q^{4c^2})_\infty (q^{2c^2}, q^{6c^2}; q^{8c^2})_\infty} = \frac{(q^{c^2}; q^{c^2})_\infty}{(q^{2c^2}; q^{2c^2})_\infty^2} = \frac{q^{\frac{c^2}{8}} \eta(c^2\tau)}{\eta(2c^2\tau)^2}. \end{aligned}$$

The identities for $\mathcal{S}(k, c; \tau)$ then follow from the definition of $A_1(u, v; \tau)$.

To establish the identities for $\tilde{\mathcal{S}}(k, c; \tau)$, we must verify that for $0 \leq k < c$ we have

$$\begin{aligned} \frac{i}{2}q^{-\frac{(1+4k-2c)^2}{8}} R\left((1+4k-2c)c\tau - \frac{c-1}{2}; 4c^2\tau\right) \\ = (-1)^{k+1}i^{1+c}e^{-\frac{\pi i}{4}c} \int_{-\bar{\tau}}^\infty \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(4c^2w)}{\sqrt{-i(w+\tau)}} dw. \end{aligned}$$

For this we use Theorem 1.6 of [23] to determine that

$$\begin{aligned} \frac{i}{2}q^{-\frac{(1+4k-2c)^2}{8}} R\left((1+4k-2c)c\tau - \frac{c-1}{2}; 4c^2\tau\right) \\ = -\frac{i}{2}q^{-\frac{(1+4k-2c)^2}{8}} \exp\left(\pi i\tau \frac{(1+4k-2c)^2}{4} - \pi i \frac{(1+4k-2c)}{4}\right) \int_{-4c^2\tau}^{i\infty} \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(z)}{\sqrt{-i(z+4c^2\tau)}} dz \\ = (-1)^{k+1}i^{1+c}e^{-\frac{\pi i}{4}c} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(4c^2w)}{\sqrt{-i(4c^2w+4c^2\tau)}} dw. \quad \square \end{aligned}$$

It is through these identities that we take the definitions of $\mathcal{S}(k, c; \tau)$ and $\tilde{\mathcal{S}}(k, c; \tau)$ when k is an arbitrary integer. That is to say, they are defined in terms of μ and $\tilde{\mu}$ rather than series and integrals.

Working with $\tilde{\mu}(u, v; \tau)$ and $\hat{A}_\ell(u, v; \tau)$ is advantageous in that the transformations under the action of the modular group $SL_2(\mathbb{Z})$ are known and are quite elegant. With this we are able to determine a simple multiplier for $\widetilde{\mathcal{R}2}(a, c; \tau)$ and $\tilde{\mathcal{S}}(k, c; \tau)$. For this reason we do not need to replace τ by 8τ to have a strong understanding of these functions, even with the factor $q^{-\frac{1}{8}}$ in the definitions of $\mathcal{R}2(a, c; \tau)$ and $\widetilde{\mathcal{R}2}(a, c; \tau)$. Additionally, the transformations of $\tilde{\mu}$ allow us to easily determine the behavior of $\widetilde{\mathcal{R}2}(a, c; \tau)$ and $\tilde{\mathcal{S}}(k, c; \tau)$ at the cusps.

4. TRANSFORMATIONS FORMULAS

For a matrix $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$, we have $\nu(B)$, the η -multiplier, defined by

$$\eta(B\tau) = \nu(B)\sqrt{\gamma\tau + \delta}\eta(\tau),$$

where $\eta(\tau)$ is Dedekind's eta-function,

$$\eta(\tau) = q^{\frac{1}{24}}(q; q)_\infty.$$

A convenient form for the η -multiplier, which can be found as Theorem 2 in Chapter 4 of [14], is

$$(4.1) \quad \nu(B) = \begin{cases} \left(\frac{\delta}{|\gamma|}\right) \exp\left(\frac{\pi i}{12}\left((\alpha + \delta)\gamma - \beta\delta(\gamma^2 - 1) - 3\gamma\right)\right) & \text{if } \gamma \text{ is odd,} \\ \left(\frac{\gamma}{\delta}\right) \exp\left(\frac{\pi i}{12}\left((\alpha + \delta)\gamma - \beta\delta(\gamma^2 - 1) + 3\delta - 3 - 3\gamma\delta\right)\right) & \text{if } \delta \text{ is odd.} \end{cases}$$

For an integer m we let

$$B_m = \begin{pmatrix} \alpha & m\beta \\ \gamma/m & \delta \end{pmatrix}.$$

The utility of B_m is in the fact that $mB\tau = B_m(m\tau)$.

Our transformation formulas for $\widetilde{\mathcal{R}2}(a, c; \tau)$ and $\widetilde{\mathcal{S}}(k, c; \tau)$ are easily deduced by the transformations of $\widetilde{\mu}(u, v; \tau)$ and $\widehat{A}_\ell(u, v; \tau)$. The following essential properties are from Theorem 1.11 of [23] and Theorem 2.2 of [22]. If k, l, m, n are integers, then

(4.2)

$$\widetilde{\mu}(u + k\tau + l, v + m\tau + n; \tau) = (-1)^{k+l+m+n} e^{\pi i \tau (k-m)^2 + 2\pi i (k-m)(u-v)} \widetilde{\mu}(u, v; \tau),$$

(4.3)

$$\widehat{A}_\ell(u + k\tau + l, v + m\tau + n; \tau) = (-1)^{\ell(k+l)} e^{2\pi i u (\ell k - m)} e^{-2\pi i v k} q^{\frac{\ell k^2}{2} - km} \widehat{A}_\ell(u, v; \tau),$$

and if $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, then

$$(4.4) \quad \widetilde{\mu}\left(\frac{u}{\gamma\tau + \delta}, \frac{v}{\gamma\tau + \delta}; B\tau\right) = \nu(B)^{-3} e^{-\frac{\pi i \gamma (u-v)^2}{\gamma\tau + \delta}} \sqrt{\gamma\tau + \delta} \widetilde{\mu}(u, v; \tau),$$

$$(4.5) \quad \widehat{A}_\ell\left(\frac{u}{\gamma\tau + \delta}, \frac{v}{\gamma\tau + \delta}; B\tau\right) = (\gamma\tau + \delta) e^{\frac{\pi i \gamma}{\gamma\tau + \delta} (-\ell u^2 + 2uv)} \widehat{A}_\ell(u, v; \tau).$$

The propositions and formulas of this section are organized as follows. We begin by investigating the transformation formula for $\widetilde{\mathcal{R}2}(a, c; \tau)$ that follows from (4.5) and refine this result by restricting to a smaller congruence subgroup of $\text{SL}_2(\mathbb{Z})$ that yields a simple multiplier. We then verify that $\widetilde{\mathcal{R}2}(a, c; \tau)$ is annihilated by $\Delta_{\frac{1}{2}}$ and that $\widetilde{\mathcal{R}2}(a, c; \tau)$ behaves correctly at the cusps. With this we will have established part (1) of Theorem 2.1. We proceed by determining a formula for the nonholomorphic part of $\widetilde{\mathcal{R}2}(a, c; \tau)$ that allows us to determine the nonholomorphic parts of the c -dissections of $\widetilde{\mathcal{R}2}(a, c; \tau)$. We then recognize $\widetilde{\mathcal{S}}(k, c; \tau)$ as the functions whose nonholomorphic parts agree with those of the c -dissections of $\widetilde{\mathcal{R}2}(a, c; \tau)$ and whose holomorphic parts are q -series with exponents taking on a single value modulo c . After this we give $\widetilde{\mathcal{S}}(k, c; \tau)$ the same treatment as $\widetilde{\mathcal{R}2}(a, c; \tau)$. Lastly, we quickly deduce the necessary transformation formulas for various generalized eta quotients using the work of Biagioli in [4].

Proposition 4.1. *Suppose a and c are integers, $c > 0$, and $c \nmid 2a$. If $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4) \cap \Gamma_0(2c) \cap \Gamma_1(c)$, then*

$$\widetilde{\mathcal{R}2}(a, c; B\tau) = \frac{\nu(B_2)}{\nu(B)\nu(B_4)} e^{\frac{\pi i a}{c} \left(\frac{2a\gamma}{c} - \frac{a\gamma\delta}{c} - \frac{\gamma}{2} + \alpha - 1 \right)} \sqrt{\gamma\tau + \delta} \widetilde{\mathcal{R}2}(a, c; \tau).$$

In particular, if $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4) \cap \Gamma_0(2c^2) \cap \Gamma_1(2c)$, then

$$\widetilde{\mathcal{R}2}(a, c; B\tau) = \frac{\nu(B_2)}{\nu(B)\nu(B_4)} \sqrt{\gamma\tau + \delta} \widetilde{\mathcal{R}2}(a, c; \tau).$$

Proof. To begin, we let

$$\widehat{F}(a, c; \tau) = \widehat{A}_2\left(\frac{a}{c}, -\tau - \frac{1}{2}; 2\tau\right)$$

so that

$$\widetilde{\mathcal{R}}2(a, c; \tau) = (\zeta_c^{-a} - 1) \frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)} \widehat{F}(a, c; \tau).$$

We note that $B_2 = \begin{pmatrix} \alpha & 2\beta \\ \gamma/2 & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, so that we may apply (4.5) to find that

$$\begin{aligned} \widehat{F}(a, c; B\tau) &= \widehat{A}_2 \left(\frac{a}{c}, -B\tau - \frac{1}{2}; B_2(2\tau) \right) \\ &= (\gamma\tau + \delta) e^{\frac{\pi i \gamma}{2(\gamma\tau + \delta)}} \left(-\frac{2a^2(\gamma\tau + \delta)^2}{c^2} - \frac{2a(\gamma\tau + \delta)}{c} ((\alpha\tau + \beta) + \frac{(\gamma\tau + \delta)}{2}) \right) \\ &\quad \times \widehat{A}_2 \left(\frac{a(\gamma\tau + \delta)}{c}, -(\alpha\tau + \beta) - \frac{(\gamma\tau + \delta)}{2}; 2\tau \right) \\ &= (\gamma\tau + \delta) e^{-\frac{\pi i a \gamma}{2c} (\frac{2a\delta}{c} + 2\beta + \delta)} q^{-\frac{a\gamma}{2c} (\frac{a\gamma}{c} + \alpha + \frac{\gamma}{2})} \\ &\quad \times \widehat{A}_2 \left(\frac{a\gamma}{c} \tau + \frac{a\delta}{c}, -\frac{(2\alpha + \gamma)}{2} \tau - \beta - \frac{\delta}{2}; 2\tau \right) \\ &= (\gamma\tau + \delta) e^{-\frac{\pi i a \gamma}{2c} (\frac{2a\delta}{c} + 1)} q^{-\frac{a\gamma}{2c} (\frac{a\gamma}{c} + \alpha + \frac{\gamma}{2})} \\ &\quad \times \widehat{A}_2 \left(\frac{a\gamma}{c} \tau + \frac{a\delta}{c}, -\frac{(2\alpha + \gamma)}{2} \tau - \beta - \frac{\delta}{2}; 2\tau \right). \end{aligned}$$

Since $\gamma \equiv 0 \pmod{2c}$, $\delta \equiv 1 \pmod{c}$, $\delta \equiv 1 \pmod{2}$, and $\alpha - 1 + \frac{\gamma}{2} \equiv 0 \pmod{2}$, we can apply (4.3) to get that

$$\begin{aligned} &\widehat{A}_2 \left(\frac{a\gamma}{c} \tau + \frac{a\delta}{c}, -\frac{2a + \gamma}{2} \tau - \beta - \frac{\delta}{2}; 2\tau \right) \\ &= \widehat{A}_2 \left(\frac{a}{c} + \frac{a\gamma}{2c} 2\tau + \frac{a(\delta - 1)}{c}, -\tau - \frac{1}{2} - \left(\frac{\alpha - 1}{2} + \frac{\gamma}{4} \right) 2\tau - \beta - \frac{(\delta - 1)}{2}; 2\tau \right) \\ &= e^{\frac{2\pi i a}{c} (\frac{a\gamma}{c} + \frac{(\alpha - 1)}{2} + \frac{\gamma}{4})} e^{2\pi i (\tau + \frac{1}{2}) \frac{a\gamma}{2c}} q^{2 \left(\frac{a^2 \gamma^2}{4c^2} + \frac{a\gamma}{2c} \left(\frac{\alpha - 1}{2} + \frac{\gamma}{4} \right) \right)} \widehat{A}_2 \left(\frac{a}{c}, -\tau - \frac{1}{2}; 2\tau \right) \\ &= e^{\frac{2\pi i a}{c} (\frac{a\gamma}{c} + \frac{(\alpha - 1)}{2} + \frac{\gamma}{2})} q^{\frac{a\gamma}{2c} (\frac{a\gamma}{c} + \alpha + \frac{\gamma}{2})} \widehat{F}(a, c; \tau) \\ &= e^{\frac{2\pi i a}{c} (\frac{a\gamma}{c} + \frac{(\alpha - 1)}{2})} q^{\frac{a\gamma}{2c} (\frac{a\gamma}{c} + \alpha + \frac{\gamma}{2})} \widehat{F}(a, c; \tau). \end{aligned}$$

For $B \in \Gamma_0(4) \cap \Gamma_0(2c) \cap \Gamma_1(c)$, this yields that

$$\widetilde{\mathcal{R}}2(a, c; B\tau) = \sqrt{\gamma\tau + \delta} \frac{\nu(B_2)}{\nu(B)\nu(B_4)} e^{\frac{\pi i a}{c} (\frac{2a\gamma}{c} - \frac{a\gamma\delta}{c} - \frac{\gamma}{2} + \alpha - 1)} \widetilde{\mathcal{R}}2(a, c; \tau).$$

To complete the proof, we simply note that

$$e^{\frac{\pi i a}{c} (\frac{2a\gamma}{c} - \frac{a\gamma\delta}{c} - \frac{\gamma}{2} + \alpha - 1)} = 1$$

for $B \in \Gamma_0(4) \cap \Gamma_0(2c^2) \cap \Gamma_1(2c)$. □

The rest of the transformation formulas have similar proofs. We apply (4.4) or (4.5) followed by (4.2) or (4.3) and reduce the various powers of q and roots of unity. For this reason we will omit the majority of these calculations, as they are routine and straightforward but rather long.

Proposition 4.2. *Suppose a and c integers, $c > 0$, and $c \nmid 2a$. Then $\widetilde{\mathcal{R}}2(a, c; \tau)$ is a harmonic weak Maass form of weight $\frac{1}{2}$ on $\Gamma_0(4) \cap \Gamma_0(2c) \cap \Gamma_1(c)$ with multiplier*

$\frac{\nu(B_2)}{\nu(B)\nu(B_4)} \cdot e^{\frac{\pi i \alpha}{c} \left(\frac{2a\gamma}{c} - \frac{a\gamma\delta}{c} - \frac{\gamma}{2} + \alpha - 1 \right)}$ for $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Furthermore, $\widetilde{\mathcal{R}2}(a, c; 8\tau)$ is a harmonic Maass form of weight $\frac{1}{2}$ on $\Gamma_0(256) \cap \Gamma_0(16c^2) \cap \Gamma_1(8) \cap \Gamma_1(2c)$.

Proof. The transformation formula for $\widetilde{\mathcal{R}2}(a, c; \tau)$ is Proposition 4.1. Since $\mathcal{R}2(a, c; \tau)$ is holomorphic in τ , to show that $\Delta_{\frac{1}{2}}$ annihilates $\widetilde{\mathcal{R}2}(a, c; \tau)$, it is sufficient to verify that $\sqrt{y} \frac{\partial}{\partial \bar{\tau}} q^{-\frac{1}{8}} R\left(\frac{2a}{c} \pm \tau; 4\tau\right)$ is holomorphic in $\bar{\tau}$. However this follows by Lemma 1.8 of [23].

While the growth condition at the cusps follows from the fact that $\frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)}$ is a modular form, the transformation formula for \widehat{A}_2 , and properties of R , we will later require a more precise bound for the holomorphic part. To establish this bound, it is more convenient to work with $\tilde{\mu}$. In particular, we suppose α and γ are integers with $\gcd(\alpha, \gamma) = 1$ and take $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. We set $g = \gcd(\gamma, 4)$, $m = \frac{4\alpha}{g}$, and $r = \frac{\gamma}{g}$. Since $\gcd(m, r) = 1$, we take $L = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and set

$$C = L^{-1} \begin{pmatrix} 4\alpha & 4\beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} g & 4\beta s - \delta n \\ 0 & 4/g \end{pmatrix}.$$

This gives that $4B\tau = LC\tau$ and $(\gamma\tau + \delta) = j(LC, \tau) = j(L, C\tau) \cdot j(C, \tau) = j(L, C\tau) \cdot \frac{4}{g}$. By (4.4) we find that

$$\begin{aligned} & (\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{\mathcal{R}2}(a, c; B\tau) \\ &= (\gamma\tau + \delta)^{-\frac{1}{2}} i (1 - \zeta_c^a) \exp\left(-\frac{\pi i LC\tau}{16}\right) \\ & \quad \times \left(\zeta_c^a \tilde{\mu}\left(\frac{2a}{c}, -\frac{LC\tau}{4}; LC\tau\right) - \tilde{\mu}\left(\frac{2a}{c}, \frac{LC\tau}{4}; LC\tau\right) \right) \\ &= \epsilon_1 \exp\left(2\pi i C\tau \left(-\frac{m^2}{32} - \frac{2a^2 r^2}{c^2} - \frac{amr}{2c}\right)\right) \tilde{\mu}\left(\frac{2arC\tau + 2as}{c}, \frac{-mC\tau - s}{4}; C\tau\right) \\ & \quad + \epsilon_2 \exp\left(2\pi i C\tau \left(-\frac{m^2}{32} - \frac{2a^2 r^2}{c^2} + \frac{amr}{2c}\right)\right) \tilde{\mu}\left(\frac{2arC\tau + 2as}{c}, \frac{mC\tau + s}{4}; C\tau\right), \end{aligned}$$

where ϵ_1 and ϵ_2 are nonzero constants. Upon noting $e^{2\pi i C\tau} = \epsilon_3 q^{\frac{g^2}{4}}$, where ϵ_3 is a fixed root of unity, we find that $(\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{\mathcal{R}2}(a, c; B\tau)$ has at most linear exponential growth as $y \rightarrow \infty$.

To determine the transformation formula for $\widetilde{\mathcal{R}2}(a, c; 8\tau)$, we first suppose $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(256) \cap \Gamma_0(16c^2) \cap \Gamma_1(8) \cap \Gamma_1(2c)$. Since $B_8 \in \Gamma_0(4) \cap \Gamma_0(2c^2) \cap \Gamma_1(2c)$, we have by Proposition 4.1 that $\widetilde{\mathcal{R}2}(a, c; 8\tau)$ has multiplier $\frac{\nu(B_{16})}{\nu(B_8)\nu(B_{32})}$. A direct calculation using (4.1) reveals that on $\Gamma_0(32)$,

$$\frac{\nu(B_{16})}{\nu(B_8)\nu(B_{32})} = \left(\frac{\gamma}{\delta}\right) \exp\left(\frac{\pi i}{4} \left(\frac{\gamma\delta}{32} - \delta + 1\right)\right).$$

However, on $\Gamma_0(256) \cap \Gamma_1(8)$ we have $\exp\left(\frac{\pi i}{4} \left(\frac{\gamma\delta}{32} - \delta + 1\right)\right) = 1$, so that $\widetilde{\mathcal{R}2}(a, c; 8\tau)$ transforms as claimed. □

Upon noting that $\Gamma_0(256) \cap \Gamma_0(16c^2) \cap \Gamma_1(8) \cap \Gamma_1(2c)$ can be abbreviated to $\Gamma_0(256t^2) \cap \Gamma_1(8t)$ where $t = \frac{c}{\gcd(4, c)}$, we have now proved part (1) of Theorem 2.1.

Proposition 4.3. *Suppose a and c are integers, $c > 0$, and $c \nmid 2a$. The non-holomorphic part of $\widetilde{\mathcal{R}2}(a, c; \tau)$ is*

$$\begin{aligned} & \frac{(1 - \zeta_c^a)\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sgn}(n + \frac{1}{4}) (\zeta_c^{-2an} - \zeta_c^{2an+a}) q^{-2n^2 - n - \frac{1}{8}} \Gamma(\frac{1}{2}, 8\pi(n + \frac{1}{4})^2 y) \\ &= \frac{i^{1-c}(1 - \zeta_c^a)\zeta_c^{-a}}{2} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) q^{-\frac{(1+4k-2c)^2}{8}} \\ & \quad \times R(c(1 + 4k - 2c)\tau + \frac{1-c}{2}; 4c^2\tau). \end{aligned}$$

Proof. The nonholomorphic part of $\widetilde{\mathcal{R}2}(a, c; \tau)$ is

$$-i(1 - \zeta_c^a)\zeta_{2c}^{-a} \left(e^{\frac{\pi i}{4}} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)}{\sqrt{-i(w + \tau)}} dw + e^{-\frac{\pi i}{4}} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)}{\sqrt{-i(w + \tau)}} dw \right).$$

We find that

$$\begin{aligned} & \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)}{\sqrt{-i(w + \tau)}} dw = \sum_{n=-\infty}^{\infty} (n + \frac{3}{4}) \int_{-\bar{\tau}}^{i\infty} \frac{e^{\pi i(n + \frac{3}{4})^2 4w + 2\pi i(n + \frac{3}{4})(\frac{1}{2} - \frac{2a}{c})}}{\sqrt{-i(w + \tau)}} dw \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} \zeta_c^{-2an} \zeta_{2c}^{-3a} e^{\frac{3\pi i}{4}}}{4\pi i(n + \frac{3}{4})} \int_{8\pi(n + \frac{3}{4})^2 y}^{\infty} \frac{e^{-t} e^{-4\pi i(n + \frac{3}{4})^2 \tau}}{\sqrt{\frac{t}{4\pi(n + \frac{3}{4})^2}}} dt \\ &= \frac{ie^{\frac{3\pi i}{4}} \zeta_{2c}^{-3a}}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \zeta_c^{-2an} q^{-2n^2 - 3n - \frac{9}{8}} \operatorname{sgn}(n + \frac{3}{4}) \Gamma(\frac{1}{2}, 8\pi(n + \frac{3}{4})^2 y), \end{aligned}$$

where we have used the substitution $w = -\frac{t}{4\pi i(n + \frac{3}{4})^2} - \tau$. Similarly, we calculate that

$$\begin{aligned} & \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)}{\sqrt{-i(w + \tau)}} dw \\ &= \frac{ie^{\frac{\pi i}{4}} \zeta_{2c}^{-a}}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \zeta_c^{-2an} q^{-2n^2 - n - \frac{1}{8}} \operatorname{sgn}(n + \frac{1}{4}) \Gamma(\frac{1}{2}, 8\pi(n + \frac{1}{4})^2 y). \end{aligned}$$

With $n \mapsto -n - 1$ we find that

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^n \zeta_c^{-2an} q^{-2n^2 - 3n - \frac{9}{8}} \operatorname{sgn}(n + \frac{3}{4}) \Gamma(\frac{1}{2}, 8\pi(n + \frac{3}{4})^2 y) \\ &= \sum_{n=-\infty}^{\infty} (-1)^n \zeta_c^{2an+2a} q^{-2n^2 - n - \frac{1}{8}} \operatorname{sgn}(n + \frac{1}{4}) \Gamma(\frac{1}{2}, 8\pi(n + \frac{1}{4})^2 y). \end{aligned}$$

Thus the nonholomorphic part of $\widetilde{\mathcal{R}2}(a, c; \tau)$ is

$$\begin{aligned} & \frac{(1 - \zeta_c^a)\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sgn}(n + \frac{1}{4})(\zeta_c^{-2an} - \zeta_c^{2an+a})q^{-2n^2-n-\frac{1}{8}}\Gamma(\frac{1}{2}, 8\pi(n + \frac{1}{4})^2y) \\ &= \frac{(1 - \zeta_c^a)\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \\ & \quad \times \sum_{n=-\infty}^{\infty} (-1)^{cn} \operatorname{sgn}(cn + k + \frac{1}{4})q^{-2(cn+k+\frac{1}{4})^2}\Gamma(\frac{1}{2}, 8\pi(cn + k + \frac{1}{4})^2y) \\ &= \frac{(1 - \zeta_c^a)\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \\ & \quad \times \sum_{n=-\infty}^{\infty} (-1)^{cn} \operatorname{sgn}(n + \frac{1+4k}{4c})q^{-2c^2(n+\frac{1+4k}{4c})^2} \int_{8\pi c^2(n+\frac{1+4k}{4c})^2y}^{\infty} \frac{e^{-t}}{\sqrt{t}} dt \\ &= -ic(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \\ & \quad \times \sum_{n=-\infty}^{\infty} (-1)^{cn} (n + \frac{1+4k}{4c}) \int_{-\bar{\tau}}^{i\infty} \frac{e^{4\pi iwc^2(n+\frac{1+4k}{4c})^2}}{\sqrt{-i(w+\tau)}} dw \\ &= -ice^{-\frac{\pi i}{4}}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(4c^2w)}{\sqrt{-i(w+\tau)}} dw \\ &= -\frac{ie^{-\frac{\pi i}{4}}(1 - \zeta_c^a)\zeta_c^{-a}}{2} \sum_{k=0}^{c-1} (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \int_{-4c^2\tau}^{i\infty} \frac{g_{\frac{1+4k-2c}{4c}+\frac{1}{2}, \frac{c-1}{2}+\frac{1}{2}}(z)}{\sqrt{-i(z+4c^2\tau)}} dz \\ &= \frac{i^{1-c}(1 - \zeta_c^a)\zeta_c^{-a}}{2} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a})q^{-\frac{(1+4k-2c)^2}{8}} \\ & \quad \times R(c(1+4k-2c)\tau + \frac{1-c}{2}; 4c^2\tau). \end{aligned}$$

We note that in the second to last equality, we have used Theorem 1.16 of [23], which is valid since $-\frac{1}{2} < \frac{1+4k-2c}{4c} < \frac{1}{2}$ for $0 \leq k \leq c-1$. \square

We now see the definitions of $\mathcal{S}(k, c; \tau)$ and $\widetilde{\mathcal{S}}(k, c; \tau)$ are well motivated. We recall that Proposition 3.3 states

$$\widetilde{\mathcal{S}}(k; c; \tau) = q^{-\frac{(1+4k-2c)^2}{8}} \widetilde{\mu} \left((1+4k-c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right).$$

From this it is apparent that we have chosen $\widetilde{\mathcal{S}}(k, c; \tau)$ so that the nonholomorphic part is $\frac{i}{2}q^{-\frac{(1+4k-2c)^2}{8}}R(c(1+4k-2c)\tau + \frac{1-c}{2}; 4c^2\tau)$, which matches with the terms of the nonholomorphic part of $\widetilde{\mathcal{R}2}(a, c; \tau)$. Furthermore, $\zeta_c^{-2ak} - \zeta_c^{2ak+a} = 0$ when $1+4k-c \equiv 0 \pmod{4c}$.

Proposition 4.4. *Suppose k and c are integers, $c > 0$, and $1+4k-c \not\equiv 0 \pmod{4c}$. If $B \in \Gamma_0(\gcd(c, 2)4c^2) \cap \Gamma_1(4c)$, then*

$$\widetilde{\mathcal{S}}(k, c; B\tau) = \frac{\nu(B_2)}{\nu(B)\nu(B_4)} \sqrt{\gamma\tau + \delta} \widetilde{\mathcal{S}}(k, c; \tau).$$

Proof. To begin, we let

$$\widehat{G}(k, c; \tau) = q^{-\frac{(3c-4k-1)(c-4k-1)}{8}} \widehat{A}_1 \left((1+4k-c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right),$$

so that

$$\widetilde{S}(k, c; B\tau) = \begin{cases} \frac{\eta(2c^2\tau)}{\eta(c^2\tau)\eta(4c^2\tau)} \widehat{G}(k, c; \tau) & \text{if } c \text{ is odd,} \\ \frac{\eta(c^2\tau)}{\eta(2c^2\tau)^2} \widehat{G}(k, c; \tau) & \text{if } c \text{ is even.} \end{cases}$$

Using (4.3), on $\Gamma_0(4c^2)$, we have that

$$\begin{aligned} & \widehat{G}(k, c; B\tau) \\ &= e^{-\frac{\pi i B\tau}{4}(3c-4k-1)(c-4k-1)} \\ & \quad \times \widehat{A}_1 \left((1+4k-c)c \frac{(\alpha\tau + \beta)}{(\gamma\tau + \delta)}, \frac{c^2(\alpha\tau + \beta)}{(\gamma\tau + \delta)} + \frac{(c-1)}{2}; B_{4c^2}(4c^2\tau) \right) \\ &= (\gamma\tau + \delta) e^{-\frac{\pi i B\tau}{4}(3c-4k-1)(c-4k-1)} \\ & \quad \times e^{\frac{\pi i \gamma \left(-(1+4k-c)^2 c^2 (\alpha\tau + \beta)^2 + 2(1+4k-c)c(\alpha\tau + \beta) \left(c^2(\alpha\tau + \beta) + \frac{(c-1)(\gamma\tau + \delta)}{2} \right) \right)}{4c^2(\gamma\tau + \delta)}} \\ & \quad \times \widehat{A}_1 \left((1+4k-c)c(\alpha\tau + \beta), c^2(\alpha\tau + \beta) + \frac{(c-1)(\gamma\tau + \delta)}{2}; 4c^2\tau \right) \\ &= e^{-\frac{\pi i \alpha \beta}{4}(3c-4k-1)(c-4k-1)} q^{-\frac{\alpha^2}{8}(3c-4k-1)(c-4k-1) + \frac{\alpha\gamma}{8c}(1+4k-c)(c-1)} \\ & \quad \times (\gamma\tau + \delta) \widehat{A}_1 \left((1+4k-c)c(\alpha\tau + \beta), c^2(\alpha\tau + \beta) + \frac{(c-1)(\gamma\tau + \delta)}{2}; 4c^2\tau \right). \end{aligned}$$

Now by (4.3) we have that

$$\begin{aligned} & \widehat{A}_1 \left((1+4k-c)c(\alpha\tau + \beta), c^2(\alpha\tau + \beta) + \frac{(c-1)(\gamma\tau + \delta)}{2}; 4c^2\tau \right) \\ &= \widehat{A}_1 \left((1+4k-c)c\tau + \frac{(1+4k-c)(\alpha-1)}{4c} 4c^2\tau + (1+4k-c)c\beta, c^2\tau + \frac{(c-1)}{2} \right. \\ & \quad \left. + \left(\frac{(\alpha-1)}{4} + \frac{(c-1)\gamma}{8c^2} \right) 4c^2\tau + c^2\beta + \frac{(c-1)(\delta-1)}{2}; 4c^2\tau \right) \\ &= (-1)^{\frac{(1+4k-c)(\alpha-1)}{4c}} e^{2\pi i(1+4k-c)c\tau \left(\frac{(1+4k-c)(\alpha-1)}{4c} - \frac{(\alpha-1)}{4} - \frac{(c-1)\gamma}{8c^2} \right)} \\ & \quad \times e^{-2\pi i \left(c^2\tau + \frac{(c-1)}{2} \right) \frac{(1+4k-c)(\alpha-1)}{4c}} \\ & \quad \times e^{8\pi i c^2\tau \left(\frac{(1+4k-c)^2(\alpha-1)^2}{32c^2} - \frac{(1+4k-c)(\alpha-1)}{4c} \left(\frac{(\alpha-1)}{4} + \frac{(c-1)\gamma}{8c^2} \right) \right)} \\ & \quad \times \widehat{A}_1 \left((1+4k-c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right) \\ &= (-1)^{\frac{(1+4k-c)(\alpha-1)}{4}} q^{-\frac{(1-\alpha^2)}{8}(3c-4k-1)(c-4k-1) - \frac{\alpha\gamma}{8c}(1+4k-c)(c-1)} \\ & \quad \times \widehat{A}_1 \left((1+4k-c)c\tau, c^2\tau + \frac{(c-1)}{2}; 4c^2\tau \right), \end{aligned}$$

and so

$$\begin{aligned} \widehat{G}(k, c; B\tau) &= (-1)^{\frac{(1+4k-c)(\alpha-1)}{4}} e^{-\frac{\pi i \alpha \beta}{4}(3c-4k-1)(c-4k-1)} (\gamma\tau + \delta) \widehat{G}(k, c; \tau) \\ &= e^{-\frac{\pi i \alpha \beta}{4}(3c-4k-1)(c-4k-1)} (\gamma\tau + \delta) \widehat{G}(k, c; \tau). \end{aligned}$$

We now must separately consider the cases when c is odd or even. When c is odd, $(3c - 4k - 1)(c - 4k - 1)$ is divisible by 8. As such,

$$\widehat{G}(k, c; B\tau) = (\gamma\tau + \delta)\widehat{G}(k, c; \tau).$$

By Theorem 1.64 of [19], $\frac{\eta(2\tau)\eta(c^2\tau)\eta(4c^2\tau)}{\eta(\tau)\eta(4\tau)\eta(2c^2\tau)}$ is a modular function on $\Gamma_0(4c^2)$ with trivial multiplier. Thus

$$\frac{\nu(B_2)}{\nu(B)\nu(B_4)} = \frac{\nu(B_{2c^2})}{\nu(B_{c^2})\nu(B_{4c^2})},$$

and the case when c is odd follows.

When c is even, we instead have that

$$\widehat{G}(k, c; B\tau) = \exp\left(-\frac{\pi i\beta}{4}(3c^2 + 1)\right)(\gamma\tau + \delta)\widehat{G}(k, c; \tau).$$

A direct, but lengthy, calculation with (4.1) reveals that on $\Gamma_0(8c^2) \cap \Gamma(4c)$, we have

$$\frac{\nu(B)\nu(B_4)\nu(B_{c^2})}{\nu(B_2)\nu(B_{2c^2})^2} = \exp\left(\frac{\pi i\beta(1 - c^2)}{4}\right).$$

With this the case when c is even follows. □

Proposition 4.5. *Suppose k and c are integers, $c > 0$, and $1 + 4k - c \not\equiv 0 \pmod{4c}$. Then $\widetilde{\mathcal{S}}(k, c; \tau)$ is a harmonic weak Maass form on $\Gamma_0(\gcd(c, 2)4c^2) \cap \Gamma_1(4c)$ with multiplier $\frac{\nu(B_2)}{\nu(B)\nu(B_4)}$.*

Proof. We first verify that $\widetilde{\mathcal{S}}(k, c; \tau)$ is annihilated by $\Delta_{\frac{1}{2}}$. For this we need only verify that

$$\sqrt{y} \frac{\partial}{\partial \bar{\tau}} q^{-\frac{(1+4k-2c)^2}{8}} R\left((1 + 4k - c)c\tau + \frac{(1-c)}{2}; 4c^2\tau\right)$$

is holomorphic in $\bar{\tau}$. However this follows from Lemma 1.8 of [23].

We next verify that $\widetilde{\mathcal{S}}(k, c; \tau)$ has at worst linear exponential growth at the cusps. For this, suppose α/γ is in reduced form. We then take $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. We set $g = \gcd(4c^2, \gamma)$, $m = 4c^2\alpha/g$, and $r = \gamma/g$. Since $\gcd(m, r) = 1$, we can take $L = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, and set

$$C = L^{-1} \begin{pmatrix} 4c^2\alpha & 4c^2\beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} g & 4c^2\beta s - \delta n \\ 0 & 4c^2/g \end{pmatrix}.$$

We note that $j(L, C\tau) = \frac{g}{4c^2}(\gamma\tau + \delta)$. Using (4.4), we compute that

$$\begin{aligned} & (\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{\mathcal{S}}(k, c; B\tau) \\ &= (\gamma\tau + \delta)^{-\frac{1}{2}} \exp\left(-\frac{\pi iLC\tau(1 + 4k - 2c)^2}{16c^2}\right) \\ & \quad \times \widetilde{\mu}\left(\frac{(1 + 4k - 2c)LC\tau}{4c}, \frac{LC\tau}{4} + \frac{(c - 1)}{2}; LC\tau\right) \\ &= \epsilon_1 \exp\left(\pi iC\tau\left(-\frac{m^2(1 + 4k - 2c)^2}{16c^2} + \frac{mr(1 + 4k - 2c)(c - 1)}{4c} - \frac{r^2(c - 1)^2}{4}\right)\right) \\ & \quad \times \widetilde{\mu}\left(\frac{(1 + 4k - c)(mC\tau + n)}{4c}, \frac{(mC\tau + n)}{4} + \frac{(c - 1)(rC\tau + s)}{2}; C\tau\right), \end{aligned}$$

where ϵ_1 is a nonzero constant. Since $e^{2\pi i C\tau} = \epsilon_2 q^{\frac{g^2}{4c^2}}$, where ϵ_2 is a fixed root of unity, we see that $(\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{\mathcal{S}}(k, c; B\tau)$ has at worst linear exponential growth as $y \rightarrow \infty$ by the definition of $\widetilde{\mu}$. \square

Since $\widetilde{\mathcal{R}2}(a, c; \tau)$ and $\widetilde{\mathcal{S}}(k, c; \tau)$ have the same weight and multiplier on $\Gamma_0(\gcd(c, 2))4c^2 \cap \Gamma_1(4c)$, taking the correct difference of these terms will cancel the nonholomorphic parts and result in a modular form. This establishes parts (2), (3), and (4) of Theorem 2.1.

To accommodate the generalized eta quotients in the dissection of $R2(\zeta_7; q)$ in Theorem 2.2, we use the following. For N a positive integer and ρ an integer with $N \nmid \rho$, we let

$$f_{N,\rho}(\tau) = q^{\frac{(N-2\rho)^2}{8N}} (q^\rho, q^{N-\rho}, q^N; q^N)_\infty.$$

We note that $f_{N,\rho}(\tau) = f_{N,-\rho}(\tau) = f_{N,\rho+N}(\tau)$, and so Lemma 2.1 of [4] can be stated as

$$(4.6) \quad f_{N,\rho}(B\tau) = (-1)^{\rho\beta + \lfloor \frac{\rho\alpha}{N} \rfloor + \lfloor \frac{\rho}{N} \rfloor} \exp\left(\frac{\pi i \alpha \beta \rho^2}{N}\right) \nu(BN)^3 \sqrt{\gamma\tau + \delta} f_{N,\rho}(\tau),$$

for $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(N)$.

Proposition 4.6. *Suppose $c > 0$ is an integer and*

$$f(\tau) = \eta(4c^2\tau)^{r_0} \prod_{k=1}^{2c} f_{4c^2, ck}(\tau)^{r_k}.$$

If $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4c^2) \cap \Gamma_1(4c)$, then

$$f(B\tau) = (-1)^{(c\beta + \frac{(\alpha-1)}{4c} + \frac{\beta(\alpha-1)}{4})S + \beta T} \exp\left(\frac{\pi i \beta S}{4}\right) \nu(B_{4c^2})^{r_0 + 3R} (\gamma\tau + \delta)^{\frac{r_0 + R}{2}} f(\tau),$$

where

$$R = \sum_{i=1}^{2c} r_i, \quad S = \sum_{\substack{i=1, \\ (\text{mod } 2)}}^{2c} r_i, \quad T = \sum_{\substack{i=1, \\ (\text{mod } 4)}}^{2c} r_i.$$

Proof. By (4.6) we have

$$f_{4c^2, ck}(B\tau) = (-1)^{kc\beta + \lfloor \frac{kc\alpha}{4c^2} \rfloor + \lfloor \frac{kc}{4c^2} \rfloor} \exp\left(\frac{\pi i \alpha \beta (kc)^2}{4c^2}\right) \nu(B_{4c^2})^3 \sqrt{\gamma\tau + \delta} f_{4c^2, ck}(\tau).$$

Since $\alpha \equiv 1 \pmod{4c}$ and $1 \leq k \leq 2c$, we can write $\alpha = 1 + \frac{(\alpha-1)}{4c}4c$ to deduce that

$$\left\lfloor \frac{kc\alpha}{4c^2} \right\rfloor = \frac{(\alpha-1)k}{4c}.$$

Thus

$$(-1)^{kc\beta + \lfloor \frac{kc\alpha}{4c^2} \rfloor + \lfloor \frac{kc}{4c^2} \rfloor} = (-1)^{k(c\beta + \frac{(\alpha-1)}{4c})}.$$

Next we have that

$$\exp\left(\frac{\pi i \alpha \beta (kc)^2}{4c^2}\right) = (-1)^{k\frac{\beta(\alpha-1)}{4}} \exp\left(\frac{\pi i \beta k^2}{4}\right).$$

So in fact

$$f_{4c^2, ck}(B\tau) = (-1)^{k(c\beta + \frac{(\alpha-1)}{4c} + \frac{\beta(\alpha-1)}{4})} \exp\left(\frac{\pi i \beta k^2}{4}\right) \nu(B_{4c^2})^3 \sqrt{\gamma\tau + \delta} f_{4c^2, ck}(\tau).$$

Thus

$$\begin{aligned}
 f(B\tau) &= \left(\prod_{k=1}^{2c} (-1)^{r_k} k^{(c\beta + \frac{(\alpha-1)}{4c} + \frac{\beta(\alpha-1)}{4})} e^{\frac{\pi i \beta k^2 r_k}{4}} \right) \nu(B_{4c^2})^{r_0+3R} (\gamma\tau + \delta)^{\frac{r_0+R}{2}} f(\tau) \\
 &= (-1)^{(c\beta + \frac{(\alpha-1)}{4c} + \frac{\beta(\alpha-1)}{4})S} (-1)^{\beta T} e^{\frac{\pi i \beta S}{4}} \nu(B_{4c^2})^{r_0+3R} (\gamma\tau + \delta)^{\frac{r_0+R}{2}} f(\tau).
 \end{aligned}$$

□

Corollary 4.7. *Suppose $c > 0$ is an odd integer,*

$$f(\tau) = \eta(4c^2\tau)^{r_0} \prod_{k=1}^{2c} f_{4c^2, ck}(\tau)^{r_k},$$

and

$$R = \sum_{i=1}^{2c} r_i, \quad S = \sum_{\substack{i=1, \\ i \equiv 1 \pmod{2}}}^{2c} r_i, \quad T = \sum_{\substack{i=1, \\ i \equiv 2 \pmod{4}}}^{2c} r_i.$$

Suppose that $r_0 + R = 1$, $R \equiv -2 \pmod{12}$, $S \equiv -1 \pmod{8}$, and $T \equiv 0 \pmod{2}$.

If $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4c^2) \cap \Gamma_1(4c)$, then

$$f(B\tau) = (-1)^{\beta + \frac{(\alpha-1)}{4} + \frac{\beta(\alpha-1)}{4}} \exp\left(-\frac{\pi i \beta}{4}\right) \nu(B_4)^{-3} \sqrt{\gamma\tau + \delta} f(\tau).$$

Furthermore, if

$$g(\tau) = \frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)} f(\tau), \quad h(\tau) = \frac{\eta(4c^2\tau)\eta(c^2\tau)}{\eta(2c^2\tau)} f(\tau),$$

then $g(\tau)$ and $h(\tau)$ are weight 1 weakly holomorphic modular forms on $\Gamma_0(4c^2) \cap \Gamma_1(4c)$.

Proof. The only simplification for the transformation formula of $f(\tau)$ that does not follow immediately from Proposition 4.6 and the conditions on R , S , and T is that $\nu(B_{4c^2})^{r_0+3R} = \nu(B_4)^{-3}$. However this follows from the fact that since c is odd, we have that $\frac{\eta(4c^2\tau)^3}{\eta(4\tau)^3}$ is a modular function on $\Gamma_0(4c^2)$ by Theorem 1.64 of [19]. From this we have that $\nu(B_{4c^2})^3 = \nu(B_4)^3$. But we also know $\nu(B_{4c^2})^3$ is a 24th root of unity and $r_0 + 3R = 1 + 2R \equiv -3 \pmod{24}$, so that $\nu(B_{4c^2})^{r_0+3R} = \nu(B_{4c^2})^{-3} = \nu(B_4)^{-3}$.

Nothing that $\frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)} = f_{4,1}(\tau)$, we use (4.6) to deduce that, on $\Gamma_1(4)$,

$$\frac{\nu(B_2)}{\nu(B)\nu(B_4)} = (-1)^{\beta + \frac{(\alpha-1)}{4} + \frac{\beta(\alpha-1)}{4}} \exp\left(\frac{\pi i \beta}{4}\right) \nu(B_4)^3.$$

As such, $g(\tau)$ has trivial multiplier on $\Gamma_0(4c^2) \cap \Gamma_1(4c)$. As noted in the proof of Proposition 4.4, on $\Gamma_0(4c^2)$ we have that

$$\frac{\nu(B_2)}{\nu(B)\nu(B_4)} = \frac{\nu(B_{2c^2})}{\nu(B_{c^2})\nu(B_{4c^2})},$$

and so $h(\tau)$ also has trivial multiplier. □

We could easily deduce an analogous result for even c , however we omit this as we would not make use of it.

5. ORDERS AT CUSPS

We recall for a modular form f on some congruence subgroup Γ , the invariant order at $i\infty$ is the least power of q appearing in the q -expansion at $i\infty$. That is, if

$$f(\tau) = \sum_{m=m_0}^{\infty} a(m) \exp(2\pi i\tau m/N)$$

and $a(m_0) \neq 0$, then the invariant order is m_0/N . For a modular form, this is always a finite number. For a harmonic weak Maass form, we cannot take such an expansion, however we can do so for the holomorphic part. As such we define the invariant order at $i\infty$ of a harmonic weak Maass form to be the least exponent of q appearing in the holomorphic part. If f is a modular form of weight k , $\gcd(\alpha, \gamma) = 1$, and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, then the invariant order of f at the cusp $\frac{\alpha}{\gamma}$ is the invariant order at $i\infty$ of $j(B, \tau)^{-k} f(B\tau)$. In the same fashion, if f is a harmonic weak Maass form, then we define the invariant order of f at the cusp $\frac{\alpha}{\gamma}$ to be the invariant order at $i\infty$ of $j(B, \tau)^{-k} f(B\tau)$. This value is independent of the choice of B .

To compute bounds on the orders of the functions handled in this article, we use Corollary 6.2 from [13]. For a real number w , we let $\lfloor w \rfloor$ denote the greatest integer less than or equal to w , and let $\{w\}$ denote the fractional part of w . That is, $w = \lfloor w \rfloor + \{w\}$, $\lfloor w \rfloor \in \mathbb{Z}$, and $0 \leq \{w\} < 1$.

Proposition 5.1. *If $f(\tau) = q^\alpha \tilde{\mu}(u_1\tau + u_2, w_1\tau + w_2; \tau)$ is a harmonic weak Maass form with $u_i, w_i \in \mathbb{R}$, then the lowest power of q appearing in the expansion of the holomorphic part of $f(\tau)$ is at least $\alpha + \tilde{\nu}(u_1, w_1)$, where*

$$\begin{aligned} \tilde{\nu}(u, w) &= \frac{1}{2} (\lfloor u \rfloor - \lfloor w \rfloor)^2 + (\lfloor u \rfloor - \lfloor w \rfloor) (\{u\} - \{w\}) + k(u, w), \\ k(u, w) &= \begin{cases} \nu(\{u\}, \{w\}) & \text{if } \{u\} - \{w\} \neq \pm \frac{1}{2}, \\ \min(\frac{1}{8}, \nu(\{u\}, \{w\})) & \text{if } \{u\} - \{w\} = \pm \frac{1}{2}, \end{cases} \\ \nu(u, w) &= \begin{cases} \frac{u+w}{2} - \frac{1}{8} & \text{if } u + w \leq 1, \\ \frac{7}{8} - \frac{u+w}{2} & \text{if } u + w > 1. \end{cases} \end{aligned}$$

In the proofs of Propositions 4.2 and 4.5 we computed $\widetilde{\mathcal{R}2}(a, c; B\tau)$ and $\widetilde{S}(k, c; B\tau)$, for $B \in \text{SL}_2(\mathbb{Z})$, in terms of $\tilde{\mu}$. With these formulas, along with Proposition 5.1, we can immediately read off bounds on the orders at cusps.

Proposition 5.2. *Suppose a and c are integers, $c > 0$, and $c \nmid 2a$. If α and γ are integers with $\gcd(\alpha, \gamma) = 1$, then the invariant order of $\widetilde{\mathcal{R}2}(a, c; \tau)$ at the cusp $\frac{\alpha}{\gamma}$ is at least*

$$\frac{g^2}{4} \left(-\frac{m^2}{32} - \frac{2a^2r^2}{c^2} + \min \left(-\frac{amr}{2c} + \tilde{\nu} \left(\frac{2ar}{c}, -\frac{m}{4} \right), \frac{amr}{2c} + \tilde{\nu} \left(\frac{2ar}{c}, \frac{m}{4} \right) \right) \right),$$

where $g = \gcd(4, \gamma)$, $m = 4\alpha/g$, and $r = \gamma/g$.

Proposition 5.3. *Suppose k and c are integers, $c > 0$, and $1+4k-c \not\equiv 0 \pmod{4c}$. If α and γ are integers with $\gcd(\alpha, \gamma) = 1$, then the invariant order of $\tilde{\mathcal{S}}(k, c; \tau)$ at the cusp $\frac{\alpha}{\gamma}$ is at least*

$$\frac{g^2}{4c^2} \left(-\frac{m^2(1+4k-2c)^2}{32c^2} + \frac{mr(1+4k-2c)(c-1)}{8c} - \frac{r^2(c-1)^2}{8} \right. \\ \left. + \tilde{\nu} \left(\frac{(1+4k-c)m}{4c}, \frac{m}{4} + \frac{(c-1)r}{2} \right) \right),$$

where $g = \gcd(4c^2, \gamma)$, $m = 4c^2\alpha/g$, and $r = \gamma/g$.

The following is Lemma 3.2 of [4].

Proposition 5.4. *Suppose α and γ are integers with $\gcd(\alpha, \gamma) = 1$. Then the invariant order of $f_{N,\rho}(\tau)$ at the cusp $\frac{\alpha}{\gamma}$ is $\frac{\gcd(N,\gamma)^2}{2N} \left(\left\{ \frac{\alpha\rho}{\gcd(N,\gamma)} \right\} - \frac{1}{2} \right)^2$.*

6. PROOF OF THEOREM 2.2

To begin we give the definitions of the remaining $R2_i(q)$ from the statement of Theorem 2.2. We let equations (6.1)–(6.6) hold.

$$(6.1) \quad R2_1(q) = \frac{A(2,1,1)qJ_0J_5J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_7J_9J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-1,0,-1)q^{-2}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}J_{14}^2} \\ + \frac{A(-8,1,-3)q^{-3}J_0J_7J_8J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(10,-7,2)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{13}^2J_{14}^2} + \frac{A(1,-18,-3)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}^2J_{14}^2} \\ + \frac{A(0,-1,0)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}^2J_{14}^2} + \frac{A(-5,11,0)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2} + \frac{A(4,10,3)q^{-3}J_0J_6J_8J_9^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\ + \frac{A(0,-1,0)q^{-3}J_0J_6J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,2,1)q^{-3}J_0J_6J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{13}^2J_{14}^2} + \frac{A(-1,1,0)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{12}J_{13}^2J_{14}^2} \\ + \frac{A(8,-15,1)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}^2J_{14}^2} + \frac{A(-2,-17,-1)q^{-3}J_0J_6J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}^2J_{14}^2} + \frac{A(-5,19,1)q^{-3}J_0J_6J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\ + \frac{A(4,-11,0)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,9,2)q^{-3}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{14}^2} + \frac{A(-11,8,-4)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{13}J_{14}^2} \\ + \frac{A(-3,-2,-2)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}^2J_{13}^2} + \frac{A(5,-11,0)q^{-3}J_0J_6J_7J_8J_{14}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2} + \frac{A(4,2,3)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9^2J_{10}J_{11}J_{12}J_{13}J_{14}^2} \\ + \frac{A(3,-11,0)q^{-2}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(3,3,1)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8^2J_9^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-3,-3,-1)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}^2J_{12}^2J_{13}^2} \\ + \frac{A(3,2,1)q^{-3}J_0J_6J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-3,-2,-2)q^{-3}J_0J_6J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{13}^2J_{14}^2} + \frac{A(-3,11,0)q^{-2}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\ + \frac{A(-5,11,0)q^{-3}J_0J_6^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-4,16,1)q^{-3}J_0J_6^2J_8J_{11}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_9J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-1,5,0)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{12}^2J_{13}^2J_{14}^2} \\ + \frac{A(4,-16,-1)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}^2J_{12}^2J_{14}^2} + \frac{A(-2,-1,-1)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}^2J_{13}J_{14}^2} + \frac{A(3,3,1)q^{-3}J_0J_6^2J_9J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8^2J_9^2J_{11}^2J_{12}J_{13}^2J_{14}^2} \\ + \frac{A(0,-1,1)q^{-3}J_0J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{12}J_{13}^2J_{14}^2} + \frac{A(-2,-4,-1)q^{-3}J_0J_6^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}^2J_{13}^2J_{14}^2} + \frac{A(3,2,2)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2} \\ + \frac{A(-1,3,-1)q^{-3}J_0J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(3,3,1)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9^2J_{11}^2J_{12}^2J_{13}J_{14}^2} + \frac{A(1,0,1)q^{-2}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9^2J_{10}^2J_{11}^2J_{12}J_{14}^2} \\ + \frac{A(-1,0,-1)q^{-3}J_0J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}^2J_{14}^2} + \frac{A(0,-1,0)q^{-3}J_0J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-1,-1,-1)J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2},$$

$$\begin{aligned}
(6.2) \quad R_{22}(q) = & \frac{A(1,1,0)qJ_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-1,-1,-1)J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(5,-1,5)q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(0,2,2)q^{-3}J_0J_7^2J_8J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-1,-1,-1)q^{-1}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(-12,-6,-16)q^{-3}J_0J_7^2J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{11}^2J_{12}^2J_{13}^2J_{14}} + \frac{A(1,-4,1)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}J_{13}J_{14}^2} + \frac{A(13,8,14)q^{-3}J_0J_7^2J_8^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{13}^2J_{14}^2} \\
& + \frac{A(3,0,6)q^{-3}J_0J_6J_8^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(2,4,1)q^{-3}J_0J_6J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(3,5,4)q^{-3}J_0J_6J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-7,-2,-11)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(4,-5,9)q^{-3}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}J_{13}J_{14}^2} + \frac{A(-11,1,-13)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(-9,-5,-9)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}J_{13}J_{14}} + \frac{A(0,-1,1)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}^2J_{11}^2J_{13}^2J_{14}} + \frac{A(13,8,14)q^{-2}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(-6,-4,-5)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}} + \frac{A(-3,-2,-2)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}^2J_{13}^2J_{14}^2} + \frac{A(10,9,7)q^{-3}J_0J_6J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(3,1,5)q^{-3}J_0J_6J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2} + \frac{A(-11,-6,-14)q^{-2}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(5,7,6)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(3,2,4)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}^2J_{12}J_{13}J_{14}} + \frac{A(2,3,1)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{12}J_{13}J_{14}} + \frac{A(-4,-7,-5)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-4,-1,-6)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}} + \frac{A(-1,1,0)q^{-3}J_0J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}J_{13}J_{14}^2} + \frac{A(-1,-6,0)q^{-3}J_0J_6^2J_8^2J_{11}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_9J_{10}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(1,-2,1)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-3,-2,-2)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(3,2,2)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(3,2,2)q^{-3}J_0J_6^2J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}^2J_{12}^2J_{14}} + \frac{A(0,-1,1)q^{-3}J_0J_6^2J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{10}^2J_{11}^2J_{13}^2J_{14}^2} + \frac{A(0,1,-1)q^{-3}J_0J_6^2J_7J_{14}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{10}J_{11}^2J_{12}^2J_{13}^2} \\
& + \frac{A(1,1,1)q^{-1}J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}} + \frac{A(-1,1,-2)q^{-3}J_0J_8J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{11}^2J_{12}^2J_{13}^2J_{14}} + \frac{A(-1,-1,0)q^{-3}J_0J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{11}^2J_{12}J_{13}^2J_{14}} \\
& + \frac{A(2,1,0)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}J_{14}} + \frac{A(-2,-2,0)q^{-2}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}^2},
\end{aligned}$$

$$\begin{aligned}
(6.3) \quad R_{23}(q) = & \frac{A(0,-1,0)q^{-2}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-9,-4,-7)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(2,2,1)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-2,0,-2)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(3,4,2)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(5,0,2)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-1,-3,0)q^{-3}J_0J_6J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-1,0,-2)q^{-3}J_0J_6J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(5,2,4)q^{-3}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{12}^2J_{13}^2J_{14}^2} + \frac{A(5,4,5)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-4,-5,0)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}J_{13}J_{14}} \\
& + \frac{A(-3,-3,-1)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(10,7,6)q^{-3}J_0J_6J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{13}^2J_{14}^2} + \frac{A(5,3,3)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(-5,-2,-4)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}J_{13}^2} + \frac{A(3,2,2)q^{-2}J_0J_6J_7^2J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,3,0)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(0,4,0)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{12}^2J_{13}^2J_{14}^2} + \frac{A(9,13,4)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(0,-4,0)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}^2J_{11}^2J_{12}J_{13}J_{14}^2} \\
& + \frac{A(-1,1,0)q^{-3}J_0J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(0,1,-1)q^{-3}J_0J_6^2J_9J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-7,-13,-2)q^{-3}J_0J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}J_{13}^2J_{14}^2} \\
& + \frac{A(-12,-11,-6)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{13}^2J_{14}^2} + \frac{A(4,1,2)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{12}J_{13}^2} + \frac{A(-3,-2,-2)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-2,1,-2)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{13}^2J_{14}^2} + \frac{A(-10,-12,-5)q^{-2}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(3,3,1)q^{-3}J_0J_6^2J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(7,11,3)q^{-2}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(0,-1,0)q^{-3}J_0J_7J_8^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6J_9J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-4,-2,-2)qJ_0J_5J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-1,1,0)q^{-3}J_0J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5J_7J_{11}^2J_{12}J_{13}^2J_{14}^2} + \frac{A(-2,-5,-1)q^{-3}J_0J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,3,0)q^{-3}J_0J_8^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5J_{10}^2J_{11}^2J_{13}^2J_{14}^2} \\
& + \frac{A(1,0,0)J_0J_6^2J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}^2},
\end{aligned}$$

$$\begin{aligned}
(6.4) \quad R_{24}(q) = & A(0, 1, -1)q^{-\frac{4}{7}}S(1, 7, \tau/7) + A(0, 1, -1)q^{-\frac{4}{7}}S(2, 7, \tau/7) \\
& + \frac{A(-4, 3, -5)qJ_0}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{11}^2 J_{13}^2} + \frac{A(7, -4, 7)q^2 J_0 J_{12}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10}^2 J_{11}^2 J_{13}^2} + \frac{A(-15, 11, -19)q^2 J_0}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{11} J_{13} J_{14}^2} \\
& + \frac{A(-1, 1, 0)q^3 J_0 J_{10}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8^2 J_9^2 J_{11}^2 J_{13}^2 J_{14}} + \frac{A(5, 13, -3)q^2 J_0 J_{12}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8^2 J_9 J_{11}^2 J_{13}^2 J_{14}^2} + \frac{A(5, -7, 8)q^2 J_0}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{12}^2 J_{13}^2 J_{14}^2} \\
& + \frac{A(15, -10, 16)qJ_0 J_{10}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{11} J_{13} J_{14}} + \frac{A(6, -2, 6)qJ_0 J_{10} J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{12}^2 J_{13}^2} + \frac{A(-13, -1, -10)q^2 J_0 J_{10}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{11}^2 J_{13}^2 J_{14}^2} \\
& + \frac{A(-3, 4, -4)J_0 J_{12}^2 J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{10}^2 J_{11}^2 J_{13}^2} + \frac{A(-5, 2, -4)q^2 J_0}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8^2 J_9 J_{11}^2 J_{13}^2 J_{14}} + \frac{A(-2, 12, -9)qJ_0 J_{12}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{13}^2 J_{14}^2} \\
& + \frac{A(-1, 1, 0)q^2 J_0 J_{10}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8^2 J_9^2 J_{11}^2 J_{12}^2 J_{13}^2} + \frac{A(3, -4, 4)q^3 J_0}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11}^2 J_{12}^2 J_{13}^2} + \frac{A(3, -4, 4)J_0 J_{14}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11}^2 J_{12}^2 J_{13}^2 J_{14}^2} \\
& + \frac{A(0, -3, 1)qJ_0 J_{12}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{10}^2 J_{13}^2} + \frac{A(4, 0, 3)q^5 J_0 J_8}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(-7, 0, -5)q^2 J_0 J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8^2 J_9^2 J_{10} J_{11}^2 J_{13}^2} \\
& + \frac{A(-13, 5, -12)qJ_0 J_{10}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9 J_{11}^2 J_{13}^2} + \frac{A(-3, -2, -2)q^2 J_0 J_{10} J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8^2 J_9 J_{11}^2 J_{12}^2 J_{13}^2} + \frac{A(0, 1, -1)q^{-1} J_0 J_{12}^2 J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{10}^2 J_{11}^2 J_{13}^2} \\
& + \frac{A(-4, 9, -9)q^2 J_0}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{11}^2 J_{12}^2 J_{13}^2} + \frac{A(2, 12, -6)q^3 J_0 J_{12}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11}^2 J_{13}^2 J_{14}^2} + \frac{A(-6, 13, -14)q^2 J_0}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11}^2 J_{12}^2 J_{13}^2} \\
& + \frac{A(0, 1, -1)q^2 J_0 J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8^2 J_9^2 J_{10}^2 J_{11}^2 J_{13}^2} + \frac{A(-4, 0, -3)q^4 J_0 J_{10}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11}^2 J_{12}^2 J_{13}^2 J_{14}^2} + \frac{A(-3, -1, -3)q^3 J_0 J_{10}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11}^2 J_{12}^2 J_{13}^2 J_{14}^2} \\
& + \frac{A(4, 1, 2)qJ_0 J_{12}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7^2 J_8^2 J_9^2 J_{10}^2 J_{11}^2 J_{14}} + \frac{A(7, 0, 5)q^4 J_0 J_8^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12}^2 J_{13}^2 J_{14}} \\
& + \frac{A(15, -11, 19)q^2 J_0 J_{10}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{12}^2 J_{13}^2 J_{14}^2} + \frac{A(7, -5, 8)q^3 J_0 J_8}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13} J_{14}} \\
& + \frac{A(2, -6, 5)qJ_0 J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11} J_{12} J_{13}^2} + \frac{A(3, -1, 3)qJ_0 J_{14}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2} \\
& + \frac{A(0, -3, 1)J_0 J_{14}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{13}^2} + \frac{A(-14, 0, -10)q^3 J_0 J_{10}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11}^2 J_{12}^2 J_{13}^2 J_{14}^2} \\
& + \frac{A(1, -1, 0)qJ_0 J_{14}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11}^2 J_{12}^2 J_{13}^2} + \frac{A(-7, -1, -4)qJ_0 J_{12}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{13}^2 J_{14}} \\
& + \frac{A(3, -4, 4)q^3 J_0 J_6 J_{14}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12}^2 J_{13}^2} + \frac{A(5, -12, 11)q^2 J_0 J_{12}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{13} J_{14}} \\
& + \frac{A(-3, 1, -3)q^2 J_0 J_{14}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12}^2 J_{13}^2} + \frac{A(-11, 5, -9)qJ_0 J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11} J_{12} J_{13}} \\
& + \frac{A(8, -8, 11)q^2 J_0 J_{14}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12}^2 J_{13}^2} + \frac{A(4, -1, 4)q^2 J_0 J_{10}}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{11} J_{12} J_{13}^2 J_{14}} \\
& + \frac{A(5, -12, 11)q^2 J_0}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{13} J_{14}} + \frac{A(7, 0, 5)qJ_0 J_{14}^2}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10}^2 J_{11}^2 J_{12} J_{13} J_{14}} \\
& + \frac{A(1, 5, -2)q^2 J_0 J_8}{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10} J_{11} J_{12} J_{13}} ,
\end{aligned}$$

$$\begin{aligned}
(6.5) \quad R_{25}(q) = & \frac{A(-2, -1, -2)J_0 J_5 J_6 J_8^2}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6^2 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(5, 2, 3)qJ_0 J_5^2 J_6^2 J_8^2}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} \\
& + \frac{A(21, 15, 5)q^{-3} J_0 J_7^2 J_8^2}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(-13, -8, -2)q^{-3} J_0 J_6 J_8^2 J_9}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(-4, 0, -2)q^{-3} J_0 J_6 J_7 J_8 J_{10}}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{11}^2 J_{12} J_{13}^2 J_{14}^2} \\
& + \frac{A(-4, -6, -1)q^{-3} J_0 J_6 J_7 J_8}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{11}^2 J_{12}^2 J_{13}^2 J_{14}} + \frac{A(23, 13, 9)q^{-3} J_0 J_6 J_7 J_8^2}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(-25, -15, -6)q^{-3} J_0 J_6 J_7 J_8}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} \\
& + \frac{A(-22, -17, -9)q^{-3} J_0 J_6 J_7 J_8}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{11}^2 J_{12} J_{13} J_{14}} + \frac{A(-13, -6, -6)q^{-3} J_0 J_6 J_7 J_8}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13} J_{14}} + \frac{A(-5, -2, -3)q^{-3} J_0 J_6 J_8 J_9}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{11}^2 J_{12} J_{13}^2 J_{14}^2} \\
& + \frac{A(-2, -1, -2)q^{-3} J_0 J_6 J_8 J_9^2}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(9, 6, 3)q^{-3} J_0 J_6^2 J_9 J_{10}^2}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(-21, -11, -7)q^{-3} J_0 J_6^2 J_8}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{11}^2 J_{12} J_{13}^2 J_{14}^2} \\
& + \frac{A(9, 8, 2)q^{-3} J_0 J_6^2 J_8}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(-8, -6, -1)q^{-3} J_0 J_6^2 J_8}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(15, 10, 5)q^{-3} J_0 J_6^2 J_8^2 J_{11}}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} \\
& + \frac{A(11, 3, 5)q^{-3} J_0 J_6^2 J_{12}^2}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(3, 2, 2)q^{-3} J_0 J_6^2 J_7 J_{10}^2}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(10, 10, 2)q^{-3} J_0 J_6^2 J_7}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} \\
& + \frac{A(-15, -14, -4)q^{-3} J_0 J_6^2 J_7}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(19, 10, 7)q^{-3} J_0 J_6^2 J_7}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2} + \frac{A(4, 1, 0)q^{-3} J_0 J_6^2 J_7 J_8}{J_1^2 J_2^2 J_3^2 J_4^2 J_5^2 J_6 J_7 J_8 J_9 J_{10} J_{11}^2 J_{12} J_{13}^2 J_{14}^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{A(0,1,2)q^{-3}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2} + \frac{A(-7,-8,-2)q^{-2}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(6,4,1)q^{-3}J_0J_6^2J_7^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-9,-7,-2)q^{-3}J_0J_6^2J_7^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(2,2,1)q^{-3}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(9,7,2)q^{-3}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2} \\
& + \frac{A(6,4,1)q^{-3}J_0J_6^2J_7^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-6,-4,-1)q^{-3}J_0J_6^2J_7^2J_{14}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2} + \frac{A(7,8,2)q^{-2}J_0J_6^2J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2} \\
& + \frac{A(-4,-4,0)q^{-3}J_0J_8J_9^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(11,8,4)q^{-3}J_0J_7J_8J_9^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-4,-1,-2)q^{-3}J_0J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(1,1,0)q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2} + \frac{A(1,-1,0)q^{-3}J_0J_6J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2} + \frac{A(-5,-2,-3)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2}, \\
(6.6) \quad R2_6(q) &= A(-1,1,0)q^{-\frac{6}{7}}S(4,7,\tau/7) + A(1,-1,0)q^{-\frac{6}{7}}S(6,7,\tau/7) \\
& + \frac{A(-30,4,-15)q^{-1}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(10,2,5)q^{-1}J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-22,3,-5)q^{-1}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(7,-1,1)q^{-1}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(34,-7,21)q^{-1}J_0J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(27,4,11)q^{-1}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(6,-1,0)q^{-1}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-3,-2,-2)q^{-1}J_0J_6^2J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-12,2,-3)q^{-1}J_0J_5J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(17,-5,7)J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-3,0,-3)q^{-1}J_0J_5J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-17,1,-10)q^{-1}J_0J_5J_6J_8J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-8,0,-6)q^{-1}J_0J_5J_6J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(2,0,2)q^{-1}J_0J_5J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-46,0,-21)q^{-1}J_0J_5J_6J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-21,-2,-10)q^{-1}J_0J_5J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-18,-2,-7)q^{-1}J_0J_5J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(5,2,5)q^{-1}J_0J_5J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(21,2,10)q^{-1}J_0J_5J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,-1,0)q^{-1}J_0J_5J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(14,1,7)q^{-1}J_0J_6^2J_7J_{12}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(11,-2,3)q^{-1}J_0J_5J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-20,3,-10)J_0J_5J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(3,2,2)q^{-1}J_0J_5J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(29,-1,12)q^{-1}J_0J_5J_6^2J_{12}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(0,-1,1)q^{-1}J_0J_5J_6^2J_{14}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(10,2,5)J_0J_6^2J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(29,-2,13)q^{-1}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,0,0)q^{-1}J_0J_5^2J_6J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-7,0,-3)J_0J_6^2J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-29,2,-13)q^{-1}J_0J_6^2J_6J_{12}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(7,0,3)q^{-1}J_0J_6^2J_6^2J_{12}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(2,-1,1)q^{-1}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-1,1,0)q^{-1}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(5,1,3)q^{-1}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(5,-2,2)q^{-1}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-2,-2,-3)q^{-1}J_0J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-1,1,0)q^{-1}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(7,0,3)q^{-1}J_0J_5^2J_8J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-5,-5,-4)q^{-1}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,0,0)q^{-1}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-11,4,-1)q^{-1}J_0J_6J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-1,-3,-3)q^{-1}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(3,3,1)q^{-1}J_0J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-2,-2,-1)qJ_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(3,2,3)q^2J_0J_5J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6^2J_7^2J_8^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2}.
\end{aligned}$$

Due to the length of the identities, we perform the calculations in Maple and only sketch the proof here. We note that while the calculations are quite long in terms of their statements, they are quite quick to compute. A Maple file containing calculations relevant to proving Theorem 2.2 can be found in the publications section of the author's website, <http://people.oregonstate.edu/~jennichr/#research>.

We multiply both sides of the identity in Theorem 2.2 by $q^{-\frac{1}{8}}$. In doing so, we find that the exponents simplify correctly to give an identity between $R2(1,7;\tau)$, various $S(k,7;\tau)$, and various quotients of $f_{196,7k}(\tau)$. When then multiply both sides by $\frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)}$ and take the difference of both sides; we call this difference $LHS - RHS$. We are to show $0 = LHS - RHS$. Upon inspection we find that the resulting generalized eta quotients are all modular forms of weight 1 on $\Gamma_0(196) \cap \Gamma_1(28)$ by

Corollary 4.7, and more importantly the roots of unity have been chosen correctly so that part (3) of Theorem 2.1 applies to give that the remaining terms also constitute a weight 1 modular form on $\Gamma_0(196) \cap \Gamma_1(28)$. We then choose one of the generalized eta quotients, call it g_1 , so that $\frac{LHS-RHS}{g_1}$ is a modular function on $\Gamma_0(196) \cap \Gamma_1(28)$. We then verify that $0 = \frac{LHS-RHS}{g_1}$ according to the valence formula.

To state the valence formula, we need to recall the concept of the order at a point with respect to a subgroup. Suppose f is a modular function on some congruence subgroup $\Gamma \subset \text{SL}_2(\mathbb{Z})$. Suppose $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. We then have a cusp $\zeta = B(\infty) = \frac{\alpha}{\gamma}$. We let $\text{ord}(f; \zeta)$ denote the invariant order of f at ζ . We define the width of ζ with respect to Γ as $\text{width}_\Gamma(\zeta) := w$, where w is the least positive integer such that $B \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} B^{-1} \in \Gamma$. We then define the order of f at ζ with respect to Γ as $\text{ORD}_\Gamma(f; \zeta) = \text{ord}(f; \zeta) \text{width}_\Gamma(\zeta)$. For $z \in \mathcal{H}$ we let $\text{ord}(f; z)$ denote the order of f at z as a meromorphic function. We then define the order of f at z with respect to Γ as $\text{ORD}_\Gamma(f; z) = \text{ord}(f; z)/m$, where m is the order of z as a fixed point of Γ (so $m = 1, 2$, or 3).

The valence formula, for modular functions, can be stated as follows. Suppose f is a modular function that is not the zero function, and suppose $\mathcal{D} \subset \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ is a fundamental domain for the action of Γ on \mathcal{H} along with a complete set of inequivalent cusps for the action. Then

$$\sum_{\zeta \in \mathcal{D}} \text{ORD}_\Gamma(f; \zeta) = 0.$$

A complete set of inequivalent cusps, along with their widths, for $\Gamma_0(196) \cap \Gamma_1(28)$ is given in Table 1. We let \mathcal{D} denote these cusps along with a fundamental region of the action of Γ .

We note $\frac{LHS-RHS}{g_1}$ has no poles in \mathcal{H} , but it may have zeros in \mathcal{H} . We take a lower bound on the orders at the noninfinite cusps by taking the minimum order of each of the individual summands, which we compute with Propositions 5.2, 5.3, and 5.4. This lower bound yields

$$\sum_{\zeta \in \mathcal{D}} \text{ORD}_\Gamma \left(\frac{LHS - RHS}{g_1}; \zeta \right) \geq \text{ord} \left(\frac{LHS - RHS}{g_1}, \infty \right) - 522.$$

However, we can expand $\frac{LHS-RHS}{g_1}$ as a series in q and find the coefficients of $\frac{LHS-RHS}{g_1}$ are zero to at least q^{523} . Thus

$$\sum_{\zeta \in \mathcal{D}} \text{ORD}_\Gamma \left(\frac{LHS - RHS}{g_1}; \zeta \right) \geq 1,$$

and so $\frac{LHS-RHS}{g_1}$ must be identically zero by the valence formula. This establishes Theorem 2.2.

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TABLE 1

0	$\frac{1}{28}$	$\frac{3}{80}$	$\frac{2}{53}$	$\frac{1}{26}$	$\frac{1}{77}$	$\frac{3}{77}$	$\frac{2}{51}$	$\frac{1}{24}$	$\frac{2}{47}$	$\frac{2}{70}$	$\frac{3}{45}$	$\frac{5}{112}$	$\frac{1}{22}$	$\frac{1}{21}$	$\frac{1}{20}$	$\frac{5}{98}$	$\frac{3}{56}$	$\frac{1}{18}$	$\frac{2}{35}$	$\frac{1}{16}$
196	1	49	196	98	4	4	196	49	196	2	2	196	98	4	49	2	1	98	4	49
1	$\frac{1}{14}$	$\frac{5}{63}$	$\frac{1}{12}$	$\frac{3}{35}$	$\frac{3}{91}$	$\frac{8}{91}$	$\frac{5}{56}$	$\frac{2}{21}$	$\frac{11}{112}$	$\frac{5}{49}$	$\frac{8}{77}$	$\frac{3}{28}$	$\frac{5}{42}$	$\frac{11}{91}$	$\frac{6}{49}$	$\frac{1}{8}$	$\frac{8}{63}$	$\frac{9}{70}$	$\frac{12}{91}$	$\frac{1}{7}$
196	2	4	49	4	4	4	1	4	1	4	4	1	2	4	4	49	4	2	4	4
12	$\frac{10}{63}$	$\frac{9}{56}$	$\frac{6}{35}$	$\frac{11}{63}$	$\frac{16}{91}$	$\frac{11}{63}$	$\frac{5}{28}$	$\frac{13}{70}$	$\frac{4}{21}$	$\frac{10}{49}$	$\frac{13}{63}$	$\frac{22}{105}$	$\frac{3}{14}$	$\frac{11}{49}$	$\frac{19}{84}$	$\frac{13}{56}$	$\frac{18}{77}$	$\frac{5}{21}$	$\frac{20}{77}$	$\frac{51}{196}$
77	4	4	4	4	4	4	1	2	4	4	4	4	4	4	1	56	4	4	4	1
15	$\frac{2}{56}$	$\frac{25}{84}$	$\frac{23}{77}$	$\frac{17}{56}$	$\frac{13}{42}$	$\frac{13}{42}$	$\frac{11}{35}$	$\frac{9}{28}$	$\frac{19}{56}$	$\frac{31}{91}$	$\frac{67}{196}$	$\frac{12}{35}$	$\frac{29}{84}$	$\frac{69}{196}$	$\frac{5}{14}$	$\frac{75}{196}$	$\frac{11}{28}$	$\frac{17}{42}$	$\frac{23}{56}$	$\frac{81}{196}$
56	7	84	77	56	42	42	35	28	56	91	196	35	84	196	14	196	28	42	56	196
1	4	1	4	1	2	4	4	1	1	4	1	4	1	1	2	1	1	2	1	1
29	$\frac{3}{7}$	$\frac{43}{98}$	$\frac{25}{56}$	$\frac{45}{98}$	$\frac{13}{28}$	$\frac{13}{28}$	$\frac{10}{21}$	$\frac{27}{56}$	$\frac{41}{84}$	$\frac{43}{84}$	$\frac{29}{56}$	$\frac{15}{28}$	$\frac{23}{42}$	$\frac{4}{7}$	$\frac{37}{63}$	$\frac{29}{49}$	$\frac{25}{42}$	$\frac{17}{28}$	$\frac{13}{21}$	$\frac{22}{35}$
70	2	4	2	1	2	1	4	1	1	1	1	1	2	4	4	4	4	2	1	4
2	4	2	1	2	1	1	4	1	1	1	1	1	2	4	4	4	2	1	4	4
9	$\frac{55}{84}$	$\frac{19}{28}$	$\frac{29}{42}$	$\frac{39}{56}$	$\frac{59}{84}$	$\frac{59}{84}$	$\frac{5}{7}$	$\frac{37}{49}$	$\frac{53}{70}$	$\frac{65}{84}$	$\frac{11}{14}$	$\frac{23}{28}$	$\frac{6}{7}$	$\frac{61}{70}$	$\frac{25}{28}$	$\frac{101}{112}$	$\frac{13}{14}$	$\frac{107}{112}$	$\frac{27}{28}$	∞
14	84	28	42	56	84	84	7	49	70	84	14	28	7	70	28	112	14	112	28	1
2	1	1	2	1	1	1	4	4	2	1	2	1	4	2	1	1	2	1	1	1

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DEPARTMENT OF MATHEMATICS, OREGON STATE UNIVERSITY, CORVALLIS, OREGON 97331
 Email address: chrisjenningsshaffer@gmail.com