

TRANSFORMATION PROPERTIES FOR DYSON'S RANK FUNCTION

F. G. GARVAN

ABSTRACT. At the 1987 Ramanujan Centenary meeting Dyson asked for a coherent group-theoretical structure for Ramanujan's mock theta functions analogous to Hecke's theory of modular forms. Many of Ramanujan's mock theta functions can be written in terms of $R(\zeta, q)$, where $R(z, q)$ is the two-variable generating function of Dyson's rank function and ζ is a root of unity. Building on earlier work of Watson, Zwegers, Gordon, and McIntosh, and motivated by Dyson's question, Bringmann, Ono, and Rhoades studied transformation properties of $R(\zeta, q)$. In this paper we strengthen and extend the results of Bringmann, Rhoades, and Ono, and the later work of Ahlgren and Treeneer. As an application we give a new proof of Dyson's rank conjecture and show that Ramanujan's Dyson rank identity modulo 5 from the Lost Notebook has an analogue for all primes greater than 3. The proof of this analogue was inspired by recent work of Jennings-Shaffer on overpartition rank differences mod 7.

1. INTRODUCTION

Let $p(n)$ denote the number of partitions of n . There are many known congruences for the partition function. The simplest and most famous were found and proved by Ramanujan:

$$\begin{aligned}p(5n + 4) &\equiv 0 \pmod{5}, \\p(7n + 5) &\equiv 0 \pmod{7}, \\p(11n + 6) &\equiv 0 \pmod{11}.\end{aligned}$$

In 1944, Dyson [19] conjectured striking combinatorial interpretations of the first two congruences. He defined the rank of a partition as the largest part minus the number of parts and conjectured that the rank mod 5 divided the partitions of $5n + 4$ into 5 equal classes and that the rank mod 7 divided the partitions of $7n + 5$ into 7 equal classes. He conjectured the existence of a partition statistic he called the *crank* which would likewise divide the partitions of $11n + 6$ into 11 equal classes. Dyson's mod 5 and 7 rank conjectures were proved by Atkin and Swinnerton-Dyer [9]. The mod 11 crank conjecture was solved by the author and Andrews [5].

Received by the editors June 17, 2016, and, in revised form, February 7, 2017.

2010 *Mathematics Subject Classification*. Primary 11F37, 11P82; Secondary 05A19, 11B65, 11F11, 11P83, 11P84, 33D15.

Key words and phrases. Dyson's rank function, Maass forms, mock theta functions, partitions, Mordell integral.

The author was supported in part by a grant from the Simons Foundation (#318714). A preliminary version of this paper was first given on May 10, 2015, at the International Conference on Orthogonal Polynomials and q -series at the University of Central Florida, Orlando, in honour of Mourad Ismail's 70th birthday.

Let $N(m, n)$ denote the number of partitions of n with rank m . We let $R(z, q)$ denote the two-variable generating function for the Dyson rank function so that

$$R(z, q) = \sum_{n=0}^{\infty} \sum_m N(m, n) z^m q^n.$$

Here and throughout the paper we use the convention that any sum \sum_m is a sum over all integers $m \in \mathbb{Z}$, unless otherwise stated.

Also throughout this paper we will use the standard q -notation:

$$\begin{aligned} (a; q)_{\infty} &= \prod_{k=0}^{\infty} (1 - aq^k), \\ (a; q)_n &= \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}, \\ (a_1, a_2, \dots, a_j; q)_{\infty} &= (a_1; q)_{\infty} (a_2; q)_{\infty} \dots (a_j; q)_{\infty}, \\ (a_1, a_2, \dots, a_j; q)_n &= (a_1; q)_n (a_2; q)_n \dots (a_j; q)_n. \end{aligned}$$

We have the following identities for the rank generating function $R(z, q)$:

$$\begin{aligned} (1.1) \quad R(z, q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq, z^{-1}q; q)_n} \\ (1.2) \quad &= \frac{1}{(q; q)_{\infty}} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1 + q^n) (1 - z) (1 - z^{-1})}{(1 - zq^n) (1 - z^{-1}q^n)} q^{\frac{1}{2}n(3n+1)} \right). \end{aligned}$$

See [21, eqs. (7.2), (7.6)].

Let $N(r, t, n)$ denote the number of partitions of n with rank congruent to $r \pmod t$, and let $\zeta_p = \exp(2\pi i/p)$. Then

$$(1.3) \quad R(\zeta_p, q) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{p-1} N(k, p, n) \zeta_p^k \right) q^n.$$

We restate the following.

Dyson’s Rank Conjecture 1.1 (1944). *For all nonnegative integers n ,*

$$(1.4) \quad N(0, 5, 5n + 4) = N(1, 5, 5n + 4) = \dots = N(4, 5, 5n + 4) = \frac{1}{5}p(5n + 4),$$

$$(1.5) \quad N(0, 7, 7n + 5) = N(1, 7, 7n + 5) = \dots = N(6, 7, 7n + 5) = \frac{1}{7}p(7n + 5).$$

Dyson’s rank conjecture was first proved by Atkin and Swinnerton-Dyer [9]. As noted in [21], [22], it can be shown that Dyson’s mod 5 rank conjecture (1.4) follows from an identity in Ramanujan’s Lost Notebook [40, p. 20], [4, eq. (2.1.17)]. We let ζ_5 be a primitive 5th root of unity. Then

$$\begin{aligned} (1.6) \quad R(\zeta_5, q) &= A(q^5) + (\zeta_5 + \zeta_5^{-1} - 2) \phi(q^5) + q B(q^5) + (\zeta_5 + \zeta_5^{-1}) q^2 C(q^5) \\ &\quad - (\zeta_5 + \zeta_5^{-1}) q^3 \left\{ D(q^5) - (\zeta_5^2 + \zeta_5^{-2} - 2) \frac{\psi(q^5)}{q^5} \right\}, \end{aligned}$$

where

$$\begin{aligned} A(q) &= \frac{(q^2, q^3, q^5; q^5)_{\infty}}{(q, q^4; q^5)_{\infty}^2}, & B(q) &= \frac{(q^5; q^5)_{\infty}}{(q, q^4; q^5)_{\infty}}, & C(q) &= \frac{(q^5; q^5)_{\infty}}{(q^2, q^3; q^5)_{\infty}}, \\ D(q) &= \frac{(q, q^4, q^5; q^5)_{\infty}}{(q^2, q^3; q^5)_{\infty}^2}, \end{aligned}$$

and

$$\phi(q) = -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q; q^5)_{n+1}(q^4; q^5)_n}, \quad \psi(q) = -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2; q^5)_{n+1}(q^3; q^5)_n}.$$

By multiplying by an appropriate power of q and substituting $q = \exp(2\pi iz)$, we recognize the functions $A(q)$, $B(q)$, $C(q)$, $D(q)$ as being modular forms. In fact, we can rewrite Ramanujan's identity (1.6) in terms of generalized eta-products:

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad q = \exp(2\pi iz),$$

and

$$(1.7) \quad \eta_{N,k}(z) = q^{\frac{N}{2}P_2(k/N)} \prod_{\substack{m>0 \\ m \equiv \pm k \pmod{N}}} (1 - q^m),$$

where $z \in \mathfrak{h}$, $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second periodic Bernoulli polynomial and $\{t\} = t - [t]$ is the fractional part of t . Here as in Robins [41], $1 \leq N \nmid k$. We have

(1.8)

$$\begin{aligned} & q^{-\frac{1}{24}} (R(\zeta_5, q) - (\zeta_5 + \zeta_5^4 - 2) \phi(q^5) + (1 + 2\zeta_5 + 2\zeta_5^4) q^{-2} \psi(q^5)) \\ &= \frac{\eta(25z) \eta_{5,2}(5z)}{\eta_{5,1}(5z)^2} + \frac{\eta(25z)}{\eta_{5,1}(5z)} + (\zeta_5 + \zeta_5^4) \frac{\eta(25z)}{\eta_{5,2}(5z)} - (\zeta_5 + \zeta_5^4) \frac{\eta(25z) \eta_{5,1}(5z)}{\eta_{5,2}(5z)^2}. \end{aligned}$$

Equation (1.6), or equivalently (1.8), gives the 5-dissection of the q -series expansion of $R(\zeta_5, q)$. We observe that the function on the right side of (1.8) is a weakly holomorphic modular form (with multiplier) of weight $\frac{1}{2}$ on the group $\Gamma_0(25) \cap \Gamma_1(5)$.

In Theorem 1.2 below we generalize the analogue of Ramanujan's result (1.8) to all primes $p > 3$. For $p > 3$ prime and $1 \leq a \leq \frac{1}{2}(p - 1)$ define

$$(1.9) \quad \Phi_{p,a}(q) := \begin{cases} \sum_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a; q^p)_{n+1}(q^{p-a}; q^p)_n}, & \text{if } 0 < 6a < p, \\ -1 + \sum_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a; q^p)_{n+1}(q^{p-a}; q^p)_n}, & \text{if } p < 6a < 3p, \end{cases}$$

and

(1.10)

$$\begin{aligned} \mathcal{R}_p(z) := & q^{-\frac{1}{24}} R(\zeta_p, q) - \chi_{12}(p) \sum_{a=1}^{\frac{1}{2}(p-1)} (-1)^a \left(\zeta_p^{3a+\frac{1}{2}(p+1)} + \zeta_p^{-3a-\frac{1}{2}(p+1)} \right. \\ & \left. - \zeta_p^{3a+\frac{1}{2}(p-1)} - \zeta_p^{-3a-\frac{1}{2}(p-1)} \right) q^{\frac{a}{2}(p-3a)-\frac{p^2}{24}} \Phi_{p,a}(q^p), \end{aligned}$$

where

$$(1.11) \quad \chi_{12}(n) := \left(\frac{12}{n} \right) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise,} \end{cases}$$

and as usual $q = \exp(2\pi iz)$ with $\Im(z) > 0$. One of our main results is the following.

Theorem 1.2. *Let $p > 3$ be prime. Then the function*

$$\eta(p^2 z) \mathcal{R}_p(z)$$

is a weakly holomorphic modular form of weight 1 on the group $\Gamma_0(p^2) \cap \Gamma_1(p)$.

Remark. The form of this result is suggested by Ramanujan’s identity (1.8). The proof of this result uses the theory of weak harmonic Maass forms and was inspired by a recent result of Jennings-Shaffer [30] on overpartition rank differences mod 7. Jennings-Shaffer was the first to prove a result of this type using the theory of weak harmonic Maass forms.

As a consequence we have the following.

Corollary 1.3. *Let $p > 3$ be prime and let $s_p = \frac{1}{24}(p^2 - 1)$. Then the function*

$$\prod_{n=1}^{\infty} (1 - q^{pn}) \sum_{n=\lceil \frac{1}{p}(s_p) \rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k, p, pn - s_p) \zeta_p^k \right) q^n$$

is a weakly holomorphic modular form of weight 1 on the group $\Gamma_1(p)$.

Remark. Corollary 1.3 is the case $m = 0$ of Proposition 6.2(i) below.

In Ramanujan’s identity (1.6) we see that the coefficient of q^{5n+4} is zero, and this implies Dyson’s rank 5 conjecture (1.4) in view of (1.3). The analogue of (1.6) for the prime 7 does not appear in Ramanujan’s Lost Notebook, although Ramanujan wrote the left side [40, p. 19] and wrote some of the functions involved in coded form on [40, p. 71]. The complete identity is given by Andrews and Berndt [4, eq. (2.1.42)]. As noted by the author [22, Theorem 4, p. 20] Ramanujan’s identity (1.6) is actually equivalent to Atkin and Swinnerton-Dyer’s [9, Theorem 4, p. 101]. In Section 6 we give a new proof of Dyson’s Conjecture 1.1 as well as some of Atkin and Hussain’s [8] results on the rank mod 11 and O’Brien’s [36] results on the rank mod 13. Until this paper, Atkin and Swinnerton-Dyer’s proof has remained the only known proof of Dyson’s Rank Conjecture 1.1.

We now explain the connection between Ramanujan’s mock theta functions and Dyson’s rank functions. In [40, p. 20] Ramanujan gives four identities for some mock theta functions of order 5. For example,

$$\chi_0(q) = 2 + 3\phi(q) - A(q),$$

where

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1}; q)_{n+1}}.$$

The other three identities correspond to [6, eqs. (3.2), (3.6), (3.7)]. These identities are the Mock Theta Conjectures which were later proved by Hickerson [26]. Thus Ramanujan’s mock theta functions of order 5 are related to $R(z, q)$ when $z = \zeta_5$. All of Ramanujan’s third order mock theta functions can also be written in terms of $R(z, q)$. For example,

$$f(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} = R(-1, q),$$

$$\omega(q) = \sum_{n \geq 0} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}^2} = g(q; q^2),$$

where

$$(1.12) \quad g(x, q) = x^{-1} \left(-1 + \frac{1}{1-x} R(x, q) \right).$$

Hickerson and Mortenson [28, Section 5] give a nice catalogue of these and other known analogous identities for all of Ramanujan's mock theta functions. These known results are originally due to Watson [43], Andrews [3], Hickerson [27], Andrews and Hickerson [7], Choi [16], [17], Berndt and Chan [10], McIntosh [35], and Gordon and McIntosh [23].

The main emphasis of this paper is transformations for Dyson's rank function $R(\zeta, q)$ and thus for Ramanujan's mock theta functions. We begin with a quote from Freeman Dyson.

The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future.

Freeman Dyson, 1987
Ramanujan Centenary Conference

In this paper we continue previous work on Dyson's challenge. First we describe the genesis of this work. Watson [43] found transformation formulas for the third order functions in terms of Mordell integrals. For example,

$$q^{-1/24} f(q) = 2\sqrt{\frac{2\pi}{\alpha}} q_1^{4/3} \omega(q_1^2) + 4\sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty e^{-3\alpha x^2/2} \frac{\sinh \alpha x}{\sinh 3\alpha x/2} dx,$$

where

$$q = \exp(-\alpha), \quad q_1 = \exp(-\beta), \quad \alpha\beta = \pi^2.$$

The derivation of transformation formulas for the other Ramanujan mock theta functions was carried out in a series of papers by Gordon and McIntosh. A summary of these results can be found in [25].

The big breakthrough came when Zwegers [46], [47] realised how Ramanujan's mock theta functions occurred as the holomorphic part of certain real analytic modular forms. An example for the third order functions $f(q)$, $\omega(q)$, $\omega(-q)$ is given in the following.

Theorem 1.4 (Zwegers [46]). *Define $F(z) = (f_0, f_1, f_2)^T$ by*

$$f_0(z) = q^{-1/24} f(q), \quad f_1(z) = 2q^{1/3} \omega(q^{1/2}), \quad f_2(z) = 2q^{1/3} \omega(-q^{1/2}),$$

where $q = \exp(2\pi iz)$, $z \in \mathfrak{h}$. Define

$$G(z) = 2i\sqrt{3} \int_{-\bar{z}}^{i\infty} \frac{(g_1(\tau), g_0(\tau), -g_2(\tau))^T}{\sqrt{-i(\tau+z)}} d\tau,$$

where

$$\begin{aligned}
 g_0(z) &= \sum_n (-1)^n \left(n + \frac{1}{3}\right) q^{3(n+1/3)^2/2}, \\
 g_1(z) &= -\sum_n \left(n + \frac{1}{6}\right) q^{3(n+1/6)^2/2}, \\
 g_2(z) &= \sum_n \left(n + \frac{1}{3}\right) q^{3(n+1/3)^2/2}.
 \end{aligned}$$

Then

$$H(z) = F(z) - G(z)$$

is a (vector valued) real analytic modular form of weight 1/2 satisfying

$$\begin{aligned}
 H(z+1) &= \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(z), \\
 \frac{1}{\sqrt{-iz}} H\left(\frac{-1}{z}\right) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z).
 \end{aligned}$$

Bringmann and Ono [13], [12], [37] extended this theorem to $R(\zeta, q)$ when ζ is a more general root of unity.

Theorem 1.5 (Bringmann and Ono, Theorem 3.4 [13]). *Let c be an odd positive integer and let $0 < a < c$. Then $q^{-1/24}R(\zeta_c^a, q)$ is the holomorphic part of a component of a vector valued weak Maass form of weight 1/2 for the full modular group $SL_2(\mathbb{Z})$.*

Remark. By component of a vector valued weak Maass form we mean an element of the set V_c defined in [13, p. 441]. We give this result explicitly below in Corollary 3.2. The definition of a vector valued Maass form of weight k for $SL_2(\mathbb{Z})$ is given in [13, p. 440].

Bringmann and Ono used this theorem to obtain a subgroup of the full modular group on which $q^{-1/24}R(\zeta, q)$ is the holomorphic part a weak Maass form of weight 1/2. We state their theorem in the case where ζ is a p th root of unity.

Theorem 1.6 (Bringmann and Ono [13]). *Let $p > 3$ be prime, and let $0 < a < p$. Define*

$$\begin{aligned}
 \theta(\alpha, \beta; z) &:= \sum_{n \equiv \alpha \pmod{\beta}} nq^{n^2} \quad q = \exp(2\pi iz), \\
 \Theta\left(\frac{a}{p}; z\right) &:= \sum_{m \pmod{2p}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{p}\right) \theta\left(6m+1, 12p, \frac{z}{24}\right), \\
 S_1\left(\frac{a}{p}; z\right) &:= -i \sin\left(\frac{\pi a}{p}\right) 2\sqrt{2}p \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{p}; 24p^2\tau\right) d\tau}{\sqrt{-i(\tau+z)}}.
 \end{aligned}$$

Then

$$D\left(\frac{a}{p}; z\right) := q^{-p^2} R\left(\zeta_p^a; q^{24p^2}\right) - S_1\left(\frac{a}{p}; z\right)$$

is a weak Maass form of weight 1/2 on $\Gamma_1(576 \cdot p^4)$.

Bringmann, Ono, and Rhoades [14] applied this theorem to prove the following.

Theorem 1.7 (Bringmann, Ono, and Rhoades, Theorem 1.1 [14]). *Suppose $t \geq 5$ is prime, $0 \leq r_1, r_2 < t$, and $0 \leq d < t$. Then the following are true:*

(i) *If $(\frac{1-24d}{t}) = -1$, then*

$$\sum_{n=0}^{\infty} (N(r_1, t, tn + d) - N(r_2, t, tn + d))q^{24(tn+d)-1}$$

is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(576 \cdot t^6)$.

(ii) *Suppose that $(\frac{1-24d}{t}) = 1$. If $r_1, r_2 \not\equiv \frac{1}{2}(\pm 1 \pm \alpha) \pmod{t}$, where α is any integer for which $0 \leq \alpha < 2t$ and $1 - 24d \equiv \alpha^2 \pmod{2t}$, then*

$$\sum_{n=0}^{\infty} (N(r_1, t, tn + d) - N(r_2, t, tn + d))q^{24(tn+d)-1}$$

is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(576 \cdot t^6)$.

Ahlgren and Treneer [1] strengthened this theorem to include the case $24d \equiv 1 \pmod{t}$.

Theorem 1.8 (Ahlgren and Treneer [1]). *Suppose $p \geq 5$ is prime and that $0 \leq r < t$. Then*

$$\sum_{n=1}^{\infty} \left(N\left(r, p, \frac{pn+1}{24}\right) - \frac{1}{t^p} \left(\frac{pn+1}{24}\right) \right) q^n$$

is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(576 \cdot p^4)$.

Remark. Ahlgren and Treneer [1] also derived many analogous results when t is not prime.

In this paper we strengthen many of the results of Ahlgren, Bringmann, Ono, Rhoades, and Treneer [1], [13], [14]. In particular:

- (i) We make Theorem 1.5 (Bringmann and Ono [13, Theorem 3.4]) more explicit. See Theorem 3.1 below. In this theorem we have corrected Bringmann and Ono's definition of the functions $\mathcal{G}_2(\frac{a}{c}; z)$ and $\mathcal{G}_2(a, b, c; z)$.
- (ii) For the case $p > 3$ prime we strengthen Theorem 1.6 (Bringmann and Ono [13, Theorem 1.2]). We show that the group $\Gamma_1(576 \cdot p^4)$ can be enlarged to $\Gamma_0(p^2) \cap \Gamma_1(p)$ but with a simple multiplier. See Corollary 4.2. We prove more. In Theorem 4.1 we basically determine the image of Bringmann and Ono's function $D\left(\frac{a}{p}; \frac{z}{24p^2}\right)$ under the group $\Gamma_0(p)$.
- (iii) We strengthen Theorem 1.7(i) (Bringmann, Ono, and Rhoades [14, Theorem 1.1(i)]) and Theorem 1.8 (Ahlgren and Treneer [1, Theorem 1.6, p. 271]). In both cases we enlarge the group $\Gamma_1(576 \cdot p^4)$ to the group $\Gamma_1(p)$ but with a simple multiplier. See Corollary 6.5 below.
- (iv) We take a different approach to Bringmann, Ono, and Rhoades's Theorem 1.7(ii). We handle the residue classes $(\frac{1-24d}{t}) = 1$ in a different way. For these residue classes we show the definition of the functions involved may be adjusted to make them holomorphic modular forms. In particular see equation (6.5), Theorem 6.3, and its Corollary 6.5.

The paper is organized as follows. In Section 2 we go over Bringmann and Ono’s transformation results for various Lambert series and Mordell integrals. We make all results explicit and correct some errors. In Section 3 we describe Bringmann and Ono’s vector valued Maass form of weight $\frac{1}{2}$, correcting some definitions and making all results explicit. In Section 4 we derive a Maass form multiplier for Bringmann and Ono’s function $\mathcal{G}_1\left(\frac{\ell}{p}; z\right)$ for the group $\Gamma_0(p)$ when $p > 3$ is prime. In Section 5 we prove Theorem 1.2, which extends Ramanujan’s Dyson rank identity (1.6) or (1.8) to all primes $p > 3$. In Section 6 we give a new proof of the Dyson Rank Conjecture (1.4)–(1.5). We extend Ahlgren, Bringmann, Ono, Rhoades, and Treener’s results, mentioned in (ii)–(iv) above, for all primes $p > 3$. We also use our method to derive Atkin and Hussain’s [8] results for the rank mod 11 and O’Brien’s [36] results for rank mod 13. In Section 7 we mention other approaches and methods of proof of Bringmann and Ono’s work, particularly Zagier’s [45] simple proof and Hickerson and Mortenson’s [29] more elementary approach.

2. PRELIMINARIES

Following [13] we define a number of functions. Suppose $0 < a < c$ are integers, and assume throughout that $q := \exp(2\pi iz)$. We define

$$M\left(\frac{a}{c}; z\right) := \frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}}}{1 - q^{n+\frac{a}{c}}} q^{\frac{3}{2}n(n+1)},$$

$$N\left(\frac{a}{c}; z\right) := \frac{1}{(q; q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1 + q^n) (2 - 2 \cos\left(\frac{2\pi a}{c}\right))}{1 - 2 \cos\left(\frac{2\pi a}{c}\right) q^n + q^{2n}} q^{\frac{1}{2}n(3n+1)} \right).$$

For integers $0 \leq a < c, 0 < b < c$ define

$$M(a, b, c; z) := \frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}}}{1 - \zeta_c^b q^{n+\frac{a}{c}}} q^{\frac{3}{2}n(n+1)},$$

where $\zeta_c := \exp(2\pi i/c)$. In addition for $\frac{b}{c} \notin \{0, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}\}$ define

$$k(b, c) := \begin{cases} 0 & \text{if } 0 < \frac{b}{c} < \frac{1}{6}, \\ 1 & \text{if } \frac{1}{6} < \frac{b}{c} < \frac{1}{2}, \\ 2 & \text{if } \frac{1}{2} < \frac{b}{c} < \frac{5}{6}, \\ 3 & \text{if } \frac{5}{6} < \frac{b}{c} < \frac{1}{6}, \end{cases}$$

and

$$N(a, b, c; z) := \frac{1}{(q; q)_\infty} \left(\frac{i\zeta_c^{-a} q^{\frac{b}{2c}}}{2(1 - \zeta_c^{-a} q^{\frac{b}{c}})} + \sum_{m=1}^{\infty} K(a, b, c, m; z) q^{\frac{m(3m+1)}{2}} \right),$$

where

$$K(a, b, c, m; z) := (-1)^m \frac{\sin\left(\frac{\pi a}{c} - \left(\frac{b}{c} + 2k(b, c)m\right)\pi z\right) + \sin\left(\frac{\pi a}{c} - \left(\frac{b}{c} - 2k(b, c)m\right)\pi z\right) q^m}{1 - 2 \cos\left(\frac{2\pi a}{c} - \frac{2\pi bz}{c}\right) q^m + q^{2m}}.$$

We need the following identities:

$$(2.1) \quad -1 + \frac{1}{1-z} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, z^{-1}q; q)_n} = \frac{z}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{3}{2}n(n+1)}}{1-zq^n}$$

and

$$(2.2) \quad \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{3}{2}n(n+1)}}{1-zq^n} = \frac{z}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^n}{1-zq^n} q^{\frac{3}{2}n(n+1)}.$$

Equation (2.1) is [21, eq. (7.10), p. 68], and (2.2) is an easy exercise. Replacing q by q^c and z by q^a in (2.1), (2.2) we find that

$$(2.3) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{q^{cn^2}}{(q^a, a^c)_{n+1} (q^{c-a}, q^c)_n} &= 1 + \frac{q^a}{(q^c, q^c)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{3c}{2}n(n+1)}}{1-q^{cn+a}} \\ &= 1 + q^a M\left(\frac{a}{c}; cz\right). \end{aligned}$$

By (1.2) we have

$$(2.4) \quad R(\zeta_c^a, q) = N\left(\frac{a}{c}; z\right).$$

Extending earlier work of Watson [43], Gordon and McIntosh [24], Bringmann and Ono [13] found a transformation formula for all these functions in terms of Mordell integrals. We define the following Mordell integrals:

$$J\left(\frac{a}{c}; \alpha\right) := \int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\left(\left(\frac{3a}{c}-2\right)\alpha x\right) + \cosh\left(\left(\frac{3a}{c}-1\right)\alpha x\right)}{\cosh(3\alpha x/2)} dx$$

and

$$J(a, b, c; \alpha) := \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2 + 3\alpha x \frac{a}{c}} \frac{(\zeta_c^b e^{-\alpha x} + \zeta_c^{2b} e^{-2\alpha x})}{\cosh\left(3\alpha \frac{x}{2} - 3\pi i \frac{b}{c}\right)} dx.$$

Following [13] we adjust the definitions of the N - and M -functions so that the transformation formulas are tidy. We define

$$(2.5) \quad \mathcal{N}\left(\frac{a}{c}; z\right) := \csc\left(\frac{a\pi}{c}\right) q^{-\frac{1}{24}} N\left(\frac{a}{c}; z\right),$$

$$(2.6) \quad \mathcal{M}\left(\frac{a}{c}; z\right) := 2q^{\frac{3a}{2c}(1-\frac{a}{c})-\frac{1}{24}} M\left(\frac{a}{c}; z\right),$$

$$(2.7) \quad \mathcal{M}(a, b, c; z) := 2q^{\frac{3a}{2c}(1-\frac{a}{c})-\frac{1}{24}} M(a, b, c; z),$$

$$(2.8)$$

$$\mathcal{N}(a, b, c; z)$$

$$:= 4 \exp\left(-2\pi i \frac{a}{c} k(b, c) + 3\pi i \frac{b}{c} \left(\frac{2a}{c} - 1\right)\right) \zeta_c^{-b} q^{\frac{b}{c}k(b, c) - \frac{3b^2}{2c^2} - \frac{1}{24}} N(a, b, c; z).$$

We now restate Bringmann and Ono's [13, Theorem 2.3, p. 435] more explicitly.

Theorem 2.1. *Suppose that c is a positive odd integer and that a and b are integers for which $0 \leq a < c$ and $0 < b < c$.*

(1) *For $z \in \mathfrak{h}$ we have*

$$(2.9) \quad \mathcal{N}\left(\frac{a}{c}; z + 1\right) = \zeta_{24}^{-1} \mathcal{N}\left(\frac{a}{c}; z\right),$$

$$(2.10) \quad \mathcal{N}(a, b, c; z + 1) = \begin{cases} \zeta_{2c^2}^{3b^2} \zeta_{24}^{-1} \mathcal{N}(a - b, b, c; z) & \text{if } a \geq b, \\ -\zeta_{2c^2}^{3b^2} \zeta_c^{-3b} \zeta_{24}^{-1} \mathcal{N}(a - b + c, b, c; z) & \text{otherwise,} \end{cases}$$

$$(2.11) \quad \mathcal{M}\left(\frac{a}{c}; z + 1\right) = \zeta_{2c}^{5a} \zeta_{2c^2}^{-3a^2} \zeta_{24}^{-1} \mathcal{M}(a, a, c; z),$$

$$(2.12) \quad \mathcal{M}(a, b, c; z + 1) = \zeta_{2c}^{5a} \zeta_{2c^2}^{-3a^2} \zeta_{24}^{-1} \begin{cases} \mathcal{M}(a, a + b, c; z) & \text{if } a + b < c, \\ \mathcal{M}\left(\frac{a}{c}; z\right) & \text{if } a + b = c, \\ \mathcal{M}(a, a + b - c, c; z) & \text{otherwise,} \end{cases}$$

where a is assumed to be positive in the first and third formulas.

(2) *For $z \in \mathfrak{h}$ we have*

$$(2.13) \quad \frac{1}{\sqrt{-iz}} \mathcal{N}\left(\frac{a}{c}; -\frac{1}{z}\right) = \mathcal{M}\left(\frac{a}{c}; z\right) + 2\sqrt{3}\sqrt{-iz} J\left(\frac{a}{c}; -2\pi iz\right),$$

$$(2.14) \quad \frac{1}{\sqrt{-iz}} \mathcal{N}\left(a, b, c; -\frac{1}{z}\right) = \mathcal{M}(a, b, c; z) + \zeta_{2c}^{-5b} \sqrt{3}\sqrt{-iz} J(a, b, c; -2\pi iz),$$

$$(2.15) \quad \frac{1}{\sqrt{-iz}} \mathcal{M}\left(\frac{a}{c}; -\frac{1}{z}\right) = \mathcal{N}\left(\frac{a}{c}; z\right) - \frac{2\sqrt{3}i}{z} J\left(\frac{a}{c}; \frac{2\pi i}{z}\right),$$

$$(2.16) \quad \frac{1}{\sqrt{-iz}} \mathcal{M}\left(a, b, c; -\frac{1}{z}\right) = \mathcal{N}(a, b, c; z) - \zeta_{2c}^{-5b} \frac{\sqrt{3}i}{z} J\left(a, b, c; \frac{2\pi i}{z}\right),$$

where again a is assumed to be positive in the first and third formulas.

We will write each of the Mordell integrals as a period integral of a theta function. Before we can do this we need some results of Shimura [42]. For integers $0 \leq k < N$ we define

$$\tilde{\theta}(k, N; z) := \sum_{m=-\infty}^{\infty} (Nm + k) \exp\left(\frac{\pi iz}{N}(Nm + k)^2\right).$$

We note that this corresponds to $\theta(z; k, N, N, P)$ in Shimura's notation [42, eq. (2.0), p. 454] (with $n = 1$, $\nu = 1$, and $P(x) = x$). For integers $0 \leq a, b < c$ we define

$$\Theta_1(a, b, c; z)$$

$$:= \zeta_{c^2}^{3ab} \zeta_{2c}^{-a} \sum_{m=0}^{6c-1} (-1)^m \sin\left(\frac{\pi}{3}(2m + 1)\right) \exp\left(\frac{-2\pi ima}{c}\right) \tilde{\theta}(2mc - 6b + c, 12c^2; z)$$

and

$$\Theta_2(a, b, c; z) := \sum_{\ell=0}^{2c-1} \left((-1)^\ell \exp\left(\frac{-\pi i b}{c}(6\ell + 1)\right) \tilde{\theta}(6c\ell + 6a + c, 12c^2; z) \right. \\ \left. + (-1)^\ell \exp\left(\frac{-\pi i b}{c}(6\ell - 1)\right) \tilde{\theta}(6c\ell + 6a - c, 12c^2; z) \right).$$

An easy calculation gives

$$(2.17) \quad \Theta_1(a, b, c; z) = 6c \zeta_{c^2}^{3ab} \zeta_{2c}^{-a} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{n}{3} + \frac{1}{6} - \frac{b}{c} \right) \sin \left(\frac{\pi}{3}(2n+1) \right) \times \exp \left(\frac{-2\pi ina}{c} \right) \exp \left(3\pi iz \left(\frac{n}{3} + \frac{1}{6} - \frac{b}{c} \right)^2 \right).$$

We calculate the action of $SL_2(\mathbb{Z})$ on each of these theta functions.

Proposition 2.2. *For integers $0 \leq a, b < c$, and $z \in \mathfrak{h}$ we have*

$$(2.18) \quad \Theta_1(a, b, c; z+1) = \zeta_{2c^2}^{-3b^2} \zeta_{24} \Theta_1(a+b, b, c; z),$$

$$(2.19) \quad (-iz)^{-\frac{3}{2}} \Theta_1 \left(a, b, c; -\frac{1}{z} \right) = -\frac{\sqrt{3}i}{2} \Theta_2(a, b, c; z),$$

$$(2.20) \quad \Theta_2(a, b, c; z+1) = \zeta_{2c^2}^{3a^2} \zeta_{24} \Theta_2(a, b-a, c; z),$$

$$(2.21) \quad (-iz)^{-\frac{3}{2}} \Theta_2 \left(a, b, c; -\frac{1}{z} \right) = \frac{2\sqrt{3}i}{3} \Theta_1(a, b, c; z).$$

Proof. Transformations (2.18) and (2.20) are an easy calculation. By [42, eq. (2.4), p. 454] we have

$$\begin{aligned} & (-iz)^{-3/2} \tilde{\theta} \left(2mc - 6b + c, 12c^2; -\frac{1}{z} \right) \\ &= (-i) (12c^2)^{-1/2} \sum_{k \pmod{12c^2}} \exp \left(\frac{\pi ik}{6c^2} (2mc + c - 6b) \right) \tilde{\theta}(k, 12c^2; z). \end{aligned}$$

Therefore

$$\begin{aligned} & (-iz)^{-3/2} \Theta_1 \left(a, b, c; -\frac{1}{z} \right) \\ &= (-i) (12c^2)^{-1/2} \zeta_{c^2}^{3ab} \zeta_{2c}^{-a} \sum_{m \pmod{6c}} (-1)^m \sin \left(\frac{\pi}{3}(2m+1) \right) \exp \left(\frac{-2\pi ima}{c} \right) \\ & \quad \times \sum_{k \pmod{12c^2}} \exp \left(\frac{\pi ik}{6c^2} (2mc + c - 6b) \right) \tilde{\theta}(k, 12c^2; z) \\ &= \frac{(-i)}{2c\sqrt{3}} \zeta_{c^2}^{3ab} \zeta_{2c}^{-a} \sum_{k \pmod{12c^2}} \tilde{\theta}(k, 12c^2; z) \\ & \quad \times \sum_{m \pmod{6c}} (-1)^m \exp \left(2\pi i \left\{ \frac{k}{12c^2} (2mc + c - 6b) - m \frac{a}{c} \right\} \right) \\ & \quad \times \frac{1}{2i} \left(\exp \left(\frac{\pi i}{3}(2m+1) \right) - \exp \left(-\frac{\pi i}{3}(2m+1) \right) \right) \\ &= \frac{(-1)}{4c\sqrt{3}} \zeta_{c^2}^{3ab} \zeta_{2c}^{-a} \sum_{k \pmod{12c^2}} \exp \left(\frac{\pi ik}{6c^2} (c - 6b) \right) \tilde{\theta}(k, 12c^2; z) \end{aligned}$$

$$\begin{aligned}
 & \times \left(\exp\left(\frac{\pi i}{3}\right) \sum_{m \pmod{6c}} \exp\left(\frac{2\pi i}{6c}(k-6a+5c)m\right) \right. \\
 & \quad \left. - \exp\left(-\frac{\pi i}{3}\right) \sum_{m \pmod{6c}} \exp\left(\frac{2\pi i}{6c}(k-6a+c)m\right) \right) \\
 & = \frac{-\sqrt{3}}{2} \zeta_{c^2}^{3ab} \zeta_{2c}^{-a} \left(\sum_{\substack{k \pmod{12c^2} \\ k \equiv 6a-5c \pmod{6c}}} \exp\left(\frac{\pi i}{3} + \frac{\pi ik}{6c^2}(c-6b)\right) \tilde{\theta}(k, 12c^2; z) \right. \\
 & \quad \left. - \sum_{\substack{k \pmod{12c^2} \\ k \equiv 6a-c \pmod{6c}}} \exp\left(-\frac{\pi i}{3} + \frac{\pi ik}{6c^2}(c-6b)\right) \tilde{\theta}(k, 12c^2; z) \right).
 \end{aligned}$$

In the sum above we let $k = 6c\ell + 6a \pm c$ where $0 \leq \ell \leq 2c - 1$, so that

$$\pi i \left(\frac{k}{6c^2}(c-6b) \pm \frac{1}{3} \right) = \pi i \left(\frac{a}{c^2}(c-6b) \pm \frac{1}{2} + \ell - \frac{b}{c}(6\ell \pm 1) \right),$$

and we find that

$$\begin{aligned}
 & (-iz)^{-3/2} \Theta_1\left(a, b, c; -\frac{1}{z}\right) \\
 & = \frac{-i\sqrt{3}}{2} \sum_{\ell=0}^{2c-1} \left((-1)^\ell \exp\left(\frac{-\pi ib}{c}(6\ell+1)\right) \tilde{\theta}(6c\ell+6a+c, 12c^2; z) \right. \\
 & \quad \left. + (-1)^\ell \exp\left(\frac{-\pi ib}{c}(6\ell-1)\right) \tilde{\theta}(6c\ell+6a-c, 12c^2; z) \right) \\
 & = -\frac{\sqrt{3}i}{2} \Theta_2(a, b, c; z),
 \end{aligned}$$

which is transformation (2.19). Transformation (2.21) follows immediately from (2.19). □

In addition we need to define
(2.22)

$$\Theta_1\left(\frac{a}{c}; z\right) := \sum_{n=-\infty}^{\infty} (-1)^n (6n+1) \sin\left(\frac{\pi a(6n+1)}{c}\right) \exp\left(3\pi iz \left(n + \frac{1}{6}\right)^2\right).$$

This coincides with Bringmann and Ono’s function $\Theta\left(\frac{a}{c}; z\right)$, which is given in [13, eq. (1.6), p. 423]. An easy calculation gives

$$\Theta_1\left(\frac{a}{c}; z\right) = -\frac{i}{2c} \Theta_2(0, -a, c; z).$$

From Proposition 2.2 we have the following.

Corollary 2.3. *Let a, b, c , and z be as in Proposition 2.2. Then*

$$(2.23) \quad \Theta_1\left(\frac{a}{c}; z + 1\right) = \zeta_{24} \Theta_1\left(\frac{a}{c}; z\right),$$

$$(2.24) \quad (-iz)^{-\frac{3}{2}} \Theta_1\left(\frac{a}{c}; -\frac{1}{z}\right) = \frac{\sqrt{3}}{3c} \Theta_1(0, -a, c; z).$$

Next we define

$$\varepsilon_1\left(\frac{a}{c}; z\right) := \begin{cases} \frac{-2}{\sqrt{-iz}} \exp\left(\frac{3\pi i}{z} \left(\frac{a}{c} - \frac{1}{6}\right)^2\right) & \text{if } 0 < \frac{a}{c} < \frac{1}{6}, \\ 0 & \text{if } \frac{1}{6} < \frac{a}{c} < \frac{5}{6}, \\ \frac{-2}{\sqrt{-iz}} \exp\left(\frac{3\pi i}{z} \left(\frac{a}{c} - \frac{5}{6}\right)^2\right) & \text{if } \frac{5}{6} < \frac{a}{c} < 1, \end{cases}$$

$$\varepsilon_1(a, b, c; z) := \begin{cases} \frac{\zeta_{2c}^b}{\sqrt{-iz}} \exp\left(\frac{3\pi i}{z} \left(\frac{a}{c} - \frac{1}{6}\right)^2\right) & \text{if } 0 \leq \frac{a}{c} < \frac{1}{6}, \\ 0 & \text{if } \frac{1}{6} < \frac{a}{c} < \frac{5}{6}, \\ \frac{\zeta_{2c}^{5b}}{\sqrt{-iz}} \exp\left(\frac{3\pi i}{z} \left(\frac{a}{c} - \frac{5}{6}\right)^2\right) & \text{if } \frac{5}{6} < \frac{a}{c} < 1, \end{cases}$$

$$\varepsilon_2\left(\frac{a}{c}; z\right) := \begin{cases} 2 \exp\left(-3\pi iz \left(\frac{a}{c} - \frac{1}{6}\right)^2\right) & \text{if } 0 < \frac{a}{c} < \frac{1}{6}, \\ 0 & \text{if } \frac{1}{6} < \frac{a}{c} < \frac{5}{6}, \\ 2 \exp\left(-3\pi iz \left(\frac{a}{c} - \frac{5}{6}\right)^2\right) & \text{if } \frac{5}{6} < \frac{a}{c} < 1, \end{cases}$$

and

$$\varepsilon_2(a, b, c; z) := \begin{cases} 2\zeta_c^{-2b} \exp\left(-3\pi iz \left(\frac{a}{c} - \frac{1}{6}\right)^2\right) & \text{if } 0 \leq \frac{a}{c} < \frac{1}{6}, \\ 0 & \text{if } \frac{1}{6} < \frac{a}{c} < \frac{5}{6}, \\ 2 \exp\left(-3\pi iz \left(\frac{a}{c} - \frac{5}{6}\right)^2\right) & \text{if } \frac{5}{6} < \frac{a}{c} < 1. \end{cases}$$

We are now ready to express each of our Mordell integrals as a period integral of a theta function.

Theorem 2.4. *Let a, b, c be as in Theorem 2.1, and in addition let $a/c \notin \{\frac{1}{6}, \frac{5}{6}\}$. Then for $z \in \mathfrak{h}$ we have*

$$(2.25) \quad \frac{2\sqrt{3}}{iz} J\left(\frac{a}{c}; \frac{2\pi i}{z}\right) = \frac{i}{\sqrt{3}} \int_0^{i\infty} \frac{\Theta_1\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau+z)}} d\tau + \varepsilon_1\left(\frac{a}{c}; z\right),$$

$$(2.26) \quad 2\sqrt{3}\sqrt{-iz} J\left(\frac{a}{c}; -2\pi iz\right) = -\frac{i}{3c} \int_0^{i\infty} \frac{\Theta_1(0, -a, c; \tau)}{\sqrt{-i(\tau+z)}} d\tau + \varepsilon_2\left(\frac{a}{c}; z\right),$$

$$(2.27) \quad \frac{\sqrt{3}}{-2iz} J\left(a, b, c; \frac{2\pi i}{z}\right) = \frac{1}{6c} \int_0^{i\infty} \frac{\Theta_1(a, b, c; \tau)}{\sqrt{-i(\tau+z)}} d\tau + \varepsilon_1(a, b, c; z),$$

$$(2.28) \quad \zeta_{2c}^{-5b} \sqrt{3}\sqrt{-iz} J(a, b, c; -2\pi iz) = -\zeta_{2c}^{-5b} \frac{i\sqrt{3}}{6c} \int_0^{i\infty} \frac{\Theta_2(a, b, c; \tau)}{\sqrt{-i(\tau+z)}} d\tau + \varepsilon_2(a, b, c; z).$$

Remark. We have corrected the results of Bringmann and Ono [13, p. 441] by including the necessary correction factors ε_1 and ε_2 .

Proof. First we prove (2.25). Assume $0 < a < c$ are integers and $a/c \notin \{\frac{1}{6}, \frac{5}{6}\}$. We proceed as in the proof of [13, Lemma 3.2, pp. 436–437]. By analytic continuation we may assume that $z = it$ and $t > 0$. We find that

$$J\left(\frac{a}{c}; \frac{2\pi}{t}\right) = t \int_0^\infty e^{-3\pi tx^2} f(x) dx = t \int_{-\infty}^\infty e^{-3\pi tx^2} g_{11}(x) dx,$$

where

$$f(z) = \frac{\cosh\left(\left(3\frac{a}{c} - 2\right) 2\pi z\right) + \cosh\left(\left(3\frac{a}{c} - 1\right) 2\pi z\right)}{\cosh(3\pi z)},$$

$$g_{11}(z) = \frac{\exp\left(\left(3\frac{a}{c} - 2\right) 2\pi z\right) + \exp\left(\left(3\frac{a}{c} - 1\right) 2\pi z\right)}{2 \cosh(3\pi z)}.$$

We note that the $f(z)$ has poles at $z = z_n = -i(\frac{1}{6} + \frac{n}{3})$, where $n \in \mathbb{Z}$. We find:

pole	residue of $f(z)$
$z_{3n-1} = -i(-\frac{1}{6} + n)$	$\frac{(-1)^n \sin(\frac{\pi a}{c}(-6n+1))}{\pi i \sqrt{3}}$
$z_{3n} = -i(\frac{1}{6} + n)$	$\frac{(-1)^{n+1} \sin(\frac{\pi a}{c}(6n+1))}{\pi i \sqrt{3}}$
$z_{3n+1} = -i(\frac{1}{2} + n)$	0

Applying the Mittag-Leffler Theory [44, pp. 134–135], we have

$$f(z) = 2 + \sum_{n \in \mathbb{Z}}^* \left(\frac{i(-1)^n \sin\left(\frac{\pi a}{c}(6n+1)\right)}{\pi \sqrt{3}} \left(\frac{1}{z + i(n + \frac{1}{6})} - \frac{1}{i(n + \frac{1}{6})} \right) \right. \\ \left. + \frac{(-i)(-1)^n \sin\left(\frac{\pi a}{c}(-6n+1)\right)}{\pi \sqrt{3}} \left(\frac{1}{z + i(n - \frac{1}{6})} - \frac{1}{i(n - \frac{1}{6})} \right) \right)$$

for $z \notin (\pm\frac{1}{6} + \mathbb{Z})$ and assuming $\frac{1}{6} < \frac{a}{c} < \frac{5}{6}$. Here we assume that $\sum_{n \in \mathbb{Z}}^* = \lim_{N \rightarrow \infty} \sum_{n=-N}^N$. We note that the convergence is uniform on any compact subset of $\mathbb{C} \setminus -i(\pm\frac{1}{6} + \mathbb{Z})$. We must consider three cases.

Case 1.1. $\frac{1}{6} < \frac{a}{c} < \frac{5}{6}$. We have

$$f(z) = 2 + \frac{(-i)}{\pi \sqrt{3}} \sum_{n \in \mathbb{Z}}^* (-1)^n \sin\left(\frac{\pi a}{c}(6n+1)\right) \left(\left(\frac{1}{z - i(n + \frac{1}{6})} + \frac{1}{i(n + \frac{1}{6})} \right) \right. \\ \left. + \left(\frac{1}{-z - i(n + \frac{1}{6})} + \frac{1}{i(n + \frac{1}{6})} \right) \right).$$

Thus

$$\int_0^\infty e^{-3\pi tx^2} f(x) dx \\ = \int_0^\infty 2e^{-3\pi tx^2} dx - \frac{i}{\pi \sqrt{3}} \int_{-\infty}^\infty e^{-3\pi tx^2} \\ \times \left(\sum_{n \in \mathbb{Z}}^* (-1)^n \sin\left(\frac{\pi a}{c}(6n+1)\right) \left(\frac{1}{x - i(n + \frac{1}{6})} + \frac{1}{i(n + \frac{1}{6})} \right) \right) dx.$$

By absolute convergence on \mathbb{R} we have

$$\begin{aligned} & \int_0^\infty e^{-3\pi tx^2} f(x) dx \\ &= \int_{-\infty}^\infty e^{-3\pi tx^2} dx \\ & \quad - \frac{i}{\pi\sqrt{3}} \sum_{n \in \mathbb{Z}}^* (-1)^n \sin\left(\frac{\pi a}{c}(6n+1)\right) \int_{-\infty}^\infty e^{-3\pi tx^2} \left(\frac{1}{x-i(n+\frac{1}{6})} + \frac{1}{i(n+\frac{1}{6})}\right) dx \\ &= -\frac{i}{\pi\sqrt{3}} \sum_{n \in \mathbb{Z}}^* (-1)^n \sin\left(\frac{\pi a}{c}(6n+1)\right) \int_{-\infty}^\infty \frac{e^{-3\pi tx^2}}{x-i(n+\frac{1}{6})} dx, \end{aligned}$$

since

$$(2.29) \quad \sum_{n \in \mathbb{Z}}^* (-1)^n \frac{\sin\left(\frac{\pi a}{c}(6n+1)\right)}{i(n+\frac{1}{6})} = \pi\sqrt{3} \quad \text{for } \frac{1}{6} < \frac{a}{c} < \frac{5}{6}.$$

We leave (2.29) as an exercise for the reader. It can be proved using [47, Lemma 1.19, p. 19]. By using the identity [46, eq. (3.8), p. 274]

$$(2.30) \quad \int_{-\infty}^\infty \frac{e^{-\pi tx^2}}{x-is} dx = \pi is \int_0^\infty \frac{e^{-\pi us^2}}{\sqrt{u+t}} du \quad \text{for } s \in \mathbb{R} \setminus \{0\},$$

we find that

$$J\left(\frac{a}{c}; \frac{2\pi}{t}\right) = \frac{t}{6\sqrt{3}} \sum_{n \in \mathbb{Z}} (-1)^n (6n+1) \sin\left(\frac{\pi a}{c}(6n+1)\right) \int_0^\infty \frac{e^{-\pi u(n+\frac{1}{6})^2}}{\sqrt{u+3t}} du.$$

Letting $u = -3i\tau$ in the integral we find that

$$J\left(\frac{a}{c}; \frac{2\pi}{t}\right) = \frac{-it}{6} \sum_{n \in \mathbb{Z}} (-1)^n (6n+1) \sin\left(\frac{\pi a}{c}(6n+1)\right) \int_0^{i\infty} \frac{e^{3\pi i\tau(n+\frac{1}{6})^2}}{\sqrt{-i(\tau+it)}} d\tau.$$

Arguing as in the proof of [46, Lemma 3.3] we may interchange summation and integration to obtain

$$\begin{aligned} & J\left(\frac{a}{c}; \frac{2\pi i}{z}\right) \\ &= -\frac{z}{6} \int_0^{i\infty} \sum_{n \in \mathbb{Z}} (-1)^n (6n+1) \sin\left(\frac{\pi a}{c}(6n+1)\right) e^{3\pi i\tau(n+\frac{1}{6})^2} \frac{d\tau}{\sqrt{-i(\tau+z)}}, \end{aligned}$$

when $z = it$, for $t > 0$. Equation (2.25) follows in this case.

Case 1.2. $0 < \frac{a}{c} < \frac{1}{6}$. Observe that in this case

$$-4\pi x < \left(3\frac{a}{c} - 2\right) 2\pi x < -3\pi x \quad (\text{for } x > 0),$$

and the Mittag-Leffler Theory does not directly apply. We simply note that

$$\frac{(1 + e^{2y})}{\cosh y} = 2e^y$$

and find that

$$\int_{-\infty}^\infty e^{-3\pi tx^2} \frac{\exp\left(\left(3\frac{a}{c} - 2\right) 2\pi x\right)}{\cosh(3\pi x)} (1 + e^{6\pi x}) dx = \frac{2}{\sqrt{3t}} \exp\left(\frac{\pi}{12t} \left(6\frac{a}{c} - 1\right)^2\right).$$

Thus we have

$$\begin{aligned} \int_0^\infty e^{-3\pi tx^2} \frac{\cosh\left(\left(3\frac{a}{c}-2\right)2\pi x\right) + \cosh\left(\left(3\frac{a}{c}-1\right)2\pi x\right)}{\cosh(3\pi x)} dx &= \int_{-\infty}^\infty e^{-3\pi tx^2} g_{11}(x) dx \\ &= \frac{1}{\sqrt{3t}} \exp\left(\frac{\pi}{12t}\left(6\frac{a}{c}-1\right)^2\right) + \int_{-\infty}^\infty e^{-3\pi tx^2} g_{12}(x) dx, \end{aligned}$$

where

$$g_{12}(z) = \frac{\exp\left(\left(3\frac{a}{c}-1\right)2\pi z\right) - \exp\left(\left(3\frac{a}{c}+1\right)2\pi z\right)}{2 \cosh(3\pi z)}.$$

We observe that the function $g_{11}(z)$ from Case 1.1 and the function $g_{12}(z)$ have the same poles and the same residue at each pole. We note we may apply the Mittag-Leffler Theory to the function $g_{12}(z)$ since in this case

$$-2\pi x < \left(3\frac{a}{c}-1\right)2\pi x < -\pi x, \quad 2\pi x < \left(3\frac{a}{c}+1\right)2\pi x < 3\pi x \quad (\text{for } x > 0).$$

The remainder of the proof is analogous to Case 1.1.

Case 1.3. $\frac{5}{6} < \frac{a}{c} < 1$. The proof is analogous to Case 1.2. This time we have

$$\int_{-\infty}^\infty e^{-3\pi tx^2} \frac{\exp\left(\left(3\frac{a}{c}-1\right)2\pi x\right)}{\cosh(3\pi x)} (1 + e^{-6\pi x}) dx = \frac{2}{\sqrt{3t}} \exp\left(\frac{\pi}{12t}\left(6\frac{a}{c}-5\right)^2\right).$$

Thus we have

$$\begin{aligned} \int_0^\infty e^{-3\pi tx^2} \frac{\cosh\left(\left(3\frac{a}{c}-2\right)2\pi x\right) + \cosh\left(\left(3\frac{a}{c}-1\right)2\pi x\right)}{\cosh(3\pi x)} dx &= \int_{-\infty}^\infty e^{-3\pi tx^2} g_{11}(x) dx \\ &= \frac{1}{\sqrt{3t}} \exp\left(\frac{\pi}{12t}\left(6\frac{a}{c}-5\right)^2\right) + \int_{-\infty}^\infty e^{-3\pi tx^2} g_{13}(x) dx, \end{aligned}$$

where

$$g_{13}(z) = \frac{\exp\left(\left(3\frac{a}{c}-2\right)2\pi z\right) - \exp\left(\left(3\frac{a}{c}-4\right)2\pi z\right)}{2 \cosh(3\pi z)}.$$

We observe that the function $g_{11}(z)$ from Case 1.1 and the function $g_{13}(z)$ have the same poles and the same residue at each pole, and we may apply the Mittag-Leffler Theory to the function $g_{13}(z)$ since in this case

$$\pi x < \left(3\frac{a}{c}-2\right)2\pi x < 2\pi x, \quad -3\pi x < \left(3\frac{a}{c}-4\right)2\pi x < -2\pi x \quad (\text{for } x > 0).$$

Equation (2.26) follows easily from equations (2.24) and (2.25).

The proof of (2.27) is analogous to that of (2.25). This time we assume that $0 \leq a < c$, $0 < b < c$ are integers, and $a/c \notin \{1/6, 5/6\}$. Again by analytic continuation we may assume $z = it$ and $t > 0$. We find that

$$J\left(a, b, c; \frac{2\pi}{t}\right) = t \int_{-\infty}^\infty e^{-3\pi tx^2} g_{21}(x) dx,$$

where

$$g_{21}(z) = \frac{\zeta_c^b e^{-2\pi z} + \zeta_c^{2b} e^{-4\pi z}}{\cosh\left(3\pi z - 3\pi i \frac{b}{c}\right)} \exp\left(6\pi z \frac{a}{c}\right).$$

We note that the $g_{21}(z)$ has poles at $z = z_n = -i(\frac{1}{6} + \frac{n}{3} - \frac{b}{c})$, where $n \in \mathbb{Z}$. We find that

$$\operatorname{Res}_{z=z_n} g_{21}(z) = \frac{2(-1)^{n+1}}{3\pi} \sin\left(\frac{\pi}{3}(2n+1)\right) \exp\left(\frac{\pi ia}{c^2}(6b-c-2nc)\right).$$

Applying the Mittag-Leffler Theory [44, pp. 134–135], we have

$$g_{21}(z) = g_{21}(0) + \sum_{n \in \mathbb{Z}}^* \frac{2(-1)^{n+1}}{3\pi} \sin\left(\frac{\pi}{3}(2n+1)\right) \exp\left(\frac{\pi ia}{c^2}(6b-c-2nc)\right) \times \left(\frac{1}{z + i(\frac{n}{3} + \frac{b}{c} - \frac{1}{6})} - \frac{1}{i(\frac{n}{3} + \frac{b}{c} - \frac{1}{6})}\right)$$

for $z \neq z_n$, $n \in \mathbb{Z}$, and assuming $\frac{1}{6} < \frac{a}{c} < \frac{5}{6}$. We note that the convergence is uniform on any compact subset of $\mathbb{C} \setminus \{z_n : n \in \mathbb{Z}\}$. Again we must consider 3 cases.

Case 2.1. $\frac{1}{6} < \frac{a}{c} < \frac{5}{6}$. Proceeding as in Case 1.1, using the analogue of (2.29), (2.31)

$$\sum_{n \in \mathbb{Z}}^* \frac{2(-1)^{n+1}}{3\pi} \sin\left(\frac{\pi}{3}(2n+1)\right) \exp\left(\frac{\pi ia}{c^2}(6b-c-2nc)\right) \frac{1}{i(\frac{n}{3} + \frac{b}{c} - \frac{1}{6})} = g_{21}(0),$$

and applying (2.30) and using (2.17) we find that

$$\begin{aligned} & J\left(a, b, c; \frac{2\pi i}{z}\right) \\ &= -\frac{2iz}{\sqrt{3}} \exp\left(\frac{6\pi iab}{c^2} - \frac{\pi ia}{c}\right) \\ &\quad \times \int_0^{i\infty} \sum_{n \in \mathbb{Z}} (-1)^n \left(\frac{n}{3} + \frac{1}{6} - \frac{b}{c}\right) \sin\left(\frac{\pi}{3}(2n+1)\right) \\ &\quad \quad \quad \times \exp\left(-2\pi in \frac{a}{c}\right) e^{3\pi i\tau(\frac{n}{3} + \frac{1}{6} - \frac{b}{c})^2} \frac{d\tau}{\sqrt{-i(\tau+z)}} \\ &= -\frac{2iz}{6\sqrt{3}c} \int_0^{i\infty} \frac{\Theta_1(a, b, c; \tau)}{\sqrt{-i(\tau+z)}} d\tau, \end{aligned}$$

which gives (2.27) for this case.

Case 2.2. $0 < \frac{a}{c} < \frac{1}{6}$. We proceed as in Case 1.2. This time we need

$$\begin{aligned} & \int_{-\infty}^{\infty} \zeta_c^{2b} e^{-3\pi tx^2 + 6\pi x \frac{a}{c} - 4\pi x} \frac{(1 + \exp(6\pi x - 6\pi i \frac{b}{c}))}{\cosh(3\pi x - 3\pi i \frac{b}{c})} dx \\ &= \frac{2}{\sqrt{3}t} \zeta_{2c}^b \exp\left(\frac{\pi}{12t} (6\frac{a}{c} - 1)^2\right). \end{aligned}$$

We have

$$J\left(a, b, c; \frac{2\pi}{t}\right) = \frac{2}{\sqrt{3}} \sqrt{t} \zeta_{2c}^b \exp\left(\frac{\pi}{12t} (6\frac{a}{c} - 1)^2\right) + t \int_{-\infty}^{\infty} e^{-3\pi tx^2} g_{22}(x) dx,$$

where

$$g_{22}(z) = \left(\frac{\zeta_c^b e^{-2\pi z} - \zeta_c^{-b} e^{2\pi z}}{\cosh(3\pi z - 3\pi i \frac{b}{c})}\right) e^{6\pi z \frac{a}{c}}.$$

We observe that the function $g_{21}(z)$ from Case 2.1 and the function $g_{22}(z)$ have the same poles and the same residue at each pole. The result follows.

Case 2.3. $\frac{5}{6} < \frac{a}{c} < 1$. We proceed as in Case 2.2. This time we need

$$\begin{aligned} & \int_{-\infty}^{\infty} \zeta_c^b e^{-3\pi t x^2 + 6\pi x \frac{a}{c} - 2\pi x} \frac{(1 + \exp(-6\pi x + 6\pi i \frac{b}{c}))}{\cosh(3\pi x - 3\pi i \frac{b}{c})} dx \\ &= \frac{2}{\sqrt{3t}} \zeta_{2c}^{5b} \exp\left(\frac{\pi}{12t} (6\frac{a}{c} - 5)^2\right). \end{aligned}$$

We have

$$J\left(a, b, c; \frac{2\pi}{t}\right) = \frac{2}{\sqrt{3}} \sqrt{t} \zeta_{2c}^{5b} \exp\left(\frac{\pi}{12t} (6\frac{a}{c} - 5)^2\right) + t \int_{-\infty}^{\infty} e^{-3\pi t x^2} g_{23}(x) dx,$$

where

$$g_{23}(z) = \left(\frac{-\zeta_c^{4b} e^{-8\pi z} + \zeta_c^{2b} e^{-4\pi z}}{\cosh(3\pi z - 3\pi i \frac{b}{c})} \right) e^{6\pi z \frac{a}{c}}.$$

We observe that the function $g_{21}(z)$ from Case 2.1 and the function $g_{23}(z)$ have the same poles and the same residue at each pole. The result follows.

Finally (2.28) follows easily from (2.19) and (2.27). □

3. VECTOR VALUED MAASS FORMS OF WEIGHT 1/2

We describe Bringmann and Ono’s vector valued Maass forms of weight $\frac{1}{2}$ making all functions and transformations explicit. Suppose $0 \leq a < c$ and $0 < b < c$ are integers where $(c, 6) = 1$. We define

$$(3.1) \quad T_1\left(\frac{a}{c}; z\right) := -\frac{i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau + z)}} d\tau,$$

$$(3.2) \quad T_2\left(\frac{a}{c}; z\right) := \frac{i}{3c} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1(0, -a, c; \tau)}{\sqrt{-i(\tau + z)}} d\tau,$$

$$(3.3) \quad T_1(a, b, c, ; z) := \frac{\zeta_{2c}^{-5b}}{3c} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1(a, b, c; \tau)}{\sqrt{-i(\tau + z)}} d\tau,$$

$$(3.4) \quad T_2(a, b, c; z) := \zeta_{2c}^{-5b} \frac{i\sqrt{3}}{6c} \int_{-\bar{z}}^{i\infty} \frac{\Theta_2(a, b, c; \tau)}{\sqrt{-i(\tau + z)}} d\tau.$$

We can now define a family of vector valued Maass forms of weight $\frac{1}{2}$:

$$(3.5) \quad \mathcal{G}_1\left(\frac{a}{c}; z\right) := \mathcal{N}\left(\frac{a}{c}; z\right) - T_1\left(\frac{a}{c}; z\right),$$

$$(3.6) \quad \mathcal{G}_2\left(\frac{a}{c}; z\right) := \mathcal{M}\left(\frac{a}{c}; z\right) + \varepsilon_2\left(\frac{a}{c}; z\right) - T_2\left(\frac{a}{c}; z\right),$$

$$(3.7) \quad \mathcal{G}_1(a, b, c; z) := \mathcal{N}(a, b, c, z) - T_1(a, b, c; z),$$

$$(3.8) \quad \mathcal{G}_2(a, b, c; z) := \mathcal{M}(a, b, c, z) + \varepsilon_2(a, b, c; z) - T_2(a, b, c; z).$$

We have corrected the definitions of $\mathcal{G}_2\left(\frac{a}{c}; z\right)$ and $\mathcal{G}_2(a, b, c; z)$ given on [13, p. 440].

Theorem 3.1. *Suppose $0 \leq a < c$ and $0 < b < c$ are integers and assume that $(c, 6) = 1$.*

(1) *For $z \in \mathfrak{h}$ we have*

$$(3.9) \quad \mathcal{G}_1\left(\frac{a}{c}; z+1\right) = \zeta_{24}^{-1} \mathcal{G}_1\left(\frac{a}{c}; z\right),$$

$$(3.10) \quad \mathcal{G}_2\left(\frac{a}{c}; z+1\right) = \zeta_{2c}^{5a} \zeta_{2c^2}^{-3a^2} \zeta_{24}^{-1} \mathcal{G}_2(a, a, c; z),$$

$$(3.11) \quad \mathcal{G}_1(a, b, c; z+1) = \begin{cases} \zeta_{2c^2}^{3b^2} \zeta_{24}^{-1} \mathcal{G}_1(a-b, b, c; z) & \text{if } a \geq b, \\ -\zeta_{2c^2}^{3b^2} \zeta_c^{-3b} \zeta_{24}^{-1} \mathcal{G}_1(a-b+c, b, c; z) & \text{otherwise,} \end{cases}$$

$$(3.12) \quad \mathcal{G}_2(a, b, c; z+1) = \zeta_{2c}^{5a} \zeta_{2c^2}^{-3a^2} \zeta_{24}^{-1} \begin{cases} \mathcal{G}_2(a, a+b, c; z) & \text{if } a+b < c, \\ \mathcal{G}_2\left(\frac{a}{c}; z\right) & \text{if } a+b = c, \\ \mathcal{G}_2(a, a+b-c, c; z) & \text{otherwise,} \end{cases}$$

where a is assumed to be positive in the first and second formulas.

(2) *For $z \in \mathfrak{h}$ we have*

$$(3.13) \quad \frac{1}{\sqrt{-iz}} \mathcal{G}_1\left(\frac{a}{c}; -\frac{1}{z}\right) = \mathcal{G}_2\left(\frac{a}{c}; z\right),$$

$$(3.14) \quad \frac{1}{\sqrt{-iz}} \mathcal{G}_2\left(\frac{a}{c}; -\frac{1}{z}\right) = \mathcal{G}_1\left(\frac{a}{c}; z\right),$$

$$(3.15) \quad \frac{1}{\sqrt{-iz}} \mathcal{G}_1\left(a, b, c; -\frac{1}{z}\right) = \mathcal{G}_2(a, b, c; z),$$

$$(3.16) \quad \frac{1}{\sqrt{-iz}} \mathcal{G}_2\left(a, b, c; -\frac{1}{z}\right) = \mathcal{G}_1(a, b, c; z),$$

where again a is assumed to be positive in the first and second formulas.

Proof. From (2.23) we have

$$\begin{aligned} T_1\left(\frac{a}{c}; z+1\right) &= -\frac{i}{\sqrt{3}} \int_{-\bar{z}-1}^{i\infty} \frac{\Theta_1\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau+z+1)}} d\tau = -\frac{i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1\left(\frac{a}{c}; \tau-1\right)}{\sqrt{-i(\tau+z)}} d\tau \\ &= -\frac{i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\zeta_{24}^{-1} \Theta_1\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau+z)}} d\tau = \zeta_{24}^{-1} T_1\left(\frac{a}{c}; z+1\right). \end{aligned}$$

Hence by (2.9), (3.5) we have (3.9). The proofs of (3.10)–(3.12) are similar.

We now prove (3.13):

$$\begin{aligned} \frac{1}{\sqrt{-iz}} \mathcal{G}_1\left(\frac{a}{c}; -\frac{1}{z}\right) &= \frac{1}{\sqrt{-iz}} \mathcal{N}\left(\frac{a}{c}; -\frac{1}{z}\right) + \frac{i}{\sqrt{3}\sqrt{-iz}} \int_{-(-1/z)}^{i\infty} \frac{\Theta_1\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau-1/z)}} d\tau, \\ \int_{-(-1/z)}^{i\infty} \frac{\Theta_1\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau-1/z)}} d\tau &= -\int_0^{-\bar{z}} \frac{\Theta_1\left(\frac{a}{c}; -1/\tau\right)}{\sqrt{-i(-1/\tau-1/z)} \tau^2} d\tau \\ &= \sqrt{-iz} \int_0^{-\bar{z}} \frac{(-i\tau)^{-3/2} \Theta_1\left(\frac{a}{c}; -1/\tau\right)}{\sqrt{-i(\tau+z)}} d\tau. \end{aligned}$$

Thus by (2.13), (2.26), (2.24) we have

$$\begin{aligned} & \frac{1}{\sqrt{-iz}} \mathcal{G}_1\left(\frac{a}{c}; -\frac{1}{z}\right) \\ &= \mathcal{M}\left(\frac{a}{c}; z\right) + 2\sqrt{3}\sqrt{-iz} J\left(\frac{a}{c}; -2\pi iz\right) + \frac{i}{\sqrt{3}} \int_0^{-\bar{z}} \frac{(-i\tau)^{-3/2} \Theta_1\left(\frac{a}{c}; -1/\tau\right)}{\sqrt{-i(\tau+z)}} d\tau \\ &= \mathcal{M}\left(\frac{a}{c}; z\right) + \varepsilon_2\left(\frac{a}{c}; z\right) - \frac{i}{3c} \int_0^{i\infty} \frac{\Theta_1(0, -a, c; \tau)}{\sqrt{-i(\tau+z)}} d\tau + \frac{i}{3c} \int_0^{-\bar{z}} \frac{\Theta_1(0, -a, c; \tau)}{\sqrt{-i(\tau+z)}} d\tau \\ &= \mathcal{M}\left(\frac{a}{c}; z\right) + \varepsilon_2\left(\frac{a}{c}; z\right) - \frac{i}{3c} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1(0, -a, c; \tau)}{\sqrt{-i(\tau+z)}} d\tau \\ &= \mathcal{G}_2\left(\frac{a}{c}; z\right), \end{aligned}$$

and we have (3.13). Equation (3.14) follows immediately from (3.13). The proofs of (3.15)–(3.16) are analogous. □

Corollary 3.2. *Suppose c is a fixed positive integer relatively prime to 6. Then*

$$\begin{aligned} \mathfrak{V}_c := & \left\{ \mathcal{G}_1\left(\frac{a}{c}; z\right), \mathcal{G}_2\left(\frac{a}{c}; z\right) : 0 < a < c \right\} \\ & \cup \left\{ \mathcal{G}_1(a, b, c; z), \mathcal{G}_2(a, b, c; z) : 0 \leq a < c \text{ and } 0 < b < c \right\} \end{aligned}$$

is a vector valued Maass form of weight $\frac{1}{2}$ for the full modular group $SL_2(\mathbb{Z})$.

4. A MAASS FORM MULTIPLIER

We will find that transformation formulas are more tractable if we modify the definition of the functions \mathcal{G}_j by multiplying by the Dedekind eta-function $\eta(z)$. For a function $F(z)$ and a weight k we define the usual stroke operator

$$(4.1) \quad F | [A]_k := (ad - bc)^{k/2} (cz + d)^{-k} F(Az), \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}),$$

where $k \in \frac{1}{2}\mathbb{Z}$, and when calculating $(cz + d)^{-k}$ we take the principal value. Our main result is the following.

Theorem 4.1. *Let $p > 3$ be prime, suppose $1 \leq \ell \leq (p - 1)$, and define*

$$(4.2) \quad \mathcal{F}_1\left(\frac{\ell}{p}; z\right) := \eta(z) \mathcal{G}_1\left(\frac{\ell}{p}; z\right).$$

Then

$$(4.3) \quad \mathcal{F}_1\left(\frac{\ell}{p}; z\right) | [A]_1 = \mu(A, \ell) \mathcal{F}_1\left(\frac{\bar{d}\ell}{p}; z\right),$$

where

$$\mu(A, \ell) = \exp\left(\frac{3\pi icd\ell^2}{p^2}\right) (-1)^{\frac{c\ell}{p}} (-1)^{\lfloor \frac{d\ell}{p} \rfloor}$$

and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p).$$

Here \bar{m} is the least nonnegative residue of $m \pmod{p}$.

Remark. The function $\mu(A, \ell)$ is reminiscent of functions that occur in transformation formulas of certain theta functions [11, Lemma 2.1] on $\Gamma_0(p)$.

Corollary 4.2. *Let $p > 3$ be prime and suppose $1 \leq \ell \leq \frac{1}{2}(p - 1)$. Then*

$$(4.4) \quad \mathcal{F}_1 \left(\frac{\ell}{p}; z \right) \Big| [A]_1 = \mathcal{F}_1 \left(\frac{\ell}{p}; z \right)$$

and

$$(4.5) \quad \mathcal{G}_1 \left(\frac{\ell}{p}; z \right) \Big| [A]_{\frac{1}{2}} = \frac{1}{\nu_\eta(A)} \mathcal{G}_1 \left(\frac{\ell}{p}; z \right),$$

where $A \in \Gamma_0(p^2) \cap \Gamma_1(p)$ and $\nu_\eta(A)$ is the eta-multiplier

$$(4.6) \quad \eta(z) \Big| [A]_{\frac{1}{2}} = \nu_\eta(A) \eta(z).$$

Remark. Equation 4.5 strengthens one of the main results [13, Theorem 1.2, p. 424] of Bringmann and Ono's paper in the case that $c = p > 3$ is prime.

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) \cap \Gamma_1(p).$$

Then $c \equiv 0 \pmod{p^2}$ and $a \equiv d \equiv 1 \pmod{p}$. So

$$\mu(A, \ell) = \exp \left(\frac{\pi i}{p^2} (3cd\ell^2 + clp + p^2 \left[\frac{d\ell}{p} \right]) \right) = 1,$$

since

$$3cd\ell^2 + clp + p^2 \left[\frac{d\ell}{p} \right] \equiv 0 \pmod{p^2}$$

and

$$d\ell = p \left[\frac{d\ell}{p} \right] + \ell \equiv \left[\frac{d\ell}{p} \right] + \ell \pmod{2},$$

so that

$$3cd\ell^2 + clp + p^2 \left[\frac{d\ell}{p} \right] \equiv cdl + cl + d\ell + \ell \equiv \ell(c + 1)(d + 1) \equiv 0 \pmod{2}$$

since either c or d is odd. Thus (4.4) follows from (4.3), and (4.5) is immediate. \square

Corollary 4.3. *Let $p > 3$ be prime, suppose $1 \leq \ell \leq (p - 1)$, and define*

$$(4.7) \quad \mathcal{F}_2 \left(\frac{\ell}{p}; z \right) := \eta(z) \mathcal{G}_2 \left(\frac{\ell}{p}; z \right).$$

Then

$$(4.8) \quad \mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right) \Big| [A]_1 = \mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right),$$

for $A \in \Gamma_0(p^2) \cap \Gamma_1(p)$.

Remark. We prove Corollary 4.3 in Section 4.2.

4.1. **Proof of Theorem 4.1.** It is well-known that the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate $SL_2(\mathbb{Z})$, and

$$(4.9) \quad \eta(z) \Big| [T]_{\frac{1}{2}} = \zeta_{24} \eta(z), \quad \eta(z) \Big| [S]_{\frac{1}{2}} = \exp\left(-\frac{\pi i}{4}\right) \eta(z).$$

We need the following.

Theorem 4.4 (Rademacher [39]). *Let p be prime. Then a set of generators for $\Gamma_0(p)$ is given by*

$$T, \quad V_k \quad (1 \leq k \leq p-1),$$

where

$$V_k = ST^k ST^{-k^*} S^{-1} = \begin{pmatrix} -k^* & -1 \\ kk^* + 1 & k \end{pmatrix}$$

and k^* is given by $1 \leq k^* \leq p-1$ and $kk^* \equiv -1 \pmod{p}$. Furthermore for $p > 3$ the number of generators can be reduced to $2 \left\lfloor \frac{p}{12} \right\rfloor + 3$.

As in (4.2), (4.7) we define functions $\mathcal{F}_j(\cdot; \cdot)$ by multiplying $\mathcal{G}_j(\cdot; \cdot)$ by $\eta(z)$:

$$(4.10) \quad \mathcal{F}_1(a, b, c; z) := \eta(z) \mathcal{G}_1(a, b, c; z),$$

$$(4.11) \quad \mathcal{F}_2(a, b, c; z) := \eta(z) \mathcal{G}_2(a, b, c; z).$$

The following follows from Theorem 3.1 and (4.9).

Theorem 4.5. *Suppose $0 \leq a < c$ and $0 < b < c$ are integers and assume $(c, 6) = 1$.*

(1) *For $z \in \mathfrak{h}$ we have*

$$\begin{aligned} \mathcal{F}_1\left(\frac{a}{c}; z\right) \Big| [T]_1 &= \mathcal{F}_1\left(\frac{a}{c}; z\right), \\ \mathcal{F}_2\left(\frac{a}{c}; z\right) \Big| [T]_1 &= \zeta_{2c}^{5a} \zeta_{2c^2}^{-3a^2} \mathcal{F}_2(a, a, c; z), \\ \mathcal{F}_1(a, b, c; z) \Big| [T]_1 &= \begin{cases} \zeta_{2c^2}^{3b^2} \mathcal{F}_1(a-b, b, c; z) & \text{if } a \geq b, \\ -\zeta_{2c^2}^{3b^2} \zeta_c^{-3b} \mathcal{F}_1(a-b+c, b, c; z) & \text{otherwise,} \end{cases} \\ \mathcal{F}_2(a, b, c; z) \Big| [T]_1 &= \zeta_{2c}^{5a} \zeta_{2c^2}^{-3a^2} \begin{cases} \mathcal{F}_2(a, a+b, c; z) & \text{if } a+b < c, \\ \mathcal{F}_2\left(\frac{a}{c}; z\right) & \text{if } a+b = c, \\ \mathcal{F}_2(a, a+b-c, c; z) & \text{otherwise,} \end{cases} \end{aligned}$$

where a is assumed to be positive in the first and second formulas.

(2) *For $z \in \mathfrak{h}$ we have*

$$\begin{aligned} \mathcal{F}_1\left(\frac{a}{c}; z\right) \Big| [S]_1 &= -i \mathcal{F}_2\left(\frac{a}{c}; z\right), \\ \mathcal{F}_2\left(\frac{a}{c}; z\right) \Big| [S]_1 &= -i \mathcal{F}_1\left(\frac{a}{c}; z\right), \\ \mathcal{F}_1(a, b, c; z) \Big| [S]_1 &= -i \mathcal{F}_2(a, b, c; z), \\ \mathcal{F}_2(a, b, c; z) \Big| [S]_1 &= -i \mathcal{F}_1(a, b, c; z), \end{aligned}$$

where again a is assumed to be positive in the first and second formulas.

Throughout this section we assume $p > 3$ is prime and $1 \leq \ell \leq p - 1$. Since $(ST)^3 = -I$ we have

$$(4.12) \quad \mathcal{F}_2(0, \ell, p; z) = i \zeta_{2p}^{-5\ell} \mathcal{F}_1\left(\frac{\ell}{p}; z\right)$$

using Theorem 4.5. We require the transformation

$$\mathcal{F}_1(a, b, c; z) \mid [T^{-1}]_1 = \begin{cases} \zeta_{2c^2}^{-3b^2} \mathcal{F}_1(a + b, b, c; z) & \text{if } a + b < c, \\ -\zeta_{2c^2}^{-3b^2} \zeta_p^{3b} \mathcal{F}_1(a + b - c, b, c; z) & \text{otherwise,} \end{cases}$$

which follows from Theorem 4.5 assuming $0 \leq a, b < c$.

Our first goal is to show that Theorem 4.1 holds when A is a generator of $\Gamma_0(p)$. The result is clearly true when $A = T$. We assume $1 \leq k \leq p - 1$. Applying Theorem 4.5 we have

$$\begin{aligned} \mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [S]_1 &= (-i) \mathcal{F}_2\left(\frac{\ell}{p}; z\right), \\ \mathcal{F}_2\left(\frac{\ell}{p}; z\right) \mid [T^k]_1 &= \zeta_{2p}^{5k\ell} \zeta_{2p^2}^{-3\ell^2 k} \mathcal{F}_2(\ell, \overline{k\ell}, p; z), \\ \mathcal{F}_2(\ell, \overline{k\ell}, p; z) \mid [S]_1 &= -i \mathcal{F}_1(\ell, \overline{k\ell}, p; z), \\ \mathcal{F}_1(\ell, \overline{k\ell}, p; z) \mid [T^{-k^*}]_1 &= \left(\zeta_{2p^2}^{-3(\overline{k\ell})^2}\right)^{k^*} \left(-\zeta_p^{3\overline{k\ell}}\right)^j \mathcal{F}_1(0, \overline{k\ell}, p; z), \end{aligned}$$

where

$$j = \frac{\ell + \overline{k\ell}k^*}{p}$$

and

$$\mathcal{F}_1(0, \overline{k\ell}, p; z) \mid [S^{-1}]_1 = i \mathcal{F}_2(0, \overline{k\ell}, p; z).$$

Putting all this together we find that

$$\mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [V_k]_1 = -i \zeta_{2p^2}^{-3(\ell^2 k + k^*(\overline{k\ell})^2)} \zeta_{2p}^{5k\ell + 6j\overline{k\ell}} (-1)^j \mathcal{F}_2(0, \overline{k\ell}, p; z).$$

Using (4.12) we have

$$\mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [V_k]_1 = \tilde{\mu}(V_k, \ell) \mathcal{F}_1\left(\frac{\overline{k\ell}}{p}; z\right),$$

where

$$\tilde{\mu}(V_k, \ell) = \zeta_{2p^2}^{-3(\ell^2 k + k^*(\overline{k\ell})^2)} \zeta_{2p}^{5(k\ell - \overline{k\ell}) + 6j\overline{k\ell}} (-1)^j$$

and

$$j = \frac{\ell + \overline{k\ell}k^*}{p}.$$

Next we must show that

$$\tilde{\mu}(V_k, \ell) = \mu(V_k, \ell).$$

This is equivalent to showing that

$$(4.13) \quad \begin{aligned} &5p(k\ell - \overline{k\ell}) - 3(\ell^2 k + k^*(\overline{k\ell})^2) + p(\ell + k^*\overline{k\ell}) + 6\overline{k\ell}(\ell + \overline{k\ell}k^*) \\ &\equiv 3(1 + kk^*)k\ell^2 + p(1 + kk^*)\ell + p^2 \left\lfloor \frac{k\ell}{p} \right\rfloor \pmod{2p^2}. \end{aligned}$$

This congruence holds mod p^2 since

$$k^*(k\ell - \overline{k\ell})^2 \equiv 0 \pmod{p^2}.$$

It remains to show that the congruence (4.13) holds mod 2. Mod 2 the congruence reduces to

$$\overline{k\ell} + \ell \equiv k\ell + \ell + \left\lfloor \frac{k\ell}{p} \right\rfloor \pmod{2},$$

which is true since when x is an integer and p is a positive odd integer

$$(4.14) \quad x = p \left\lfloor \frac{x}{p} \right\rfloor + \overline{x} \quad \text{and} \quad \left\lfloor \frac{x}{p} \right\rfloor \equiv x + \overline{x} \pmod{2}.$$

Thus we have shown that Theorem 4.1 holds when A is one of Rademacher's generators for $\Gamma_0(p)$.

In the next part of the proof we show that Theorem 4.1 holds when A is the inverse of any of Rademacher's generators. The result clearly holds for $A = T^{-1}$. Let $1 \leq k \leq p - 1$. We find that

$$V_k^{-1} = \begin{pmatrix} k & 1 \\ -kk^* - 1 & -k^* \end{pmatrix} = - \begin{pmatrix} -k & -1 \\ kk^* + 1 & k^* \end{pmatrix} = -V_{k^*}.$$

We know that

$$\mathcal{F}_1 \left(\frac{\ell}{p}; z \right) | [V_{k^*}]_1 = \mu(V_{k^*}, \ell) \mathcal{F}_1 \left(\frac{\overline{k^*\ell}}{p}; z \right).$$

Hence

$$\begin{aligned} \mathcal{F}_1 \left(\frac{\ell}{p}; z \right) | [V_k^{-1}]_1 &= \mathcal{F}_1 \left(\frac{\ell}{p}; z \right) | [-V_{k^*}]_1 \\ &= -\mathcal{F}_1 \left(\frac{\ell}{p}; z \right) | [V_{k^*}]_1 = -\mu(V_{k^*}, \ell) \mathcal{F}_1 \left(\frac{\overline{k^*\ell}}{p}; z \right) \\ &= -\mu(V_{k^*}, \ell) \mathcal{F}_1 \left(\frac{-\overline{k^*\ell}}{p}; z \right) \\ &= \exp \left(\frac{3\pi i(1 + kk^*)k^*\ell^2}{p^2} \right) (-1)^{\frac{(1+kk^*)\ell}{p}} (-1)^{\lfloor \frac{k^*\ell}{p} \rfloor + 1} \mathcal{F}_1 \left(\frac{-\overline{k^*\ell}}{p}; z \right) \\ &= \exp \left(\frac{3\pi i(1 + kk^*)k^*\ell^2}{p^2} \right) (-1)^{\frac{(1+kk^*)\ell}{p}} (-1)^{\lfloor \frac{-k^*\ell}{p} \rfloor} \mathcal{F}_1 \left(\frac{-\overline{k^*\ell}}{p}; z \right) \\ &\hspace{15em} (\text{since } \lfloor x \rfloor + 1 = -\lfloor -x \rfloor \text{ when } x \notin \mathbb{Z}) \\ &= \mu(V_k^{-1}, z) \mathcal{F}_1 \left(\frac{-\overline{k^*\ell}}{p}; z \right), \end{aligned}$$

and Theorem 4.1 holds for $A = V_k^{-1}$.

Finally we need to show that if Theorem 4.1 holds for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

then it also holds for

$$AB = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

We must show that

$$\mathcal{F}_1 \left(\frac{\ell}{p}; z \right) | [AB]_1 = \mu(AB, \ell) \mathcal{F}_1 \left(\frac{\overline{(cb' + dd')\ell}}{p}; z \right).$$

Now

$$\begin{aligned} \mathcal{F}_1\left(\frac{\ell}{p}; z\right) |[AB]_1 &= \left(\mathcal{F}_1\left(\frac{\ell}{p}; z\right) |[A]_1\right) |[B]_1 \\ &= \mu(A, \ell) \mathcal{F}_1\left(\frac{\overline{d\ell}}{p}; z\right) |[B]_1 \quad (\text{since Theorem 4.1 holds for the matrix } A) \\ &= \mu(B, \overline{d\ell}) \mu(A, \ell) \mathcal{F}_1\left(\frac{\overline{dd'\ell}}{p}; z\right) = \mu(A, \ell) \mu(B, \overline{d\ell}) \mathcal{F}_1\left(\frac{\overline{(cb' + dd')\ell}}{p}; z\right), \end{aligned}$$

since $c \equiv 0 \pmod{p}$ and Theorem 4.1 holds for the matrix B . This means we need only verify that

$$\mu(A, \ell) \mu(B, \overline{d\ell}) = \mu(AB, \ell).$$

This is equivalent to showing that

$$\begin{aligned} (4.15) \quad & 3(ca' + dc')(cb' + dd')\ell^2 + p\ell(ca' + dc') + p^2 \left[\frac{(cb' + dd')\ell}{p} \right] \\ & \equiv 3cd\ell^2 + p\ell + p^2 \left[\frac{d\ell}{p} \right] + 3c'd'(\overline{d\ell})^2 + pc'\overline{d\ell} + p^2 \left[\frac{d'\overline{d\ell}}{p} \right] \pmod{2p^2}. \end{aligned}$$

This can be easily verified mod p^2 using the congruences $c \equiv c' \equiv 0 \pmod{p}$ and $a'd' \equiv 1 \pmod{p}$. It remains to verify that the congruence holds mod 2; i.e.,

$$\begin{aligned} & (ca' + dc')(cb' + dd')\ell + \ell(ca' + dc') + cb'\ell + \left[\frac{dd'\ell}{p} \right] \\ & \equiv cd\ell + c\ell + \left[\frac{d\ell}{p} \right] + c'd'(\overline{d\ell}) + c'\overline{d\ell} + \left[\frac{d'\overline{d\ell}}{p} \right] \pmod{2}. \end{aligned}$$

Using (4.14) we see that this is equivalent to showing that

$$\begin{aligned} & (ca' + dc')(cb' + dd' + 1)\ell + cb'\ell + dd'\ell + \overline{dd'\ell} \\ & \equiv cd\ell + c\ell + d\ell + \overline{d\ell} + c'd'(\overline{d\ell}) + c'\overline{d\ell} + d'\overline{d\ell} + \overline{d'\overline{d\ell}} \pmod{2} \end{aligned}$$

or

$$\begin{aligned} & ((ca' + dc')(cb' + dd' + 1) + cb' + dd')\ell \\ & \equiv ((c + 1)(d + 1) - 1)\ell + (c' + 1)(d' + 1)\overline{d\ell} \pmod{2} \end{aligned}$$

or

$$(ca' + dc')(cb' + dd' + 1) + cb' + dd' \equiv 1 \pmod{2},$$

since at least one of c, d is odd and at least one of c', d' is odd. But

$$\begin{aligned} & (ca' + dc')(cb' + dd' + 1) + cb' + dd' \\ & \equiv c(a'b' + a' + b') + d(c'd' + c' + d') + dc(a'd' + c'b') \pmod{2} \\ & \equiv c + d + dc \equiv 1 \pmod{2}, \end{aligned}$$

since at least one of a', b' is odd and $a'b' + a' + b \equiv 1 \pmod{2}$. Similarly $c'd' + c' + d' \equiv 1 \pmod{2}$, $a'd' + c'b' \equiv 1 \pmod{2}$, and at least one of c, d is odd. Thus (4.15) holds mod 2, mod p^2 , and hence mod $2p^2$.

We have shown that Theorem 4.1 holds for generators of $\Gamma_0(p)$, inverses, and products of generators, and hence for all matrices in $\Gamma_0(p)$, which completes the proof.

4.2. **Proof of Corollary 4.3.** Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) \cap \Gamma_1(p), \quad \text{and recall } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

From Theorem 4.5 we have

$$\mathcal{F}_2\left(\frac{\ell}{p}; z\right) = -i \mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [S]_1.$$

We let

$$P = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and find that

$$S P A = B S P,$$

where

$$B = \begin{pmatrix} d & -c/p^2 \\ -p^2b & a \end{pmatrix} \in \Gamma_0(p^2) \cap \Gamma_1(p).$$

Therefore

$$\begin{aligned} \left(\mathcal{F}_2\left(\frac{\ell}{p}; z\right) \mid [P]_1\right) \mid [A]_1 &= -i \mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [SPA]_1 \\ &= -i \mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [BSP]_1 \\ &= -i \mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [SP]_1 \quad (\text{by (4.4)}) \\ &= -\mathcal{F}_2\left(\frac{\ell}{p}; z\right) \mid [P]_1, \end{aligned}$$

and we have (4.8).

5. EXTENDING RAMANUJAN'S DYSON RANK FUNCTION IDENTITY

Equation (1.6) is Ramanujan's identity for the 5-dissection of $R(\zeta_5, q)$. In equation (1.8) we showed how this identity could be written in terms of generalized eta-functions. In this section we show that there is an analogous result for the p -dissection of $R(\zeta_p, q)$ when p is any prime greater than 3.

We assume $p > 3$ is prime and define

$$\begin{aligned} &\mathcal{J}\left(\frac{1}{p}; z\right) \\ (5.1) \quad &:= \eta(p^2z) \left(\mathcal{N}\left(\frac{1}{p}; z\right) - 2 \chi_{12}(p) \sum_{\ell=1}^{\frac{1}{2}(p-1)} (-1)^\ell \sin\left(\frac{6\ell\pi}{p}\right) \right. \\ &\quad \left. \times \left(\mathcal{M}\left(\frac{\ell}{p}; p^2z\right) + \varepsilon_2\left(\frac{\ell}{p}; p^2z\right) \right) \right), \end{aligned}$$

where $\chi_{12}(n)$ is defined in (1.11). Using (2.3), (2.4) we find that

$$\eta(p^2z) \mathcal{R}_p(z) = \sin\left(\frac{\pi}{p}\right) \mathcal{J}\left(\frac{1}{p}; z\right).$$

Thus we rewrite one of our main results, Theorem 1.2, in the equivalent form.

Theorem 5.1. *Let $p > 3$ be prime. Then the function $\mathcal{J}\left(\frac{1}{p}; z\right)$, defined in (5.1), is a weakly holomorphic modular form of weight 1 on the group $\Gamma_0(p^2) \cap \Gamma_1(p)$.*

This theorem leads to our analogue of Ramanujan's identity (1.6) or (1.8).

Corollary 5.2. *Let $p > 3$ be prime. Then the function (5.2)*

$$\mathcal{R}_p(z) = q^{-\frac{1}{24}} R(\zeta_p, q) - 4q^{-\frac{1}{24}} \chi_{12}(p) \sum_{a=1}^{\frac{1}{2}(p-1)} (-1)^a \sin\left(\frac{\pi}{p}\right) \sin\left(\frac{6a\pi}{p}\right) q^{\frac{a}{2}(p-3a) - \frac{1}{24}(p^2-1)} \Phi_{p,a}(q^p)$$

is a modular form of weight $\frac{1}{2}$ on $\Gamma_0(p^2) \cap \Gamma_1(p)$ with multiplier. In particular

$$(5.3) \quad \mathcal{R}_p(z) \Big| [A]_{\frac{1}{2}} = \frac{1}{\nu_\eta(p^2 A)} \mathcal{R}_p(z),$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) \cap \Gamma_1(p)$ and where

$$p^2 A = \begin{pmatrix} a & p^2 b \\ c/p^2 & d \end{pmatrix}.$$

Remark. Equation (5.2) follows easily from the definition of $\mathcal{R}_p(z)$, which is given in (1.10). Equation (5.3) follows from Theorem 5.1 and (4.6).

This result will follow from Corollary 4.2 and Proposition 5.3.

Proposition 5.3. *Let $p > 3$ be prime. Then*

$$(5.4) \quad \Theta_1\left(\frac{1}{p}; z\right) = -\frac{2}{\sqrt{3}} \chi_{12}(p) \sum_{a=1}^{\frac{1}{2}(p-1)} (-1)^a \sin\left(\frac{6a\pi}{p}\right) \Theta_1(0, -a, p; p^2 z).$$

Proof. From (2.22) we recall that

$$\Theta_1\left(\frac{1}{p}; z\right) = \sum_{n=-\infty}^{\infty} (-1)^n (6n+1) \sin\left(\frac{\pi(6n+1)}{p}\right) \exp\left(3\pi iz \left(n + \frac{1}{6}\right)^2\right).$$

We assume $p > 3$ is prime and consider two cases.

Case 1. $p \equiv 1 \pmod{6}$. We let $p_1 = \frac{1}{6}(p-1)$ so that $6p_1 + 1 = p$. We note that each integer n satisfying $6n + 1 \not\equiv 0 \pmod{p}$ can be written uniquely as

$$(i) \quad n = p(2pm + \ell_1) + a + p_1, \quad \text{where } 1 \leq a \leq \frac{1}{2}(p-1), 0 \leq \ell_1 < 2p, \text{ and } m \in \mathbb{Z},$$

or

$$(ii) \quad n = p(-2pm - \ell_1) - a + p_1, \quad \text{where } 1 \leq a \leq \frac{1}{2}(p-1), 1 \leq \ell_1 \leq 2p, \text{ and } m \in \mathbb{Z}.$$

If $n = p(2pm + \ell_1) + a + p_1$, then

$$6n + 1 = 12p^2 m + 2p\ell + 6a + p, \quad \text{where } \ell = 3\ell_1 \text{ and } 0 \leq \ell < 6p, \\ \sin\left(\frac{\pi(6n+1)}{p}\right) = -\sin\left(\frac{6a\pi}{p}\right), \quad \sin\left(\frac{\pi}{3}(2\ell+1)\right) = \frac{\sqrt{3}}{2}, \quad (-1)^n = (-1)^{\ell+a+p_1}.$$

If $n = p(-2pm - \ell_1) - a + p_1$, then

$$6n + 1 = -(12p^2m + 2p\ell + 6a + p), \quad \text{where } \ell = 3\ell_1 - 1 \text{ and } 0 < \ell \leq 6p - 1,$$

$$\sin\left(\frac{\pi(6n + 1)}{p}\right) = \sin\left(\frac{6a\pi}{p}\right), \quad \sin\left(\frac{\pi}{3}(2\ell + 1)\right) = -\frac{\sqrt{3}}{2}, \quad (-1)^n = -(-1)^{\ell+a+p_1}.$$

Hence we have

$$\begin{aligned} \Theta_1\left(\frac{1}{p}; z\right) &= \sum_{\substack{n=-\infty \\ 6n+1 \not\equiv 0 \pmod{p}}}^{\infty} (-1)^n (6n + 1) \sin\left(\frac{\pi(6n + 1)}{p}\right) \exp\left(3\pi iz \left(n + \frac{1}{6}\right)^2\right) \\ &= - \sum_{a=1}^{\frac{1}{2}(p-1)} \sum_{\ell=0}^{6p-1} (-1)^{\ell+a+p_1} \sin\left(\frac{\pi}{3}(2\ell + 1)\right) \frac{2}{\sqrt{3}} \sin\left(\frac{6a\pi}{p}\right) \\ &\quad \times \sum_{m=-\infty}^{\infty} (12p^2m + 2\ell p + 6a + p) \exp\left(\frac{\pi iz}{12}(12p^2m + 2\ell p + 6a + p)^2\right) \\ &= -\frac{2}{\sqrt{3}} \chi_{12}(p) \sum_{a=1}^{\frac{1}{2}(p-1)} (-1)^a \sin\left(\frac{6a\pi}{p}\right) \Theta_1(0, -a, p; p^2z), \end{aligned}$$

since $(-1)^{p_1} = \chi_{12}(p)$.

Case 2. $p \equiv -1 \pmod{6}$. We proceed as in Case 1 except this time we let $p_1 = \frac{1}{6}(p + 1)$ so that $6p_1 - 1 = p$, and we find that each integer n satisfying $6n + 1 \not\equiv 0 \pmod{p}$ can be written uniquely as

(i) $n = p(2pm + \ell_1) + a - p_1$, where $1 \leq a \leq \frac{1}{2}(p - 1), 1 \leq \ell_1 \leq 2p$, and $m \in \mathbb{Z}$,

or

(ii) $n = p(-2pm - \ell_1) - a - p_1$, where $1 \leq a \leq \frac{1}{2}(p - 1), 0 \leq \ell_1 < 2p$, and $m \in \mathbb{Z}$.

The result (5.4) follows as in Case 1. □

5.1. Proof of Theorem 5.1: Part 1 – Transformations. First we show that

$$(5.5) \quad \mathcal{J}\left(\frac{1}{p}; z\right) = \mathcal{J}^*\left(\frac{1}{p}; z\right),$$

where $\mathcal{J}\left(\frac{1}{p}; z\right)$ is defined in (5.1) and

$$(5.6) \quad \mathcal{J}^*\left(\frac{1}{p}; z\right) := \frac{\eta(p^2z)}{\eta(z)} \left(\mathcal{F}_1\left(\frac{1}{p}; z\right) - 2\chi_{12}(p) \sum_{\ell=1}^{\frac{1}{2}(p-1)} (-1)^\ell \sin\left(\frac{6\ell\pi}{p}\right) \mathcal{F}_2\left(\frac{\ell}{p}; p^2z\right) \right).$$

From (3.2) we have

$$T_2\left(\frac{\ell}{p}; p^2z\right) = \frac{i}{3} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1(0, -\ell, c; p^2\tau)}{\sqrt{-i(\tau + z)}} d\tau.$$

Therefore using (3.1) and Proposition 5.3 we have

$$\begin{aligned} T_1\left(\frac{1}{p}; z\right) &= \frac{-i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1\left(\frac{1}{p}; \tau\right)}{\sqrt{-i(\tau+z)}} d\tau \\ &= \frac{2i}{3} \chi_{12}(p) \sum_{\ell=1}^{\frac{1}{2}(p-1)} (-1)^\ell \sin\left(\frac{6\ell\pi}{p}\right) \int_{-\bar{z}}^{i\infty} \frac{\Theta_1(0, -\ell, c; p^2\tau)}{\sqrt{-i(\tau+z)}} d\tau \end{aligned}$$

and

$$(5.7) \quad T_1\left(\frac{1}{p}; z\right) = 2 \chi_{12}(p) \sum_{\ell=1}^{\frac{1}{2}(p-1)} (-1)^\ell \sin\left(\frac{6\ell\pi}{p}\right) T_2\left(\frac{\ell}{p}; p^2z\right).$$

From (5.6), (3.5), and (3.6) we have

$$\begin{aligned} \mathcal{J}^*\left(\frac{1}{p}; z\right) &= \eta(p^2z) \left(\mathcal{G}_1\left(\frac{1}{p}; z\right) - 2 \chi_{12}(p) \sum_{\ell=1}^{\frac{1}{2}(p-1)} (-1)^\ell \sin\left(\frac{6\ell\pi}{p}\right) \mathcal{G}_2\left(\frac{\ell}{p}; p^2z\right) \right) \\ &= \eta(p^2z) \left(\mathcal{N}\left(\frac{1}{p}; z\right) - T_1\left(\frac{1}{p}; z\right) \right. \\ &\quad \left. - 2 \chi_{12}(p) \sum_{\ell=1}^{\frac{1}{2}(p-1)} (-1)^\ell \sin\left(\frac{6\ell\pi}{p}\right) \left(\mathcal{M}\left(\frac{\ell}{p}; p^2z\right) + \varepsilon_2\left(\frac{\ell}{p}; p^2z\right) - T_2\left(\frac{\ell}{p}; p^2z\right) \right) \right) \\ &= \mathcal{J}\left(\frac{1}{p}; z\right), \end{aligned}$$

by (5.7). By Corollaries 4.2, 4.3, and equation (5.6) we have

$$\mathcal{J}^*\left(\frac{1}{p}; z\right) \Big|_{[A]_1} = \mathcal{J}^*\left(\frac{1}{p}; z\right),$$

for $A \in \Gamma_0(p^2) \cap \Gamma_1(p)$, using the well-known result that $\frac{\eta(p^2z)}{\eta(z)}$ is a modular function on $\Gamma_0(p^2)$ when $p > 3$ is prime. Hence

$$\mathcal{J}\left(\frac{1}{p}; z\right) \Big|_{[A]_1} = \mathcal{J}\left(\frac{1}{p}; z\right),$$

for $A \in \Gamma_0(p^2) \cap \Gamma_1(p)$.

5.2. Proof of Theorem 5.1: Part 2 – Checking cusp conditions. We note that the function $\mathcal{J}\left(\frac{1}{p}; z\right)$ is holomorphic on \mathfrak{h} . In Section 5.1 we showed that

$$\mathcal{J}\left(\frac{1}{p}; z\right) \Big|_{[A]_1} = \mathcal{J}\left(\frac{1}{p}; z\right), \quad \text{for all } A \in \Gamma_0(p^2) \cap \Gamma_1(p).$$

By [33, p. 125] this implies that the function $\mathcal{J}\left(\frac{1}{p}; z\right) \Big|_{[A]_1}(z)$ has period p^2 for each $A \in \text{SL}_2(\mathbb{Z})$. Hence the function $\mathcal{J}\left(\frac{1}{p}; z\right)$ has a Fourier expansion in powers of $q_{p^2} = \exp(2\pi iz/p^2)$. To complete the proof of Theorem 5.1 we need to show that this expansion has only finitely many negative powers of q_{p^2} .

We need the following.

Lemma 5.4. *Suppose the functions*

$$f_1(z), f_2(z), \dots, f_n(z)$$

are holomorphic on an open connected set \mathfrak{D} and linearly independent over \mathbb{C} . Suppose the functions

$$G_1(z), G_2(z), \dots, G_n(z)$$

are holomorphic on $-\overline{\mathfrak{D}} = \{-\bar{d} : d \in \mathfrak{D}\}$, and

$$\sum_{j=1}^n f_j(z) G_j(-\bar{z}) = 0$$

on \mathfrak{D} . Then

$$G_1(z) = G_2(z) = \dots = G_n(z) = 0$$

on $-\overline{\mathfrak{D}}$.

Proof. We proceed by induction on n . The result is clearly true for $n = 1$. Now suppose the result is true for $n = m$ where $m \geq 1$ is fixed. Suppose the functions

$$f_1(z), f_2(z), \dots, f_{m+1}(z)$$

are holomorphic on an open connected set \mathfrak{D} and linearly independent over \mathbb{C} . Suppose the functions

$$G_1(z), G_2(z), \dots, G_{m+1}(z)$$

are holomorphic on $-\overline{\mathfrak{D}} = \{-\bar{d} : d \in \mathfrak{D}\}$, and

$$\sum_{j=1}^{m+1} f_j(z) G_j(-\bar{z}) = 0$$

on \mathfrak{D} . Let

$$\mathfrak{D}_1 = \mathfrak{D} \setminus \{z_0 : f_{m+1}(z_0) = 0\}.$$

Then \mathfrak{D}_1 is an open connected set, and on \mathfrak{D}_1 the functions $F_j(z) = \frac{f_j(z)}{f_{m+1}(z)}$ are holomorphic, linearly independent over \mathbb{C} , and on \mathfrak{D}_1 ,

$$(5.8) \quad \sum_{j=1}^m F_j(z) G_j(-\bar{z}) = -G_{m+1}(-\bar{z}).$$

Since $G_{m+1}(z)$ is holomorphic,

$$\frac{\partial}{\partial z} \sum_{j=1}^m F_j(z) G_j(-\bar{z}) = -\frac{\partial}{\partial z} G_{m+1}(-\bar{z}) = 0,$$

and

$$(5.9) \quad \sum_{j=1}^m F'_j(z) G_j(-\bar{z}) = 0$$

on \mathfrak{D}_1 . We next show that the $F'_j(z)$ are linearly independent over \mathbb{C} . Suppose there are complex numbers a_1, a_2, \dots, a_m such that

$$a_1 F'_1(z) + a_2 F'_2(z) + \dots + a_m F'_m(z) = 0$$

on \mathfrak{D}_1 . Then

$$a_1 F_1(z) + a_2 F_2(z) + \dots + a_m F_m(z) = a_{m+1}$$

for some constant a_{m+1} . But then

$$a_1 f_1(z) + a_2 f_2(z) + \cdots + a_m f_m(z) - a_{m+1} f_{m+1}(z) = 0,$$

on \mathfrak{D}_1 and hence on \mathfrak{D} . This implies that

$$a_1 = a_2 = \cdots = a_m = a_{m+1} = 0$$

by the linear independence of the f_j , thus that the $F'_j(z)$ are linearly independent over \mathbb{C} .

This together with (5.9) and the fact that the $F'_j(z)$ are holomorphic on \mathfrak{D}_1 implies that

$$G_1(z) = G_2(z) = \cdots = G_m(z) = 0$$

on \mathfrak{D}_1 , by the induction hypothesis. By (5.8) we have

$$G_1(z) = G_2(z) = \cdots = G_m(z) = G_{m+1}(z) = 0$$

on \mathfrak{D}_1 and hence on \mathfrak{D} , and the result is true for $n = m + 1$, thus completing the induction proof. \square

We define

$$(5.10) \quad \mathfrak{W}_p := \left\{ \mathcal{F}_1\left(\frac{a}{p}; z\right), \mathcal{F}_2\left(\frac{a}{p}; z\right) : 0 < a < p \right\} \\ \cup \left\{ \mathcal{F}_1(a, b, p; z), \mathcal{F}_2(a, b, p; z) : 0 \leq a < p \text{ and } 0 < b < p \right\}.$$

Now let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \quad \text{so that } A(\infty) = a/c \text{ is a cusp.}$$

As mentioned above we must show that $\mathcal{J}\left(\frac{1}{p}; z\right) |[A]_1$ expanded as a series in q_{p^2} has only finitely many terms with negative exponents. We examine each of the functions

$$\frac{\eta(p^2 z)}{\eta(z)} \mathcal{F}_1\left(\frac{1}{p}; z\right), \quad \mathcal{F}_2\left(\frac{\ell}{p}; p^2 z\right) \quad (1 \leq \ell \leq \frac{1}{2}(p-1)),$$

which occur on the right side of (5.6). By Theorem 4.5 we have

$$\mathcal{F}_1\left(\frac{1}{p}; z\right) |[A]_1 = \varepsilon_A \mathcal{F}_A(z),$$

for some $\mathcal{F}_A \in \mathfrak{W}_p$ and some root of unity ε_A . Thus

$$\frac{\eta(p^2 z)}{\eta(z)} \mathcal{F}_1\left(\frac{1}{p}; z\right) |[A]_1 = S_A(z) + W_A(z) \int_{-\bar{z}}^{i\infty} \frac{g_A(\tau)}{\sqrt{-i(\tau+z)}} d\tau,$$

for some functions S_A, W_A , and g_A holomorphic on \mathfrak{h} . The function $S_A(z)$ is the product of a constant, the function

$$\frac{\eta(p^2 z)}{\eta(z)} |[A]_1,$$

the function $\eta(z)$, and one of the following:

$$\mathcal{N}\left(\frac{a}{p}; z\right), \mathcal{M}\left(\frac{a}{p}; z\right) + \varepsilon_2 \left(\frac{a}{p}; z\right), \mathcal{N}(a, b, p, z) \mathcal{M}(a, b, p, z) + \varepsilon_2(a, b, p; z).$$

Using the fact that $\frac{\eta(p^2z)}{\eta(z)}$ is a modular function on $\Gamma_0(p^2)$ and by examining (2.5)–(2.8) we find that $S_A(z)$ has only finitely many terms with negative exponents when expanded as a series in q_{p^2} .

We let $1 \leq \ell \leq \frac{1}{2}(p-1)$ and show an analogous result holds for $\mathcal{F}_2\left(\frac{\ell}{p}; z\right) \mid [A]_1$.

Case 1. $c \not\equiv 0 \pmod{p}$. From Theorem 4.5 we have

$$\mathcal{F}_2\left(\frac{\ell}{p}; z\right) = i \mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [S]_1.$$

We choose b' so that $b'c \equiv d \pmod{p^2}$ and

$$\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Y \begin{pmatrix} 1 & b' \\ 0 & p^2 \end{pmatrix},$$

where

$$SY = \begin{pmatrix} -c & \frac{1}{p^2}(b'c - d) \\ p^2a & b - b'a \end{pmatrix} \in \Gamma_0(p^2).$$

Hence by Theorem 4.1 we have

$$\begin{aligned} \mathcal{F}_2\left(\frac{\ell}{p}; p^2z\right) \mid [A]_1 &= i p^{-1} \mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [S]_1 \mid \left[\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}\right]_1 \mid [A]_1 \\ &= i p^{-1} \mathcal{F}_1\left(\frac{\ell}{p}; z\right) \mid [SY]_1 \mid \left[\begin{pmatrix} 1 & b' \\ 0 & p^2 \end{pmatrix}\right]_1 \\ &= i p^{-1} \mu(SY, \ell) \mathcal{F}_1\left(\frac{\ell'}{p}; z\right) \mid \left[\begin{pmatrix} 1 & b' \\ 0 & p^2 \end{pmatrix}\right]_1 \\ &= i \mu(SY, \ell) \mathcal{F}_1\left(\frac{\ell'}{p}; \frac{z+b'}{p^2}\right), \end{aligned}$$

for some integer ℓ' . Hence

$$\mathcal{F}_2\left(\frac{\ell}{p}; p^2z\right) \mid [A]_1 = S_{\ell,A}(z) + W_{\ell,A}(z) \int_{-\frac{z+b'}{p^2}}^{i\infty} \frac{g_{\ell,A}(\tau)}{\sqrt{-i\left(\tau + \frac{z+b'}{p^2}\right)}} d\tau,$$

for some functions $S_{\ell,A}$, $W_{\ell,A}$, and $g_{\ell,A}$ holomorphic on \mathfrak{h} . By considering the transformation $\tau \mapsto \frac{\tau-b'}{p^2}$ we find that

$$\mathcal{F}_2\left(\frac{\ell}{p}; p^2z\right) \mid [A]_1 = S_{\ell,A}(z) + W_{\ell,A}(z) \int_{-\bar{z}}^{i\infty} \frac{\tilde{g}_{\ell,A}(\tau)}{\sqrt{-i(\tau+z)}} d\tau,$$

for some holomorphic function $\tilde{g}_{\ell,A}$. This time the function $S_{\ell,A}(z)$ is a product of a constant and

$$\eta(z) \mathcal{N}\left(\frac{a}{p}; z\right),$$

with z replaced by $\frac{z+b'}{p^2}$, and thus has only finitely many terms with negative exponents when expanded as a series in q_{p^2} . In fact in this case the exponents are nonnegative.

Case 2. $c \equiv 0 \pmod{p^2}$. Then

$$\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} A = Y' \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$Y' = \begin{pmatrix} a & p^2 b \\ cp^{-2} & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

By Theorem 4.5 we have

$$\begin{aligned} \mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right) | [A]_1 &= p^{-1} \mathcal{F}_2 \left(\frac{\ell}{p}; z \right) \left| \left[\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \right]_1 \right| [A]_1 \\ &= p^{-1} \mathcal{F}_2 \left(\frac{\ell}{p}; z \right) | [Y]_1 \left| \left[\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \right]_1 \right| \\ &= \varepsilon_{\ell,A} \mathcal{F}_{\ell,A}(p^2 z), \end{aligned}$$

for some $\mathcal{F}_{\ell,A} \in \mathfrak{W}_p$ and some root of unity $\varepsilon_{\ell,A}$. As in Case 1 we find that

$$\mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right) | [A]_1 = S_{\ell,A}(z) + W_{\ell,A}(z) \int_{-\bar{z}}^{i\infty} \frac{\tilde{g}_{\ell,A}(\tau)}{\sqrt{-i(\tau+z)}} d\tau,$$

for some functions $S_{\ell,A}$, $W_{\ell,A}$, and $\tilde{g}_{\ell,A}$ holomorphic on \mathfrak{h} , and for which $S_A(z)$ has only finitely many terms with negative exponents when expanded as a series in q_{p^2} .

Case 3. $c \equiv 0 \pmod{p}$ and $c \not\equiv 0 \pmod{p^2}$. We choose b'' so that $b''c \equiv dp \pmod{p^2}$ and

$$\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Y'' \begin{pmatrix} p & b'' \\ 0 & p \end{pmatrix},$$

where

$$Y'' = \begin{pmatrix} pa & bp - ab'' \\ c/p & \frac{1}{p^2}(dp - cb'') \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Again by Theorem 4.5 it follows that

$$\mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right) | [A]_1 = \varepsilon_{\ell,A} \mathcal{F}_{\ell,A} \left(z + \frac{b''}{p} \right).$$

As in Case 2 we find that

$$\mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right) | [A]_1 = S_{\ell,A}(z) + W_{\ell,A}(z) \int_{-\bar{z}}^{i\infty} \frac{\tilde{g}_{\ell,A}(\tau)}{\sqrt{-i(\tau+z)}} d\tau,$$

for some functions $S_{\ell,A}$, $W_{\ell,A}$, and $\tilde{g}_{\ell,A}$ holomorphic on \mathfrak{h} and for which $S_A(z)$ has only finitely many terms with negative exponents when expanded as a series in q_{p^2} .

From (5.5), (5.6) we have

$$\begin{aligned} \mathcal{J} \left(\frac{1}{p}; z \right) | [A]_1 &= \mathcal{J}^* \left(\frac{1}{p}; z \right) | [A]_1 \\ &= \sum_{\ell=0}^{\frac{1}{2}(p-1)} \tilde{S}_{\ell,A}(z) + \sum_{\ell=0}^{\frac{1}{2}(p-1)} \tilde{W}_{\ell,A}(z) \times \int_{-\bar{z}}^{i\infty} \frac{\tilde{g}_{\ell,A}(\tau)}{\sqrt{-i(\tau+z)}} d\tau \end{aligned}$$

where

$$\begin{aligned} \tilde{S}_{0,A}(z) &= S_A(z), & \tilde{W}_{0,A}(z) &= W_A(z), \\ \tilde{S}_{\ell,A}(z) &= 2 \chi_{12}(p) (-1)^{\ell+1} \sin\left(\frac{6\ell\pi}{p}\right) S_{\ell,A}(z), \\ \tilde{W}_{\ell,A}(z) &= 2 \chi_{12}(p) (-1)^{\ell+1} \sin\left(\frac{6\ell\pi}{p}\right) W_{\ell,A}(z), \end{aligned}$$

for $1 \leq \ell \leq \frac{1}{2}(p-1)$. We claim that the sum

$$\sum_{\ell=0}^{\frac{1}{2}(p-1)} \tilde{W}_{\ell,A}(z) \int_{-\bar{z}}^{i\infty} \frac{\tilde{g}_{\ell,A}(\tau)}{\sqrt{-i(\tau+z)}} d\tau$$

is identically zero. Hence we may suppose that not all the functions

$$\tilde{W}_{0,A}(z), \tilde{W}_{1,A}(z), \dots, \tilde{W}_{\frac{1}{2}(p-1),A}(z)$$

are identically zero. We take a maximal linearly independent subset of them, say,

$$\tilde{W}_{1,A}^*(z), \tilde{W}_{2,A}^*(z), \dots, \tilde{W}_{m,A}^*(z).$$

Then for each ℓ , $0 \leq \ell \leq \frac{1}{2}(p-1)$, there exist constants $\beta_{j,\ell}$ such that

$$\tilde{W}_{\ell,A}(z) = \sum_{j=1}^m \beta_{j,\ell} \tilde{W}_{j,A}^*(z),$$

and we have

$$\begin{aligned} \sum_{\ell=0}^{\frac{1}{2}(p-1)} \tilde{W}_{\ell,A}(z) \int_{-\bar{z}}^{i\infty} \frac{\tilde{g}_{\ell,A}(\tau)}{\sqrt{-i(\tau+z)}} d\tau &= \sum_{\ell=0}^{\frac{1}{2}(p-1)} \sum_{j=1}^m \beta_{j,\ell} \tilde{W}_{j,A}^*(z) \int_{-\bar{z}}^{i\infty} \frac{\tilde{g}_{\ell,A}(\tau)}{\sqrt{-i(\tau+z)}} d\tau \\ &= \sum_{j=1}^m \tilde{W}_{j,A}^*(z) \int_{-\bar{z}}^{i\infty} \sum_{\ell=0}^{\frac{1}{2}(p-1)} \beta_{j,\ell} \tilde{g}_{\ell,A}(\tau) \frac{d\tau}{\sqrt{-i(\tau+z)}} \\ &= \sum_{j=1}^m \tilde{W}_{j,A}^*(z) \int_{-\bar{z}}^{i\infty} \frac{g_j^*(\tau) d\tau}{\sqrt{-i(\tau+z)}}, \end{aligned}$$

where

$$g_j^*(\tau) = \sum_{\ell=0}^{\frac{1}{2}(p-1)} \beta_{j,\ell} \tilde{g}_{\ell,A}(\tau)$$

is holomorphic on \mathfrak{h} . Applying $\frac{\partial}{\partial \bar{z}}$ to $\mathcal{J}^*\left(\frac{1}{p}; z\right) | [A]_1$ gives

$$\begin{aligned} 0 &= \frac{\partial}{\partial \bar{z}} \sum_{j=1}^m \tilde{W}_{j,A}^*(z) \int_{-\bar{z}}^{i\infty} \frac{g_j^*(\tau) d\tau}{\sqrt{-i(\tau+z)}} \\ &= \sum_{j=1}^m \frac{\tilde{W}_{j,A}^*(z) g_j^*(-\bar{z})}{\sqrt{-i(z+\bar{z})}}, \end{aligned}$$

since $\mathcal{J}^* \left(\frac{1}{p}; z \right) = \mathcal{J} \left(\frac{1}{p}; z \right)$ and the $S_{\ell,A}(z)$ are holomorphic. Thus

$$\sum_{j=1}^m \tilde{W}_{j,A}(z) g_j^*(-\bar{z}) = 0,$$

and so all the g_j^* are identically zero by Lemma 5.4. Hence

$$\mathcal{J} \left(\frac{1}{p}; z \right) | [A]_1 (z) = \sum_{\ell=0}^{\frac{1}{2}(p-1)} S_{\ell,A}(z).$$

Each of the functions $S_{\ell,A}(z)$ has only finitely many terms with negative exponents when expanded as a series in q_{p^2} . Thus $\mathcal{J} \left(\frac{1}{p}; z \right)$ is a weakly holomorphic modular form of weight 1 on $\Gamma_0(p^2) \cap \Gamma_1(p)$, which completes the proof of Theorem 5.1.

6. DYSON'S RANK CONJECTURE AND BEYOND

In this section we give a new proof of Dyson's rank conjecture and related results for the rank mod 11 due to Atkin and Hussain and for the rank mod 13 due to O'Brien.

We define the (weight k) Atkin U_p operator by

$$(6.1) \quad F | [U_p]_k := \frac{1}{p} \sum_{r=0}^{p-1} F \left(\frac{z+r}{p} \right) = p^{\frac{k}{2}-1} \sum_{n=0}^{p-1} F | [T_r]_k,$$

where

$$T_r = \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix},$$

and the more general $U_{p,m}$ defined by

$$(6.2) \quad \begin{aligned} F | [U_{p,m}]_k &:= \frac{1}{p} \sum_{r=0}^{p-1} \exp \left(-\frac{2\pi irm}{p} \right) F \left(\frac{z+r}{p} \right) \\ &= p^{\frac{k}{2}-1} \sum_{r=0}^{p-1} \exp \left(-\frac{2\pi irm}{p} \right) F | [T_r]_k. \end{aligned}$$

We note that $U_p = U_{p,0}$. In addition, if

$$F(z) = \sum_n a(n)q^n = \sum_n a(n) \exp(2\pi izn),$$

then

$$F | [U_{p,m}]_k = q^{m/p} \sum_n a(pn+m) q^n = \exp(2\pi imz/p) \sum_n a(pn+m) \exp(2\pi inz).$$

Definition 6.1. For $p > 3$ prime and $0 \leq m \leq p-1$ define

$$(6.3) \quad \mathcal{K}_{p,m}(z) := \sin \left(\frac{\pi}{p} \right) \mathcal{J} \left(\frac{1}{p}; z \right) | [U_{p,m}]_1,$$

where $\mathcal{J} \left(\frac{1}{p}; z \right)$ is defined in (5.1).

Then a straightforward calculation gives the following.

Proposition 6.2. For $p > 3$ prime and $0 \leq m \leq p - 1$ we have the following:

(i) For $m = 0$ or $\left(\frac{-24m}{p}\right) = -1$ we have

$$(6.4) \quad \mathcal{K}_{p,m}(z) = q^{m/p} \prod_{n=1}^{\infty} (1 - q^{pn}) \sum_{n=\lceil \frac{1}{p}(s_p - m) \rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k, p, pn + m - s_p) \zeta_p^k \right) q^n,$$

where $s_p = \frac{1}{24}(p^2 - 1)$ and $q = \exp(2\pi iz)$.

(ii) If $\left(\frac{-24m}{p}\right) = 1$ we choose $1 \leq a \leq \frac{1}{2}(p - 1)$ so that

$$-24m \equiv (6a)^2 \pmod{p},$$

and we have

(6.5)

$$\begin{aligned} &\mathcal{K}_{p,m}(z) \\ &= q^{m/p} \prod_{n=1}^{\infty} (1 - q^{pn}) \left(\sum_{n=\lceil \frac{1}{p}(s_p - m) \rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k, p, pn + m - s_p) \zeta_p^k \right) q^n \right. \\ &\quad \left. - \chi_{12}(p) (-1)^a \left(\zeta_p^{3a + \frac{1}{2}(p+1)} + \zeta_p^{-3a - \frac{1}{2}(p+1)} - \zeta_p^{3a + \frac{1}{2}(p-1)} - \zeta_p^{-3a + \frac{1}{2}(p-1)} \right) \right. \\ &\quad \left. \times q^{\frac{1}{p}(\frac{a}{2}(p-3a) - m)} \Phi_{p,a}(q) \right). \end{aligned}$$

Dyson’s rank conjecture is equivalent to showing that

$$\mathcal{K}_{5,0}(z) = \mathcal{K}_{7,0}(z) = 0.$$

In this section we will give a new proof of Dyson’s rank conjecture and much more. We will prove the following.

Theorem 6.3. Let $p > 3$ be prime and suppose $0 \leq m \leq p - 1$. Then:

- (i) $\mathcal{K}_{p,0}(z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma_1(p)$.
- (ii) If $1 \leq m \leq (p - 1)$, then $\mathcal{K}_{p,m}(z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma(p)$. In particular,

$$\mathcal{K}_{p,m}(z) \mid [A]_1 = \exp\left(\frac{2\pi ibm}{p}\right) \mathcal{K}_{p,m}(z),$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p)$.

Definition 6.4. For $p > 3$ prime and $0 \leq m \leq p - 1$ define

$$\mathcal{R}_{p,m}(z) := \frac{1}{\eta(pz)} \mathcal{K}_{p,m}(z).$$

From Theorem 6.3 and (4.6) we have the following.

Corollary 6.5. Let $p > 3$ be prime and suppose $0 \leq m \leq p - 1$. Then $\mathcal{R}_{p,m}(z)$ is a weakly holomorphic modular form of weight $1/2$ on $\Gamma_1(p)$ with multiplier. In particular,

$$\mathcal{R}_{p,m}(z) \mid [A]_{\frac{1}{2}} = \frac{\exp\left(\frac{2\pi ibm}{p}\right)}{\nu_{\eta}(pA)} \mathcal{R}_{p,m}(z),$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p)$ and where

$${}^p A = \begin{pmatrix} a & pb \\ c/p & d \end{pmatrix}.$$

Remark. When $m = 0$ or $\left(\frac{-24m}{p}\right) = -1$,

$$\mathcal{R}_{p,m}(z) = \sum_{n=\lceil \frac{1}{p}(s_p-m) \rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k,p,pn+m-s_p) \zeta_p^k \right) q^{n+\frac{m}{p}-\frac{p}{24}},$$

and we note that in these cases Corollary 6.5 greatly strengthens a theorem of Ahlgren and Treener [1, Theorem 1.6, p. 271] and a theorem of Bringmann, Ono, and Rhoades [14, Theorem 1.1(i)].

Remark. When $\left(\frac{-24m}{p}\right) = 1$ we have

$$\begin{aligned} &\mathcal{R}_{p,m}(z) \\ &= \sum_{n=\lceil \frac{1}{p}(s_p-m) \rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k,p,pn+m-s_p) \zeta_p^k \right) q^{n+\frac{m}{p}-\frac{p}{24}} \\ &\quad - \chi_{12}(p) (-1)^a \left(\zeta_p^{3a+\frac{1}{2}(p+1)} + \zeta_p^{-3a-\frac{1}{2}(p+1)} - \zeta_p^{3a+\frac{1}{2}(p-1)} - \zeta_p^{-3a+\frac{1}{2}(p-1)} \right) \\ &\qquad \qquad \qquad \times q^{\frac{1}{p}\left(\frac{a}{2}(p-3a)-\frac{p^2}{24}\right)} \Phi_{p,a}(q). \end{aligned}$$

The result that this is a weakly holomorphic modular form is new. Here a is defined as in Proposition 6.2(ii).

6.1. Proof of Theorem 6.3. We let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p)$$

so that $a \equiv d \equiv 1 \pmod{p}$ and $c \equiv 0 \pmod{p}$. Let $0 \leq k \leq p-1$. We take $k' \equiv b+k \pmod{p}$ so that

$$T_k A = B_k T_{k'}$$

and

$$B_k = T_k A T_{k'}^{-1} = \begin{pmatrix} a+ck & \frac{1}{p}(-k'(a+kc)+b+kd) \\ pc & d-k'c \end{pmatrix} \in \Gamma_0(p^2) \cap \Gamma_1(p).$$

Firstly

$$\mathcal{K}_{p,m}(z) = \frac{1}{\sqrt{p}} \sin\left(\frac{\pi}{p}\right) \sum_{k=0}^{p-1} \exp\left(-\frac{2\pi i k m}{p}\right) \mathcal{J}\left(\frac{1}{p}; z\right) | [T_r]_1.$$

Thus

$$\begin{aligned}
 & \mathcal{K}_{p,m}(z) \mid [A]_1 \\
 &= \frac{1}{\sqrt{p}} \sin\left(\frac{\pi}{p}\right) \sum_{k=0}^{p-1} \exp\left(-\frac{2\pi i k m}{p}\right) \mathcal{J}\left(\frac{1}{p}; z\right) \mid [(T_k A)]_1 \\
 &= \frac{1}{\sqrt{p}} \sin\left(\frac{\pi}{p}\right) \sum_{k=0}^{p-1} \exp\left(-\frac{2\pi i(k' - b)m}{p}\right) \mathcal{J}\left(\frac{1}{p}; z\right) \mid [(B_k T_{k'})]_1 \\
 &= \frac{1}{\sqrt{p}} \sin\left(\frac{\pi}{p}\right) \exp\left(\frac{2\pi i b m}{p}\right) \sum_{k'=0}^{p-1} \exp\left(-\frac{2\pi i k' m}{p}\right) \mathcal{J}\left(\frac{1}{p}; z\right) \mid [T_{k'}]_1 \\
 & \hspace{15em} \text{(by Theorem 5.1 since } B_k \in \Gamma_0(p^2) \cap \Gamma_1(p)) \\
 &= \exp\left(\frac{2\pi i b m}{p}\right) \mathcal{K}_{p,m}(z),
 \end{aligned}$$

as required. Thus each function $\mathcal{K}_{p,m}(z)$ has the desired transformation property. It is clear that each $\mathcal{K}_{p,m}(z)$ is holomorphic on \mathfrak{h} . The cusp conditions follow by a standard argument. For $m = 0$ we will examine orders at each cusp in more detail in the next section.

6.2. Orders at cusps. Recall from Corollary 3.2 that

$$\begin{aligned}
 (6.6) \quad \mathfrak{Y}_p &:= \left\{ \mathcal{G}_1\left(\frac{a}{p}; z\right), \mathcal{G}_2\left(\frac{a}{p}; z\right) : 0 < a < p \right\} \\
 & \cup \{ \mathcal{G}_1(a, b, p; z), \mathcal{G}_2(a, b, p; z) : 0 \leq a < p \text{ and } 0 < b < p \}
 \end{aligned}$$

is a vector valued Maass form of weight $\frac{1}{2}$ for the full modular group $SL_2(\mathbb{Z})$ and that the action of $SL_2(\mathbb{Z})$ on each element is given explicitly by Theorem 3.1. Also for each $\mathcal{G} \in \mathfrak{Y}_p$ there are unique holomorphic functions $\mathcal{G}_{\text{holo}}(z)$ and $\mathcal{G}_{\text{shadow}}(z)$ such that

$$\mathcal{G}(z) = \mathcal{G}_{\text{holo}}(z) + \int_{-\bar{z}}^{i\infty} \frac{\mathcal{G}_{\text{shadow}}(\tau) d\tau}{\sqrt{-i(\tau + z)}}.$$

Also each $\mathcal{G}_{\text{holo}}(z)$ has a q -expansion

$$\mathcal{G}_{\text{holo}}(z) = \sum_{m \geq m_0} a(m) \exp\left(2\pi i z \frac{m}{24p^2}\right),$$

where $a(m_0) \neq 0$. We define

$$\text{ord}_{\text{holo}}(\mathcal{G}; \infty) := \frac{m_0}{24p^2}.$$

For any cusp $\frac{a}{c}$ with $(a, c) = 1$ we define

$$\text{ord}_{\text{holo}}\left(\mathcal{G}; \frac{a}{c}\right) := \text{ord}_{\text{holo}}(\mathcal{G} \mid [A]_1; \infty),$$

where $A \in SL_2(\mathbb{Z})$ and $A\infty = \frac{a}{c}$. We note that $\mathcal{G} \mid [A]_1 \in \mathfrak{Y}_p$. One can easily check that this definition does not depend on the choice of A so that ord_{holo} is well-defined. We also note that when $\mathcal{G}(z)$ is a weakly holomorphic modular form this definition coincides with the definition of invariant order at a cusp [15, p. 2319],

[11, p. 275]. The order of each function $\mathcal{F}(z) = \eta(z) \mathcal{G}(z) \in \eta(z) \cdot \mathfrak{Y}_p$ is defined in the natural way, i.e.,

$$\text{ord}_{\text{holo}} \left(\mathcal{F}; \frac{a}{c} \right) := \text{ord} \left(\eta(z); \frac{a}{c} \right) + \text{ord}_{\text{holo}} \left(\mathcal{G}; \frac{a}{c} \right),$$

where $\text{ord} \left(\eta(z); \frac{a}{c} \right)$ is the usual invariant order of $\eta(z)$ at the cusp $\frac{a}{c}$ [34, p. 34].

We determine $\text{ord}_{\text{holo}}(\mathcal{F}; \infty)$ for each $\mathcal{F}(z) = \eta(z) \mathcal{G}(z) \in \mathfrak{Y}_p = \eta(z) \cdot \mathfrak{Y}_p$. After some calculation we find the following.

Proposition 6.6. *Let $p > 3$ be prime. Then*

$$\text{ord}_{\text{holo}} \left(\mathcal{F}_1 \left(\frac{a}{p}; z \right); \infty \right) = 0,$$

$$\text{ord}_{\text{holo}}(\mathcal{F}_1(a, b, p; z); \infty) = \begin{cases} \frac{b}{2p} - \frac{3b^2}{2p^2} & \text{if } 0 \leq \frac{b}{p} < \frac{1}{6}, \\ \frac{3b}{2p} - \frac{3b^2}{2p^2} & \text{if } \frac{1}{6} < \frac{b}{p} < \frac{1}{2}, \\ \frac{5b}{2p} - \frac{3b^2}{2p^2} & \text{if } \frac{1}{2} < \frac{b}{p} < \frac{5}{6}, \\ \frac{7b}{2p} - \frac{3b^2}{2p^2} & \text{if } \frac{5}{6} < \frac{b}{p} < 1, \end{cases}$$

$$\text{ord}_{\text{holo}} \left(\mathcal{F}_2 \left(\frac{a}{p}; z \right); \infty \right) = \text{ord}_{\text{holo}}(\mathcal{F}_2(a, b, p; z); \infty) = \begin{cases} \frac{a}{2p} - \frac{3a^2}{2p^2} & \text{if } 0 \leq \frac{a}{p} < \frac{1}{6}, \\ \frac{3a}{2p} - \frac{3a^2}{2p^2} & \text{if } \frac{1}{6} < \frac{a}{p} < \frac{5}{6}, \\ \frac{5a}{2p} - \frac{3a^2}{2p^2} - 1 & \text{if } \frac{5}{6} < \frac{a}{p} < 1. \end{cases}$$

We also need [34, Corollary 2.2].

Proposition 6.7. *Let $N \geq 1$ and let*

$$F(z) = \prod_{m|N} \eta(mz)^{r_m},$$

where each $r_m \in \mathbb{Z}$. Then for $(a, c) = 1$,

$$\text{ord} \left(F(z); \frac{a}{c} \right) = \sum_{m|N} \frac{(m, c)^2 r_m}{24m}.$$

From [18, Corollary 4, p. 930] we have the following.

Proposition 6.8. *Let $p > 3$ be prime. Then a set of inequivalent cusps for $\Gamma_1(p)$ is given by*

$$\infty, 0, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{\frac{1}{2}(p-1)}, \frac{2}{p}, \frac{3}{p}, \dots, \frac{\frac{1}{2}(p-1)}{p}.$$

We next calculate lower bounds of the invariant order of $\mathcal{K}_{p,0}(z)$ at each cusp of $\Gamma_1(p)$.

Theorem 6.9. *Let $p > 3$ be prime, and suppose $2 \leq m \leq \frac{1}{2}(p-1)$. Then*

(i)

$$\text{ord}(\mathcal{K}_{p,0}(z); 0) \begin{cases} \geq 0 & p = 5 \text{ or } 7, \\ = -\frac{1}{24p}(p-5)(p-7) & \text{if } p > 7; \end{cases}$$

(ii)

$$\text{ord} \left(\mathcal{K}_{p,0}(z); \frac{1}{m} \right) \begin{cases} = -\frac{3}{2p} \left(\frac{1}{6}(p-1) - m \right) \left(\frac{1}{6}(p+1) - m \right) & \text{if } 2 \leq m < \frac{1}{6}(p-1), \\ \geq 0 & \text{otherwise;} \end{cases}$$

and

(iii)

$$\text{ord} \left(\mathcal{K}_{p,0}(z); \frac{m}{p} \right) \geq \left(\frac{p^2 - 1}{24p} \right).$$

Proof. We derive lower bounds for $\text{ord}(\mathcal{K}_{p,0}(z); \zeta)$ for each cusp ζ of $\Gamma_1(p)$ not equivalent to ∞ .

First we show that

$$\mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right) \mid [U_p]_1 = 0,$$

for $1 \leq \ell \leq \frac{1}{2}(p-1)$. From Theorem 4.5

$$\mathcal{F}_2 \left(\frac{\ell}{p}; z + p \right) = \zeta_p^{\ell'} \mathcal{F}_2 \left(\frac{\ell}{p}; z \right),$$

where $\ell' \equiv \frac{3}{2}(p-1)\ell^2 \pmod{p}$. We have

$$\begin{aligned} \mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right) \mid [U_p]_1 &= \frac{1}{p} \sum_{r=0}^{p-1} \mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z + rp \right) \\ &= \frac{1}{p} \mathcal{F}_2 \left(\frac{\ell}{p}; p^2 z \right) \sum_{r=0}^{p-1} \left(\zeta_p^{\ell'} \right)^r \\ &= 0. \end{aligned}$$

Thus from (5.5), (5.6), (6.3) we have

$$\mathcal{K}_{p,0}(z) = \sin \left(\frac{\pi}{p} \right) \mathcal{J} \left(\frac{1}{p}; z \right) \mid [U_{p,0}]_1 = \sin \left(\frac{\pi}{p} \right) \mathcal{F}_1^* \left(\frac{1}{p}; z \right) \mid [U_p]_1,$$

where

$$(6.7) \quad \mathcal{F}_1^* \left(\frac{1}{p}; z \right) = \frac{\eta(p^2 z)}{\eta(z)} \mathcal{F}_1 \left(\frac{1}{p}; z \right).$$

Next we calculate

$$\mathcal{F}_1^* \left(\frac{1}{p}; z \right) \mid [T_k A]_1,$$

for each $0 \leq k \leq p-1$ and each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$.

Case 1. $a + kc \not\equiv 0 \pmod{p}$. Choose $0 \leq k' \leq p-1$ such that

$$(a + kc)k' \equiv (b + kd) \pmod{p}.$$

Then

$$T_k A = C_k T_{k'},$$

where

$$C_k = T_k A T_{k'}^{-1} = \begin{pmatrix} a + ck & \frac{1}{p}(-k'(a + kc) + b + kd) \\ pc & d - k'c \end{pmatrix} \in \Gamma_0(p).$$

From Theorem 4.1 we have

$$(6.8) \quad \begin{aligned} \mathcal{F}_1^* \left(\frac{1}{p}; z \right) \mid [T_k A]_1 &= \mathcal{F}_1^* \left(\frac{1}{p}; z \right) \mid [C_k T_{k'}]_1 \\ &= (F_p(z) \mid [C_k T_{k'}]_0) \left(\mu(C_k, 1) \mathcal{F}_1 \left(\frac{\ell}{p}; z \right) \mid [T_{k'}]_1 \right), \end{aligned}$$

where

$$(6.9) \quad F_p(z) = \frac{\eta(p^2 z)}{\eta(z)}.$$

Case 2. $a + kc \equiv 0 \pmod{p}$. In this case we find that

$$T_k A = D_k P,$$

where

$$P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_k = \begin{pmatrix} \frac{1}{p}(a + kc) & b + kd \\ c & pd \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

and

$$E_k = D_k S = \begin{pmatrix} b + kd & \frac{-1}{p}(a + kc) \\ pd & -c \end{pmatrix} \in \Gamma_0(p).$$

From Theorem 4.1 we have

$$\mathcal{F}_1 \left(\frac{1}{p}; z \right) \mid [E_k]_1 = \mu(E_k, 1) \mathcal{F}_1 \left(\frac{\overline{-c}}{p}; z \right).$$

By Theorem 4.5(2) we have

$$\mathcal{F}_1 \left(\frac{1}{p}; z \right) \mid [D_k]_1 = \mu(E_k, 1) \mathcal{F}_1 \left(\frac{\overline{-c}}{p}; z \right) \mid [S^{-1}]_1 = i\mu(E_k, 1) \mathcal{F}_2 \left(\frac{\overline{-c}}{p}; z \right)$$

and

$$\mathcal{F}_1 \left(\frac{1}{p}; z \right) \mid [T_k A]_1 = i\mu(E_k, 1) \mathcal{F}_2 \left(\frac{\overline{-c}}{p}; z \right) \mid [P]_1,$$

so that

$$(6.10) \quad \mathcal{F}_1^* \left(\frac{1}{p}; z \right) \mid [T_k A]_1 = (F_p(z) \mid [D_k P]_0) \left(i\mu(E_k, 1) \mathcal{F}_2 \left(\frac{\overline{-c}}{p}; z \right) \mid [P]_1 \right).$$

Now we are ready to examine each cusp ζ of $\Gamma_1(p)$. We choose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \quad \text{so that } A(\infty) = \frac{a}{c} = \zeta.$$

(i) $\zeta = 0$. Here $a = 0, c = 1$, and we assume that $0 \leq k \leq p - 1$. If $k \neq 0$, then applying (6.8) we have

$$\begin{aligned} \text{ord}_{\text{holo}} \left(\mathcal{F}_1^* \left(\frac{1}{p}; \frac{z+k}{p} \right); 0 \right) &= \frac{1}{p} \text{ord} \left(F_p(z); \frac{k}{p} \right) + \frac{1}{p} \text{ord}_{\text{holo}} \left(\mathcal{F}_1 \left(\frac{\ell}{p}; z \right); \infty \right) \\ &= 0 + 0 = 0, \end{aligned}$$

by Propositions 6.7 and 6.6. Now applying (6.10) with $k = 0$ we have

$$\begin{aligned} \text{ord}_{\text{holo}} \left(\mathcal{F}_1^* \left(\frac{1}{p}; \frac{z}{p} \right); 0 \right) &= p \text{ord} (F_p(z); 0) + p \text{ord}_{\text{holo}} \left(\mathcal{F}_2 \left(\frac{1}{p}; z \right); \infty \right) \\ &= -\frac{1}{24p}(p^2 - 1) + \begin{cases} \frac{6}{5} & \text{if } p = 5, \\ \frac{1}{2p}(p - 3) & \text{if } p > 5 \end{cases} \\ &= \begin{cases} 1 & \text{if } p = 5, \\ -\frac{1}{24p}(p - 5)(p - 7) & \text{if } p > 5, \end{cases} \end{aligned}$$

again by Propositions 6.7 and 6.6. The result (i) follows since

$$\text{ord}(\mathcal{K}_{p,0}(z); 0) \geq \min_{0 \leq k \leq p-1} \text{ord}_{\text{holo}} \left(\mathcal{F}_1^* \left(\frac{1}{p}; \frac{z+k}{p} \right); 0 \right).$$

(ii) $\zeta = \frac{1}{m}$, where $2 \leq m \leq \frac{1}{2}(p - 1)$. Let $A = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ so that $A(\infty) = 1/m$. If $km \not\equiv -1 \pmod{p}$ we apply (6.8) with $C_k = \begin{pmatrix} 1 + km & * \\ pm & 1 - k'm \end{pmatrix}$ and find that

$$\begin{aligned} \text{ord}_{\text{holo}} \left(\mathcal{F}_1^* \left(\frac{1}{p}; \frac{z+k}{p} \right); \frac{1}{m} \right) &= \frac{1}{p} \text{ord} \left(F_p(z); \frac{1+km}{pm} \right) \\ &\quad + \frac{1}{p} \text{ord}_{\text{holo}} \left(\mathcal{F}_1 \left(\frac{\ell}{p}; z \right); \infty \right) \\ &= 0 + 0 = 0. \end{aligned}$$

Now we assume $km \equiv -1 \pmod{p}$ and we will apply (6.10). We have

$$\begin{aligned} \text{ord}_{\text{holo}} \left(\mathcal{F}_1^* \left(\frac{1}{p}; \frac{z+k}{p} \right); \frac{1}{m} \right) &= p \text{ord} \left(F_p(z); \frac{(1+km)/p}{m} \right) \\ &\quad + p \text{ord}_{\text{holo}} \left(\mathcal{F}_2 \left(\frac{m}{p}; z \right); \infty \right) \\ &= -\frac{1}{24p}(p^2 - 1) + \begin{cases} \frac{m}{2} - \frac{3m^2}{2p} & \text{if } 2 \leq m < \frac{p}{6}, \\ \frac{3m}{2} - \frac{3m^2}{2p} & \text{if } \frac{p}{6} < m \leq \frac{p-1}{2} \end{cases} \\ &\quad \begin{cases} = -\frac{3}{2p} \left(\frac{1}{6}(p-1) - m \right) \left(\frac{1}{6}(p+1) - m \right) & \text{if } 2 \leq m \leq \frac{1}{6}(p-1), \\ > 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and the result (ii) follows.

(iii) $\zeta = \frac{m}{p}$, where $2 \leq m \leq \frac{1}{2}(p - 1)$. Choose b, d so that $A = \begin{pmatrix} m & b \\ p & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $A(\infty) = m/p$. Since $m \not\equiv 0 \pmod{p}$ we may apply (6.8) for each k . We find

that $C_k = \begin{pmatrix} m + kp & * \\ p^2 & d - k'p \end{pmatrix}$ and

$$\begin{aligned} \text{ord}_{\text{holo}} \left(\mathcal{F}_1^* \left(\frac{1}{p}; \frac{z+k}{p} \right); \frac{m}{p} \right) &= \frac{1}{p} \text{ord} \left(F_p(z); \frac{m+kp}{p^2} \right) \\ &\quad + \frac{1}{p} \text{ord}_{\text{holo}} \left(\mathcal{F}_1 \left(\frac{\ell}{p}; z \right); \infty \right) \\ &= \frac{p^2 - 1}{24p} + 0 = \frac{p^2 - 1}{24p}. \end{aligned}$$

The result (iii) follows. □

Since

$$\text{ord}(\mathcal{K}_{p,0}(z); 0) = -\frac{1}{24p}(p-5)(p-7) < 0,$$

for $p > 7$ we have

Corollary 6.10. *The analogue of the Dyson Rank Conjecture does not hold for any prime $p > 7$. In other words $\mathcal{K}_{p,0}(z) \not\equiv 0$ for any prime $p > 7$.*

6.3. Proof of Dyson's rank conjecture. As noted before, Dyson's rank conjecture is equivalent to showing that

$$(6.11) \quad \mathcal{K}_{5,0}(z) = \mathcal{K}_{7,0}(z) = 0.$$

The first proof was given by Atkin and Swinnerton-Dyer [9]. Their proof involved finding and proving identities for basic hypergeometric functions, theta functions, and Lerch-type series using the theory of elliptic functions. It also involved identifying the generating functions for rank differences $N(0, p, pn + r) - N(k, p, pn + r)$ for $p = 5, 7$ for each $1 \leq k \leq \frac{1}{2}(p - 1)$ and each $r = 0, 1, \dots, p - 1$. Atkin and Swinnerton-Dyer note [9, p. 84] that they are unable to simplify their proof so as to obtain only Dyson's results. In particular to prove the result for $(p, r) = (5, 4)$ or $(p, r) = (7, 5)$ they must simultaneously prove identities for all r with $0 \leq r \leq p - 1$. Here we show how to avoid this difficulty.

To prove (6.11) we use the following.

Theorem 6.11 (The Valence Formula [38, p. 98]). *Let $f \neq 0$ be a modular form of weight k with respect to a subgroup Γ of finite index in $\Gamma(1) = SL_2(\mathbb{Z})$. Then*

$$(6.12) \quad \text{ORD}(f, \Gamma) = \frac{1}{12} \mu k,$$

where μ is index $\widehat{\Gamma}$ in $\widehat{\Gamma(1)}$,

$$\text{ORD}(f, \Gamma) := \sum_{\zeta \in R^*} \text{ORD}(f, \zeta, \Gamma),$$

R^* is a fundamental region for Γ , and

$$\text{ORD}(f; \zeta; \Gamma) = n(\Gamma; \zeta) \text{ord}(f; \zeta),$$

for a cusp ζ and $n(\Gamma; \zeta)$ denotes the fan width of the cusp $\zeta \pmod{\Gamma}$.

Remark. For $\zeta \in \mathfrak{h}$, $\text{ORD}(f; \zeta; \Gamma)$ is defined in terms of the invariant order $\text{ord}(f; \zeta)$, which is interpreted in the usual sense. See [38, p. 91] for details of this and the notation used.

$p = 5$.

cusps ζ	$n(\Gamma_1(5); \zeta)$	$\text{ord}(\mathcal{K}_{5,0}(z); \zeta)$	$\text{ORD}(\mathcal{K}_{5,0}(z), \Gamma_1(5), \zeta)$
∞	1	≥ 1	≥ 1
0	5	≥ 0	≥ 0
$\frac{1}{2}$	5	≥ 0	≥ 0
$\frac{2}{5}$	1	≥ 1	≥ 1

Hence $\text{ORD}(\mathcal{K}_{5,0}(z); \Gamma_1(5)) \geq 2$. But $\mu k = \frac{12}{12} = 1$. The Valence Formula implies that $\mathcal{K}_{5,0}(z)$ is identically zero, which proves Dyson’s conjecture for $p = 5$.

$p = 7$.

cusps ζ	$n(\Gamma_1(7); \zeta)$	$\text{ord}(\mathcal{K}_{7,0}(z); \zeta)$	$\text{ORD}(\mathcal{K}_{7,0}(z), \Gamma_1(7), \zeta)$
∞	1	≥ 1	≥ 1
0	7	≥ 0	≥ 0
$\frac{1}{2}$	7	≥ 0	≥ 0
$\frac{1}{3}$	7	≥ 0	≥ 0
$\frac{2}{7}$	1	≥ 1	≥ 1
$\frac{3}{7}$	1	≥ 1	≥ 1

Hence $\text{ORD}(\mathcal{K}_{7,0}(z); \Gamma_1(7)) \geq 3$. But $\mu k = \frac{24}{12} = 2$. The Valence Formula implies that $\mathcal{K}_{7,0}(z)$ is identically zero, which proves Dyson’s conjecture for $p = 7$.

6.4. The rank mod 11. Atkin and Hussain [8] studied the rank mod 11. In this section we find an identity for $\mathcal{K}_{11,0}(z)$ in terms of generalized eta-functions, which were defined in (1.7).

We will prove that

$$(6.13) \quad (q^{11}; q^{11})_\infty \sum_{n=1}^\infty \left(\sum_{k=0}^{10} N(k, 11, 11n - 5) \zeta_{11}^k \right) q^n = \sum_{k=1}^5 c_{11,k} j_{11,k}(z),$$

where

$$j_{11,k}(z) = \frac{\eta(11z)^4}{\eta(z)^2} \frac{1}{\eta_{11,4k}(z) \eta_{11,5k}(z)^2},$$

and

$$\begin{aligned} c_{11,1} &= 2 \zeta_{11}^9 + 2 \zeta_{11}^8 + \zeta_{11}^7 + \zeta_{11}^4 + 2 \zeta_{11}^3 + 2 \zeta_{11}^2 + 1, \\ c_{11,2} &= -(\zeta_{11}^9 + \zeta_{11}^8 + 2 \zeta_{11}^7 + \zeta_{11}^6 + \zeta_{11}^5 + 2 \zeta_{11}^4 + \zeta_{11}^3 + \zeta_{11}^2 + 1), \\ c_{11,3} &= 2 \zeta_{11}^8 + 2 \zeta_{11}^7 + 2 \zeta_{11}^4 + 2 \zeta_{11}^3 + 3, \\ c_{11,4} &= 4 \zeta_{11}^9 + \zeta_{11}^8 + 2 \zeta_{11}^7 + 2 \zeta_{11}^6 + 2 \zeta_{11}^5 + 2 \zeta_{11}^4 + \zeta_{11}^3 + 4 \zeta_{11}^2 + 4, \\ c_{11,5} &= -(\zeta_{11}^9 + 2 \zeta_{11}^8 - \zeta_{11}^7 + 2 \zeta_{11}^6 + 2 \zeta_{11}^5 - \zeta_{11}^4 + 2 \zeta_{11}^3 + \zeta_{11}^2 + 3). \end{aligned}$$

By Theorem 6.3 we know that the left side of (6.13) is a weakly holomorphic modular form of weight 1 on $\Gamma_1(11)$. We prove (6.13) by showing that the right side is also a weakly holomorphic modular form of weight 1 on $\Gamma_1(11)$ and using the Valence Formula (6.12). Following Biagioli [11] we define

$$(6.14) \quad f_{N,\rho}(z) := q^{(N-2\rho)^2/(8N)} (q^\rho, q^{N-\rho}, q^N; q^N)_\infty.$$

Then

$$f_{N,\rho}(z) = f_{N,N+\rho}(z) = f_{N,-\rho}(z),$$

and

$$f_{N,\rho}(z) = \eta(Nz) \eta_{N,\rho}(z).$$

We observe that

$$j_{11,k}(z) = \frac{\eta(11z)^7}{\eta(z)^2} \frac{1}{f_{11,4k}(z) f_{11,5k}(z)^2},$$

for $1 \leq k \leq 5$.

Theorem 6.12 (Biagioli [11]). *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Then*

$$f_{N,\rho}(z) \Big| [A]_{\frac{1}{2}} = (-1)^{\rho b + \lfloor \rho a/N \rfloor + \lfloor \rho/N \rfloor} \exp\left(\frac{\pi i a b}{N} \rho^2\right) \nu_{\theta_1}({}^N A) f_{N,\rho a}(z),$$

where

$${}^N A = \begin{pmatrix} a & bN \\ c/N & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

and $\nu_{\theta_1} = \nu_{\theta}^3$ is the theta-multiplier.

Biagioli has also calculated the orders of these functions at the cusps.

Proposition 6.13. *Let $1 \leq N \nmid \rho$, and let $(a, c) = 1$. Then*

$$\text{ord}\left(f_{N,\rho}(z), \frac{a}{c}\right) = \frac{g}{2N} \left(\frac{a\rho}{g} - \left\lfloor \frac{a\rho}{g} \right\rfloor - \frac{1}{2}\right)^2,$$

where $g = (N, c)$.

We need Knopp's [32, Theorem 2, p. 51] formula for the eta-multiplier given in the following.

Theorem 6.14. *For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we have*

$$\eta(z) \Big| [A]_{\frac{1}{2}} = \nu_{\eta}(A) \eta(z),$$

where

$$\nu_{\eta}(A) = \begin{cases} \left(\frac{d}{c}\right)^* \exp\left(\frac{\pi i}{12}((a+d)c - bd(c^2 - 1) - 3c)\right) & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right)_* \exp\left(\frac{\pi i}{12}((a+d)c - bd(c^2 - 1) + 3d - 3 - 3cd)\right) & \text{if } d \text{ is odd,} \end{cases}$$

$$\left(\frac{d}{c}\right)^* = \left(\frac{c}{|d|}\right), \quad \left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right) (-1)^{(\text{sgnc}-1)(\text{sgnd}-1)/4},$$

and (\cdot) is the Jacobi symbol.

We now show that each $j_{11,k}(z)$ is a weakly holomorphic function of weight 1 on $\Gamma_1(11)$. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(11)$. Then

$$\eta(11Az) = \eta({}^{11}A(11z)) = \nu_{\eta}({}^{11}A) \sqrt{cz + d} \eta(11z),$$

and

$$\begin{aligned}
 & j_{11,k} | [A]_1 \\
 &= \nu_\eta({}^{11}A)^7 \nu_\eta(A)^{-2} \nu_{\theta_1}^{-3}({}^{11}A) (-1)^{\lfloor 4ka/11 \rfloor + \lfloor 4k/11 \rfloor} \exp\left(\frac{-\pi i ab}{11}(66k^2)\right) \\
 &\quad \times \frac{\eta(11z)^7}{\eta(z)^2} \frac{1}{f_{11,4ka}(z) f_{11,5ka}(z)^2} \\
 &= \nu_\eta({}^{11}A)^{-2} \nu_\eta(A)^{-2} j_{11,k},
 \end{aligned}$$

since $a \equiv 1 \pmod{11}$ and

$$\lfloor 4ka/11 \rfloor \equiv \lfloor 4k/11 \rfloor \pmod{2}.$$

So we must show that

$$\nu_\eta({}^{11}A)^2 \nu_\eta(A)^2 = 1.$$

Case 1. c is odd. For $p > 3$ prime we find that

$$\nu_\eta({}^pA)^2 \nu_\eta(A)^2 = \exp\left(\frac{\pi i}{6p}(p+1)(-3c - bdc^2 + bdp + ca + cd)\right) = 1$$

in this case since $p \mid c$ and $p \equiv 11 \pmod{12}$.

Case 2. d is odd. For $p > 3$ prime we find that

$$\nu_\eta({}^pA)^2 \nu_\eta(A)^2 = \exp\left(\frac{\pi i}{6p}(p+1)(-acd^2 - ac + dbp - dc)\right) = 1$$

in this case since $p \mid c$ and $p \equiv 11 \pmod{12}$. It follows that each $j_{11,k}(z)$ is a weakly holomorphic function of weight 1 on $\Gamma_1(11)$.

Next we calculate orders at cusps. From Propositions 6.7 and 6.13 we have

	ord($j_{11,k}, \zeta$)					
cusps	ζ	$j_{11,1}$	$j_{11,2}$	$j_{11,3}$	$j_{11,4}$	$j_{11,5}$
∞		3	1	2	2	2
$1/m$		-1/11	-1/11	-1/11	-1/11	-1/11
$2/11$		1	2	2	2	3
$3/11$		2	2	1	3	2
$4/11$		2	2	3	2	1
$5/11$		2	3	2	1	2

where $2 \leq m \leq 5$.

Now we calculate lower bounds of orders at cusps of both sides of equation (6.13).

cusps ζ	$n(\Gamma_1(11); \zeta)$	ord(LHS; ζ)	ord(RHS; ζ)	ORD(LHS - RHS; ζ)
∞	1			
0	11	$\geq -1/11$	$\geq -1/11$	≥ -1
$1/m$	11	≥ 0	$\geq -1/11$	≥ -1
$m/11$	1	≥ 1	1	≥ 1

where $2 \leq m \leq 5$. But $\mu k = 5$. The result follows from the Valence Formula (6.12) provided we can show that $\text{ORD}(LHS - RHS, \infty) \geq 7$. This is easily carried out using Maple.

6.5. **The rank mod 13.** O'Brien [36] has studied the rank mod 13. Using the methods of the previous section we may obtain an identity for $\mathcal{K}_{13,0}(z)$. We state the identity and omit the details:

$$(6.15) \quad (q^{13}; q^{13})_\infty \sum_{n=1}^\infty \left(\sum_{k=0}^{10} N(k, 13, 13n - 7) \zeta_{13}^k \right) q^n \\ = \sum_{k=1}^6 \left(c_{13,k} + d_{13,k} 13 \frac{\eta(13z)^2}{\eta(z)^2} \right) j_{13,k}(z),$$

where

$$j_{13,k}(z) = \frac{\eta(13z)^3}{\eta(z)} \frac{1}{\eta_{13,2k}(z)^2 \eta_{13,3k}(z) \eta_{13,4k}(z) \eta_{13,5k}(z) \eta_{13,6k}(z)^2},$$

and

$$\begin{aligned} c_{13,1} &= 5\zeta_{13}^{11} + \zeta_{13}^{10} + 5\zeta_{13}^9 + 2\zeta_{13}^8 + 3\zeta_{13}^7 + 3\zeta_{13}^6 + 2\zeta_{13}^5 + 5\zeta_{13}^4 + \zeta_{13}^3 + 5\zeta_{13}^2 + 6, \\ c_{13,2} &= -\zeta_{13}^{11} + 2\zeta_{13}^9 + 2\zeta_{13}^8 - \zeta_{13}^7 - \zeta_{13}^6 + 2\zeta_{13}^5 + 2\zeta_{13}^4 - \zeta_{13}^2 + 3, \\ c_{13,3} &= \zeta_{13}^{11} + 2\zeta_{13}^{10} + 2\zeta_{13}^9 + 2\zeta_{13}^4 + 2\zeta_{13}^3 + \zeta_{13}^2 - 1, \\ c_{13,4} &= 3\zeta_{13}^{11} + 3\zeta_{13}^{10} + 5\zeta_{13}^8 + \zeta_{13}^7 + \zeta_{13}^6 + 5\zeta_{13}^5 + 3\zeta_{13}^3 + 3\zeta_{13}^2 + 5, \\ c_{13,5} &= \zeta_{13}^{11} - 3\zeta_{13}^{10} - \zeta_{13}^9 + 2\zeta_{13}^8 - 2\zeta_{13}^7 - 2\zeta_{13}^6 + 2\zeta_{13}^5 - \zeta_{13}^4 - 3\zeta_{13}^3 + \zeta_{13}^2 - 2, \\ c_{13,6} &= -\zeta_{13}^{11} - \zeta_{13}^{10} - 2\zeta_{13}^9 - \zeta_{13}^8 - 2\zeta_{13}^7 - 2\zeta_{13}^6 - \zeta_{13}^5 - 2\zeta_{13}^4 - \zeta_{13}^3 - \zeta_{13}^2 - 1, \\ d_{13,1} &= 2\zeta_{13}^{11} + \zeta_{13}^9 + \zeta_{13}^8 + \zeta_{13}^7 + \zeta_{13}^6 + \zeta_{13}^5 + \zeta_{13}^4 + 2\zeta_{13}^2 + 2, \\ d_{13,2} &= -\zeta_{13}^{11} - \zeta_{13}^{10} - \zeta_{13}^7 - \zeta_{13}^6 - \zeta_{13}^3 - \zeta_{13}^2, \\ d_{13,3} &= \zeta_{13}^{11} + \zeta_{13}^{10} + \zeta_{13}^9 + \zeta_{13}^8 + \zeta_{13}^5 + \zeta_{13}^4 + \zeta_{13}^3 + \zeta_{13}^2 + 1, \\ d_{13,4} &= \zeta_{13}^{10} - \zeta_{13}^9 + \zeta_{13}^8 + \zeta_{13}^5 - \zeta_{13}^4 + \zeta_{13}^3 + 1, \\ d_{13,5} &= -\zeta_{13}^{10} - \zeta_{13}^9 - \zeta_{13}^7 - \zeta_{13}^6 - \zeta_{13}^4 - \zeta_{13}^3 - 2, \\ d_{13,6} &= -\zeta_{13}^8 - \zeta_{13}^5. \end{aligned}$$

7. CONCLUDING REMARKS

In this section we discuss other approaches to the work of Bringmann and Ono [13] on Dyson's rank. Zagier [45] found a simplification of Bringmann and Ono's theorem [13, Theorem 1.2, p. 424] (see also Theorem 1.6) that $q^{-1/24}R(\zeta, q)$ is the holomorphic part of a weak Maass form of weight 1/2 when $\zeta \neq 1$ is any odd order root of unity. He did this by using the identity

$$(7.1) \quad R(z, q) = \frac{1 - z}{(q; q)_\infty} \sum_n \frac{(-1)^n q^{\frac{n}{2}(3n+1)}}{1 - zq^n}$$

to write $q^{-1/24}R(\zeta, q)$ in terms of Zwegers' μ -function [47, Proposition 1.4, p. 8]. Zwegers [47, Theorem 1.11, p. 15] showed how such a function could be completed using an explicit nonholomorphic function to become a weak Maass form of weight 1/2. See [45, Theorem 2.1]. We also mention that Kang [31] extended Zagier's and Zwegers' results to other rank-type functions.

We note that equation (7.1) follows easily from (2.1). Bringmann and Ono prove the result only if the order of ζ is odd. There is no such restriction on the order of ζ for Zagier's result. Zagier does not give the congruence subgroup for the

corresponding Maass form. In the case where ζ is a p th root of unity for $p > 3$ prime we showed that the relevant group is $\Gamma_0(p^2) \cap \Gamma_1(p)$.

Hickerson and Mortenson [29] have an alternative approach to the work of Bringmann, Ono, and Rhoades [13], [14] on Dyson's rank. Using [28, Theorems 3.5 and 3.9] they show how to make the results of Bringmann, Ono, and Rhoades more explicit. For integers $0 \leq a < M$ they define

$$(7.2) \quad D(a, M) = D(a, M, q) := \sum_{n=0}^{\infty} \left(N(a, M, n) - \frac{p(n)}{M} \right) q^n.$$

Their main theorem decomposes (7.2) into modular and mock modular components

$$(7.3) \quad D(a, M, q) = d(a, M, q) + T_{a,M}(q),$$

where $d(a, M, q)$ the mock modular part is given explicitly in terms of Appell-Lerch series, and $T_{a,M}(q)$ are certain linear combinations of theta-quotients. Hickerson and Mortenson do not consider modular transformation properties of the Dyson rank function. Our approach is to extend Bringmann and Ono's results. Hickerson and Mortenson's results depend on properties for Appell-Lerch series. It is not clear that Hickerson and Mortenson's results imply our Theorem 1.2. It would also be interesting to extend our theorem to $p = 2, 3$ and prime powers. The case $p = 2$ is related to Theorem 1.4, which is due to Zwegers [46]. When $M = p > 3$ is prime it should be possible to use Theorem 1.2 to obtain a result like (7.3), except $T_{a,M}(q)$ would not be given explicitly but rather as an element of an explicit space of modular forms.

Finally we note that Andersen [2] has applied the results of this paper as well as results of Zwegers [47] to give a new proof of Ramanujan's mock theta conjectures [6]. As mentioned earlier, the first proof of the mock theta conjectures was given by Hickerson [26]. Folsom [20] showed how the mock theta conjectures could be proved using the theory of Maass forms. However this involved verifying an identity to over 10^{13} coefficients, which is clearly beyond the limits of computation. Andersen's proof involves nonholomorphic vector valued modular forms and does not rely on any computational verification.

ACKNOWLEDGMENTS

I would like to thank Chris Jennings-Shaffer for his careful reading of this paper and for his suggestions and corrections. I would also like to thank Eric Mortenson for reminding me about [28] and [29] and Nick Andersen for [2]. Finally, I thank the referee for valuable comments, suggestions, and corrections.

REFERENCES

- [1] Scott Ahlgren and Stephanie Treneer, *Rank generating functions as weakly holomorphic modular forms*, Acta Arith. **133** (2008), no. 3, 267–279, DOI 10.4064/aa133-3-5. MR2434604
- [2] Nickolas Andersen, *Vector-valued modular forms and the mock theta conjectures*, Res. Number Theory **2** (2016), Art. 32, 14, DOI 10.1007/s40993-016-0062-6. MR3575840
- [3] George E. Andrews, *Mordell integrals and Ramanujan's "lost" notebook*, Analytic number theory (Philadelphia, Pa., 1980), Lecture Notes in Math., vol. 899, Springer, Berlin-New York, 1981, pp. 10–18. MR654518
- [4] George E. Andrews and Bruce C. Berndt, *Ramanujan's lost notebook. Part III*, Springer, New York, 2012. MR2952081
- [5] George E. Andrews and F. G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. (N.S.) **18** (1988), no. 2, 167–171, DOI 10.1090/S0273-0979-1988-15637-6. MR929094

- [6] George E. Andrews and F. G. Garvan, *Ramanujan's "lost" notebook. VI. The mock theta conjectures*, Adv. in Math. **73** (1989), no. 2, 242–255, DOI 10.1016/0001-8708(89)90070-4. MR987276
- [7] George E. Andrews and Dean Hickerson, *Ramanujan's "lost" notebook. VII. The sixth order mock theta functions*, Adv. Math. **89** (1991), no. 1, 60–105, DOI 10.1016/0001-8708(91)90083-J. MR1123099
- [8] A. O. L. Atkin and S. M. Hussain, *Some properties of partitions. II*, Trans. Amer. Math. Soc. **89** (1958), 184–200, DOI 10.2307/1993139. MR0103872
- [9] A. O. L. Atkin and P. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. (3) **4** (1954), 84–106, DOI 10.1112/plms/s3-4.1.84. MR0060535
- [10] Bruce C. Berndt and Song Heng Chan, *Sixth order mock theta functions*, Adv. Math. **216** (2007), no. 2, 771–786, DOI 10.1016/j.aim.2007.06.004. MR2351377
- [11] Anthony J. F. Biagioli, *A proof of some identities of Ramanujan using modular forms*, Glasgow Math. J. **31** (1989), no. 3, 271–295, DOI 10.1017/S0017089500007850. MR1021804
- [12] Kathrin Bringmann, *Mock theta functions, weak Maass forms, and applications*, Modular forms on Schiermonnikoog, Cambridge Univ. Press, Cambridge, 2008, pp. 29–56, DOI 10.1017/CBO9780511543371.004. MR2512355
- [13] Kathrin Bringmann and Ken Ono, *Dyson's ranks and Maass forms*, Ann. of Math. (2) **171** (2010), no. 1, 419–449, DOI 10.4007/annals.2010.171.419. MR2630043
- [14] Kathrin Bringmann, Ken Ono, and Robert C. Rhoades, *Eulerian series as modular forms*, J. Amer. Math. Soc. **21** (2008), no. 4, 1085–1104, DOI 10.1090/S0894-0347-07-00587-5. MR2425181
- [15] Kathrin Bringmann and Holly Swisher, *On a conjecture of Koike on identities between Thompson series and Rogers-Ramanujan functions*, Proc. Amer. Math. Soc. **135** (2007), no. 8, 2317–2326, DOI 10.1090/S0002-9939-07-08735-7. MR2302552
- [16] Youn-Seo Choi, *Tenth order mock theta functions in Ramanujan's lost notebook*, Invent. Math. **136** (1999), no. 3, 497–569, DOI 10.1007/s002220050318. MR1695205
- [17] Youn-Seo Choi, *Tenth order mock theta functions in Ramanujan's lost notebook. II*, Adv. Math. **156** (2000), no. 2, 180–285, DOI 10.1006/aima.2000.1948. MR1808245
- [18] Bumkyu Cho, Ja Kyung Koo, and Yoon Kyung Park, *Arithmetic of the Ramanujan-Göllnitz-Gordon continued fraction*, J. Number Theory **129** (2009), no. 4, 922–947, DOI 10.1016/j.jnt.2008.09.018. MR2499414
- [19] F. J. Dyson, *Some guesses in the theory of partitions*, Eureka **8** (1944), 10–15. MR3077150
- [20] Amanda Folsom, *A short proof of the mock theta conjectures using Maass forms*, Proc. Amer. Math. Soc. **136** (2008), no. 12, 4143–4149, DOI 10.1090/S0002-9939-08-09434-3. MR2431026
- [21] F. G. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11*, Trans. Amer. Math. Soc. **305** (1988), no. 1, 47–77, DOI 10.2307/2001040. MR920146
- [22] F. G. Garvan, *Combinatorial interpretations of Ramanujan's partition congruences*, Ramanujan revisited (Urbana-Champaign, Ill., 1987), Academic Press, Boston, MA, 1988, pp. 29–45. MR938958
- [23] Basil Gordon and Richard J. McIntosh, *Some eighth order mock theta functions*, J. London Math. Soc. (2) **62** (2000), no. 2, 321–335, DOI 10.1112/S0024610700008735. MR1783627
- [24] Basil Gordon and Richard J. McIntosh, *Modular transformations of Ramanujan's fifth and seventh order mock theta functions*, Rankin memorial issues, Ramanujan J. **7** (2003), no. 1-3, 193–222, DOI 10.1023/A:1026299229509. MR2035802
- [25] Basil Gordon and Richard J. McIntosh, *A survey of classical mock theta functions*, Partitions, q -series, and modular forms, Dev. Math., vol. 23, Springer, New York, 2012, pp. 95–144, DOI 10.1007/978-1-4614-0028-8_9. MR3051186
- [26] Dean Hickerson, *A proof of the mock theta conjectures*, Invent. Math. **94** (1988), no. 3, 639–660, DOI 10.1007/BF01394279. MR969247
- [27] Dean Hickerson, *On the seventh order mock theta functions*, Invent. Math. **94** (1988), no. 3, 661–677, DOI 10.1007/BF01394280. MR969248
- [28] Dean R. Hickerson and Eric T. Mortenson, *Hecke-type double sums, Appell-Lerch sums, and mock theta functions, I*, Proc. Lond. Math. Soc. (3) **109** (2014), no. 2, 382–422, DOI 10.1112/plms/pdu007. MR3254929
- [29] Dean Hickerson and Eric Mortenson, *Dyson's ranks and Appell-Lerch sums*, Math. Ann. **367** (2017), no. 1-2, 373–395, DOI 10.1007/s00208-016-1390-5. MR3606444

- [30] C. Jennings-Shaffer, *Overpartition rank differences modulo 7 by Maass forms*, J. Number Theory **163** (2016), 331–358, DOI 10.1016/j.jnt.2015.11.018. MR3459574
- [31] Soon-Yi Kang, *Mock Jacobi forms in basic hypergeometric series*, Compos. Math. **145** (2009), no. 3, 553–565, DOI 10.1112/S0010437X09004060. MR2507741
- [32] Marvin I. Knopp, *Modular functions in analytic number theory*, Markham Publishing Co., Chicago, Ill., 1970. MR0265287
- [33] Neal Koblitz, *Introduction to elliptic curves and modular forms*, 2nd ed., Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1993. MR1216136
- [34] Günter Köhler, *Eta products and theta series identities*, Springer Monographs in Mathematics, Springer, Heidelberg, 2011. MR2766155
- [35] Richard J. McIntosh, *Second order mock theta functions*, Canad. Math. Bull. **50** (2007), no. 2, 284–290, DOI 10.4153/CMB-2007-028-9. MR2317449
- [36] J. N. O'Brien, *Some properties of partitions with special reference to primes other than 5, 7 and 11*, Ph.D. thesis, Univ. of Durham, England, 1966.
- [37] Ken Ono, *Mock theta functions, ranks, and Maass forms*, Surveys in number theory, Dev. Math., vol. 17, Springer, New York, 2008, pp. 119–141. MR2462948
- [38] Robert A. Rankin, *Modular forms and functions*, Cambridge University Press, Cambridge-New York-Melbourne, 1977. MR0498390
- [39] Hans Rademacher, *Über die erzeugenden von kongruenzuntergruppen der modulgruppe* (German), Abh. Math. Sem. Univ. Hamburg **7** (1929), no. 1, 134–148, DOI 10.1007/BF02941169. MR3069525
- [40] Srinivasa Ramanujan, *The lost notebook and other unpublished papers*, with an introduction by George E. Andrews, Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, 1988. MR947735
- [41] Sinai Robins, *Generalized Dedekind η -products*, The Rademacher legacy to mathematics (University Park, PA, 1992), Contemp. Math., vol. 166, Amer. Math. Soc., Providence, RI, 1994, pp. 119–128, DOI 10.1090/conm/166/01645. MR1284055
- [42] Goro Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481, DOI 10.2307/1970831. MR0332663
- [43] G. N. Watson, *The final problem : An account of the mock theta functions*, J. London Math. Soc. **11** (1936), no. 1, 55–80, DOI 10.1112/jlms/s1-11.1.55. MR1573993
- [44] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions; reprint of the fourth (1927) edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996. MR1424469
- [45] Don Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)*, Séminaire Bourbaki. Vol. 2007/2008, Astérisque **326** (2009), Exp. No. 986, vii–viii, 143–164 (2010). MR2605321
- [46] S. P. Zwegers, *Mock θ -functions and real analytic modular forms, q -series with applications to combinatorics, number theory, and physics* (Urbana, IL, 2000), Contemp. Math., vol. 291, Amer. Math. Soc., Providence, RI, 2001, pp. 269–277, DOI 10.1090/conm/291/04907. MR1874536
- [47] S. P. Zwegers, *Mock theta functions* Ph.D. thesis, Universiteit Utrecht, 2002, 96 pp.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611-8105
 Email address: fgarvan@ufl.edu