

RELATIVE CLUSTER TILTING OBJECTS IN TRIANGULATED CATEGORIES

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ABSTRACT. Assume that \mathcal{D} is a Krull-Schmidt, Hom-finite triangulated category with a Serre functor and a cluster-tilting object T . We introduce the notion of relative cluster tilting objects, and $T[1]$ -cluster tilting objects in \mathcal{D} , which are a generalization of cluster-tilting objects. When \mathcal{D} is 2-Calabi–Yau, the relative cluster tilting objects are cluster-tilting. Let $\Lambda = \text{End}_{\mathcal{D}}^{\text{op}}(T)$ be the opposite algebra of the endomorphism algebra of T . We show that there exists a bijection between $T[1]$ -cluster tilting objects in \mathcal{D} and support τ -tilting Λ -modules, which generalizes a result of Adachi–Iyama–Reiten [τ -tilting theory, *Compos. Math.* **150** (2014), no. 3, 415–452]. We develop a basic theory on $T[1]$ -cluster tilting objects. In particular, we introduce a partial order on the set of $T[1]$ -cluster tilting objects and mutation of $T[1]$ -cluster tilting objects, which can be regarded as a generalization of “cluster-tilting mutation”. As an application, we give a partial answer to a question posed in Adachi–Iyama–Reiten, *loc. cit.*

1. INTRODUCTION

Cluster-tilting objects in a triangulated category \mathcal{D} were introduced in [BMRRT, BMR, IY, KR, KZ]. When \mathcal{D} is a cluster category or more general, a 2-Calabi–Yau (2-CY for short) triangulated category, they play a crucial role in the categorification of cluster algebras, and they correspond to the clusters [K2].

Cluster algebras were introduced by Fomin and Zelevinsky in [FZ]. There has been a vast literature to establish connections with representation theory of finite dimensional algebras. Marsh, Reineke, and Zelevinsky made a first attempt to understand cluster algebras in terms of the representation theory of quivers in [MRZ]. Immediately following this, Buan, Marsh, Reiten, Reineke, and Todorov in [BMRRT, K1] invented cluster categories (see also [CCS] for type A_n). This led to developing a theory, namely cluster-tilting theory, and yielded a categorification of acyclic cluster algebras. At the same time, Geiß, Leclerc, and Schröer [GLS1, GLS2] studied cluster-tilting objects in module categories over preprojective algebras and gave a categorification of certain cyclic cluster algebras. Cluster categories and the stable module categories of preprojective algebras of Dynkin quivers are examples of 2-CY triangulated categories.

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Cluster-tilting objects have many nice properties. A fruitful theory about them has been developed in the last ten years; see for example, [KR], [BIRS] for cluster-tilting in 2-CY triangulated categories and [KZ], [IY], [B] for cluster-tilting in triangulated categories. One of the important properties of cluster-tilting objects in 2-CY triangulated categories is that when we remove some direct summand T_i from cluster-tilting the object $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ to get $T/T_i = \bigoplus_{j \neq i} T_j$ (which is called an almost complete cluster-tilting object), then there is exactly one indecomposable object T_i^* such that $T_i^* \not\cong T_i$ and $T/T_i \oplus T_i^*$ is a cluster-tilting object, which is called the mutation of T at T_i . Mutation of cluster-tilting objects in 2-CY triangulated categories were defined in [BMRRT, IY] and studied by many authors after them, and it corresponds to the mutation of clusters in the categorification of cluster algebras. But the mutation of cluster-tilting objects in triangulated categories which are not 2-CY, is not always possible; see for example Section III in [BIRS], and see examples in Section 5. To make mutation always possible, it is desirable to enlarge the class of cluster-tilting objects to get the more general property that almost complete ones always have two complements.

Cluster-tilted algebras were introduced by Buan, Marsh, and Reiten in [BMR], which are by definition, the endomorphism algebras of cluster-tilting objects in cluster categories. It was proved that the module category of a cluster-tilted algebra is equivalent to the quotient category of cluster category by the cluster-tilting object [BMR]. One can also study the endomorphism algebra of a cluster-tilting object in a triangulated category, the equivalence above still holds in this general case [KR, KZ, IY]. Under this equivalence, one can pose the problem of whether a tilting module over the endomorphism algebra of a cluster-tilting object can be lifted to a cluster-tilting object in the triangulated category. Smith [Smi] and Fu–Liu [FL] proved that it is always true for cluster categories and 2-CY triangulated categories. Holm–Jørgensen [HJ] and Beligiannis [B] proved it is true when the global dimension of endomorphism algebra is finite. More recently, Adachi–Iyama–Reiten [AIR] introduced the τ -tilting modules for any finite dimensional algebra. Assume that \mathcal{D} is a 2-CY triangulated category with a cluster-tilting objects T . In [AIR], the authors established a bijection between cluster-tilting objects in \mathcal{D} and support τ -tilting modules in $\text{mod End}_{\mathcal{D}}^{op}(T)$ (see also [CZZ, YZZ] for various versions of the bijection). Unfortunately, many examples (see for example Section 5, and see Example 2.16) indicate that this result does not hold if \mathcal{D} is not 2-CY. It is then reasonable to find a class of objects in \mathcal{D} which correspond to support τ -tilting modules in $\text{mod End}_{\mathcal{D}}^{op}(T)$ bijectively in a more general setting.

For these purposes, we introduce the notion of relative cluster tilting objects in a triangulated category \mathcal{D} , which are a generalization of cluster-tilting objects. For an object M in \mathcal{D} , we use $[M](X, Y)$ to denote the subgroup of $\text{Hom}_{\mathcal{D}}(X, Y)$ consisting of the morphisms from X to Y factoring through $\text{add } M$. In this way, we define an ideal $[M[1]](-, -)$ of \mathcal{D} if M is a cluster-tilting object, which is called ghost ideal of \mathcal{D} in [B].

Definition 1.1. Let \mathcal{D} be a triangulated category with cluster-tilting objects.

- An object X in \mathcal{D} is called *relative rigid* if there exists a cluster-tilting object T such that $[T[1]](X, X[1]) = 0$. In this case, X is also called $T[1]$ -*rigid*.
- An object X in \mathcal{D} is called *relative cluster tilting* (respectively, *almost relative cluster tilting*) if there exists a cluster-tilting object T such that X is

$T[1]$ -rigid and $|X| = |T|$ (respectively, $|X| = |T| - 1$), where $|X|$ denotes the number of non-isomorphic indecomposable direct summands of X . In this case, X is also called $T[1]$ -cluster tilting (respectively, almost $T[1]$ -cluster tilting).

For a cluster-tilting object T , we introduce a partial order on the set of basic $T[1]$ -cluster tilting objects and get the first main result of this paper.

Theorem 1.2 (see Theorem 3.6 and Proposition 4.4 for details). *Let \mathcal{D} be a triangulated category with a Serre functor and a cluster-tilting object T , and let $\Lambda = \text{End}_{\mathcal{D}}^{\text{op}}(T)$. Then there is an order-preserving bijection between the set of isomorphism classes of basic $T[1]$ -cluster tilting objects in \mathcal{D} and the set of isomorphism classes of basic support τ -tilting Λ -modules.*

If \mathcal{D} is a 2-CY triangulated category, then it turns out that $T[1]$ -cluster tilting objects are precisely cluster-tilting objects. Thus this theorem improves a result in [AIR]. Furthermore, we introduce mutation of relative cluster tilting objects. The second main result of this paper is the following, which is a generalization of a result in [BMRRT, IY].

Theorem 1.3 (see Corollary 3.7 and Theorem 4.10 for details). *Let \mathcal{D} be a triangulated category with a Serre functor and a cluster-tilting object T . Then any basic almost $T[1]$ -cluster tilting object in \mathcal{D} has exactly two non-isomorphic indecomposable complements, and they are related by exchange triangles.*

In the last part of this paper, we give an application. In [AIR], the authors gave a method to calculate left mutation of support τ -tilting modules by exchange sequences and raised a question about exchange sequences (Question 2.28 in [AIR]). For this question, we first give a relationship between exchange sequences and the exchange triangles in Theorem 1.3 and then give a partial answer.

The paper is organized as follows. In Section 2, we review some elementary definitions and facts that we need to use, including cluster-tilting objects and support τ -tilting modules. In Section 3, we first give some basic properties of relative cluster tilting objects, then we state and prove our first main result. In Section 4, we introduce mutation of relative cluster tilting objects and prove our second main result. As an application, we give a partial answer to Question 2.28 in [AIR]. In the last section, we present some examples.

We end this section with some conventions. Throughout this article, k is an algebraically closed field and $D = \text{Hom}_k(-, k)$ is the k -duality. All modules we consider in this paper are left modules. For a finite dimensional algebra Λ , $\text{mod}\Lambda$ denotes the category of finitely generated left Λ -modules, and $\text{proj}\Lambda$ denotes the subcategory of $\text{mod}\Lambda$ consisting of projective Λ -modules. For any triangulated category \mathcal{D} , we assume that it is k -linear, Hom-finite, and satisfies the Krull-Remak-Schmidt property [H]. In \mathcal{D} , we denote the shift functor by $[1]$ and define $\text{Ext}_{\mathcal{D}}^i(X, Y) := \text{Hom}_{\mathcal{D}}(X, Y[i])$ for any objects X and Y . For simplicity, we use $\mathcal{D}(X, Y)$ or (X, Y) to denote the set of morphisms from X to Y in \mathcal{D} . If \mathcal{T} is a subcategory of \mathcal{D} , then we always assume that \mathcal{T} is a full subcategory which is closed under taking isomorphisms, direct sums and direct summands. For three objects M, X , and Y in \mathcal{D} , we denote by $\text{add}M$ the full subcategory of \mathcal{D} consisting of direct summands of direct sum of finitely many copies of M , and denote by $[M](X, Y)$ the subgroup of $\text{Hom}_{\mathcal{D}}(X, Y)$ consisting of morphisms which factor through objects in $\text{add}M$. The

quotient category $\mathcal{D}/[M]$ of \mathcal{D} is a category with the same objects as \mathcal{D} and the space of morphisms from X to Y is the quotient of group of morphisms from X to Y in \mathcal{D} by the subgroup consisting of morphisms factor through objects in $\text{add}M$. For two morphisms $f : M \rightarrow N$ and $g : N \rightarrow L$, the composition of f and g is denoted by $gf : M \rightarrow L$.

2. PRELIMINARIES

In this section, we recall some definitions and results that will be used in this paper.

2.1. Support τ -tilting modules. Let Λ be a finite dimensional k -algebra and τ be the Auslander-Reiten translation. Support τ -tilting modules were introduced by Adachi, Iyama, and Reiten [AIR], which can be regarded as a generalization of tilting modules.

Definition 2.1. Let (X, P) be a pair with $X \in \text{mod}\Lambda$ and $P \in \text{proj}\Lambda$.

- (1) We say that (X, P) is *basic* if X and P are basic.
- (2) X is called *τ -rigid* if $\text{Hom}_\Lambda(X, \tau X) = 0$. (X, P) is called a *τ -rigid pair* if X is τ -rigid and $\text{Hom}_\Lambda(P, X) = 0$.
- (3) X is called *τ -tilting* if X is τ -rigid and $|X| = |\Lambda|$.
- (4) A τ -rigid pair (X, P) is said to be a *support τ -tilting* (respectively, *almost support τ -tilting*) pair if $|X| + |P| = |\Lambda|$ (respectively, $|X| + |P| = |\Lambda| - 1$). If (X, P) is a support τ -tilting pair, then X is called a *support τ -tilting module*.
- (5) Let (X', P') be a pair with $X' \in \text{mod}\Lambda$ and $P' \in \text{proj}\Lambda$. We say that (X, P) is a direct summand of (X', P') if X is a direct summand of X' and P is a direct summand of P' .

Throughout this paper, we denote by $\tau\text{-tilt}\Lambda$ (respectively, $s\tau\text{-tilt}\Lambda$) the set of isomorphism classes of basic τ -tilting (respectively, support τ -tilting) Λ -modules, and by $\tau\text{-rigid}\Lambda$ the set of isomorphism classes of basic τ -rigid pairs of Λ . The following observation is basic in τ -tilting theory.

Proposition 2.2 ([AIR, Proposition 2.3]). *Let (X, P) be a basic pair with $X \in \text{mod}\Lambda$ and $P = \Lambda e \in \text{proj}\Lambda$, where e is an idempotent of Λ .*

- (a) *(X, P) is a τ -rigid (respectively, support τ -tilting) pair for Λ if and only if X is a τ -rigid (respectively, τ -tilting) $(\Lambda/\Lambda e\Lambda)$ -module.*
- (b) *Let (X, P) be a support τ -tilting pair for Λ . Then P is determined by X uniquely. This means that if (X, P) and (X, Q) are basic support τ -tilting pairs for Λ , then $P \simeq Q$.*

For τ -tilting modules, we have the analog of the Bongartz completion of tilting modules.

Theorem 2.3 ([AIR, Theorem 2.9]). *Let X be a τ -rigid Λ -module. Then there exists a Λ -module V such that $X \oplus V$ is a τ -tilting Λ -module.*

The notion of mutation was also introduced in [AIR].

Definition 2.4. Two basic support τ -tilting pairs (T, P) and (T', P') for Λ are said to be *mutation* of each other if there exists a basic almost support τ -tilting pair (U, Q) which is a direct summand of (T, P) and (T', P') . In this case we write

$T' = \mu_X(T)$ if X is an indecomposable Λ -module satisfying either $T = U \oplus X$ or $P = Q \oplus X$.

The following result shows that support τ -tilting modules “complete” tilting modules from the viewpoint of mutation.

Theorem 2.5 ([AIR, Theorem 2.17]). *Let Λ be a finite dimensional k -algebra. Then any basic almost support τ -tilting pair (U, Q) for Λ is a direct summand of exactly two basic support τ -tilting pairs (T, P) and (T', P') for Λ . Moreover, we have $\{\text{Fac}T, \text{Fac}T'\} = \{\text{Fac}U, {}^\perp(\tau U) \cap Q^\perp\}$.*

2.2. Functorially finite torsion classes. Let Λ be a finite dimensional k -algebra and τ be the Auslander-Reiten translation. We denote by $\mathbf{K}^b(\text{proj}\Lambda)$ the homotopy category of bounded complexes of finitely generated projective Λ -modules. We recall the definition of functorially finite torsion classes.

We say that a full subcategory \mathcal{T} of $\text{mod}\Lambda$ is a *torsion class* if it is closed under factor modules and extensions and an object X in \mathcal{T} is *Ext-projective* if $\text{Ext}_\Lambda^1(X, -)|_{\mathcal{T}} = 0$. We denote by $P(\mathcal{T})$ the direct sum of one copy of each of the indecomposable Ext-projective objects in \mathcal{T} up to isomorphism [Ho, Sma].

Let X be a module in $\text{mod}\Lambda$. A morphism $f : T_0 \rightarrow X$ is called a *right \mathcal{T} -approximation* of X if $T_0 \in \mathcal{T}$ and $\text{Hom}_\Lambda(-, f)|_{\mathcal{T}}$ is surjective. If any module in $\text{mod}\Lambda$ has a right \mathcal{T} -approximation, we call \mathcal{T} *contravariantly finite* in $\text{mod}\Lambda$. Dually, a *left \mathcal{T} -approximation* and a *covariantly finite subcategory* are defined. We say that \mathcal{T} is *functorially finite* if it is both covariantly finite and contravariantly finite.

We denote by $\text{f-tors}\Lambda$ the set of functorially finite torsion classes in $\text{mod}\Lambda$. The following result gives a relationship between support τ -tilting Λ -modules and functorially finite torsion classes in $\text{mod}\Lambda$.

Theorem 2.6 ([AIR, Theorem 2.6]). *There is a bijection*

$$s\tau\text{-tilt}\Lambda \longleftrightarrow \text{f-tors}\Lambda$$

given by $s\tau\text{-tilt}\Lambda \ni M \mapsto \text{Fac}M \in \text{f-tors}\Lambda$ and $\text{f-tors}\Lambda \ni \mathcal{T} \mapsto P(\mathcal{T}) \in s\tau\text{-tilt}\Lambda$, where $\text{Fac}M$ is the subcategory of $\text{mod}\Lambda$ consisting of all objects which are factor modules of finite direct sums of copies of M .

Under this bijection, we can give a partial order on $s\tau\text{-tilt}\Lambda$.

Definition 2.7. For $M, N \in s\tau\text{-tilt}\Lambda$, we write $M \geq N$ if $\text{Fac}M \supseteq \text{Fac}N$.

The relation \geq gives a partial order on $s\tau\text{-tilt}\Lambda$. The following proposition is very important for mutation of support τ -tilting modules.

Proposition 2.8 ([AIR, Definition-Proposition 2.26]). *Let $T = X \oplus U$ and T' be support τ -tilting Λ -modules such that $T' = \mu_X(T)$ for some indecomposable Λ -module X . Then either $T > T'$ or $T' > T$. Moreover, $T > T'$ (respectively, $T < T'$) if and only if $X \notin \text{Fac}U$ (respectively, $X \in \text{Fac}U$).*

2.3. Serre functors. Following Bondal and Kapranov [BK], we give the definition of Serre functors.

Definition 2.9. Let \mathcal{D} be a k -linear triangulated category with finite dimensional Hom-spaces. A Serre functor $\mathbb{S} : \mathcal{D} \rightarrow \mathcal{D}$ is a k -linear equivalence with bifunctorial isomorphisms

$$\text{Hom}_{\mathcal{D}}(A, B) \simeq D\text{Hom}_{\mathcal{D}}(B, \mathbb{S}A)$$

for any $A, B \in \mathcal{D}$, where D is the duality over k .

In [RVdB], Reiten and Van den Bergh proved that if \mathcal{D} has a Serre functor \mathbb{S} , then \mathcal{D} has Auslander-Reiten triangles. Moreover, if $\tau_{\mathcal{D}}$ is the Auslander-Reiten translation in \mathcal{D} , then $\mathbb{S} \simeq \tau_{\mathcal{D}}[1]$. We say that a triangulated category \mathcal{D} with a Serre functor \mathbb{S} is *n-Calabi–Yau* if $\mathbb{S} \simeq [n]$.

2.4. Cluster-tilting objects. Let \mathcal{D} be a k -linear, Hom-finite triangulated category with a Serre functor \mathbb{S} . An important class of objects in \mathcal{D} are the cluster-tilting objects, which have many nice properties. Following Iyama and Yoshino [IY], we give the definitions of cluster-tilting objects and related objects as follows.

Definition 2.10.

- (1) An object T in \mathcal{D} is called *rigid* if $\text{Ext}_{\mathcal{D}}^1(T, T) = 0$.
- (2) An object T in \mathcal{D} is called *maximal rigid* if it is rigid and maximal with respect to the property: $\text{add}T = \{X \in \mathcal{D} \mid \text{Ext}_{\mathcal{D}}^1(T \oplus X, T \oplus X) = 0\}$.
- (3) An object T in \mathcal{D} is called *cluster-tilting* if

$$\text{add}T = \{X \in \mathcal{D} \mid \text{Ext}_{\mathcal{D}}^1(T, X) = 0\} = \{X \in \mathcal{D} \mid \text{Ext}_{\mathcal{D}}^1(X, T) = 0\}.$$

Throughout this paper, we denote by $\text{rigid}\mathcal{D}$ (respectively, $\text{c-tilt}\mathcal{D}$) the set of isomorphism classes of basic rigid (respectively, cluster-tilting) objects in \mathcal{D} . For two objects X and Y in \mathcal{D} , we define by $X * Y$ a full subcategory of \mathcal{D} whose objects are all such $M \in \mathcal{D}$ with triangles

$$X_0 \longrightarrow M \longrightarrow Y_0 \longrightarrow X_0[1],$$

where $X_0 \in \text{add}X$ and $Y_0 \in \text{add}Y$. Let $\tau_{\mathcal{D}}$ be the Auslander-Reiten translation in \mathcal{D} and denote by $F = \tau_{\mathcal{D}}^{-1}[1]$. We have the following results, which will be used frequently in this paper.

Theorem 2.11 ([IY, KZ]). *Let T be a cluster-tilting object in \mathcal{D} . Then we have the following:*

- (a) $\mathcal{D} = T * T[1]$.
- (b) F is an auto-equivalence of \mathcal{D} satisfying $FT \cong T$.

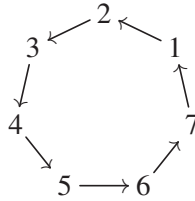
Theorem 2.12 ([KR, KZ]). *Let T be a cluster-tilting object in \mathcal{D} , and let $\Lambda = \text{End}_{\mathcal{D}}^{\text{op}}(T)$. Then the functor $\overline{(-)} := \text{Hom}_{\mathcal{D}}(T, -) : \mathcal{D} \rightarrow \text{mod}\Lambda$ induces an equivalence,*

$$(1) \quad \mathcal{D}/[T[1]] \xrightarrow{\sim} \text{mod}\Lambda,$$

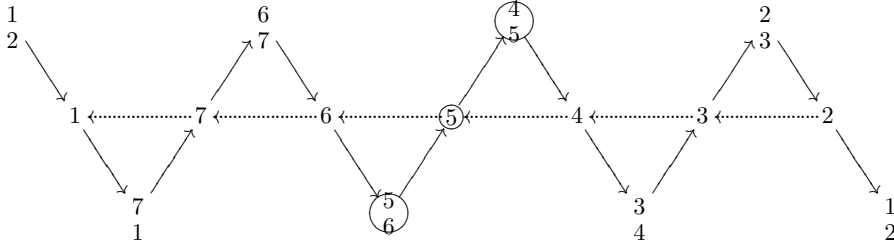
and Λ is a Gorenstein algebra of Gorenstein dimension at most 1.

This theorem tells us that all Λ -modules have projective dimension zero, one or infinity. In order to characterize the Λ -modules of infinite projective dimension, Beaudet, Brüstle, and Todorov [BBT] introduced the following definition.

Definition 2.13. Let X be an object in \mathcal{D} . The ideal of $\text{End}_{\mathcal{D}}^{\text{op}}(T[1])$ given by all endomorphisms that factor through X is called *factorization ideal* through X , denoted by I_X .



The AR-quiver of $\text{mod } \Lambda$ is as follows:



It is easy to see that

$$M = \begin{matrix} 4 & 5 \\ 5 & \oplus & 5 & \oplus & 5 \\ & & 6 & & \end{matrix}$$

is a support τ -tilting Λ -module, but the object in \mathcal{D} corresponding to M is

$$1[1] \oplus \begin{matrix} 1 & 1 \\ 2 & 3 \end{matrix} [1] \oplus 2[1] \oplus 3 \oplus \begin{matrix} 2 & 1 \\ 3 & 3 \end{matrix} \oplus 1 \oplus 2[1],$$

which is not a cluster-tilting object since it has self-extensions.

In next section, we shall investigate an important class of objects in \mathcal{D} , which correspond to support τ -tilting Λ -modules bijectively.

3. RELATIVE CLUSTER TILTING OBJECTS AND SUPPORT τ -TILTING MODULES

In this section, we study the following objects in triangulated categories.

Definition 3.1. Let \mathcal{D} be a triangulated category with cluster-tilting objects.

- An object X in \mathcal{D} is called *relative rigid* if there exists a cluster-tilting object T such that $[T[1]](X, X[1]) = 0$. In this case, X is also called *$T[1]$ -rigid*.
- An object X in \mathcal{D} is called *relative cluster tilting* (respectively, *almost relative cluster tilting*) if there exists a cluster-tilting object T such that X is $T[1]$ -rigid and $|X| = |T|$ (respectively, $|X| = |T| - 1$), where $|X|$ denotes the number of non-isomorphic indecomposable direct summands of X . In this case, X is also called *$T[1]$ -cluster tilting* (respectively, *almost $T[1]$ -cluster tilting*).

Remark 3.2. Any rigid object in \mathcal{D} is relative rigid.

The following easy observation shows that relative cluster tilting objects can be regarded as a generalization of cluster-tilting objects.

Proposition 3.3. *If \mathcal{D} admits a Serre functor, then for any cluster-tilting object T , cluster-tilting objects in \mathcal{D} are $T[1]$ -cluster tilting.*

Proof. Let M be a cluster-tilting object in \mathcal{D} . Clearly, it is $T[1]$ -rigid. By Corollary 2.15 in [YZZ], all basic cluster-tilting objects have the same number of indecomposable direct summands. Thus, $|M| = |T|$. Hence M is $T[1]$ -cluster tilting. \square

Throughout this section, we assume that \mathcal{D} is a k -linear, Hom-finite triangulated category with cluster-tilting objects and a Serre functor \mathbb{S} . Let T be a cluster-tilting object in \mathcal{D} and $\Lambda = \text{End}_{\mathcal{D}}^{op}(T)$ the opposite algebra of the endomorphism algebra of T . Our aim in this section is to show that there is a close relationship between $T[1]$ -cluster tilting objects in \mathcal{D} and support τ -tilting Λ -modules.

Let $\tau_{\mathcal{D}}$ be the Auslander-Reiten translation in \mathcal{D} . We first give the following proposition, which indicates that $T[1]$ -rigid objects and rigid objects coincide in some cases.

Proposition 3.4. *For two objects M and N in \mathcal{D} , $[T[1]](M, N[1]) = 0$ and $[T[1]](N, \tau_{\mathcal{D}}M) = 0$ if and only if $\text{Hom}_{\mathcal{D}}(M, N[1]) = 0$. In particular, if \mathcal{D} is 2-CY, then X is $T[1]$ -rigid if and only if X is rigid.*

Proof. We show the “if” part. If $\text{Hom}_{\mathcal{D}}(M, N[1]) = 0$, then $[T[1]](M, N[1]) = 0$. By Serre duality, we have

$$\text{Hom}_{\mathcal{D}}(N, \tau_{\mathcal{D}}M) \simeq \text{Hom}_{\mathcal{D}}(N[1], \mathbb{S}M) \simeq D\text{Hom}_{\mathcal{D}}(M, N[1]) = 0.$$

Thus we obtain $[T[1]](N, \tau_{\mathcal{D}}M) = 0$.

We show the “only if” part. Since T is a cluster-tilting object, by Theorem 2.11, we know there exists a triangle

$$T_0 \xrightarrow{g} N \xrightarrow{f} T_1[1] \xrightarrow{h} T_0[1]$$

with $T_0, T_1 \in \text{add}T$. Thus we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} (T_1[1], \tau_{\mathcal{D}}M) & \xrightarrow{f} & (N, \tau_{\mathcal{D}}M) & \longrightarrow & (T_0, \tau_{\mathcal{D}}M) & \longrightarrow & (T_1, \tau_{\mathcal{D}}M) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ D(M, T_1[2]) & \xrightarrow{D(f[1]\cdot)} & D(M, N[1]) & \xrightarrow{D(g[1]\cdot)} & D(M, T_0[1]) & \xrightarrow{D(h\cdot)} & D(M, T_1[1]). \end{array}$$

Since $\text{Im}(\cdot f) = \{af \mid a \in \text{Hom}_{\mathcal{D}}(T_1[1], \tau_{\mathcal{D}}M)\} \subseteq [T[1]](N, \tau_{\mathcal{D}}M) = 0$, we know that

$$(3) \quad \text{Ker}D(g[1]\cdot) = \text{Im}D(f[1]\cdot) \simeq \text{Im}(\cdot f) = 0.$$

$$\begin{array}{ccccccc} N & \xrightarrow{f} & T_1[1] & \xrightarrow{h} & T_0[1] & \xrightarrow{g[1]} & N[1] \\ & & \swarrow & \nearrow \text{ } \exists c & \uparrow b & & \\ & & \tau_{\mathcal{D}}M & & M & & \end{array}$$

Take any $b \in \text{Hom}_{\mathcal{D}}(M, T_0[1])$. Since $[T[1]](M, N[1]) = 0$, we have $g[1]b = 0$. Thus there exists $c : M \rightarrow T_1[1]$ such that $b = hc$, which implies that

$$\begin{array}{ccc} (h\cdot) : \text{Hom}_{\mathcal{D}}(M, T_1[1]) & \longrightarrow & \text{Hom}_{\mathcal{D}}(M, T_0[1]) \\ c & \longmapsto & hc \end{array}$$

is surjective. Therefore, $D(h\cdot)$ is injective and

$$\text{Im}D(g[1]\cdot) = \text{Ker}D(h\cdot) = 0.$$

Combining this with (3), we know that $D\text{Hom}_{\mathcal{D}}(M, N[1]) = 0$ and $\text{Hom}_{\mathcal{D}}(M, N[1])$ vanishes. \square

The following lemma plays an important role in this paper. This was proved by Palu [P] in case \mathcal{D} is a 2-CY category, and proved in [YZZ] for general case. For the convenience of the readers, we give a simple proof in the following.

Lemma 3.5. *Let \mathcal{D} be a triangulated category with a Serre functor \mathbb{S} and a cluster-tilting object T . Then for any objects X and Y in \mathcal{D} , there is a bifunctorial isomorphism*

$$\text{Hom}_{\mathcal{D}/[T]}(\tau_{\mathcal{D}}^{-1}Y, X) \simeq D[T](X[-1], Y).$$

Proof. Since T is a cluster-tilting object, by Theorem 2.11, we know there exists a triangle

$$T_1 \longrightarrow T_0 \longrightarrow X \xrightarrow{\xi} T_1[1]$$

in \mathcal{D} with T_0 and T_1 in $\text{add}T$. Applying $\text{Hom}_{\mathcal{D}}(-, Y)$ to it, we have a map

$$\begin{aligned} \varphi : \text{Hom}_{\mathcal{D}}(T_1, Y) &\longrightarrow \text{Hom}_{\mathcal{D}}(X[-1], Y) \\ f &\longmapsto f\xi[-1]. \end{aligned}$$

It is easy to see that $\text{Im}\varphi \simeq [T](X[-1], Y)$. Since the category \mathcal{D} has a Serre functor \mathbb{S} , we have isomorphisms

$$\begin{aligned} D\text{Hom}_{\mathcal{D}}(T_1, Y) &\simeq \text{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}Y, T_1) \simeq \text{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, T_1[1]), \\ D\text{Hom}_{\mathcal{D}}(X[-1], Y) &\simeq \text{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}Y, X[-1]) \simeq \text{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, X). \end{aligned}$$

$$\begin{array}{ccc} D\text{Hom}_{\mathcal{D}}(X[-1], Y) & \xrightarrow{D\varphi} & D\text{Hom}_{\mathcal{D}}(T_1, Y) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, X) & \xrightarrow{\exists\phi} & \text{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, T_1[1]) \end{array}$$

Thus, $D\varphi$ is isomorphic to a map

$$\begin{aligned} \phi : \text{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, X) &\longrightarrow \text{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, T_1[1]) \\ g &\longmapsto \xi g. \end{aligned}$$

Note that $\text{Ker}\phi = [T](\tau_{\mathcal{D}}^{-1}Y, X)$. Hence, we have isomorphisms

$$\begin{aligned} D[T](X[-1], Y) &\simeq D\text{Im}\varphi \simeq \text{Im}D\varphi \simeq \text{Im}\phi \\ &\simeq \text{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, X)/\text{Ker}\phi \\ &\simeq \text{Hom}_{\mathcal{D}}/[T](\tau_{\mathcal{D}}^{-1}Y, X). \end{aligned} \quad \square$$

We keep the notation of bijection (2). Let X be an object in \mathcal{D} , we define

$$|\tilde{X}| = |(\overline{X'}, \overline{X''[-1]})| := |\overline{X'}| + |\overline{X''[-1]}|,$$

it is easy to see that $|\tilde{X}| = |X|$.

From now on, we denote by $T[1]\text{-rigid}\mathcal{D}$ (respectively, $T[1]\text{-tilt}\mathcal{D}$) the set of isomorphism classes of basic $T[1]\text{-rigid}$ (respectively, $T[1]\text{-cluster tilting}$) objects in \mathcal{D} ,

by $T[1]$ -tilt $_T\mathcal{D}$ the set of objects in $T[1]$ -tilt \mathcal{D} which do not have non-zero direct summands in $\text{add}T[1]$, and by

$$T[1]\text{-tilt}_T^0\mathcal{D} := \{ X \in T[1]\text{-tilt}_T\mathcal{D} \mid I_X = 0 \}.$$

On the other hand, we denote by $\text{tilt}\Lambda$ the set of isomorphism classes of basic tilting modules in $\text{mod}\Lambda$. The following correspondences are our main result in this section.

Theorem 3.6. *Let \mathcal{D} be a k -linear, Hom-finite triangulated category with a Serre functor \mathbb{S} and a cluster-tilting object T , and let $\Lambda = \text{End}_{\mathcal{D}}^{\text{op}}(T)$. Then the bijection $(-)$ in (2) induces the following bijections*

$$\begin{aligned} T[1]\text{-rigid}\mathcal{D} &\xleftrightarrow{(a)} \tau\text{-rigid}\Lambda, \\ T[1]\text{-tilt}\mathcal{D} &\xleftrightarrow{(b)} s\tau\text{-tilt}\Lambda, \\ T[1]\text{-tilt}_T\mathcal{D} &\xleftrightarrow{(c)} \tau\text{-tilt}\Lambda, \\ \text{and } T[1]\text{-tilt}_T^0\mathcal{D} &\xleftrightarrow{(d)} \text{tilt}\Lambda. \end{aligned}$$

Proof. By Corollary 6.5(3)(iii) in [IY], if $M \in \mathcal{D}$ has no nonzero indecomposable direct summands of $T[1]$, then

$$\overline{\tau_{\mathcal{D}}M} \simeq \tau\overline{M}.$$

Combining this with Lemma 3.5, for any object X in \mathcal{D} , we have

$$\begin{aligned} \text{Hom}_{\Lambda}(\overline{X'}, \tau\overline{X'}) &= \text{Hom}_{\Lambda}(\overline{X'}, \overline{\tau_{\mathcal{D}}X'}) \simeq \text{Hom}_{\mathcal{D}/[T[1]]}(X, \tau_{\mathcal{D}}X') \\ &\simeq D[T[1]](\tau_{\mathcal{D}}X'[-1], \tau_{\mathcal{D}}X). \end{aligned}$$

Further, by Theorem 2.11, we have

$$(4) \quad \text{Hom}_{\Lambda}(\overline{X'}, \tau\overline{X'}) \simeq D[FT[1]](F\tau_{\mathcal{D}}X'[-1], F\tau_{\mathcal{D}}X) \simeq D[T[1]](X', X[1]).$$

In a similar way, we know

$$\text{Hom}_{\Lambda}(\overline{X''[-1]}, \overline{X'}) \simeq D[T[1]](X[-1], \tau_{\mathcal{D}}X''[-1]) = D[T[1]](X[-1], F^{-1}X'').$$

Note that $F^{-1}X'' \in \text{add}T[1]$. Then

$$\begin{aligned} \text{Hom}_{\Lambda}(\overline{X''[-1]}, \overline{X'}) &\simeq D\text{Hom}_{\mathcal{D}}(X[-1], F^{-1}X'') \\ &\simeq \text{Hom}_{\mathcal{D}}(F^{-1}X'', \mathbb{S}X[-1]) \quad (\text{Serre duality}) \\ &\simeq [T[1]](F^{-1}X'', \tau_{\mathcal{D}}X) \\ (5) \quad &\simeq [T[1]](X'', X[1]). \quad (\text{Theorem 2.11}) \end{aligned}$$

(a) By equalities (4) and (5), we know that X is a $T[1]$ -rigid object in \mathcal{D} if and only if

$$[T[1]](X', X[1]) = [T[1]](X'', X[1]) = 0$$

if and only if $(\overline{X'}, \overline{X''[-1]})$ is a τ -rigid pair for Λ .

(b) Note that $|\tilde{X}| = |(\overline{X'}, \overline{X''[-1]})| = |X|$ and $|\Lambda| = |T|$. We know that (a) induces a bijection between $T[1]$ -cluster tilting objects in \mathcal{D} and support τ -tilting pairs for Λ . By Proposition 2.2(b), we have proved the assertion.

(c) This assertion is clear.

(d) By Theorem 2.14, we only need to show that τ -tilting modules whose projective dimension are at most one are precisely tilting modules. This is immediate from the fact that if the projective dimension of a Λ -module M is at most one, then

M is τ -rigid if and only if it is rigid, i.e., $\text{Ext}_\Lambda^1(M, M) = 0$ (using AR duality; see [ASS06]). □

We define

$$\text{c-tilt}_T^0 \mathcal{D} := \{ X \in \text{c-tilt}_T \mathcal{D} \mid I_X = 0 \}$$

and end this section with the following direct consequences.

Corollary 3.7. *Let \mathcal{D} be a k -linear, Hom-finite triangulated category with a Serre functor and a cluster-tilting object T , and let $\Lambda = \text{End}_{\mathcal{D}}^{\text{op}}(T)$. Then we have the following.*

- (1) *For any $T[1]$ -rigid object X in \mathcal{D} , $|X| \leq |T|$. In particular, for any maximal rigid object X in \mathcal{D} , $|X| \leq |T|$.*
- (2) *Any $T[1]$ -rigid object in \mathcal{D} is a direct summand of some $T[1]$ -cluster tilting object in \mathcal{D} .*
- (3) *Any basic almost $T[1]$ -cluster tilting object in \mathcal{D} is a direct summand of exactly two basic $T[1]$ -cluster tilting objects in \mathcal{D} .*
- (4) *If \mathcal{D} is 2-CY, then we have bijections*

$$\begin{aligned} \text{rigid} \mathcal{D} &\longleftrightarrow \tau\text{-rigid} \Lambda, \\ \text{c-tilt} \mathcal{D} &\longleftrightarrow \text{s}\tau\text{-tilt} \Lambda, \\ \text{c-tilt}_T \mathcal{D} &\longleftrightarrow \tau\text{-tilt} \Lambda, \\ \text{and c-tilt}_T^0 \mathcal{D} &\longleftrightarrow \text{tilt} \Lambda. \end{aligned}$$

Proof. (1) This is immediate from the bijection (a) in Theorem 3.6.

(2) Let X be a $T[1]$ -rigid object in \mathcal{D} , then $\tilde{X} = (\overline{X'}, \overline{X''[-1]})$ is a τ -rigid pair for Λ . We may assume that $X''[-1] = \Lambda e$, where e is an idempotent of Λ . By Proposition 2.2, we know $\overline{X'}$ is a τ -rigid $(\Lambda/\langle e \rangle)$ -module. Using Theorem 2.3, we know there exists a τ -tilting $(\Lambda/\langle e \rangle)$ -module S such that $\overline{X'}$ is a direct summand of S . Thus, $(S, \Lambda e)$ is a support τ -tilting pair for Λ and $(\overline{X'}, \Lambda e)$ is a direct summand of $(S, \Lambda e)$. Hence, it follows from Theorem 3.6 and Proposition 2.2 that X is a direct summand of some $T[1]$ -cluster tilting object in \mathcal{D} .

(3) This assertion follows from Theorem 2.5 and the bijection (b) in Theorem 3.6 immediately.

(4) This assertion follows from Proposition 3.4 and the fact that an object X in \mathcal{D} is cluster-tilting if and only if it is rigid and $|X| = |T|$ (see [ZZ]). □

Remark 3.8.

- In a triangulated category \mathcal{D} with cluster-tilting objects and a Serre functor \mathbb{S} , any cluster-tilting object is maximal rigid. Conversely, any maximal rigid object M satisfying $\mathbb{S}(M) \simeq M[2]$ is cluster-tilting (see [AIR, ZZ, YZZ]). But we do not know whether any maximal rigid object in \mathcal{D} is cluster-tilting.
- The first 3 bijections in Corollary 3.7(4) are known by [AIR].

4. MUTATION OF $T[1]$ -CLUSTER TILTING OBJECTS AND AN APPLICATION

As previously, we assume that \mathcal{D} is a k -linear, Hom-finite triangulated category with a cluster-tilting object T and a Serre functor \mathbb{S} , and $\Lambda = \text{End}_{\mathcal{D}}^{\text{op}}(T)$ is the opposite algebra of the endomorphism algebra of T . In this section, we first introduce a natural partial order on the set of $T[1]$ -cluster tilting objects, then prove our main result on complements for almost $T[1]$ -cluster tilting objects in \mathcal{D} . As an application, we give a partial answer to a question posed in [AIR].

4.1. **A partial order.** For an object M in \mathcal{D} , we denote by $M * [T[1]]$ a full subcategory of \mathcal{D} whose objects are all such $X \in \mathcal{D}$ with triangles

$$M_X \longrightarrow X \xrightarrow{\eta_X} C_X \longrightarrow M_X[1],$$

where $M_X \in \text{add}M$ and the morphism η_X factors through $\text{add}T[1]$.

Definition 4.1. For $M, N \in T[1]\text{-tilt}\mathcal{D}$, we write $M \geq N$ if $M * [T[1]] \supseteq N * [T[1]]$.

The main result in this subsection is the following.

Theorem 4.2. *The relation \geq gives a partial order on $T[1]\text{-tilt}\mathcal{D}$.*

Proof. We only need to show that for $M, N \in T[1]\text{-tilt}\mathcal{D}$, $M \geq N$ and $N \geq M$ imply $M = N$. We assume $M * [T[1]] = N * [T[1]]$. Since $N \in N * [T[1]] = M * [T[1]]$, there exists a triangle $M_N \xrightarrow{a} N \xrightarrow{\eta_N} C_N \longrightarrow M_N[1]$, where $M_N \in \text{add}M$ and η_N factors through $\text{add}T[1]$

$$\begin{array}{ccccccc}
 M_N & \xrightarrow{a} & N & \xrightarrow{\eta_N} & C_N & \longrightarrow & M_N[1] \\
 & \swarrow \text{dotted } b & \uparrow f_2 & & & & \\
 & & T_0 & & & & \\
 & & \uparrow f_1 & & & & \\
 & & M[-1] & & & &
 \end{array}$$

for all $f \in [T](M[-1], N)$, there are two morphisms $f_1 : M[-1] \rightarrow T_0$ and $f_2 : T_0 \rightarrow N$ such that $f = f_2 f_1$, where $T_0 \in \text{add}T$. Thus, $\eta_N f_2 = 0$ and there exists $b : T_0 \rightarrow M_N$ such that $f_2 = ab$. Since M is $T[1]$ -cluster tilting, we have $f = f_2 f_1 = a(b f_1) = 0$. This implies that $[T[1]](M, N[1]) = 0$. Dually, $[T[1]](N, M[1]) = 0$. Hence $M \oplus N$ is $T[1]$ -rigid. By Corollary 3.7(1), we have

$$|T| = |M| = |N| \leq |M \oplus N| \leq |T|,$$

which forces that $|M \oplus N| = |M| = |N|$. Therefore, $M = N$. □

The following observation is crucial in this subsection.

Lemma 4.3. *For any two objects M and X in \mathcal{D} , $X \in M * [T[1]]$ if and only if $\overline{X'} \in \text{Fac}\overline{M'}$ in $\text{mod}\Lambda$.*

Proof. (1) If $X \in M * [T[1]]$, then there exists a triangle

$$(6) \quad M_X \xrightarrow{g} X \xrightarrow{\eta_X} C_X \longrightarrow M_X[1],$$

where $M_X \in \text{add}M$ and the morphism η_X factors through $\text{add}T[1]$. Applying $\overline{(\quad)}$ to (6), we have an exact sequence

$$\overline{M_X} \xrightarrow{\overline{g}} \overline{X'} \xrightarrow{\overline{\eta_X}} \overline{C_X} \longrightarrow \overline{M_X[1]},$$

Since η_X factors through $\text{add}T[1]$, we have $\text{Im}\overline{\eta_X} = 0$. Thus $\text{Im}\overline{g} = \text{Ker}\overline{\eta_X} = \overline{X'}$, i.e., \overline{g} is surjective. Because $\overline{M_X} \in \text{add}\overline{M'}$, we obtain $\overline{X'} \in \text{Fac}\overline{M'}$.

(2) Conversely, let $\overline{X'} \in \text{Fac} \overline{M'}$ in $\text{mod} \Lambda$, then there is a surjection $\overline{M'}^n \xrightarrow{h} \overline{X'} \rightarrow 0$. By Theorem 2.12, we have an equivalence

$$(7) \quad \mathcal{D}/[T[1]] \xrightarrow{\sim} \text{mod} \Lambda,$$

thus there exists a morphism $f : (M')^n \rightarrow X'$ in \mathcal{D} such that $\overline{f} = h$. Take a triangle $(M')^n \xrightarrow{f} X' \xrightarrow{g} C \rightarrow (M')^n[1]$. Applying $\overline{(\)}$ to it, we get an exact sequence

$$\overline{M'}^n \xrightarrow{h} \overline{X'} \xrightarrow{\overline{g}} \overline{C} \rightarrow \overline{(M')^n[1]}.$$

Because h is surjective, we have $\text{Im} \overline{g} = 0$. Since T is a cluster-tilting object in \mathcal{D} , by Theorem 2.11, we have the following triangles

$$\begin{array}{ccccccc}
 & & T_0 & & & & \\
 & & \downarrow a & & & & \\
 (M')^n & \xrightarrow{f} & X' & \xrightarrow{g} & C & \longrightarrow & (M')^n[1], \\
 & & \downarrow b & \nearrow \exists c & & & \\
 & & T_1[1] & & & & \\
 & & \downarrow & & & & \\
 & & T_0[1] & & & &
 \end{array}$$

where $T_0, T_1 \in \text{add} T$. Since the image of $\overline{g} : \text{Hom}_{\mathcal{D}}(T, X') \rightarrow \text{Hom}_{\mathcal{D}}(T, C)$ is zero, we get $ga = 0$. Thus g factors through $T_1[1]$. Since $(M')^n \in \text{add} M$, we know $X' \in M * [T[1]]$, so is X . \square

This lemma gives the following direct consequence.

Proposition 4.4. *For any two objects M and N in \mathcal{D} , $M * [T[1]] \subseteq N * [T[1]]$ in \mathcal{D} if and only if $\text{Fac} \overline{M'} \subseteq \text{Fac} \overline{N'}$ in $\text{mod} \Lambda$. In particular, the bijection (b) in Theorem 3.6 induces an isomorphism of two partially ordered sets.*

We introduce the following notion in triangulated categories, which is an analog of Ext-projective modules in module categories.

Definition 4.5. Let \mathcal{D} be a triangulated category and \mathcal{T} be a subcategory of \mathcal{D} . We say that an object $X \in \mathcal{T}$ is *relative cluster projective* if there exists a cluster-tilting object T such that $[T[1]](X, \mathcal{T}[1]) = 0$. In this case, X is also called *$T[1]$ -projective*.

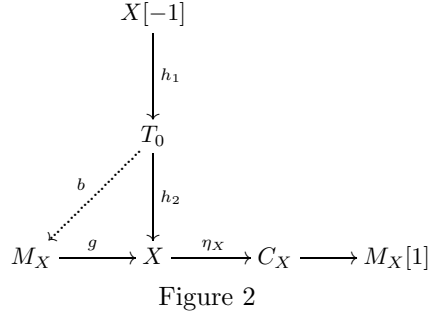
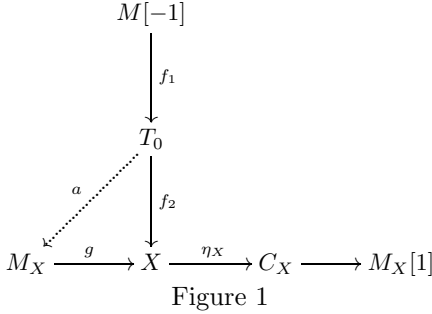
We first give the following interesting observation.

Lemma 4.6. *Let T be a cluster-tilting object and M be a basic $T[1]$ -cluster tilting object in \mathcal{D} . Any indecomposable $T[1]$ -projective object in $M * [T[1]]$ is a direct summand of M .*

Proof. Let X be an indecomposable $T[1]$ -projective object in $M * [T[1]]$. Then

$$(8) \quad \begin{aligned} [T[1]](X, M[1]) &= 0, \\ \text{and } [T[1]](X, X[1]) &= 0. \end{aligned}$$

Since $X \in M * [T[1]]$, we have a triangle $M_X \xrightarrow{g} X \xrightarrow{\eta_X} C_X \rightarrow M_X[1]$, where $M_X \in \text{add}M$ and η_X factors through $\text{add}T[1]$. For all $f \in [T](M[-1], X)$, there are two morphisms $f_1 : M[-1] \rightarrow T_0$ and $f_2 : T_0 \rightarrow X$ such that $f = f_2 f_1$, where $T_0 \in \text{add}T$.



Since η_X factors through $\text{add}T[1]$, we have $\eta_X f_2 = 0$. Thus, there exists $a : T_0 \rightarrow M_X$ such that $f_2 = ga$. Because M is $T[1]$ -rigid, we have $f = f_2 f_1 = g(a f_1) = 0$. Therefore

$$(9) \quad [T[1]](M, X[1]) = 0.$$

It follows from the equalities (8) and (9) that $[T[1]](M \oplus X, (M \oplus X)[1]) = 0$, i.e., $M \oplus X$ is $T[1]$ -rigid. By Corollary 3.7(1), we know that $X \in \text{add}M$. \square

Thanks to this lemma, we can consider the direct sum of one copy of each of the indecomposable $T[1]$ -projective objects in $M * [T[1]]$ up to isomorphism and denote it by $P_{T[1]}(M * [T[1]])$.

We put

$$T[1]\text{-tilt}\mathcal{D} * T[1] := \{M * [T[1]] \mid M \in T[1]\text{-tilt}\mathcal{D}\}.$$

Then we have the following result.

Proposition 4.7.

- (1) For any object M in $T[1]\text{-tilt}\mathcal{D}$, $P_{T[1]}(M * [T[1]]) = M$.
- (2) There is a one-to-one correspondence

$$(10) \quad T[1]\text{-tilt}\mathcal{D} * T[1] \longrightarrow T[1]\text{-tilt}\mathcal{D}$$

given by $T[1]\text{-tilt}\mathcal{D} * T[1] \ni \mathcal{T} \mapsto P_{T[1]}(\mathcal{T}) \in T[1]\text{-tilt}\mathcal{D}$. The inverse is given by $M \mapsto M * [T[1]]$.

Proof. (1) Let X be an indecomposable $T[1]$ -projective object in $M * [T[1]]$. By Lemma 4.6, we know $X \in \text{add}M$. On the other hand, we can use similar arguments as in the proof of Lemma 4.6 to show that each direct summand of M is a $T[1]$ -projective object in $M * [T[1]]$.

(2) The assertion follows from (1) immediately. \square

With the notation of the above discussion, we give the following result.

Theorem 4.8. *The bijection in Proposition 4.7 is compatible with the bijection in Theorem 2.6. In other words, we have a commutative diagram*

$$\begin{array}{ccc}
 T[1]\text{-tilt}\mathcal{D} * T[1] & \xrightarrow[\text{1:1}]{P_{T[1]}(-)} & T[1]\text{-tilt}\mathcal{D} \\
 \text{Hom}_{\mathcal{D}}(T, -) \downarrow \text{1:1} & & \downarrow \text{1:1 Hom}_{\mathcal{D}}(T, -) \\
 \text{f-tors}\Lambda & \xrightarrow[\text{1:1}]{P(-)} & \text{s}\tau\text{-tilt}\Lambda
 \end{array}$$

in which each map is a bijection. The upper horizontal map is given in Proposition 4.7, the lower horizontal map is given in Theorem 2.6 and the right vertical map is given in Theorem 3.6.

Proof. We consider the left vertical map. For any $M * [T[1]]$ in $T[1]\text{-tilt}\mathcal{D} * T[1]$, where $M \in T[1]\text{-tilt}\mathcal{D}$, by Lemma 4.3, we get

$$\text{Hom}_{\mathcal{D}}(T, M * [T[1]]) = \text{Fac}\overline{M'} \in \text{f-tors}\Lambda.$$

By Theorem 2.6, we know that the left vertical map is bijective. It is easy to see that this diagram commutes. \square

4.2. Mutation of $T[1]$ -cluster tilting objects. Let \mathcal{D} be a k -linear, Hom-finite triangulated category with a cluster-tilting object T and a Serre functor \mathbb{S} , and let $\Lambda = \text{End}_{\mathcal{D}}^{op}(T)$ be the opposite algebra of the endomorphism algebra of T .

Recall that for an almost $T[1]$ -cluster tilting object U in \mathcal{D} , by Corollary 3.7 we know that there are two non-isomorphic $T[1]$ -cluster tilting objects $M = U \oplus X$ and $N = U \oplus Y$ in \mathcal{D} . Under bijections in Theorem 3.6, we get two support τ -tilting pairs $\widetilde{M} = (\overline{M'}, \overline{M''[-1]})$ and \widetilde{N} for Λ which have \widetilde{U} as a direct summand. By Proposition 2.8, we know that either $\overline{M'} > \overline{N'}$ or $\overline{M'} < \overline{N'}$. Using Proposition 4.4, we know that either $M > N$ or $M < N$. Thus, we can introduce the following notion.

Definition 4.9. For an almost $T[1]$ -cluster tilting object U in \mathcal{D} , by Corollary 3.7 and Proposition 4.4, we know that there are two $T[1]$ -cluster tilting objects $M = U \oplus X$ and $N = U \oplus Y$ in \mathcal{D} satisfying $M > N$, where X and Y are indecomposable. In this case, we call (M, N) an U -mutation pair and X and Y two complements to U . In this section, we also say that N is a left mutation of M and M is a right mutation of N and we write $N = \mu_X^L(M)$ and $M = \mu_Y^R(N)$.

Given an almost $T[1]$ -cluster tilting object in \mathcal{D} , the main result in this subsection shows that starting with a complement, we can calculate the other one by an exchange triangle, which is constructed from a left approximation or a right approximation.

Theorem 4.10. *Let $M = U \oplus X$ be a basic $T[1]$ -cluster tilting object in \mathcal{D} , where X is indecomposable. Then we have the following.*

(1) *If $X \in U * [T[1]]$, we take a triangle*

$$(\star) \quad Y \xrightarrow{g} U_1 \xrightarrow{f} X \xrightarrow{h} Y[1],$$

where f is a minimal right $(\text{add}U)$ -approximation. In this case, Y is another complement to U and $U \oplus Y > M$.

(2) If $X \notin U * [T[1]]$, we take a triangle

$$(\star\star) \quad X \xrightarrow{g} U_2 \xrightarrow{f} Y \xrightarrow{h} X[1],$$

where g is a minimal left $(\text{add}U)$ -approximation. In this case, Y is another complement to U and $U \oplus Y < M$.

To prove this theorem, we need some preparation. First we need the following easy observation.

Lemma 4.11. *The map $h : X \rightarrow Y[1]$ in (\star) factors through $\text{add}T[1]$.*

Proof. Since $X \in U * [T[1]]$, we have a triangle $U_X \xrightarrow{c_X} X \xrightarrow{\eta_X} C_X \rightarrow U_X[1]$, where $U_X \in \text{add}U$ and the morphism η_X factors through $\text{add}T[1]$. Because the map f in (\star) is a right $(\text{add}U)$ -approximation, there exists $i : U_X \rightarrow U_1$ such that $c_X = fi$. By the octahedral axiom, we have a commutative diagram

$$\begin{array}{ccccccc}
 & & U_X & \xlongequal{\quad} & U_X & & \\
 & & \vdots & & \downarrow c_X & & \\
 & & \exists i & & & & \\
 Y & \xrightarrow{g} & U_1 & \xrightarrow{f} & X & \xrightarrow{h} & Y[1] \\
 \parallel & & \downarrow & & \downarrow \eta_X & & \parallel \\
 Y & \longrightarrow & X' & \longrightarrow & C_X & \longrightarrow & Y[1] \\
 & & \downarrow & & \downarrow & & \\
 & & U_X[1] & \xlongequal{\quad} & U_X[1] & &
 \end{array}$$

of triangles. Thus h factors through η_X , which implies that h factors through $\text{add}T[1]$. □

The following lemma plays an important role in the proof of Theorem 4.10.

Lemma 4.12. *The map $g : Y \rightarrow U_1$ in (\star) is a minimal left $(\text{add}M)$ -approximation. In particular, g is a minimal left $(\text{add}U)$ -approximation.*

Proof. Take any map $a : Y \rightarrow M_0$, where $M_0 \in \text{add}M$. By Lemma 4.11, we may assume that there are two morphisms $h_1 : X[-1] \rightarrow T_0$ and $h_2 : T_0 \rightarrow Y$ such that $h[-1] = h_2h_1$, where $T_0 \in \text{add}T$:

$$\begin{array}{ccccccc}
 X[-1] & \xrightarrow{h[-1]} & Y & \xrightarrow{g} & U_1 & \xrightarrow{f} & X \xrightarrow{h} Y[1] \\
 \downarrow h_1 & \nearrow h_2 & & & \downarrow & & \\
 T_0 & & & & M_0 & &
 \end{array}$$

Noticing that

$$ah[-1] = (ah_2)h_1 \in [T](X[-1], M_0)$$

and $M = U \oplus X$ is $T[1]$ -rigid, we get $ah[-1] = 0$. This implies that a factors through g , and hence g is a left $(\text{add}M)$ -approximation.

Now we show that g is a left minimal map. If this were not true, then there would be a decomposition $U_1 = U_{11} \oplus U_{12}$ such that

$$g = \begin{pmatrix} g_0 \\ 0 \end{pmatrix} : Y \longrightarrow U_{11} \oplus U_{12}.$$

Consider the following triangles:

$$\begin{aligned} Y &\xrightarrow{g_0} U_{11} \longrightarrow Y' \xrightarrow{h} Y[1], \\ 0 &\longrightarrow U_{12} \longrightarrow U_{12} \longrightarrow 0. \end{aligned}$$

Thus we get a triangle

$$(11) \quad Y \xrightarrow{g = \begin{pmatrix} g_0 \\ 0 \end{pmatrix}} U_{11} \oplus U_{12} \longrightarrow Y' \oplus U_{12} \longrightarrow Y[1],$$

Comparing the triangle (\star) , we obtain that $X \simeq Y' \oplus U_{12}$. Since X is indecomposable, we would have $X \simeq U_{12} \in \text{add}U$. This is a contradiction and our claim follows. \square

The following results are also crucial.

Lemma 4.13. *The object Y in (\star) is indecomposable and it is not in $\text{add}M$.*

Proof. Suppose that there is a decomposition $Y = Y_1 \oplus Y_2$ with Y_1 and Y_2 nonzero. Since $\text{add}U$ is functorially finite in \mathcal{D} , we can get two triangles,

$$Y_1 \xrightarrow{g_1} U_{13} \xrightarrow{f_1} X_1 \longrightarrow Y_1[1] \quad \text{and} \quad Y_2 \xrightarrow{g_2} U_{14} \xrightarrow{f_2} X_2 \longrightarrow Y_2[1],$$

where g_1 and g_2 are two minimal left $(\text{add}U)$ -approximation. Thus by Lemma 4.12 the direct sum of these two triangles is $Y \xrightarrow{g} U_1 \xrightarrow{f} X \xrightarrow{h} Y[1]$, which implies that $X = X_1 \oplus X_2$. Since X is indecomposable, we may assume that $X_1 = 0$ and $X_2 = X$. Thus $U_{13} \simeq Y_1$ and

$$f = (0, f_2) : U_{13} \oplus U_{14} \longrightarrow X.$$

This is a contradiction because f is a right minimal map. Therefore Y is indecomposable.

Now we show that Y is not in $\text{add}M$. If this were not true, we would have $U_1 \simeq Y$ and $X = 0$ by Lemma 4.12. This is a contradiction and our claim follows. \square

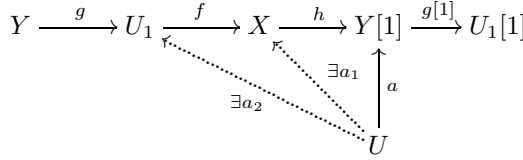
Lemma 4.14. *Assume that X and Z are two non-isomorphic complements to an almost $T[1]$ -cluster tilting object U in \mathcal{D} . Then $U \oplus X > U \oplus Z$ if and only if $X \notin U * [T[1]]$.*

Proof. By Proposition 4.4, we know that $U \oplus X > U \oplus Z$ if and only if $\overline{U \oplus X} > \overline{U \oplus Z}$. This is equivalent to $\overline{X'} \notin \text{Fac}\overline{U'}$ by Proposition 2.8. Thanks to Lemma 4.3, this is equivalent to $X \notin U * [T[1]]$. \square

Now we are ready to prove Theorem 4.10.

(1) We first show that Y is another complement to U . Using Lemma 4.13, we only need to show that $U \oplus Y$ is $T[1]$ -rigid. Take any map $a \in [T[1]](U, Y[1])$. Since

a factors through $\text{add}T[1]$ and U is $T[1]$ -rigid, we have $g[1]a = 0$:



Thus there exists $a_1 : U \rightarrow X$ such that $a = ha_1$. Observing that f is a right $(\text{add}U)$ -approximation, we know there exists $a_2 : U \rightarrow U_1$ such that $a_1 = fa_2$. Thus $a = ha_1 = (hf)a_2 = 0$ and hence

$$[T[1]](U, Y[1]) = 0.$$

For any morphism $b \in [T[1]](Y, U[1])$, we know that there are two morphisms $b_1 : Y \rightarrow T_1[1]$ and $b_2 : T_1[1] \rightarrow U[1]$ such that $b = b_2b_1$, where $T_1 \in \text{add}T$:

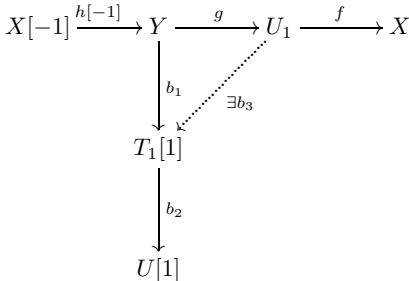


Figure 3

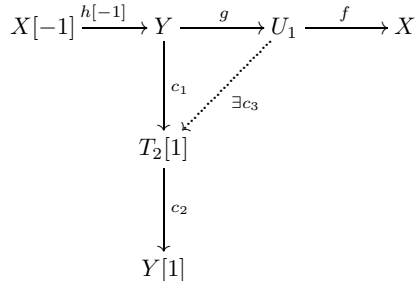


Figure 4

By Lemma 4.11, we know $h[-1]$ factors through $\text{add}T$, which implies that $b_1h[-1] = 0$. Thus there exists $b_3 : U_1 \rightarrow T_1[1]$ such that $b_1 = b_3g$. Since U is $T[1]$ -rigid, we have $b = b_2b_1 = (b_2b_3)g = 0$. Hence

$$[T[1]](Y, U[1]) = 0.$$

It remains to show that Y is $T[1]$ -rigid. In the similar way (see Figure 4), we know that, for any map $c \in [T[1]](Y, Y[1])$, $c = (c_2c_3)g$. Since $c_2c_3 \in [T[1]](U, Y[1]) = 0$, we have $c = 0$ and $[T[1]](Y, Y[1]) = 0$. Therefore, Y is another complement to U .

In order to show that $U \oplus Y > M$, by Lemma 4.14, we need to show that $Y \notin U * [T[1]]$. If this were not true, by Lemma 4.3, we would have $\overline{Y'} \in \text{Fac}\overline{U'}$. Since $X \in U * [T[1]]$, similarly, we have $\overline{X'} \in \text{Fac}\overline{U'}$. This contradicts with Proposition 2.8 and our claim follows.

(2) Let Y be another complement to U . Since $X \notin U * [T[1]]$, we have that $U \oplus Y < M$ and $Y \in U * [T[1]]$. By Lemma 4.14. Using (1) and Lemma 4.12, we know the assertion follows immediately. \square

4.3. An application. We end this section with an application of mutation. We first recall the following definition.

Definition 4.15 ([AIR]). Let Λ be a finite dimensional k -algebra and U be a basic τ -rigid Λ -module. We call $P^\perp(\tau U)$ the *Bongartz completion* of U .

Note that $U \in P(\perp(\tau U))$ and thus any τ -rigid Λ -module is a direct summand of some τ -tilting Λ -module. The following observation is needed in this subsection.

Proposition 4.16 ([AIR, Definition-Proposition 2.26]). *Let Λ be a finite dimensional k -algebra and let $T = X \oplus U$ and T' be support τ -tilting Λ -modules such that $T' = \mu_X(T)$ for some indecomposable Λ -module X . If T is a Bongartz completion of U , then $T > T'$ and $X \notin \text{Fac}U$.*

In [AIR], the authors gave the following result to calculate left mutation of support τ -tilting modules by exchange sequences.

Theorem 4.17. *Assume that Λ is a finite dimensional k -algebra. Let $M_\Lambda = X_\Lambda \oplus U_\Lambda$ be a basic τ -tilting Λ -module which is the Bongartz completion of U_Λ , where X_Λ is indecomposable. Let*

$$(\star\star\star) \quad X_\Lambda \xrightarrow{g_\Lambda} U'_\Lambda \xrightarrow{f_\Lambda} Y_\Lambda \longrightarrow 0$$

be an exact sequence, where g_Λ is a minimal left $(\text{add}U_\Lambda)$ -approximation. Then we have the following.

- (a) *If U_Λ is not sincere, then $Y_\Lambda = 0$. In this case $U_\Lambda = \mu_{X_\Lambda}^L(M_\Lambda)$ holds and this is a basic support τ -tilting Λ -module which is not τ -tilting.*
- (b) *If U_Λ is sincere, then Y_Λ is a direct sum of copies of an indecomposable Λ -module Y_1 and is not in $\text{add}M_\Lambda$. In this case $Y_1 \oplus U_\Lambda = \mu_{X_\Lambda}^L(M_\Lambda)$ holds and this is a basic τ -tilting Λ -module.*

Furthermore, the authors posed the following question.

Question 4.18. Is Y_Λ always indecomposable in Theorem 4.17(b)?

In this subsection, we give a positive answer to this question when Λ is an endomorphism algebra of a cluster-tilting object. More precisely, there is a cluster-tilting object T in a triangulated category \mathcal{D} with a Serre functor \mathbb{S} such that $\Lambda = \text{End}_{\mathcal{D}}^{\text{op}}(T)$.

Since $M_\Lambda = X_\Lambda \oplus U_\Lambda$ is a τ -tilting Λ -module, we know there exists a $T[1]$ -cluster tilting object $M = X \oplus U$ in \mathcal{D} such that $\overline{X} = X_\Lambda$ and $\overline{U} = U_\Lambda$ by Theorem 3.6. Note that $M_\Lambda = X_\Lambda \oplus U_\Lambda$ is the Bongartz completion of U_Λ . By Proposition 4.16, we know $X_\Lambda \notin \text{Fac}U_\Lambda$. By Lemma 4.3, we have $X \notin U * [T[1]]$. Thus we can use the triangle (11) to obtain another complement Y to U . Our main result of this subsection is the following.

Theorem 4.19. *The exact sequence $(\star\star\star)$ in $\text{mod}\Lambda$ is induced from the triangle (11) in \mathcal{D} .*

Proof. Applying $\overline{(\)}$ to (11) and using Lemma 4.11, we have an exact sequence:

$$(12) \quad \overline{X} \xrightarrow{\overline{g}} \overline{U}_2 \xrightarrow{\overline{f}} \overline{Y} \longrightarrow 0.$$

Since g is a left $(\text{add}U)$ -approximation, we know that \bar{g} is a left $(\text{add}U_\Lambda)$ -approximation. Now we show that \bar{g} is a left minimal map. If this were not true, then there would be a decomposition $\overline{U_2} = W_1 \oplus W_2$ and an exact sequence

$$\overline{X} \xrightarrow{\bar{g} = \begin{pmatrix} g_0 \\ \varphi \end{pmatrix}} W_1 \oplus W_2 \xrightarrow{\bar{f}} \overline{Y} \rightarrow 0$$

in $\text{mod}\Lambda$. Thus W_2 would be a direct summand of \overline{Y} . Since $Y \notin \text{add}U$, we know this is a contradiction and our claim follows. Hence the exact sequences (12) and $(\star\star\star)$ are coincident. \square

The following consequence is direct, which gives a partial answer to Question 4.18.

Corollary 4.20. *Let \mathcal{D}, Λ be as above. Then Y_Λ is always indecomposable in Theorem 4.17(b).*

5. EXAMPLES

Example 5.1. Let $A = kQ/I$ be a self-injective algebra given by the quiver

$$Q : 1 \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{matrix} 2$$

and $I = \langle \alpha\beta\alpha\beta, \beta\alpha\beta\alpha \rangle$. Let \mathcal{D} be the stable module category $\underline{\text{mod}}A$ of A . This is a triangulated category with a Serre functor (it is not 2-CY). We describe the AR-quiver of $\underline{\text{mod}}A$ in the following picture:

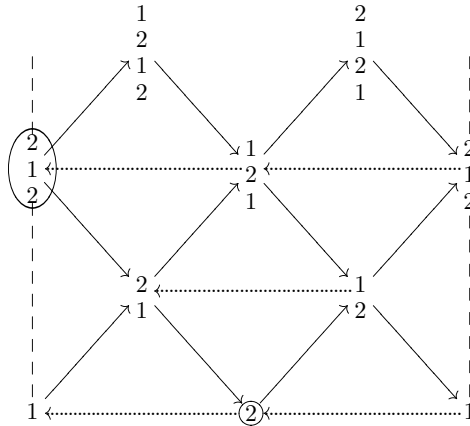


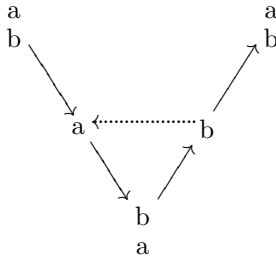
Figure 5

where the leftmost and rightmost columns are identified. Thus, we also get the AR-quiver of $\underline{\text{mod}}A$ by deleting the first row in Figure 5. The direct sum

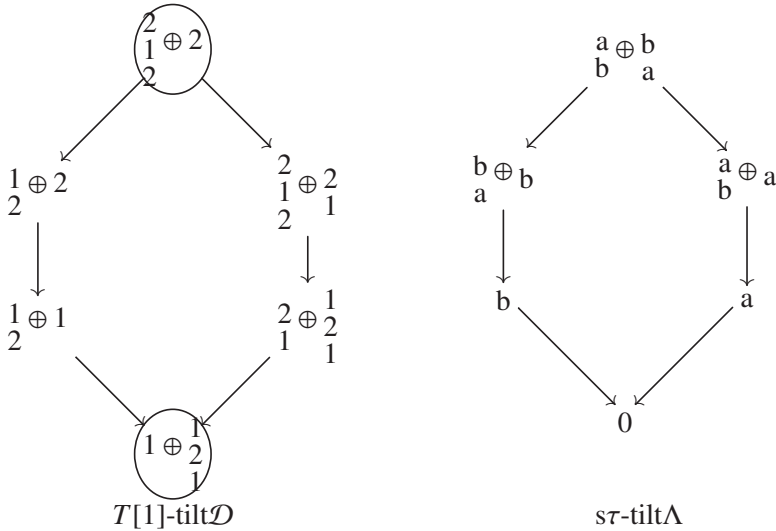
$$T = \begin{matrix} 2 \\ 2 \oplus 1 \\ 2 \end{matrix}$$

is a cluster-tilting object in \mathcal{D} . The opposite algebra of endomorphism algebra $\Lambda = \text{End}_{\mathcal{D}}^{op}(T) = kQ'/I'$ is given by the quiver $Q' : a \begin{matrix} \xrightarrow{\gamma} \\ \xleftarrow{\delta} \end{matrix} b$ and $I' = \langle \gamma\delta, \delta\gamma \rangle$.

The AR-quiver of $\text{mod } \Lambda$ is



We depict $T[1]$ -cluster tilting objects in \mathcal{D} and support τ -tilting modules in $\text{mod } \Lambda$ as follows (the encircled objects are cluster-tilting objects)



Example 5.2. Let Q be the quiver $1 \xrightarrow{\alpha} 2$. Assume that τ_Q is the Auslander-Reiten translation in $D^b(kQ)$. We consider the repetitive cluster category $\mathcal{D} = D^b(kQ)/\langle \tau_Q^{-2}[2] \rangle$ introduced by Zhu in [Z11], whose objects are the same in $D^b(kQ)$, and whose morphisms are given by

$$\text{Hom}_{D^b(kQ)/\langle \tau_Q^{-2}[2] \rangle}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(kQ)}(X, (\tau_Q^{-2}[2])^i Y).$$

It is shown in [Z11] that \mathcal{D} is a triangulated category with a Serre functor \mathbb{S} . Note that it is not 2-CY (but it is a fractional Calabi–Yau category with CY-dimension $\frac{4}{2}$). The AR-quiver of \mathcal{D} is as in Figure 6 below.

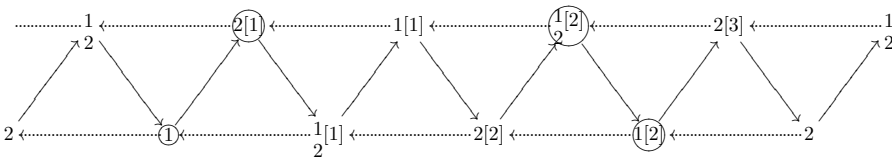


Figure 6

The direct sum

$$T = 1 \oplus 2[1] \oplus \begin{matrix} 1 \\ 2 \end{matrix} [2] \oplus 1[2]$$

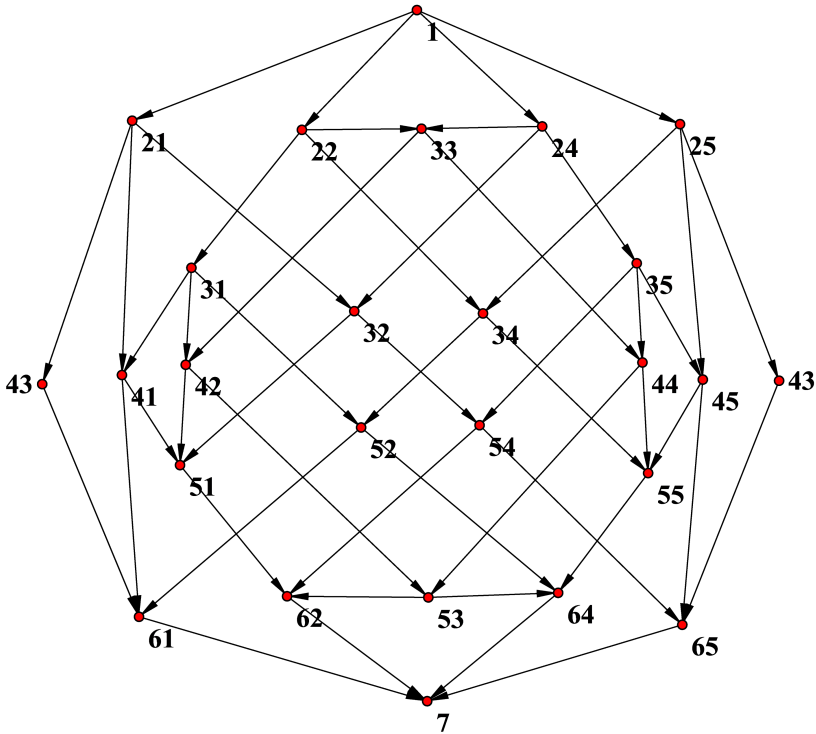
of the encircled indecomposable objects gives a cluster-tilting object. Note that the opposite algebra of the endomorphism algebra $\Lambda = \text{End}_{\mathcal{D}}^{op}(T)$ is not connected, it is given by the following disconnected quiver,

$$a \longrightarrow b \qquad c \longrightarrow d$$

with no relations. The AR-quiver of $\text{mod}\Lambda$ is



We use the following picture to describe $T[1]$ -cluster tilting objects in \mathcal{D} and support τ -tilting modules in $\text{mod}\Lambda$.



where the leftmost and rightmost points are identified, and the cluster-tilting objects are marked by ♣.

| Points | $T[1]$ -cluster tilting objects | support τ -tilting modules |
|--------|-----------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|
| 1♣ | $\frac{1}{2}[2] \oplus 1[2] \oplus 1 \oplus 2[1]$ | Λ |
| 21 | $\frac{1}{2}[2] \oplus 1[2] \oplus 1 \oplus 2[2]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus d \oplus b$ |
| 22 | $\frac{1}{2}[2] \oplus 1[2] \oplus \frac{1}{2}[1] \oplus 2[1]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus d \oplus a \oplus \begin{smallmatrix} a \\ b \end{smallmatrix}$ |
| 24 | $2[3] \oplus 1[2] \oplus 1 \oplus 2[1]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus c \oplus \begin{smallmatrix} a \\ b \end{smallmatrix} \oplus b$ |
| 25 | $\frac{1}{2}[2] \oplus \frac{1}{2} \oplus 1 \oplus 2[1]$ | $\begin{smallmatrix} a \\ b \end{smallmatrix} \oplus b \oplus d$ |
| 31 | $\frac{1}{2}[1] \oplus 1[2] \oplus 1[1] \oplus \frac{1}{2}[2]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus d \oplus a$ |
| 32 | $2[2] \oplus 1[2] \oplus 1 \oplus 2[3]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus b \oplus c$ |
| 33♣ | $\frac{1}{2}[1] \oplus 1[2] \oplus 2[3] \oplus 2[1]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus c \oplus \begin{smallmatrix} a \\ b \end{smallmatrix} \oplus a$ |
| 34 | $\frac{1}{2}[1] \oplus \frac{1}{2}[2] \oplus \frac{1}{2} \oplus 2[1]$ | $\begin{smallmatrix} a \\ b \end{smallmatrix} \oplus a \oplus d$ |
| 35 | $1 \oplus 2[1] \oplus 2 \oplus 2[3]$ | $\begin{smallmatrix} a \\ b \end{smallmatrix} \oplus b \oplus c$ |
| 41 | $\frac{1}{2}[2] \oplus 1[2] \oplus 1[1] \oplus 2[2]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus d$ |
| 42 | $\frac{1}{2}[1] \oplus 1[2] \oplus 1[1] \oplus 2[3]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus a \oplus c$ |
| 43♣ | $\frac{1}{2}[2] \oplus 2[2] \oplus 1 \oplus \frac{1}{2}$ | $b \oplus d$ |
| 44 | $\frac{1}{2}[1] \oplus 2[3] \oplus 2 \oplus 2[1]$ | $\begin{smallmatrix} a \\ b \end{smallmatrix} \oplus a \oplus c$ |
| 45 | $\frac{1}{2} \oplus 2[1] \oplus 1 \oplus 2$ | $\begin{smallmatrix} a \\ b \end{smallmatrix} \oplus b$ |
| 51 | $2[3] \oplus 1[2] \oplus 1[1] \oplus 2[2]$ | $\begin{smallmatrix} c \\ d \end{smallmatrix} \oplus c$ |
| 52 | $\frac{1}{2} \oplus \frac{1}{2}[1] \oplus \frac{1}{2}[2] \oplus 1[1]$ | $a \oplus d$ |
| 53♣ | $\frac{1}{2}[1] \oplus 1[1] \oplus 2 \oplus 2[3]$ | $a \oplus c$ |
| 54 | $2 \oplus 2[2] \oplus 1 \oplus 2[3]$ | $b \oplus c$ |
| 55 | $\frac{1}{2}[1] \oplus 2[1] \oplus 2 \oplus \frac{1}{2}$ | $\begin{smallmatrix} a \\ b \end{smallmatrix} \oplus a$ |
| 61 | $\frac{1}{2}[2] \oplus 1[1] \oplus \frac{1}{2} \oplus 2[2]$ | d |
| 62 | $1[1] \oplus 2[2] \oplus 2[3] \oplus 2$ | c |
| 64 | $\frac{1}{2}[1] \oplus 1[1] \oplus 2 \oplus \frac{1}{2}$ | a |
| 65 | $\frac{1}{2} \oplus 1 \oplus 2 \oplus 2[2]$ | b |
| 7♣ | $\frac{1}{2} \oplus 1[1] \oplus 2 \oplus 2[2]$ | 0 |

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