

COUPLING BY REFLECTION AND HÖLDER REGULARITY FOR NON-LOCAL OPERATORS OF VARIABLE ORDER

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ABSTRACT. We consider the non-local operator of variable order as follows:

$$Lf(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) \frac{n(x, z)}{|z|^{d+\alpha(x)}} dz,$$

where $\alpha(x) \in [\alpha_0, \alpha_2]$ for any $x \in \mathbb{R}^d$ and some constants $0 < \alpha_0 \leq \alpha_2 < 2$ and $n(x, z)$ is a positive Borel measurable function bounded from above and below. Under mild continuity conditions on $\alpha(x)$ and $n(x, z)$, we establish the Hölder regularity for the associated semigroups. The proof is based on a new construction of the coupling by reflection for non-local operator L , and the results successfully apply to both stable-like processes in the sense of Bass and time-change of symmetric stable processes.

1. INTRODUCTION AND MAIN RESULTS

In the last few years, more and more people have tended to use non-local integral operators (or, equivalently, processes with jumps) to model problems in mathematical physics and finance, since in many applications jump processes are more realistic models than continuous processes. While there is a rich literature concerned with the regularity for diffusion semigroups generated by second order elliptic differential operators (see e.g. [19, 29, 35] and the references therein), the study of regularity properties of the semigroups associated to non-local operators is far from complete.

In this paper we aim to establish the Hölder regularity for Markov semigroups associated with the infinitesimal generator

$$(1.1) \quad Lf(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) \frac{n(x, z)}{|z|^{d+\alpha(x)}} dz.$$

This is a reasonably general integro-differential operator which includes, for example, many of the operators considered by probabilists. In probabilistic terms, when both $\alpha(x)$ and $n(x, z)$ are constant functions, the process associated with L given by (1.1) is essentially a (rotationally) symmetric stable process. In our setting,

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however, the operator L given by (1.1) can be of variable order; i.e., $\alpha(x)$ is a function of x . Therefore, in general the corresponding process should behave like a symmetric stable process at each point x , but vary from point to point. Though it is known that the semigroups of symmetric stable processes enjoy nice smooth properties, there are many essential differences between symmetric stable processes and those processes associated with L above, even in the case that $\alpha(x)$ is a constant function; see for instance [3, 9, 10]. The purpose of our paper is to investigate what conditions on $\alpha(x)$ and $n(x, z)$ are needed to guarantee the Hölder continuity of the corresponding semigroups. The functions $\alpha(x)$ and $n(x, z)$ are supposed to be at least uniformly continuous; see Assumption **(H)** below for the precise conditions. We stress that we are able to treat the case where the index function $\alpha(x)$ is logarithmically uniformly continuous and the coefficient $n(x, z)$ enjoys the symmetry property, i.e., $n(x, z) = n(x, -z)$ for each $x, z \in \mathbb{R}^d$, or the case that the function $\alpha(x)$ is logarithmically uniformly continuous such that $\inf_{x \in \mathbb{R}^d} \alpha(x) > 1$. In particular, we will establish Hölder regularity for Markov semigroups of stable-like processes in the sense of Bass [2].

To state the main result, we first introduce the following assumptions on the operator L .

Assumption (H).

- (H1) The martingale problem associated with the operator L has a unique solution, and the associated unique L -process enjoys the strong Markov property and does not explode starting from any $x \in \mathbb{R}^d$.
- (H2) There exist two constants $0 < \alpha_0 \leq \alpha_2 < 2$ such that for all $x \in \mathbb{R}^d$, $\alpha(x) \in [\alpha_0, \alpha_2]$. The function $\alpha(x)$ satisfies that

$$(1.2) \quad \lim_{|x-y| \rightarrow 0} |\alpha(x) - \alpha(y)| \log \frac{1}{|x-y|} = 0.$$

- (H3) There are two positive constants c_1 and c_2 such that for all $x, z \in \mathbb{R}^d$,

$$c_1 \leq n(x, z) \leq c_2.$$

The function $(x, z) \mapsto n(x, z)$ is continuous on \mathbb{R}^{2d} in the sense that

$$(1.3) \quad \lim_{r \rightarrow 0} \left[\sup_{x \in \mathbb{R}^d, |z_1 - z_2| \leq r} |n(x, z_1) - n(x, z_2)| + \sup_{|z| \leq 1, |x-y| \leq r} |n(x, z) - n(y, z)| \right] = 0.$$

- (H4) The functions $\alpha(x)$ and $n(x, z)$ satisfy that

$$(1.4) \quad \lim_{r \rightarrow 0} \sup_{|x-y| \leq r} |x-y|^{\alpha(x) \wedge \alpha(y) - 2} \int_{\{\frac{|x-y|}{2} < |z| \leq 1\}} \langle x-y, z \rangle \left[\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z)}{|z|^{d+\alpha(x)}} \right] dz = 0.$$

Let us make some comments on Assumption **(H)**.

Remark 1.1.

(1) There are a few papers that handle martingale problems for non-local operators of variable order; e.g. see [2, 30, 34, 37, 43, 44] and the references therein. Among the first is [2] by Bass, which proved in the one-dimensional situation well-posedness of the martingale problem for $-(-\Delta)^{\alpha(x)}$ under weak assumptions on $\alpha(x)$; see Example 1.3 below for more details. Following [2], we call the associated pure jump type Markov process the stable-like process with exponent $\alpha(x)$. In recent years there has been considerable interest in operators whose jump kernel is

of the form $\frac{n(x,z)}{|z|^{d+\alpha(x)}} dz$; e.g. see [1, 5, 6, 8]. The corresponding martingale problem was considered in [43]. See [13] and the references therein for more details.

(2) The assumption (H2) on the index function $\alpha(x)$ is standard to ensure the uniqueness of the solution to the martingale problem for L ; see [2, Corollary 2.3] and [43, Assumption 2.2]. In particular, (1.2) holds when $\alpha(x)$ is Hölder continuous.

(3) According to [10, Assumption 1.1(1)] or [43, Assumption 2.1], the condition that the function n is bounded from above and below is natural for the existence of solutions to the martingale problem of L . This is the analogue of uniform ellipticity and boundedness conditions in the theory of second order elliptic differential operators. On the other hand, [10, Assumption 1.1(2)] or [43, Assumption 2.2] shows that condition (1.3) is quite mild for the uniqueness of the solution to the martingale problem for L . The paper [1] by Barlow et al. indicates that uniqueness may fail if one only requires that $n(x, z)$ be bounded from above and below; see [1, Section 6] for more details. In particular, (1.3) is satisfied when the function $(x, z) \mapsto n(x, z)$ is uniformly continuous on \mathbb{R}^{2d} .

(4) The assumption (H4) is our technique condition. It is clear that (1.4) holds if $n(x, z) = n(x, -z)$ for all $x, z \in \mathbb{R}^d$. Note that the symmetry of the function $n(x, z)$ with respect to z is a commonly assumed condition for non-local operators; e.g. see [8, 16, 25]. We will see from Proposition 1.8 below that (1.4) is also satisfied in many situations, in particular, when $\inf_{x \in \mathbb{R}^d} \alpha(x) > 1$.

Let $((\mathbb{P}^x)_{x \in \mathbb{R}^d}, (X_t)_{t \geq 0})$ be a strong Markov process such that for each $x \in \mathbb{R}^d$, the probability measure \mathbb{P}^x is the unique solution to the martingale problem for L starting at x . Let \mathbb{E}^x be the expectation of the process $(X_t)_{t \geq 0}$ starting from $x \in \mathbb{R}^d$. For any $x \in \mathbb{R}^d$ and $t > 0$, let

$$P_t f(x) := \mathbb{E}^x(f(X_t)), \quad f \in B_b(\mathbb{R}^d).$$

Now, it is time to present the main contribution of our paper.

Theorem 1.2. *Under Assumption (H), for any $\beta \in (0, \alpha_0 \wedge 1)$ there exists a constant $C := C(\beta, \alpha(x), n(x, z)) > 0$ such that for all $f \in B_b(\mathbb{R}^d)$ and $t > 0$,*

$$\sup_{x \neq y} \frac{|P_t f(y) - P_t f(x)|}{|y - x|^\beta} \leq \frac{C \|f\|_\infty}{(t \wedge 1)^{\beta/\alpha_0}}.$$

In particular, for any $f \in B_b(\mathbb{R}^d)$ and $t > 0$, $P_t f$ is β -Hölder continuous with any $\beta \in (0, \alpha_0 \wedge 1)$.

Hölder continuity and Harnack inequality of bounded functions that are harmonic in a domain with respect to pure jump non-local operators with variable orders are established in [5, 6]. In fact, the setting of [6] is more general in the sense that the jump measure of the corresponding operator is not required to have a density with respect to the Lebesgue measure. In particular, according to [6, Theorem 2.2 and Example 2], under assumptions (H1)–(H3) and the additional condition that either $\inf_{x \in \mathbb{R}^d} \alpha(x) > 1$ or $n(x, z) = n(x, -z)$ for all $x, z \in \mathbb{R}^d$ (which implies that (H4) holds true), we know that any bounded function which is harmonic in a domain with respect to the operator L given by (1.1) is Hölder continuous. This, along with [5, Proposition 3.1] and [8, Section 4], implies that the associated resolvents are Hölder continuous too; see e.g. [7, Proposition 3.3]. If moreover the associated Markov process is symmetric and possesses the bounded transition density function, then, according to [7, Proposition 3.4] and by using the spectral

theory, we can get the Hölder continuity of the corresponding semigroups. However, we can see from [28, Section 5] or [40, Section 4] that the process generated by the operator L is in general non-symmetric. On the other hand, consider, for example, the shift semigroup $P_t f(x) := f(x + t)$, which is certainly not strong Feller but has a strong Feller resolvent; that is, the resolvent maps $B_b(\mathbb{R})$ into $C_b(\mathbb{R})$. Indeed, for any $\lambda > 0$ and $f \in B_b(\mathbb{R})$,

$$U_\lambda f(x) = \int_0^\infty e^{-\lambda t} f(x + t) dt = e^{\lambda x} \int_x^\infty e^{-\lambda t} f(t) dt$$

is globally Lipschitz continuous on \mathbb{R} with Lipschitz constant $2\|f\|_\infty$. This counterexample indicates that in the non-symmetric situation, the Hölder continuity of resolvent operators does not imply that of the corresponding semigroups. Therefore, Theorem 1.2 is not a direct or a simple consequence of the existing results on the Hölder continuity of bounded functions that are harmonic in a domain. Finally, we should mention that there is a huge amount of interest among analysts in studying Harnack inequalities and related regularity properties for non-local operators; see [15, 17, 18, 24, 27, 31, 32, 41] and the references therein.

As applications of Theorem 1.2, we consider the following two examples.

Example 1.3. Consider the integro-differential operator

$$Lf(x) = w(x) \int_{\mathbb{R}^d \setminus \{0\}} \left(f(x + z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) |z|^{-d-\alpha(x)} dz$$

for $f \in C_c^\infty(\mathbb{R}^d)$. The weight function $w(x)$ is chosen in such a way that

$$w(x) = \alpha(x) 2^{\alpha(x)-1} \frac{\Gamma((\alpha(x) + d)/2)}{\pi^{d/2} \Gamma(1 - \alpha(x)/2)}.$$

Then $Le_\xi(x) = -|\xi|^{\alpha(x)} e_\xi(x)$, where $e_\xi(x) = e^{i\langle x, \xi \rangle}$; see e.g. [11, Exercise 18.23, p. 184]. With this choice of $w(x)$, L can be written as a pseudo-differential operator $-p(x, D)$ with the symbol $-|\xi|^{\alpha(x)}$,

$$Lf(x) = \int e^{i\langle x, \xi \rangle} |\xi|^{\alpha(x)} \widehat{f}(\xi) d\xi = -(-\Delta)^{\alpha(x)} f(x),$$

which implies that $L = -(-\Delta)^{\alpha(x)}$ is a stable-like operator in the sense of Bass [2].

For any $r > 0$, let

$$\rho(r) := \sup_{|x-y| \leq r} |\alpha(x) - \alpha(y)|$$

be the modulus of continuity of the index function α . Suppose the following conditions hold:

- (i) $0 < \alpha_0 := \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \sup_{x \in \mathbb{R}^d} \alpha(x) =: \alpha_2 < 2$;
- (ii) $\lim_{r \rightarrow 0} \rho(r) |\log r| = 0$, and $\int_0^1 \rho(r)/r dr < \infty$.

Then, according to [2, Corollary 2.3 and Remark 7.1], the martingale problem associated with the operator L is well-posed. It is clear that assumptions (H2) and (H4) are satisfied. In particular, in this case $n(x, z) = w(x)$ is independent of $z \in \mathbb{R}^d$. On the other hand, since the function

$$r \mapsto r 2^{r-1} \frac{\Gamma((r + d)/2)}{\pi^{d/2} \Gamma(1 - r/2)}$$

is smooth on $[\alpha_0, \alpha_2] \subset (0, 2)$, we know from the definition of $w(x)$ and the assumption $\lim_{r \rightarrow 0} \rho(r) |\log r| = 0$ that the function $w(x)$ is bounded from above and below and is uniformly continuous on \mathbb{R}^d . Thus, assumption (H3) also holds. Therefore, by Theorem 1.2, the associated semigroup of the stable-like process is β -Hölder continuous for any $\beta \in (0, \alpha_0 \wedge 1)$.

Example 1.4. Suppose that $A(x) = (a_{i,j}(x))_{1 \leq i,j \leq d}$ is a bounded continuous $(d \times d)$ -matrix-valued function on \mathbb{R}^d that is non-degenerate at every $x \in \mathbb{R}^d$ and $(Z_t)_{t \geq 0}$ is a (rotationally) symmetric α -stable process on \mathbb{R}^d for some $0 < \alpha < 2$. It is shown in [4, Theorem 7.1] that for every $x \in \mathbb{R}^d$ the stochastic differential equation (SDE)

$$(1.5) \quad dX_t = A(X_{t-}) dZ_t, \quad X_0 = x$$

has a unique weak solution. Although it is assumed in [4] that $d \geq 2$, the results here are valid for $d = 1$ as well. In particular, for $d = 1$, $(X_t)_{t \geq 0}$ is a time-change of a symmetric α -stable process. Using the Itô formula, one deduces (see the formula above [4, (7.2)]) that $(X_t)_{t \geq 0}$ has generator

$$Lf(x) = \int \left(f(x + A(x)u) - f(x) - \langle \nabla f(x), A(x)u \rangle \mathbf{1}_{\{|u| \leq 1\}} \right) \frac{c_{d,\alpha}}{|u|^{d+\alpha}} du,$$

where $c_{d,\alpha}$ is a positive constant depending on d and α . A change of variable formula $z = A(x)u$ yields that

$$Lf(x) = \int \left(f(x + z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \frac{k(x, z)}{|z|^{d+\alpha}} dz,$$

where

$$k(x, z) = \frac{c_{d,\alpha}}{|\det A(x)|} \left(\frac{|z|}{|A(x)^{-1}z|} \right)^{d+\alpha}.$$

Here, $\det A(x)$ is the determinant of the matrix $A(x)$, and $A(x)^{-1}$ is the inverse of $A(x)$. In particular, according to the argument in [4, Section 7], we know that the martingale problem for the operator L is well-posed. It is also easy to see that assumptions (H2) and (H4) hold true.

Next, suppose furthermore that $A(x) = (a_{i,j}(x))_{1 \leq i,j \leq d}$ is uniformly continuous, bounded, and elliptic (that is, there are positive constants λ_1 and λ_2 such that $\lambda_1 \text{Id} \leq A(x) \leq \lambda_2 \text{Id}$ for every $x \in \mathbb{R}^d$). Then, the assumption (H3) is also satisfied by the definition of $k(x, z)$. Therefore, according to Theorem 1.2, we conclude that the semigroup corresponding to the SDE (1.5) is β -Hölder continuous for any $\beta \in (0, \alpha \wedge 1)$.

When $A(x) = (a_{i,j}(x))_{1 \leq i,j \leq d}$ is Hölder continuous, Example 1.4 has been studied in [25, Corollary 1.3]. Indeed, in this case sharp two-sided estimates on the transition density of the SDE (1.5) are presented there, and Hölder estimates (for all $\alpha \in (0, 2)$) and gradient estimates (for $\alpha \in [1, 2)$) on the heat kernel are obtained; see [25, (1.9) and (1.15)]. In particular, when $\alpha \in [1, 2)$, the associated semigroups are even Lipschitz continuous. We also refer the reader to [46] for the recent study of gradient estimates for SDEs driven by multiplicative Lévy noise. Though the assertion of Example 1.4 is much weaker than [25, Corollary 1.3], we want to stress that our approach is completely different from that of [25], and we only require the uniform continuity of the coefficient $A(x) = (a_{i,j}(x))_{1 \leq i,j \leq d}$. This

fact convinces us that the coupling method could yield regularity of semigroups associated with non-local operators with coefficients of low regularity. Indeed, when $\alpha_0 := \inf_{x \in \mathbb{R}^d} \alpha(x) > 1$ and the functions $\alpha(x)$ and $n(x, z)$ fulfill stronger continuity, the coupling method allows us to establish better regularity properties (including the Lipschitz continuity) for the Markov semigroups associated with the operator L given by (1.1). For this, we need the following two classes of reference functions:

$$\begin{aligned} \mathcal{D} &= \{ \varphi \in C^2((0, 2]) \cap L^1((0, 2]; dx) : \varphi > 0, \varphi' < 0, \text{ and } \varphi'' > 0 \}, \\ \mathcal{D}_\theta &= \left\{ \varphi \in \mathcal{D} : \lim_{r \rightarrow 0} \frac{r^{\theta-2}}{\varphi'(r)} = 0 \text{ and } \limsup_{r \rightarrow 0} \frac{\varphi''(r)r}{\varphi'(r)} < \theta - 2 \right\}, \quad \theta \in (0, 2). \end{aligned}$$

Theorem 1.5. *Assume that $\alpha_0 := \inf_{x \in \mathbb{R}^d} \alpha(x) > 1$ and assumptions (H1)–(H3) hold. Let $\varphi \in \mathcal{D}_{\alpha_0}$ and*

$$\Psi(r) := -\frac{\varphi'(2r)r}{\varphi(r)}, \quad r \in (0, 1].$$

For any $x, y \in \mathbb{R}^d$, set

$$\begin{aligned} A(x, y) &:= |\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) + \sup_{|z| \leq 1} |n(x, z) - n(y, z)| \\ &\quad + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x - y|} |n(z, z_1) - n(z, z_2)|. \end{aligned}$$

If

$$(1.6) \quad \lim_{|x-y| \rightarrow 0} \frac{A(x, y)}{\Psi(|x - y|)} = 0,$$

then there exist constants $\varepsilon_0 \in (0, 1)$ and $C > 0$ such that for all $f \in B_b(\mathbb{R}^d)$, $t > 0$, and $|x - y| \leq 1$,

$$\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{\int_0^{|x-y|} \varphi(s) ds} \leq C \|f\|_\infty \inf_{\varepsilon \in (0, \varepsilon_0]} \left[\frac{1}{\int_0^\varepsilon \varphi(s) ds} - \frac{1}{t\varphi'(2\varepsilon)\varepsilon^{2-\alpha_0}} \right].$$

As applications of Theorem 1.5, we have the following two typical examples. The first one deals with the Lipschitz continuity, and it follows from Theorem 1.5 by taking $\varphi(r) = \exp(-\int_0^r \frac{h(u)}{u} du)$; while the second one treats the log-Lipschitz continuity, and it is a consequence of Theorem 1.5 by taking $\varphi(r) = \log^\beta(6/r)$ for any $\beta > 0$.

Example 1.6. Assume that $\alpha_0 := \inf_{x \in \mathbb{R}^d} \alpha(x) > 1$ and assumptions (H1)–(H3) hold.

- (1) Let $h : [0, \infty) \rightarrow [0, \infty)$ be an increasing function such that $h(0) = 0$ and it is differentiable on $(0, \infty)$,

$$\limsup_{r \rightarrow 0} \frac{h'(r)r}{h(r)} < \alpha_0 - 1, \quad \lim_{r \rightarrow 0} \frac{r^{\alpha_0-1}}{h(r)} = 0, \quad \text{and} \quad \int_0^1 \frac{h(r)}{r} dr < \infty.$$

If

$$(1.7) \quad \lim_{|x-y| \rightarrow 0} \frac{A(x, y)}{h(|x - y|)} = 0,$$

then there exist constants $\varepsilon_0 \in (0, 1]$ and $C > 0$ such that for all $f \in B_b(\mathbb{R}^d)$ and $t > 0$,

$$\sup_{x \neq y} \frac{|P_t f(y) - P_t f(x)|}{|y - x|} \leq C \|f\|_\infty \inf_{\varepsilon \in (0, \varepsilon_0]} \left(\frac{1}{\varepsilon} + \frac{\varepsilon^{\alpha_0 - 1}}{t h(\varepsilon)} \right).$$

In particular, for any $f \in B_b(\mathbb{R}^d)$ and $t > 0$, $P_t f$ is Lipschitz continuous.

(2) If

$$\lim_{|x-y| \rightarrow 0} \left(\log \frac{1}{|x-y|} \right) A(x, y) = 0,$$

then, for any $\beta > 0$, there exists a constant $C > 0$ such that for all $f \in B_b(\mathbb{R}^d)$ and $t > 0$,

$$\sup_{x \neq y} \frac{|P_t f(y) - P_t f(x)|}{|y - x| \cdot |\log |x - y||^\beta} \leq \frac{C \|f\|_\infty}{(t \wedge e^{-1})^{1/\alpha_0} |\log(t \wedge e^{-1})|^{-1/\alpha_0 + \beta}}.$$

Remark 1.7. Comparing Example 1.6(1) with Theorem 1.2 and Example 1.6(2), we can find that in order to yield the Lipschitz continuity of the Markov semigroups, some additional (but mild) continuity assumptions on the functions $\alpha(x)$ and $n(x, z)$ are required. For instance, by Example 1.6(1), the associated semigroup is Lipschitz continuous if

$$(1.8) \quad \lim_{|x-y| \rightarrow 0} \left[\log \frac{1}{|x-y|} \left(\log \log \frac{1}{|x-y|} \right)^2 \right] A(x, y) = 0$$

or

$$\lim_{|x-y| \rightarrow 0} \left[\log \frac{1}{|x-y|} \left(\log \log \frac{1}{|x-y|} \right) \left(\log \log \log \frac{1}{|x-y|} \right)^2 \right] A(x, y) = 0$$

and so on. In particular, under (1.8), we have

$$\sup_{x \neq y} \frac{|P_t f(y) - P_t f(x)|}{|y - x|} \leq \frac{C \|f\|_\infty}{(t \wedge e^{-2})^{1/\alpha_0} |\log(t \wedge e^{-2})|^{-1/\alpha_0} |\log |\log(t \wedge e^{-2})||^{-2/\alpha_0}}.$$

Note that if there exists some constant $\delta \in (0, 1]$ such that

$$\begin{aligned} \lim_{|x-y| \rightarrow 0} \frac{1}{|x-y|^\delta} \left[|\alpha(x) - \alpha(y)| + \sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right. \\ \left. + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x-y|} |n(z, z_1) - n(z, z_2)| \right] = 0, \end{aligned}$$

i.e., the functions $\alpha(x)$ and $n(x, z)$ are δ -Hölder continuous, then (1.7) holds with $h(r) = r^\theta$ for some $\theta \in (0, \delta \wedge (\alpha_0 - 1))$ small enough.

Quite a lot of progress has been made during the last decades on heat kernel estimates for non-local operators; see e.g. [14, 21–23, 25] and the references therein. According to the remark above and motivated by [25], one may wish to construct the fundamental solution for the operator L defined by (1.1) and to establish sharp two-sided estimates as well as its fractional derivative and gradient estimates, which immediately lead to some regularity properties of the associated semigroups. However, as mentioned above, since the operator L has variable order and is generally non-symmetric, it remains a big challenge to search for heat kernel estimates on it. To the authors' knowledge, there are very few results on this topic; see [33]. This

once again illustrates the power of the coupling method for non-local operators with variable order. To show clearly the practicality of Theorem 1.2, we present the following sufficient conditions for assumption (H4).

Proposition 1.8. *Assume that (H2) and (H3) hold. Then, we have the following two statements.*

- (1) *If $\alpha_0 := \inf_{x \in \mathbb{R}^d} \alpha(x) > 1$, then (1.4) is fulfilled.*
- (2) *If $\alpha_2 := \sup_{x \in \mathbb{R}^d} \alpha(x) < 1$ and*

$$(1.9) \quad \lim_{r \rightarrow 0} \sup_{|x-y| \leq r} \frac{|\alpha(x) - \alpha(y)| + \sup_{|z| \leq 1} |n(x, z) - n(y, z)|}{|x - y|^{1 - \alpha(x) \wedge \alpha(y)}} = 0,$$

then (1.4) is satisfied. In particular, if the two functions $\alpha(x)$ and $x \mapsto n(x, z)$ are Lipschitz continuous (uniformly in $z \in B(1) := \{y \in \mathbb{R}^d : |y| \leq 1\}$), then (1.9) holds true.

To prove Theorem 1.2, the main technique we adopt here is the coupling method as in [20, 26, 38] for diffusion processes. However, there are two fundamental differences for which we require new approaches and ideas.

- (1) The coupling method, in particular the construction of coupling processes in terms of the coupling operator, for diffusions is now well studied. Recently, there has been some progress on the coupling property of Lévy processes; see e.g. [12, 39, 45]. All these papers rely heavily on the particular characterization of Lévy processes. Since the corresponding integro-differential operator L given by (1.1) is of variable order, our techniques differ significantly from the papers cited above. In particular, our new coupling is constructed in the following way: by reflection for small jumps and by parallel displacement (called march coupling in [20]) for large jumps; see the next section for more details. Let us mention that, so far, our paper seems to be the first attempt to study regularity properties of the semigroups of purely non-local Lévy type operators by using the coupling method.
- (2) Another point for obtaining regularity of diffusion semigroups by using the coupling method is the choice of proper auxiliary functions, which are based on explicit expressions for solutions to some equations associated with second order elliptic differential operators; see e.g. [38, Case I in the proof of Theorem 3.4]. However, we are unable to use this approach here, since the operator L given by (1.1) is non-local. For this reason, we need a new choice of test function for the coupling operator associated with L , which is completely different from the diffusion case and depends heavily on the modulus of continuity of the associated semigroups. In addition, some more delicate calculations are needed to get the desired assertion. See Section 3.1 for details.

We also point out that Theorems 1.2 and 1.5 are not simple analogies with the diffusion case. The reason is as follows. The relation between the Laplacian (i.e., the generator of Brownian motion) and the diffusion operator is similar to that between the fractional Laplacian and the integro-differential operator L given by (1.1) with $\alpha(x) = \alpha$ for all $x \in \mathbb{R}^d$ and some constant $\alpha \in (0, 2)$; i.e., it has a fixed order. So, a novel point of our results and techniques lies in the fact that we can deal with non-local operators of variable order. On the other hand, even in the case that $\alpha(x) = \alpha$ for all $x \in \mathbb{R}^d$ and some constant $\alpha \in (0, 2)$, Theorems 1.2 and 1.5

are also a little different from the diffusion case, since by the non-local property one can only expect in some situations Hölder continuity of the associated semigroups for non-local operators, not Lipschitz continuity.

The remainder of this paper is arranged as follows. In the next section, we shall construct a new Markov coupling (i.e., the coupling by reflection) operator \tilde{L} of L and prove that there exists a coupling process associated with it. The last section is devoted to the proofs of Theorem 1.2 and other assertions in Section 1.

2. COUPLING BY REFLECTION FOR NON-LOCAL OPERATORS WITH VARIABLE ORDER

We begin with the construction of a Markov coupling operator for the generator L given by (1.1). First, for any x, y , and $z \in \mathbb{R}^d$, set

$$\varphi_{x,y}(z) := \begin{cases} z - \frac{2\langle x-y, z \rangle}{|x-y|^2}(x-y), & x \neq y; \\ -z, & x = y. \end{cases}$$

It is clear that $\varphi_{x,y} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ enjoys the following three properties:

- (A1) $\varphi_{x,y}(z) = \varphi_{y,x}(z)$ and $\varphi_{x,y}^2(z) = z$, i.e., $\varphi_{x,y}^{-1}(z) = \varphi_{x,y}(z)$;
- (A2) $|\varphi_{x,y}(z)| = |z|$;
- (A3) $(z - \varphi_{x,y}(z)) \parallel (x - y)$ and $(z + \varphi_{x,y}(z)) \perp (x - y)$.

Next, for any x, y , and $z \in \mathbb{R}^d$, let

$$\tilde{n}(x, y, z) := n(x, z) \wedge n(y, z) \wedge n(x, \varphi_{x,y}(z)) \wedge n(y, \varphi_{x,y}(z)),$$

and for any $f \in C_b^2(\mathbb{R}^{2d})$, let

$$\nabla_x f(x, y) := \left(\frac{\partial f(x, y)}{\partial x_i} \right)_{1 \leq i \leq d}, \quad \nabla_y f(x, y) := \left(\frac{\partial f(x, y)}{\partial y_i} \right)_{1 \leq i \leq d}.$$

Now, for any $f \in C_b^2(\mathbb{R}^{2d})$, we define

$$\begin{aligned} \tilde{L}_1 f(x, y) := & \frac{1}{2} \left[\int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x+z, y+\varphi_{x,y}(z)) - f(x, y) - \langle \nabla_x f(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right. \right. \\ & \left. \left. - \langle \nabla_y f(x, y), \varphi_{x,y}(z) \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \right. \\ & + \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x+\varphi_{x,y}(z), y+z) - f(x, y) - \langle \nabla_y f(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right. \\ & \left. \left. - \langle \nabla_x f(x, y), \varphi_{x,y}(z) \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \right] \\ & + \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x+z, y) - f(x, y) - \langle \nabla_x f(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ & \quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & + \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x, y+z) - f(x, y) - \langle \nabla_y f(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ & \quad \times \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \end{aligned}$$

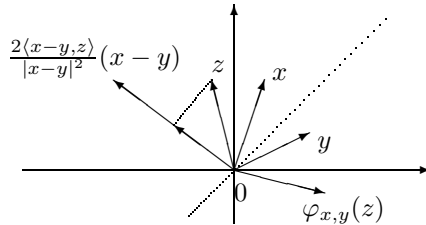
and

$$\begin{aligned} \tilde{L}_2 f(x, y) &:= \int_{\{|z| > \frac{|x-y|}{2}\}} \left(f(x+z, y+z) - f(x, y) - \langle \nabla_x f(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right. \\ &\quad \left. - \langle \nabla_y f(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &+ \int_{\{|z| > \frac{|x-y|}{2}\}} \left(f(x+z, y) - f(x, y) - \langle \nabla_x f(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &+ \int_{\{|z| > \frac{|x-y|}{2}\}} \left(f(x, y+z) - f(x, y) - \langle \nabla_y f(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz. \end{aligned}$$

Finally, for any $f \in C_b^2(\mathbb{R}^{2d})$, define

$$(2.10) \quad \tilde{L}f(x, y) := \tilde{L}_1 f(x, y) + \tilde{L}_2 f(x, y).$$

Remark 2.1. Here we give an intuitive interpretation for the construction of the coupling operator \tilde{L} . The operator \tilde{L} consists of two parts, \tilde{L}_1 and \tilde{L}_2 , which deal with the jumps smaller than $|x - y|/2$ and those bigger than $|x - y|/2$ respectively. The term \tilde{L}_1 is just the coupling by reflection, which is the main part of our constructed coupling operator. Its name on the one hand comes from the mappings $\varphi_{x,y}$ for any $x \neq y$, which represent the reflection with respect to the hyperplane perpendicular to $x - y$. Here is a figure which illustrates the reflection mapping $\varphi_{x,y}$:



On the other hand, according to the figure above and the proofs of Theorems 1.2 and 1.5 below, one can see that the coupling by reflection operator \tilde{L}_1 is analogous with that for Brownian motion or the diffusion process; see e.g. [20, 36]. We should mention that it seems to be the first time the coupling by reflection for non-local operators has been presented. The second part, \tilde{L}_2 , stands for the coupling by parallel displacement, where the components maintain at each step the same length of jumps (that is, from (x, y) to $(x+z, y+z)$) at the common rate $\frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}}$.

First we prove the following lemma.

Lemma 2.2. *The operator \tilde{L} defined by (2.10) is the coupling operator of the operator L given by (1.1); i.e., for any $g, h \in C_b^2(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,*

$$\tilde{L}H(x, y) = Lg(x) + Lh(y),$$

where $H(x, y) = g(x) + h(y)$.

Proof. Since \tilde{L} is a linear operator, it suffices to verify that

$$\tilde{L}f(x) = Lf(x), \quad f \in C_b^2(\mathbb{R}^d),$$

where, on the left hand side, f is regarded as a bivariate function on \mathbb{R}^{2d} .

First we have

$$\begin{aligned} \tilde{L}_1 f(x) &= \frac{1}{2} \left[\int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \right. \\ &\quad \times \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &\quad + \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x + \varphi_{x,y}(z)) - f(x) - \langle \nabla f(x), \varphi_{x,y}(z) \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \left. \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \right] \\ &\quad + \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz. \end{aligned}$$

By (A1) and the definition of $\tilde{n}(x, y, z)$, we know that the kernel $\frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz$ is invariant under the transformation $z \mapsto \varphi_{x,y}(z)$. This, along with (A2) and the equality above, leads to

$$\begin{aligned} \tilde{L}_1 f(x) &= \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &\quad + \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &= \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \frac{n(x, z)}{|z|^{d+\alpha(x)}} dz. \end{aligned}$$

Next

$$\begin{aligned} \tilde{L}_2 f(x) &= \int_{\{|z| > \frac{|x-y|}{2}\}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &+ \int_{\{|z| > \frac{|x-y|}{2}\}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &= \int_{\{|z| > \frac{|x-y|}{2}\}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \frac{n(x, z)}{|z|^{d+\alpha(x)}} dz. \end{aligned}$$

Combining the above two equalities finishes the proof. □

In the remainder of this section, we will construct a coupling process associated with the coupling operator \tilde{L} defined by (2.10). For any $x, y \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^{2d})$, set

$$\begin{aligned} \mu(x, y, A) &:= \frac{1}{2} \int_{\{(z, \varphi_{x,y}(z)) \in A, |z| \leq \frac{|x-y|}{2}\}} \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &+ \frac{1}{2} \int_{\{(\varphi_{x,y}(z), z) \in A, |z| \leq \frac{|x-y|}{2}\}} \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &+ \int_{\{(z, 0) \in A, |z| \leq \frac{|x-y|}{2}\}} \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &+ \int_{\{(0, z) \in A, |z| \leq \frac{|x-y|}{2}\}} \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &+ \int_{\{(z, z) \in A, |z| > \frac{|x-y|}{2}\}} \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &+ \int_{\{(z, 0) \in A, |z| > \frac{|x-y|}{2}\}} \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &+ \int_{\{(0, z) \in A, |z| > \frac{|x-y|}{2}\}} \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz. \end{aligned}$$

Then, by (2.10), for any $x, y \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^{2d})$, we have

$$\begin{aligned} \tilde{L}f(x, y) &= \int_{\mathbb{R}^{2d}} \left[f((x, y) + (u_1, u_2)) - f(x, y) \right. \\ &\quad \left. - \langle (\nabla_x f(x, y), \nabla_y f(x, y)), (u_1, u_2) \rangle \mathbf{1}_{\{|u_1| \leq 1, |u_2| \leq 1\}} \right] \mu(x, y, du_1, du_2). \end{aligned}$$

For any $h \in C_b(\mathbb{R}^{2d})$, by (A2),

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} h(u) \frac{|u|^2}{1+|u|^2} \mu(x, y, du) \\ &= \int_{\{|z| \leq \frac{|x-y|}{2}\}} h(z, \varphi_{x,y}(z)) \frac{|z|^2}{1+2|z|^2} \cdot \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &+ \int_{\{|z| \leq \frac{|x-y|}{2}\}} h(\varphi_{x,y}(z), z) \frac{|z|^2}{1+2|z|^2} \cdot \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &+ \int_{\{|z| \leq \frac{|x-y|}{2}\}} h(z, 0) \frac{|z|^2}{1+|z|^2} \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &+ \int_{\{|z| \leq \frac{|x-y|}{2}\}} h(0, z) \frac{|z|^2}{1+|z|^2} \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &+ 2 \int_{\{|z| > \frac{|x-y|}{2}\}} h(z, z) \frac{|z|^2}{1+2|z|^2} \cdot \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} dz \\ &+ \int_{\{|z| > \frac{|x-y|}{2}\}} h(z, 0) \frac{|z|^2}{1+|z|^2} \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &+ \int_{\{|z| > \frac{|x-y|}{2}\}} h(0, z) \frac{|z|^2}{1+|z|^2} \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz, \end{aligned}$$

which, along with the facts that $\alpha(x)$ and $n(x, z)$ are continuous and strictly positive, implies that $(x, y) \mapsto \int h(u) \frac{|u|^2}{1+|u|^2} \mu(x, y, du)$ is a continuous function on \mathbb{R}^{2d} .

According to [42, Theorem 2.2], there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$ and an \mathbb{R}^{2d} -valued process $(\tilde{X}_t)_{t \geq 0}$ such that $(\tilde{X}_t)_{t \geq 0}$ is $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ -progressively measurable, and for every $f \in C_b^2(\mathbb{R}^{2d})$,

$$\left\{ f(\tilde{X}_t) - \int_0^{t \wedge e} \tilde{L}f(\tilde{X}_u) du, t \geq 0 \right\}$$

is an $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ -local martingale, where e is the explosion time of $(\tilde{X}_t)_{t \geq 0}$, i.e.,

$$e = \lim_{n \rightarrow \infty} \inf \left\{ t \geq 0 : |X'_t| + |X''_t| \geq n \right\}.$$

Here, $(\tilde{X}_t)_{t \geq 0} := (X'_t, X''_t)_{t \geq 0}$, and $(X'_t)_{t \geq 0}$ and $(X''_t)_{t \geq 0}$ are two stochastic processes on \mathbb{R}^d . Since \tilde{L} is the coupling operator of L , the generator of both marginal processes $(X'_t)_{t \geq 0}$ and $(X''_t)_{t \geq 0}$ is just the operator L , and hence both are solutions to the martingale problem of L . In particular, by (H1), the processes $(X'_t)_{t \geq 0}$ and $(X''_t)_{t \geq 0}$ are non-explosive; hence one has $e = \infty$ a.s. Therefore, the coupling operator \tilde{L} generates a non-explosive process $(\tilde{X}_t)_{t \geq 0}$.

Let T be the coupling time of $(X'_t)_{t \geq 0}$ and $(X''_t)_{t \geq 0}$, i.e.,

$$T = \inf \{ t \geq 0 : X'_t = X''_t \}.$$

Then T is an $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ -stopping time. Define a new process $(Y'_t)_{t \geq 0}$ as follows:

$$Y'_t = \begin{cases} X''_t, & t < T; \\ X'_t, & t \geq T. \end{cases}$$

For any $f \in C_b^2(\mathbb{R}^d)$ and $t > 0$,

$$\begin{aligned} f(Y'_t) - \int_0^t Lf(Y'_s) ds &= f(Y'_{t \wedge T}) - \int_0^{t \wedge T} Lf(Y'_s) ds \\ &\quad + f(Y'_t) - f(Y'_{t \wedge T}) - \int_{t \wedge T}^t Lf(Y'_s) ds \\ &= f(X''_{t \wedge T}) - \int_0^{t \wedge T} Lf(X''_s) ds \\ &\quad + f(X'_t) - f(X'_{t \wedge T}) - \int_{t \wedge T}^t Lf(X'_s) ds \\ &=: M_t^1 + M_t^2. \end{aligned}$$

By the optimal stopping theorem and the facts that both $(X'_t)_{t \geq 0}$ and $(X''_t)_{t \geq 0}$ are solutions to the martingale problem of L and $(M'_t)_{t \geq 0}$ and $(M''_t)_{t \geq 0}$ are martingales, then so is $(Y'_t)_{t \geq 0}$ (see e.g. [38, Section 3.1, p. 251]). Since the martingale problem for the operator L is well-posed, $(Y'_t)_{t \geq 0}$ and $(X''_t)_{t \geq 0}$ are equal in the distribution. Therefore, we conclude that $(X'_t, Y'_t)_{t \geq 0}$ is also a non-explosive coupling process of the process generated by the operator L such that $X'_t = Y'_t$ for any $t \geq T$ and the generator of $(X'_t, Y'_t)_{t \geq 0}$ before the coupling time T is just the coupling operator \tilde{L} given by (2.10). On the other hand, according to (H1) and [20, Lemma 2.1], we know that for any $x, y \in \mathbb{R}^d$ and $f \in B_b(\mathbb{R}^d)$,

$$P_t f(x) = \mathbb{E}^x f(X'_t) = \tilde{\mathbb{E}}^{(x,y)} f(X'_t)$$

and

$$P_t f(y) = \mathbb{E}^y f(X''_t) = \tilde{\mathbb{E}}^{(x,y)} f(Y'_t).$$

3. PROOFS

This section consists of two parts. We present in the first subsection the proofs of Theorem 1.2 and Proposition 1.8, and then we prove Theorem 1.5 in subsection 3.2.

3.1. Proof of Theorem 1.2. To prove Theorem 1.2, we fix $\beta \in (0, \alpha_0 \wedge 1)$. For any $n \geq 1$, define an increasing function $f_n \in C_b^2([0, \infty))$ such that $f_n(r) \leq r^\beta$ for all $r \geq 0$ and

$$f_n(r) = \begin{cases} r^2, & 0 \leq r \leq 1/(n+1); \\ r^\beta, & 1/n \leq r \leq 1. \end{cases}$$

The following estimate is crucial for the proof of Theorem 1.2.

Proposition 3.1. *There exist three positive constants C_1, C_2 , and C_3 such that for all $n \geq 1$ and for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,*

$$\begin{aligned} (3.11) \quad &\tilde{L}f_n(|x - y|) \\ &\leq -C_1|x - y|^{\beta - \alpha(x) \wedge \alpha(y)} + C_2 \left[1 + |x - y|^{\beta - \alpha(x) \wedge \alpha(y)} A(x, y) \right] \\ &\quad + C_3|x - y|^{\beta - 2} \int_{\left\{ \frac{|x-y|}{2} < |z| < 1 \right\}} \langle x - y, z \rangle \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z)}{|z|^{d+\alpha(x)}} \right) dz, \end{aligned}$$

where

$$A(x, y) := |\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} + \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x - y|} |n(z, z_1) - n(z, z_2)| \right).$$

Proof. By the definition (2.10) of the coupling operator \tilde{L} , we shall estimate the two terms $\tilde{L}_1 f_n(|x - y|)$ and $\tilde{L}_2 f_n(|x - y|)$ separately.

(1) First, for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$, by (A3),

$$(3.12) \quad \langle \nabla_x f_n(|x - y|), z + \varphi_{x,y}(z) \rangle = 0 \quad \text{and} \quad \langle \nabla_y f_n(|x - y|), z + \varphi_{x,y}(z) \rangle = 0.$$

We have for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} & \tilde{L}_1 f_n(|x - y|) \\ & \leq \frac{1}{2} \left[\int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(\left| (x - y) + \frac{2\langle x - y, z \rangle}{|x - y|^2} (x - y) \right|^\beta \right. \right. \\ & \quad \left. \left. + \left| (x - y) - \frac{2\langle x - y, z \rangle}{|x - y|^2} (x - y) \right|^\beta - 2|x - y|^\beta \right) \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \right] \\ & + \left[\int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(|x - y + z|^\beta - |x - y|^\beta - \beta|x - y|^{\beta-2} \langle x - y, z \rangle \right) \right. \\ & \quad \left. \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \right. \\ & + \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(|x - y - z|^\beta - |x - y|^\beta + \beta|x - y|^{\beta-2} \langle x - y, z \rangle \right) \\ & \quad \left. \times \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \right] \\ & =: \tilde{L}_{1,1} f_n(|x - y|) + \tilde{L}_{1,2} f_n(|x - y|). \end{aligned}$$

On the one hand, using the facts that

$$(1 + s)^\beta + (1 - s)^\beta \leq 2 + \beta(\beta - 1)s^2, \quad 0 \leq s < 1, \quad \beta \in (0, 1),$$

and $n(x, z) \geq c_1$ for all $x, z \in \mathbb{R}^d$, we deduce that there exists a constant $c_{1,1} > 0$ such that for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} \tilde{L}_{1,1} f_n(|x - y|) & \leq 2\beta(\beta - 1)|x - y|^{\beta-2} \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{|\langle x - y, z \rangle|^2}{|x - y|^2} \cdot \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\ & \leq 2c_1\beta(\beta - 1)|x - y|^{\beta-2} \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{|\langle x - y, z \rangle|^2}{|x - y|^2} \cdot \frac{1}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\ & = 2c_1\beta(\beta - 1)|x - y|^{\beta-2} \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{|z_1|^2}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\ & \leq -c_{1,1}|x - y|^{\beta-\alpha(x) \wedge \alpha(y)}, \end{aligned}$$

where in the equality above we have used the rotational invariance property of the kernel $\frac{1}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz$; and z_1 is the first coordinate of z , i.e., $z = (z_1, z_2, \dots, z_d)$; and in the last inequality we have used that $\alpha(x) \wedge \alpha(y) \geq \alpha_0 > 0$ for all $x, y \in \mathbb{R}^d$.

On the other hand, applying the inequality

$$b^\beta - a^\beta \leq \beta a^{\beta-1}(b - a), \quad a, b > 0, \beta \in (0, 1),$$

we know that for all x, y , and $z \in \mathbb{R}^d$,

$$|x - y + z|^\beta - |x - y|^\beta \leq \beta |x - y|^{\beta-1} (|x - y + z| - |x - y|).$$

Moreover, for all x, y , and $z \in \mathbb{R}^d$,

$$\begin{aligned} & |x - y| \cdot |x - y + z| - |x - y|^2 - \langle x - y, z \rangle \\ (3.13) \quad &= \frac{1}{2} \left(2|x - y| \cdot |x - y + z| - 2|x - y|^2 - |x - y + z|^2 + |z|^2 + |x - y|^2 \right) \\ &= \frac{1}{2} \left(|z|^2 - (|x - y + z| - |x - y|)^2 \right) \leq \frac{|z|^2}{2}. \end{aligned}$$

Thus for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$, we arrive at

$$\begin{aligned} \tilde{L}_{1,2} f_n(|x - y|) &\leq \frac{\beta}{2} |x - y|^{\beta-2} \int_{\{|z| \leq \frac{|x-y|}{2}\}} |z|^2 \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \\ &\quad + \frac{\beta}{2} |x - y|^{\beta-2} \int_{\{|z| \leq \frac{|x-y|}{2}\}} |z|^2 \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz. \end{aligned}$$

Furthermore, for all x, y , and $z \in \mathbb{R}^d$,

$$\begin{aligned} (3.14) \quad & \frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \\ &= \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) + \left(\frac{n(x, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right). \end{aligned}$$

For the first term, we note that $n(x, z) \leq c_2$ for all $x, z \in \mathbb{R}^d$, and by the mean value theorem, for all $z \in \mathbb{R}^d$ with $|z| \leq 1$,

$$\begin{aligned} (3.15) \quad & \frac{1}{|z|^{d+\alpha(x)}} - \frac{1}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \\ &= \frac{1}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \left[\left(\frac{1}{|z|} \right)^{\alpha(x) - \alpha(x) \wedge \alpha(y)} - 1 \right] \\ &\leq \frac{1}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \left(\frac{1}{|z|} \right)^{\alpha(x) - \alpha(x) \wedge \alpha(y)} \left(\log \frac{1}{|z|} \right) [\alpha(x) - \alpha(x) \wedge \alpha(y)] \\ &\leq \frac{1}{|z|^{d+\alpha(x)}} \left(\log \frac{1}{|z|} \right) |\alpha(x) - \alpha(y)|. \end{aligned}$$

The estimate of the second term follows from the definition of the function $\tilde{n}(x, y, z)$:

$$\begin{aligned} (3.16) \quad & \frac{n(x, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \\ &\leq \frac{1}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \left[\sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right. \\ &\quad \left. + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x - y|} |n(z, z_1) - n(z, z_2)| \right]. \end{aligned}$$

Hence, we get that there exists a constant $c_{1,2} > 0$ such that for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} & \tilde{L}_{1,2} f_n(|x - y|) \\ & \leq \frac{c_2 \beta}{2} |x - y|^{\beta-2} |\alpha(x) - \alpha(y)| \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{|z|^2}{|z|^{d+\alpha(x)}} \left(\log \frac{1}{|z|} \right) dz \\ & \quad + \frac{c_2 \beta}{2} |x - y|^{\beta-2} |\alpha(x) - \alpha(y)| \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{|z|^2}{|z|^{d+\alpha(y)}} \left(\log \frac{1}{|z|} \right) dz \\ & \quad + \beta |x - y|^{\beta-2} \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x-y|} |n(z, z_1) - n(z, z_2)| \right) \\ & \quad \quad \times \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{|z|^2}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\ & \leq c_{1,2} |x - y|^{\beta-\alpha(x) \wedge \alpha(y)} \left[|\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} \right. \\ & \quad \left. + \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x-y|} |n(z, z_1) - n(z, z_2)| \right) \right], \end{aligned}$$

where in the last inequality we have used the fact that $\alpha(x) \vee \alpha(y) \leq \alpha_2$ for all $x, y \in \mathbb{R}^d$.

(2) Secondly, for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} & \tilde{L}_2 f_n(|x - y|) \\ & \leq \int_{\{|z| > \frac{|x-y|}{2}\}} \left(|x - y + z|^\beta - |x - y|^\beta - \beta |x - y|^{\beta-2} \langle x - y, z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ & \quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & \quad + \int_{\{|z| > \frac{|x-y|}{2}\}} \left(|x - y - z|^\beta - |x - y|^\beta + \beta |x - y|^{\beta-2} \langle x - y, z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \\ & \quad \times \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & \leq \int_{\{|z| > \frac{|x-y|}{2}\}} |z|^\beta \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & \quad + \int_{\{|z| > \frac{|x-y|}{2}\}} |z|^\beta \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & \quad - \beta |x - y|^{\beta-2} \int_{\{\frac{|x-y|}{2} < |z| \leq 1\}} \langle x - y, z \rangle \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(y, z)}{|z|^{d+\alpha(y)}} \right) dz, \end{aligned}$$

where in the second inequality we have used the inequality that

$$(a + b)^\beta \leq a^\beta + b^\beta, \quad a, b > 0, \beta \in (0, 1).$$

On the one hand, it is easy to see that

$$\begin{aligned} & \int_{\{|z| \geq 1\}} |z|^\beta \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & \quad + \int_{\{|z| \geq 1\}} |z|^\beta \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & \leq c_2 \int_{\{|z| \geq 1\}} \left(\frac{1}{|z|^{d+\alpha(x)-\beta}} + \frac{1}{|z|^{d+\alpha(y)-\beta}} \right) dz \\ & \leq 2c_2 \int_{\{|z| \geq 1\}} \frac{1}{|z|^{d+\alpha_0-\beta}} dz =: c_{2,1} < \infty. \end{aligned}$$

On the other hand, following the argument for the estimate of $\tilde{L}_{1,2}f_n(|x - y|)$, we can find a constant $c_{2,2} > 0$ such that for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} & \int_{\{\frac{|x-y|}{2} < |z| \leq 1\}} |z|^\beta \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & \quad + \int_{\{\frac{|x-y|}{2} < |z| \leq 1\}} |z|^\beta \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ & \leq c_{2,2} |x - y|^{\beta-\alpha(x) \wedge \alpha(y)} \left[|\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} \right. \\ & \quad \left. + \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right) \right]. \end{aligned}$$

Thus, for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} & \tilde{L}_2 f_n(|x - y|) \\ & \leq c_{2,1} + c_{2,2} |x - y|^{\beta-\alpha(x) \wedge \alpha(y)} \left[|\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} \right. \\ & \quad \left. + \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right) \right] \\ & \quad - \beta |x - y|^{\beta-2} \int_{\{\frac{|x-y|}{2} < |z| \leq 1\}} \langle x - y, z \rangle \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(y, z)}{|z|^{d+\alpha(y)}} \right) dz. \end{aligned}$$

(3) The required assertion (3.11) immediately follows from the above two estimates on $\tilde{L}_1 f_n(|x - y|)$ and $\tilde{L}_2 f_n(|x - y|)$. □

Corollary 3.2. *Under assumptions (H2), (H3), and (H4), there exist two constants $\varepsilon_0 := \varepsilon_0(\beta, \alpha(x), n(x, z)) \in (0, 1)$ and $C_0 := C_0(\beta, \alpha(x), n(x, z)) > 0$, such that for all $n \geq 1$, $\varepsilon \in (0, \varepsilon_0]$, and any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| < \varepsilon$,*

$$(3.17) \quad \tilde{L}f_n(|x - y|) \leq -C_0 \varepsilon^{\beta-\alpha_0} =: -A_{\beta,\alpha} < 0.$$

Proof. By (1.2),

$$(3.18) \quad \lim_{|x-y| \rightarrow 0} |\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} = 0.$$

Due to (1.3),

$$\lim_{|x-y| \rightarrow 0} \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x - y|} |n(z, z_1) - n(z, z_2)| \right) = 0.$$

Noticing that for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$,

$$|x - y|^{\beta - \alpha(x) \wedge \alpha(y)} \geq |x - y|^{\beta - \alpha_0},$$

we obtain from (1.3) and (3.11) that there is a constant $\varepsilon_0 := \varepsilon_0(\beta, \alpha(x), n(x, z)) \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq \varepsilon$,

$$\tilde{L}f_n(|x - y|) \leq -\frac{C_1}{2}|x - y|^{\beta - \alpha(x) \wedge \alpha(y)} \leq -\frac{C_1\varepsilon^{\beta - \alpha_0}}{2}.$$

This proves the desired assertion. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We will use the coupling process $(X'_t, Y'_t)_{t \geq 0}$ constructed in Section 2. Denote by $\tilde{\mathbb{P}}^{(x,y)}$ and $\tilde{\mathbb{E}}^{(x,y)}$ the distribution and the expectation of $(X'_t, Y'_t)_{t \geq 0}$ starting from (x, y) , respectively. Recall that $\varepsilon_0 > 0$ is given in Corollary 3.2. For any $n \geq 1$ and $\varepsilon \in (0, \varepsilon_0]$, we set

$$\begin{aligned} S_\varepsilon &:= \inf\{t \geq 0 : |X'_t - Y'_t| > \varepsilon\}, \\ T_n &:= \inf\{t \geq 0 : |X'_t - Y'_t| \leq 1/n\}, \\ T_{n,\varepsilon} &:= T_n \wedge S_\varepsilon. \end{aligned}$$

Furthermore, we will still use the coupling time defined by

$$T := \inf\{t \geq 0 : X'_t = Y'_t\}.$$

Note that $T_n \uparrow T$ as $n \uparrow \infty$.

For any $x, y \in \mathbb{R}^d$ with $0 < |x - y| < \varepsilon \leq \varepsilon_0$, we take n large enough such that $|x - y| > 1/n$. Then, by (3.17),

$$\begin{aligned} 0 &\leq \tilde{\mathbb{E}}^{(x,y)} f_n(|X'_{t \wedge T_{n,\varepsilon}} - Y'_{t \wedge T_{n,\varepsilon}}|) \\ &= f_n(|x - y|) + \tilde{\mathbb{E}}^{(x,y)} \left(\int_0^{t \wedge T_{n,\varepsilon}} \tilde{L}f_n(|X'_s - Y'_s|) ds \right) \\ &\leq f_n(|x - y|) - A_{\beta,\alpha} \tilde{\mathbb{E}}^{(x,y)}(t \wedge T_{n,\varepsilon}). \end{aligned}$$

Therefore

$$\tilde{\mathbb{E}}^{(x,y)}(t \wedge T_{n,\varepsilon}) \leq \frac{f_n(|x - y|)}{A_{\beta,\alpha}} = \frac{|x - y|^\beta}{A_{\beta,\alpha}}.$$

Letting $t \rightarrow \infty$ and then $n \rightarrow \infty$, we arrive at

$$\tilde{\mathbb{E}}^{(x,y)}(T \wedge S_\varepsilon) \leq \frac{|x - y|^\beta}{A_{\beta,\alpha}}.$$

On the other hand, again by (3.17), for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| < \varepsilon \leq \varepsilon_0$,

$$\begin{aligned} &\tilde{\mathbb{E}}^{(x,y)} f_n(|X'_{t \wedge T_{n,\varepsilon}} - Y'_{t \wedge T_{n,\varepsilon}}|) \\ &= f_n(|x - y|) + \tilde{\mathbb{E}}^{(x,y)} \left(\int_0^{t \wedge T_{n,\varepsilon}} \tilde{L}f_n(|X'_u - Y'_u|) du \right) \\ &\leq f_n(|x - y|). \end{aligned}$$

This yields that

$$f_n(\varepsilon) \tilde{\mathbb{P}}^{(x,y)}(S_\varepsilon < T_n \wedge t) \leq f_n(|x - y|).$$

Letting $t \rightarrow \infty$ and then $n \rightarrow \infty$ leads to

$$\tilde{\mathbb{P}}^{(x,y)}(T > S_\varepsilon) \leq \frac{|x - y|^\beta}{\varepsilon^\beta}.$$

Therefore, for any $x, y \in \mathbb{R}^d$ with $0 < |x - y| < \varepsilon \leq \varepsilon_0$,

$$\begin{aligned} \tilde{\mathbb{P}}^{(x,y)}(T \geq t) &\leq \tilde{\mathbb{P}}^{(x,y)}(T \wedge S_\varepsilon > t) + \tilde{\mathbb{P}}^{(x,y)}(T > S_\varepsilon) \\ &\leq \frac{\tilde{\mathbb{E}}^{(x,y)}(T \wedge S_\varepsilon)}{t} + \frac{|x - y|^\beta}{\varepsilon^\beta} \\ &\leq \frac{|x - y|^\beta}{tA_{\beta,\alpha}} + \frac{|x - y|^\beta}{\varepsilon^\beta} \\ &= |x - y|^\beta \left[\frac{1}{\varepsilon^\beta} + \frac{1}{tA_{\beta,\alpha}} \right]. \end{aligned}$$

Hence, for any $f \in B_b(\mathbb{R}^d)$, $t > 0$, and any $x, y \in \mathbb{R}^d$ with $0 < |x - y| < \varepsilon \leq \varepsilon_0$,

$$\begin{aligned} |P_t f(x) - P_t f(y)| &= |\mathbb{E}^x f(X'_t) - \mathbb{E}^y f(Y'_t)| \\ &= |\tilde{\mathbb{E}}^{(x,y)}(f(X'_t) - f(Y'_t))| \\ &= |\tilde{\mathbb{E}}^{(x,y)}(f(X'_t) - f(Y'_t))\mathbf{1}_{\{T \geq t\}}| \\ &\leq 2\|f\|_\infty \tilde{\mathbb{P}}^{(x,y)}(T \geq t) \\ &\leq 2\|f\|_\infty |x - y|^\beta \left[\frac{1}{\varepsilon^\beta} + \frac{1}{tA_{\beta,\alpha}} \right], \end{aligned}$$

which immediately yields that

$$\sup_{|x-y| \leq \varepsilon} \frac{|P_t f(x) - P_t f(y)|}{|x - y|^\beta} \leq 2\|f\|_\infty \left[\frac{1}{\varepsilon^\beta} + \frac{1}{tA_{\beta,\alpha}} \right].$$

This along with the fact that

$$\sup_{|x-y| \geq \varepsilon} \frac{|P_t f(x) - P_t f(y)|}{|x - y|^\beta} \leq 2\|f\|_\infty \varepsilon^{-\beta}$$

further gives us that for all $\varepsilon \in (0, \varepsilon_0]$,

$$\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{|x - y|^\beta} \leq 2\|f\|_\infty \left[\frac{1}{\varepsilon^\beta} + \frac{1}{tA_{\beta,\alpha}} \right] = 2\|f\|_\infty \left[\frac{1}{\varepsilon^\beta} + \frac{\varepsilon^{\alpha_0 - \beta}}{C_0 t} \right],$$

where $C_0 := C_0(\beta, \alpha(x), n(x, z))$ is a positive constant independent of ε ; e.g. see (3.17). In particular, we have

$$(3.19) \quad \sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{|x - y|^\beta} \leq 2\|f\|_\infty \inf_{\varepsilon \in (0, \varepsilon_0]} \left[\frac{1}{\varepsilon^\beta} + \frac{\varepsilon^{\alpha_0 - \beta}}{C_0 t} \right].$$

Therefore, we complete the proof by taking $\varepsilon = t^{1/\alpha_0} \wedge \varepsilon_0$ in (3.19). □

At the end of this subsection, we would like to present the following proof.

Proof of Proposition 1.8. (1) For all $z \in \mathbb{R}^d$ with $|z| \leq 1$,

$$\begin{aligned}
 & \left| \int_{\left\{ \frac{|x-y|}{2} < |z| \leq 1 \right\}} \langle x-y, z \rangle \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z)}{|z|^{d+\alpha(x)}} \right) dz \right| \\
 & \leq |x-y| \int_{\left\{ \frac{|x-y|}{2} < |z| \leq 1 \right\}} |z| \left| \frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z)}{|z|^{d+\alpha(x)}} \right| dz \\
 (3.20) \quad & \leq |x-y| \int_{\left\{ \frac{|x-y|}{2} < |z| \leq 1 \right\}} |z| \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \\
 & \quad + |x-y| \int_{\left\{ \frac{|x-y|}{2} < |z| \leq 1 \right\}} |z| \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \\
 & \quad + |x-y| \int_{\left\{ \frac{|x-y|}{2} < |z| < 1 \right\}} |z| \frac{|n(x, z) - n(y, z)|}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz.
 \end{aligned}$$

On the one hand, there exists a constant $c_{1,1} > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
 & \int_{\left\{ \frac{|x-y|}{2} < |z| \leq 1 \right\}} |z| \frac{|n(x, z) - n(y, z)|}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\
 & \leq \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right) \int_{\left\{ \frac{|x-y|}{2} < |z| \leq 1 \right\}} \frac{|z|}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\
 & \leq c_{1,1} |x-y|^{1-\alpha(x) \wedge \alpha(y)} \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right),
 \end{aligned}$$

where in the last inequality we have used the fact that $\inf_{x \in \mathbb{R}^d} \alpha(x) > 1$. On the other hand, by (3.15), there exists a constant $c_{1,2} > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
 & \int_{\left\{ \frac{|x-y|}{2} < |z| \leq 1 \right\}} |z| \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \\
 & \leq c_2 |\alpha(x) - \alpha(y)| \int_{\left\{ \frac{|x-y|}{2} < |z| \leq 1 \right\}} \frac{|z|}{|z|^{d+\alpha(x)}} \left(\log \frac{1}{|z|} \right) dz \\
 & \leq c_{1,2} |\alpha(x) - \alpha(y)| |x-y|^{1-\alpha(x)} \log \frac{1}{|x-y|} \\
 & \leq c_{1,2} |\alpha(x) - \alpha(y)| |x-y|^{1-\alpha(x) \wedge \alpha(y)} |x-y|^{-|\alpha(x) - \alpha(y)|} \log \frac{1}{|x-y|},
 \end{aligned}$$

where in the second inequality we have used again that $\inf_{x \in \mathbb{R}^d} \alpha(x) > 1$. Similarly, we have that

$$\begin{aligned}
 & \int_{\left\{ \frac{|x-y|}{2} < |z| < 1 \right\}} |z| \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \\
 & \leq c_{1,3} |\alpha(x) - \alpha(y)| |x-y|^{-\alpha(x) \wedge \alpha(y) + 1} |x-y|^{-|\alpha(x) - \alpha(y)|} \log \frac{1}{|x-y|}
 \end{aligned}$$

holds for some constant $c_{1,3} > 0$ and all $x, y \in \mathbb{R}^d$.

Combining all the conclusions above with (1.3) and (3.18), we get (1.4).

(2) We mainly follow the arguments of part (1), and here we only sketch the proof.

Since $\sup_{x \in \mathbb{R}^d} \alpha(x) < 1$, there exist constants $c_{2,1}, c_{2,2} > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\int_{\left\{\frac{|x-y|}{2} < |z| \leq 1\right\}} |z| \frac{|n(x, z) - n(y, z)|}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \leq c_{2,1} \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right)$$

and

$$\int_{\left\{\frac{|x-y|}{2} < |z| \leq 1\right\}} |z| \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \leq c_{2,2} |\alpha(x) - \alpha(y)|.$$

This, along with (3.20) and (1.9), gives us the required assertion. □

3.2. Proof of Theorem 1.5. In this part, suppose that $\alpha_0 = \inf_{x \in \mathbb{R}^d} \alpha(x) > 1$. We first give an elementary lemma.

Lemma 3.3. *Let $\varphi \in \mathcal{D}$ and $f(r) = \int_0^r \varphi(s) ds$ for all $r \in (0, 2]$. Then, for any $0 \leq \delta \leq a \leq 1$,*

$$f(a + \delta) + f(a - \delta) - 2f(a) \leq \varphi'(2a)\delta^2.$$

Proof. For any $0 \leq \delta \leq a$,

$$\begin{aligned} f(a + \delta) + f(a - \delta) - 2f(a) &= \int_a^{a+\delta} ds \int_{s-\delta}^s \varphi'(r) dr \\ &\leq \varphi'(a + \delta)\delta^2 \leq \varphi'(2a)\delta^2, \end{aligned}$$

where in the two inequalities we have used the fact that $\varphi'' > 0$. □

Let $\varphi \in \mathcal{D}$ and f be the function defined in Lemma 3.3. For any $n \geq 1$, define an increasing function $f_n \in C_b^2([0, \infty))$ such that $f_n(r) \leq f(r)$ for all $0 < r \leq 2$ and

$$f_n(r) = \begin{cases} f(r), & 1/n \leq r \leq 2; \\ f(2) + 1, & r \geq 3. \end{cases}$$

The next result is analogous to Proposition 3.1.

Proposition 3.4. *Assume that assumptions (H2) and (H3) hold. There exist three positive constants C_1, C_2 , and C_3 such that for all $n \geq 1$ and for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,*

$$\begin{aligned} &\tilde{L}f_n(|x - y|) \\ &\leq C_1 \varphi'(2|x - y|) |x - y|^{2-\alpha(x) \wedge \alpha(y)} + C_2 \\ &\quad + C_3 \varphi(|x - y|) |x - y|^{1-\alpha(x) \wedge \alpha(y)} \left[|\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} \right. \\ &\quad \left. + \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x - y|} |n(z, z_1) - n(z, z_2)| \right) \right]. \end{aligned}$$

Proof. We follow the line of arguments for proving Proposition 3.1.

(1) First, for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$, by (3.12) and the facts that $f_n(r) \leq f(r)$ for all $r \in (0, 2]$ and $f_n(r) = f(r)$ for all $r \in [1/n, 2]$,

$$\begin{aligned} & \tilde{L}_1 f_n(|x - y|) \\ & \leq \frac{1}{2} \left[\int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(|x - y + (z - \varphi_{x,y}(z))|) + f(|x - y - (z - \varphi_{x,y}(z))|) \right. \right. \\ & \quad \left. \left. - 2f(|x - y|) \right) \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \right] \\ & \quad + \left[\int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(|x - y + z|) - f(|x - y|) - f'(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \right) \right. \\ & \quad \quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \\ & \quad \quad + \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left(f(|x - y - z|) - f(|x - y|) + f'(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \right) \\ & \quad \quad \times \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \left. \right] \\ & =: \tilde{L}_{1,1} f(|x - y|) + \tilde{L}_{1,2} f(|x - y|). \end{aligned}$$

According to the definition of $\varphi_{x,y}(z)$ and Lemma 3.3, for all $x, y, z \in \mathbb{R}^d$ with $|x - y| \leq 1$ and $|z| \leq |x - y|/2$, we have

$$\begin{aligned} & f(|x - y + (z - \varphi_{x,y}(z))|) + f(|x - y - (z - \varphi_{x,y}(z))|) - 2f(|x - y|) \\ & \leq 4\varphi'(2|x - y|) \frac{\langle x - y, z \rangle^2}{|x - y|^2}. \end{aligned}$$

As a result, since $\varphi'(r) < 0$ for $r \in (0, 2]$ and $n(x, z) \geq c_1$ for all $x, z \in \mathbb{R}^d$, there exists a constant $c_{1,1} > 0$ such that for all $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} (3.21) \quad \tilde{L}_{1,1} f(|x - y|) & \leq 2\varphi'(2|x - y|) \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{\langle x - y, z \rangle^2}{|x - y|^2} \cdot \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\ & \leq 2c_{1,1}\varphi'(2|x - y|) \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{\langle x - y, z \rangle^2}{|x - y|^2 |z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\ & \leq c_{1,1}\varphi'(2|x - y|) |x - y|^{2-\alpha(x) \wedge \alpha(y)}, \end{aligned}$$

where in the last inequality we have used the fact that $\alpha(x) \wedge \alpha(y) \geq \alpha_0$ for all $x, y \in \mathbb{R}^d$.

Next we estimate the term $\tilde{L}_{1,2} f(|x - y|)$. Since $\varphi'(r) < 0$ for all $r \in (0, 2]$,

$$(3.22) \quad f(b) - f(a) \leq \varphi(a)(b - a), \quad a, b \in (0, 2].$$

We have for any $x, y, z \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$ and $|z| \leq |x - y|/2$ that

$$\begin{aligned} & f(|x - y + z|) - f(|x - y|) - f'(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \\ & \leq \varphi(|x - y|) \left[(|x - y + z| - |x - y|) - \frac{\langle x - y, z \rangle}{|x - y|} \right]. \end{aligned}$$

Thus, by (3.13), for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$, we arrive at

$$\begin{aligned} \tilde{L}_{1,2}f(|x - y|) &\leq \frac{\varphi(|x - y|)}{2|x - y|} \int_{\{|z| \leq \frac{|x-y|}{2}\}} |z|^2 \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz \\ &\quad + \frac{\varphi(|x - y|)}{2|x - y|} \int_{\{|z| \leq \frac{|x-y|}{2}\}} |z|^2 \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{\tilde{n}(x, y, z)}{|z|^{d+\alpha(x) \wedge \alpha(y)}} \right) dz. \end{aligned}$$

This, along with (3.14)–(3.16) and the fact that $n(x, z) \leq c_2$ for all $x, z \in \mathbb{R}^d$, yields the existence of a constant $c_{1,2} > 0$ such that for all $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} &\tilde{L}_{1,2}f(|x - y|) \\ &\leq \frac{c_2|\alpha(x) - \alpha(y)|\varphi(|x - y|)}{2|x - y|} \int_{\{|z| \leq \frac{|x-y|}{2}\}} \left[\frac{|z|^2}{|z|^{d+\alpha(x)}} + \frac{|z|^2}{|z|^{d+\alpha(y)}} \right] \left(\log \frac{1}{|z|} \right) dz \\ &\quad + \frac{\varphi(|x - y|)}{|x - y|} \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x-y|} |n(z, z_1) - n(z, z_2)| \right) \\ &\quad \times \int_{\{|z| \leq \frac{|x-y|}{2}\}} \frac{|z|^2}{|z|^{d+\alpha(x) \wedge \alpha(y)}} dz \\ &\leq c_{1,2}|x - y|^{1-\alpha(x) \wedge \alpha(y)} \varphi(|x - y|) \left[|\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} \right. \\ &\quad \left. + \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| + \sup_{z \in \mathbb{R}^d, |z_1 - z_2| \leq |x-y|} |n(z, z_1) - n(z, z_2)| \right) \right], \end{aligned}$$

where in the last inequality we have used the fact that $\alpha(x) \vee \alpha(y) \leq \alpha_2$ for all $x, y \in \mathbb{R}^d$.

(2) Secondly, for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned} &\tilde{L}_2f_n(|x - y|) \\ &= \int_{\{|z| > \frac{|x-y|}{2}\}} \left(f_n(|x - y + z|) - f_n(|x - y|) - f'_n(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\ &\quad + \int_{\{|z| > \frac{|x-y|}{2}\}} \left(f_n(|x - y - z|) - f_n(|x - y|) + f'_n(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \mathbf{1}_{\{|z| \leq 1\}} \right) \\ &\quad \times \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz. \end{aligned}$$

We separate each one of the above two integrals into two parts: one part with integral domain $\{|z| > 1\}$ and the other with $\{\frac{|x-y|}{2} \leq |z| \leq 1\}$, and then we sum

the corresponding terms with the same integral domain. In this way we get two quantities I_1 and I_2 :

$$\begin{aligned}
 I_1 &= \int_{\{|z|>1\}} \left(f_n(|x - y + z|) - f_n(|x - y|) \right) \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\
 &\quad + \int_{\{|z|>1\}} \left(f_n(|x - y - z|) - f_n(|x - y|) \right) \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz, \\
 I_2 &= \int_{\{\frac{|x-y|}{2} < |z| \leq 1\}} \left(f_n(|x - y + z|) - f_n(|x - y|) - f'_n(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \right) \\
 &\quad \times \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\
 &\quad + \int_{\{\frac{|x-y|}{2} < |z| \leq 1\}} \left(f_n(|x - y - z|) - f_n(|x - y|) + f'_n(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \right) \\
 &\quad \times \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz.
 \end{aligned}$$

On the one hand, noticing that $f_n(r) \leq f(2) + 1$ for all $r > 0$ and $n(x, z) \leq c_2$ for all $x, z \in \mathbb{R}^d$, we have

$$\begin{aligned}
 I_1 &\leq (f(2) + 1) \int_{\{|z|>1\}} \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\
 &\quad + (f(2) + 1) \int_{\{|z|>1\}} \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \vee |z|^{d+\alpha(y)}} \right) dz \\
 &\leq c_2(f(2) + 1) \int_{\{|z|\geq 1\}} \left(\frac{1}{|z|^{d+\alpha(x)}} + \frac{1}{|z|^{d+\alpha(y)}} \right) dz \\
 &\leq c_2(f(2) + 1) \int_{\{|z|\geq 1\}} \left(\frac{1}{|z|^{d+\alpha_0}} + \frac{1}{|z|^{d+\alpha_0}} \right) dz =: c_{2,1} < \infty.
 \end{aligned}$$

On the other hand, by the definition of f_n and (3.22), for all $x, y, z \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$ and $|z| \leq 1$,

$$\begin{aligned}
 &f_n(|x - y + z|) - f_n(|x - y|) - f'_n(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \\
 &\leq f(|x - y + z|) - f(|x - y|) - f'(|x - y|) \frac{\langle x - y, z \rangle}{|x - y|} \\
 &\leq \varphi(|x - y|) \left(|x - y + z| - |x - y| - \frac{\langle x - y, z \rangle}{|x - y|} \right) \\
 &\leq 2\varphi(|x - y|)|z|,
 \end{aligned}$$

where in the last inequality we have used the fact that $\varphi(r) > 0$ for all $r \in (0, 2]$. Therefore, following the argument of $\tilde{L}_{1,2}f(|x - y|)$, we get that for any $x, y \in \mathbb{R}^d$

with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned}
 I_2 &\leq 2\varphi(|x - y|) \int_{\{|x-y|/2 < |z| \leq 1\}} |z| \left(\frac{n(x, z)}{|z|^{d+\alpha(x)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \sqrt{|z|^{d+\alpha(y)}}} \right) dz \\
 &\quad + 2\varphi(|x - y|) \int_{\{|x-y|/2 < |z| \leq 1\}} |z| \left(\frac{n(y, z)}{|z|^{d+\alpha(y)}} - \frac{n(x, z) \wedge n(y, z)}{|z|^{d+\alpha(x)} \sqrt{|z|^{d+\alpha(y)}}} \right) dz \\
 &\leq c_{2,2}\varphi(|x - y|)|x - y|^{1-\alpha(x) \wedge \alpha(y)} \left[|\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} \right. \\
 &\quad \left. + \left(\sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right) \right],
 \end{aligned}$$

where in the last inequality we have used the fact that $\alpha(x) \wedge \alpha(y) \geq \alpha_0 > 1$ for all $x, y \in \mathbb{R}^d$.

Combining the estimates on I_1 and I_2 , we obtain for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq 1$,

$$\begin{aligned}
 \tilde{L}_2 f_n(|x - y|) &\leq c_{2,1} + c_{2,2}\varphi(|x - y|)|x - y|^{1-\alpha(x) \wedge \alpha(y)} \left[\sup_{|z| \leq 1} |n(x, z) - n(y, z)| \right. \\
 &\quad \left. + |\alpha(x) - \alpha(y)| \left(\log \frac{1}{|x - y|} \right) |x - y|^{-|\alpha(x) - \alpha(y)|} \right].
 \end{aligned}$$

(3) The required assertion immediately follows from the above two estimates on $\tilde{L}_1 f_n(|x - y|)$ and $\tilde{L}_2 f_n(|x - y|)$. □

As a consequence of Proposition 3.4, we have Corollary 3.5.

Corollary 3.5. *Assume that assumptions (H2) and (H3) hold. If (1.6) holds for some $\varphi \in \mathcal{D}_{\alpha_0}$, then there exist constants $\varepsilon_0 \in (0, 1)$ and $C_0 > 0$ such that for all $n \geq 1$, $\varepsilon \in (0, \varepsilon_0]$, and any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| < \varepsilon$,*

$$(3.23) \quad \tilde{L} f_n(|x - y|) \leq C_0 \varphi'(2\varepsilon) \varepsilon^{2-\alpha_0} < 0.$$

Proof. Due to $\varphi \in \mathcal{D}_{\alpha_0}$ and the fact that $\varphi'(r) < 0$ for $r \in (0, 1]$,

$$\lim_{r \rightarrow 0} \varphi'(r) r^{2-\alpha_0} = -\infty,$$

and so there exists a constant $\varepsilon_1 \in (0, 1)$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq \varepsilon_1$,

$$\varphi'(2|x - y|)|x - y|^{2-\alpha(x) \wedge \alpha(y)} \leq \varphi'(2|x - y|)|x - y|^{2-\alpha_0} \leq -2C_2/C_1,$$

where $C_1, C_2 > 0$ are the constants in Proposition 3.4. On the other hand, by (1.2), we can find constants $\varepsilon_2 \in (0, \varepsilon_1]$ and $C_4 > 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq \varepsilon_2$,

$$|x - y|^{-|\alpha(x) - \alpha(y)|} \leq C_4.$$

Combining both estimates above with Proposition 3.4, we conclude that there exists a constant $C_5 > 0$ such that for all $n \geq 1$, $\varepsilon \in (0, \varepsilon_2]$, and any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| < \varepsilon_2$,

$$\tilde{L} f_n(|x - y|) \leq \frac{C_1}{2} \varphi'(2|x - y|)|x - y|^{2-\alpha_0} \left(1 - C_5 \frac{A(x, y)}{\Psi(|x - y|)} \right).$$

This along with (1.6) in turn yields that there exist constants $\varepsilon_3 \in (0, \varepsilon_2]$ and $C_6 > 0$ such that for all $n \geq 1$, $\varepsilon \in (0, \varepsilon_3]$, and any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| < \varepsilon$,

$$\tilde{L} f_n(|x - y|) \leq C_6 \varphi'(2|x - y|)|x - y|^{2-\alpha_0}.$$

Again by $\varphi \in \mathcal{D}_{\alpha_0}$, we know that the function $r \mapsto \varphi'(2r)r^{2-\alpha_0}$ is increasing near the origin. Combining it with the inequality above, we prove the required assertion. \square

Having Corollary 3.5 at hand, we can follow the argument of Theorem 1.2 to prove Theorem 1.5. Here, we only present the following sketch.

Sketch of the proof of Theorem 1.5. We use the notation in the proof of Theorem 1.2. By (3.23), for any $\varepsilon \in (0, \varepsilon_0]$ and $x, y \in \mathbb{R}^d$ with $|x - y| \leq \varepsilon$,

$$\widetilde{\mathbb{E}}^{(x,y)}(T \wedge S_\varepsilon) \leq \frac{\int_0^{|x-y|} \varphi(s) ds}{-C_0 \varphi'(2\varepsilon) \varepsilon^{2-\alpha_0}}$$

and

$$\widetilde{\mathbb{P}}^{(x,y)}(T > S_\varepsilon) \leq \frac{\int_0^{|x-y|} \varphi(s) ds}{\int_0^\varepsilon \varphi(s) ds}.$$

This further gives us that for any $h \in B_b(\mathbb{R}^d)$ and $t > 0$,

$$\sup_{x \neq y} \frac{|P_t h(x) - P_t h(y)|}{\int_0^{|x-y|} \varphi(s) ds} \leq 2 \|h\|_\infty \left[\frac{1}{\int_0^\varepsilon \varphi(s) ds} - \frac{1}{t C_0 \varphi'(2\varepsilon) \varepsilon^{2-\alpha_0}} \right],$$

which in turn yields the desired assertion. \square

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