TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 371, Number 1, 1 January 2019, Pages 641–679
http://dx.doi.org/10.1090/tran/7285
Article electronically published on June 26, 2018

LOCAL RESTRICTION THEOREM AND MAXIMAL BOCHNER-RIESZ OPERATORS FOR THE DUNKL TRANSFORMS

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ABSTRACT. For the Dunkl transforms associated with the weight functions $h_\kappa^2(x) = \prod_{j=1}^d |x_j|^{2\kappa_j}$, $\kappa_1, \cdots, \kappa_d \geq 0$ on $\mathbb{R}^d$, it is proved that if $p \geq 2 + \frac{1}{\lambda_\kappa}$ and $\lambda_\kappa := \frac{d-1}{2} + \sum_{j=1}^d \kappa_j$, the maximal Bochner-Riesz operator $B_*^\delta(h_\kappa^2; f)$ order $\delta > 0$ is bounded on the space $L^p(\mathbb{R}^d; h_\kappa^2 dx)$ if and only if $\delta > \delta_\kappa(p) := \max\{(2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, 0\}$. This extends a well known result of M. Christ for the classical Fourier transforms (Proc. Amer. Math. Soc. **95** (1985), 16–20). The proof relies on a new local restriction theorem for the Dunkl transforms, which is stronger than the corresponding global restriction theorem, but significantly more difficult to prove.

1. Introduction and main results

The classical Fourier transform, initially defined on $L^1(\mathbb{R}^d)$, extends to an isometry of $L^2(\mathbb{R}^d)$ and commutes with the rotation group. For a family of weight functions h_κ invariant under a reflection group, there is a similar isometry of $L^2(\mathbb{R}^d; h_\kappa^2 dx)$, called the Dunkl transform, which has applications in physics for the analysis of quantum many body systems of Calogero-Moser-Sutherland type (see, for instance, [20,21], [10, Section 11.6], [17]). From the mathematical analysis point of view, the importance of the Dunkl transform lies in that it generalizes the classical Fourier transform and plays a similar role as the Fourier transform in classical Fourier analysis.

The weight functions that will be considered in this paper are product functions of the form

$$h_\kappa^2(x) = \prod_{j=1}^d |x_j|^{2\kappa_j}, \quad \kappa = (\kappa_1, \cdots, \kappa_d), \quad \kappa_j \geq 0, \quad j = 1, \cdots, d. \tag{1.1}$$

These weights are invariant under the Abelian reflection group $G = \mathbb{Z}_2^d$ generated by the reflections $\sigma_1, \cdots, \sigma_d$, where σ_j denotes the reflection with respect to the coordinate plane $x_j = 0$; that is,

$$x\sigma_j = (x_1, \cdots, x_{j-1}, -x_j, x_{j+1}, \cdots, x_d), \quad x \in \mathbb{R}^d.$$

Received by the editors December 4, 2016, and, in revised form, May 9, 2017.
2010 *Mathematics Subject Classification.* Primary 42B10, 42B15; Secondary 40A10, 33C10.
Key words and phrases. Dunkl transforms, local restriction theorem, Bochner-Riesz means, almost everywhere convergence, weighted Littlewood-Paley inequality.
This work was supported by NSERC Canada under grant RGPIN 04702 Dai. It was conducted when the second author was a PhD student at the University of Alberta.

For $1 \leqslant p \leqslant \infty$, denote by $L^p(\mathbb{R}^d; h_\kappa^2)$ or $L^p(\mathbb{R}^d; h_\kappa^2 dx)$ the L^p-space defined with respect to the measure $h_\kappa^2(x)dx$ on $\mathbb{R}^d$ and by $\|\cdot\|_{\kappa,p}$ the norm of $L^p(\mathbb{R}^d; h_\kappa^2)$. For a set $E \subset \mathbb{R}^d$, write $\mathrm{meas}_\kappa(E) := \int_E h_\kappa^2(x)\,dx$. Denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the Euclidean inner product on $\mathbb{R}^d$, respectively. The notation $B(x, r)$ stands for the ball $\{y \in \mathbb{R}^d : \ \|y - x\| \le r\}$ with center $x \in \mathbb{R}^d$ and radius $r > 0$.

The Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$\mathcal{F}_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(-\mathbf{i}x, y) h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d, \tag{1.2}$$

where $c_\kappa^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(y) e^{-\|y\|^2/2}\,dy$, $E_\kappa(-\mathbf{i}x, y) = V_\kappa\Big[e^{-\mathbf{i}\langle x, \cdot\rangle}\Big](y)$ is the weighted analogue of the character $e^{-\mathbf{i}\langle x,y\rangle}$ on $\mathbb{R}^d$, and $V_\kappa : \ C(\mathbb{R}^d) \to C(\mathbb{R}^d)$ denotes the Dunkl intertwining operator. In the case of $\kappa = 0$ (i.e., the unweighted case), V_κ is simply the identity operator on $C(\mathbb{R}^d)$, and the Dunkl transform $\mathcal{F}_\kappa f$ becomes the classical Fourier transform. Furthermore, in the case when $f(\cdot) = f_0(\|\cdot\|)$ is a radial function, $\mathcal{F}_\kappa f$ becomes the Hankel transform of f_0.

The Dunkl transform enjoys many properties similar to those of the classical Fourier transform (see, for instance, [14, 20, 21]). For example, each function $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is uniquely determined by its Dunkl transform $\mathcal{F}_\kappa f$, and every function $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ can be recovered from its Dunkl transform via the Bochner-Riesz means defined as

$$B_R^\delta(h_\kappa^2; f)(x) = c_\kappa \int_{\|y\| \le R} \left(1 - \frac{\|y\|^2}{R^2}\right)^\delta \mathcal{F}_\kappa f(y) E_\kappa(\mathbf{i}x, y) h_\kappa^2(y)\,dy, \quad x \in \mathbb{R}^d,$$

where $R > 0$, $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, and $\delta > -1$ is the order of the Bochner-Riesz means. As in classical Fourier analysis, $B_R^\delta(h_\kappa^2; f)(x)$ can be expressed as an integral

$$B_R^\delta(h_\kappa^2; f)(x) = c_\kappa \int_{\mathbb{R}^d} f(y) K_R^\delta(h_\kappa^2; x, y) h_\kappa^2(y)\,dy, \quad x \in \mathbb{R}^d, \tag{1.3}$$

which further extends $B_R^\delta(h_\kappa^2; f)$ to a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$ for all $1 \le p < \infty$ and $R > 0$.

Summability of the Bochner-Riesz means $B_R^\delta(h_\kappa^2; f)$ for the Dunkl transform was studied in the pioneering paper by Thangavelu and Xu [20, Theorem 5.5], where it was proved that[1]

$$\sup_{R>0} \|B_R^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p}, \quad \forall f \in L^p(\mathbb{R}^d; h_\kappa^2) \tag{1.4}$$

for all $1 \le p \le \infty$ if and only if $\delta > \lambda_\kappa := \frac{d-1}{2} + |\kappa|$, where $|\kappa| = \sum_{j=1}^d \kappa_j$. Moreover, it was proved in [6, Theorem 4.3] that if $1 \le p \le \infty$ and $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2\lambda_\kappa + 2}$, then (1.4) holds for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ if and only if

$$\delta > \delta_\kappa(p) := \max\Big\{(2\lambda_\kappa + 1)\Big|\frac{1}{p} - \frac{1}{2}\Big| - \frac{1}{2}, 0\Big\}. \tag{1.5}$$

This last result extends the celebrated Thomas-Stein theorem for the Fourier transforms [22].

Thangavelu and Xu [20] also studied the maximal Bochner-Riesz operators:

$$B_*^\delta(h_\kappa^2; f)(x) := \sup_{R>0} |B_R^\delta(h_\kappa^2; f)(x)|, \quad x \in \mathbb{R}^d.$$

[1]The result of [20] holds for more general weights invariant under a finite reflection group as well.

They proved that for $1 < p \le \infty$ and $\delta > \lambda_\kappa$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \le C\|f\|_{\kappa,p}. \tag{1.6}$$

In this paper, we shall give a necessary and sufficient condition on the order δ of the Bochner-Riesz means for which (1.6) holds for a given $p \ge 2 + \frac{2}{\lambda_\kappa}$. Our main result is stated as follows.

Theorem 1.1. *Assume that* $\frac{1}{2} - \frac{1}{p} \ge \frac{1}{2\lambda_\kappa+2}$. *Then* (1.6) *holds for all* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ *if and only if* $\delta > \delta_\kappa(p) := \max\{(2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, 0\}$.

Clearly, Theorem 1.1, in particular, improves the analogue of the Thomas-Stein theorem on the uniform L^p-boundedness of the Bochner-Riesz means as stated above (i.e., Theorem 4.3 of [6]). We point out that in the case of the Fourier transform (i.e., the case of $\kappa = 0$), Theorem 1.1 is due to M. Christ [4].

Note that Theorem 1.1 implies that if $\frac{1}{2} - \frac{1}{p} \ge \frac{1}{2\lambda_\kappa+2}$, $\delta > \delta_\kappa(p)$, and $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, then

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f)(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^d. \tag{1.7}$$

It turns out, however, that the condition $\frac{1}{2} - \frac{1}{p} \ge \frac{1}{2\lambda_\kappa+2}$ for the almost everywhere (a.e.) convergence here can be dropped. More precisely, it can be shown that if $p \ge 2$ and $\delta > \delta_\kappa(p)$, then (1.7) holds almost everywhere for all $f \in L^p(\mathbb{R}^d, h_\kappa^2)$. In the case of the classical Fourier transform, this result is well known and is due to A. Carbery, J. L. R. De Francia, and L. Vega [3]. The proof of this result for general Dunkl transforms will appear elsewhere.

One of the main difficulties in the proof of Theorem 1.1 comes from the fact that the measure $h_\kappa^2(x)\,dx$ is neither translation invariant nor rotation invariant, which makes many global estimates in classical analysis insufficient for our purposes. As a result, more delicate local estimates of certain highly oscillated kernels are required. For example, for the integral kernel $K_R^\delta(h_\kappa^2; x, y)$ of the Bochner-Riesz means $B_R^\delta(h_\kappa^2; f)(x)$, we shall prove the following fairly nontrivial estimates: for $\delta > 0$, $R > 0$, and $x, y \in \mathbb{R}^d$,

$$|K_R^\delta(h_\kappa^2; x, y)| \le C\frac{R^d \prod_{j=1}^d (|x_j y_j| + R^{-2} + R^{-1}\|\bar{x} - \bar{y}\|)^{-\kappa_j}}{(1 + R\|\bar{x} - \bar{y}\|)^{\frac{d+1}{2}+\delta}}, \tag{1.8}$$

where $\bar{x} = (|x_1|, \cdots, |x_d|)$ for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$.

The main tool for the proof of Theorem 1.1 is a local restriction theorem for the Dunkl transforms, as stated in Theorem 1.2 below. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^d$ equipped with the usual surface Lebesgue measure $d\sigma$. Given $1 \le p \le \infty$, we denote by $L^p(\mathbb{S}^{d-1}; h_\kappa^2)$ the L^p-space defined with respect to the measure $h_\kappa^2(x)d\sigma(x)$ on $\mathbb{S}^{d-1}$.

Theorem 1.2. *Let* $c_0 \in (0, 1)$ *be a constant depending only on* d *and* κ, *and let* B *be a closed ball in* $\mathbb{R}^d$ *with radius* $\theta \ge c_0 > 0$. *If* $1 \le p \le p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$ *and* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ *is supported in the ball* B, *then*

$$\|\mathcal{F}_\kappa f\|_{L^2(\mathbb{S}^{d-1}; h_\kappa^2)} \le C\Big(\frac{\theta^{2\lambda_\kappa+1}}{\text{meas}_\kappa(B)}\Big)^{\frac{1}{p}-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^d; h_\kappa^2)}.$$

This local restriction theorem is stronger than the corresponding global restriction theorem. In fact, letting $\theta \to \infty$ in Theorem 1.2, we obtain the following.

Corollary 1.3. *If* $1 \le p \le p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$ *and* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, *then*

$$\|\mathcal{F}_\kappa f\|_{L^2(\mathbb{S}^{d-1};h_\kappa^2)} \le C\|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}.$$

It turns out, however, that unlike the classical case of Fourier transforms, this global restriction theorem is not enough for analysis near the zeros of the weight h_κ^2. The proof of the local restriction theorem (Theorem 1.2), on the other hand, is rather involved and significantly more difficult. It will be given in Section 4 of this paper.

To the best of our knowledge, both Theorems 1.1 and 1.2 seem new even for the Hankel transforms in one variable.

In addition to the local restriction theorem, the proof of Theorem 1.1 also requires weighted Littlewood-Paley inequalities with A_p-weights in the Dunkl setting, which will be established in Section 5 of this paper. The crucial step in the proof of these weighted inequalities is to establish various pointwise kernel estimates in the Dunkl setting.

This paper is organized as follows. Section 2 contains some preliminary materials on analysis associated with the Dunkl transform. Section 3 is devoted to the proof of a global restriction theorem, which is relatively easier and, in fact, close to that of the classical Fourier restriction theorem. After that, in Section 4, we turn to our main estimate, the local restriction theorem (Theorem 1.2). The proof of this local estimate relies partially on the global estimate established in Section 3. Several useful corollaries of the local restriction theorem are also given in Section 4. Section 5 is devoted to the proof of the weighted Littlewood-Paley inequality associated with the Dunkl transform. Finally, in Section 6, we prove our main result, Theorem 1.1. The proof relies on the local restriction theorem and the weighted Littlewood-Paley inequality established in previous sections.

2. Preliminaries

In this section, we shall present some preliminary materials on Dunkl analysis associated with the group $\mathbb{Z}_2^d$ and the weight $h_\kappa^2(x)$ given in (1.1), most of which can be found in [7–10, 17, 20, 21].

2.1. **Dunkl intertwining operators.** The Dunkl operators $\mathcal{D}_{\kappa,j}$ are defined by

$$\mathcal{D}_{\kappa,j} := \frac{\partial}{\partial x_j} + \kappa_j E_j, \quad j = 1, \cdots, d, \tag{2.1}$$

where E_j denotes the difference operator given by $E_j f(x) := \frac{f(x)-f(x\sigma_j)}{x_j}$. These operators mutually commute and map $\mathbb{P}_n^d$ to $\mathbb{P}_{n-1}^d$, where $\mathbb{P}_n^d$ denotes the space of homogeneous polynomials of degree n in d variables. A fundamental result in the Dunkl theory states that there exists a linear operator V_κ on the space of all algebraic polynomials on $\mathbb{R}^d$, called the Dunkl intertwining operator, determined uniquely by the following conditions:

$$V_\kappa(\mathbb{P}_n^d) \subset \mathbb{P}_n^d, \quad V_\kappa(1) = 1, \quad \text{and} \quad \mathcal{D}_{\kappa,i} V_\kappa = V_\kappa \frac{\partial}{\partial x_i}, \quad 1 \le i \le d. \tag{2.2}$$

It was proved by Xuan Xu [23] that

$$V_\kappa f(x) = \int_{[-1,1]^d} f(x_1 t_1, \cdots, x_d t_d) \prod_{j=1}^d d\mu_j(t_j), \tag{2.3}$$

where $d\mu_j(s) = c_{\kappa_j}(1+s)(1-s^2)^{\kappa_j-1}ds$ with $c_{\kappa_j} := \frac{\Gamma(\kappa_j+1/2)}{\sqrt{\pi}\Gamma(\kappa_j)}$ if $\kappa_j > 0$, and $d\mu_j$ is the Dirac measure supported at $s = 1$ if $\kappa_j = 0$. In particular, the formula (2.3) extends V_κ to a positive operator on the space of continuous functions on $\mathbb{R}^d$.

For the Dunkl intertwining operator associated with a general finite reflection group, we refer to the work of Rösler [15].

2.2. **Dunkl transforms.** The Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$\mathcal{F}_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(-\mathbf{i}x, y) h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d, \tag{2.4}$$

where $c_\kappa^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(y) e^{-\|y\|^2/2}\, dy$,

$$E_\kappa(-\mathbf{i}x, y) = V_\kappa\Big[e^{-\mathbf{i}\langle x,\cdot\rangle}\Big](y) = \prod_{j=1}^d c_{\kappa_j}\Big[\frac{J_{\kappa_j-\frac12}(x_jy_j)}{(x_jy_j)^{\kappa_j-\frac12}} - \mathbf{i}x_jy_j\frac{J_{\kappa_j+\frac12}(x_jy_j)}{(x_jy_j)^{\kappa_j+\frac12}}\Big],$$

and J_α denotes the Bessel function of order α of the first kind:

$$J_\alpha(t) = \Big(\frac{t}{2}\Big)^\alpha \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\Big(\frac{t}{2}\Big)^{2n}, \quad t \in \mathbb{R}. \tag{2.5}$$

If $\kappa = 0$, then the Dunkl transform coincides with the usual Fourier transform.

The Dunkl transform $\mathcal{F}_\kappa$ restricted on the class $\mathcal{S}(\mathbb{R}^d)$ of Schwarz functions on $\mathbb{R}^d$ satisfies the Plancherel formula $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$. Thus, it extends to an isometric isomorphism on the space $L^2(\mathbb{R}^d; h_\kappa^2)$. As a result, the Dunkl transform can also be defined on the space $L^1 + L^2 \supset \bigcup_{1\le p\le 2} L^p$.

The Dunkl transforms have properties similar to those of the classical Fourier transform. We collect some of these properties in the following lemma.

Lemma 2.1 ([14, 16])**.**

(i) *If $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, then $\mathcal{F}_\kappa f \in C(\mathbb{R}^d)$ and $\lim_{\|\xi\|\to\infty} \mathcal{F}_\kappa f(\xi) = 0$.*

(ii) *The Dunkl transform $\mathcal{F}_\kappa$ is an isomorphism of the Schwarz class $\mathcal{S}(\mathbb{R}^d)$ onto itself, and $\mathcal{F}_\kappa^2 f(x) = f(-x)$ for $f \in \mathcal{S}(\mathbb{R}^d)$.*

(iii) *If f and $\mathcal{F}_\kappa f$ are both in $L^1(\mathbb{R}^d; h_\kappa^2)$, then the following inverse formula holds:*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(\mathbf{i}x, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d.$$

(iv) *If $f, g \in L^1(\mathbb{R}^d; h_\kappa^2)$, then*

$$\int_{\mathbb{R}^d} \mathcal{F}_\kappa f(x) g(x)\, h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa g(x)\, h_\kappa^2(x)\, dx. \tag{2.6}$$

(v) *If $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then*

$$\|\mathcal{F}_\kappa f\|_{\kappa,p'} \le \|f\|_{\kappa,p}. \tag{2.7}$$

(vi) *Given $\varepsilon > 0$, let $f_\varepsilon(x) = \varepsilon^{-(2\lambda_\kappa+1)} f(\varepsilon^{-1}x)$ with $\lambda_\kappa := \frac{d-1}{2} + |\kappa|$. Then $\mathcal{F}_\kappa f_\varepsilon(\xi) = \mathcal{F}_\kappa f(\varepsilon\xi)$.*

(vii) *If* $f(x) = f_0(\|x\|)$ *is a radial function in* $L^p(\mathbb{R}^d; h_\kappa^2)$ *with* $1 \le p \le 2$, *then* $\mathcal{F}_\kappa f(\xi) = H_{\lambda_\kappa - \frac{1}{2}} f_0(\|\xi\|)$ *is again a radial function, where* H_α *denotes the Hankel transform defined by*

$$H_\alpha g(s) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty g(r) \frac{J_\alpha(rs)}{(rs)^\alpha} r^{2\alpha+1}\, dr, \quad \alpha > -\frac{1}{2}.$$

We will also consider the Dunkl transform on the space $\mathcal{M}$ of finite Borel measures on $\mathbb{R}^d$:

$$\mathcal{F}_\kappa(d\mu)(\xi) = c_\kappa \int_{\mathbb{R}^d} E(-i\xi, y) h_\kappa^2(y) d\mu(y), \quad \xi \in \mathbb{R}^d, \quad \mu \in \mathcal{M}.$$

Many identities in this paper are interpreted in a distributional sense. Denote by $\mathcal{S}'(\mathbb{R}^d)$ the space of all tempered distributions on $\mathbb{R}^d$. We shall identify a function f in $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p \le \infty$, with a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$ given by

$$(f, \varphi) := \int_{\mathbb{R}^d} f(x)\varphi(x) h_\kappa^2(x)\, dx, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

According to (2.6), we may define the distributional Dunkl transform $\mathcal{F}_\kappa f$ of $f \in \bigcup_{p>2} L^p(\mathbb{R}^d, h_\kappa^2)$ via

$$(\mathcal{F}_\kappa f, \varphi) := (f, \mathcal{F}_\kappa \varphi) \equiv \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa \varphi(x) h_\kappa^2(x)\, dx, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

For later applications, we also record here some useful facts about the Bessel functions.

Lemma 2.2.

(i) ([19, (1.71.1), (1.71.5)]). *For each* $\alpha \in \mathbb{C}$ *with* $Re\, \alpha > -1$, $z^{-\alpha} J_\alpha(z)$ *is an even entire function of* $z \in \mathbb{C}$ *and*

$$\frac{d}{dz}\left[z^{-\alpha} J_\alpha(z)\right] = -z^{-\alpha} J_{\alpha+1}(z). \tag{2.8}$$

(ii) ([19, (1.71.1), (1.71.11)]). *For each* $\alpha = \sigma + i\tau \in \mathbb{C}$ *with* $\sigma > -1$,

$$|x^{-\alpha} J_\alpha(x)| \le C e^{c|\tau|} (1 + |x|)^{-\sigma - \frac{1}{2}}, \quad x \in \mathbb{R}. \tag{2.9}$$

(iii) ([2, p. 218, (4.11.12)]). *Set* $j_\alpha(t) = \Gamma(\alpha+1)\frac{J_\alpha(t)}{t^\alpha}$. *If* $Re\ \alpha > -1$ *and* $Re\ \beta > 0$, *then*

$$\int_0^\infty j_{\alpha+\beta}(s) j_\alpha(st) s^{2\alpha+1}\, ds = \frac{\Gamma(\alpha+\beta+1)\Gamma(\alpha+1)}{2^{\beta-1}\Gamma(\beta)} (1-t^2)_+^{\beta-1}, \quad t \in \mathbb{R}. \tag{2.10}$$

2.3. Generalized translations on $\mathcal{S}(\mathbb{R}^d)$.

Definition 2.3 ([16, 17, 20])**.** Given $y \in \mathbb{R}^d$, the generalized translation $T^y f$ of $f \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$T^y f(x) := c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(\xi) E_\kappa(-\mathbf{i}y, \xi) E_\kappa(\mathbf{i}\xi, x) h_\kappa^2(\xi)\, d\xi, \quad x \in \mathbb{R}^d. \tag{2.11}$$

According to the inverse formula, if $f \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, then

$$\mathcal{F}_\kappa(T^y f)(x) = E_\kappa(-\mathbf{i}x, y) \mathcal{F}_\kappa f(x). \tag{2.12}$$

Moreover, it was shown in [16, Lemma 2.2] that T^y is a linear operator from the space $\mathcal{S}(\mathbb{R}^d)$ to itself, and for any $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} T^y f(x) g(x) h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} f(x) T^{-y} g(x) h_\kappa^2(x)\, dx. \tag{2.13}$$

The generalized translation has the following integral representation on the space $\mathcal{S}(\mathbb{R}^d)$.

Lemma 2.4 ([20, Theorem 7.1]). *For $y = (y_1, \cdots, y_d), x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$T^y f(x) = T_{1,y_1} T_{2,y_2} \cdots T_{d,y_d} f(x). \tag{2.14}$$

Here the operators T_{j,y_j} are defined by

$$T_{j,y_j} f(x) := \int_{-1}^{1} \Big[f_{j,e}(u_j(x,y,t)) + \frac{(x_j - y_j) f_{j,o}(u_j(x,y,t))}{v_j(x_j,y_j,t)} \Big]\, d\mu_j(t), \tag{2.15}$$

where

$$v_j(x_j, y_j, t) := \sqrt{x_j^2 + y_j^2 - 2x_j y_j t}, u_j(x,y,t) := \Big(x_1, \cdots, x_{j-1}, v_j(x_j,y_j,t), x_{j+1}, \cdots x_d \Big),$$

$f_{j,e}(x) := \frac{1}{2}\Big(f(x) + f(x\sigma_j) \Big)$, $f_{j,o}(x) = \frac{1}{2}\Big(f(x) - f(x\sigma_j) \Big)$, *and the measure $d\mu_j$ is the same as in* (2.3).

According to (2.14), we have

$$T^y f(x) := \int_{\mathbb{R}^d} f(z)\, d\mu_{x,y}(z), \quad x, y \in \mathbb{R}^d, \tag{2.16}$$

where $d\mu_{x,y}$ is a signed Borel measure supported on

$$\Big\{ z = (z_1, \cdots, z_d) \in \mathbb{R}^d : \quad \big| |x_i| - |y_i| \big| \le |z_i| \le |x_i| + |y_i|, \quad i = 1, 2, \cdots, d \Big\}.$$

2.4. **Generalized translations on L^p-spaces.** In this subsection, we will clarify the definition of generalized translations on L^p-spaces with $1 \le p \le \infty$. The clarification relies on the following proposition.

Proposition 2.5. *The integral representation* (2.14) *extends T^y to a bounded operator on the spaces $L^p(\mathbb{R}^d; h_\kappa^2)$ for all $1 \le p \le \infty$ with*

$$\sup_{y \in \mathbb{R}^d} \|T^y f\|_{\kappa,p} \le C_d \|f\|_{\kappa,p}, \quad 1 \le p \le \infty. \tag{2.17}$$

Throughout the paper, we will use (2.14) as the definition of the generalized translation $T^y f$ of $f \in \bigcup_{p \ge 1} L^p(\mathbb{R}^d; h_\kappa^2)$. This definition is justified by Proposition 2.5 and the fact that $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d; h_\kappa^2)$.

We point out that a proof of (2.17) was given in [20] using a duality argument and the Plancherel and the Riesz-Thorin interpolation theorems. However, it seems necessary to give some clarification on this duality argument because $\mathcal{S}(\mathbb{R}^d)$ is not dense in L^∞, and it is not very clear from the definition why the definition (2.14) is the same as the definition (2.12) on the space $L^\infty \cap L^2(\mathbb{R}^d; h_\kappa^2)$. Since Proposition 2.5 plays an indispensable role in this paper, we sketch its proof as follows.

Proof of Proposition 2.5. First, we note that for $t \in [-1,1]$ and $x, y \in \mathbb{R}$,

$$\sqrt{x^2+y^2-2xyt} \geq \max\{|x-yt|, |xt-y|\} \geq \frac{1}{2}|x-y|(1+t).$$

Thus, by Fubini's theorem and (2.15), we reduce to showing that

$$\|\widetilde{T^y} f\|_{L^p(\mathbb{R};|x|^{2\kappa}dx)} \leq C_\kappa \|f\|_{L^p(\mathbb{R};|x|^{2\kappa}dx)}, \quad 1 \leq p \leq \infty, \tag{2.18}$$

where $\widetilde{T^y}$ is defined as

$$\widetilde{T^y} f(x) := \int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt})(1-t^2)^{\kappa-1}\,dt, \quad f \in L^p(\mathbb{R};|x|^{2\kappa})$$

with $\kappa > 0$. Indeed, a change of variable $z = \sqrt{x^2+y^2-2xyt}$ yields that $\widetilde{T^y} f(x) = \int_{\mathbb{R}} f(z) W(x,y,z)|z|^{2\kappa}\,dz$, where for $x, y \in \mathbb{R}$ with $xy \neq 0$,

$$W(x,y,z) = \begin{cases} \dfrac{\Big(\Delta(x,y,z)\Big)^{\kappa-1}}{2^{2\kappa-2}|xyz|^{2\kappa-1}} & \text{if } \big||x|-|y|\big| \leq z \leq |x|+|y|, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Delta(x,y,z) = (|x|+|y|+z)(|x|+|y|-z)(z+|x|-|y|)(z-|x|+|y|).$$

The desired inequality (2.18) then follows since $\sup_{y,z\in\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} W(x,y,z)|x|^{2\kappa}\,dx \leq C < \infty$, which can be verified through straightforward calculations. □

As a direct consequence of Proposition 2.5, we obtain the following.

Corollary 2.6. *If $f(x) = f_0(\|x\|)$ is a radial function in $L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 \leq p \leq \infty$, then for each $y \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$,*

$$T^y f(x) = c_\kappa \int_{[-1,1]^d} f_0\big(z(x,y,t)\big) \prod_{j=1}^{d} d\mu_j(t_j), \tag{2.19}$$

where $z(x,y,t) = \sqrt{\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j y_j t_j}$, and the measure $\prod_{j=1}^d d\mu_j$ is the same as in (2.3).

In the case when f is a radial Schwarz function, (2.19) is a direct consequence of (2.3) and a formula of Rösler [16] that is applicable to the general case of finite reflection groups. The main point in Corollary 2.6 lies in that (2.19) holds under the weaker condition $f = f_0(\|\cdot\|) \in \bigcup_{p\geq 1} L^p(\mathbb{R}^d; h_\kappa^2)$, which is very important in our later applications. It is worthwhile to point out that a limit argument using the result of [16] doesn't seem to yield (2.19) under the weaker condition.

The property (2.12) of the generalized translation operators carries over to L^p spaces as well.

Proposition 2.7. *If $y \in \mathbb{R}^d$ and $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 \leq p \leq \infty$, then* (2.12) *holds almost everywhere* (*a.e.*) *on $\mathbb{R}^d$ for $1 \leq p \leq 2$ and holds in a distributional sense for $1 \leq p \leq \infty$:*

$$\Big(\mathcal{F}_\kappa(T^y f), \varphi\Big) = \Big(\mathcal{F}_\kappa f, E_\kappa(-\mathbf{i}y,\cdot)\varphi\Big), \ \forall \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.20}$$

2.5. **Generalized convolutions.** We start with the definition on the space $\mathcal{S}(\mathbb{R}^d)$.

Definition 2.8. The generalized convolution of $f, g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y) T^y g(x) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d. \tag{2.21}$$

The generalized convolution has the following basic property: for $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}_\kappa(f *_\kappa g)(\xi) = \mathcal{F}_\kappa f(\xi) \mathcal{F}_\kappa g(\xi), \quad \xi \in \mathbb{R}^d. \tag{2.22}$$

Since the generalized translation operators are uniformly bounded on L^p-spaces with $1 \le p \le \infty$, the following Young's inequality can be established (see [20, Proposition 7.2]):

$$\|f *_\kappa g\|_{\kappa,r} \le \|f\|_{\kappa,p} \|f\|_{\kappa,q}, \tag{2.23}$$

where $1 \le p, q, r \le \infty$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. This, in particular, implies that the generalized convolution $f *_\kappa g$ can be defined for $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^q(\mathbb{R}^d; h_\kappa^2)$ with $1 \le p, q \le \infty$, and $\frac{1}{p} + \frac{1}{q} \ge 1$. The following corollary can be easily verified.

Corollary 2.9. *If $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p \le \infty$, and $g \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\mathcal{F}_\kappa(f *_\kappa g)(\xi) = \mathcal{F}_\kappa f(\xi) \mathcal{F}_\kappa g(\xi) \tag{2.24}$$

holds in a distributional sense.

2.6. **Spaces of homogeneous type.** A straightforward calculation shows that

$$\mathrm{meas}_\kappa(B(x,r)) \sim r^d \prod_{j=1}^d (|x_j| + r)^{2\kappa_j}, \quad x \in \mathbb{R}^d, \quad r > 0. \tag{2.25}$$

This, in particular, implies that the measure $d\mu_\kappa(x) = h_\kappa^2(x)\, dx$ satisfies the doubling condition on the space $\mathbb{R}^d$, and hence $(\mathbb{R}^d, d\mu_\kappa)$ is a space of homogeneous type in the sense of Coifman and Weiss [5]. In this subsection, we will review briefly some basic definitions and facts on homogeneous spaces, most of which can be found in [5, 13].

Definition 2.10. A space of homogeneous type (X, ρ, μ) is a topological space X endowed with a quasi-metric ρ and a nonnegative Borel measure μ such that

(i) ρ is continuous on $X \times X$ and every ball $B_\rho(x,r) := \{y \in X : \ \rho(x,y) < r\}$ is open in X;

(ii) the measure μ satisfies the doubling condition:

$$\mu(B(x,2r)) \le C\mu(B(x,r)), \quad \forall x \in X, \quad \forall r > 0;$$

(iii) $0 < \mu(B_\rho(x,r)) < \infty$ for every $x \in X$ and $r > 0$.

Definition 2.11. A weight w (any nonnegative measurable function) on a homogeneous space (X, ρ, μ) satisfies the A_p condition for $1 \le p < \infty$ if

$$[w]_{A_p} := \sup_B \Big(\frac{1}{\mu(B)} \int_B w\, d\mu \Big) \Big(\frac{1}{\mu(B)} \int_B w^{-\frac{1}{p-1}}\, d\mu \Big)^{p-1} < \infty,$$

where the supremum is taken over all the balls B in X, and in the case of $p = 1$, we replace $\left(\frac{1}{\mu(B)} \int_B w^{-\frac{1}{p-1}}\, d\mu \right)^{p-1}$ with $\|w^{-1}\|_\infty$.

The A_p classes are increasing with respect to p; that is, $A_{p_1} \subset A_{p_2}$ for $1 \le p_1 < p_2$. As a result, we can define the A_∞ class by $A_\infty = \bigcup_{p>1} A_p$.

Definition 2.12. A function $K : \{(x, y) : x \neq y \in X\} \to \mathbb{R}$ is a Calderón-Zygmund kernel if there exist $\delta \in (0, 1]$ and $C < \infty$ such that

$$|K(x, y)| \leq \frac{C}{\mu(B_\rho(x, \rho(x, y)))}, \quad \forall x \neq y \in X,$$

and for all $x, y, z \in X$ with $x \neq y$ and $\rho(x, z) \leq \frac{1}{2}\rho(x, y)$,

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \leq C\Big(\frac{\rho(x, z)}{\rho(x, y)}\Big)^\delta \frac{1}{\mu(B_\rho(x, \rho(x, y)))}.$$

Definition 2.13. An operator T is associated with a Calderón-Zygmund kernel K if for every compactly supported continuous function f on X,

$$Tf(x) = \int_X f(y)K(x, y)\, d\mu(y), \quad x \notin \operatorname{supp}(f).$$

If, in addition, T is bounded on $L^2(X, \mu)$, then T is called a Calderón-Zygmund operator.

The following result is a direct consequence of [1, Corollary 3.6] and [13, Theorem 1.2].

Theorem 2.14. *Let T be a Calderón-Zygmund operator. If $1 < p < \infty$ and $w \in A_p$, then*

$$\|Tf\|_{L^p(w\, d\mu)} \leq C_{p,w}\|f\|_{L^p(w\, d\mu)}.$$

The weighted theory can be repeated for vector-valued Calderón-Zygmund operators, and hence Theorem 2.14 remains true in this case (see, for instance, [11, Chapter 4]).

3. Global restriction theorem for the Dunkl transforms

Recall that $d\sigma$ stands for the surface Lebesgue measure on $\mathbb{S}^{d-1}$. For $f \in L^1(\mathbb{S}^{d-1}; h_\kappa^2)$, the Dunkl transform of $fd\sigma$ is defined by

$$\mathcal{F}_\kappa(fd\sigma)(\xi) := \int_{\mathbb{S}^{d-1}} f(y)E_\kappa(-\mathbf{i}\xi, y)h_\kappa^2(y)\, d\sigma(y), \quad \xi \in \mathbb{R}^d.$$

It is known that (see, for instance, [7, Proposition 6.1.9])

$$\mathcal{F}_\kappa(d\sigma)(\xi) = c_{\kappa,d}\|\xi\|^{-(\frac{d-2}{2}+|\kappa|)}J_{\frac{d-2}{2}+|\kappa|}(\|\xi\|) = cj_{\lambda_\kappa-\frac{1}{2}}(\|\xi\|). \tag{3.1}$$

In this section, we will prove the following global restriction theorem for the Dunkl transforms, which will be used in our proof of the local restriction theorem (Theorem 1.2).

Theorem 3.1. *If $1 \leq p \leq p_\kappa := \frac{2\lambda_\kappa+2}{\lambda_\kappa+2}$, then*

$$\|f *_\kappa (\mathcal{F}_\kappa(d\sigma))\|_{\kappa,p'} \leq C\|f\|_{\kappa,p}.$$

The proof of Theorem 3.1 is very close to the proof for the classical Fourier transform. For completeness, we include it below.

Proof. By (2.5), the function

$$K_z(x) := \frac{1}{\Gamma(\lambda_\kappa + \frac{1}{2} + z)} j_{\lambda_\kappa-\frac{1}{2}+z}(\|x\|) = \frac{J_{\lambda_\kappa-\frac{1}{2}+z}(\|x\|)}{\|x\|^{\lambda_\kappa-\frac{1}{2}+z}}, \quad x \in \mathbb{R}^d,$$

is analytic in z on the domain $\{z \in \mathbb{C} : \text{ Re } z > -\frac{1}{2} - \lambda_\kappa\}$, whereas by (2.9),

$$|K_{\sigma+\mathbf{i}\tau}(x)| \le C_\sigma e^{c|\tau|}(1+\|x\|)^{-(\lambda_\kappa+\sigma)}, \quad x \in \mathbb{R}^d, \quad \lambda_\kappa + \sigma \ge 0. \tag{3.2}$$

Furthermore, according to (2.11), the function K_z has the following distributional Dunkl transform:

$$\mathcal{F}_\kappa K_z(\xi) = \frac{c_\kappa}{2^z \Gamma(z+1)}(1-\|\xi\|^2)_+^{z-1}, \quad \text{Re } z > 0, \quad \xi \in \mathbb{R}^d. \tag{3.3}$$

Next, define

$$R_z f(x) := f *_\kappa K_z(x) = c \int_{\mathbb{R}^d} f(y) T^y K_z(x) h_\kappa^2(y)\, dy, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

On one hand, by (3.2) and Young's inequality (2.23), we have that

$$\|R_{-\lambda_\kappa+\mathbf{i}\tau} f\|_\infty \le \|K_z\|_\infty \|f\|_{\kappa,1} \le C e^{c|\tau|} \|f\|_{\kappa,1}.$$

On the other hand, by (3.3) and (2.24), it follows that

$$\|R_{1+\mathbf{i}\tau} f\|_{\kappa,2} \le \|\mathcal{F}_\kappa K_{1+\mathbf{i}\tau}\|_\infty \|f\|_{\kappa,2} \le C e^{c|\tau|} \|f\|_{\kappa,2}.$$

Thus, using Stein's interpolation theorem for analytic families of operators, we conclude that

$$\|f *_\kappa \mathcal{F}_\kappa d\sigma\|_{\kappa,p_\kappa'} = c\|R_0 f\|_{\kappa,p_\kappa'} \le C\|f\|_{\kappa,p_\kappa}.$$

Finally, by (3.1) and Young's inequality (2.23),

$$\|f *_\kappa \mathcal{F}_\kappa d\sigma\|_\infty \le C\|f\|_{\kappa,1} \|\mathcal{F}_\kappa d\sigma\|_\infty \le C\|f\|_{\kappa,1}.$$

Theorem 3.1 then follows by the Riesz-Thorin theorem. □

For $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^1(\mathbb{S}^{d-1}; h_\kappa^2 d\sigma)$, we define $Rf = \mathcal{F}_\kappa f\Big|_{\mathbb{S}^{d-1}}$ and

$$R^* g(x) = \int_{\mathbb{S}^{d-1}} g(\xi) E_\kappa(\mathbf{i}x, \xi) h_\kappa^2(\xi)\, d\sigma(\xi), \quad x \in \mathbb{R}^d.$$

It is easy to verify that

$$\langle Rf, g\rangle_{L^2(\mathbb{S}^{d-1}; h_\kappa^2)} = \langle f, R^* g\rangle_{L^2(\mathbb{R}^d; h_\kappa^2)}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \quad \forall g \in C(\mathbb{S}^{d-1}),$$

where the notation $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product of a given Hilbert space $\mathcal{H}$. Moreover, a straightforward calculation shows that for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$R^* R f(x) = c \int_{\mathbb{R}^d} f(z) T^x (\mathcal{F}_\kappa d\sigma)(z)\, h_\kappa^2(z)\, dz = c_\kappa' f *_\kappa (\mathcal{F}_\kappa d\sigma)(x). \tag{3.4}$$

Thus, Theorem 3.1 implies that $R^* R$ extends to a bounded operator from $L^p(\mathbb{R}^d; h_\kappa^2)$ to $L^{p'}(\mathbb{R}^d; h_\kappa^2)$ for $1 \le p \le p_\kappa$. Finally, observing that

$$\begin{aligned}\|R\|^2_{L^p(\mathbb{R}^d;h_\kappa^2)\to L^2(\mathbb{S}^{d-1};h_\kappa^2)} &= \|R^*\|^2_{L^2(\mathbb{S}^{d-1};h_\kappa^2)\to L^{p'}(\mathbb{R}^d;h_\kappa^2)} \\ &= \|R^* R\|_{L^p(\mathbb{R}^d;h_\kappa^2)\to L^{p'}(\mathbb{R}^d;h_\kappa^2)},\end{aligned}$$

we deduce the following.

Corollary 3.2. *If $1 \le p \le p_\kappa$, then R extends to a bounded operator from $L^p(\mathbb{R}^d; h_\kappa^2)$ to $L^2(\mathbb{S}^{d-1}; h_\kappa^2)$, and R^* extends to a bounded operator from $L^2(\mathbb{S}^{d-1}; h_\kappa^2)$ to $L^{p'}(\mathbb{R}^d; h_\kappa^2)$.*

4. Local restriction theorem for the Dunkl transform

The main purpose of this section is to show the following local restriction theorem.

Theorem 4.1. *Let $c_0 \in (0,1)$ be a parameter depending only on d and κ. If $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ is supported in a ball $B \subset \mathbb{R}^d$ of radius $\theta \ge c_0$ and $1 \le p \le p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$, then*

$$\Bigl(\int_B |f *_\kappa (\mathcal{F}_\kappa d\sigma)(x)|^{p'} h_\kappa^2(x)\,dx\Bigr)^{\frac{1}{p'}} \le C\Bigl(\frac{1}{\theta^{2\lambda_\kappa+1}}\int_B h_\kappa^2(y)\,dy\Bigr)^{1-\frac{2}{p}} \|f\|_{\kappa,p}, \tag{4.1}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Theorem 1.2, as stated in the introduction, is a direct consequence of Theorem 4.1 (see Corollary 4.2 below).

The proof of Theorem 4.1 is much more involved than that of the global restriction theorem. Indeed, a direct application of the techniques used in the proof of the classical Stein-Thomas restriction theorem would yield (4.1) for a smaller range of p only.

For the moment, we take Theorem 4.1 for granted and deduce the following corollary.

Corollary 4.2. *Let $c_0 \in (0,1)$ be a constant depending only on d and κ, and let B be the ball $B(\omega,\theta)$ centered at $\omega \in \mathbb{R}^d$ and having radius $\theta \ge c_0 > 0$.*

(i) *If $1 \le p \le p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$ and $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ is supported in the ball B, then*

$$\|\mathcal{F}_\kappa f\|_{L^2(\mathbb{S}^{d-1};h_\kappa^2)} \le C\Bigl(\frac{\theta^{2\lambda_\kappa+1}}{\int_B h_\kappa^2(y)\,dy}\Bigr)^{\frac{1}{p}-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}.$$

(ii) *If $2+\frac{2}{\lambda_\kappa} \le q \le \infty$ and $f \in L^2(\mathbb{S}^{d-1}; h_\kappa^2)$, then*

$$\Bigl\|\int_{\mathbb{S}^{d-1}} f(\xi)E_\kappa(\mathbf{i}\xi,\cdot)h_\kappa^2(\xi)\,d\sigma(\xi)\Bigr\|_{L^q(B;h_\kappa^2)} \le C\Bigl(\frac{\theta^{2\lambda_\kappa+1}}{\int_B h_\kappa^2(y)\,dy}\Bigr)^{\frac{1}{2}-\frac{1}{q}} \|f\|_{L^2(\mathbb{S}^{d-1};h_\kappa^2)},$$

where $L^q(B; h_\kappa^2)$ denotes the L^q-space defined with respect to the measure $h_\kappa^2(x)dx$ on the ball B.

Proof. Consider the operator $Tf := \Bigl((f\chi_B) *_\kappa (\mathcal{F}_\kappa d\sigma)\Bigr)\chi_B$. According to Theorem 4.1, T is a bounded operator from $L^p(\mathbb{R}^d; h_\kappa^2)$ to $L^{p'}(\mathbb{R}^d; h_\kappa^2)$ satisfying

$$\|Tf\|_{\kappa,p'} \le C\Bigl(\frac{1}{\theta^{2\lambda_\kappa+1}}\int_B h_\kappa^2(y)\,dy\Bigr)^{1-\frac{2}{p}} \|f\|_{\kappa,p} \tag{4.2}$$

for $1 \le p \le p_\kappa$. Next, define

$$Rf(x) := c_\kappa \int_B f(y)E_\kappa(-ix,y)h_\kappa^2(y)\,dy, \quad x \in \mathbb{S}^{d-1}, \quad f \in L^1(B; h_\kappa^2),$$

and

$$R^*f(x) = c_\kappa \int_{\mathbb{S}^{d-1}} f(y)E_\kappa(\mathbf{i}x,y)h_\kappa^2(y)\,d\sigma(y), \quad x \in B, \quad f \in L^1(h_\kappa^2; \mathbb{S}^{d-1}).$$

A straightforward calculation shows that

$$\langle Rf, g\rangle_{L^2(\mathbb{S}^{d-1};h_\kappa^2)} = \langle f, R^*g\rangle_{L^2(B;h_\kappa^2)}, \ \forall f \in L^1(B; h_\kappa^2), \ g \in L^1(\mathbb{S}^{d-1}; h_\kappa^2). \tag{4.3}$$

We further claim that

$$R^*Rf(x) = c'_\kappa Tf(x), \quad x \in B, \quad f \in L^1(B; h_\kappa^2), \tag{4.4}$$

where c'_κ is a positive constant depending only on d and κ. Indeed, for $f \in L^1(B; h_\kappa^2)$ and $x \in B$,

$$\begin{aligned} R^*Rf(x) &= c_\kappa \int_{\mathbb{S}^{d-1}} Rf(z) E_\kappa(\mathbf{i}x, z) h_\kappa^2(z)\, d\sigma(z) \\ &= c_\kappa^2 \int_B f(y) h_\kappa^2(y) \Big[\int_{\mathbb{S}^{d-1}} E_\kappa(x, \mathbf{i}z) E_\kappa(-y, \mathbf{i}z) h_\kappa^2(z)\, d\sigma(z) \Big]\, dy. \end{aligned}$$

Since (see, for instance, [7, p. 77])

$$\begin{aligned} &\int_{\mathbb{S}^{d-1}} E_\kappa(x, \mathbf{i}z) E_\kappa(-y, \mathbf{i}z) h_\kappa^2(z)\, d\sigma(z) \\ &= c''_\kappa V_\kappa \Big[j_{\lambda_\kappa - \frac{1}{2}} \big(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, \cdot \rangle}\big) \Big](y) = c''_\kappa T^y(\mathcal{F}_\kappa d\sigma)(x), \quad x, y \in \mathbb{R}^d, \end{aligned}$$

it then follows that

$$R^*Rf(x) = c'_\kappa \int_B f(y) T^y(\mathcal{F}_\kappa d\sigma)(x) h_\kappa^2(y)\, dy = c'_\kappa Tf(x), \quad x \in B,$$

which proves the claim (4.4).

Now using (4.2), (4.3), (4.4), and a standard duality argument, we obtain that for $1 \le p \le p_\kappa$,

$$\begin{aligned} \|R\|^2_{L^p(B; h_\kappa^2) \to L^2(\mathbb{S}^{d-1}; h_\kappa^2)} &= \|R^*\|^2_{L^2(\mathbb{S}^{d-1}; h_\kappa^2) \to L^{p'}(B; h_\kappa^2)} \\ &= c'_\kappa \|T\|_{L^p(B; h_\kappa^2) \to L^{p'}(B; h_\kappa^2)} < \infty. \end{aligned}$$

This proves the corollary. □

4.1. **Proof of Theorem 4.1.** Assume that $B = B(\omega, \theta)$ with $\omega = (\omega_1, \cdots, \omega_d) \in \mathbb{R}^d$. Set $I = \{j \in \{1, \cdots, d\} : |\omega_j| \le 4\theta\}$ and $I^c = \{1, \cdots, d\} \setminus I$. Let $\gamma = \gamma_B := \sum_{j \in I} \kappa_j$. We consider the following two cases:

Case 1 ($\gamma = |\kappa|$). In this case, $\kappa_j = 0$ whenever $|\omega_j| > 4\theta$. Thus,

$$\int_B h_\kappa^2(y)\, dy \sim \theta^d \prod_{j=1}^d (|\omega_j| + \theta)^{2\kappa_j} \sim \theta^{2\lambda_\kappa + 1},$$

which implies that $\Big(\frac{1}{\theta^{2\lambda_\kappa + 1}} \int_B h_\kappa^2(y)\, dy \Big)^{1 - \frac{2}{p}} \sim 1$. Thus, the stated estimate in this case follows directly from Theorem 3.1 proved in the last section.

Case 2 ($\gamma < |\kappa|$). In this case, there exists $1 \le j \le d$ such that $|\omega_j| \ge 4\theta$ and $\kappa_j > 0$. The proof in this case is more involved. Our goal is to show the estimate (4.2) with $Tf := \Big((f\chi_B) *_\kappa \mathcal{F}_\kappa d\sigma \Big) \chi_B$.

Let ξ_0 be an even C^∞-function on $\mathbb{R}$ that equals 1 on $[-1, 1]$ and equals zero outside the interval $[-2, 2]$. Let $\xi(x) = \xi_0(x) - \xi_0(2x)$. Define $\xi_j(x) = \xi(2^{-j}x) = \xi_0(2^{-j}x) - \xi_0(2^{-j+1}x)$ for $j \ge 1$ and $x \in \mathbb{R}$. Then $\sum_{j=0}^\infty \xi_j(x) = 1$ for all $x \in \mathbb{R}$.

Recall that

$$Tf(x) = c_\kappa \int_B f(y) K(x, y) h_\kappa^2(y)\, dy, \quad x \in B,$$

where

$$K(x,y) = T^y(\mathcal{F}_\kappa(d\sigma))(x) = c'_\kappa T^y\Big[j_{\lambda_\kappa-\frac12}(\|\cdot\|)\Big](x).$$

Thus, we may decompose the operator T as $T=\sum_{n=0}^\infty T_n$, where

$$T_nf(x) = \Big[\int_B f(y)K_n(x,y)h_\kappa^2(y)\,dy\Big]\chi_B(x) \tag{4.5}$$

and

$$K_n(x,y) = T^y\Big[(j_{\lambda_\kappa-\frac12}\xi_n)(\|\cdot\|)\Big](x). \tag{4.6}$$

First, we show that

$$\|T_nf\|_\infty \le C2^{-n(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}\Big(\int_B h_\kappa^2(y)\,dy\Big)^{-1}\|f\|_{\kappa,1}. \tag{4.7}$$

To this end, we need the following kernel estimates.

Lemma 4.3. *For $\alpha>-1$ and $n=0,1,\cdots$, set*

$$K_{\alpha,n}(x,y) := T^y\Big[(j_\alpha\xi_n)(\|\cdot\|)\Big](x), \quad x,y\in\mathbb{R}^d.$$

Then for $x,y\in\mathbb{R}^d$,

$$|K_{\alpha,n}(x,y)| \le C2^{-n(\alpha+\frac12-|\kappa|)}\prod_{j=1}^d(|x_jy_j|+2^n)^{-\kappa_j}. \tag{4.8}$$

The proof of Lemma 4.3 is long, so we postpone it until the next subsection. For the moment, we take it for granted and proceed with the proof of (4.7).

To show (4.7), we note that $|y_j|\sim|\omega_j|$ for $j\in I^c$ whenever $y\in B$. Thus, using Lemma 4.3 with $\alpha=\lambda_\kappa-\frac12$, we obtain that for $x,y\in B$,

$$\begin{aligned}|K_n(x,y)| &\le C2^{-n(\frac{d-1}{2})}\prod_{j=1}^d(|x_jy_j|+2^n)^{-\kappa_j}\\ &\le C2^{-n\frac{d-1}{2}}\Big[\prod_{j\in I^c}(|\omega_j|^2+\theta^2)^{-\kappa_j}\Big]\Big(\prod_{j\in I}2^{-n\kappa_j}\Big)\\ &\le C2^{-n(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}\Big(\int_B h_\kappa^2(z)\,dz\Big)^{-1}.\end{aligned}$$

Inequality (4.7) then follows by (4.5).

Next, we show that for $n\ge0$,

$$\|T_nf\|_{\kappa,2}\le C2^n\|f\|_{\kappa,2}. \tag{4.9}$$

To this end, we write

$$T_nf(x) = \Big[(f\chi_B)*_\kappa G_n\Big]\chi_B,$$

where

$$G_n(x) = cj_{\lambda_\kappa-\frac12}(\|x\|)\xi_n(\|x\|) = (\mathcal{F}_\kappa d\sigma)(x)\xi_n(\|x\|).$$

Let ψ be a radial Schwarz function on $\mathbb{R}^d$ such that $\mathcal{F}_\kappa\psi_{2^{-n}}(x)=\xi_n(x)$, where $\psi_{2^{-n}}(x):=2^{n(2\lambda_\kappa+1)}\psi(2^nx)$. Then

$$\mathcal{F}_\kappa G_n(x) = c_\kappa\int_{\mathbb{S}^{d-1}}T^y(\psi_{2^{-n}})(x)h_\kappa^2(y)\,d\sigma(y). \tag{4.10}$$

The proof of (4.9) relies on the following lemma, which gives an estimate of this last integral.

Lemma 4.4. *Assume that* $\varphi(x) = \varphi_0(\|x\|)$ *is a radial Schwarz function on* $\mathbb{R}^d$, *and let* $\varphi_{2^{-n}}(x) = 2^{n(2\lambda_\kappa+1)}\varphi(2^n x)$ *for* $n \in \mathbb{N}$. *Then for a.e.* $x \in \mathbb{R}^d$,

$$\Big|\int_{\mathbb{S}^{d-1}} \Big[T^y \varphi_{2^{-n}}(x)\Big] h_\kappa^2(y)\, d\sigma(y)\Big| \le C2^n.$$

The proof of Lemma 4.4 will be given in subsection 4.3.

By (4.10) and Lemma 4.4, it follows that for a.e. $x \in \mathbb{R}^d$,

$$|\mathcal{F}_\kappa G_n(x)| = c_\kappa \Big|\int_{\mathbb{S}^{d-1}} T^y \psi_{2^{-n}}(x) h_\kappa^2(y)\, d\sigma(y)\Big| \le C2^n.$$

Thus,

$$\|T_n f\|_{\kappa,2} \le \|\mathcal{F}_\kappa(f\chi_B)\mathcal{F}_\kappa(G_n)\|_{\kappa,2} \le C2^n \|f\|_{\kappa,2}.$$

Using (4.7), (4.9), and the Riesz-Thorin interpolation theorem, we obtain that

$$\|T_n f\|_{\kappa,p'} \le C2^{-n\left((\frac{d+1}{2}+\gamma)t-1\right)}\theta^{(2\gamma+d)t}A^{-t}\|f\|_{\kappa,p}, \tag{4.11}$$

where $A = \int_B h_\kappa^2(y)\, dy$, $t = \frac{1}{1+\lambda_\kappa} = \frac{2}{p} - 1$, and $p = p_\kappa = \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$.

On the other hand, using (4.7) and Hölder's inequality, we obtain that

$$\begin{aligned} \|T_n f\|_{\kappa,p'} &\le A^{\frac{1}{p'}}\|T_n f\|_\infty \le C2^{-n(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}A^{-\frac{1}{p}}\|f\|_{\kappa,1} \\ &\le C2^{-n(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}A^{1-\frac{2}{p}}\|f\|_{\kappa,p}. \end{aligned} \tag{4.12}$$

Finally, recalling that $Tf = \sum_{n=0}^\infty T_n f$, we obtain

$$\begin{aligned} \|Tf\|_{\kappa,p'} &\le \sum_{n=0}^\infty \|T_n f\|_{\kappa,p'} = \sum_{2^n \le \theta^2} + \cdots + \sum_{2^n > \theta^2} \cdots \\ &=: \Sigma_1 + \Sigma_2. \end{aligned}$$

For the first sum Σ_1, noticing that

$$1 - (\frac{d+1}{2} + \gamma)t = \frac{1}{1+\lambda_\kappa}(|\kappa| - \gamma) > 0,$$

we use (4.11) to obtain

$$\begin{aligned} \Sigma_1 &\le C\theta^{(2\gamma+d)t}A^{-t}\|f\|_{\kappa,p} \sum_{2^n \le \theta^2} 2^{n\left(-(\frac{d+1}{2}+\gamma)t+1\right)} \\ &\le C\theta^{\frac{2}{1+\lambda_\kappa}(|\kappa|-\gamma)}\theta^{\frac{2\gamma+d}{1+\lambda_\kappa}}A^{-\frac{1}{1+\lambda_\kappa}}\|f\|_{\kappa,p} = C\theta^{\frac{2\lambda_\kappa+1}{1+\lambda_\kappa}}A^{1-\frac{2}{p}}\|f\|_{\kappa,p}. \end{aligned}$$

For the second sum Σ_2, we use (4.12) and obtain

$$\begin{aligned} \Sigma_2 &\le C\sum_{2^n > \theta^2} 2^{-n(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}A^{1-\frac{2}{p}}\|f\|_{\kappa,p} \\ &\le C\theta A^{1-\frac{2}{p}}\|f\|_{\kappa,p} \le C\theta^{\frac{2\lambda_\kappa+1}{1+\lambda_\kappa}}A^{1-\frac{2}{p}}\|f\|_{\kappa,p}, \end{aligned}$$

where the last step uses the assumption $\theta \ge c_0 > 0$. This completes the proof of Theorem 4.1.

4.2. **Proof of Lemma 4.3.** Without loss of generality, we may assume that $\kappa_{\min} := \min_{1\le j\le d}\kappa_j > 0$. (The proof in this subsection with slight modifications works equally well for the case of $\kappa_{\min} = 0$). Let η denote either the function ξ_0 or the function ξ on $\mathbb{R}$ depending on whether $n = 0$ or $n \ge 1$. Then η is an even C_c^∞-function on $\mathbb{R}$ which is constant near the origin. According to (2.3) and (2.19), we have

(4.13)
$$K_{\alpha,n}(x,y) = c\int_{[-1,1]^d} j_\alpha(z(x,y,t))\eta(2^{-n}z(x,y,t))\prod_{j=1}^d (1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j,$$

where $z(x,y,t) = \sqrt{\|x\|^2+\|y\|^2-2\sum_{j=1}^d x_jy_jt_j}$.

Next, let $G_\alpha(u) = (\sqrt{u})^{-\alpha}J_\alpha(\sqrt{u}) = j_\alpha(\sqrt{u})$. Fix $x, y \in \mathbb{R}^d$ and set $F_\alpha(t) = G_\alpha\Bigl(u(x,y,t)\Bigr) = j_\alpha(z(x,y,t))$, where $u(x,y,t) = z(x,y,t)^2$ and $t = (t_1,\cdots,t_d) \in [-1,1]^d$. By (2.8) and (2.9), it is easily seen that for $\alpha \in \mathbb{R}$,

$$\frac{\partial}{\partial t_j}F_{\alpha-1}(t) = x_jy_jF_\alpha(t), \quad t = (t_1,\cdots,t_d)\in[-1,1]^d \tag{4.14}$$

and

$$|F_\alpha(t)| \le C(1+u(x,y,t))^{-\frac{\alpha}{2}-\frac14}, \quad t\in[-1,1]^d. \tag{4.15}$$

By (4.13), we may write

$$K_{\alpha,n}(x,y) = c_\kappa\int_{[-1,1]^d} F_\alpha(t)\widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right)\prod_{j=1}^d(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j,$$

where $\widetilde{\eta}(z) = \eta(\sqrt{|z|})$ for $z\in\mathbb{R}$. Since η is constant near the origin, it is easily seen that $\widetilde{\eta}\in C_c^\infty(\mathbb{R})$. Without loss of generality, we may assume that $|x_jy_j| \ge 2^n$, $j = 1,\cdots,m$ and $|x_jy_j| < 2^n$, $j = m+1,\cdots,d$ for some $1\le m\le d$ (otherwise, we re-index the sequence $\{x_jy_j\}_{j=1}^d$). Fix temporarily $t_{m+1},\cdots,t_d\in[-1,1]$, and set

$$\begin{aligned} &I(t_{m+1},\cdots,t_d)\\ &:= c_\kappa\int_{[-1,1]^m} F_\alpha(t)\widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right)\prod_{j=1}^m(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j. \end{aligned} \tag{4.16}$$

By Fubini's theorem, we then have

$$K_{\alpha,n}(x,y) = \int_{[-1,1]^{d-m}} I(t_{m+1},\cdots,t_d)\prod_{j=m+1}^d(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j.$$

Thus, for the proof of (4.8), it suffices to show that for each $t_{m+1},\cdots,t_d\in[-1,1]$,

$$|I(t_{m+1},\cdots,t_d)| \le C2^{-n(\alpha+\frac12-\sum_{j=1}^m\kappa_j)}\prod_{j=1}^m|x_jy_j|^{-\kappa_j}. \tag{4.17}$$

To show (4.17), let $\eta_0\in C^\infty(\mathbb{R})$ be such that $\eta_0(s) = 1$ for $|s|\le\frac12$ and $\eta_0(s) = 0$ for $|s|\ge1$, and let $\eta_1(s) = 1-\eta_0(s)$. Set $B_j := \frac{2^n}{|x_jy_j|}$ for $j = 1,\cdots,m$. Given $\varepsilon := (\varepsilon_1,\cdots,\varepsilon_m)\in\{0,1\}^m$, we define $\psi_\varepsilon : [-1,1]^m\to\mathbb{R}$ by

$$\psi_\varepsilon(t) := \widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right)\prod_{j=1}^m\eta_{\varepsilon_j}\Bigl(\frac{1-t_j^2}{B_j}\Bigr)(1+t_j)(1-t_j^2)^{\kappa_j-1},$$

where $t = (t_1, \cdots, t_m)$. We then split the integral in (4.16) into a finite sum to obtain

$$I(t_{m+1}, \cdots, t_d) = \sum_{\varepsilon \in \{0,1\}^m} \int_{[-1,1]^m} F_\alpha(t)\psi_\varepsilon(t)\, dt_1 \cdots dt_m =: \sum_{\varepsilon \in \{0,1\}^m} J_\varepsilon,$$

where

$$J_\varepsilon \equiv J_\varepsilon(t_{m+1}, \cdots, t_d) := \int_{[-1,1]^m} F_\alpha(t)\psi_\varepsilon(t)\, dt_1 \cdots dt_m. \tag{4.18}$$

Thus, it suffices to prove the estimate (4.17) with $I(t_{m+1}, \cdots, t_d)$ replaced by J_ε for each $\varepsilon \in \{0,1\}^m$, namely,

$$|J_\varepsilon(t_{m+1}, \cdots, t_d)| \le C 2^{-n(\alpha + \frac{1}{2} - \sum_{j=1}^m \kappa_j)} \prod_{j=1}^m |x_j y_j|^{-\kappa_j}. \tag{4.19}$$

By symmetry and Fubini's theorem, we need only prove (4.19) for the case of $\varepsilon_1 = \cdots = \varepsilon_{m_1} = 0$ and $\varepsilon_{m_1+1} = \cdots = \varepsilon_m = 1$ with m_1 being an integer in $[0, m]$. Write

$$\psi_\varepsilon(t) = \varphi(t) \prod_{j=1}^{m_1} \eta_0\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\kappa_j - 1} \tag{4.20}$$

with

$$\varphi(t) := \widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right) \prod_{j=m_1+1}^{m} \eta_1\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\kappa_j - 1}.$$

Since the support set of each $\eta_1\left(\frac{1-t_j^2}{B_j}\right)$ is a subset of $\{t_j : |t_j| \le 1 - \frac{1}{4}B_j\}$, we can use (4.14) and integration by parts $|\mathbf{l}| = \sum_{j=m_1+1}^m \ell_j$ times to obtain

$$\begin{aligned} &\Big|\int_{[-1,1]^{m-m_1}} F_\alpha(t)\varphi(t)\, dt_{m_1+1} \cdots dt_m\Big| \\ &\quad = c \prod_{j=m_1+1}^{m} |x_j y_j|^{-\ell_j} \Big|\int_{[-1,1]^{m-m_1}} F_{\alpha - |\mathbf{l}|}(t) \frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1} \cdots \partial^{\ell_m} t_m}\, dt\Big| \\ &\quad \le c \prod_{j=m_1+1}^{m} |x_j y_j|^{-\ell_j} \int_{[-1,1]^{m-m_1}} \Big|F_{\alpha - |\mathbf{l}|}(t)\Big| \Big|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1} \cdots \partial^{\ell_m} t_m}\Big|\, dt, \end{aligned}$$

where $\mathbf{l} = (\ell_{m_1+1}, \cdots, \ell_m) \in \mathbb{N}^{m-m_1}$ satisfies $\ell_j > \kappa_j$ for all $m_1 < j \le m$. Since $\widetilde{\eta}$ is supported in $(-4, 4)$, $\varphi(t)$ is zero unless

$$\begin{aligned} 4^{n+1} &\ge \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d |x_j y_j t_j| \\ &\ge \|\bar{x} - \bar{y}\|^2 + 2|x_j y_j|(1 - |t_j|) \ge 2|x_j y_j|(1 - |t_j|), \end{aligned} \tag{4.21}$$

for all $m_1 + 1 \le j \le m$; that is, $\frac{|x_j y_j|}{4^n} \le 2(1 - |t_j|)^{-1}$ for $j = m_1 + 1, \cdots, m$. On the other hand, note that the derivative of the function $\eta_1\Big(\frac{1-t_j^2}{B_j}\Big)$ in the variable

t_j is supported in $\{t_j : \frac{1}{2}B_j \le 1 - t_j^2 \le B_j\}$. Consequently, by the Lebnitz rule, we conclude that

$$\Big|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\Big| \le c\prod_{j=m_1+1}^{m}(1-|t_j|)^{\kappa_j-\ell_j-1}.$$

Finally, recall that $\widetilde{\eta} \in C_c^\infty(\mathbb{R})$ for all $n \ge 0$ and that $\widetilde{\eta}$ is zero near the origin for $n \ge 1$. This implies that for all $n \ge 0$, $\widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right) = 0$ unless $c_1 4^n < 1+u(x,y,t) < c_2 4^n$ for some absolute constants $c_1, c_2 > 0$. It then follows by (4.15) that

$$\Big|F_{\alpha-|\mathbf{l}|}(t)\Big| \le c(1+u(x,y,t))^{-\frac{\alpha-|\mathbf{l}|}{2}-\frac{1}{4}} \sim 2^{-n(\alpha-|\mathbf{l}|+\frac{1}{2})}.$$

Thus,

$$\begin{aligned}
&\int_{[-1,1]^{m-m_1}}\Big|F_{\alpha-|\mathbf{l}|}(t)\Big|\Big|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\Big|\,dt_{m_1+1}\cdots dt_m\\
&\quad\le c2^{-n(\alpha-|\mathbf{l}|+\frac{1}{2})}\prod_{j=m_1+1}^{m}\int_0^{1-\frac{B_j}{4}}(1-t_j)^{\kappa_j-\ell_j-1}\,dt_j\\
&\quad\le c2^{-n(\alpha-|\mathbf{l}|+\frac{1}{2})}\prod_{j=m_1+1}^{m}B_j^{\kappa_j-\ell_j} \le c2^{-n(\alpha+\frac{1}{2}-\sum_{j=m_1+1}^m\kappa_j)}\prod_{j=m_1+1}^{m}|x_jy_j|^{\ell_j-\kappa_j}.
\end{aligned} \tag{4.22}$$

Thus, using (4.18) and Fubini's theorem, we obtain that

$$\begin{aligned}
|J_\varepsilon| &\le \int_{[-1,1]^{m_1}}\Big|\int_{[-1,1]^{m-m_1}}F_\alpha(t)\varphi(t)\,dt_{m_1+1}\cdots dt_m\Big|\\
&\quad\times\prod_{j=1}^{m_1}\eta_0\left(\frac{1-t_j^2}{B_j}\right)(1+t_j)(1-t_j^2)^{\kappa_j-1}dt_j\\
&\le c2^{-n(\alpha+\frac{1}{2}-\sum_{j=m_1+1}^m\kappa_j)}\prod_{j=m_1+1}^{m}|x_jy_j|^{-\kappa_j}\prod_{j=1}^{m_1}\int_{1-B_j\le|t_j|\le1}(1-|t_j|)^{\kappa_j-1}\,dt_j\\
&\le c2^{-n(\alpha+\frac{1}{2}-\sum_{j=1}^m\kappa_j)}\prod_{j=1}^{m}|x_jy_j|^{-\kappa_j},
\end{aligned}$$

where we used (4.22) and the fact that $\eta_0\left(\frac{1-t_j^2}{B_j}\right)$ is supported in $\{t_j : 1 - B_j \le |t_j| \le 1\}$ for $1 \le j \le m_1$ in the second step. This yields the desired estimate (4.19) and hence completes the proof of Lemma 4.3.

We conclude this subsection with the following useful corollary.

Corollary 4.5. *For $\alpha > |\kappa| - \frac{1}{2}$ and a.e. $x, y \in \mathbb{R}^d$,*

$$\Big|T^y\big(j_\alpha(\|\cdot\|)\big)(x)\Big| \le C\frac{\prod_{j=1}^d(|x_jy_j|+1+\|\bar{x}-\bar{y}\|)^{-\kappa_j}}{(1+\|\bar{x}-\bar{y}\|)^{\alpha+\frac{1}{2}-|\kappa|}}.$$

Proof. Set $K_\alpha(x,y) = T^y\Big[j_\alpha(\|\cdot\|)\Big](x)$. We then write

$$K_\alpha(x,y) = \sum_{n=0}^\infty T^y\Big[(j_\alpha\xi_n)(\|\cdot\|)\Big](x) =: \sum_{n=0}^\infty K_{\alpha,n}(x,y). \tag{4.23}$$

It is easily seen that $K_{\alpha,n}(x,y)$ is supported in $\{(x,y) : \|\bar{x}-\bar{y}\| \le 2^{n+1}\}$. Thus, by (4.23) and (4.8), it follows that

$$\begin{aligned} |K_\alpha(x,y)| &\le C \sum_{2^{n+1}\ge \max\{\|\bar{x}-\bar{y}\|,1\}} 2^{-n(\alpha+\frac{1}{2}-|\kappa|)} \prod_{j=1}^d (|x_j y_j| + 2^n)^{-\kappa_j} \\ &\le C(1+\|\bar{x}-\bar{y}\|)^{-(\alpha+\frac{1}{2}-|\kappa|)} \prod_{j=1}^d (1+\|\bar{x}-\bar{y}\| + |x_j y_j|)^{-\kappa_j}, \end{aligned}$$

where the last step uses the assumption that $\alpha > |\kappa| - \frac{1}{2}$. □

4.3. **Proof of Lemma 4.4.** For the proof of Lemma 4.4, we need an additional lemma.

Lemma 4.6. *Assume that $\varphi(x) = \phi(\|x\|)$ is a radial function on $\mathbb{R}^d$ satisfying that*

$$|\varphi(x)| \le C(1+\|x\|)^{-\ell-|\kappa|-\varepsilon}, \quad \forall x \in \mathbb{R}^d,$$

for some $\ell > 0$ and $\varepsilon > 0$. Then for a.e. $x, y \in \mathbb{R}^d$ and any $t > 0$,

$$\begin{aligned} |T^y \varphi_t(x)| &\le \frac{C_\varepsilon t^{-d} \prod_{j=1}^d (|x_j y_j| + t^2 + \|\bar{x}-\bar{y}\|^2)^{-\kappa_j}}{\left(1+t^{-1}\|\bar{x}-\bar{y}\|\right)^\ell} \\ &\le C_\varepsilon \left(1+t^{-1}\|\bar{x}-\bar{y}\|\right)^{-\ell} t^{-d} \prod_{j=1}^d (|y_j|+t)^{-2\kappa_j}, \end{aligned} \tag{4.24}$$

where $\varphi_t(x) = t^{-2\lambda_\kappa-1}\varphi(t^{-1}x)$.

Proof. The second inequality in (4.24) can be deduced from the first one by straightforward calculations. Hence, it suffices to show the first inequality in (4.24). Without loss of generality, we may assume that $\kappa_j > 0$ for each $j = 1, \cdots, d$. (All the general constants are uniform in κ when $\kappa_j \to 0$.)

Note that for $t > 0$,

$$\begin{aligned} (T^y \varphi_t)(x) &= t^{-2\lambda_\kappa-1} V_\kappa \Big[\phi\big(\sqrt{\|t^{-1}x\|^2 + \|t^{-1}y\|^2 - 2\langle t^{-1}y, \cdot\rangle}\big)\Big](t^{-1}x) \\ &= t^{-2\lambda_\kappa-1}\Big(T^{t^{-1}y}\varphi\Big)(t^{-1}x). \end{aligned} \tag{4.25}$$

Thus, it suffices to show (4.24) for $t = 1$.

Indeed,

$$\begin{aligned} |T^y\varphi(x)| &= c_\kappa \Big| \int_{[-1,1]^d} \phi\Big(\sqrt{\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j y_j t_j}\Big) \prod_{j=1}^d (1-t_j^2)^{\kappa_j-1}(1+t_j)\, dt_j \Big| \\ &\le C \int_{[-1,1]^d} \Big(1+\|x\|^2+\|y\|^2-2\sum_{j=1}^d x_j y_j t_j\Big)^{-\ell-|\kappa|-\varepsilon} \prod_{j=1}^d (1-t_j^2)^{\kappa_j-1}(1+t_j)\, dt_j. \end{aligned}$$

Since for each fixed $t = (t_1, \cdots, t_d) \in [-1,1]^d$,

$$\begin{aligned}\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j y_j t_j &\geq \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d |x_j y_j||t_j| \\ &= \|\bar{x} - \bar{y}\|^2 + 2\sum_{j=1}^d (1 - |t_j|)|x_j y_j| \\ &\geq \|\bar{x} - \bar{y}\|^2 + 2\max_{1\leq j\leq d} |x_j y_j|(1 - |t_j|),\end{aligned}$$

it follows that

$$\begin{aligned}&\Big(1 + \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j y_j t_j\Big)^{-\ell-|\kappa|-\varepsilon} \\ &\leq C(1 + \|\bar{x} - \bar{y}\|^2)^{-\ell} \prod_{j=1}^d \Big(1 + \|\bar{x} - \bar{y}\|^2 + 2|x_j y_j|(1 - |t_j|)\Big)^{-\ell_j},\end{aligned}$$

where $\ell_j = \kappa_j + \varepsilon_j$, $\varepsilon_j > 0$, and $\sum_{j=1}^d \varepsilon_j = \varepsilon$. This implies that

$$|T^y \varphi(x)| \leq C(1 + \|\bar{x} - \bar{y}\|^2)^{-\ell} \prod_{j=1}^d \int_0^1 (1 + \|\bar{x} - \bar{y}\|^2 + 2|x_j y_j| s)^{-\ell_j} s^{\kappa_j - 1}\, ds.$$

To complete the proof, we just need to observe that

$$\begin{aligned}&c_{\kappa_j} \int_0^1 (1 + \|\bar{x} - \bar{y}\|^2 + 2|x_j y_j| s)^{-\ell_j} s^{\kappa_j - 1}\, ds \\ &\quad= c'|x_j y_j|^{-\kappa_j} \int_0^{|x_j y_j|} (1 + \|\bar{x} - \bar{y}\|^2 + 2s)^{-\ell_j} s^{\kappa_j - 1}\, ds \\ &\quad\leq C \min\Big\{(1 + \|\bar{x} - \bar{y}\|)^{-\kappa_j}, |x_j y_j|^{-\kappa_j}\Big\} \leq C(1 + |x_j y_j| + \|\bar{x} - \bar{y}\|)^{-\kappa_j}.\end{aligned}$$

□

We are now in a position to show Lemma 4.4.

Proof of Lemma 4.4. By Lemma 4.6, for any $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}&\Big|\int_{\mathbb{S}^{d-1}} \Big[T^y \varphi_{2^{-n}}(x)\Big] h_\kappa^2(y)\, d\sigma(y)\Big| \\ &\quad\leq C \sum_{\sigma\in\mathbb{Z}_2^d} \int_{\mathbb{S}^{d-1}} \Big(1 + 2^n\|x\sigma - y\|\Big)^{-\ell} 2^{nd} \Big(\prod_{j=1}^d (|y_j| + 2^{-n})^{-2\kappa_j}\Big)\ h_\kappa^2(y)\, d\sigma(y) \\ &\quad\leq C \sum_{\sigma\in\mathbb{Z}_2^d} 2^{nd} \int_{\mathbb{S}^{d-1}} \Big(1 + 2^n\|x\sigma - y\|\Big)^{-\ell} d\sigma(y).\end{aligned}$$

Thus, it is sufficient to show that for a sufficiently large ℓ (say, $\ell \geq d+1$) and any $x \in \mathbb{R}^d$,

$$2^{nd} \int_{\mathbb{S}^{d-1}} \Big(1 + 2^n\|x - y\|\Big)^{-\ell} d\sigma(y) \leq C2^n. \tag{4.26}$$

Without loss of generality, we may assume that $\frac{1}{2} \le \|x\| \le 2$, since otherwise the desired estimate (4.26) holds trivially. Writing $x = \|x\|x'$, we have that for $y \in \mathbb{S}^{d-1}$,

$$\|x-y\|^2 = (\|x\|-1)^2 + 2\|x\|(1-\langle x', y\rangle) \ge 1 - \langle x', y\rangle.$$

Thus,

$$\begin{aligned} 2^{nd}\int_{\mathbb{S}^{d-1}} \Big(1+2^n\|x-y\|\Big)^{-\ell} d\sigma(y) &\le C2^{nd}\int_{\mathbb{S}^{d-1}} \Big(1+4^n(1-\langle x', y\rangle)\Big)^{-\ell/2} d\sigma(y) \\ &\le C2^n. \end{aligned}$$

□

5. Weighted Littlewood-Paley inequality

We start with the following definition.

Definition 5.1. Assume that Ψ is a radial Schwarz function on $\mathbb{R}^d$ such that $\mathcal{F}_\kappa\Psi(0) = 0$. Let $\Psi_j(x) = 2^{j(2\lambda_\kappa+1)}\Psi(2^j x)$ for $j \in \mathbb{Z}$. The square function $Lf \equiv L_\Psi f$ associated with the function Ψ is defined by

$$Lf(x) \equiv L_\Psi f(x) := \Big(\sum_{j\in\mathbb{Z}} |f *_\kappa \Psi_j(x)|^2\Big)^{\frac{1}{2}}, \quad x \in \mathbb{R}^d, \quad f \in \mathcal{S}'(\mathbb{R}^d). \tag{5.1}$$

Next, we recall that $d\mu_\kappa = h_\kappa^2(x)dx$ is a Borel measure on $\mathbb{R}^d$ satisfying

$$\mu_\kappa(B(x,r)) \sim r^d \prod_{j=1}^d (|x_j|+r)^{2\kappa_j}, \quad \forall x \in \mathbb{R}^d, \quad \forall r > 0. \tag{5.2}$$

A weight function w on $\mathbb{R}^d$ is said to be $\mathbb{Z}_2^d$-invariant if $w(x\sigma) = w(x)$ for all $\sigma \in \mathbb{Z}_2^d$ and $x \in \mathbb{R}^d$.

This section is devoted to the proof of the following Littlewood-Paley inequality.

Theorem 5.2. *Let $L \equiv L_\Psi$ denote the square function associated with a radial Schwarz function Ψ as defined in Definition 5.1. Assume that $1 < p < \infty$ and w is an A_p-weight on the homogeneous space $(\mathbb{R}^d, d\mu_\kappa)$ that is $\mathbb{Z}_2^d$–invariant. Then*

$$\|Lf\|_{L^p(wd\mu_\kappa)} \leqslant C\|f\|_{L^p(wd\mu_\kappa)},$$

where C depends only on Ψ and the A_p constant of w. If, in addition,

$$\sum_{j\in\mathbb{Z}} |(\mathcal{F}_\kappa\Psi)(2^{-j}\xi)|^2 = 1 \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\},$$

then

$$\|Lf\|_{L^p(wd\mu_\kappa)} \sim \|f\|_{L^p(wd\mu_\kappa)}.$$

We will also need the following modified Hardy-Littlewood maximal function on the homogeneous space $(\mathbb{R}^d, d\mu_\kappa)$:

$$M_\kappa f(x) := \sup_B \frac{1}{\mu_\kappa(B)} \int_{\widetilde{B}} |f(y)| d\mu_\kappa(y),$$

where the supremum is taken over all the balls B in $\mathbb{R}^d$ such that $x \in \widetilde{B} = \bigcup_{\sigma\in\mathbb{Z}_2^d}\{x\sigma : x \in B\}$. Since μ_κ is a $\mathbb{Z}_2^d$-invariant doubling measure on $\mathbb{R}^d$, it follows that

$$\mu_\kappa\Big\{x \in \mathbb{R}^d : M_\kappa f(x) > \alpha\Big\} \le C_\kappa \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0, \tag{5.3}$$

and

$$\|M_\kappa f\|_{\kappa,p} \le C_p \|f\|_{\kappa,p}, \quad 1 < p \le \infty. \tag{5.4}$$

As a consequence of Theorem 5.2, we have the following.

Corollary 5.3. *Let g be a function on $\mathbb{R}^d$ such that $M_\kappa g(x) < \infty$ for a.e. $x \in \mathbb{R}^d$, and let $Lf = L_\Psi f$ be the square function as defined in* (5.1). *If $0 < \delta < 1$ and $1 < p < \infty$, then*

$$\|Lf\|_{L^p((M_\kappa g)^\delta d\mu_\kappa)} \leqslant C \|f\|_{L^p((M_\kappa g)^\delta d\mu_\kappa)},$$

where the constant C is independent of f.

5.1. **Proof of Theorem 5.2.** Recall that

$$f *_\kappa \Psi_j(x) = \int_{\mathbb{R}^d} f(y) T^y \Psi_j(x) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d, \quad j \in \mathbb{Z},$$

with T^y denoting the generalized translation operator. The proof of Theorem 5.2 relies on the following estimates of the vector-valued kernel $K(x,y) = \{T^y(\Psi_j)(x)\}_{j\in\mathbb{Z}}$.

Lemma 5.4.

(i) *For $x, y \in \mathbb{R}^d$,*

$$\sum_{j\in\mathbb{Z}} |T^y(\Psi_j)(x)| \le \frac{C}{\mathrm{meas}_\kappa(B(x, \|\bar{x} - \bar{y}\|))}. \tag{5.5}$$

(ii) *If $x, y, z \in \mathbb{R}^d$ and $\|x - z\| \le \frac{1}{2}\|\bar{x} - \bar{y}\|$, then*

$$\sum_{j\in\mathbb{Z}} \Big| T^y(\Psi_j)(z) - T^y(\Psi_j)(x) \Big| \le C \frac{\|x - z\|}{\|\bar{x} - \bar{y}\|} \frac{1}{\mathrm{meas}_\kappa(B(x, \|\bar{x} - \bar{y}\|))}. \tag{5.6}$$

The proof of Lemma 5.4 will be given in subsection 5.2. For the moment, we take it for granted and proceed with the proof of Theorem 5.2.

Proof of Theorem 5.2. We start with the proof of the upper estimate:

$$\|Lf\|_{L^p(w d\mu_\kappa)} \leqslant C \|f\|_{L^p(w d\mu_\kappa)}. \tag{5.7}$$

We will use Lemma 5.4 and the theory of vector-valued Calderón-Zygmund operators, viewing the square function Lf as the ℓ^2-norm of a vector-valued operator $\vec{L}f := \{Lf(x,j)\}_{j\in\mathbb{Z}}$ given by

$$\vec{L}f(x,j) = \int_{\mathbb{R}^d} f(y) K_j(x,y)\, d\mu_\kappa(y), \quad x \in \mathbb{R}^d, \quad j \in \mathbb{Z},$$

with $K_j(x,y) = T^y \Psi_j(x)$. The only problem lies in that the function $(x,y) \mapsto \|\bar{x} - \bar{y}\|$ is not a quasi-metric on $\mathbb{R}^d$, and hence Lemma 5.4 cannot be applied to conclude that the kernel $\{K_j(x,y)\}_{j\in\mathbb{Z}}$ is a vector-valued Calderón-Zygmund operator on the homogeneous space $(\mathbb{R}^d, d\mu_\kappa)$.

To overcome the difficulty, we set $R_\sigma = \{x\sigma : \ x \in \mathbb{R}^d_+\}$, $\sigma \in \mathbb{Z}_2^d$ with $\mathbb{R}^d_+ = [0,\infty)^d$, and write

$$\begin{aligned}
\vec{L}f(x,j) &= \sum_{\sigma'\in\mathbb{Z}_2^d} \sum_{\sigma\in\mathbb{Z}_2^d} \Big(\int_{R_\sigma} K_j(x,y) f(y)\, d\mu_\kappa(y) \Big) \chi_{R_{\sigma'}}(x) \\
&= \sum_{\sigma'\in\mathbb{Z}_2^d} \sum_{\sigma\in\mathbb{Z}_2^d} \Big(\int_{\mathbb{R}^d_+} K_j(x\sigma', y\sigma\sigma') f(y\sigma)\, d\mu_\kappa(y) \Big) \chi_{\mathbb{R}^d_+}(x\sigma'),
\end{aligned}$$

where the last step uses the $\mathbb{Z}_2^d$-invariance of the kernel $K_j(x,y)$ (that is, $K_j(x\sigma, y\sigma) = K_j(x,y)$ for all $x, y \in \mathbb{R}^d$ and all $\sigma \in \mathbb{Z}_2^d$). Thus, setting $f_\sigma(y) = f(y\sigma)$ for $\sigma \in \mathbb{Z}_2^d$, we have

$$\vec{L}f(x,j) = \sum_{\sigma' \in \mathbb{Z}_2^d} \sum_{\sigma \in \mathbb{Z}_2^d} (\vec{L}_{\sigma\sigma'} f_\sigma)(x\sigma', j),$$

where

$$\begin{aligned} \vec{L}_\sigma f(x,j) &= \Big(\int_{\mathbb{R}_+^d} K_j(x, y\sigma) f(y)\, d\mu_\kappa(y) \Big) \chi_{\mathbb{R}_+^d}(x) \\ &= (f_\sigma \chi_{R_\sigma}) *_\kappa \Psi_j(x) \chi_{\mathbb{R}_+^d}(x), \quad x \in \mathbb{R}^d, \quad \sigma \in \mathbb{Z}_2^d. \end{aligned} \tag{5.8}$$

Thus, by Minkowskii's inequality and the $\mathbb{Z}_2^d$-invariance of w, we reduce the proof of (5.7) to showing that for each $\sigma \in \mathbb{Z}_2^d$,

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |\vec{L}_\sigma f(\cdot, j)|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}_+^d; w d\mu_\kappa)} \le C_p \|f\|_{L^p(\mathbb{R}_+^d; w d\mu_\kappa)}. \tag{5.9}$$

Let $\rho(x,y) = \|x-y\|$ denote the Euclidean metric on $\mathbb{R}^d$. Since both w and $d\mu_\kappa$ are $\mathbb{Z}_2^d$-invariant on $\mathbb{R}^d$, it is easily seen that $(\mathbb{R}_+^d, \rho, d\mu_\kappa)$ is also a homogeneous space on which w is an A_p-weight. Thus, for the proof of (5.9), it suffices to show that $\vec{L}_\sigma$ is an $\ell^2(\mathbb{Z})$-valued Calderón-Zygmund operator on the homogeneous space $(\mathbb{R}_+^d, \rho, d\mu_\kappa)$.

To see this, note first that $K_j(x, y\sigma) = K_j(y, x\sigma)$ and $\|\bar{x} - \bar{y}\| = \|x-y\|$ for any $x, y \in \mathbb{R}_+^d$ and $\sigma \in \mathbb{Z}_2^d$. Thus, using Lemma 5.4 and the fact that $\|\cdot\|_{\ell^2(\mathbb{Z})} \le \|\cdot\|_{\ell^1(\mathbb{Z})}$, we conclude that $\{K_j(x,y)\}_{j\in\mathbb{Z}}$ is an $\ell^2(\mathbb{Z})$-valued Calderón-Zygmund kernel on the homogeneous space $(\mathbb{R}_+^d, \rho, d\mu_\kappa)$. On the other hand, since $\sup_{\xi\in\mathbb{R}^d} \sum_{j\in\mathbb{Z}} |\mathcal{F}_\kappa \Psi(\xi)|^2 \le C < \infty$, using (5.8) and Plancherel's theorem, we conclude that $\vec{L}_\sigma$ is bounded on $L^2(\mathbb{R}_+^d; d\mu_\kappa)$; that is,

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |\vec{L}_\sigma f(\cdot, j)|^2 \Big)^{1/2} \Big\|_{L^2(\mathbb{R}_+^d; d\mu_\kappa)} \le C_p \|f\|_{L^2(\mathbb{R}_+^d; d\mu_\kappa)}.$$

Thus, $\vec{L}_\sigma$ is an ℓ^2-valued Calderón-Zygmund operator on the homogeneous space $(\mathbb{R}_+^d; d\mu_\kappa)$. This proves (5.9) and hence the upper estimate (5.7).

Finally, we point out that the inverse inequality

$$\|Lf\|_{L^p(w d\mu_\kappa)} \ge C \|f\|_{L^p(w d\mu_\kappa)} \tag{5.10}$$

follows by a duality argument. Indeed, Plancherel's theorem and the identity $\sum_{j\in\mathbb{Z}} |\mathcal{F}_\kappa \Psi(\xi)|^2 = 1, \;\; \xi \in \mathbb{R}^d \setminus \{0\}$, imply that for all $f, g \in L^2(\mathbb{R}^d, d\mu_\kappa)$,

$$\int_{\mathbb{R}^d} \langle \vec{L}f(x), \vec{L}g(x) \rangle_{\ell^2(\mathbb{Z})}\, d\mu_\kappa(x) = \langle f, g \rangle_{L^2(d\mu_\kappa)}. \tag{5.11}$$

Thus, for $g \in L^{p'}(w d\mu_\kappa) \cap \mathcal{S}(\mathbb{R}^d)$ with $\|g\|_{L^{p'}(w d\mu_\kappa)} = 1$,

$$\begin{aligned} |\langle f, gw \rangle_{L^2(d\mu_\kappa)}| &= \Big| \int_{\mathbb{R}^d} \langle \vec{L}f, \vec{L}(gw) \rangle_{\ell^2(\mathbb{Z})}\, d\mu_\kappa \Big| \\ &\le \int_{\mathbb{R}^d} \|\vec{L}f\|_{\ell^2} \|\vec{L}(gw)\|_{\ell^2} w\, d\mu_\kappa \\ &\le C \|Lf\|_{L^p(w\, d\mu_\kappa)} \|L(gw)\|_{L^{p'}(w^{-\frac{1}{p-1}} d\mu_\kappa)}. \end{aligned}$$

Since $w^{-\frac{1}{p-1}}$ is a $\mathbb{Z}_2^d$–invariant $A_{p'}$ weight on $(\mathbb{R}^d, d\mu_\kappa)$, it follows by the already proven inequality (5.7) that

$$\|L(gw)\|_{L^{p'}(w^{-\frac{1}{p-1}}d\mu_\kappa)} \le C\|gw\|_{L^{p'}(w^{-\frac{1}{p-1}}d\mu_\kappa)} = C\|g\|_{L^{p'}(w\,d\mu_\kappa)} = C.$$

Thus, taking the supremum over all $g \in L^{p'}(wd\mu_\kappa) \cap \mathcal{S}(\mathbb{R}^d)$ with $\|g\|_{L^{p'}(wd\mu_\kappa)} = 1$ yields the stated inverse inequality (5.7). $\square$

5.2. **Proof of Lemma 5.4.** For simplicity, we set $\rho := \|\bar{x} - \bar{y}\|$ for $x, y \in \mathbb{R}^d$. Throughout the proof, the letter ℓ denotes a large positive number (say, $\ell > d+2|\kappa|$) whose exact value is not important.

We start with the proof of (i). Using Lemma 4.6, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|T^y\Psi_j(x)| &= \sum_{j\in\mathbb{Z}}|T^{-x}\Psi_j(-y)| \leqslant C\sum_{j\in\mathbb{Z}}\frac{2^{jd}\prod_{i=1}^d(|x_i|+2^{-j})^{-2\kappa_i}}{(1+2^j\|\bar{x}-\bar{y}\|)^\ell}\\ &= C\Big[\sum_{\{j\in\mathbb{Z}:2^j\rho\le1\}} + \sum_{\{j\in\mathbb{Z}:2^j\rho>1\}}\Big] =: C\Big[\Sigma_1+\Sigma_2\Big].\end{aligned}$$

For the first sum, a straightforward calculation shows that

$$\begin{aligned}\Sigma_1 &\le C\sum_{\{j\in\mathbb{Z}:2^j\rho\le1\}} 2^{jd}\prod_{i=1}^d(|x_i|+\rho)^{-2\kappa_i}\\ &\le C\rho^{-d}\prod_{i=1}^d(|x_i|+\rho)^{-2\kappa_i} \le \frac{C}{\mathrm{meas}_\kappa(B(x,\|\bar{x}-\bar{y}\|))}.\end{aligned}$$

To estimate the second sum, we set $J = \{j : \ 1 \leqslant j \leqslant d,\ |x_j| \geqslant \rho\}$, $J^c = \{1, 2, \cdots, d\} \setminus J$, and write $|\kappa|_{J^c} = \sum_{j\in J^c}\kappa_j$. We then obtain

$$\begin{aligned}\Sigma_2 &\leqslant C\sum_{\{j\in\mathbb{Z}:2^j\rho>1\}} 2^{jd}2^{2j|\kappa|_{J^c}}(2^j\rho)^{-\ell}\Big(\prod_{i\in J}(|x_i|+\rho)^{-2\kappa_i}\Big)\\ &\le C\rho^{-d}\rho^{-2|\kappa|_{J^c}}\Big(\prod_{i\in J}(|x_i|+\rho)^{-2\kappa_i}\Big)\\ &\le C\rho^{-d}\prod_{i=1}^d(|x_i|+\rho)^{-2\kappa_i} \le C\frac{C}{\mathrm{meas}_\kappa(B(x,\|\bar{x}-\bar{y}\|))}.\end{aligned}$$

Putting the above together completes the proof of assertion (i).

Next, we show assertion (ii). Without loss of generality, we may assume that $\kappa_j > 0$ for each $j \in \{1, \cdots, d\}$. Write $\Psi(x) = \Phi(\|x\|)$ with $\Phi \in \mathcal{S}(\mathbb{R})$, and set

$$u(x,y,t) = \sqrt{\|x\|^2+\|y\|^2-2\sum_{j=1}^d x_jy_jt_j},\quad x,y\in\mathbb{R}^d,\quad t\in[-1,1]^d.$$

We then use Corollary 2.6 to obtain that for $n = 1, \cdots, d$,

$$\frac{\partial}{\partial x_n}\Big(T^y(\Psi)(x)\Big) = c_\kappa\int_{[-1,1]^d}\Phi'\big(u(x,y,t)\big)\frac{x_n-y_nt_n}{u}\prod_{j=1}^d(1-t_j^2)^{\kappa_j-1}(1+t_j)dt_j.$$

Since for each $t=(t_1,\cdots,t_d)\in[-1,1]^d$ and $n\in\{1,\cdots,d\}$,

$$u^2(x,y,t)=\sum_{j=1}^{d}(x_j^2+y_j^2-2x_jy_jt_j)\geqslant x_n^2+y_n^2-2x_ny_nt_n\geqslant(x_n-y_nt_n)^2,$$

it follows that $|x_n-y_nt_n|\leqslant|u(x,y,t)|$ for all $x,y\in\mathbb{R}^d$ and $t\in[-1,1]^d$. Thus, using Lemma 4.6, we conclude that for any $\ell>0$,

$$\begin{aligned}\left|\frac{\partial}{\partial x_n}T^y(\Psi)(x)\right|&\leqslant C_\kappa\int_{[-1,1]^d}|\Phi'\big(u(x,y,t)\big)|\prod_{j=1}^{d}(1-t_j^2)^{\kappa_j-1}(1+t_j)dt_j\\&\leqslant C(1+\|\bar{x}-\bar{y}\|)^{-\ell}\prod_{i=1}^{d}(|y_i|+1)^{-2\kappa_i}.\end{aligned}$$

By (4.25), this further implies that

$$\left|\frac{\partial}{\partial x_n}T^y(\Psi_j)(x)\right|\leqslant C2^{j(d+1)}\left(1+2^j\|\bar{x}-\bar{y}\|\right)^{-\ell}\prod_{i=1}^{d}(|y_i|+2^{-j})^{-2\kappa_i}.$$

It then follows by the mean value theorem that

$$\begin{aligned}|T^y(\Psi_j)(z)-T^y(\Psi_j)(x)|&\leqslant\|x-z\|\|\nabla(T^y(\Psi_j))(\xi)\|\\&\leqslant C2^{j(d+1)}\|x-z\|\left(1+2^j\|\bar{\xi}-\bar{y}\|\right)^{-\ell}\prod_{i=1}^{d}(|y_i|+2^{-j})^{-2\kappa_i},\end{aligned}$$

for some $\xi=\theta x+(1-\theta)z$ with $\theta\in[0,1]$. Since $\|x-z\|\le\frac{1}{2}\|\bar{x}-\bar{y}\|$, we have

$$\|\bar{\xi}-\bar{y}\|\geqslant\|\bar{x}-\bar{y}\|-\|\bar{x}-\bar{\xi}\|\geqslant\|\bar{x}-\bar{y}\|-\|x-\xi\|\geqslant\|\bar{x}-\bar{y}\|-\|x-z\|\geqslant\frac{1}{2}\|\bar{x}-\bar{y}\|.$$

This implies that

$$|T^y(\Psi_j)(z)-T^y(\Psi_j)(x)|\leqslant C2^{j(d+1)}\|x-z\|\left(1+2^j\rho\right)^{-\ell}\prod_{i=1}^{d}(|y_i|+2^{-j})^{-2\kappa_i}.$$

Thus,

$$\sum_{j\in\mathbb{Z}}|T^y(\Psi_j)(z)-T^y(\Psi_j)(x)|\le C\|x-z\|\sum_{j\in\mathbb{Z}}2^{j(d+1)}\left(1+2^j\rho\right)^{-\ell}\prod_{i=1}^{d}(|y_i|+2^{-j})^{-2\kappa_i},$$

which, following the proof in part (i), is estimated by

$$\frac{C\|x-z\|}{\rho\,\mathrm{meas}_\kappa(B(y,\rho))}.$$

This completes the proof of (ii).

5.3. **Proof of Corollary 5.3.** Corollary 5.3 is a direct consequence of Theorem 5.2 and the following lemma.

Lemma 5.5. *Let $f:\mathbb{R}^d\to\mathbb{R}$ be such that $M_\kappa f(x)<\infty$ for a.e. $x\in\mathbb{R}^d$. If $0<\delta<1$, then $(M_\kappa f)^\delta$ is a $\mathbb{Z}_2^d$-invariant A_1 weight on the homogeneous space $(\mathbb{R}^d,h_\kappa^2\,dx)$.*

The proof of this lemma follows along the same lines as that of the unweighted case (see, for instance, [12, Theorem 7.7, p. 140]). We omit the details here.

6. Maximal Bochner-Riesz operators

This section is devoted to the proof of Theorem 1.1. The necessity part of this theorem follows directly from Theorem 4.3 of [6], as stated in the introduction. It remains to show the sufficiency.

The proof follows along the same lines as that of Christ [4] for the classical Fourier transforms. It has several components, some of which are quite technical. One of the most important tools is the square function defined below.

Definition 6.1. Let Φ be a compactly supported C^∞-function on $\mathbb{R}$ with $\operatorname{supp}\Phi \subset [-\frac12, \frac12]$. For a parameter $\alpha \in (0, 1/2]$, set

$$\phi(x) \equiv \phi(x;\alpha) := \Phi\Big(\frac{1-\|x\|^2}{\alpha}\Big), \quad x \in \mathbb{R}^d.$$

Define the square function G_α by

$$G_\alpha f(x) = \Big(\int_0^\infty |f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x)|^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in \mathbb{R}^d,$$

where $(\mathcal{F}_\kappa\phi)_{t^{-1}}(x) = t^{2\lambda_\kappa+1}\mathcal{F}_\kappa\phi(tx) = \mathcal{F}_\kappa(\phi(t^{-1}\cdot))(x)$.

Clearly, the function ϕ in the above definition satisfies the conditions

$$\operatorname{supp}\phi \subset \Big\{x \in \mathbb{R}^d : 1 - \frac{\alpha}{2} \le \|x\| \le 1 + \frac{\alpha}{4}\Big\} \tag{6.1}$$

and

$$\sup_{x\in\mathbb{R}^d} |\nabla^\ell \phi(x)| \le C_\ell \alpha^{-\ell}, \quad \ell = 0, 1, \cdots. \tag{6.2}$$

The proof of Theorem 1.1 can be reduced to the following estimate.

Theorem 6.2. *If $2 \le p < 2 + \frac{2}{\lambda_\kappa}$ and $\alpha \in (0, \frac12]$, then*

$$\|G_\alpha f\|_{\kappa,p} \le C\alpha^{\frac{1}{2(\lambda_\kappa+1)}} \|f\|_{\kappa,p},$$

where the constant C is independent of the parameter α.

For the moment, we take Theorem 6.2 for granted and show how to deduce Theorem 1.1 from Theorem 6.2.

Proof of Theorem 1.1. The proof follows along the standard technique in [18]. For completeness, we write it below.

Let ξ be a C^∞-function on $\mathbb{R}$ satisfying that $\xi(x) = 1$ for $|x| \le 1$ and $\xi(x) = 0$ for $|x| \ge 2$, and set $\theta(x) = \xi(x) - \xi(2x)$. Clearly, $\operatorname{supp}\theta \subset \{x : \frac12 \le |x| \le 2\}$, and $\sum_{j=0}^\infty \theta(2^j x) = \xi(x) = 1$ for all $x \in [-1,1]\setminus\{0\}$. Thus, we can write, for $\beta > -1/2$,

$$(1-\|\xi\|^2)_+^\beta = (1-\|\xi\|^2)_+^\beta \sum_{j=0}^\infty \theta\Big(2^j(1-\|\xi\|^2)\Big) = \sum_{j=0}^\infty 2^{-j\alpha} m_j^\beta(\xi),$$

where $m_j^\beta(\xi) = 2^{j\beta}\theta\big(2^j(1-\|\xi\|^2)\big)(1-\|\xi\|^2)_+^\beta$. As a result, the operator $B_R^\beta(h_\kappa^2; f)$ can be decomposed as

$$B_R^\beta(h_\kappa^2; f)(x) = \sum_{j=0}^\infty 2^{-j\beta} T_{R,j}^\beta f(x), \tag{6.3}$$

where the operators $T^{\beta}_{R,j}$ are given by $\mathcal{F}_\kappa T^{\beta}_{R,j} f(\xi) = m^{\beta}_j(R^{-1}\xi)\mathcal{F}_\kappa f(\xi)$. Since $m^{\beta}_0, m^{\beta}_1, m^{\beta}_2 \in \mathcal{S}(\mathbb{R}^d)$, by Lemma 4.6, it's easily seen that

$$\sup_{R>0} |T^{\beta}_{R,j} f(x)| \le C M_\kappa f(x), \quad j = 0, 1, 2. \tag{6.4}$$

Next, for $j \ge 3$, we write $m^{\beta}_j(\xi) = \Phi^\beta(2^{j-2}(1-\|\xi\|^2))$, with $\Phi^\beta(t) = \theta(4t)|4t|^\beta$, $t \in \mathbb{R}$. Since Φ^β is a C^∞-function supported on $[-\frac{1}{2}, \frac{1}{2}]$, it follows by Theorem 6.2 with $\alpha = 2^{-j+2}$ that for $2 \le p < 2 + \frac{2}{\lambda_\kappa}$,

$$\Big\| \Big(\int_0^\infty |T^{\beta}_{R,j} f(x)|^2 \frac{dR}{R} \Big)^{\frac{1}{2}} \Big\|_{\kappa,p} \le C_{\kappa,\beta} 2^{-\frac{j}{2(\lambda_\kappa+1)}} \|f\|_{\kappa,p}, \quad j = 3, 4, \cdots. \tag{6.5}$$

On the other hand, according to [18, (5.9), p. 279], we have that for any $\varepsilon > 0$ and $\delta > 0$,

$$B^\delta_*(f; h^2_\kappa)(x) \le C_{\delta,\varepsilon} \Big(\sup_{R>0} \frac{1}{R} \int_0^R |B^{\delta-\frac{1}{2}-\varepsilon}_t (h^2_\kappa; f)(x)|^2 \, dt \Big)^{\frac{1}{2}}.$$

This combined with (6.3) and (6.4) yields that

$$\begin{aligned} B^\delta_*(f; h^2_\kappa)(x) &\le C M_\kappa f(x) + C \Big(\sup_{R>0} \frac{1}{R} \int_0^R |\sum_{j=3}^\infty 2^{-j(\delta-\frac{1}{2}-\varepsilon)} T^{\delta-\frac{1}{2}-\varepsilon}_{t,j}(f)(x)|^2 \, dt \Big)^{\frac{1}{2}} \\ &\le C M_\kappa f(x) + C \Big(\int_0^\infty |\sum_{j=3}^\infty 2^{-j(\delta-\frac{1}{2}-\varepsilon)} T^{\delta-\frac{1}{2}-\varepsilon}_{t,j}(f)(x)|^2 \, \frac{dt}{t} \Big)^{\frac{1}{2}} \\ &\le C M_\kappa f(x) + C \sum_{j=3}^\infty 2^{-j(\delta-\frac{1}{2}-\varepsilon)} \Big(\int_0^\infty |T^{\delta-\frac{1}{2}-\varepsilon}_{t,j}(f)(x)|^2 \, \frac{dt}{t} \Big)^{\frac{1}{2}}. \end{aligned}$$

Thus, using (6.5), we prove that

$$\|B^\delta_*(h^2_\kappa; f)\|_{\kappa,p} \le C\|f\|_{\kappa,p}, \quad \text{for } 2 \le p < 2 + \frac{2}{\lambda_\kappa} \text{ and } \delta > \frac{\lambda_\kappa}{2(\lambda_\kappa+1)}. \tag{6.6}$$

Finally, the stated conclusion of Theorem 1.1 follows by applying Stein's interpolation theorem for a family of analytic operators, using (1.6) for the already proven case of $\delta = \lambda_\kappa + \varepsilon$ and $p = \infty$ and (6.6) with $\delta = \frac{\lambda_\kappa}{2(\lambda_\kappa+1)} + \varepsilon$ and $p = \frac{2\lambda_\kappa+2}{\lambda_\kappa} - \varepsilon$ for a sufficiently small $\varepsilon > 0$. This shows Theorem 1.1. □

The proof of Theorem 6.2 is long and technical; therefore, we break it into several steps in the next three subsections.

6.1. **A locality lemma.** Denote by $\mathcal{D}_j$ the collection of all dyadic cubes in $\mathbb{R}^d$ with side length 2^j. Let T be a sublinear operator with the following local property: for any function f supported in a cube $Q \in \mathcal{D}_j$, Tf is supported in a fixed dilate $Q^* = c\widetilde{Q}$ of $\widetilde{Q} = \bigcup_{\sigma\in\mathbb{Z}^d_2} Q\sigma$. By (2.16), it is easily seen that if K is a kernel supported in $B(0, c2^j)$, then $Tf = f *_\kappa K$ has the above local property.

Lemma 6.3. *Let $p_0 > 2$ and $r = (p_0/2)' = \frac{p_0}{p_0-2}$. Let T be a sublinear operator having the above stated local property for some $j \in \mathbb{Z}$ and satisfying the following condition: for any function f that is supported in a dyadic cube Q of side length 2^j,*

$$\|Tf\|_{\kappa,p_0} \le A \Big(\frac{2^{j(2\lambda_\kappa+1)}}{\text{meas}_\kappa(Q)} \Big)^{\frac{1}{2}-\frac{1}{p_0}} \|f\|_{\kappa,2}. \tag{6.7}$$

Then for any f defined on $\mathbb{R}^d$ and any nonnegative function g on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |Tf(x)|^2 g(x) h_\kappa^2(x)\,dx \le CA^2 2^{j(2\lambda_\kappa+1)/r} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\,dx, \tag{6.8}$$

where $M_{\kappa,r} g = (M_\kappa(|g|^r))^{1/r}$.

Proof. Assume momentarily that f is supported in a dyadic cube $Q \in \mathcal{D}_j$. By the local property of T and Hölder's inequality,

$$\int_{\mathbb{R}^d} |Tf(x)|^2 g(x) h_\kappa^2(x)\,dx = \int_{Q^*} |Tf(x)|^2 g(x) h_\kappa^2(x)\,dx$$
$$\le \Big\| |Tf|^2 \Big\|_{\kappa,\frac{p_0}{2}} \Big(\int_{Q^*} |g(x)|^r h_\kappa^2(x)\,dx \Big)^{1/r} \le C(\mathrm{meas}_\kappa(Q))^{\frac{1}{r}} \|Tf\|_{\kappa,p_0}^2 \inf_{x\in Q} M_{\kappa,r} g(x),$$

which, using (6.7), is estimated above by

$$CA^2 2^{j(2\lambda_\kappa+1)/r} \int_Q |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\,dx.$$

This proves (6.8) for f supported in a dyadic cube $Q \in \mathcal{D}_j$.

In general, we decompose a function f as $f = \sum_{Q\in\mathcal{D}_j} f\chi_Q = \sum_{Q\in\mathcal{D}_j} f_Q$. Since T is sublinear, we have, by the local property of T, $|Tf| \le \sum_{Q\in\mathcal{D}_j} |T(f_Q)|\chi_{Q^*}$, which in turn implies that $|Tf|^2 \le C\sum_{Q\in\mathcal{D}_j} |T(f_Q)|^2$. It then follows by the already proven case of (6.8) that

$$\int_{\mathbb{R}^d} |Tf(x)|^2 g(x) h_\kappa^2(x)\,dx \le C \sum_{Q\in\mathcal{D}_j} \int_{\mathbb{R}^d} |T(f_Q)|^2 g(x) h_\kappa^2(x)\,dx$$
$$\le CA^2 2^{j(2\lambda_\kappa+1)/r} \int_{\mathbb{R}^d} |f|^2 M_{\kappa,r}(g)(x) h_\kappa^2(x)\,dx.$$

This completes the proof of the lemma. □

6.2. **A pointwise kernel estimate.** Let $\alpha \in (0, \frac{1}{2}]$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ be as given in Definition 6.1. Let i be a positive integer such that $2^{-i-1} < \alpha \le 2^{-i}$. Let η be a C^∞ radial function on $\mathbb{R}^d$ such that $\eta(x) = 1$ for $\|x\| \le 1$ and $\eta(x) = 0$ for $\|x\| \ge 2$. Define $\eta_i(x) := \eta(2^{-i}x)$ and $\eta_j(x) := \eta(2^{-j}x) - \eta(2^{-j+1}x)$ for $j > i$. Clearly,

$$\sum_{j=i}^{\infty} \eta_j(\xi) = \lim_{j\to\infty} \eta(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d. \tag{6.9}$$

The main goal in this subsection is to prove the following technical lemma, which plays a crucial role in the proof of Theorem 6.2.

Lemma 6.4. *With the above notation, we have that for $\frac{1}{2} \le t \le 4$, $j \ge i$, and any $N \in \mathbb{N}$,*

$$|(\mathcal{F}_\kappa \eta_j) *_\kappa \phi^t(x)| \le C_{N,\eta} \Big(\max_{0\le j\le N} \|\Phi^{(j)}\|_\infty \Big)$$
$$\times \begin{cases} 2^{(i-j)N} \big(1 + 2^i \big|\|x\| - t\big|\big)^{-N} & \text{if } \frac{1}{32} \le \|x\| \le 16; \\ 2^{(i-j)N} 2^{-iN} (1 + \|x\|)^{-N} & \text{otherwise,} \end{cases} \tag{6.10}$$

where $\phi^t(x) = \phi(t^{-1}x)$.

Proof. Let $\psi \equiv \psi^j$ denote the radial Schwarz function on $\mathbb{R}^d$ whose Dunkl transform is equal to either the function $\eta(\xi)$ or the function $\eta(\xi)-\eta(2\xi)$ depending on whether $j=i$ or $j>i$. Then $\mathcal{F}_\kappa \eta_j(x) = 2^{j(2\lambda_\kappa+1)}\psi(2^j x) =: \psi_j(x)$ and

$$\mathcal{F}_\kappa \eta_j *_\kappa \phi^t(x) = \psi_j *_\kappa \phi^t(x) = c_\kappa 2^{j(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} \psi(2^j y) T^y \phi^t(x) h_\kappa^2(y)\, dy.$$

Since $T^y\phi^t(x) = T^{y/t}\phi(x/t)$, we have

$$\mathcal{F}_\kappa \eta_j *_\kappa \phi^t(x) = c(2^j t)^{(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} \psi(2^j t y) T^y \phi(x/t) h_\kappa^2(y)\, dy = ((\psi_{t^{-1}})_j *_\kappa \phi)(x/t),$$

where $\psi_{t^{-1}}(x) = t^{2\lambda_\kappa+1}\psi(xt)$ and $(\psi_{t^{-1}})_j(x) = 2^{j(2\lambda_\kappa+1)}\psi^t(2^j x)$. Thus, it suffices to prove the estimates (6.10) for $t=1$.

For simplicity, we shall assume throughout the proof that $\kappa_{\min} := \min_{1\le j\le d}\kappa_j > 0$. The proof below with slight modifications works equally well for the case $\kappa_{\min}=0$.

We first claim that for any $x, y \in \mathbb{R}^d$,

$$|T^y\phi(x)| \le \frac{C\|\Phi\|_\infty}{\operatorname{meas}_\kappa(B(x,1))}\chi_E(x,y), \tag{6.11}$$

where E is the set of all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$1 - 2^{-i-1} \le \|\bar{x} + \bar{y}\| \qquad \text{and} \qquad \|\bar{x} - \bar{y}\| \le 1 + 2^{-i-1}. \tag{6.12}$$

Note that (6.12) implies that

$$\big|\|x\| - 1\big| < \|y\| + 2^{-i-1} \qquad \text{and} \qquad \big|\|y\| - 1\big| < \|x\| + 2^{-i-1}. \tag{6.13}$$

To show the claim (6.11), we use Corollary 2.6 to obtain

$$T^y\phi(x) = c_\kappa \int_{[-1,1]^d} \phi(u(x,y,t)) \prod_{j=1}^d (1-t_j^2)^{\kappa_j-1}(1+t_j)\, dt_j, \tag{6.14}$$

where $u(x,y,t) := \sqrt{\|x\|^2 + \|y\|^2 - \sum_{j=1}^d 2x_j y_j t_j}$. Here we use a slight abuse of the notation that $\phi(x) = \phi(\|x\|)$. We also recall that

$$\operatorname{supp}\phi \subset \{\xi \in \mathbb{R}^d : 1 - 2^{-i-2} \le \|\xi\| \le 1 + 2^{-i-1}\}. \tag{6.15}$$

Since $\|\bar{x}-\bar{y}\| \le u(x,y,t) \le \|\bar{x}+\bar{y}\|$ for all $t \in [-1,1]^d$ and $x, y \in \mathbb{R}^d$, it follows that $T^y\phi(x) = 0$ unless (6.12) is satisfied.

Next, using (6.14) and (6.15), we obtain

(6.16)

$$|T^y\phi(x)| \le C\|\Phi\|_\infty \int_{\{t\in[-1,1]^d:\ \ u(x,y,t)\le 1+2^{-i-1}\}} \prod_{j=1}^d (1-t_j^2)^{\kappa_j-1}(1+t_j)\, dt_j.$$

Observe that for any $t \in [-1,1]^d$,

(6.17)

$$\begin{aligned} u(x,y,t)^2 &\ge \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d |x_j y_j||t_j| = \|\bar{x}-\bar{y}\|^2 + 2\sum_{j=1}^d |x_j y_j|(1-|t_j|) \\ &\ge \|\bar{x}-\bar{y}\|^2 + 2\max_{1\le j\le d}|x_j y_j|(1-|t_j|). \end{aligned}$$

It follows from (6.16) that

$$|T^y\phi(x)| \le C\|\Phi\|_\infty \prod_{j=1}^d \int_{\{t_j\in[-1,1]:2|x_jy_j|(1-|t_j|)\le(1+2^{-i-1})^2\}} (1-|t_j|)^{\kappa_j-1}\,dt_j$$

$$\le C\|\Phi\|_\infty \prod_{j=1}^d (|x_jy_j|+1)^{-\kappa_j}. \tag{6.18}$$

On the other hand, according to (6.12), if $T^y\phi(x)\neq 0$, then $\big||x_jy_j|-|x_j|^2\big| \le 2|x_j| \le \frac12(|x_j|^2+4)$ for $1\le j\le d$, and hence $|x_jy_j|+4 \sim |x_j|^2+4$ for all $1\le j\le d$. Thus, using (6.18), we deduce that

$$|T^y\phi(x)| \le C\|\Phi\|_\infty \prod_{j=1}^d (|x_j|^2+1)^{-\kappa_j} \sim \frac{\|\Phi\|_\infty}{\mathrm{meas}_\kappa(B(x,1))},$$

which completes the proof of the claim (6.11).

Now we turn to the proof of the estimate (6.10) for $t=1$. If $\|x\|\le \frac{1}{32}$ and $T^y\phi(x)\neq 0$, then by (6.13),

$$\|y\| = \|x\| + \|y\| - \|x\| \ge 1 - 2^{-i-1} - \frac{1}{32} \ge \frac14,$$

which implies for $\|x\|\le\frac{1}{32}$ that

$$\begin{aligned}|\psi_j *_\kappa \phi(x)| &\le c2^{j(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} |\psi(2^jy)||T^y\phi(x)| h_\kappa^2(y)\,dy \\ &\le C\|\Phi\|_\infty 2^{j(2\lambda_\kappa+1)} \int_{\|y\|\ge\frac14} (2^j\|y\|)^{-2\lambda_\kappa-1-N} h_\kappa^2(y)\,dy \\ &\le C\|\Phi\|_\infty 2^{(i-j)N} 2^{-iN}(1+\|x\|)^{-N}.\end{aligned}$$

Similarly, if $\|x\|\ge 16$ and $T^y\phi(x)\neq 0$, then by (6.12), $\|\bar{x}-\bar{y}\|\le\frac32\le\frac{\|x\|}{2}$, and hence $\frac{\|x\|}{2}\le\|y\|\le\frac32\|x\|$. By (6.11), this implies that for $\|x\|\ge 16$,

$$\begin{aligned}|\psi_j *_\kappa \phi(x)| &\le C\|\Phi\|_\infty 2^{j(2\lambda_\kappa+1)} \int_{\|y\|\sim\|x\|} (2^j\|y\|)^{-2\lambda_\kappa-2-N} h_\kappa^2(y)\,dy \\ &\le C\|\Phi\|_\infty (2^j\|x\|)^{-N} \int_{\mathbb{R}^d} (1+\|y\|)^{-2\lambda_\kappa-2} h_\kappa^2(y)\,dy \\ &\le C\|\Phi\|_\infty 2^{(i-j)N} 2^{-iN}(1+\|x\|)^{-N}.\end{aligned}$$

Thus, it remains to show that for all $N\in\mathbb{Z}_+$,

$$|\psi_j *_\kappa \phi(x)| \le C\Big(\max_{0\le j\le N}\|\Phi^{(j)}\|_\infty\Big)2^{(i-j)N}\Big(1+2^i\Big|1-\|x\|\Big|\Big)^{-N} \tag{6.19}$$

whenever $\frac{1}{32}\le\|x\|\le 16$.

For the rest of the proof, we assume that $\frac{1}{32}\le\|x\|\le16$. We first prove that

$$|\psi_j *_\kappa \phi(x)| \le C\Big(\max_{0\le j\le N}\|\Phi^{(j)}\|_\infty\Big)2^{(i-j)N}, \tag{6.20}$$

which will give (6.19) for $|1-\|x\||\le 2^{-i+5}$.

If $j = i$, then (6.20) holds trivially since $|\psi_j *_\kappa \phi(x)| \le C\|\Phi\|_\infty \|\psi_j\|_{1,\kappa} \le C\|\Phi\|_\infty$. Now assume that $j > i$. Since $\mathcal{F}_\kappa \psi$ is zero near the origin when $j > i$, it follows from (2.1) that $\mathcal{D}_\kappa^\gamma \mathcal{F}_\kappa \psi(0) = 0$ for any $\gamma \in \mathbb{Z}_+^d$. Thus, by Lemma 2.1,

$$0 = \mathcal{D}_\kappa^\gamma \mathcal{F}_\kappa \psi(0) = c\mathcal{F}_\kappa x^\gamma \psi(0) = c\int_{\mathbb{R}^d} x^\gamma \psi(x) h_\kappa^2(x)\, dx, \quad \forall \gamma \in \mathbb{Z}_+^d.$$

This implies that for every polynomial P on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \psi_j(x) P(x) h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} \psi(x) P(2^{-j}x) h_\kappa^2(x)\, dx = 0. \tag{6.21}$$

Fix temporarily $x \in \{z \in \mathbb{R}^d : \frac{1}{32} \le \|z\| \le 16\}$ and $t = (t_1, \cdots, t_d) \in [-1,1]^d$. Set $F_{x,t}(y) = \phi(u(x,y,t))$, where $u(x,y,t) := \sqrt{\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j y_j t_j}$. Recall that $\phi(u(x,y,t)) = 0$ unless $1 - 2^{-i-1} \le u(x,y,t) \le 1 + 2^{-i-1}$. Thus, by (6.13), (6.12), and (6.17), $F_{x,t}(y)$ is a C^∞- function of y supported in a set where

$$\sum_{j=1}^d |x_j||y_j|(1 - |t_j|) < 2, \quad \big|\|x\| - 1\big| < \|y\| + 2^{-i} \qquad \text{and} \qquad \big|\|y\| - 1\big| < \|x\| + 2^{-i}. \tag{6.22}$$

Furthermore, by (6.2),

$$\|\nabla^n F_{x,t}\|_\infty \le C_{n,\Phi} 2^{in}, \quad \forall n = 0, 1, \cdots, \tag{6.23}$$

where $C_{n,\Phi} = C\max_{0\le \ell \le n} \|\Phi^{(\ell)}\|_\infty$. Now using Taylor's theorem, we obtain that given any $N \in \mathbb{Z}_+$,

$$\phi(u(x,y,t)) = \sum_{|\gamma|\le N-1} \frac{\partial^\gamma F_{x,t}(0)}{\gamma!} y^\gamma + \sum_{|\gamma|=N} \frac{\partial^\gamma F_{x,t}(\theta y)}{\gamma!} y^\gamma,$$

for some $\theta = \theta(x,y,t) \in [0,1]$. It then follows by (6.14) that

$$\begin{aligned} T^y\phi(x) = c &\sum_{|\gamma|\le N-1} \frac{y^\gamma}{\gamma!} \int_{[-1,1]^d} \partial^\gamma F_{x,t}(0) \prod_{j=1}^d (1 - t_j^2)^{\kappa_j - 1}(1 + t_j)\, dt_j \\ &+ c\sum_{|\gamma|=N} \frac{y^\gamma}{\gamma!} \int_{[-1,1]^d} \partial^\gamma F_{x,t}(\theta(x,y,t)y) \prod_{j=1}^d (1 - t_j^2)^{\kappa_j - 1}(1 + t_j)\, dt_j. \end{aligned} \tag{6.24}$$

Thus, using (6.21) and (6.23), we conclude that

$$\begin{aligned} |\psi_j *_\kappa \phi(x)| &= \Big|\int_{\mathbb{R}^d} \psi_j(y) T^y\phi(x) h_\kappa^2(y)\, dy\Big| \le C\int_{\mathbb{R}^d} |\psi_j(y)|\|y\|^N \\ &\quad \times \Big[\int_{[-1,1]^d} \|\nabla^N F_{x,t}(\theta(x,y,t)y)\| \prod_{j=1}^d (1 - t_j^2)^{\kappa_j - 1}(1 + t_j) dt_j\Big] h_\kappa^2(y) dy \\ &\le C_{N,\Phi} 2^{iN} 2^{j(2\lambda_\kappa + 1)} \int_{\mathbb{R}^d} (1 + 2^j\|y\|)^{-N-2\lambda_\kappa - 2} \|y\|^N h_\kappa^2(y)\, dy \\ &\le C_{N,\Phi} 2^{(i-j)N}. \end{aligned} \tag{6.25}$$

This proves (6.20) and hence (6.19) for $\Big|1 - \|x\|\Big| \le 2^{-i+5}$.

Finally, we prove (6.19) under the assumption $\Big|\|x\|-1\Big| \geq 2^{-i+5}$. First, we observe that if $\Big|\|x\|-1\Big| \geq 2^{-i+5}$, then by (6.22), $\partial^\gamma F_{x,t}(0)=0$ for all $\gamma\in\mathbb{Z}_+^d$, and hence, according to Taylor's theorem, (6.25) holds for all $j\geq i$. Second, observe that if $\|y\|\leq \frac12\Big|1-\|x\|\Big|$, then $\Big|\|x\|-1\Big| > \|y\|+2^{-i}\geq \|\theta(x,y,t)y\|+2^{-i}$, which, by (6.22), implies that $\partial^\gamma F_{x,t}(\theta(x,y,t)y)=0$ for all $\gamma\in\mathbb{Z}_+^d$. Thus, (6.25) implies that for all $j\geq i$,

$$\begin{aligned}|\psi_j *_\kappa \phi(x)| &\leq C\int_{\|y\|\geq \frac12\left|1-\|x\|\right|} |\psi_j(y)|\|y\|^N\\ &\qquad\times\Big[\int_{[-1,1]^d}\|\nabla^N F_{x,t}(\theta(x,y,t)y)\|\prod_{j=1}^d(1-t_j^2)^{\kappa_j-1}(1+t_j)dt_j\Big]h_\kappa^2(y)dy\\ &\leq C_{N,\Phi}2^{(i-j)N}\Big(1+2^i\big|1-\|x\|\big|\Big)^{-N}.\end{aligned}$$

This completes the proof of (6.19). □

6.3. **Proof of Theorem 6.2.** In this subsection, we will prove Theorem 6.2. Recall that $p_\kappa=\frac{2\lambda_\kappa+2}{\lambda_\kappa+2}$. Let $q_\kappa=p_\kappa'=2+\frac{2}{\lambda_\kappa}$. By a simple duality argument, the proof of Theorem 6.2 can be reduced to the following theorem.

Theorem 6.5. *Let $q_\kappa=2+\frac{2}{\lambda_\kappa}$ and $r=(\frac12 q_\kappa)'=\lambda_\kappa+1$. Then for any nonnegative function g on $\mathbb{R}^d$,*

$$\int_{\mathbb{R}^d}|G_\alpha f(x)|^2 g(x)h_\kappa^2(x)\,dx\leq C_\kappa\alpha^{\frac{1}{\lambda_\kappa+1}}\int_{\mathbb{R}^d}|f(x)|^2M_{\kappa,r}(g)(x)h_\kappa^2(x)\,dx,$$

where $M_{\kappa,r}g=(M_\kappa(g^r))^{1/r}$.

The rest of this subsection is devoted to the proof of Theorem 6.5.

First, we define, for $2^{-1}\leq t\leq 4$, $Tf(x,t):=f*_\kappa(\mathcal{F}_\kappa\phi)_{t^{-1}}(x)$. Let i be a positive integer such that $2^{-i-1}<\alpha\leq 2^{-i}$, and let $\{\eta_j\}_{j=i}^\infty$ be a sequence of radial Schwarz functions on $\mathbb{R}^d$ as defined in subsection 6.2. We then decompose $Tf(x,t)$ as

$$Tf(x,t)=\sum_{j=i}^\infty f*_\kappa((\mathcal{F}_\kappa\phi)_{t^{-1}}\eta_j)(x)=:\sum_{j=i}^\infty T_jf(x,t),\quad x\in\mathbb{R}^d, \tag{6.26}$$

where $T_jf(x,t)=f*_\kappa(\eta_j(\mathcal{F}_\kappa\phi)_{t^{-1}})(x)$. We consider each T_j as a vector-valued operator $T_j:\ L^2(\mathbb{R}^d;h_\kappa^2)\to L^{q_\kappa}(L^2[2^{-1},4])$ with the norm of $L^{q_\kappa}(L^2[2^{-1},4])$ being defined as

$$\|F\|_{L^{q_\kappa}(L^2[2^{-1},4])}:=\Big(\int_{\mathbb{R}^d}\|F(x,t)\|^{q_\kappa}_{L^2([2^{-1},4],dt)}h_\kappa^2(x)\,dx\Big)^{1/q_\kappa},$$

where $\|F(x,\cdot)\|_{L^2[2^{-1},4]}:=\Big(\int_{2^{-1}}^4|F(x,t)|^2\,dt\Big)^{\frac12}$.

We break the proof of Theorem 6.5 into several lemmas.

Lemma 6.6. *Let $B=B(\omega,c2^j)$ denote a ball with center at $\omega\in\mathbb{R}^d$ and radius $c2^j$ for some $c>0$, and let $\widetilde{B}=\bigcup_{\sigma\in\mathbb{Z}_2^d}B(\omega\sigma,c2^j)$. Then for $j\geq i\geq 1$,*

$$\Big\|\Big(\int_{2^{-1}}^4|T_jf(\cdot,t)|^2\,dt\Big)^{1/2}\Big\|_{L^{q_\kappa}(\widetilde{B};h_\kappa^2dx)}\leq C2^{-j}\Big(\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac12-\frac{1}{q_\kappa}}\|f\|_{\kappa,2}.$$

Proof. Write $f = f_1 + f_2$ with $\mathcal{F}_\kappa f_1(\xi) = \mathcal{F}_\kappa f(\xi)\chi_{4^{-1}\le\|\xi\|\le 8}(\xi)$, $\mathcal{F}_\kappa f_2(\xi) = \mathcal{F}_\kappa f(\xi)\chi_I(\|\xi\|)$, and $I := [0,\frac14)\bigcup(8,\infty)$. We then reduce to showing that for $n=1,2$,

$$\Big\|\Big(\int_{2^{-1}}^4 |T_j f_n(\cdot,t)|^2\,dt\Big)^{1/2}\Big\|_{L^{q_\kappa}(\widetilde{B};h_\kappa^2 dx)} \le C2^{-j}\Big(\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac12-\frac1{q_\kappa}}\|f_n\|_{\kappa,2}. \tag{6.27}$$

We first show (6.27) for $n=1$. Note that

$$\Big(T_j f(\cdot,t)\Big)^\wedge(\xi) = \mathcal{F}_\kappa f(\xi)(\eta_j(\mathcal{F}_\kappa\phi)_{t^{-1}})^\wedge(\xi) = \mathcal{F}_\kappa f(\xi)(\mathcal{F}_\kappa\eta_j *_\kappa \phi^t)(\xi), \tag{6.28}$$

where $\phi^t(\xi)=\phi(\xi/t)$. It follows by the Fourier inverse formula that

$$\begin{aligned} T_j f(x,t) &= c_\kappa\int_{\mathbb{R}^d}\mathcal{F}_\kappa f(\xi)(\mathcal{F}_\kappa\eta_j *_\kappa \phi^t)(\xi)E_\kappa(\mathbf{i}\xi,x)h_\kappa^2(\xi)\,d\xi\\ &= c\int_0^\infty\int_{\mathbb{S}^{d-1}}\mathcal{F}_\kappa f(\rho\xi)(\mathcal{F}_\kappa\eta_j *_\kappa \phi^t)(\rho\xi)E_\kappa(\mathbf{i}\rho\xi,x)h_\kappa^2(\xi)\,d\sigma(\xi)\rho^{2\lambda_\kappa}\,d\rho. \end{aligned} \tag{6.29}$$

Thus, using (6.29) and Minkowskii's inequality, we obtain that for $t\in[2^{-1},4]$,

$$\begin{aligned} &\|T_j f_1(\cdot,t)\|^2_{L^{q_\kappa}(\widetilde{B};h_\kappa^2)}\\ &\le C\Big\|\int_{4^{-1}}^8\rho^{2\lambda_\kappa}\Big[\int_{\mathbb{S}^{d-1}}\mathcal{F}_\kappa f(\rho\xi)(\mathcal{F}_\kappa\eta_j *_\kappa \phi^t)(\rho\xi)E_\kappa(\mathbf{i}\rho\cdot,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\Big]\,d\rho\Big\|^2_{L^{q_\kappa}(\widetilde{B};h_\kappa^2)}\\ &\le C\Big(\int_{4^{-1}}^8\Big\|\int_{\mathbb{S}^{d-1}}\mathcal{F}_\kappa f(\rho\xi)(\mathcal{F}_\kappa\eta_j *_\kappa \phi^t)(\rho\xi)E_\kappa(\mathbf{i}\cdot,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\Big\|_{L^{q_\kappa}(\widetilde{B}_\rho;h_\kappa^2)}\,d\rho\Big)^2, \end{aligned}$$

where $\widetilde{B}_\rho=\bigcup_{\sigma\in\mathbb{Z}_2^d}B(\rho(\omega\sigma),c2^j\rho)$. By the Cauchy-Schwarz inequality, the term on the right hand side of this last inequality can be controlled by a constant multiple of

$$\begin{aligned} 2^{-i}\Big(\int_{4^{-1}}^8\Big\|\int_{\mathbb{S}^{d-1}}\mathcal{F}_\kappa f(\rho\xi)(\mathcal{F}_\kappa\eta_j *_\kappa \phi^t)(\rho\xi)E_\kappa(\mathbf{i}\cdot,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\Big\|^2_{L^{q_\kappa}(\widetilde{B}_\rho;h_\kappa^2)}&\\ \times(1+2^i|\rho-t|)^2\,d\rho\Big),& \end{aligned}$$

which, using the restriction theorem (Corollary 4.2(ii)), is bounded above by

$$\begin{aligned} C2^{-i}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{1-\frac{2}{q_\kappa}}\int_{4^{-1}}^8(1+2^i|\rho-t|)^2&\\ \times\int_{\mathbb{S}^{d-1}}|\mathcal{F}_\kappa\eta_j *_\kappa \phi^t(\rho\xi)|^2|\mathcal{F}_\kappa f(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho.& \end{aligned}$$

Here we used the fact that for $B_\rho=B(\rho\omega,c2^j\rho)$ with $\rho>0$,

$$\frac{(2^j\rho)^{2\lambda_\kappa+1}}{\mathrm{meas}_\kappa(\widetilde{B}_\rho)}\sim\frac{(2^j\rho)^{2\lambda_\kappa+1}}{\mathrm{meas}_\kappa(B_\rho)}\sim\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)}\sim\frac{2^{2j|\kappa|}}{\prod_{n=1}^d(|\omega_n|+2^j)^{2\kappa_n}}. \tag{6.30}$$

Thus, by Lemma 6.4, it follows that for any $t\in[2^{-1},4]$,

$$\begin{aligned} &\|T_j f_1(\cdot,t)\|^2_{L^{q_\kappa}(\widetilde{B};h_\kappa^2)}\\ &\le C2^{-i}4^{i-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{1-\frac{2}{q_\kappa}}\int_{4^{-1}}^8(1+2^i|\rho-t|)^{-N}\int_{\mathbb{S}^{d-1}}|\mathcal{F}_\kappa f(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho. \end{aligned}$$

Here and throughout the proof, N denotes a sufficiently large number depending only on κ and d. Now using Minkowskii's inequality again and taking into account the fact that $q_\kappa > 2$, we deduce that

$$\begin{aligned}
&\Big\|\Big(\int_{2^{-1}}^4 |T_j f_1(\cdot,t)|^2\,dt\Big)^{\frac12}\Big\|^2_{L^{q_\kappa}(\widetilde{B};h_\kappa^2)} \le \int_{2^{-1}}^4 \Big\|T_j f_1(\cdot,t)\Big\|^2_{L^{q_\kappa}(\widetilde{B};h_\kappa^2)}\,dt\\
&\le C2^{-i}4^{i-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\text{meas}_\kappa(B)}\Big)^{1-\frac{2}{q_\kappa}}\int_{4^{-1}}^8\int_{\mathbb{S}^{d-1}}|\mathcal{F}_\kappa f(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\\
&\quad\times\Big[\int_{2^{-1}}^4 (1+2^i|\rho-t|)^{-N}\,dt\Big]\,d\rho\\
&\le C4^{-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\text{meas}_\kappa(B)}\Big)^{1-\frac{2}{q_\kappa}}\int_{4^{-1}}^8 \rho^{2\lambda_\kappa}\int_{\mathbb{S}^{d-1}}|\mathcal{F}_\kappa f(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho\\
&\le C4^{-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\text{meas}_\kappa(B)}\Big)^{1-\frac{2}{q_\kappa}}\|f_1\|^2_{\kappa,2}.
\end{aligned}$$

This shows (6.27) for $n=1$.

Next, we show (6.27) for $n=2$. By Minkowskii's inequality, it suffices to show that for any $t\in[2^{-1},4]$ and $p=q_\kappa$,

$$\|T_j f_2(\cdot,t)\|_{L^p(\widetilde{B};h_\kappa^2)} \le C2^{-jN}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\text{meas}_\kappa(B)}\Big)^{\frac12-\frac1p}\|f_2\|_{\kappa,2}. \tag{6.31}$$

By the log-convexity of the L^p-norm,

$$\|T_j f_2(\cdot,t)\|_{L^{q_\kappa}(\widetilde{B};h_\kappa^2)} \le \|T_j f_2(\cdot,t)\|^{1-\frac{2}{q_\kappa}}_{L^\infty(\widetilde{B};h_\kappa^2)}\|T_j f_2(\cdot,t)\|^{\frac{2}{q_\kappa}}_{L^2(\widetilde{B};h_\kappa^2)}.$$

Thus, it suffices to show (6.31) for $p=\infty$ and $p=2$.

To show (6.31) for $p=\infty$, we observe that by (6.29), for $t\in[2^{-1},4]$ and $x\in\widetilde{B}$,

$$\begin{aligned}
|T_j f_2(x,t)| &= \Big|\int_I \rho^{2\lambda_\kappa}\int_{\mathbb{S}^{d-1}}\mathcal{F}_\kappa f(\rho\xi)(\mathcal{F}_\kappa\eta_j *_\kappa \phi^t)(\rho\xi)E_\kappa(\mathbf{i}\rho x,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho\Big|\\
&\le \int_I \rho^{2\lambda_\kappa}\sup_{y\in\widetilde{B}_\rho}\Big|\int_{\mathbb{S}^{d-1}}\mathcal{F}_\kappa f(\rho\xi)(\mathcal{F}_\kappa\eta_j *_\kappa \phi^t)(\rho\xi)E_\kappa(\mathbf{i}y,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\Big|\,d\rho,
\end{aligned}$$

which, using the fact (6.30) and Corollary 4.2(ii) with $q=\infty$, is estimated above by

$$C\Big(\frac{2^{j(2\lambda_\kappa+1)}}{\text{meas}_\kappa(B)}\Big)^{\frac12}\int_I \rho^{2\lambda_\kappa}\Big(\int_{\mathbb{S}^{d-1}}|\mathcal{F}_\kappa\eta_j *_\kappa \phi^t(\rho\xi)|^2|\mathcal{F}_\kappa f(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\Big)^{\frac12}\,d\rho.$$

Thus, by Lemma 6.4, it follows that

$$\begin{aligned}
&\sup_{x\in\widetilde{B}}|T_j f_2(x,t)|\\
&\quad\le C2^{-jN}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\text{meas}_\kappa(B)}\Big)^{\frac12}\int_I (1+|\rho|)^{-N}\rho^{2\lambda_\kappa}\Big(\int_{\mathbb{S}^{d-1}}|\mathcal{F}_\kappa f(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\Big)^{\frac12}\,d\rho\\
&\quad\le C2^{-jN}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\text{meas}_\kappa(B)}\Big)^{\frac12}\Big(\int_I \rho^{2\lambda_\kappa}\int_{\mathbb{S}^{d-1}}|\mathcal{F}_\kappa f(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho\Big)^{\frac12}\\
&\quad\le C2^{-jN}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\text{meas}_\kappa(B)}\Big)^{\frac12}\|f_2\|_{\kappa,2},
\end{aligned}$$

where the second step uses the Cauchy-Schwarz inequality. This shows (6.31) for $p = \infty$.

To show (6.31) for $p = 2$, we use Plancherel's theorem and Lemma 6.4 to obtain

$$\begin{aligned}\|T_j f_2(\cdot, t)\|_{\kappa,2}^2 &= c\int_{\mathbb{R}^d} |\mathcal{F}_\kappa f_2(\xi)|^2 |\mathcal{F}_\kappa \eta_j *_\kappa \phi^t(\xi)|^2 h_\kappa^2(\xi)\, d\xi \\ &\le c4^{-jN} \int_{\mathbb{R}^d} |\mathcal{F}_\kappa f_2(\xi)|^2 (1 + \|\xi\|)^{-2N} h_\kappa^2(\xi)\, d\xi \le C4^{-jN} \|f_2\|_{\kappa,2}^2.\end{aligned}$$

Putting the above estimates together, we obtain (6.31) for $p = q_\kappa$ and hence complete the proof of the lemma. ☐

Lemma 6.7. *For every nonnegative function g on $\mathbb{R}^d$,*

$$\int_{2^{-1}}^4 \int_{\mathbb{R}^d} |f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x)|^2 g(x) h_\kappa^2(x)\, dx\, dt \le C\alpha^{\frac{1}{\lambda_\kappa+1}} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\, dx.$$

Proof. Recall that $T_j : L^2(\mathbb{R}^d; h_\kappa^2) \to L^{q_\kappa}(L^2[2^{-1}, 4])$ is a vector-valued operator given by

$$T_j f(x,t) = f *_\kappa (\eta_j (\mathcal{F}_\kappa \phi)_{t^{-1}})(x) = c_\kappa \int_{\mathbb{R}^d} f(y) T^y ((\mathcal{F}_\kappa \phi)_{t^{-1}} \eta_j)(x) h_\kappa^2(y)\, dy, \qquad t \in [2^{-1}, 4].$$

Since η_j is supported in the ball $B_{2^{j+2}}(0)$, we conclude from (2.16) that

$$T^y(\eta_j(\mathcal{F}_\kappa \phi)_{t^{-1}})(x) = 0$$

unless $\big||x_n| - |y_n|\big| \le 2^{j+2}$ for $n = 1, 2, \cdots, d$. Thus, for each fixed $y \in \mathbb{R}^d$, the function $x \mapsto T^y(\eta_j(\mathcal{F}_\kappa\phi)_{t^{-1}})(x)$ is supported in the set $\bigcup_{\sigma\in\mathbb{Z}_2^d} B(y\sigma, \sqrt{d}2^{j+2})$. This implies that for every function f that is supported in a ball $B = B(\omega, c2^j)$, $T_j f(\cdot, t)$ is supported in the set $\widetilde{B} = \bigcup_{\sigma\in\mathbb{Z}_2^d} B(\omega\sigma, c'2^j)$. This means that the operator T_j has the locality property stated in Lemma 6.3. On the other hand, if f is supported in the ball B, then by Lemma 6.6,

$$\begin{aligned}\Big(\int_{\mathbb{R}^d} \|T_j f(x,\cdot)\|_{L^2([2^{-1},4])}^{q_\kappa} h_\kappa^2(x)\, dx\Big)^{\frac{1}{q_\kappa}} &= \Big(\int_{\widetilde{B}} \|T_j f(x,\cdot)\|_{L^2([2^{-1},4])}^{q_\kappa} h_\kappa^2(x)\, dx\Big)^{\frac{1}{q_\kappa}} \\ &\le C2^{-j} \Big(\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac12 - \frac{1}{q_\kappa}} \|f\|_{\kappa,2}.\end{aligned}$$

Thus, using Lemma 6.3, we conclude that for every test function f on $\mathbb{R}^d$ (which is not necessarily compactly supported) and any nonnegative function g,

$$\begin{aligned}&\int_{\mathbb{R}^d} \|T_j f(x,\cdot)\|_{L^2([2^{-1},4])}^2 g(x) h_\kappa^2(x)\, dx \\ &\le C4^{-j} 2^{j(2\lambda_\kappa+1)/r} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\, dx, \quad j = i, i+1, \cdots,\end{aligned} \tag{6.32}$$

with $r = (q_\kappa/2)' = \lambda_\kappa + 1$.

Let $\varepsilon \in (0,1)$ be such that $0 < \varepsilon < 2 - \frac{2\lambda_\kappa+1}{r} = \frac{1}{\lambda_\kappa+1}$. Using (6.26) and the Cauchy-Schwarz inequality, we have

$$\|Tf(x,\cdot)\|^2_{L^2([2^{-1},4])} \le \Big(\sum_{j=i}^{\infty} \|T_j f(x,\cdot)\|_{L^2([2^{-1},4])}\Big)^2$$
$$\le C2^{-i\varepsilon} \sum_{j=i}^{\infty} 2^{j\varepsilon} \|T_j f(x,\cdot)\|^2_{L^2([2^{-1},4])}.$$

It then follows from (6.32) that

$$\int_{2^{-1}}^{4} \int_{\mathbb{R}^d} |f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x)|^2 g(x) h_\kappa^2(x)\, dx\, dt = \int_{\mathbb{R}^d} \|Tf(x,\cdot)\|^2_{L^2([2^{-1},4])} g(x) h_\kappa^2(x)\, dx$$
$$\le C2^{-i\varepsilon} \sum_{j=i}^{\infty} 2^{j\varepsilon} \int_{\mathbb{R}^d} \|T_j f(x,\cdot)\|^2_{L^2([2^{-1},4])} g(x) h_\kappa^2(x)\, dx$$
$$\le C2^{-i\varepsilon} \sum_{j=i}^{\infty} 2^{j\varepsilon} 2^{-2j} 2^{j(2\lambda_\kappa+1)/r} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h^2(x)\, dx$$
$$\le C\alpha^{\frac{1}{1+\lambda_\kappa}} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\, dx.$$

This completes the proof of the lemma. □

Lemma 6.8. *For any $k \in \mathbb{Z}$ and any function $g \ge 0$,*

$$\int_{2^{k-1}}^{2^{k+2}} \int_{\mathbb{R}^d} |f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x)|^2 g(x) h_\kappa^2(x)\, dx \frac{dt}{t} \le C\alpha^{\frac{1}{1+\lambda_\kappa}} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\, dx,$$

where $\phi^t(x) = \phi(x/t)$.

Proof. For simplicity, we define the dilation operator Dil_u^p for $1 \le p \le \infty$ and $u > 0$ by

$$\mathrm{Dil}_u^p f(x) = u^{-(2\lambda_\kappa+1)/p} f(x/u), \quad x \in \mathbb{R}^d.$$

Clearly, Dil_u^p is an isometry on the space $L^p(\mathbb{R}^d; h_\kappa^2)$ and satisfies $\mathcal{F}_\kappa \mathrm{Dil}_u^p = \mathrm{Dil}_{u^{-1}}^{p'} \mathcal{F}_\kappa$.

Note that for any $u > 0$,

$$(f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}})^\wedge(\xi) = \mathcal{F}_\kappa f(\xi)\phi(t^{-1}\xi) = \mathrm{Dil}_{u^{-1}}^\infty \Big[(\mathrm{Dil}_u^\infty \mathcal{F}_\kappa f)(\mathrm{Dil}_{ut}^\infty \phi)\Big](\xi)$$
$$= \mathcal{F}_\kappa \mathrm{Dil}_u^1 \Big[(\mathrm{Dil}_{u^{-1}}^1 f) *_\kappa (\mathrm{Dil}_{(ut)^{-1}}^1 \mathcal{F}_\kappa \phi)\Big](\xi).$$

It follows that for any test function f on $\mathbb{R}^d$, any $x \in \mathbb{R}^d$, and $u > 0$,

(6.33)
$$f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x) = \mathrm{Dil}_u^1 (\mathrm{Dil}_{u^{-1}}^1 f) *_\kappa (\mathcal{F}_\kappa \phi_{(ut)^{-1}})(x) = (\mathrm{Dil}_{u^{-1}}^\infty f) *_\kappa \mathcal{F}_\kappa \phi_{(ut)^{-1}}(u^{-1}x).$$

Thus, using Lemma 6.7, we obtain

$$\begin{aligned}
&\int_{2^{k-1}}^{2^{k+2}} \int_{\mathbb{R}^d} |f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x)|^2 g(x) h_\kappa^2(x)\, dx\, \frac{dt}{t} \\
&\quad = \int_{2^{-1}}^{4} \int_{\mathbb{R}^d} |f *_\kappa \mathcal{F}_\kappa \phi_{(2^k t)^{-1}}(x)|^2 g(x) h_\kappa^2(x)\, dx \frac{dt}{t} \\
&\quad = \int_{2^{-1}}^{4} \int_{\mathbb{R}^d} |(\mathrm{Dil}_{2^k}^\infty f) *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(2^k x)|^2 g(x) h_\kappa^2(x)\, dx\, \frac{dt}{t} \\
&\quad = 2^{-k(2\lambda_\kappa+1)} \int_{2^{-1}}^{4} \int_{\mathbb{R}^d} |(\mathrm{Dil}_{2^k}^\infty f) *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x)|^2 g(2^{-k}x) h_\kappa^2(x)\, dx\, \frac{dt}{t} \\
&\quad \le C\alpha^{\frac{1}{1+\lambda_\kappa}} 2^{-k(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} |f(2^{-k}x)|^2 M_{\kappa,r} g(2^{-k}x) h_\kappa^2(x)\, dx \\
&\quad = C\alpha^{\frac{1}{1+\lambda_\kappa}} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\, dx,
\end{aligned}$$

where we used (6.33) with $u = 2^{-k}$ in the second step, and Lemma 6.7 and the fact that $M_{\kappa,r}\mathrm{Dil}_u^\infty = \mathrm{Dil}_u^\infty M_{\kappa,r}$ for any $u > 0$ in the fourth step. This completes the proof of the lemma. □

We are now in a position to prove Theorem 6.5.

Proof of Theorem 6.5. Without loss of generality, we may assume that $M_\kappa(g^r)(x) < \infty$ a.e. on $\mathbb{R}^d$. Let η be a C^∞-radial function on $\mathbb{R}^d$ such that $\eta(x) = 1$ for $\|x\| \le 1$ and $\eta(x) = 0$ for $\|x\| \ge 2$. Let $\theta(x) = \eta(x) - \eta(2x)$. Clearly, θ is a C^∞ radial function supported on the set $\{x \in \mathbb{R}^d : 2^{-1} \le |x| \le 2\}$ and satisfying that $\sum_{k\in\mathbb{Z}} \theta(2^k x) = 1$ for all $x \in \mathbb{R}^d \setminus \{0\}$. Let Ψ denote the radial Schwarz function on $\mathbb{R}^d$ such that $\mathcal{F}_\kappa \Psi = \theta$. For $j \in \mathbb{Z}$, define $L_k f(x) = f *_\kappa \Psi_{-k}(x)$ with $\Psi_{-k}(x) = 2^{-k(2\lambda_\kappa+1)}\Psi(2^{-k}x)$. Then

$$f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x) = \sum_{k\in\mathbb{Z}} (L_k f) *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x).$$

Note that

$$((L_k f) *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}})^\wedge(\xi) = \theta(2^k \xi)\mathcal{F}_\kappa f(\xi)\phi(t^{-1}\xi)$$

is supported on a set where

$$2^{k-2} \le \frac{3}{4}\|\xi\|^{-1} \le t^{-1} \le \frac{5}{4}\|\xi\|^{-1} \le 2^{k+2}.$$

This implies that

$$f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x) = \sum_{\{k\in\mathbb{Z}: 2^{-k-2} \le t \le 2^{-k+2}\}} (L_k f) *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x),$$

and hence

$$|f *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x)|^2 \le C \sum_{\{k\in\mathbb{Z}:\ 2^{-k-2} \le t \le 2^{-k+2}\}} |(L_k f) *_\kappa (\mathcal{F}_\kappa \phi)_{t^{-1}}(x)|^2.$$

Thus, setting $d\mu_\kappa(x) = h_\kappa^2(x)\,dx$, we obtain

$$\int_{\mathbb{R}^d}\Big(\int_0^\infty |f *_\kappa (\mathcal{F}_\kappa\phi)_{t^{-1}}(x)|^2 \frac{dt}{t}\Big) g(x) d\mu_\kappa(x)$$
$$\leq C\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^{-k-2}}^{2^{-k+2}} |(L_k f) *_\kappa (\mathcal{F}_\kappa\phi)_{t^{-1}}(x)|^2 \frac{dt}{t} g(x) d\mu_\kappa(x),$$

which, using Lemma 6.8, is estimated above by

$$C\alpha^{\frac{1}{1+\lambda_\kappa}}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d} |L_k f|^2 M_{\kappa,r} g(x) d\mu_\kappa(x)$$
$$= C\alpha^{\frac{1}{1+\lambda_\kappa}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|L_k f|^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^d,(M_\kappa(|g|^r))^{1/r}d\mu_\kappa)}.$$

However, by the weighted Littlewood-Paley inequality (i.e., Corollary 5.3), we have

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|L_k f|^2\Big)^{1/2}\Big\|^2_{L^2((M_\kappa(|g|^r))^{1/r}d\mu_\kappa)} \leq C\|f\|^2_{L^2((M_\kappa(|g|^r))^{1/r}d\mu_\kappa)}$$
$$= C\int_{\mathbb{R}^d}|f(x)|^2 M_{\kappa,r} g(x)\, d\mu_\kappa(x).$$

This completes the proof of the theorem. □

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