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A RESTRICTED MAGNUS PROPERTY FOR PROFINITE SURFACE GROUPS

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Abstract. Magnus proved in 1930 that, given two elements x and y of a finitely generated free group F with equal normal closures $\langle x\rangle^F = \langle y\rangle^F$, x is conjugated either to y or y^{-1}. More recently, this property, called the Magnus property, has been generalized to oriented surface groups.

In this paper, we consider an analogue property for profinite surface groups. While the Magnus property, in general, does not hold in the profinite setting, it does hold in some restricted form. In particular, for $\mathscr{S}$ a class of finite groups, we prove that if x and y are *algebraically simple* elements of the pro-$\mathscr{S}$ completion $\widehat{\Pi}^{\mathscr{S}}$ of an orientable surface group Π such that, for all $n \in \mathbb{N}$, there holds $\langle x^n\rangle^{\widehat{\Pi}^{\mathscr{S}}} = \langle y^n\rangle^{\widehat{\Pi}^{\mathscr{S}}}$, then x is conjugated to y^s for some $s \in (\widehat{\mathbb{Z}}^{\mathscr{S}})^*$. As a matter of fact, a much more general property is proved and further extended to a wider class of profinite completions.

The most important application of the theory above is a generalization of the description of centralizers of profinite Dehn twists given in [Marco Boggi, Trans. Amer. Math. Soc. 366 (2014), 5185–5221] to profinite Dehn multitwists.

1. Introduction

Let Π be an oriented surface group, that is to say, the fundamental group of an oriented Riemann surface of finite type S.

Definition 1.1. *A class of finite groups* (cf. [2, Definition 3.1]) is a full subcategory $\mathscr{S}$ of the category of finite groups which is closed under taking subgroups, homomorphic images, and extensions (meaning that a short exact sequence of finite groups is in $\mathscr{S}$ whenever its exterior terms are). We always assume that $\mathscr{S}$ contains a non-trivial group.

For $\mathscr{S}$ a class of finite groups, *the pro-$\mathscr{S}$ completion* $\widehat{\Pi}^{\mathscr{S}}$ *of* Π is the inverse limit of the finite quotients of Π which belong to $\mathscr{S}$. The profinite group $\widehat{\Pi}^{\mathscr{S}}$ is also called *a pro-$\mathscr{S}$ surface group.*

Let us give some examples. Fixed a non-empty set of primes Λ, then let $\mathscr{S}$ be the category of finite Λ-groups, i.e., finite groups whose orders are product of primes in Λ. In this case, the corresponding profinite completion is denoted by $\widehat{\Pi}^{\Lambda}$ and called the *pro-Λ completion of* Π. The two cases of interest are usually when Λ is the set of all primes, in which case $\widehat{\Pi}^{\Lambda}$ is just the profinite completion $\widehat{\Pi}$ of Π, and when Λ consists of only one prime p, in which case $\widehat{\Pi}^{\Lambda}$ is the pro-p completion of Π and is denoted by $\widehat{\Pi}^{(p)}$. Another important example of class of finite groups is the class of finite solvable groups.

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The main purpose of this paper is to prove for the pro-$\mathscr{S}$ surface group $\widehat{\Pi}^{\mathscr{S}}$ some properties which are analogous to the Magnus property proved for Π in [6] and [7].

For a given element x of a (profinite) group G, let us denote by $\langle x\rangle$ and $\langle x\rangle^G$, respectively, the (closed) subgroup and the (closed) normal subgroup generated (topologically) by x in G. The Magnus property for the discrete group Π says that if, for two given elements $x, y \in \Pi$, the normal subgroup $\langle x\rangle^{\Pi}$ equals the normal subgroup $\langle y\rangle^{\Pi}$, then x is conjugated either to y or to y^{-1}. This property cannot be transported literally to the profinite case since $\widehat{\mathbb{Z}}$ has more units than just $\{\pm 1\}$ and so the property would fail already for $\widehat{\mathbb{Z}}$. Moreover, even if we take this into account, there are counterexamples to the analogue property which can be formulated for the profinite completion $\widehat{\Pi}^{\mathscr{S}}$.

A counterexample for a free profinite group of finite rank is the following. Let us denote by $M(G)$ the intersection of all maximal normal subgroups of a group G. Then let U be a normal subgroup of a finitely generated free profinite group F such that $U/M(U)$ is a direct product of non-abelian simple groups (for instance, let U be the kernel of the natural epimorphism of F onto the maximal prosolvable quotient of F).

By [23, Proposition 8.3.6], the subgroup U is the normal closure of an element $u \in U$ if and only if $U/M(U)$ is the normal closure of $u \cdot M(U)$. Now, in an infinite direct product of non-abelian simple groups, there are plenty of elements and groups generated by them which are non-conjugate but normally generate $U/M(U)$. For instance, any element with non-trivial projection to every direct simple factor of $U/M(U)$ has this property. However, having different order in some of the projections, such elements are not conjugate.

A counterexample for the pro-p case is instead the following. Let $F = F(x, y)$ be the free pro-p group in two generators. Then, the normal closure of x has generators which are not conjugated to powers of x. Indeed, this follows from the fact that $\langle x\rangle^F$, modulo its Frattini subgroup, identifies with the completed group ring $\mathbb{F}_p[[\langle y\rangle]]$, and this has more units than just the powers of y.

For this reason, the profinite analogue of the Magnus property should rather be formulated by saying that if $x, y \in \widehat{\Pi}^{\mathscr{S}}$ satisfy the stronger condition $\langle x^n\rangle^{\widehat{\Pi}^{\mathscr{S}}} = \langle y^n\rangle^{\widehat{\Pi}^{\mathscr{S}}}$, for all n which are a product of primes in $\Lambda_{\mathscr{S}}$, then x is conjugated to a power y^s, where s is a unit of the standard pro-$\mathscr{S}$ cyclic group $\widehat{\mathbb{Z}}^{\mathscr{S}}$, i.e., the pro-$\mathscr{S}$ completion of $\mathbb{Z}$. The latter group is more explicitly described as follows. Let $\Lambda_{\mathscr{S}}$ be the set of primes that occur as orders of groups in $\mathscr{S}$. There is then a natural isomorphism $\widehat{\mathbb{Z}}^{\mathscr{S}} \cong \prod_{p\in\Lambda_{\mathscr{S}}} \mathbb{Z}_p$.

A first instance of the profinite analogue of the Magnus property for a free profinite group of finite rank follows from a deep theorem of Wise (cf. Section 2), but with the further restriction that one of the two elements be abstract.

Theorem 1.2. *Let $\widehat{F}$ be a free profinite group of finite rank and let $x, y \in \widehat{F}$, with y an element contained in an abstract dense free subgroup F of $\widehat{F}$. If, for all $n \in \mathbb{N}^+$, there is a $k_n \in \mathbb{N}^+$ such that $x^{k_n} \in \langle y^n\rangle^{\widehat{F}}$, then x is conjugated to y^s for some $s \in \widehat{\mathbb{Z}}$.*

We do not know in which generality the Magnus property holds for profinite surface groups. In what follows, we will restrict to the case where one of the two elements satisfies some geometric conditions similar to those considered in [6].

Definition 1.3.

(i) Let us fix a presentation of Π as the fundamental group of a Riemann surface S. A subset of non-trivial elements $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \Pi$ is *simple* if there is a set of disjoint simple closed curves (briefly, *s.c.c.*'s) $\tilde{\sigma} = \{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_h\}$ on S, such that they are two by two non-isotopic and $\tilde{\gamma}_i$ belongs to the free homotopy class of γ_i for $i = 1, \ldots, h$. An s.c.c. on S is *peripheral* if it bounds a 1-punctured disc.

(ii) Let $\mathscr{S}$ be a class of finite groups. *A pro-$\mathscr{S}$ surface group* $\widehat{\Pi}^{\mathscr{S}}$ is the pro-$\mathscr{S}$ completion of a surface group Π. It is endowed with a natural monomorphism with dense image $\Pi \hookrightarrow \widehat{\Pi}^{\mathscr{S}}$. Let Γ be the group of mapping classes of self-homeomorphisms of S fixing the base point of Π and let $\widehat{\Gamma}^{\mathscr{S}}$ be its closure in $\mathrm{Aut}(\widehat{\Pi}^{\mathscr{S}})$. Then, a subset of elements $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \widehat{\Pi}^{\mathscr{S}}$ is *simple* if it is in the orbit of the image of a simple set $\sigma' \subset \Pi$ for the action of $\widehat{\Gamma}^{\mathscr{S}}$.

(iii) A subset of elements $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \widehat{\Pi}^{\mathscr{S}}$ is *algebraically simple* if it is in the $\mathrm{Aut}(\widehat{\Pi}^{\mathscr{S}})$–orbit of the image of a set $\sigma' \subset \Pi$ which is simple for some presentation of Π as the fundamental group of a Riemann surface.

Remark 1.4. Let Π be a free group of rank n and let $\widehat{\Pi}^{\mathscr{S}}$ be either its pro-Λ completion, for some non-empty set of primes Λ, or its pro-solvable completion. Then, given a minimal set $\{\alpha_1, \ldots, \alpha_n\}$ of topological generators for $\widehat{\Pi}^{\Lambda}$, any element of this set and any product of commutators $\prod_{i=1}^{k}[\alpha_i, \alpha_{n-i}]$ and of commutators and generators of the form $\prod_{i=1}^{k}[\alpha_i, \alpha_{n-i}]\alpha_{i+1}\alpha_{i+2}\ldots\alpha_j$, for $1 \leq k \leq [n/2]$ and $i+1 \leq j \leq n-k-1$, are algebraically simple.

Part (i) of the above definition can be rephrased, group-theoretically, saying that there is a graph of groups $\mathscr{G}$ whose vertex groups are finitely generated free groups of rank at least 2, together with an isomorphism $\pi_1(\mathscr{G}) \cong \Pi$ which identifies the set of edge groups of $\mathscr{G}$ with the set of cyclic subgroups of Π generated by non-peripheral elements of σ.

Part (iii) just says that, modulo automorphisms, the profinite group $\widehat{\Pi}^{\mathscr{S}}$ can be realized as a profinite completion of a discrete group Π of the above type.

In this setting, we are actually going to prove (cf. Section 3) the following stronger statement.

Theorem 1.5. *Let $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \widehat{\Pi}^{\mathscr{S}}$ be an algebraically simple subset and let us denote by $\sigma_n^{\widehat{\Pi}^{\mathscr{S}}}$, for $n \in \mathbb{N}^+$, the closed normal subgroup of $\widehat{\Pi}^{\mathscr{S}}$ generated by n-th powers of elements of σ. Let $y \in \widehat{\Pi}^{\mathscr{S}}$ be an element such that, for all n a product of primes in $\Lambda_{\mathscr{S}}$, there exists a $k_n \in \mathbb{N}^+$ with the property that $y^{k_n} \in \sigma_n^{\widehat{\Pi}^{\mathscr{S}}}$. Then, for some $s \in \widehat{\mathbb{Z}}^{\mathscr{S}}$ and $i \in \{1, \ldots, h\}$, the element y is conjugated to the element γ_i^s.*

An immediate corollary of Theorem 1.5 is the restricted Magnus property for profinite surface groups, mentioned above.

Corollary 1.6. *Let $x \in \widehat{\Pi}^{\mathscr{S}}$ be an algebraically simple element and let $y \in \widehat{\Pi}^{\mathscr{S}}$ be an arbitrary element. Let us denote by $\langle x^n \rangle^{\widehat{\Pi}^{\mathscr{S}}}$ and $\langle y^n \rangle^{\widehat{\Pi}^{\mathscr{S}}}$ the closed normal subgroups of $\widehat{\Pi}^{\mathscr{S}}$ generated respectively by x^n and y^n, for $n \in \mathbb{N}^+$. If, for all n a product of primes in $\Lambda_{\mathscr{S}}$, there holds $\langle x^n \rangle^{\widehat{\Pi}^{\mathscr{S}}} = \langle y^n \rangle^{\widehat{\Pi}^{\mathscr{S}}}$, then, for some $s \in (\widehat{\mathbb{Z}}^{\mathscr{S}})^*$, the element y is conjugated to x^s.*

The restricted Magnus property has applications to Grothendieck-Teichmüller theory. In order to show this, we need more definitions. A simple closed curve on a hyperbolic Riemann surface S is described by an unordered pair $\{\gamma, \gamma^{-1}\}$ in the set of conjugacy classes $\Pi/\sim$ of elements of the fundamental group Π of S. Let $\mathscr{L}$ be the set of non-peripheral simple closed curves on S. We then define the set of (non-peripheral) *profinite simple closed curves* $\widehat{\mathscr{L}}$ as the closure of $\mathscr{L}$ in the profinite set of pairs $\mathscr{P}_2(\widehat{\Pi}/\sim)$ in the set of conjugacy classes $\widehat{\Pi}/\sim$. In [4, Theorem 4.2], it was proved that the profinite set $\widehat{\mathscr{L}}$ parameterizes the set of *profinite Dehn twists* in the procongruence Teichmüller group $\check{\Gamma}(S)$ associated to S (cf. Section 5). This is the completion of the mapping class group $\Gamma(S)$ associated to the surface S with respect to the congruence topology.

For K a normal open subgroup of $\widehat{\Pi}$, let $p_K\colon\ S_K \to S$ be the associated covering with covering transformation group $G_K := \widehat{\Pi}/K$. We can naturally associate to an element $\gamma \in \widehat{\mathscr{L}}$ a subspace $V_{K,\gamma}$ of $H_1(\overline{S}_K, \mathbb{Q}_\ell)$, where $\overline{S}_K$ is the closed Riemann surface obtained from S_K filling in the punctures, in the following way. Let us also denote by $\gamma \in \widehat{\Pi}$ an element in the class of $\gamma \in \widehat{\mathscr{L}}$ and let γ^{ν_K} be the smallest positive power of γ contained in K. We then let $V_{K,\gamma}$ be the subspace of $H_1(\overline{S}_K, \mathbb{Q}_\ell)$ generated by the G_K-orbit of the image of γ^{ν_K} in $H_1(\overline{S}_K, \mathbb{Q}_\ell)$.

The main result of Section 5 (cf. Theorem 5.6) is that the sets $\{V_{K,\gamma}\}_{K \lhd_o \widehat{\Pi}}$ separate elements of $\widehat{\mathscr{L}}$. This is a non-trivial result even if restricted to the subset of simple closed curves $\mathscr{L}$ of $\widehat{\mathscr{L}}$. For $\mathscr{L}$, this result was proved, with different techniques, in [5] (cf. Theorem 5.1 therein).

The above results are then used to generalize the description of centralizers of profinite Dehn twists in the procongruence Teichmüller group $\check{\Gamma}(S)$, given in [4], to profinite multitwists. A multitwist in the mapping class group $\Gamma(S)$ is the product of a set of Dehn twists along disjoint simple closed curves on S. We show, in particular, that the centralizer in $\check{\Gamma}(S)$ of a multitwist of $\Gamma(S)$ is the closure of the centralizer of the same element in $\Gamma(S)$. The result we prove is actually stronger, but we refer to Section 6 for the precise statement. So far, this is probably the main application of the restricted Magnus property.

In Section 7, we give a linear version of some classical faithfulness results on Galois representations associated to projective hyperbolic curves over number fields which appeared in [16] and [4].

2. The proof of Theorem 1.2

In order to prove Theorem 1.2, we need a preliminary result of independent interest, which follows from the work of Wise [27].

Theorem 2.1. *Let $G = \langle F \mid r^n \rangle$ be a one-relator group with torsion and let $y \in \widehat{G}$ be a torsion element of its profinite completion. Then, y is conjugate in $\widehat{G}$ to a power of the image $\bar{r}$ of r in G.*

Proof. The result is well-known for the discrete group G (see for example [1, Theorem 9.3]). Hence it is enough to show that a torsion element $y \in \widehat{G}$ is conjugate to an element of G. Every one-relator group G embeds naturally into a free product $G' = G * \mathbb{Z}$ which is an HNN extension $HNN(H, M, t)$ of a one-relator group H with shorter relator (length of the reduced word), where M is a free subgroup generated by subsets of the generators of the presentation of G (cf. the Magnus-Moldavanskii construction in [27, Section 18.b]). The hierarchy is finite; i.e., continuing further the splitting into such HNN-extensions, we terminate at a virtually free group of the form $\mathbb{Z}/n * F$, where F is free. Let us use induction on the length of the relator to prove that y is conjugate to an element of G. If the length is zero, this means that $r = 1$ and G is free and then $\widehat{G}$ is torsion free. In this case, the claim trivially holds.

According to [27, Theorem 18.1], the Magnus-Moldavanskii hierarchy is quasi-convex for any one-relator group with torsion; i.e., for one-relator groups with torsion, the subgroups H,M,M^t are quasi-convex at each level of the hierarchy. Wise showed that G has a finite index subgroup G_0 that embeds as a quasi-convex subgroup of a right-angled Artin group. It follows that every quasi-convex subgroup of G is a virtual retract and is hence separable (cf. [14, Theorem 7.3]).

Thus the hierarchy is separable (including finite index subgroups of the groups of the hierarchy), and so the profinite completion functor extends the hierarchy on G to a hierarchy on $\widehat{G}$. In particular $\widehat{G}' = \widehat{G} \amalg \widehat{\mathbb{Z}} = HNN(\widehat{H}, \widehat{M}, t)$. By [28, Theorem 3.10], any torsion element of a profinite HNN-extension is conjugate to an element of the base group. We may then assume that our torsion element y is in $\widehat{H}$ and use the induction hypothesis to conclude that it is conjugated to an element of H and so of G'. Since $\widehat{G}$ is a free factor of $\widehat{G}'$ and $y \in \widehat{G}$, it follows that y is actually conjugated to an element of G. □

Let us recall that an abstract group G is *good* if the natural homomorphism $G \to \widehat{G}$ of the group to its profinite completion induces an isomorphism on cohomology with finite coefficients (cf. [24, Exercises, §2.6]). From the proof of Theorem 2.1 and [13, Theorem 1.4], Theorem 2.2 then follows.

Theorem 2.2. *One-relator groups with torsion are good.*

Proof. Let G be a one-relator group with torsion. As in the proof of Theorem 2.1, we use induction on the length of the relator of G. We also use the notation of that proof. For $G = \mathbb{Z}/n * F$, the result is clear, and this provides the base for the induction. The subgroup H of $G' = G * \mathbb{Z} = HNN(H, M, t)$ satisfies the induction hypothesis, and so by [13, Proposition 3.5] combined with the last paragraph of the preceding proof we deduce that HNN-extension $G' = HNN(H, M, t)$ is good. But the cohomology of a free product is the sum of cohomologies of the factors in both the abstract and the profinite situation. Therefore the isomorphism $H^i(\widehat{G}', M) \to H^i(G', M)$ restricts to the required isomorphism $H^i(\widehat{G}, M) \to H^i(G, M)$, for $i \geq 0$. □

Proof of Theorem 1.2. By Theorem 2.1, the element $x^{k_n}\langle y^n\rangle^{\widehat{F}}/\langle y^n\rangle^{\widehat{F}}$ of the quotient group $\widehat{F}/\langle y^n\rangle^{\widehat{F}}$ is conjugated to a power of the element $y\langle y^n\rangle^{\widehat{F}}/\langle y^n\rangle^{\widehat{F}}$ for every $n \in \mathbb{N}^+$. The result then follows taking the inverse limit of all these quotients for $n \in \mathbb{N}^+$. □

3. A geometric proof of Theorem 1.5

An (*orientation-preserving*) *Fuchsian group* Π is a group which admits a presentation of the form

$$\Pi = \langle \alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g, x_1, \dots, x_d, y_1, \dots, y_s | \; x_1 \dots x_d \cdot y_1 \dots y_s \cdot \prod_{i=1}^{g} [\alpha_i, \beta_i]; x_1^{m_1}, \dots, x_d^{m_d}, \text{ for } m_1, \dots, m_d \in \mathbb{N}^+\rangle.$$

The *measure* $\mu(\Pi)$ of such a Fuchsian group is defined by

$$\mu(\Pi) = 2g - 2 + \sum_{i=1}^{d}(1 - \frac{1}{m_i}) + s.$$

The Fuchsian group Π is *hyperbolic* if $\mu(\Pi) > 0$. Geometrically, this can be reformulated by saying that Π is a finitely generated non-elementary discrete group of isometries of the hyperbolic plane. Hyperbolic Fuchsian groups arise as topological fundamental groups of complex hyperbolic orbicurves (see, for instance, [21] for a definition).

The integer g is the genus of an orbifold Riemann surface S whose fundamental group has the standard presentation given to the hyperbolic Fuchsian group Π. Let us then observe that the same Fuchsian group as an abstract group can be given distinct presentations corresponding to orbifold Riemann surfaces of different genuses.

Following [21], the *order* of the Fuchsian group Π is the least common multiple of the integers $m_1, \ldots, m_d$, i.e., of the orders of the cyclic subgroups generated by $x_1, \ldots, x_d$ in Π. A finite subgroup of Π in the conjugacy class of the cyclic subgroup $\langle x_i \rangle$, for $i = 1, \ldots, d$, is called *a decomposition group.*

Let S be an orbifold Riemann surface whose fundamental group can be identified with the Fuchsian group Π described above. Then, to a finite index subgroup K of Π is associated an unramified covering $S_K \to S$. If K is a normal subgroup, the covering is normal with covering transformation group the quotient $G_K := \Pi/K$. The orbifold Riemann surface S_K is representable; i.e., the decomposition groups of its points are all trivial if and only if $\langle x_i \rangle^a \cap K = \{1\}$, for all $i = 1, \ldots, d$ and all elements $a \in \Pi$. If K is a normal subgroup, then it is enough to ask that $\langle x_i \rangle \cap K = \{1\}$, for all $i = 1, \ldots, d$.

The following is a slight generalization, in different terminology, of [21, Lemma 2.11], of which, however, we prefer to give an independent proof.

Lemma 3.1. *Let* Π *be a hyperbolic Fuchsian group with the presentation given above and let* $\mathscr{S}$ *be a class of finite groups. Let us assume that* Π *contains a torsion free normal subgroup* H *such that* $\Pi/H \in \mathscr{S}$. *Let* $\widehat{\Pi}^{\mathscr{S}}$ *be the pro-*$\mathscr{S}$ *completion of* Π. *Then, there hold:*

(i) $\langle x_i \rangle \cap \langle x_j \rangle^h \neq \{1\}$ *if and only if* $i = j$ *and* $h \in \langle x_j \rangle$. *In particular, the subgroup* $\langle x_i \rangle$ *is self-normalizing in the profinite group* $\widehat{\Pi}^{\mathscr{S}}$, *for* $i = 1, \ldots, d$.

(ii) *Every finite non-trivial subgroup* C *of* $\widehat{\Pi}^{\mathscr{S}}$ *is contained in a decomposition group, i.e., in a subgroup* $\langle x_i \rangle^h$ *for some* $i \in \{1, \ldots, d\}$ *and* $h \in \widehat{\Pi}^{\mathscr{S}}$.

Proof. Let S be an orbifold Riemann surface such that $\Pi = \pi_1(S)$ and let $S_H \to S$ be the covering associated to the subgroup H of Π. By hypothesis, S_H is a Riemann surface. *The* $\mathscr{S}$*–solenoid* $\mathbb{S}^{\mathscr{S}}$ (cf. [5] for more details on this construction) is defined to be the inverse limit space $\mathbb{S}^{\mathscr{S}} := \varprojlim_{\Pi/K \in \mathscr{S}} S_K$ of the coverings $S_K \to S$ associated to normal subgroups K of Π such that $\Pi/K \in \mathscr{S}$. Let us observe that $\mathbb{S}^{\mathscr{S}} \cong \mathbb{S}^{\mathscr{S}}_H$, where the latter space is the inverse limit of the coverings $S_K \to S_H$ associated to normal subgroups K of H such that $H/K \in \mathscr{S}$. There is then a series of natural isomorphisms:

$$H^k(\mathbb{S}^{\mathscr{S}}, \mathbb{Z}/p) \cong H^k(\mathbb{S}^{\mathscr{S}}_H, \mathbb{Z}/p) := \varinjlim_{H/K \in \mathscr{S}} H^k(S_K, \mathbb{Z}/p) \cong \varinjlim_{H/K \in \mathscr{S}} H^k(K, \mathbb{Z}/p).$$

It is well known that a surface group is p-good for all primes p (cf. Lemma 5.12 (iii) in [10], for instance). This implies that $\varinjlim_{H/K\in\mathscr{S}} H^k(K,\mathbb{Z}/p)=0$, for $k>0$ and all primes $p\in\Lambda_{\mathscr{S}}$. It follows that, for all $p\in\Lambda_{\mathscr{S}}$, there hold $H^k(\mathbb{S}^{\mathscr{S}},\mathbb{Z}/p)=\{0\}$, for $k>0$, and $H^0(\mathbb{S}^{\mathscr{S}},\mathbb{Z}/p)=\mathbb{Z}/p$.

There is a natural continuous action of $\widehat{\Pi}^{\mathscr{S}}$ on the $\mathscr{S}$–solenoid $\mathbb{S}^{\mathscr{S}}$, and the decompositions groups of $\widehat{\Pi}^{\mathscr{S}}$ naturally identify with the stabilizers of points of $\mathbb{S}^{\mathscr{S}}$ in the inverse image of points of the orbifold Riemann surface S which have non-trivial isotropy groups.

In order to prove (i), it is enough to show that the intersection of the stabilizers of two points P_1 and P_2 contains a cyclic subgroup C_p of prime order $p\in\Lambda_{\mathscr{S}}$ if and only if it holds that $P_1=P_2$.

Let us observe that the solenoid $\mathbb{S}^{\mathscr{S}}$ can be triangulated by a simplicial profinite set in such a way that the inverse images of the orbifold points of S with non-trivial isotropy group are realized inside the set of 0-simplices. Therefore, it is possible to apply to the action of C_p on the solenoid $\mathbb{S}^{\mathscr{S}}$ the results of [25].

By the results of Section 5 in [25], item (b) of Theorem 10.5, Chap. VII in [9] generalizes to profinite spaces. Therefore, since the profinite space $\mathbb{S}^{\mathscr{S}}$ is p-acyclic, for all $p\in\Lambda_{\mathscr{S}}$, it follows that $(\mathbb{S}^{\mathscr{S}})^{C_p}$ is also p-acyclic, where $(\mathbb{S}^{\mathscr{S}})^{C_p}$ is the fixed point set of the action of the p-group C_p on the $\mathscr{S}$–solenoid $\mathbb{S}^{\mathscr{S}}$. In particular, $(\mathbb{S}^{\mathscr{S}})^{C_p}$ is connected and thus consists of only one point. In particular, it holds that $P_1=P_2$.

Let us now prove (ii). Here, we basically proceed as in the proof of [21, Lemma 2.11]. So, let C be a finite subgroup of $\widehat{\Pi}^{\mathscr{S}}$. It then holds that $C\in\mathscr{S}$. Let us assume moreover that C is solvable. Then, by induction on the order of C, we can further assume that either:

(a) C is of prime order $p\in\Lambda_{\mathscr{S}}$ or
(b) C is an extension of a group of prime order $p\in\Lambda_{\mathscr{S}}$ by a non-trivial subgroup $C_1\subseteq C$ which is contained in the decomposition group A.

If (a) is satisfied, then, since the space $\mathbb{S}^{\mathscr{S}}$ is p-acyclic, there holds $(\mathbb{S}^{\mathscr{S}})^C\neq\emptyset$; i.e., the subgroup C is contained in a decomposition group of $\widehat{\Pi}^{\mathscr{S}}$.

If (b) holds, by replacing the profinite group $\widehat{\Pi}^{\mathscr{S}}$ with its open subgroup $C_1\cdot\widehat{H}^{\mathscr{S}}$, we can actually assume that C_1 is a decomposition group. But then, by (i), it is also self-normalizing and it holds that $C_1=C$.

For C any finite subgroup of $\widehat{\Pi}^{\mathscr{S}}$, the above arguments show that the Sylow subgroups of C are cyclic. By a classical result of group theory, the group C is then solvable, and we are reduced to the case already treated. □

Remark 3.2. A consequence of Lemma 3.1 is that two (orientation-preserving) hyperbolic Fuchsian groups are isomorphic if and only if their pro-$\mathscr{S}$ completions are isomorphic, where $\mathscr{S}$ is a class of groups containing their torsion subgroups (e.g. we can take for $\mathscr{S}$ the class of solvable groups). In fact, it is not difficult to see that in order to reconstruct a Fuchsian group Π of the above type, we need the following information:

(i) the rank of the abelianization of Π;
(ii) whether Π is virtually free or not;
(iii) the conjugacy classes of torsion subgroups of Π.

The first two items can be recovered from the homology of $\widehat{\Pi}^{\mathscr{S}}$ because the group Π is $\mathscr{S}$-good, since it contains a surface group N as a normal finite index subgroup such that $\Pi/N \in \mathscr{S}$ and surface groups are $\mathscr{S}$-good. As for the third item, this can be recovered from $\widehat{\Pi}^{\mathscr{S}}$ by Lemma 3.1. In particular, this provides a generalization of [8, Theorem 1.4] in the hyperbolic case.

The next step is to generalize Lemma 3.1 to the fundamental group of a graph of hyperbolic Fuchsian groups. More precisely, let $(\mathscr{G}, Y)$ be a graph of groups such that the vertex groups G_v, for every vertex $v \in v(Y)$, are hyperbolic Fuchsian groups and the edge groups G_e identify with maximal finite subgroups of the vertex groups, for every edge $e \in e(Y)$. Then, we say that $\pi_1(\mathscr{G}, Y)$ is *a nodal Fuchsian group.*

Nodal Fuchsian groups can be characterized as topological fundamental groups of *orbifold nodal Riemann surface*, i.e., nodally degenerate orbifold Riemann surfaces.

As above, if X is an orbifold nodal Riemann surface, then its fundamental group is virtually torsion free if and only if X admits a finite étale (equivalently finite étale Galois) covering $Y \to X$, where Y is a connected nodal Riemann surface.

Definition 3.3.

(i) Let $\mathscr{S}$ be a class of finite groups, i.e., closed by taking subgroups, homomorphic images, and extensions. *A pro-$\mathscr{S}$ nodal Fuchsian group* $\widehat{\Pi}^{\mathscr{S}}$ is the pro-$\mathscr{S}$ completion of a nodal Fuchsian group Π.

(ii) We say that a finite subgroup D of a nodal Fuchsian group $\pi_1(\mathscr{G}, Y)$ or of its pro-$\mathscr{S}$ completion $\widehat{\pi}_1^{\mathscr{S}}(\mathscr{G}, Y)$ is *a decomposition group of type I* if it is in the conjugacy class of a decomposition subgroup I of a vertex group of $(\mathscr{G}, Y)$.

The following result is a result of independent interest we need in order to generalize Lemma 3.1 to pro-$\mathscr{S}$ nodal Fuchsian groups.

Lemma 3.4. *Let $\mathscr{S}$ be a class of finite groups and let $(\mathscr{G}, Y)$ be a finite graph of (discrete) groups, with finite edge groups, such that $G = \pi_1(\mathscr{G}, Y)$ is residually $\mathscr{S}$. Then, the pro-$\mathscr{S}$ completion $\widehat{G}^{\mathscr{S}}$ of G is isomorphic to the pro-$\mathscr{S}$ fundamental group $\widehat{\pi}_1^{\mathscr{S}}(\widehat{\mathscr{G}}^{\mathscr{S}}, Y)$ of the finite graph of pro-$\mathscr{S}$ groups $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y)$ obtained from $(\mathscr{G}, Y)$ by taking the pro-$\mathscr{S}$ completion of each vertex group and the fact that the vertex groups of $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y)$ embed in $\widehat{G}^{\mathscr{S}}$.*

Proof. Since G is residually $\mathscr{S}$ one can find an open (in the pro-$\mathscr{S}$ topology) subgroup H of G that intersects trivially all the edge groups. Then it suffices to show that the pro-$\mathscr{S}$ topology of G or equivalently of H induces the full pro-$\mathscr{S}$ topology on $H \cap \mathscr{G}(v)$. But $H \cap \mathscr{G}(v)$ is a free factor of H, so this statement is just Corollary 3.1.6 in [23]. □

Let us then extend Lemma 3.1 to pro-$\mathscr{S}$ nodal Fuchsian groups.

Lemma 3.5. *Let $\Pi := \pi_1(\mathscr{G}, Y)$ be a nodal Fuchsian group and let $\mathscr{S}$ be a class of finite groups. Let us assume that Π contains a torsion free normal subgroup H such that $\Pi/H \in \mathscr{S}$. Then let $\widehat{\Pi}^{\mathscr{S}}$ be the pro-$\mathscr{S}$ completion of Π. Let D_1 and D_2 be decomposition groups of $\widehat{\Pi}^{\mathscr{S}}$ of type I_1 and I_2, respectively. Then, there hold:*

(i) *The profinite group $\widehat{\Pi}^{\mathscr{S}}$ is virtually torsion free.*

(ii) $\widehat{\Pi}^{\mathscr{S}} = \widehat{\pi}_1^{\mathscr{S}}(\widehat{\mathscr{G}}^{\mathscr{S}}, Y)$ *and the vertex groups of $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y)$ embed in $\widehat{\Pi}^{\mathscr{S}}$.*

(iii) $D_1 \cap D_2 \neq \{1\}$ *if and only if* $D_1 = D_2$ *and* I_1, I_2 *are contained and conjugated in a vertex group* G_v *for some vertex* $v \in v(Y)$.
(iv) *The decomposition groups of* $\widehat{\Pi}^{\mathscr{S}}$ *are self-normalizing.*
(v) *Every finite non-trivial subgroup* C *of* $\widehat{\Pi}^{\mathscr{S}}$ *is contained in a decomposition group.*

Proof. Since the subgroup H is torsion free, it is a free product of surface groups. Moreover, since $\Pi/H \in \mathscr{S}$, the closure of the subgroup H in $\widehat{\Pi}^{\mathscr{S}}$ coincides with its pro-$\mathscr{S}$ completion $\widehat{H}^{\mathscr{S}}$ and so is a free pro-$\mathscr{S}$ product of pro-$\mathscr{S}$ surface groups. By [28, Theorem 3.10 and Remark 3.18], any finite subgroup of the profinite group $\widehat{H}^{\mathscr{S}}$ is contained in a free factor of $\widehat{H}^{\mathscr{S}}$. As observed in the proof of Lemma 3.4, such a free factor is the pro-$\mathscr{S}$ completion of the corresponding free factor of the group H; hence it is a pro-$\mathscr{S}$ surface group, which is torsion free. It follows that $\widehat{H}^{\mathscr{S}}$ is torsion free. This proves (i).

In particular, by the above proof, the group $\Pi = \pi_1(\mathscr{G}, Y)$ is residually $\mathscr{S}$. Item (ii) then follows from Lemma 3.4.

By [28, Theorem 3.10 and Remark 3.18], any finite subgroup of the profinite group $\widehat{\Pi}^{\mathscr{S}}$ is contained in a vertex group of $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y)$. Moreover, by [28, Theorem 3.12], the intersection of the conjugates of two distinct vertex groups is conjugate to a subgroup of an edge group.

Since the edge groups of $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y)$ identify with decomposition groups of the vertex groups, if $D_1 \cap D_2 \neq \{1\}$, then they are both conjugated to the same decomposition group of some vertex group of $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y)$. Thus, items (iii), (iv), and (v) follow from Lemma 3.1. □

Proof of Theorem 1.5. It is not restrictive to assume that $\sigma = \{\gamma_1, \ldots, \gamma_h\}$ is a simple subset of Π. Let σ_n^{Π}, for $n \in \mathbb{N}^+$, be the closed normal subgroup of Π generated by n-th powers of elements of σ.

Let us denote by S_n the orbifold nodal Riemann surface obtained topologically from S glueing a disc to each s.c.c. in σ with an attaching map of degree n. Then, there is a natural isomorphism $\pi_1(S_n) \cong \Pi/\sigma_n^{\Pi}$.

Therefore, the quotient group Π/σ_n^{Π} is a nodal Fuchsian group whose decomposition groups are the conjugacy classes of the subgroups generated by the images of the elements in σ, and the quotient group $\widehat{\Pi}^{\mathscr{S}}/\sigma_n^{\widehat{\Pi}^{\mathscr{S}}}$ is the pro-$\mathscr{S}$ completion of Π/σ_n^{Π}.

Let us prove that if n is a product of primes in $\Lambda_{\mathscr{S}}$, then the nodal Fuchsian group Π/σ_n^{Π} satisfies the hypotheses of Lemma 3.5. It is enough to show that there is a torsion free, normal subgroup H of the quotient group Π/σ_n^{Π} of index a product of powers of primes in $\Lambda_{\mathscr{S}}$ and such that the quotient of Π/σ_n^{Π} by H is a metabelian group or, equivalently, that there exists a normal, metabelian $\Lambda_{\mathscr{S}}$-covering $S_n'' \to S_n$ such that S_n'' is representable.

Let $S' \to S$ be the abelian $\Lambda_{\mathscr{S}}$-covering associated to the characteristic subgroup $[\Pi, \Pi]\Pi^n$ of Π. This covering has the property that its restriction to every s.c.c. of S', which covers either a non-separating s.c.c. or, in case there is more than one puncture on S, a peripheral s.c.c. on S, has degree n.

Since $\sigma_n^{\Pi} < [\Pi, \Pi]\Pi^n$, there is an induced $\Lambda_{\mathscr{S}}$-covering $S_n' \to S_n$ which ramifies with order n over the orbifold points of S_n corresponding to the non-separating and, in case there is more than one puncture on S, the peripheral s.c.c.'s in σ. Therefore the orbifold Riemann surface S_n' is representable over those points.

From the same argument used in the proof of [3, Lemma 3.10], it follows that an s.c.c. contained in the inverse image in S' of a non-peripheral separating s.c.c. γ in the set σ is non-separating and its image in S'_n is homologically non-trivial. In case there is only one peripheral s.c.c. in σ, an s.c.c. contained in its inverse image in S' also has a homologically non-trivial image in S'_n.

Let $\Pi' := [\Pi,\Pi]\Pi^n$. The metabelian $\Lambda_{\mathscr{S}}$-covering $S'' \to S$ associated to the normal subgroup $\sigma_n^{\Pi}[\Pi',\Pi'](\Pi')^n$ of Π then has the property that its restriction to every s.c.c. of S'', lying above an s.c.c. of σ, has degree n. Therefore, the induced metabelian $\Lambda_{\mathscr{S}}$-covering $S''_n \to S_n$ is such that S''_n is representable.

For an element $a \in \widehat{\Pi}^{\mathscr{S}}$, let us denote by $\bar{a}$ its image in the quotient group $\widehat{\Pi}^{\mathscr{S}}/\sigma_n^{\widehat{\Pi}^{\mathscr{S}}}$. The image $\bar{y}$ of the given y then has finite order.

From Lemma 3.5, it follows that $\bar{y} \in \langle\bar{\gamma}_i\rangle^{\bar{x}}$, for some $i \in \{1,\dots,h\}$ and $\bar{x} \in \widehat{\Pi}^{\mathscr{S}}/\sigma_n^{\widehat{\Pi}^{\mathscr{S}}}$. Since this holds for all n which are a product of primes in $\Lambda_{\mathscr{S}}$, by an inverse limit argument, it actually holds that $y \in \langle\gamma_i\rangle^x$, for some $i \in \{1,\dots,h\}$ and $x \in \widehat{\Pi}^{\mathscr{S}}$. □

We say that an element x of a profinite group G is *full* if the p-component $\langle x\rangle^{(p)}$ of the pro-cyclic group generated by x is non-trivial for every prime p dividing the order of G.

By cohomological methods, it is possible to show that normalizers of full elements in a non-abelian pro-$\mathscr{S}$ surface group are pro-cyclic. For algebraically simple elements in the pro-$\mathscr{S}$ completion of a surface group, this also follows from an argument similar to the one given in the proof of Theorem 1.5.

Proposition 3.6. *Let $\mathscr{S}$ be a class of finite groups and let $\widehat{\Pi}^{\mathscr{S}}$ be the pro-$\mathscr{S}$ completion of a non-abelian surface group Π. Let x be an algebraically simple element of $\widehat{\Pi}^{\mathscr{S}}$. Then, for all $n \in \widehat{\mathbb{Z}}^{\mathscr{S}} \smallsetminus \{0\}$, there holds:*

$$N_{\widehat{\Pi}^{\mathscr{S}}}(\langle x^n\rangle) = N_{\widehat{\Pi}^{\mathscr{S}}}(\langle x\rangle) = \langle x\rangle.$$

Proof. We can assume that x is a simple element of Π. The quotient group $\Xi_h := \Pi/\langle x^h\rangle^{\Pi}$ is a nodal Fuchsian group which satisfies the hypotheses of Lemma 3.5. Let x_h be the image of x in Ξ_h. Then, the cyclic group C_h generated by x_h is a decomposition group of Ξ_h. From Lemma 3.5, it follows that, in the pro-$\mathscr{S}$ completion $\widehat{\Xi}_h^{\mathscr{S}}$ of Ξ_h, for $n \in \mathbb{N}^+$ and $h > n$, it holds that $N_{\widehat{\Xi}_h^{\mathscr{S}}}(C_h) = N_{\widehat{\Xi}_h^{\mathscr{S}}}(C_h^n) = C_h$. The conclusion of the proposition then follows taking the inverse limit for $h \to \infty$. □

A multicurve (cf. Definition 5.2) σ on a Riemann surface S is a set $\{\gamma_0,\dots,\gamma_k\}$ of disjoint, non-trivial, non-peripheral s.c.c.'s on S, such that they are two by two non-isotopic. The complement $S \smallsetminus \sigma$ is then a disjoint union of hyperbolic Riemann surfaces $\coprod_{i=0}^h S_i$, and, for some choices of base points and a path between them, the fundamental group $\Pi_i := \pi_1(S_i)$, for $i = 0,\dots,h$, identifies with a subgroup of $\Pi := \pi_1(S)$.

Let Y_σ be the graph which has for vertices the connected components of $S \smallsetminus \sigma$ and for edges the elements of σ, where two vertices S_i and S_j are joined by the edge γ_l if γ_l is a boundary component of both S_i and S_j. Then, the group Π is naturally isomorphic to the fundamental group of the graph of groups $(\mathscr{G}, Y_\sigma)$ with vertex groups $\mathscr{G}_i = \Pi_i$, for $i = 0,\dots,h$, and with edge groups the cyclic groups C_j, for $j = 0,\dots k$, generated by representatives in Π of the s.c.c.'s in σ.

Lemma 3.5 (ii) and an argument similar to that of the proof of Theorem 1.5 yield the following description of the pro-$\mathscr{S}$ completion of Π in terms of the graph of groups described above.

Theorem 3.7. *Let σ be a multicurve on a Riemann surface S, let $S \smallsetminus \sigma = \coprod_{i=0}^{h} S_i$ be the decomposition in connected components, and let Π_i be the fundamental group of S_i, for $i = 0, \ldots, h$. The fundamental group Π of S is naturally isomorphic to the fundamental group of the graph of groups $(\mathscr{G}, Y_\sigma)$, described above, with vertex groups the groups Π_i, for $i = 0, \ldots, h$. Let $\mathscr{S}$ be a class of finite groups. Then, the pro-$\mathscr{S}$ completion $\widehat{\Pi}^{\mathscr{S}}$ of Π is the pro-$\mathscr{S}$ fundamental group of the graph of profinite groups $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y_\sigma)$ whose vertex and edge groups are the pro-$\mathscr{S}$ completions of vertex and edge groups of the graph of groups $(\mathscr{G}, Y_\sigma)$. Moreover, the vertex groups $\widehat{\Pi}_i^{\mathscr{S}}$, for $i = 0, \ldots, h$, embed in $\widehat{\Pi}^{\mathscr{S}}$ and are their own normalizers in this group.*

Proof. Let us also denote by σ a set of representatives in Π of the s.c.c.'s in σ. As in the proof of Theorem 1.5, let us consider the quotient group Π/σ_n^{Π}, which is a nodal Fuchsian group satisfying the hypotheses of Lemma 3.5. This group is the fundamental group of the graph of groups $(\mathscr{G}_n, Y_\sigma)$ whose vertex groups are the fundamental groups of the orbifolds obtained from the connected components of the manifold with boundary obtained cutting S in σ, attaching discs to their boundary components with an attaching map of degree n.

By Lemma 3.5 (ii), its pro-$\mathscr{S}$ completion $\widehat{\Pi}^{\mathscr{S}}/\sigma_n^{\widehat{\Pi}^{\mathscr{S}}}$ is the pro-$\mathscr{S}$ fundamental group $\widehat{\Pi}_1^{\mathscr{S}}(\widehat{\mathscr{G}}, Y_\sigma)$ of the finite graph of pro-$\mathscr{S}$ groups $(\widehat{\mathscr{G}}_n^{\mathscr{S}}, Y_\sigma)$ obtained from $(\mathscr{G}_n, Y)$ by taking the pro-$\mathscr{S}$ completion of each vertex group of $(\mathscr{G}_n, Y_\sigma)$, and the vertex groups of $(\widehat{\mathscr{G}}_n^{\mathscr{S}}, Y_\sigma)$ embed in $\widehat{\Pi}^{\mathscr{S}}/\sigma_n^{\widehat{\Pi}^{\mathscr{S}}}$.

The inverse limit, for $n \to \infty$, of the graphs of pro-$\mathscr{S}$ groups $(\widehat{\mathscr{G}}_n^{\mathscr{S}}, Y_\sigma)$ is the graph of groups $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y_\sigma)$, whose vertex groups are the pro-$\mathscr{S}$ completions of the fundamental groups of the connected components of $S \smallsetminus \sigma$. The pro-$\mathscr{S}$ fundamental group of $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y_\sigma)$ is the inverse limit, for $n \to \infty$, of the groups $\widehat{\Pi}^{\mathscr{S}}/\sigma_n^{\widehat{\Pi}^{\mathscr{S}}}$, i.e., the pro-$\mathscr{S}$ completion $\widehat{\Pi}^{\mathscr{S}}$ of Π. Therefore, the vertex groups of $(\widehat{\mathscr{G}}^{\mathscr{S}}, Y_\sigma)$ embed in $\widehat{\Pi}^{\mathscr{S}}$. The last statement in the theorem then follows from [28, Corollary 3.13 and Remark 3.18]. □

4. Relative versions of the restricted Magnus property

For the applications given in Section 5, we need a finer result than Theorem 1.5. Let us give the following definition.

Definition 4.1.

(i) For a given profinite group G (possibly finite) and a given prime $p \geq 2$, let us denote respectively by $G^{(p)}$ and G^{nil} its maximal pro-p and pro-nilpotent quotients. It is clear that if the group G is pro-nilpotent, it holds that $G = \prod G^{(p)}$ and the pro-p group $G^{(p)}$ is naturally identified with the p-Sylow subgroup of G.

(ii) For K a normal open subgroup of a profinite group G and a given prime $p \geq 2$, let $G_K^{(p)}$ and G_K^{nil} be, respectively, the quotients of G by the kernels of the natural epimorphisms $K \to K^{(p)}$ and $K \to K^{nil}$. Let us call them, respectively, the relative maximal pro-p and pro-nilpotent quotients of G with respect to the subgroup K.

(iii) For K a normal finite index subgroup of a discrete group G and a given prime $p \geq 2$, let us denote by $\widehat{G}_K^{(p)}$ and $\widehat{G}_K^{nil}$, respectively, the quotients of the profinite completion $\widehat{G}$ by the kernels of the natural epimorphisms $\widehat{K} \to \widehat{K}^{(p)}$ and $\widehat{K} \to \widehat{K}^{nil}$. Let us call $\widehat{G}_K^{(p)}$ and $\widehat{G}_K^{nil}$, respectively, the relative pro-p and pro-nilpotent completion of G with respect to the subgroup K.

Mochizuki's Lemma 3.1 admits the following generalization to relative pro-p completions of hyperbolic Fuchsian groups.

Lemma 4.2. *Let Π be a hyperbolic Fuchsian group with the presentation given in Section 3. Let K be a finite index normal subgroup of Π which contains a torsion free normal subgroup H of index a power of p, for a prime $p \geq 2$. Then let $\widehat{\Pi}_K^{(p)}$ be the relative pro-p completion of Π with respect to the subgroup K. Let D_i be a decomposition group of $\widehat{\Pi}_K^{(p)}$ in the conjugacy class of the cyclic subgroup $\langle x_i \rangle$, for $i = 1, \ldots, n$. Then, there hold:*

(i) *For all $x \in \widehat{\Pi}_K^{(p)}$, there holds $D_i^{(p)} \cap (D_j^{(p)})^x \neq \{1\}$ if and only if $i = j$ and $D_i = (D_i)^x$. In particular, for all $i = 1, \ldots, n$ such that $D_i^{(p)} \neq \{1\}$, there is a series of identities:*

$$N_{\widehat{\Pi}_K^{(p)}}(D_i) = N_{\widehat{\Pi}_K^{(p)}}(D_i^{(p)}) = D_i.$$

(ii) *Every finite nilpotent subgroup C of $\widehat{\Pi}_K^{(p)}$ such that $C^{(p)} \neq \{1\}$ is contained in a decomposition group of $\widehat{\Pi}_K^{(p)}$.*

(iii) *Therefore, the decomposition groups D of $\Pi_K^{(p)}$ such that $D^{(p)} \neq \{1\}$ may be characterized as the maximal finite nilpotent subgroups M of the profinite group $\Pi_K^{(p)}$ such that $M^{(p)} \neq \{1\}$.*

Proof. Let S be a hyperbolic orbifold Riemann surface such that $\Pi \cong \pi_1(S)$ and let $S_K \to S$ be the Galois unramified covering associated to the normal finite index subgroup K of Π. Let us also denote by $S_H \to S_K$ the normal unramified p-covering associated to the subgroup H of K. By hypothesis, S_H is a hyperbolic Riemann surface and the p-adic solenoid $\mathbb{S}_K^{(p)}$ of S_K, i.e., the inverse limit of all unramified p-coverings of S_K, identifies with the p-adic solenoid $\mathbb{S}_H^{(p)}$ of S_H. Let $\{L \lhd_o H\}$ be the set of subgroups of H open for the pro-p topology. There is then a series of natural isomorphisms:

$$H^k(\mathbb{S}_K^{(p)}, \mathbb{Z}/p) \cong H^k(\mathbb{S}_H^{(p)}, \mathbb{Z}/p) := \varinjlim_{L \lhd_o H} H^k(S_L, \mathbb{Z}/p) \cong \varinjlim_{L \lhd_o H} H^k(L, \mathbb{Z}/p).$$

Since H is p-good, there hold $H^k(\mathbb{S}_K^{(p)}, \mathbb{Z}/p) = \{0\}$, for $k > 0$, and $H^0(\mathbb{S}_K^{(p)}, \mathbb{Z}/p) = \mathbb{Z}/p$.

There is a natural continuous action of $\Pi_K^{(p)}$ on the p-adic solenoid $S_K^{(p)}$, and the decompositions groups of $\Pi_K^{(p)}$ identify with the stabilizers of points of $S_K^{(p)}$ in the inverse image of points of the orbifold Riemann surface S which have non-trivial isotropy groups.

The proof then proceeds exactly like the proof of Lemma 3.1. In order to prove (i), it is enough to show that the intersection of the stabilizers of two such points P_1 and P_2 contains a cyclic subgroup C_p of order p if and only if $P_1 = P_2$.

Since the profinite space $\mathbb{S}_K^{(p)}$ is p-acyclic, it follows that $(\mathbb{S}_K^{(p)})^{C_p}$ is also p-acyclic, where $(\mathbb{S}_K^{(p)})^{C_p}$ is the fixed point set of the action of the p-group C_p on the p-adic solenoid $\mathbb{S}_K^{(p)}$. In particular, $(\mathbb{S}_K^{(p)})^{C_p}$ is connected and thus consists of only one point. In particular, it holds that $P_1 = P_2$.

The proof of item (ii) essentially follows from the same arguments used in the proof of item (ii) of Lemma 3.1. Since C is nilpotent, it has a unique normal and, by hypothesis, non-trivial p-Sylow subgroup $C^{(p)}$. Proceeding by induction on the order of C, we can assume either that C is of order p or that C is an extension of a group of prime order by a non-trivial subgroup $C_1 \leq C$ which is contained in a decomposition group D and such that $C^{(p)} \leq C_1$.

In the first case, since the profinite space $\mathbb{S}_K^{(p)}$ is p-acyclic and of finite dimension, it follows that $(\mathbb{S}_K^{(p)})^C$ is non-empty; i.e., C is contained in a decomposition group.

In the second case, replacing $\widehat{\Pi}_K^{(p)}$ by its open subgroup $\widehat{H}^{(p)} \cdot C$, which is also the relative pro-p completion of a hyperbolic Fuchsian group with respect to some normal subgroup, we may assume that actually $C_1 = D \leq C$. Since C_1 is normal in C and D is self-normalizing, it follows that $C = D$.

Item (iii) is just a reformulation of (ii). □

The next step is to generalize Lemma 4.2 to nodal Fuchsian groups.

Definition 4.3.

(i) *A virtual pro-p nodal Fuchsian group* $\widehat{\Pi}_K^{(p)}$ is the relative pro-p completion of a nodal Fuchsian group Π with respect to a given finite index normal subgroup K.

(ii) We say that a finite subgroup D of a virtual pro-p nodal Fuchsian group $\widehat{\pi}_1(\mathscr{G}, Y)_K^{(p)}$ is *a decomposition group of type I* if it is in the conjugacy class of a decomposition subgroup I of a vertex group of $(\mathscr{G}, Y)$.

(iii) Sets of simple and algebraically simple elements of $\widehat{\Pi}_K^{(p)}$ are defined as in Definition 1.3. Thus, a subset of elements $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \widehat{\Pi}_K^{(p)}$ is *algebraically simple* if it is in the $\mathrm{Aut}(\widehat{\Pi}_K^{(p)})$–orbit of the image of a set $\sigma' \subset \Pi$ which is simple for some presentation of Π as the fundamental group of a Riemann surface.

Lemma 4.4. *Let* $\Pi := \pi_1(G, Y)$ *be a nodal Fuchsian group. Let* K *be a finite index normal subgroup of* Π *and let* $p \geq 2$ *be a prime such that, for some normal subgroup* H *of* K *of index a power of* p *and for all vertices* v *of* Y*, the subgroups* $H \cap G_v$ *are torsion free. Then let* $\widehat{\Pi}_K^{(p)}$ *be the relative pro-p completion of* Π *with respect to the subgroup* K*. Let* D_1 *and* D_2 *be decomposition groups of* $\widehat{\Pi}_K^{(p)}$ *of type* I_1 *and* I_2*, respectively, such that their p-Sylow subgroups are non-trivial. Then, there hold:*

(i) *The profinite group* $\widehat{\Pi}_K^{(p)}$ *is virtually torsion free.*

(ii) *It holds that* $D_1^{(p)} \cap D_2^{(p)} \neq \{1\}$ *if and only if* $D_1 = D_2$ *and* I_1*,* I_2 *are contained and conjugated in a vertex group* G_v *for some vertex* $v \in v(Y)$*. In particular, if* D *is a decomposition group of* $\widehat{\Pi}_K^{(p)}$ *such that* $D^{(p)} \neq \{1\}$*, there is a series of identities:*

$$N_{\widehat{\Pi}_K^{(p)}}(D) = N_{\widehat{\Pi}_K^{(p)}}(D^{(p)}) = D.$$

(iii) *Every finite nilpotent subgroup C of $\widehat{\Pi}_K^{(p)}$ such that $C^{(p)} \neq \{1\}$ is contained in a decomposition group.*

(iv) *Therefore, the decomposition groups of $\widehat{\Pi}_K^{(p)}$ with a non-trivial p-component may be characterized as the maximal finite nilpotent subgroups of $\widehat{\Pi}_K^{(p)}$ with a non-trivial p-component.*

Proof. From the same argument used to prove item (i) of Lemma 3.5, it follows that $\widehat{H}^{(p)}$ is a torsion free group.

In order to prove (ii), let us consider the Galois unramified covering $S_K \to S$ associated to the normal finite index subgroup K of Π and let $S_H \to S_K$ be the p-covering associated to the subgroup H of K. As in the proof of Lemma 4.2 (i), the p-adic solenoid $\mathbb{S}_K^{(p)}$ of S_K, i.e., the inverse limit of all unramified p-coverings of S_K, identifies with the p-adic solenoid $\mathbb{S}_H^{(p)}$ of S_H. Since H is a free product of surface groups, it is p-good. Therefore, as in the proof of Lemma 4.2, there hold $H^k(\mathbb{S}_K^{(p)}, \mathbb{Z}/p) = H^k(\mathbb{S}_H^{(p)}, \mathbb{Z}/p) = \{0\}$, for $k > 0$, and $H^0(\mathbb{S}_K^{(p)}, \mathbb{Z}/p) = \mathbb{Z}/p$. Then, the proof proceeds exactly as for Lemma 4.2. □

Definition 4.5. For an element $a \in \widehat{\Pi}_K^{(p)}$, let $\nu_K(a)$ be the minimal natural number such that it holds that $a^{\nu_K(a)} \in K^{(p)}$. For $\sigma = \{\gamma_1, \dots, \gamma_h\} \subset \widehat{\Pi}_K^{(p)}$ an algebraically simple subset, then let $\sigma_{K,n}^{\widehat{\Pi}_K^{(p)}}$, for $n \in \mathbb{N}^+$, be the closed normal subgroup of $\widehat{\Pi}_K^{(p)}$ generated by the elements $\gamma_1^{n\cdot\nu_K(\gamma_1)}, \dots, \gamma_h^{n\cdot\nu_K(\gamma_h)}$.

We then have the following generalization of Theorem 1.5.

Theorem 4.6. *Given an algebraically simple subset $\sigma = \{\gamma_1, \dots, \gamma_h\} \subset \widehat{\Pi}_K^{(p)}$, let $y \in \widehat{\Pi}_K^{(p)}$ be an element such that $y^{\nu_K(y)}$ is a generator of the pro-p group $K^{(p)}$ and, for every $n = p^t$, with $t \in \mathbb{N}^+$, there exists a $k_n \in \mathbb{N}^+$ with the property that there holds $y^{k_n} \in \sigma_{K,n}^{\widehat{\Pi}_K^{(p)}}$. Then, for some $s \in \widehat{\mathbb{Z}} \smallsetminus \{0\}$ and $i \in \{1, \dots, h\}$, the element y is conjugated to γ_i^s.*

Proof. It is not restrictive to assume that $\sigma = \{\gamma_1, \dots, \gamma_h\}$ is a simple subset of Π. Let us then denote by $\sigma_{K,n}^{\Pi}$ the normal subgroup of Π generated by the elements

$$\gamma_1^{n\cdot\nu_K(\gamma_1)}, \dots, \gamma_h^{n\cdot\nu_K(\gamma_h)}.$$

The quotient group $\Pi/\sigma_{K,n}^{\Pi}$ is a nodal Fuchsian group whose decomposition groups are the conjugacy classes of the subgroups generated by the images of the elements in σ.

By the same argument used in the proof of Theorem 1.5, the group $\Pi/\sigma_{K,n}^{\Pi}$ contains a normal, torsion free subgroup, which is contained as a subgroup of index a power of p in the image of the subgroup K. Its relative pro-p completion with respect to the image of K is then exactly the quotient group $\widehat{\Pi}_K^{(p)}/\sigma_{K,n}^{\widehat{\Pi}_K^{(p)}}$.

For an element $a \in \Pi_K^{(p)}$, let us denote by $\bar{a}$ its image in the quotient group $\widehat{\Pi}_K^{(p)}/\sigma_{K,n}^{\widehat{\Pi}_K^{(p)}}$.

Since the natural epimorphism $\widehat{K}^{(p)} \twoheadrightarrow H_1(K, \mathbb{Z}/p)$ factors through the quotient group $\widehat{K}^{(p)}/\sigma_{K,n}^{\widehat{\Pi}_K^{(p)}}$ and, by hypotheses, the lift $y^{\nu_K(y)}$ is a generator of the pro-p group $\widehat{K}^{(p)}$, the image $\bar{y}$ of y then has finite order divisible by p.

From Lemma 4.4, it then follows that, for all $n = p^t$, there holds $\bar{y} \in \langle\bar{\gamma}_i\rangle^{\bar{x}}$, for some $i \in \{1,\dots,h\}$ and some $\bar{x} \in \widehat{\Pi}_K^{(p)}/\sigma_{K,n}^{\widehat{\Pi}_K^{(p)}}$. By an inverse limit argument, we conclude that there holds $y \in \langle\gamma_i\rangle^x$, for some $i \in \{1,\dots,h\}$ and some $x \in \widehat{\Pi}_K^{(p)}$. □

Corollary 4.7. *Let K be a normal finite index subgroup of Π such that, for every algebraically simple element $x \in \widehat{\Pi}_K^{(p)}$, the lift $x^{\nu_K(y)} \in \widehat{K}^{(p)}$ is a generator. Given an algebraically simple subset $\sigma = \{\gamma_1,\dots,\gamma_h\} \subset \widehat{\Pi}_K^{(p)}$, let $y \in \widehat{\Pi}_K^{(p)}$ be an algebraically simple element such that, for every $n = p^t$, with $t \in \mathbb{N}^+$, there exists a $k_n \in \mathbb{N}^+$ with the property that there holds $y^{k_n} \in \sigma_{K,n}^{\widehat{\Pi}_K^{(p)}}$. Then, for some $s \in \widehat{\mathbb{Z}}^*$ and $i \in \{1,\dots,h\}$, the element y is conjugated to γ_i^s.*

Remark 4.8. In [3, Lemma 3.10], there is defined a characteristic finite index subgroup K_ℓ of Π such that, for every simple element $x \in \Pi$, the lift $x^{\nu_{K_\ell}(y)} \in K_\ell$ is a generator. The definition of this group can be rephrased group-theoretically as follows. Let K be a finite index characteristic subgroup of Π such that, for all simple (and so for all algebraically simple) elements $\gamma \in \Pi$, there holds $\gamma \notin K$. For a given integer $\ell > 1$, we then let $K_\ell := [K,K]K^\ell$. For any algebraically simple element $x \in \widehat{\Pi}_{K_\ell}^{(p)}$, the lift $x^{\nu_{K_\ell}(y)} \in \widehat{K}_\ell^{(p)}$ is then a generator. It follows that any normal finite index subgroup N of Π, contained in K_ℓ, satisfies the hypothesis of Corollary 4.7.

Corollary 4.7 and Remark 4.8 yield a substantial refinement of Theorem 1.5. We need to fix some more notation.

Definition 4.9. For an open normal subgroup K of $\widehat{\Pi}$ and $a \in \widehat{\Pi}$, let $\nu_K(a)$ be the minimal natural number such that $a^{\nu_K(a)} \in K$. For an algebraically simple subset $\sigma = \{\gamma_1,\dots,\gamma_h\} \subset \widehat{\Pi}$, then let $\sigma_{K,n}^{\widehat{\Pi}}$, for $n \in \mathbb{N}^+$, be the closed normal subgroup of $\widehat{\Pi}$ generated by the elements $\gamma_1^{n\cdot\nu_K(\gamma_1)},\dots,\gamma_h^{n\cdot\nu_K(\gamma_h)}$.

Let $\widehat{\Pi}$ be the profinite completion of a hyperbolic surface group Π and let $\widehat{\Pi}_K^{(p)}$ be its relative pro-p completion with respect to some normal finite index subgroup K. There is then a natural epimorphism:

$$\psi_K^{(p)}\colon \widehat{\Pi} \to \widehat{\Pi}_K^{(p)}.$$

From an inverse limit argument, Corollary 4.7, and Remark 4.8, Corollary 4.10 follows.

Corollary 4.10. *Let $\sigma = \{\gamma_1,\dots,\gamma_h\} \subset \widehat{\Pi}$ be an algebraically simple subset and let p be a fixed prime. Let $y \in \widehat{\Pi}$ be an algebraically simple element such that, for a cofinal system of open normal subgroups $\{K\}$ of $\widehat{\Pi}$ and every $n = p^t$, with $t \in \mathbb{N}^+$, there exists a $\mu_{K,n} \in \mathbb{N}^+$ with the property that there holds $\psi_K^{(p)}(y)^{\mu_{K,n}} \in \psi_K^{(p)}(\sigma_{K,n}^{\widehat{\Pi}})$. Then, for some $s \in \widehat{\mathbb{Z}}^*$ and $i \in \{1,\dots,h\}$, the element y is conjugated to γ_i^s.*

The assumption that the element y is algebraically simple can be dropped, reformulating Corollary 4.10 for relative pro-nilpotent, instead of relative pro-p, completions. Let us denote by $\psi_K^{nil}\colon \widehat{\Pi} \to \widehat{\Pi}_K^{nil}$ the natural epimorphism. The following holds.

Theorem 4.11. *Let $\sigma = \{\gamma_1,\dots,\gamma_h\} \subset \widehat{\Pi}$ be an algebraically simple subset. Let $y \in \widehat{\Pi}$ be an element such that, for a cofinal system of open normal subgroups*

$\{K\}$ *of* $\widehat{\Pi}$ *and every* $n \in \mathbb{N}^+$*, there exists a* $\mu_{K,n} \in \mathbb{N}^+$ *with the property that* $\psi_K^{nil}(y)^{\mu_{K,n}} \in \psi_K^{nil}(\sigma_{K,n}^{\widehat{\Pi}})$*. Then, for some* $s \in \widehat{\mathbb{Z}}$ *and* $i \in \{1, \ldots, h\}$*, the element* y *is conjugated to* γ_i^s*.*

Proof. Let us assume that $y \neq 1$. Since y has infinite order, a Sylow p-subgroup Y_p of $\overline{\langle y \rangle}$ is infinite for some prime p. Let y_p be a generator of Y_p and let U be an open subgroup of $\widehat{\Pi}$ such that y_p is a generator of the maximal pro-p quotient $U^{(p)}$ of U. Such a U exists because Y_p is the intersection of all open subgroups containing it. Let K_U be a subgroup in the cofinal system $\{K\}$ contained in U. Then its image $\tilde{K}_U$ in $U^{(p)}$ intersects $\overline{\langle y_p \rangle}$ non-trivially and is such that $\tilde{K}_U \cap \overline{\langle y_p \rangle} \not\leq \Phi(\tilde{K}_U)$. Therefore, since $\tilde{K}_U$ is a quotient of $K_U^{(p)}$, the image of $Y_p \cap K_U$ in $K_U^{(p)}$ is not contained in $\Phi(K_U^{(p)})$. This means that $\psi_{K_U}^{(p)}(y)^{\nu_{K_U}(y)}$ is a generator of $K_U^{(p)}$. Then, the conclusion follows from Theorem 4.6 and the usual inverse limit argument. ☐

Proceeding as in the proof of Theorem 4.6, we can also describe normalizers of algebraically simple elements in the relative pro-p completion of a surface group.

Proposition 4.12. *Let* $\widehat{\Pi}_K^{(p)}$ *be the relative pro-p completion of a non-abelian surface group* Π *with respect to some normal finite index subgroup* K *and let* x *be an algebraically simple element of* $\widehat{\Pi}_K^{(p)}$*. Then, for all* $n \in \widehat{\mathbb{Z}} \smallsetminus \{0\}$*, there holds:*

$$N_{\widehat{\Pi}_K^{(p)}}(\langle x^n \rangle) = N_{\widehat{\Pi}_K^{(p)}}(\langle x \rangle) = \langle x \rangle.$$

Proof. We can assume that x is a simple element of Π. Then let $k > 0$ be the smallest integer such that $x^k \in K$. The quotient group $\Xi_h := \Pi / \langle x^{p^{hk}} \rangle^{\Pi}$ is a nodal Fuchsian group which satisfies the hypotheses of Lemma 4.4 with respect to the normal subgroup K_h, the image of K in the quotient group Ξ_h. Let x_h be the image of x in Ξ_h. Then, the cyclic group C_h generated by x_h is a decomposition group of Ξ_h with non-trivial p-component for $h > 0$.

From Lemma 4.4, it follows that in the relative pro-p completion $\Xi_{K_h}^{(p)}$ of Ξ_h with respect to the subgroup K_h, for $h > 0$ and $p^h \nmid n$, there holds:

$$N_{\Xi_{K_h}^{(p)}}(C_h) = N_{\Xi_{K_h}^{(p)}}(C_h^n) = C_h.$$

The conclusion of the proposition then follows taking the inverse limit for $h \to \infty$. ☐

5. A linearization of the complex of profinite curves

Let S_g be a closed orientable Riemann surface of genus g and let $\{P_1, \ldots, P_n\}$ be a set of distinct points on S_g. The Teichmüller modular group $\Gamma_{g,n}$, for $2g - 2 + n > 0$, is defined to be the group of isotopy classes of diffeomorphisms or, equivalently, of homeomorphisms of the surface S_g which preserve the orientation and the given ordered set $\{P_1, \ldots, P_n\}$ of marked points:

$$\Gamma_{g,n} := \mathrm{Diff}^+(S_g, n) / \mathrm{Diff}_0(S_g, n) \cong \mathrm{Hom}^+(S_g, n) / \mathrm{Hom}_0(S_g, n),$$

where $\mathrm{Diff}_0(S_g, n)$ and $\mathrm{Hom}_0(S_g, n)$ denote the connected components of the identity in the respective topological groups.

Forgetting the last marked point, P_{n+1} induces an epimorphism of Teichmüller modular groups $p_{n+1} \colon \Gamma_{g,n+1} \to \Gamma_{g,n}$.

Let $S_{g,n}$ be the differentiable surface obtained removing the points $P_1,\dots,P_n$ from S_g and let $\Pi_{g,n} := \pi_1(S_{g,n}, P_{n+1})$. The homomorphism p_{n+1} induces a short exact sequence of Teichmüller modular groups, called *the Birman exact sequence*:

$$1 \to \Pi_{g,n} \stackrel{i}{\to} \Gamma_{g,n+1} \stackrel{p_{n+1}}{\to} \Gamma_{g,n} \to 1.$$

The monomorphism $i\colon \Pi_{g,n} \hookrightarrow \Gamma_{g,n+1}$ sends the isotopy class of a P_{n+1}-pointed oriented closed curve γ to the isotopy class of the homeomorphism $i(\gamma)$ defined by pushing the base point P_{n+1} all along the path γ in the direction given by the orientation of γ. It is clear that $i(\gamma)$ is isotopic to the identity for isotopies which are allowed to move the base point P_{n+1} and then that $p_{n+1}(i(\gamma)) = 1$.

There are natural faithful representations, induced by the action of homeomorphisms on the fundamental group of the Riemann surface $S_{g,n}$:

$$\rho_{g,n}\colon \Gamma_{g,n} \hookrightarrow \mathrm{Out}(\Pi_{g,n}) \qquad \text{and} \qquad \rho'_{g,n+1}\colon \Gamma_{g,n+1} \hookrightarrow \mathrm{Aut}(\Pi_{g,n}).$$

Since the automorphism $\rho'_{g,n+1}(i(\gamma))$ is the inner automorphism $\mathrm{inn}\,\gamma$, induced by γ, for all elements $\gamma \in \Pi_{g,n}$, the exactness of the Birman sequence then follows from the exactness of the standard group-theoretical short exact sequence:

$$1 \to \Pi_{g,n} \stackrel{\mathrm{inn}}{\to} \mathrm{Aut}(\Pi_{g,n}) \to \mathrm{Out}(\Pi_{g,n}) \to 1.$$

It also follows that the representations $\rho_{g,n}$ and $\rho'_{g,n+1}$ can be recovered, algebraically, from the Birman exact sequence and the action by restriction of inner automorphisms of $\Gamma_{g,n+1}$ on its normal subgroup $\Pi_{g,n}$.

Let us now switch to the profinite setting. Let, as usual, $\widehat{\Pi}_{g,n}$ be the profinite completion of the fundamental group $\Pi_{g,n}$. Since the profinite group $\widehat{\Pi}_{g,n}$ is center-free, there is also a natural short exact sequence:

$$1 \to \widehat{\Pi}_{g,n} \stackrel{\mathrm{inn}}{\to} \mathrm{Aut}(\widehat{\Pi}_{g,n}) \to \mathrm{Out}(\widehat{\Pi}_{g,n}) \to 1.$$

Let us mention here a fundamental result of Nikolov and Segal [22] which asserts that any finite index subgroup of any topologically finitely generated profinite group G is open. Since such a profinite group G also has a basis of neighborhoods of the identity consisting of open characteristic subgroups, it follows that all automorphisms of G are continuous and that $\mathrm{Aut}(G)$ is a profinite group as well.

Let $\widehat{\Gamma}_{g,n}$, for $2g-2+n>0$, be the profinite completion of the Teichmüller modular group. From the universal property of the profinite completion, it follows that there are natural representations:

$$\hat{\rho}_{g,n}\colon \widehat{\Gamma}_{g,n} \to \mathrm{Out}(\widehat{\Pi}_{g,n}) \qquad \text{and} \qquad \hat{\rho}'_{g,n+1}\colon \widehat{\Gamma}_{g,n+1} \to \mathrm{Aut}(\widehat{\Pi}_{g,n}).$$

Let us then recall a few definitions from [4].

Definition 5.1. For $2g-2+n>0$, let the profinite groups $\tilde{\Gamma}_{g,n+1}$ and $\check{\Gamma}_{g,n}$ be, respectively, the image of $\hat{\rho}'_{g,n+1}$ in $\mathrm{Aut}(\widehat{\Pi}_{g,n})$ and of $\hat{\rho}_{g,n}$ in $\mathrm{Out}(\widehat{\Pi}_{g,n})$. For all $n \geq 0$, there is a natural isomorphism $\check{\Gamma}_{g,n+1} \stackrel{\sim}{\to} \tilde{\Gamma}_{g,n+1}$ (cf. [15, Lemma 20]). We then call $\check{\Gamma}_{g,n}$ *the congruence completion of the Teichmüller group* or, more simply, *the procongruence Teichmüller group.*

One of the most important objects in Teichmüller theory is the complex of curves.

Definition 5.2. A simple closed curve (s.c.c.) γ on the Riemann surface $S_{g,n}$ is *non-peripheral* if it does not bound a disc with less than two punctures. *A multicurve* σ on $S_{g,n}$ is a set of disjoint, non-trivial, non-peripheral s.c.c.'s on $S_{g,n}$

such that they are two by two non-isotopic. The complex of curves $C(S_{g,n})$ is the abstract simplicial complex whose simplices are isotopy classes of multicurves on $S_{g,n}$.

It is easy to check that the combinatorial dimension of $C(S_{g,n})$ is $n-4$ for $g=0$ and $3g-4+n$ for $g \geq 1$. There is a natural simplicial action of $\Gamma_{g,n}$ on $C(S_{g,n})$.

In order to construct a profinite version of the complex of curves, we need to reformulate its definition in more algebraic terms.

Let $\mathscr{L}_{g,n} = C(S_{g,n})_0$, for $2g-2+n>0$, be the set of isotopy classes of non-peripheral simple closed curves on $S_{g,n}$. Let $\Pi_{g,n}/\sim$ be the set of conjugacy classes of elements of $\Pi_{g,n}$ and let $\mathscr{P}_2(\Pi_{g,n}/\sim)$ be the set of unordered pairs of elements of $\Pi_{g,n}/\sim$.

For a given $\gamma \in \Pi_{g,n}$, let us denote by $\gamma^{\pm 1}$ the set $\{\gamma, \gamma^{-1}\}$ and by $[\gamma^{\pm 1}]$ its equivalence class in $\mathscr{P}_2(\Pi_{g,n}/\sim)$. Let us then define the natural embedding $\iota\colon \mathscr{L}_{g,n} \hookrightarrow \mathscr{P}_2(\Pi_{g,n}/\sim)$, choosing, for an element $\gamma \in \mathscr{L}_{g,n}$, an element $\vec{\gamma}_* \in \Pi_{g,n}$ whose free homotopy class contains γ and letting $\iota(\gamma) := [\vec{\gamma}_*^{\pm 1}]$.

Let $\widehat{\Pi}_{g,n}/\sim$ be the set of conjugacy classes of elements of $\widehat{\Pi}_{g,n}$ and let $\mathscr{P}_2(\widehat{\Pi}_{g,n}/\sim)$ be the profinite set of unordered pairs of elements of $\widehat{\Pi}_{g,n}/\sim$. Since $\Pi_{g,n}$ is conjugacy separable (cf. [26]) the set $\Pi_{g,n}/\sim$ embeds in the profinite set $\widehat{\Pi}_{g,n}/\sim$. So, let us define the set of *non-peripheral profinite s.c.c.*'s $\widehat{\mathscr{L}_{g,n}}$ on $S_{g,n}$ to be the closure of the set $\iota(\mathscr{L}_{g,n})$ inside the profinite set $\mathscr{P}_2(\widehat{\Pi}_{g,n}/\sim)$. When it is clear from the context, we omit the subscripts and denote these sets simply by $\mathscr{L}$ and $\widehat{\mathscr{L}}$.

An ordering of the set $\{\alpha, \alpha^{-1}\}$ is preserved by the conjugacy action and defines *an orientation* for the associated equivalence class $[\alpha^{\pm 1}] \in \widehat{\mathscr{L}}$.

For all $k \geq 0$, there is a natural embedding of the set $C(S_{g,n})_{k-1}$ of isotopy classes of multicurves on $S_{g,n}$ of cardinality k into the profinite set $\mathscr{P}_k(\widehat{\mathscr{L}})$ of unordered subsets of k elements of $\widehat{\mathscr{L}}$. Let us then define the set of *profinite multicurves* on $S_{g,n}$ as the union of the closures of the sets $C(S_{g,n})_{k-1}$ inside the profinite sets $\mathscr{P}_k(\widehat{\mathscr{L}})$, for all $k>0$.

Let us observe that the sets of elements of $\widehat{\Pi}_{g,n}$ in the class of a profinite multicurve on $S_{g,n}$ are simple in the sense of Definition 1.3. However, in general, the class in $\mathscr{P}_k(\widehat{\mathscr{L}})$ of a simple subset of k elements of $\widehat{\Pi}_{g,n}$ is not a profinite multicurve because it may contain peripheral elements.

A *simplicial profinite complex* is an abstract simplicial complex whose set of vertices is endowed with a profinite topology such that the sets of k-simplices, with the induced topologies, are compact and then profinite, for all $k \geq 0$. For these simplicial complexes, the procedure which associates to an abstract simplicial complex and an ordering of its vertex set a simplicial set produces a simplicial profinite set.

Definition 5.3 (cf. [4])**.** Let $L(\widehat{\Pi}_{g,n})$, for $2g-2+n>0$, be the abstract simplicial profinite complex whose simplices are the profinite multicurves on $S_{g,n}$. The abstract simplicial profinite complex $L(\widehat{\Pi}_{g,n})$ is called *the complex of profinite curves on $S_{g,n}$*.

For $2g-2+n>0$, there is a natural continuous action of the procongruence Teichmüller group $\check{\Gamma}_{g,n}$ on the complex of profinite curves $L(\widehat{\Pi}_{g,n})$. There are finitely many orbits of $\check{\Gamma}_{g,n}$ in $L(\widehat{\Pi}_{g,n})_k$, each containing an element of $C(S_{g,n})_k$, for $k \geq 0$, and, by the results of [4, Section 4], these orbits correspond to the possible topological types of a surface $S_{g,n} \smallsetminus \sigma$, for σ a multicurve on $S_{g,n}$.

The main result of [4, §4] (cf. Theorem 4.2) was that, for all $k \geq 0$, the profinite set $L(\widehat{\Pi}_{g,n})_k$ is the $\check{\Gamma}_{g,n}$-completion of the discrete $\Gamma_{g,n}$-set $C(S_{g,n})_k$; i.e., there is a natural continuous isomorphism of $\check{\Gamma}_{g,n}$-sets:

$$L(\widehat{\Pi}_{g,n})_k \cong \varprojlim_{\lambda\in\Lambda} C(S_{g,n})_k \,/\Gamma^{\lambda} \ ,$$

where $\{\Gamma^{\lambda}\}_{\lambda\in\Lambda}$ is a tower of finite index normal subgroup of $\Gamma_{g,n}$ which forms a fundamental system of neighborhoods of the identity for the congruence topology.

The set $\mathscr{L}$ of isotopy classes of non-peripheral s.c.c.'s on $S_{g,n}$ parametrizes the set of Dehn twists of $\Gamma_{g,n}$, which is the standard set of generators for this group. In other words, the assignment $\gamma \mapsto \tau_\gamma$, for $\gamma \in \mathscr{L}$, defines an embedding $d\colon\ \mathscr{L} \hookrightarrow \Gamma_{g,n}$, for $2g-2+n>0$.

The set $\{\tau_\gamma\}_{\gamma\in\mathscr{L}}$ of all Dehn twists of $\Gamma_{g,n}$ is closed under conjugation and falls in a finite set of conjugacy classes which are in bijective correspondence with the possible topological types of the Riemann surface $S_{g,n} \smallsetminus \gamma$. So, let us define, for the congruence completion $\check{\Gamma}_{g,n}$, the set of *profinite Dehn twists* to be the closure of the image of the set $\{\tau_\gamma\}_{\gamma\in\mathscr{L}}$ inside $\check{\Gamma}_{g,n}$. This is the same as the union of the conjugacy classes in $\check{\Gamma}_{g,n}$ of the images of the Dehn twists of $\Gamma_{g,n}$.

There is a natural $\Gamma_{g,n}$-equivariant map $d_k\colon\ \mathscr{L} \to \check{\Gamma}_{g,n}$, defined by the assignment $\gamma \mapsto \tau_\gamma^k$, where $\Gamma_{g,n}$ acts by conjugation on $\check{\Gamma}_{g,n}$. From the universal property of the $\check{\Gamma}_{g,n}$-completion and [4, Theorem 4.2], it then follows that the map d_k extends to a continuous $\check{\Gamma}_{g,n}$-equivariant map $\hat{d}_k\colon\ \widehat{\mathscr{L}} \to \check{\Gamma}_{g,n}$, whose image is the set of k-th powers of profinite Dehn twists.

In particular, a profinite s.c.c. $\gamma \in \widehat{\mathscr{L}}$ determines a profinite Dehn twist, which we denote by τ_γ, in the procongruence Teichmüller group $\check{\Gamma}_{g,n}$. The main result of [4, §5] (cf. Theorem 5.1) is that this provides a parametrization of the set of profinite Dehn twists of the procongruence Teichmüller group.

Theorem 5.4. *For $2g-2+n>0$ and any $k \in \widehat{\mathbb{Z}} \smallsetminus 0$, there is a natural injective map $\hat{d}_k\colon\ \widehat{\mathscr{L}} \hookrightarrow \check{\Gamma}_{g,n}$ which assigns to a profinite s.c.c. $\gamma \in \widehat{\mathscr{L}}$ the k-th power of the profinite Dehn twist τ_γ.*

The purpose of this section is to show how the complex of profinite curves $L(\widehat{\Pi}_{g,n})$ can be "linearized" and then to extract, as a consequence of this process, both a new proof and a generalization of [4, Theorem 5.1] to profinite multitwists. This linearization result can also be considered a generalization to the profinite case of [5, Theorem 5.1].

In order to make this statement more precise, we need to introduce more notation and definitions. Let K be an open normal subgroup of $\widehat{\Pi}_{g,n}$ and let $p_K\colon\ S_K \to S_{g,n}$ be the associated normal unramified covering of Riemann surfaces with covering transformation group $G_K := \widehat{\Pi}_{g,n}/K$. Then let $\overline{S}_K$ be the closed Riemann surface obtained from S_K filling in its punctures, and, for a commutative unitary ring of coefficients A, let $H_1(\overline{S}_K, A)$ be its first homology group. There is a natural map $\psi_K\colon\ K \to H_1(\overline{S}_K, A)$.

Definition 5.5. For a given $\gamma \in \widehat{\mathscr{L}}$, let us denote by the same letter an element of the profinite group $\widehat{\Pi}_{g,n}$ in the class of the given profinite s.c.c. Let $\nu_K(\gamma)$ be the smallest positive integer such that $\gamma^{\nu_K(\gamma)} \in K$. For a profinite multicurve $\sigma \in L(\widehat{\Pi}_{g,n})$, let us also denote by σ a simple subset of $\widehat{\Pi}_{g,n}$ in the class of the given multicurve. Then let $V_{K,\sigma}$ be the primitive A-submodule of $H_1(\overline{S}_K, A)$ generated by the G_K-orbit of the subset $\{\psi_K(\gamma^{\nu_K(\gamma)})\}_{\gamma\in\sigma}$.

For $\sigma = \{\gamma_1, \dots, \gamma_h\} \subset \widehat{\Pi}_{g,n}$ a simple subset, this is the same as the A-submodule generated by the image $\psi_K(\sigma_{K,1}^{\widehat{\Pi}_{g,n}})$ in the homology group $H_1(\overline{S}_K, A)$ (cf. Theorem 4.6).

Then let $\mathrm{Gr}(H_1(\overline{S}_K, A))$ be the absolute Grassmanian of primitive A-submodules of the homology group $H_1(\overline{S}_K, A)$, that is to say, the disjoint union of the Grassmanians of primitive, k-dimensional, A-submodules of the homology group $H_1(\overline{S}_K, A)$, for all $1 \leq k \leq \operatorname{rank} H_1(\overline{S}_K, A)$.

For A_p equal to the ring of p-adic integers $\mathbb{Z}_p$ or the finite field $\mathbb{F}_p$, the absolute Grassmanian $\mathrm{Gr}(H_1(\overline{S}_K, A_p))$ has a natural structure of profinite space, while for $A_p = \mathbb{Q}_p$, it is a locally compact totally disconnected Hausdorff space. In all cases, for $\sigma \in L(\widehat{\Pi}_{g,n})$, the assignment $\sigma \mapsto V_{K,\sigma}$ defines a natural continuous

$$\Psi_{K,p}\colon\ L(\widehat{\Pi}_{g,n}) \longrightarrow \mathrm{Gr}(H_1(\overline{S}_K, A_p)).$$

Theorem 5.6. *For $p > 0$ a prime number, let $A_p = \mathbb{F}_p$, $\mathbb{Z}_p$, or $\mathbb{Q}_p$. For $2g-2+n > 0$, there is a natural continuous injective map,*

$$\widehat{\Psi}_p := \prod_{K \trianglelefteq \widehat{\Pi}_{g,n}} \Psi_{K,p}\colon\ L(\widehat{\Pi}_{g,n}) \hookrightarrow \prod_{K \trianglelefteq \widehat{\Pi}_{g,n}} \mathrm{Gr}(H_1(\overline{S}_K, A_p)),$$

where $\{K\}$ is a cofinal system of open normal subgroups of the profinite group $\widehat{\Pi}_{g,n}$.

Proof. For $A_p = \mathbb{Z}_p$, the submodule $V_{K,\sigma}$ of $H_1(\overline{S}_K, \mathbb{Z}_p)$ is primitive. Therefore, it is enough to prove the theorem for $A_p = \mathbb{F}_p$.

Let $u_1, \dots, u_n \in \Pi_{g,n}$ be simple loops around the punctures on the surface $S_{g,n}$ labeled by the points $P_1, \dots, P_n$, respectively.

For K an open normal subgroup of $\widehat{\Pi}_{g,n}$, a simple set of elements $\sigma = \{\gamma_1, \dots, \gamma_h\}$, and $s \in \mathbb{N}^+$, let us denote by $\tilde{\sigma}_{K,s}^{\widehat{\Pi}_{g,n}}$ the closed normal subgroup generated by the set of elements

$$\gamma_1^{s\cdot\nu_K(\gamma_1)}, \dots, \gamma_h^{s\cdot\nu_K(\gamma_h)}, u_1^{s\cdot\nu_K(u_1)}, \dots, u_n^{s\cdot\nu_K(u_n)}.$$

For $\sigma = \{\gamma_1, \dots, \gamma_h\} \in L(\widehat{\Pi}_{g,n})$, let $\tilde{V}_{K,\sigma}$ be the subspace of the $\mathbb{F}_p$-vector space $K_p^{ab} := H_1(K, \mathbb{F}_p)$ which is the image of the normal subgroup $\tilde{\sigma}_{K,s}^{\widehat{\Pi}_{g,n}}$ of K by the natural epimorphism $K \to K_p^{ab}$. The assignment $\gamma \mapsto \tilde{V}_{K,\sigma}$ then defines a natural continuous map:

$$\tilde{\Psi}_{K,p}\colon\ L(\widehat{\Pi}_{g,n}) \longrightarrow \mathrm{Gr}(K_p^{ab}).$$

For $\xi = \{\delta_1, \dots, \delta_h\} \in L(\widehat{\Pi}_{g,n})$, let us also denote by $\tilde{\xi}_{K,s}^{\widehat{\Pi}_{g,n}}$ the closed normal subgroup generated by the set of elements

$$\delta_1^{s\cdot\nu_K(\gamma_1)}, \dots, \delta_h^{s\cdot\nu_K(\gamma_h)}, u_1^{s\cdot\nu_K(u_1)}, \dots, u_n^{s\cdot\nu_K(u_n)}.$$

Let us show that if $\sigma \neq \xi \in L(\widehat{\Pi}_{g,n})$, there is an open normal subgroup K of $\widehat{\Pi}_{g,n}$ such that there holds $\tilde{\Psi}_{K,p}(\sigma) \neq \tilde{\Psi}_{K,p}(\xi)$. This obviously implies that there holds as well $\Psi_{K,p}(\sigma) \neq \Psi_{K,p}(\xi)$, proving Theorem 5.6.

A consequence of [4, Theorem 4.2] is also that not every element of σ is conjugated to a $\widehat{\mathbb{Z}}$-power of an element in ξ. Since no element of σ is conjugated to u_k, for $k = 1, \dots, n$, from Corollary 4.10, it then follows that there is an open normal subgroup H of $\widehat{\Pi}_{g,n}$ and an $s = p^t$, for $t \in \mathbb{N}$, such that the images of the subgroups $\tilde{\sigma}_{H,s}^{\widehat{\Pi}_{g,n}}$ and $\tilde{\xi}_{H,s}^{\widehat{\Pi}_{g,n}}$ in the maximal pro-p quotient $H^{(p)}$ of the profinite group H are distinct. Moreover, by Remark 4.8, we can assume that all elements $\gamma_i^{\nu_K(\gamma_i)}$ and $\delta_j^{\nu_K(\delta_j)}$, for $i, j = 1, \dots, h$, are generators in H.

Then let L be an open normal subgroup of $\widehat{\Pi}_{g,n}$ contained in H and of index a power of p in H such that there holds $\nu_L(x) = s\nu_H(x)$, for $x = \gamma_i, \delta_j$, or u_k, for $i, j = 1, \dots, h$ and $k = 1, \dots, n$. Then, the images of the subgroups $\tilde{\sigma}_{L,1}^{\widehat{\Pi}_{g,n}} = \tilde{\sigma}_{H,s}^{\widehat{\Pi}_{g,n}}$ and $\tilde{\xi}_{L,1}^{\widehat{\Pi}_{g,n}} = \tilde{\xi}_{H,s}^{\widehat{\Pi}_{g,n}}$ in the maximal pro-p quotient $L^{(p)}$ of L are also distinct. We need one more lemma.

Lemma 5.7. *Let $L^{(p)}$ be a pro-p group and let $N_1 \neq N_2$ be normal subgroups of $L^{(p)}$ invariant for the action of a finite subgroup G of $\mathrm{Out}(L^{(p)})$. Then, there exists an open normal subgroup U of $L^{(p)}$, containing N_1, N_2 and invariant for the action of G, such that $N_1\Phi(U) \neq N_2\Phi(U)$, where, for a given group H, we denote by $\Phi(H)$ its Frattini subgroup.*

Proof. Note that $N_1 \subsetneq N_1N_2$. Hence, it suffices to prove the existence of an open normal subgroup U of $L^{(p)}$ containing N_1N_2 and invariant for the action of G such that there holds $N_1\Phi(U) \neq N_1N_2\Phi(U)$.

Let $\{V_\upsilon\}_{\upsilon\in\Upsilon}$ be the set of all open normal subgroups of $L^{(p)}$ containing N_1N_2 and invariant for the action of G. It then holds that $\bigcap_{\upsilon\in\Upsilon} V_\upsilon = N_1N_2$ and $\bigcap_{\upsilon\in\Upsilon} \Phi(V_\upsilon) = \Phi(N_1N_2)$.

If $N_1\Phi(V_\upsilon) = N_1N_2\Phi(V_\upsilon)$, for all $\upsilon \in \Upsilon$, then there holds as well:

$$N_1 = N_1\Phi(N_1) = \bigcap_{\upsilon\in\Upsilon} N_1\Phi(V_\upsilon) = \bigcap_{\upsilon\in\Upsilon} N_1N_2\Phi(V_\upsilon) = N_1N_2\Phi(N_1N_2) = N_1N_2.$$

Therefore, since $N_1 \neq N_1N_2$, there holds $N_1\Phi(V_\upsilon) \neq N_1N_2\Phi(V_\upsilon)$, for some $\upsilon \in \Upsilon$. □

In order to complete the proof, we apply Lemma 5.7 with $G = \widehat{\Pi}_{g,n}/L$, $N_1 = \psi_K^{(p)}(\tilde{\sigma}_{L,1}^{\widehat{\Pi}_{g,n}})$, and $N_2 = \psi_K^{(p)}(\tilde{\xi}_{L,1}^{\widehat{\Pi}_{g,n}})$. Then, let U be an open normal subgroup of $L^{(p)}$ as in the statement of Lemma 5.7, let U' be its inverse image in $\widehat{\Pi}_{g,n}$, which is also a normal subgroup, and let $K := L \cap U'$. Since K contains both $\tilde{\sigma}_{L,1}^{\widehat{\Pi}_{g,n}}$ and $\tilde{\xi}_{L,1}^{\widehat{\Pi}_{g,n}}$, it holds that $\tilde{\Psi}_{K,p}(\sigma) \neq \tilde{\Psi}_{K,p}(\xi)$. □

6. Centralizers of profinite multitwists in the procongruence Teichmüller group

The results of the previous section have interesting implications for the combinatorial structure of the procongruence Teichmüller group. We will improve the results both of [4] and of [17].

As a first application of Theorem 5.6, it is now possible to give a partial parametrization also to profinite multitwists, i.e., products of powers of commuting profinite Dehn twists. Let us observe that, for $\sigma = \{\gamma_0, \dots, \gamma_k\}$ a profinite multicurve on $S_{g,n}$, the set $\{\tau_{\gamma_0}, \dots, \tau_{\gamma_k}\}$ is a set of commuting profinite Dehn twists.

Theorem 6.1. *For $2g-2+n>0$, let $\sigma = \{\gamma_1, \dots, \gamma_s\}$ and $\sigma' = \{\delta_1, \dots, \delta_t\}$ be two profinite multicurves on $S_{g,n}$. Suppose that there is an identity in $\check{\Gamma}_{g,n}$:*

$$\tau_{\gamma_1}^{h_1}\tau_{\gamma_2}^{h_2}\cdot\ldots\cdot\tau_{\gamma_s}^{h_s} = \tau_{\delta_1}^{k_1}\tau_{\delta_2}^{k_2}\cdot\ldots\cdot\tau_{\delta_t}^{k_t},$$

for $h_i \in m_\sigma \cdot \mathbb{N}^+$ and $k_j \in m_{\sigma'} \cdot \mathbb{N}^+$, with $m_\sigma, m_{\sigma'} \in \widehat{\mathbb{Z}}^*$. Then, there hold:

(i) *$t = s$;*
(ii) *there is a permutation $\phi \in \Sigma_s$ such that $\delta_i = \gamma_{\phi(i)}$ and $k_i = h_{\phi(i)}$, for $i = 1, \dots, s$.*

Proof. If $\sigma \neq \sigma'$, by Theorem 5.6, there is an open characteristic subgroup K of $\widehat{\Pi}_{g,n}$ such that there holds $\Psi_{K,p}(\sigma) \neq \Psi_{K,p}(\sigma')$ in the $\mathbb{Q}_p$-vector space $H_1(\overline{S}_K, \mathbb{Q}_p)$.

Let $\check{\Gamma}^K$ be the geometric level associated to K, i.e., the kernel of the natural representation $\check{\Gamma}_{g,n} \to \mathrm{Out}(\widehat{\Pi}_{g,n}/K)$. Then (cf. [4, §2]), there is a natural representation $\rho_{K,(p)}\colon \check{\Gamma}^K \to Z_{\mathrm{Sp}(H_1(\overline{S}_K,\mathbb{Q}_p))}(G_K)$.

Let $r \in \mathbb{N}^+$ be such that, for every profinite Dehn twist $\tau_\gamma \in \check{\Gamma}_{g,n}$, there holds $\tau_\gamma^r \in \check{\Gamma}^K$. From the results of [4, §5], it follows that it is possible to recover the subspaces $\Psi_{K,p}(\sigma)$ and $\Psi_{K,p}(\sigma')$ as the *cores* (cf. remarks preceding [4, Lemma 5.11]) of the symmetric bilinear forms on $H_1(\overline{S}_K, \mathbb{Q}_p)$ associated to the multitransvections $\rho_{K,(p)}(\tau_{\gamma_1}^{r\cdot h_1}\dots\tau_{\gamma_s}^{r\cdot h_s})$ and $\rho_{K,(p)}(\tau_{\delta_1}^{r\cdot k_1}\dots\tau_{\delta_t}^{r\cdot k_t})$, respectively. Therefore, the hypotheses of the theorem imply that $\sigma = \sigma'$, but then items (i) and (ii) follow immediately. □

An immediate consequence of Theorem 6.1 is a description of centralizers of profinite multitwists of the procongruence Teichmüller group, generalizing Theorems D and E in [17], in which this result was proved for maximal multicurves.

Corollary 6.2.

(i) *For $2g-2+n>0$, let $\sigma = \{\gamma_1, \dots, \gamma_s\}$ be a profinite multicurve on $S_{g,n}$ and $(h_1, \dots, h_k) \in (m_\sigma \cdot \mathbb{N}^+)^k$ a multiindex, with $m_\sigma \in \widehat{\mathbb{Z}}^*$. Then, there holds:*

$$Z_{\check{\Gamma}_{g,n}}(\tau_{\gamma_1}^{h_1}\cdot\ldots\cdot\tau_{\gamma_k}^{h_k}) = N_{\check{\Gamma}_{g,n}}(\langle\tau_{\gamma_1}^{h_1}\cdot\ldots\cdot\tau_{\gamma_k}^{h_k}\rangle) = N_{\check{\Gamma}_{g,n}}(\langle\tau_{\gamma_1}, \dots, \tau_{\gamma_k}\rangle).$$

(ii) *Let us assume that $\sigma = \{\gamma_1, \dots, \gamma_s\}$ is a multicurve on $S_{g,n}$ such that there holds $S_{g,n} \smallsetminus \{\gamma_1, \dots, \gamma_k\} \cong S_{g_1,n_1} \amalg \ldots \amalg S_{g_h,n_h}$. Then, the centralizer in the procongruence Teichmüller modular group $\check{\Gamma}_{g,n}$ of the multitwist $\tau_{\gamma_1}^{h_1}\cdot\ldots\cdot\tau_{\gamma_k}^{h_k}$ is the closure, inside $\check{\Gamma}_{g,n}$, of the stabilizer $\Gamma_\sigma < \Gamma_{g,n}$. Therefore, it is described by the exact sequences*

$$1 \to \check{\Gamma}_{\vec{\sigma}} \to Z_{\check{\Gamma}_{g,n}}(\tau_{\gamma_1}^{h_1}\cdot\ldots\cdot\tau_{\gamma_k}^{h_k}) \to \mathrm{Sym}^{\pm}(\sigma),$$

$$1 \to \bigoplus_{i=1}^k \widehat{\mathbb{Z}}\cdot\tau_{\gamma_i} \to \check{\Gamma}_{\vec{\sigma}} \to \check{\Gamma}_{g_1,n_1}\times\cdots\times\check{\Gamma}_{g_h,n_h} \to 1,$$

where $\mathrm{Sym}^{\pm}(\sigma)$ is the group of signed permutations on the set σ.

Proof. As we already observed, for $f \in \check{\Gamma}_{g,n}$, there holds the identity

$$f \cdot (\tau_{\gamma_1}^{h_1} \cdot \ldots \cdot \tau_{\gamma_k}^{h_k}) \cdot f^{-1} = \tau_{f(\gamma_1)}^{h_1} \cdot \ldots \cdot \tau_{f(\gamma_k)}^{h_k}.$$

The conclusion then follows from Theorem 6.1 and [4, Corollary 6.4 and Theorem 6.6]. □

7. Galois actions on hyperbolic curves

Let C be a hyperbolic curve defined over a number field $\Bbbk$. Let us fix an embedding $\Bbbk \subset \overline{\mathbb{Q}}$ and a $\overline{\mathbb{Q}}$-valued point $\tilde{\xi} \in C$. The structural morphism $C \to \mathrm{Spec}(\Bbbk)$ induces a short exact sequence of algebraic fundamental groups:

$$1 \to \pi_1(C \times_{\Bbbk} \overline{\mathbb{Q}}, \tilde{\xi}) \to \pi_1(C, \tilde{\xi}) \to G_{\Bbbk} \to 1,$$

where $G_{\Bbbk}$ is the absolute Galois group and the group $\pi_1(C \times_{\Bbbk} \overline{\mathbb{Q}})$ is isomorphic to the profinite completion of a hyperbolic surface group. Associated to the above short exact sequence is the outer Galois representation:

$$\rho_C \colon\ G_{\Bbbk} \to \mathrm{Out}(\pi_1(C \times_{\Bbbk} \overline{\mathbb{Q}}, \tilde{\xi})).$$

By [20, Theorem 2.2], [16, Corollary 6.3], and [4, Theorem 7.7], the representation ρ_C is faithful. In this section, as an application of the restricted Magnus property for profinite surface groups, we are going to prove some refinements of these results.

Theorem 7.1. *Let C be a smooth n-punctured, genus g curve, defined over a number field $\Bbbk$. For $2g-2+n>0$ and $3g-3+n>0$, the faithful outer Galois representation ρ_C induces a faithful representation:*

$$\omega_{g,n} \colon\ G_{\Bbbk} \hookrightarrow \mathrm{Aut}(L(\widehat{\Pi}_{g,n})).$$

Proof. By the definition of profinite simple closed curves, it is clear that the representation ρ_C induces a natural representation $G_{\Bbbk} \to \mathrm{Aut}(\widehat{\mathscr{L}_{g,n}})$, which induces the natural representation $\omega_{g,n}$. The faithfulness of the representation $\omega_{g,n}$ then follows from [4, Theorem 7.2 and Corollary 7.6]. □

Let $\{C^\lambda\}_{\lambda \in \Lambda}$ be the tower of Galois étale connected covering of C associated to characteristic subgroups of $\widehat{\Pi} := \pi_1(C \times_{\Bbbk} \overline{\mathbb{Q}}, \tilde{\xi})$ and let us denote by G^λ the covering transformation group of the covering $C^\lambda \to C$. For $\lambda \in \Lambda$, let us also denote by $\overline{C}^\lambda$ the smooth projective curve obtained from C filling in its punctures.

For a G-vector space V, let us denote by $\mathrm{Gr}_G(V)$ the absolute Grassmanian of G-invariant subspaces of V. The outer Galois representation ρ_C then induces, for every $\lambda \in \Lambda$, a natural representation $G_{\Bbbk} \to \mathrm{Aut}\,(\mathrm{Gr}_{G^\lambda}(H^1_{ét}(\overline{C}^\lambda, \mathbb{Q}_\ell)))$.

Let $H^1_{ét}(\overline{C}^\infty, \mathbb{Q}_\ell) := \varinjlim_{\lambda \in \Lambda} H^1_{ét}(\overline{C}^\lambda, \mathbb{Q}_\ell)$. This space is endowed with a natural continuous action of the profinite group $\widehat{\Pi}$. Then let $\mathrm{Gr}_{\widehat{\Pi}}(H^1_{ét}(\overline{C}^\infty, \mathbb{Q}_\ell))$ be the absolute Grassmanian of $\widehat{\Pi}$-invariant subspaces of $H^1_{ét}(\overline{C}^\infty, \mathbb{Q}_\ell)$.

From Theorem 7.1 and Theorem 5.6, Corollary 7.2 follows.

Corollary 7.2. *Let C be a hyperbolic curve defined over a number field $\Bbbk$ with nontrivial moduli space. The associated outer Galois representation ρ_C then induces a natural faithful representation:*

$$G_{\Bbbk} \hookrightarrow \mathrm{Aut}\,(\mathrm{Gr}_{\widehat{\Pi}}(H^1_{ét}(\overline{C}^\infty, \mathbb{Q}_\ell))).$$

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