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FUNDAMENTAL GROUP AND ANALYTIC DISKS

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ABSTRACT. Let W be a domain in a connected complex manifold M and let $w_0 \in W$. Let $\mathcal{A}_{w_0}(W,M)$ be the space of all continuous mappings of a closed unit disk $\overline{\mathbb{D}}$ into M that are holomorphic on the interior of $\overline{\mathbb{D}}$, and let $f(\partial\mathbb{D}) \subset W$ and $f(1) = w_0$. On the homotopic equivalence classes $\eta_1(W,M,w_0)$ of $\mathcal{A}_{w_0}(W,M)$ we introduce a binary operation $\star$ so that $\eta_1(W,M,w_0)$ becomes a semigroup and the natural mappings $\iota_1 : \eta_1(W,M,w_0) \to \pi_1(W,w_0)$ and $\delta_1 : \eta_1(W,M,w_0) \to \pi_2(M,W,w_0)$ are homomorphisms.

We show that if W is a complement of an analytic variety in M and if $S = \delta_1(\eta_1(W,M,w_0))$, then $S \cap S^{-1} = \{e\}$ and any element $a \in \pi_2(M,W,w_0)$ can be represented as $a = bc^{-1} = d^{-1}g$, where $b,c,d,g \in S$.

Let $\mathcal{R}_{w_0}(W,M)$ be the space of all continuous mappings of $\overline{\mathbb{D}}$ into M such that $f(\partial\mathbb{D}) \subset W$ and $f(1) = w_0$. We describe its open dense subset $\mathcal{R}^{\pm}_{w_0}(W,M)$ such that any connected component of $\mathcal{R}^{\pm}_{w_0}(W,M)$ contains at most one connected component of $\mathcal{A}_{w_0}(W,M)$.

1. Introduction

An *analytic disk* in a complex manifold M is a continuous mapping f of the closed unit disk $\overline{\mathbb{D}}$ into M holomorphic on $\mathbb{D}$. We will denote the set of all such disks by $\mathcal{A}(M)$. For a domain W in M we introduce the space $\mathcal{A}(W,M)$ of all continuous mappings f of the unit circle $\mathbb{T} = \partial\mathbb{D}$ into W such that f extends to a mapping $\hat{f} \in \mathcal{A}(M)$. If $w_0 \in W$, then we denote by $\mathcal{A}_{w_0}(W,M)$ the subset of all $f \in \mathcal{A}(W,M)$ such that $f(1) = w_0$.

We let $\eta_1(W,M)$ be the set of all connected components of $\mathcal{A}(W,M)$ and let $\eta_1(W,M,w_0)$ be the set of all connected components of $\mathcal{A}_{w_0}(W,M)$. There is a natural mapping ι_1 of the set $\eta_1(W,M)$ or $\eta_1(W,M,w_0)$ into the set $\pi_1(W)$ or $\pi_1(W,w_0)$, respectively.

In this paper we study the mapping ι_1, its injectivity and its image. These questions originated in [9], where L. Rudolph showed that if B is the braid group (the fundamental group of the complement of some set A of hyperplanes in $\mathbb{C}^n$), then $S = \iota_1((\eta_1(\mathbb{C}^n \setminus A, \mathbb{C}^n, w_0)))$ is a semigroup, $S \cap S^{-1} = \{e\}$ and any element $a \in B$ can be represented as $a = bc^{-1} = d^{-1}g$, where $b,c,d,g \in S$. He called the elements of S *quasipositive*.

In the recent paper [5] J. Kollár and A. Némethi showed under additional assumptions that if M is an algebraic variety with an isolated singularity O and $W = M \setminus O$, then $\iota_1 : \eta_1(W,M) \to \pi_1(W)$ is an injection. They used this result to obtain more information about the singularity.

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These results do not hold in general. For example, when $M = \mathbb{CP}^2$ and A is an algebraic variety in M such that $\pi_1(M \setminus A, w_0) = \mathbb{Z}_p$ and p is prime, then Rudolph's result evidently fails and the result of Kollár and Némethi fails because the set $\eta_1(W, M)$ is infinite due to the homotopic invariance of the intersection index.

However Rudolph's result stays true if we change ingredients. We introduce on the set $\eta_1(W, M, w_0)$ a binary operation $\star$. With this operation $\eta_1(W, M, w_0)$ becomes a semigroup with unity. The natural mapping $\delta_1 : \eta_1(W, M, w_0) \to \pi_2(M, W, w_0)$ is a homomorphism and we show that its image S has the same properties as in Rudolph's result anytime when W is a complement to an analytic variety in a connected complex manifold. When $\pi_1(M, w_0) = \pi_2(M, w_0) = 0$ we obtain a complete analogy but in more general settings.

The problem of injectivity is more interesting and looks more difficult. To advance in this direction we consider the space $\mathcal{R}(W, M, w_0)$ of continuous mappings f of $\overline{\mathbb{D}}$ into M such that $f(\mathbb{T}) \subset W$ and $f(1) = w_0$. We show that there is an open dense set $\mathcal{R}^{\pm}_{w_0}(W, M)$ in $\mathcal{R}_{w_0}(W, M)$ such that the natural mapping δ_1 of $\eta_1(W, M, w_0)$ into the set $\rho^{\pm}_1(W, M, w_0)$ of all connected components of $\mathcal{R}^{\pm}_{w_0}(W, M)$ is an injection.

For future purposes we need to consider not domains $W \subset M$ but Riemann domains W over M. In Section 2 we prove basic facts about them. Since our constructions require more complicated compact sets than $\overline{\mathbb{D}}$, in Section 3 we introduce an operator $I_{K,\gamma}$ that maps homotopic equivalence classes of holomorphic mappings of compact sets into homotopic equivalence classes of holomorphic mappings of the closed disk. The properties of this operator allow us in Section 4 to introduce on $\eta_1(W, M, w_0)$ the structure of a semigroup. In Section 5 we establish major algebraic properties of $\eta_1(W, M, w_0)$. In particular, we obtain the description of the set $\eta_1(W, M)$ as the set of all π_1-conjugacy classes in $\eta_1(W, M, w_0)$.

In Section 6 we introduce the group $\rho_1(W, M, w_0)$ and prove its basic properties. In Section 7 we consider the case when W is the complement to an analytic variety in M and Π is an identity. The generalization of Rudolph's result is one of the theorems in this section. The last Section 8 is devoted to the problem of injectivity.

2. Basic notions and facts

In this paper $\mathbb{D}(a, r)$ is an open disk of radius r centered at a and $\mathbb{T}(0, r)$ is its boundary. We let $\mathbb{D} = \mathbb{D}(0, 1)$ and $\mathbb{T} = \mathbb{T}(0, 1)$.

A *Riemann domain* over a complex manifold M is a pair (W, Π), where W is a connected Hausdorff complex manifold and Π is a locally biholomorphic mapping of W into M. Let $\hat{d}$ be a Riemann metric on M and let d be its lifting to W.

Let K be a connected compact set in $\mathbb{C}$ with connected complement. We denote by $\mathcal{A}(K, M)$ the set of all continuous mappings of K into M that are holomorphic on the interior K^o of K. By $\mathcal{A}(K, W, M)$ we denote the set of all continuous mappings f of ∂K into W such that there is a mapping $\hat{f} \in \mathcal{A}(K, M)$ coinciding with $\Pi \circ f$ on ∂K. The mapping $\hat{f}$ is unique. If $\zeta_0 \in \partial K$ and $w_0 \in W$, then the space $\mathcal{A}_{\zeta_0, w_0}(K, W, M)$ is the set of all $f \in \mathcal{A}(K, W, M)$ such that $f(\zeta_0) = w_0$.

The space $\mathcal{T}(W, M)$ consists of all pairs (K, f), where $K \subset \mathbb{C}$ is a connected compact set with connected complement and $f \in \mathcal{A}(K, W, M)$. If $(K, f), (L, g) \in \mathcal{T}(W, M)$ we define the distance $d((K, f), (L, g))$ between (K, f) and (L, g) as the sum of the Hausdorff distances between the graphs of f and g on ∂K and

∂L, respectively, and between the graphs of $\hat{f}$ and $\hat{g}$ on K and L, respectively. (The distance between points (ζ_1, w_1) and (ζ_2, w_2) on $\mathbb{C} \times M$ is defined as $\max\{|\zeta_1 - \zeta_2|, \hat{d}(w_1, w_2)\}$.) Since the graphs are compact, d is a metric on $\mathcal{T}(W, M)$ and the topology on $\mathcal{T}(W, M)$ is induced by this metric. Clearly, this topology does not depend on the choice of $\hat{d}$.

The set $\mathcal{T}_{\zeta_0,w_0}(W, M) \subset \mathcal{T}(W, M)$ consists of all pairs (K, f) such that $f \in \mathcal{A}_{\zeta_0,w_0}(K, W, M)$. On this set and the sets $\mathcal{A}(K, W, M)$ and $\mathcal{A}_{\zeta_0,w_0}(K, W, M)$ we define the topology relative to the topology imposed on $\mathcal{T}(W, M)$.

Let $\mathcal{T}(M, M)$ be the set of pairs (K, f), where $f \in \mathcal{A}(K, M)$. We define the mapping Π_1 of $\mathcal{T}(W, M)$ into the set $\mathcal{T}(M, M)$ as $\Pi_1(K, f) = (K, \hat{f})$. Clearly, Π_1 is open and locally isometric.

Suppose that $K \subset \mathbb{C}$ is a compact set, $f \in \mathcal{A}(K, M)$ and the graph Γ_f of f on K has a Stein neighborhood U in $\mathbb{C} \times M$. Let F be an imbedding of U into $\mathbb{C}^N$ as a complex submanifold. By [4, Theorem 8.C.8] there are an open neighborhood V of $F(\Gamma_f)$ in $\mathbb{C}^N$ and a holomorphic retraction P of V onto $F(U)$.

Let (L, g) be a pair, where $L \subset \mathbb{C}$ is a compact set, $g \in \mathcal{A}(L, M)$ and $\Gamma_g \subset U$. Then we let $\Phi(L, g)$ be the pair (L, h), where $h(\zeta) = F(\zeta, g(\zeta))$. Conversely, if (L, h) is a pair, where $L \subset \mathbb{C}$ is a compact set, $h \in \mathcal{A}(L, \mathbb{C}^N)$ and $\Gamma_h \subset V$, then we let $\Psi(L, h)$ be the pair (L, g), where $g = P_M \circ F^{-1} \circ P \circ h$ and P_M is a projection of $\mathbb{C} \times M$ onto M. Clearly, the mappings Φ and Ψ are continuous and $\Psi \circ \Phi$ is the identity.

This construction leads to the following lemma.

Lemma 2.1. *Let K be a connected compact set in $\mathbb{C}$ with connected complement. For every $\varepsilon > 0$ there is $\delta > 0$ such that:*

(1) *If $f \in \mathcal{A}(K, W, M)$ and pairs (L, g_0) and (L, g_1) lie in the δ-neighborhood of (K, f) in $\mathcal{T}(W, M)$, then there is a continuous path (L, g_t) in the ε-neighborhood of (K, f) in $\mathcal{T}(W, M)$, $t \in [0, 1]$, connecting (L, g_0) and (L, g_1). Moreover, if, additionally, a compact set $L' \subset L$ and $g_0|_{L'} = g_1|_{L'}$, then we can assume that $g_t|_{L'} = g_0|_{L'}$ for all $t \in [0, 1]$.*

(2) *If $0 \le t \le 1$ and (K, f_t), (L_t, g_t), w_t and $\xi_t \in L_t$ are continuous paths in $\mathcal{T}(W, M)$, $\mathcal{T}(W, M)$, W and $\mathbb{C}$, respectively, and for all $0 \le t \le 1$ the pairs (L_t, g_t) lie in the δ-neighborhood of (K, f_t) and $d(g_t(\xi_t), w_t) < \delta$, then there is another continuous path (L_t, h_t) in $\mathcal{T}(W, M)$ such that $h_t(\xi_t) = w_t$ and the pairs (L_t, h_t) lie in the ε-neighborhood of (K, f_t) for all $0 \le t \le 1$. Moreover, if $g_t(\xi_t) = w_t$ for some $0 \le t \le 1$, then we can assume that $h_t = g_t$.*

Proof. (1) It was shown in [7, Theorem 3.1] that the graph of $\hat{f}$ has a basis of Stein neighborhoods in $\mathbb{C} \times M$. If $M = \mathbb{C}^N$, then we connect $(L, \hat{g}_0)$ and $(L, \hat{g}_1)$ by the path $(L, \hat{g}_t)$, where

$$\hat{g}_t(\zeta) = (1 - t)\hat{g}_0(\zeta) + t\hat{g}_1(\zeta).$$

In the general case, we take $\varepsilon > 0$ so that Π_1^{-1} is defined on the ε-neighborhood of $(K, \hat{f})$. We choose $\delta > 0$ so small that if we take pairs $\Phi(L, \hat{g}_0)$ and $\Phi(L, \hat{g}_1)$, connect them in $\mathbb{C}^N$ by (L, h_t) as above and let $(L, \hat{g}_t) = \Psi(L, h_t)$, then the path $(L, \hat{g}_t)$ lies in the ε-neighborhood of $(K, \hat{f})$. Finally, we let $(L, g_t) = \Pi_1^{-1}(L, \hat{g}_t)$.

(2) For the proof of the second part we note that by [7, Theorem 4.1] the set $\widetilde{\Gamma} = \{(t, \zeta, \hat{f}_t(\zeta)) : \zeta \in K, 0 \le t \le 1\}$ has a Stein neighborhood in $\mathbb{C} \times \mathbb{C} \times M$ and then the proof follows the same pattern as above. □

By part (1) of this theorem the spaces $\mathcal{A}(W,M)$ and $\mathcal{A}_{\zeta_0,w_0}(W,M)$ are locally path-connected and, therefore, their connected components are path-connected.

3. Operator $I_{K,\gamma}$

Throughout this section K will denote a connected compact set in $\mathbb{C}$ with the connected complement. Let $\zeta_0 \in \partial K$ and let a base point $w_0 \in W$. We say that $f,g \in \mathcal{A}_{\zeta_0,w_0}(K,W,M)$ are *h-homotopic* or $f \sim^h g$ if there is a continuous path connecting f and g in $\mathcal{A}_{\zeta_0,w_0}(K,W,M)$. The relation $\sim^h$ is evidently an equivalence and we denote the equivalence class of f by $[f]_{\zeta_0,w_0}$ or $[f]$ if ζ_0 and w_0 are fixed. The set of equivalence classes will be denoted by $\mathcal{H}_{\zeta_0,w_0}[K,W,M]$ or $\mathcal{H}_{\zeta_0,w_0}[K]$. It follows from Lemma 2.1(1) that the equivalence classes are closed in $\mathcal{A}_{\zeta_0,w_0}(K,W,M)$.

Our goal is to construct a mapping of the set $\mathcal{H}_{\zeta_0,w_0}[K]$ into the set $\mathcal{H}_{1,w_0}[\overline{\mathbb{D}}]$. First we do it when K is the closure of a Jordan domain, i.e., K is bounded by a Jordan curve (a homeomorphic image of a circle). Let e_{1,ζ_0} be a conformal mapping of $\overline{\mathbb{D}}$ onto K that maps 1 onto ζ_0. We define the mapping I_{K,ζ_0} as $[f \circ e_{1,\zeta_0}]_{1,w_0}$. Since the group of conformal automorphisms of the unit disk with a fixed point on the boundary is connected, this mapping does not depend on the choice of e.

To define the mapping of $\mathcal{H}_{\zeta_0,w_0}[K]$ into $\mathcal{H}_{1,w_0}[\overline{\mathbb{D}}]$ for a general K we will approximate $f \in \mathcal{A}_{\zeta_0,w_0}(K,W,M)$ by mappings on Jordan domains Ω containing K. To determine a point in $\partial\Omega$ where approximations are equal to w_0 we need an *access curve* to K at ζ_0, i.e., a continuous curve $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = \zeta_0$ and $\gamma(t) \in \mathbb{C} \setminus K$ when $t > 0$.

Let Ω be a smooth Jordan domain containing K whose boundary meets γ. We let $\zeta_{\Omega,\gamma} = \gamma(s_{\Omega,\gamma})$, where $s_{\Omega,\gamma} = \inf\{t : \gamma(t) \in \partial\Omega\}$. A pair $(\overline{\Omega}, g) \in \mathcal{T}(W,M)$ is an *ε-approximation* of $(K,f) \in \mathcal{T}_{\zeta_0,w_0}(W,M)$ with respect to γ if $K \subset \Omega$, $g(\zeta_{\Omega,\gamma}) = w_0$ and $(\overline{\Omega}, g)$ lies in the ε-neighborhood of (K,f).

The following proposition asserts the existence of ε-approximations for every $\varepsilon > 0$.

Proposition 3.1. *Let $f \in \mathcal{A}_{\zeta_0,w_0}(K,W,M)$ and let γ be an access curve to K at ζ_0. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that for any Jordan domain Ω containing K and lying in the δ-neighborhood of K and any point $\zeta \in \gamma \cap \partial\Omega$ there is a mapping $g \in \mathcal{A}_{\zeta,w_0}(\overline{\Omega},W,M)$ such that the pair $(\overline{\Omega}, g)$ lies in the ε-neighborhood of (K,f).*

Proof. First, we prove this proposition when $f \in \mathcal{A}_{\zeta_0,w_0}(K,M,M)$ so $\hat{f} = f$. For the given $\varepsilon > 0$ we denote by η the δ from Lemma 2.1(2). The mapping f is uniformly continuous on K. So there is $\delta_1 > 0$ such that $d(f(\zeta_1), f(\zeta_2)) < \eta/2$ when $|\zeta_1 - \zeta_2| < \delta_1$. We assume that $\delta_1 < \eta$.

By Corollary 4.4 from [7] the compact sets $K \subset \mathbb{C}$ have the Mergelyan property, i.e., there are a neighborhood U of K and a holomorphic mapping $h : U \to M$ such that $d((K,f),(K,h)) < \delta_1/4$. Let us take $\delta > 0$ with the following properties: 1) any smooth Jordan neighborhood Ω of K that lies in the δ-neighborhood of K compactly belongs to U; 2) the diameter of $\gamma \cap \Omega$ is less than $\delta_1/4$; 3) the restriction of h to $\overline{\Omega}$ that we denote also by h lies in the $\delta_1/2$-neighborhood of (K,f).

Let ζ be any point in $\gamma \cap \partial\Omega$. Since $(\overline{\Omega}, h)$ lies in the $\delta_1/2$-neighborhood of (K,f) there is a point $\xi \in K$ such that $|\zeta - \xi| < \delta_1/2$ and $d(h(\zeta), f(\xi)) < \delta_1/2$. Hence $|\zeta_0 - \xi| < \delta_1$ because $|\zeta - \zeta_0| < \delta_1/4$. Thus $d(f(\zeta_0), f(\xi)) < \eta/2$ and $d(h(\zeta), w_0) < \eta$.

By Lemma 2.1(2) we can shift h so that for the shifted mapping g we have $g(\zeta) = w_0$ and $(\overline{\Omega}, g)$ lies in the ε-neighborhood of (K, f).

If $f \in \mathcal{A}_{\zeta_0,w_0}(K, W, M)$, then we take $\varepsilon > 0$ so small that Π^{-1} is defined on the ε-neighborhood of $(K, \hat{f})$, approximate $(K, \hat{f})$ in this neighborhood and compose it with Π^{-1}. $\square$

To continue we need the notion of *Radó continuity.* A family of Jordan domains $\Omega_t \subset \mathbb{C}$, $0 \le t \le 1$, is called *Radó continuous at* $t_0 \in [0, 1]$ if for some $\varepsilon > 0$ a neighborhood of some point $\zeta \in \mathbb{C}$ belongs to the intersection of all Ω_t, $t_0 - \varepsilon < t < t_0 + \varepsilon$, and the family of conformal mappings ϕ_t of $\mathbb{D}$ onto Ω_t such that $\phi_t(0) = \zeta$ and $\phi'_t(0) > 0$ converges uniformly on $\overline{\mathbb{D}}$ to ϕ_{t_0} as $t \to t_0$. (By a theorem of Carathéodory the mappings ϕ_t extend to $\overline{\mathbb{D}}$ as its homeomorphisms onto $\overline{\Omega}_t$.) Such a family is *Radó continuous* if it is Radó continuous at every t. A result of Radó (see [8] or [3, Theorem II.5.2]) claims, in particular, that a family of Jordan domains $\Omega_t \subset \mathbb{C}$ is Radó continuous if and only if for every $t_0 \in [0, 1]$ there are homeomorphisms Ψ_t of $\partial\Omega_{t_0}$ onto $\partial\Omega_t$ converging uniformly to identity on $\partial\Omega_{t_0}$ as $t \to t_0$.

The significance of Radó continuity is in the following lemma.

Lemma 3.2. *If the family of Jordan domains Ω_t, $t \in [0, 1]$, is Radó continuous, $\zeta_t \in \partial\Omega_t$ is a continuous path in $\mathbb{C}$ and $(\overline{\Omega}_t, f_t)$ is a continuous path in $\mathcal{T}(W, M)$ such that $f_t(\zeta_t) = w_0$, then $I_{\overline{\Omega}_t,\zeta_t}(f_t) \equiv \text{const}$.*

Proof. Suppose that ζ is a common point of all Ω_t when t is near t_0 and ϕ_t are the conformal mappings from the definition of Radó continuity. Let $\xi_t = \phi_t^{-1}(\zeta_t) \in \mathbb{T}$ and let α_t be the rotations of $\overline{\mathbb{D}}$ moving 1 to ξ_t. Since the family Ω_t is Radó continuous, then ξ_t and α_t are continuous in t. If $\psi_t = \phi_t \circ \alpha_t$, then $\psi_t(1) = \zeta_t$ and ψ_t is also continuous on $\overline{\mathbb{D}} \times [0, 1]$. Hence $(\overline{\mathbb{D}}, f_t \circ \psi_t)$ is a continuous path in $\mathcal{A}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ and $I_{\overline{\Omega}_t,\zeta_t}(f_t) \equiv \text{const}$. $\square$

We need the following basic lemma.

Lemma 3.3. *Let $w_0 \in W$ and $(K, f) \in \mathcal{T}_{\zeta_0,w_0}(W, M)$. There is $\delta > 0$ such that if:*

(1) *$\Omega_0 \Subset \Omega_1$ are smooth Jordan domains;*
(2) *pairs $(\overline{\Omega}_1, g_1)$ and $(\overline{\Omega}_0, g_0)$ lie in the δ-neighborhood of (K, f) in $\mathcal{T}(W, M)$;*
(3) *$\zeta_0 \in \partial\Omega_0$ and $\zeta_1 \in \partial\Omega_1$ and $g_0(\zeta_0) = g_1(\zeta_1) = w_0$;*
(4) *there is a continuous curve $\gamma : [0, 1] \to \overline{\Omega}_1$ such that $\gamma(t) \in \Omega_1$, $0 \le t < 1$, $\gamma(0) = \zeta_0$, $\gamma(1) = \zeta_1$ and $\gamma \subset \mathbb{D}(\zeta_0, \delta)$,*

then $I_{\overline{\Omega}_0,\zeta_0}(g_0) = I_{\overline{\Omega}_1,\zeta_1}(g_1)$.

Proof. Let us show that if $(K, f) \in \mathcal{T}_{\zeta_0,w_0}(M, M)$, then for any $\varepsilon > 0$ the δ can be chosen in such a way that we can connect the pairs $(\overline{\Omega}_0, g_0)$ and $(\overline{\Omega}_0, g_1)$ in the ε-neighborhood of (K, f). Let us fix $\varepsilon > 0$ and find $0 < \delta_1 < \varepsilon$ such that Lemma 2.1(1) holds with $\delta = \delta_1$. Let us denote by η the δ in Lemma 2.1(2) for which this lemma holds when $\varepsilon = \delta_1$. The mapping f is uniformly continuous on K. So there is $\delta > 0$ such that $d(f(\zeta), f(\xi)) < \eta/2$ when $|\zeta - \xi| < 2\delta$. We assume that $\delta < \eta/2 < \delta_1$.

Since $(\overline{\Omega}_1, g_1)$ lies in the δ-neighborhood of (K, f) for any $\zeta \in \gamma$ there is a point $\xi \in K$ such that $|\zeta - \xi| < \delta$ and $d(g_1(\zeta), f(\xi)) < \delta$. Hence $|\xi - \zeta_0| < 2\delta$ because $\gamma \subset \mathbb{D}(\zeta_0, \delta)$. Thus $d(f(\zeta_0), f(\xi)) < \eta/2$ and $d(g_1(\zeta), w_0) < \eta/2 + \delta < \eta$.

Let Θ be a conformal mapping of $\overline{\Omega}_1 \setminus \Omega_0$ onto an annulus $A(r_0, 1) = \{\zeta \in \mathbb{C} : r_0 \le |\zeta| \le 1\}$ that maps $\partial\Omega_1$ onto the unit circle. We define the intermediate domains Ω_t, $0 \le t \le 1$, as bounded domains with boundaries equal to $\Theta^{-1}(\{|\zeta| = (1 - r_0)t + r_0\})$. The domains Ω_t are simply connected and the family Ω_t is Radó continuous because as homeomorphisms Ψ_t of $\partial\Omega_t$ onto $\partial\Omega_{t_0}$ we can take preimages under the mapping Θ of the radial correspondences between circles in $A(r_0, 1)$.

We will reparameterize this family letting $G_t := \Omega_s$, $\gamma(t) \in \partial\Omega_s$, $t \in [0, 1]$. Then the new family is still Radó continuous. For $t \in [0, 1]$ we define the pairs $(\overline{G}_t, h_t)$, where h_t is the restriction of g_1 to $\overline{G}_t$. This family still lies in the δ-neighborhood of (K, f). Now $h_t(\gamma(t)) = g_1(\gamma(t))$ so $d(h_t(\gamma(t)), w_0) < \eta$. By Lemma 2.1(2) we can shift h_t to get mappings p_t so that $p_t(\gamma(t)) = w_0$ and pairs $(\overline{G}_t, p_t)$ lie in the δ_1-neighborhood of (K, f). Note that $G_1 = \Omega_1$, $G_0 = \Omega_0$ and by the same lemma we can assume that $p_1 = g_1$.

The pairs $(\overline{\Omega}_0, p_0)$ and $(\overline{\Omega}_0, g_0)$ are in the δ_1-neighborhood of (K, f) and by our choice of δ_1 we can connect them by a continuous path in the intersection of the ε-neighborhood of (K, f) with $\mathcal{A}_{\zeta_0,w_0}(\overline{\Omega}_0, W, M)$. Consequently we can connect the pairs $(\overline{\Omega}_0, g_0)$ and $(\overline{\Omega}_1, g_1)$ in the ε-neighborhood of (K, f).

If $(K, f) \in \mathcal{T}_{\zeta_0,w_0}(W, M)$, then we take $\varepsilon > 0$ such that Π_1^{-1} is defined and continuous on the ε-neighborhood of $(K, \hat{f})$ in $\mathcal{T}(M, M)$. We find $\delta > 0$ for $(K, \hat{f})$ such that the pairs $(\overline{\Omega}_0, \hat{g}_0)$ and $(\overline{\Omega}_1, \hat{g}_1)$ can be connected by a continuous path $(\overline{\Omega}_t, h_t)$, $0 \le t \le 1$, in the ε-neighborhood of $(K, \hat{f})$ and the family of Jordan domains Ω_t is Radó continuous. Then the continuous path $\Pi_1^{-1}((\overline{\Omega}_t, h_t))$ connects $(\overline{\Omega}_0, g_0)$ and $(\overline{\Omega}_1, g_1)$. Hence by Lemma 3.2 $I_{\overline{\Omega}_1,\zeta_1}(g_1) = I_{\overline{\Omega}_0,\zeta_0}(p_0)$. ☐

Now we prove that close approximations have the same homotopic type.

Proposition 3.4. *Let $f \in \mathcal{A}_{\zeta_0,w_0}(K, W, M)$ and let γ be an access curve to K at ζ_0. There is $\delta > 0$ such that if $(\overline{\Omega}_1, g_1)$ and $(\overline{\Omega}_2, g_2)$ are δ-approximations of (K, f) with respect to γ, then $I_{\overline{\Omega}_2,\zeta_{\Omega_2,\gamma}}(g_2) = I_{\overline{\Omega}_1,\zeta_{\Omega_1,\gamma}}(g_1)$.*

Proof. We take as δ the δ in Lemma 3.3. By Proposition 3.1 there are a Jordan domain Ω_0 containing K such that $\overline{\Omega}_0 \subset \Omega_1 \cap \Omega_2$ and, given any point $\zeta_1 \in \gamma \cap \partial\Omega_0$, a mapping $g_0 \in \mathcal{A}(\overline{\Omega}_0, W, M)$ such that the pair $(\overline{\Omega}_0, g_0)$ lies in the δ-neighborhood of (K, f) and $g_0(\zeta_1) = w_0$. Let $t_0 = \sup\{t : t < s_{\Omega_1,\gamma}, \gamma(t) \in \Omega_0\}$ and let $\zeta_1 = \gamma(t_0)$. By Lemma 3.3 $I_{\overline{\Omega}_1,\zeta_{\Omega_1,\gamma}}(g_1) = I_{\overline{\Omega}_0,\zeta_1}(g_0)$ and by the same argument $I_{\overline{\Omega}_2,\zeta_{\Omega_2,\gamma}}(g_2) = I_{\overline{\Omega}_0,\zeta_1}(g_0)$. ☐

Let γ be an access curve to K at ζ_0. We define the mapping

$$I_{K,\gamma} = I_\gamma : \mathcal{H}_{\zeta_0,w_0}[K, W, M] \to \mathcal{H}_{1,w_0}[\overline{\mathbb{D}}, W, M] = \eta_1(W, M, w_0)$$

as $I_{K,\gamma}(f) = I_{\overline{\Omega},\zeta_{\Omega,\gamma}}(g)$, where $(\overline{\Omega}, g)$ is a sufficiently close approximation of (K, f). By Proposition 3.4 this mapping is well defined.

If $f \in \mathcal{A}_{w_0}(W, M)$ let $\iota(f)$ be the loop $f|_{\mathbb{T}}$ in W. Clearly, if $[f]_{1,w_0} = [g]_{1,w_0}$ in $\eta_1(W, M, w_0)$, then $\iota(f)$ and $\iota(g)$ are homotopic in $\pi_1(W, w_0)$. Hence the mapping

$$\iota_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0)$$

is also well defined.

The following result shows that $I_{K,\gamma}(f)$ continuously depends on (K, f).

Theorem 3.5. *For any pair* $(K,f)\in\mathcal{T}_{\zeta_0,w_0}(W,M)$ *and an access curve* γ *to* K *at* ζ_0 *there is* $\delta>0$ *such that* $I_{K,\gamma}(f)=I_{L,\gamma}(g)$ *for any pair* $(L,g)\in\mathcal{T}_{\zeta_0,w_0}(W,M)$ *that lies in the* δ*-neighborhood of* (K,f) *and has* γ *as an access curve to* L *at* ζ_0*.*

Proof. Let δ_1 be the δ from Lemma 3.3. Let $(\overline{\Omega}_1,g_1)$ be a δ_1-approximation of (K,f) such that the restriction of γ to $[0,s_{\overline{\Omega}_1,\gamma}]$ lies in $\mathbb{D}(\zeta_0,\delta_1)$. Let r be the minimal distance between points on $\partial\Omega_1$ and K. We take $\delta=\min\{r,\delta_1\}/2$.

If a pair (L,g) lies in the δ-neighborhood of (K,f), then $L\subset\Omega_1$. We take a δ-approximation $(\overline{\Omega}_0,g_0)$ of (L,g) such that $\Omega_0\Subset\Omega_1$ and $I_{\overline{\Omega}_0,\zeta_{\Omega_0,\gamma}}(g_0)=I_{L,\gamma}(g)$. The pair $(\overline{\Omega}_0,g_0)$ lies in the δ_1-neighborhood of (K,f) so by Lemma 3.3

$$I_{K,\gamma}(f)=I_{\overline{\Omega}_1,\zeta_{\Omega_1,\gamma}}(g_1)=I_{\overline{\Omega}_0,\xi_{\Omega_0,\gamma}}(g_0)=I_{L,\gamma}(g).$$

□

The following technical lemma will be used several times later.

Lemma 3.6. *Suppose that* K *consists of a simple curve* α *connecting* ζ_0 *and* ζ_1 *and the closure of a smooth Jordan domain* Ω_1 *such that* $\overline{\Omega}_1\cap\alpha=\{\zeta_1\}$*. Let* $(K,f)\in\mathcal{T}_{\zeta_0,w_0}(W,M)$ *and let* γ *be an access curve to* K *at* ζ_0*. Let* $\Omega_0\subset\Omega_1$ *be another smooth Jordan domain such that* $\partial\Omega_1\cap\partial\Omega_0=\{\zeta_1\}$ *and let* $L=\alpha\cup\overline{\Omega}_0$*. Then there is a mapping* $g\in\mathcal{A}_{\zeta_0,w_0}(L,W,M)$ *such that* $g=f$ *on* α*,* $I_{\overline{\Omega}_1,\zeta_1}(f)=I_{\overline{\Omega}_0,\zeta_1}(g)$ *and* $I_{K,\gamma}(f)=I_{L,\gamma}(g)$*.*

Proof. We take a conformal mapping Φ of $\Omega_1\setminus\overline{\Omega}_0$ onto the strip $\{0<\mathbf{Re}\,\zeta<1\}$. This mapping extends smoothly to the boundary and we assume that $\Phi(\partial\Omega_1)=\{\mathbf{Re}\,\zeta=1\}$ and $\Phi(\partial\Omega_0)=\{\mathbf{Re}\,\zeta=0\}$. Since $\Phi^{-1}(\zeta)$ converges to ζ_1 when $\mathbf{Re}\,\zeta\to\pm\infty$ the domains Ω_t bounded by curves $\Phi^{-1}(\{\mathbf{Re}\,\zeta=t\})$ and ζ_1 are Jordan domains. Moreover the family $\{\Omega_t\}$ is Radó continuous because as homeomorphisms of $\partial\Omega_t$ onto $\partial\Omega_{t_0}$ we can take preimages of mappings $x+it\to x+it_0$.

Let Ψ_t be a continuous family of conformal mappings of Ω_t onto Ω_1 such that $\Psi_t(\zeta_1)=\zeta_1$ and let $K_t=\alpha\cup\overline{\Omega}_t$. We define f_t as f on α and as $f\circ\Psi_t$ on $\overline{\Omega}_t$. Thus we obtain a continuous path in $\mathcal{T}_{\zeta_0,w_0}(W,M)$ and letting $g=f_0$ we get our lemma. □

Two access curves γ_1 and γ_2 are *equivalent* if for every $\varepsilon>0$ there is $\delta>0$ such that if $0<t_1,t_2<\delta$, then the points $\gamma_1(t_1)$ and $\gamma_2(t_2)$ can be connected by a continuous curve α in $\mathbb{D}(\zeta_0,\varepsilon)\setminus K$. In the terminology of the prime ends theory (see [1]) this means that curves γ_1 and γ_2 determine the same prime end.

The following result shows that $I_{K,\gamma_1}=I_{K,\gamma_2}$ when γ_1 and γ_2 are equivalent. In particular, if K is the closure of a Jordan domain, then by a theorem of Carathéodory all access curves at any point of ∂K are equivalent and $I_{K,\gamma}$ is determined only by ζ_0 so we can write $I_{K,\gamma}=I_{K,\zeta_0}$.

Proposition 3.7. *Let* $f\in\mathcal{A}_{\zeta_0,w_0}(K,W,M)$ *and* $\zeta_0\in\partial K$*. If* γ_1 *and* γ_2 *are equivalent access curves to* K *at* ζ_0*, then* $I_{K,\gamma_1}(f)=I_{K,\gamma_2}(f)$*.*

Proof. We take $\delta>0$ that is less than the δs in Lemma 3.3 and Propositions 3.1 and 3.4. Then we find δ-approximations $(\overline{\Omega},g_1)$ of (K,f) with respect to γ_1 and $(\overline{\Omega},g_2)$ of (K,f) with respect to γ_2. The domain Ω has been chosen so that the restrictions of curves γ_1 and γ_2 to $[0,s_{\Omega,\gamma_1}]$ and $[0,s_{\Omega,\gamma_2}]$ lie in $\mathbb{D}(\zeta_0,\delta)$. By Proposition 3.4 $I_{K,\gamma_1}(f)=I_{\overline{\Omega},\zeta_{\Omega,\gamma_1}}(g_1)$ and $I_{K,\gamma_2}(f)=I_{\overline{\Omega},\zeta_{\Omega,\gamma_2}}(g_2)$. We take $0<\sigma<\delta$ such that

$\mathbb{D}(\zeta_0, \sigma) \Subset \Omega$ and find $t_1, t_2 > 0$ such that $\gamma_1(t_1)$ and $\gamma_2(t_2)$ can be connected by a continuous path γ_3 in $\mathbb{D}(\zeta_0, \sigma) \setminus K$.

We take a Jordan domain $\Omega_0 \Subset \Omega$ such that $K \subset \Omega_0$ and $\gamma_3 \subset \Omega \setminus \overline{\Omega}_0$. Let $t_3 = \sup\{t : \gamma_2(t) \in \Omega_0\}$. By Proposition 3.1 there is a δ-approximation $(\overline{\Omega}_0, h)$ of (K, f) such that $g(\gamma_2(t_3)) = w_0$. Let γ be a curve in $\overline{\Omega} \setminus \Omega_0$ that follows γ_1 to $\gamma_1(t_1)$, then γ_3 until $\gamma_2(t_2)$ and then γ_2 to ζ_0. Since $\gamma_3 \subset \mathbb{D}(\zeta_0, \delta)$ by Lemma 3.3 $I_{\overline{\Omega},\zeta_{\Omega,\gamma}}(g_1) = I_{\overline{\Omega}_0,\gamma_2(t_3)}(h)$. By the same lemma $I_{\overline{\Omega},\zeta_{\Omega,\gamma_2}}(g) = I_{\overline{\Omega}_0,\gamma_2(t_3)}(h)$ and we are done. □

4. Holomorphic fundamental semigroup of Riemann domains

If $f \in \mathcal{A}_{w_0}(W, M)$ we will denote $[f]_{1,w_0}$ by $[f]$. To introduce on $\eta_1(W, M, w_0)$ a semigroup structure compatible with ι_1 we need an additional construction since in the standard definition the concatenation of two loops cannot be realized as a boundary of an analytic disk.

Suppose that $f_1, f_2 \in \mathcal{A}_{w_0}(W, M)$ are representatives of equivalence classes $[f_1]$ and $[f_2]$, respectively, in $\eta_1(W, M, w_0)$. Let $K \subset \mathbb{C}$ be the union of $K_1 = \{|\zeta - 1| \le 1\}$ and $K_2 = \{|\zeta + 1| \le 1\}$ and let $\gamma(t) = -it$, $0 \le t \le 1$. Then γ is an access curve for K to 0. We define the mapping

$$h_{f_1,f_2}(\zeta) = \begin{cases} f_1(1-\zeta),\ \zeta \in \partial K_1, \\ f_2(1+\zeta),\ \zeta \in \partial K_2, \end{cases}$$

of ∂K into W. The mapping $\hat{h}_{f_1,f_2}$ maps K into M so $h_{f_1,f_2} \in \mathcal{A}_{0,w_0}(K, W, M)$.

We let $[f_1] \star [f_2] = I_{K,\gamma}(h_{f_1,f_2})$. If f_1 and f_2 are h-homotopic to g_1 and g_2, respectively, in $\mathcal{A}_{w_0}(W, M)$, then evidently h_{f_1,f_2} is h-homotopic to h_{g_1,g_2} in $\mathcal{A}_{0,w_0}(K, W, M)$. Hence the class $[f_1] \star [f_2]$ is well defined.

The following notion of stars is similar to the notion of stars in [10]. A *star* is a plane compact set K that consists of n simple curves $\alpha_j : [0, 1] \to \mathbb{C}$ starting at the same point ζ_0 called the *center* of the star and n disjoint closed disks D_j such that $\zeta_j = \alpha_j(1) \in \partial D_j$. It is required that curves α_j and α_i meet only at ζ_0 when $i \ne j$ and $D_j \cap (\bigcup_{i=1}^n \alpha_i) = \{\zeta_j\}$ for all j. We let $K_j = \alpha_j \cup D_j$ and call them the *arms* of a star. Note that for any star there is a homeomorphism of the plane that transforms this star into a *straight star*, i.e., a star where all curves α_j are intervals.

The conformal mapping $\phi : \mathbb{D} \to \mathbb{CP}^1 \setminus K$ extends continuously to the boundary. This happens because the natural mapping of the prime ends space of $\mathbb{CP}^1 \setminus K$ onto ∂K is continuous. If γ is an access curve to K at ζ_0 and $\phi(1) = \zeta_0$, then the numeration of K_j is chosen in such a way that as ζ travels by $\mathbb{T}$ clockwise starting at 1 the point $\phi(\zeta)$ travels first by ∂K_1, then ∂K_2 and so on.

Proposition 4.1. *Suppose that K is a star with arms $K_j = \alpha_j \cup D_j$, $1 \le j \le n$, ζ_0 is the center of K and γ is an access curve to K at ζ_0. Let L_2 be the star with arms K_j, $2 \le j \le n$, and let L_1 be the star with arms K_j, $1 \le j \le n-1$. If $f \in \mathcal{A}_{\zeta_0,w_0}(K, W, M)$ and $f_j = f|_{K_j}$, then*

$$I_{K,\gamma}(f) = I_{K_1,\gamma}(f_1) \star I_{L_2,\gamma}(f|_{L_2}) = I_{L_1,\gamma}(f|_{L_1}) \star I_{K_n,\gamma}(f_n).$$

Proof. We assume that $\zeta_0 = 0$. Take some small $t_0 > 0$ and redefine f on each α_j letting it be w_0 on $\alpha_j([0, t_0])$ while we set the mappings as $f(\alpha_j((t - t_0)/(1 - t_0)))$ on $[t_0, 1]$. We don't change f on disks and preserve for new mappings the same notation. Since this operation can be achieved by a continuous family of deformations by Theorem 3.5 the h-homotopic classes of (K, f) and all (K_j, f_j) will not change.

In the next step for $s \in [0,1]$ we squeeze starting intervals of curves $\alpha_2, \dots, \alpha_n$ considering a continuous family of continuous curves $\alpha_2^s, \dots, \alpha_n^s$ on $[0,1]$ such that each of these curves is simple, $\alpha_j^0 = \alpha_j$, $\alpha_j^s(t) = \alpha_j(t)$ for $t > t_0$ and $\alpha_j^1(t) = \alpha_i^1(t)$ for all $2 \le i, j \le n$ and $t \in [0, t_1]$ for some $0 < t_1 < t_0$. We set $f_j^s(\alpha_j^s(t)) = f_j(\alpha_j(t))$ and do not change f_j on D_j. It is easy to see that such a family can be found for straight stars and, consequently, for all stars. Let N_2 be the union of curves α_j^1 and disks D_j, $2 \le j \le n$, and let g_2 be the mapping of N_2 equal to f_j^1 on $\alpha_j^1 \cup D_j$. Let $N = K_1 \cup N_2$ and let $g \in \mathcal{A}_{\zeta_0, w_0}(N, W, M)$ be equal to f_1 on K_1 and to g_2 on N_2. It follows from Theorem 3.5 that $I_{K,\gamma}(f) = I_{N,\gamma}(g)$ and $I_{L_2,\gamma}(f) = I_{N_2,\gamma}(g_2)$.

Let $\beta_1 = \alpha_1|_{[0,t_0]}$, $N_1' = K_1 \setminus \beta_1$ and $g_1 = f_1|_{N_1'}$. Let $\beta_2 = \alpha_2^1|_{[0,t_1]}$, $N_2' = N_2 \setminus \beta_2$ and $h_2 = g_2|_{N_2'}$. We take close approximations of (N_j', f_j) by $(\overline{U}_j, p_j)$, $j = 1, 2$, where U_j are smooth Jordan domains. Proposition 3.1 gives us a lot of freedom of choices for U_j so we may assume that U_j meet β_j only once at points $\zeta_j = \beta_j(s_j)$ and $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Let $\gamma_j = \beta_j|_{[0,s_j]}$, $A_j = \gamma_j \cup \overline{U}_j$, $A = A_1 \cup A_2$ and $q_j = p_j$ on $\overline{U}_j$ and w_0 on γ_j. By our choice $I_{A_j,\gamma}(q_j) = I_{N_j,\gamma}(g_j)$ and if approximations are close enough $I_{A,\gamma}(q) = I_{N,\gamma}(g)$, where $q = q_j$ on A_j.

Using Lemma 3.6 we replace Jordan domains U_j with small disjoint disks $V_j \subset U_j$ attached to γ_j at ζ_j and the mapping q on them with a mapping r preserving all involved h-homotopy classes. Then we take continuous deformations γ_j^s of γ_j, $s \in [0,1]$, so that $\gamma_j^s(0) = \zeta_0 = 0$ and $\gamma_1^1(t) = t$ while $\gamma_2^1(t) = -t$. We may assume that if $V_j^s = V_j + \gamma_j^s(1) - \zeta_j$, $M_j^s = \gamma_j^s \cup V_j^s$ and M^s is the union of M_j^s, then M^s are stars for all s. Let $r_j^s(\gamma_j^s(t)) = r_j(t)$ and $r_j^s(\zeta) = r_j(\zeta - \gamma_j^s(1) + \zeta_j)$ on V_j^s. We let r^s be equal to q_j^s on M_j^s.

Applying a continuous family of rotation and dilations we can make the disks V_j^1 perpendicular to the real axis and of radius 1. The mapping r^1 will follow these changes as compositions with rotations and dilations.

The obtained compact set L that consists of intervals $[0,1]$ and $[-1,0]$ and disks $\overline{\mathbb{D}}(2,1)$ and $\overline{\mathbb{D}}(-2,1)$ has two prime ends at 0: one of them is equivalent to $[-i,0]$ and another to $[0,i]$. The chosen numeration tells us that our access curve is $\gamma = [-i, 0]$. Now we shrink intervals $[0,1]$ and $[-1,0]$ to 0 simultaneously translating disks V_j^1. We can do this because $r_j^1 \equiv w_0$ on these intervals. As the result we obtain the figure from the definition of the $\star$ operation and we see that $I_{K,\gamma}(f) = I_{K_1,\gamma}(f_1) \star I_{L_2,\gamma}(f|_{L_1})$.

The second equality in the proposition is proved similarly. □

This proposition leads to the following theorem.

Theorem 4.2. *The operation $\star$ induces on $\eta_1(W, M, w_0)$ the structure of a semigroup with unity.*

Proof. The unity is the class of the constant mapping equal to w_0 on $\mathbb{T}$. If, say, in the definition of the $\star$ operation $f_1 \equiv w_0$, then continuously shrinking K_1 to the origin leaving the functions equal to w_0 we will get a continuous path in $\mathcal{T}_{0,w_0}(W, M)$ which ends at $(K_2, f_2(1+\zeta))$. By Theorem 3.5 $I_\gamma(h_{f_1,f_2}) = [f_2]$.

To prove that the operation $\star$ is associative we consider a compact set L consisting of three intervals $I_1 = [0,1]$, $I_2 = [0,i]$, $I_3 = [-1,0]$ and three closed disks $D_1 = \{|\zeta - 2| \le 1\}$, $D_2 = \{|\zeta - 2i| \le 1\}$ and $D_3 = \{|\zeta + 2| \le 1\}$. The access curve $\gamma = [-i, 0]$. Given $f_1, f_2, f_3 \in \mathcal{A}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ we define the mapping f on L to be equal to w_0 on intervals I_1, I_2, I_3 and $f_1(2-\zeta)$ on D_1, $f_2(2+i\zeta)$ on D_2 and $f_3(2+\zeta)$ on D_3.

By Proposition 4.1

$$I_{L,\gamma}(f) = I_{L_1,\gamma}(f_1) \star (I_{L_2,\gamma}(f_2) \star I_{L_3,\gamma}(f_3)) = (I_{L_1,\gamma}(f_1) \star I_{L_2,\gamma}(f_2)) \star I_{L_3,\gamma}(f_3).$$

□

Using induction and Proposition 4.1 and the previous theorem we get

Theorem 4.3. *Under the assumptions of Proposition* 4.1

$$I_{K,\gamma}(f) = I_{K_1,\gamma}(f_1) \star \cdots \star I_{K_n,\gamma}(f_n).$$

We finish this section with two examples of the semigroup η_1 when $W = A_{s,r} = \{s < |z| < r\}$, where $0 < s < 1 < r$, and $M = \mathbb{CP}^1$ or $M = \mathbb{D}(0, x)$, where $r \le x \le \infty$. We fix $\Pi(z) = z$ and $w_0 = 1$. The examples below show that the mapping $\iota_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0)$ need not be injective or surjective.

If $f \in \mathcal{A}_{w_0}(A_{s,r}, \mathbb{CP}^1)$, then we can write it as $\hat{f} = gB_1B_2^{-1}$, where B_1 and B_2 are Blaschke products and B_1 contains all zeros of $\hat{f}$ while B_2 contains all poles. The function g has no zeros and poles, $g(1) = 1$ and $s < |g| < r$ on $\mathbb{T}$. Hence g maps $\overline{\mathbb{D}}$ into $A_{s,r}$ and is h-homotopic to the constant mapping equal to 1.

We change B_1 by dragging its zeros to 0 by continuous curves and then change B_2 by dragging its zeros to some $a \ne 0$ at $\mathbb{D}$. Thus f is h-homotopic to

$$h(\zeta) = (-a)^n \zeta^m \left(\frac{1 - \bar{a}\zeta}{\zeta - a}\right)^n.$$

Thus we obtained a mapping $[f] \to (m, n)$ and, clearly, it is a homomorphism and it is injective because continuous deformations of analytic disks do not change the numbers of zeros and poles. Hence the semigroup $\eta_1(A_{s,r}, \mathbb{CP}^1, w_0)$ is isomorphic to $\mathbb{N}_0 \oplus \mathbb{N}_0$, where $\mathbb{N}_0$ is the semigroup by addition of non-negative integers.

A similar argument shows that the semigroup $\eta_1(A_{s,r}, \mathbb{D}(0, x), w_0)$ is isomorphic to $\mathbb{N}_0$. The isomorphism is given by the mapping $[f] \to m$, where m is the number of zeros of $\hat{f}$ counted with multiplicity.

5. Properties of holomorphic fundamental semigroups

Let (W_1, Π_1) and (W_2, Π_2) be two Riemann domains over two complex manifolds M_1 and M_2, respectively. Suppose $w_1 \in W_1$, $w_2 \in W_2$ and there are holomorphic mappings $\phi : W_1 \to W_2$ such that $\phi(w_1) = w_2$ and $\psi : M_1 \to M_2$ which satisfy $\psi \circ \Pi_1 = \Pi_2 \circ \phi$. Then for any $f \in \mathcal{A}(K, W_1, M_1)$ we have $\Pi_2 \circ \phi \circ f = \psi \circ \Pi_1 \circ f = \psi \circ \hat{f}$. So $\widehat{\phi \circ f} = \psi \circ \hat{f}$ and we get a continuous mapping from $\mathcal{T}(W_1, M_1)$ to $\mathcal{T}(W_2, M_2)$ which maps a pair (K, f) to $(K, \phi \circ f)$. Hence, first, the mapping from $\mathcal{A}_{w_1}(W_1, M_1)$ to $\mathcal{A}_{w_2}(W_2, M_2)$ induces a well-defined mapping ϕ_* from $\eta_1(W_1, M_1, w_1)$ to $\eta_1(W_2, M_2, w_2)$ given by $\phi_*([f]) = [\phi \circ f]$. Second, if γ is an access curve to K, then $\phi_*(I_{K,\gamma}(f)) = I_{K,\gamma}(\phi \circ f)$. In particular, if (K, h_{f_1,f_2}) is the pair in the definition of the $\star$ operation, then

$$\begin{aligned}&\phi_*([f_1] \star [f_2]) = \phi_*(I_{K,\gamma}(h_{f_1,f_2}))\\ &= I_{K,\gamma}(\phi \circ h_{f_1,f_2}) = I_{K,\gamma}(h_{\phi\circ f_1,\phi\circ f_2}) = [\phi \circ f_1] \star [\phi \circ f_2] = \phi_*[f_1] \star \phi_*[f_2].\end{aligned}$$

This leads us to the following proposition.

Proposition 5.1. *The induced mapping* $\phi_* : \eta_1(W_1, M_1, w_1) \to \eta_1(W_2, M_2, w_2)$ *is a homomorphism.*

Clearly $(W_1 \times W_2, (\Pi_1, \Pi_2))$ is a Riemann domain over $M_1 \times M_2$. As in the classical homotopy theory Proposition 5.1 leads to the following corollary.

Corollary 5.2. *If* (W_1, Π_1) *and* (W_2, Π_2) *are two Riemann domains over two complex manifolds* M_1 *and* M_2*, respectively, then*

$$\eta_1(W_1 \times W_2, M_1 \times M_2, (w_1, w_2)) \cong \eta_1(W_1, M_1, w_1) \times \eta_1(W_2, M_2, w_2).$$

Another corollary describes the powers in $\eta_1(W, M, w_0)$.

Corollary 5.3. *Let* $f \in \mathcal{A}_{w_0}(W, M)$ *and let* $[f]^{\star k}$ *be the product of* k *classes* $[f]$*. Then* $[f]^{\star k} = [f(\zeta^k)]$*.*

Proof. We may assume that $\hat{f}$ is defined on $\mathbb{D}(0, r)$, $r > 1$, and f maps $A_{r^{-1},r} = \{\zeta \in \mathbb{C} : r^{-1} < |\zeta| < r\}$ into W. Set $W_1 = A_{r^{-1},r}$, $M_1 = \mathbb{D}(0, r)$ and $w_1 = 1$. Let $\phi = f$. By Proposition 5.1 and an example at the end of the previous section we have

$$[f(\zeta^k)] = \phi_*([\zeta^k]) = \phi_*([\zeta]^{\star k}) = \phi_*([\zeta])^{\star k} = [f]^{\star k}.$$

□

Let $\alpha(t)$, $t \in [0, 1]$, be a continuous curve in W with $\alpha(0) = w_0$ and $\alpha(1) = w_1$. Let L be a compact set on the plane consisting of the interval $I = [0, 1]$ and the disk $D = \{|\zeta - 2| \le 1\}$. Given a mapping $f \in \mathcal{A}_{1,w_1}(\overline{\mathbb{D}}, W, M)$ we define a mapping $\widetilde{f}$ on L to be equal to α on I and to $f(2 - \zeta)$ on ∂D. Clearly, $\widetilde{f} \in \mathcal{A}_{0,w_0}(L, W, M)$.

We take the access curve $\gamma(t) = -it$, $0 \le t \le 1$, to L at the origin. Clearly, if $[f]_{1,w_0} = [g]_{1,w_0}$, then $I_{L,\gamma}(\widetilde{f}) = I_{L,\gamma}(\widetilde{g})$. Hence we have a well-defined mapping $F_\alpha(f) = F_\alpha([f]) = I_{L,\gamma}(\widetilde{f})$ from $\eta_1(W, M, w_1)$ into $\eta_1(W, M, w_0)$.

By Theorem 3.5 any curve connecting w_0 to w_1 which is homotopic to α will give us the same mapping F_α. Thus F_α depends only on the homotopy class $\{\alpha\}$ of α in $\pi_1(W, w_0, w_1)$.

We let α^{-1} be the curve $(\alpha^{-1})(t) = \alpha(1 - t)$ for $0 \le t \le 1$ and, if a curve β connects w_1 and $w_2 \in W$, denote by $\alpha\beta$ the curve on $[0, 1]$ defined as $\alpha(2t)$ when $0 \le t \le 1/2$ and as $\beta(2t - 1)$ when $1/2 \le t \le 1$.

Theorem 5.4. *Let* w_0, w_1, w_2 *be points of* W*, and let continuous curves* α *and* β *connect* w_0 *with* w_1 *and* w_1 *with* w_2*, respectively. Then:*

(1) $F_{\alpha\beta} = F_\alpha \circ F_\beta$;
(2) F_α *is an isomorphism of* $\eta_1(W, M, w_1)$ *onto* $\eta_1(W, M, w_0)$*.*

Proof. (1) Let $K = [0, 2] \cup \overline{\mathbb{D}}(3, 1)$ and let $f \in \mathcal{A}_{w_2}(W, M)$. We define the mapping g on K as α on $[0, 1]$, $\beta(t - 1)$ on $[1, 2]$ and $f(3 - \zeta)$ on $\overline{\mathbb{D}}(3, 1)$. All access curves to K at 0 are equivalent; we take any such γ and $I_{K,\gamma}(g) = F_{\alpha\beta}(f)$.

We take a Jordan domain Ω containing $K_1 = [1, 2] \cup \overline{\mathbb{D}}(3, 1)$ in its closure such that $1 \in \partial\Omega$ and a close approximation h_1 of $g|_{K_1}$ on $\overline{\Omega}$ so that $I_{\overline{\Omega},1}(h_1) = I_{K_1,1}(g|_{K_1}) = F_\beta(f)$. If the mapping h is defined on $K_2 = [0, 1] \cup \overline{\Omega}$ as h_1 on $\overline{\Omega}$ and α on $[0, 1]$, then also $I_{K,\gamma}(f) = I_{K_2,0}(h)$.

Then using Lemma 3.6 we deform Ω to a small disk attached to 1 and then to the disk $\mathbb{D}(2,1)$ while h changes to p_1. If the mapping p is defined on $K_3 = [0,1] \cup \overline{\mathbb{D}}(2,1)$ as p_1 on $\overline{\mathbb{D}}(2,1)$ and as α on $[0,1]$, then also $I_{K,\gamma}(f) = I_{K_3,\gamma}(p)$. Since $I_{\overline{\mathbb{D}}(2,1),1}(p) = I_{\overline{\Omega},1}(h_1)$ we see that $I_{K_3,\gamma}(p) = F_\alpha(p_1) = F_\alpha(F_\beta(f))$.

(2) Let us show that F_α is a homomorphism, i.e., if $[f] = [f_1] \star [f_2]$, then $F_\alpha(f) = F_\alpha(f_1) \star F_\alpha(f_2)$. Let K be the compact set from the definition of the $\star$ operation, $K' = [-i, 0] \cup K$ and the mapping h is equal to h_{f_1,f_2} on K and as $\alpha(-it + 1)$ on $[-i, 0]$.

We take the interval $[-2i, -i]$ as an access curve γ to K' at $-i$ and consider compact sets $N_1 = [-i, 0] \cup K_1$ and $N_2 = [-i, 0] \cup K_2$. Then we rotate N_1 by a small positive angle around $-i$ and rotate N_2 by a small negative angle around $-i$. The mapping h follows these rotations. The obtained set is a star (P, q) with two arms (P_1, q_1) and (P_2, q_2) such that $I_{P_1,\gamma}(q_1) = F_\alpha(f_1)$ and $I_{P_2,\gamma}(q_2) = F_\alpha(f_2)$. By Theorem 3.5 $I_{P,\gamma}(q) = I_{K',\gamma}(h) = F_\alpha(f)$ and by Theorem 4.3

$$I_{P,\gamma}(q) = I_{P_1,\gamma}(q_1) \star I_{P_2,\gamma}(q_2) = F_\alpha(f_1) \star F_\alpha(f_2).$$

Thus F_α is a homomorphism. Since by (1) $F_{\alpha^{-1}} \circ F_\alpha$ is an identity mapping, F_α is an isomorphism. □

If $g \in \mathcal{A}_{w_0}(W, M)$, then we let the mapping $F_g = F_\alpha$, where $\alpha(t) = g(e^{2\pi it})$, and if α is a loop in W starting at w_0 we denote by $\{\alpha\}$ the equivalence class of α in $\pi_1(W, w_0)$.

Theorem 5.5. *The mapping $\Phi : \{\alpha\} \to F_\alpha$ establishes a homomorphism of $\pi_1(W, w_0)$ into $\mathrm{Aut}(\eta_1(W, M, w_0))$. Moreover, if $g \in \mathcal{A}_{w_0}(\overline{\mathbb{D}}, W, M)$, then $F_g([f]) \star [g] = [g] \star [f]$ and $F_g([g]) = [g]$.*

Proof. The first part of the theorem is a direct consequence of Theorem 5.4. Let us show that $F_g([f]) \star [g] = [g] \star [f]$.

Consider a compact set K consisting of the disks $D^1 = \{|\zeta| \le 1\}$ and $D^2 = \{|\zeta - 3| \le 1\}$ and the interval $I = [1, 2]$. We define a mapping $h(\zeta)$ on D^1 as $g(\zeta)$ and on D^2 as $f(3 - \zeta)$. Let $h(t) = g(e^{2\pi it})$ on I. Let $\gamma = [-i + 1, 1]$ be an access curve to K at 1. Then $I_{K,\gamma}(h) = F_g([f]) \star [g]$.

Consider the continuous family of compact sets K_s, $0 \le s \le 1/2$, consisting of the disk D^1, an interval $I_s = [e^{2\pi si}, (2 - s)e^{2\pi si}]$ and the closed unit disk D^2_s attached normally to $(2-s)e^{2\pi si}$. The mapping h_s on K_s is defined as h on D^1 and as $h(s+|\zeta|)$ when $\zeta \in I_s$. The mapping h_s on D^2_s is defined as a composition of h on D^2 and a conformal mapping that maps D^2_s onto D^2 moving $(2 - s)e^{2\pi si}$. Simply speaking we rotate $I \cup D^2$ around D^1 leaving one end of I_s attached normally to D^1. Clearly, the pairs (K_s, h_s) form a continuous path and $I_{K_s,\gamma}(h_s) = F_g([f]) \star [g]$.

When $s = 1/2$ the set $K_{1/2}$ consists of D^1, $I_{1/2} = [-1, -3/2]$ and the disk $D^2_{1/2}$. Since all access curves to $K_{1/2}$ at 1 are equivalent we replace γ with $\gamma' = [i + 1, 1]$. Still $I_{K_{1/2},\gamma'}(h_{1/2}) = F_g([f]) \star [g]$.

Then we continue the process described above for $1/2 \le s \le 1$. Finally, K_1 will consist of D^1 and $D^2_1 = \{|\zeta - 2| \le 1\}$. The mapping h_1 is equal to g on D^1 and to $f(2 - \zeta)$ on D^2_1. Now it is clear that $I_{K_1,\gamma'}(h_1) = [g] \star [f]$.

To show that $F_g([g]) = [g]$ we start with the compact set K_1 consisting of the interval $I = [0, 1]$ and the unit disk $D_1 = \{|\zeta - 2| \le 1\}$. The mapping f_1 on K_1 is defined as $g(e^{2\pi it})$ on I and as $g(2 - \zeta)$ on D_1. If the access curve $\gamma = [-i, 0]$, then $I_{K_1,\gamma}(f_1) = F_g([g])$.

For $0 \le s \le 1$ we define compact sets K_s consisting of the intervals $I_s = [0, s]$ and the disks $D_s = \{|\zeta - (1+s)| \le 1\}$. The mapping f_s is defined as $g(e^{2\pi it})$ on I_s and as $g(e^{2\pi is}(1 + s - \zeta))$ on D_s. The pairs (K_s, f_s) form a continuous path and $I_{K_s,\gamma}(f_s) = F_g([g])$. Since K_0 consists of the disk $\{|\zeta - 1| \le 1\}$ and the mapping $f_0(\zeta) = g(1 - \zeta)$ we see that $I_{K_1,\gamma}(f_1) = [g]$. □

We recall that a semigroup S is *cancellative* if $ab = ac$ or $ba = ca$ imply $b = c$ for any $a, b, c \in S$ and it is *right(left)-reversible* if $Sa \cap Sb \ne \emptyset$ $(aS \cap bS)$ for any $a, b \in S$ and *reversible* if it is both right- and left-reversible.

Corollary 5.6. *Let $f, g \in \mathcal{A}_{w_0}(W, M)$. Then:*

1. *If $\iota_1([f]) = \iota_1([g])$, then $[f] \star [g] = [g] \star [f]$.*
2. *The semigroup η_1 is reversible.*
3. *The semigroup η_1 is embeddable into a group if and only if it is cancellative.*
4. *The image of $\eta_1(W, M, w_0)$ in $\pi_1(W, w_0)$ under the mapping ι_1 is invariant with respect to the inner automorphisms in $\pi_1(W, w_0)$.*

Proof. Since $F_f = F_g$, $[f]\star[g] = [g]\star[f]$ and we get (1) by the second part of Theorem 5.5. (2) follows because $F_g([f])\star[g] = [g]\star[f]$ and $[f]\star[g] = [g]\star F_{g^{-1}}([f])$. (3) follows from Ore's theorem ([2, 1.10]) which says that any right-reversible cancellative semigroup can be embedded into a group.

To show that the image is invariant with respect to the inner automorphisms we take $f \in \mathcal{A}_{w_0}(W, M)$ and the representative α of $\{\alpha\} \in \pi_1(W, w_0)$. Since $\{\alpha\}\{f\}\{\alpha\}^{-1} = \iota_1(F_\alpha([f]))$ the invariance follows. □

We say that elements $[f_0]$ and $[f_1]$ in $\eta_1(W, M, w_0)$ are π_1-*conjugate* if $F_\alpha([f_0]) = [f_1]$ for some $\alpha \in \pi_1(W, w_0)$. The sets of all elements of $\eta_1(W, M, w_0)$ that are π_1-conjugate of $[f]$ is said to be the π_1-*conjugacy class* of $[f]$. We denote the set of all π_1-conjugacy classes by $\mathcal{C}(W, M, w_0)$.

Proposition 5.7. *Mappings $f_0, f_1 \in \mathcal{A}_{w_0}(W, M)$ belong to the same connected component of $\mathcal{A}(W, M)$ if and only if they are π_1-conjugate.*

Proof. If mappings $f_0, f_1 \in \mathcal{A}_{w_0}(W, M)$ belong to the same connected component of $\mathcal{A}(W, M)$, then there is a continuous curve f_t, $0 \le t \le 1$, in $\mathcal{A}(W, M)$ that connects f_0 and f_1. For $0 \le t \le 1$ define $\alpha(t) = f_t(1)$, $K_t = [0, t] \cup \overline{\mathbb{D}}(1 + t, 1)$ and the mappings g_t to be equal to α on $[0, t]$ and to $f_t(1 + t - \zeta)$ on $\overline{\mathbb{D}}(1 + t, 1)$. The access curve γ is $[-1, 0]$. Note that the curve α is closed, $I_{K_1,\gamma}([g_1]) = F_\alpha([f_1])$ and $I_{K_0,\gamma}([g_0]) = [f_0]$. By Theorem 3.5 $F_\alpha([f_1]) = [f_0]$.

If $F_\alpha([f_1]) = [f_0]$ for some $\alpha \in \pi_1(W, w_0)$, then we let $K = [0, 1] \cup \overline{\mathbb{D}}(2, 1)$ and let g be equal to α on $[0, 1]$ and to $f_1(2 - \zeta)$ on $\overline{\mathbb{D}}(2, 1)$. The access curve γ is $[-i, 0]$. Then $I_{K,\gamma}(g) = [f_0]$ and $I_{\overline{\mathbb{D}}(2,1),1}(g) = [f_1]$.

Let $(\overline{\Omega}, h)$ be a close approximation of (K, g) with the following properties: $\overline{\Omega}$ is symmetric with respect to the real axis and intersects the real axis only at 0 and $x_0 > 3$, $h(0) = h(1) = w_0$, $I_{\Omega,0}(h) = [f_0]$ and $I_{\overline{\mathbb{D}}(2,1),1}(h) = [f_1]$. We take a conformal mapping Φ of $\Omega \setminus \overline{\mathbb{D}}(2, 1)$ onto an annulus $A = \{s \le |\zeta| \le r\}$ such that $\Phi(\bar{z}) = \bar{\Phi}(z)$ and Φ moves $[0, 1]$ to $[s, r]$. Define domains Ω_t as domains bounded by $\Phi^{-1}(\mathbb{T}(0, t))$, $s \le t \le r$. The domains Ω_t are Jordan and if e_t are conformal mappings of $\overline{\mathbb{D}}$ onto Ω_t such that $e_t(1) = \Phi^{-1}(t)$ and $e_t'(0) > 0$, then the mappings $h \circ e_t$ form a continuous path in $\mathcal{A}(W, M)$ while $[h \circ e_r] = [f_1]$ and $[h \circ e_s] = [f_0]$. □

It follows from this proposition that the mapping Ψ assigning to each class in $\mathcal{C}(W, M, w_0)$ the connected component of $\mathcal{A}(W, M)$ containing representatives of elements in this class is well defined.

Theorem 5.8. *The mapping Ψ is a bijection.*

Proof. By Proposition 5.7 Ψ is an injection. If U is a connected component of $\mathcal{A}(W, M)$, then there are a point $w_1 \in W$ and a mapping $f \in \mathcal{A}_{w_1}(W, M)$ such that $f \in U$. Let α be a path in W that connects w_0 to w_1. By Theorem 5.4 $F_\alpha([f]) \in \eta_1(W, M, w_0)$ and the same argument as in the proof of the "only if" part in Proposition 5.7 shows that the representatives of $F_\alpha([f])$ are in U. So Ψ is surjective. $\square$

6. The group $\rho_1(W, M, w_0)$

By the analogy with the complex case we introduce the space $\mathcal{T}^{\mathbb{R}}(W, M)$ of pairs (K, f), where K is a connected compact set on the plane with connected complement and f is a continuous mapping of ∂K into W such that $\hat{f} = \Pi \circ f$ extends to a continuous mapping of K into M. If (K, f) and (L, g) are in $\mathcal{T}^{\mathbb{R}}(W, M)$ we define the distance between (K, f) and (L, g) similar to the definition of the distance d on $\mathcal{T}(W, M)$. That makes the imbedding of $\mathcal{T}(W, M)$ into $\mathcal{T}^{\mathbb{R}}(W, M)$ an isometry.

For such a compact set K, a point $\zeta_0 \in \partial K$ and a point $w_0 \in W$ let us denote by $\mathcal{R}_{\zeta_0, w_0}(K, W, M)$ the subset of all pairs $(K, f) \in \mathcal{T}(W, M)$ such that $f(\zeta_0) = w_0$. If $K = \overline{\mathbb{D}}$, then $\mathcal{R}_{w_0}(W, M) = \mathcal{R}_{1, w_0}(\overline{\mathbb{D}}, W, M)$. We say that $f_0, f_1 \in \mathcal{R}_{\zeta_0, w_0}(K, W, M)$ are equivalent if they belong to the same connected component of $\mathcal{R}_{\zeta_0, w_0}(K, W, M)$ and denote the set of all equivalence classes by $\mathcal{H}^{\mathbb{R}}_{\zeta_0, w_0}(K, W, M)$ and let $[f]_\rho$ be the equivalence class containing f.

If γ is an access curve to K at ζ_0, then, similar to the complex case, we can introduce the mapping

$$I^{\mathbb{R}}_{K,\gamma} : \mathcal{H}^{\mathbb{R}}_{\zeta_0, w_0}(K, W, M) \to \mathcal{H}^{\mathbb{R}}_{1, w_0}(\overline{\mathbb{D}}, W, M) = \rho_1(W, M, w_0).$$

Similar to the $\star$ operation introduced earlier we can define the $\star$ operation on $\rho_1(W, M, w_0)$ and a similar but simpler reasoning shows that $\rho_1(W, M, w_0)$ with the $\star$ operation is a semigroup with unity. All properties of the operator $I_{K,\gamma}$ and the semigroup η_1, proved in the previous sections, stay true for their analogs $I^{\mathbb{R}}_{K,\gamma}$ and ρ_1 but ρ_1 is a group.

Theorem 6.1. *The operation $\star$ induces on $\rho_1(W, M, w_0)$ the structure of a group: if $[f]_\rho \in \rho_1(W, M, w_0)$, then $[f(\bar{\zeta})]_\rho = [f]_\rho^{-1}$.*

Proof. Let $f_1 \in \mathcal{R}_{w_0}(W, M)$ and let $f_2(\zeta) = f_1(\bar{\zeta})$. Let $K_1 = \{|\zeta - 1| \le 1\}$ and $K_2 = \{|\zeta + 1| \le 1\}$ be the sets from the definition of the $\star$ operation. Let $K_1^t = K_1 - t$ and $K_2^t = K_2 + t$ and let $L_t = K_1^t \cup K_2^t$ when $0 \le t \le 1$ and $L_t = K_1^t \cap K_2^t$ when $1 \le t \le 2$. Since $\hat{f}_2(1 + \zeta) = \hat{f}_1(1 + \bar{\zeta}) = \hat{f}_1(1 - (-\bar{\zeta}))$ the mappings $\hat{f}_1(1 - \zeta)$ and $\hat{f}_2(1 + \zeta)$ are symmetric with respect to the imaginary axis. Hence the mappings g_t that are equal to $\hat{f}_1(1 - \zeta + t)$ on $\partial L_t \cap K_1^t$ and to $\hat{f}_2(1 + \bar{\zeta} - t)$ on $\partial L_t \cap K_2^t$ are continuous and $\hat{g}_t$ extends to the continuous mapping of L_t.

To preserve the base points we shift L_t and f_t upward by $i\alpha(t)$, where $\alpha(t) = \cos^{-1}(1 - t)$, and let $L'_t = (L_t + i\alpha(t)) \cup [0, i\alpha(t)]$. We set g'_t as g_t shifted upward and also let $g'_t(iy) = f_1(e^{i\alpha(y)})$ for $0 \le y \le \alpha(t)$. Note $g_t(0) = f_1(1) = w_0$ and $g_t(i\alpha(t)) = f_1(e^{i\alpha(t)})$.

The path (L'_t, g'_t), $0 \le t \le 2$, is continuous in $\mathcal{T}^{\mathbb{R}}(W, M)$ and by the real analog of Theorem 3.5 we see that $[f_1]_\rho \star [f_2]_\rho = [g'_2]_\rho$. But L'_2 is the interval $[0, i\pi]$ so $[g'_2]_\rho = e$. □

There are natural homomorphisms $\delta_1 : \eta_1(W, M, w_0) \to \rho_1(W, M, w_0)$ and $\delta_2 : \rho_1(W, M, w_0) \to \pi_1(W, w_0)$ such that $\delta_2 \circ \delta_1 = \iota_1$.

Theorem 6.2. *Let $W \subset M$. Then in the notation above:*

(1) *if $\pi_1(M, w_0) = 0$, then δ_2 is onto;*
(2) *if $\pi_2(M, w_0) = 0$, then δ_2 is one-to-one.*

Proof. (1) is evident. To show (2) we take an element $[f]_\rho \in \ker \delta_2$ and let $\alpha = f|_{\mathbb{T}}$. Then $\{\alpha\} = \delta_2([f]_\rho)$. There is a continuous mapping $g : \overline{\mathbb{D}} \to W$ such that $g|_{\mathbb{T}} = \alpha$. It means that $[g]_\rho = e$. If we realize $\hat{f}$ as a mapping of the upper hemisphere of the unit ball in $\mathbb{R}^3$ and g as the mapping of the lower one, then we obtain the mapping h of the sphere S^2 into M equal to α on the equator. We may assume that $h(1, 0, 0) = w_0$. Since $\pi_2(M, w_0) = 0$ the mapping h can be continuously extended to the ball as a mapping into M. Thus $[f]_\rho = [g]_\rho = e$. □

As simple consequences of Corollary 5.6(1) and Theorem 6.2 we obtain

Corollary 6.3. *Let $W \subset M$. Then:*

(1) *The kernel of δ_1 is a commutative semigroup.*
(2) *If $\pi_1(M, w_0) = \pi_2(M, w_0) = 0$, then $\rho_1(W, M, w_0) = \pi_1(W, w_0)$.*

If $W \subset M$ and Π is an inclusion map, then $\rho_1(W, M, w_0)$ is, of course, the relative homotopy group $\pi_2(M, W, w_0)$. So for the examples in Section 4 we get that $\rho_1(A_{s,r}, \mathbb{CP}^1, w_0) = \mathbb{Z} \oplus \mathbb{Z}$ while $\rho_1(A_{s,r}, \mathbb{C}, w_0) = \mathbb{Z}$.

7. Complements to analytic varieties

Let A be an analytic set in a connected complex manifold M and let $W = M \setminus A$. We assume that A is the union of irreducible components A_j of pure codimension 1. (Analytic sets of codimension 2 and higher do not influence groups η_1 and ρ_1.)

If $f \in \mathcal{R}_{w_0}(W, M)$ and the set $A(f) = \{\zeta \in \mathbb{D} : f(\zeta) \in A\}$ is finite, then we define the index $\operatorname{ind}(f, A_j)$ as the intersection index of $f(\overline{\mathbb{D}})$ and A_j. If $\zeta \in A_j(f)$ and ϕ is a defining function of A_j on a neighborhood of $f(\zeta)$, then we define a local index $\operatorname{ind}_\zeta(f, A_j)$ of f at ζ as the index of $\phi \circ f$ at ζ. Hence

$$\operatorname{ind}(f, A_j) = \sum_{f(\zeta_k) \in A_j} \operatorname{ind}_{\zeta_k}(f, A_j).$$

A general $f \in \mathcal{R}_{w_0}(W, M)$ can be approximated by such mappings and close approximations have the same indexes so $\operatorname{ind}(f, A_i)$ is defined for all $f \in \mathcal{R}_{w_0}(W, M)$. The index is a homotopic invariant so if $f_0, f_1 \in \mathcal{R}_{w_0}(W, M)$ and $[f_0]_\rho = [f_1]_\rho$, then $\operatorname{ind}(f_0, A_j) = \operatorname{ind}(f_1, A_j)$. Thus the mapping ind is well defined on ρ_1. Also $\operatorname{ind}(f, A_j) = 0$ for all j if $[f]_\rho = e$.

It follows directly from the definition of the $\star$ operation that $\operatorname{ind}([f]_\rho \star [g]_\rho, A_j) = \operatorname{ind}([f]_\rho, A_j) + \operatorname{ind}([g]_\rho, A_j)$. In particular, the group ρ_1 has no idempotents.

Suppose that $f \in \mathcal{R}_{w_0}(W, M)$ and the set $A(f)$ is finite. Let K be a star in $\overline{\mathbb{D}}$ with its center at 1 such that its arms K_j consist of simple curves $\alpha_j \in \overline{\mathbb{D}} \setminus A(f)$

that meet $\partial\overline{\mathbb{D}}$ only at 1 and of closed disjoint disks $D_j \subset \mathbb{D}$, $1 \le j \le k$, such that the set $\partial D_j \cap A(f)$ is empty, each D_j contains exactly one point of $A(f)$ and $A(f)$ is covered by disks D_j. Let $\widetilde{f}$ be the restriction of f to K. We will call $(K, \widetilde{f})$ the *factorization* of f.

Theorem 7.1. *Suppose that $f \in \mathcal{R}_{w_0}(W, M)$ and $f(\mathbb{D})$ meets A only at finitely many points $\zeta_1, \dots, \zeta_k$. Let $K = \bigcup_{j=1}^k K_j$ be a factorization of f. Then*

$$[f]_\rho = I^{\mathbb{R}}_{K,1}(\widetilde{f}) = \prod_{j=1}^k I^{\mathbb{R}}_{K_j,1}(\widetilde{f}).$$

Proof. Since in this case $\rho_1(W, M, w_0) = \pi_2(M, W, w_0)$ and K is a homotopic retract of $\overline{\mathbb{D}}$, the first equality follows. The second equality follows from the real analog of Theorem 4.3. □

Now we can prove the general analog of the result of L. Rudolph in [9]. Let $S = \delta_1(\eta_1(W, M, w_0))$ and let S^{-1} be the semigroup consisting of all $a \in \rho_1(W, M, w_0)$ such that $a = b^{-1}$ and $b \in S$.

Theorem 7.2. *The semigroup S has the following properties:*

(1) *S is reversible;*
(2) *$S \cap S^{-1} = \{e\}$;*
(3) *any element $a \in \rho_1(W, M, w_0)$ is expressible in the form bc^{-1}, $b, c \in S$;*
(4) *any element $a \in \rho_1(W, M, w_0)$ is expressible in the form $d^{-1}f$, $d, f \in S$.*

Proof. (1) holds because $\eta_1(W, M, w_0)$ is reversible. To show (2) we suppose that $a = b^{-1}$, $a = \delta_1([f_0]) \ne e$, and $b = \delta_1([f_1])$, $f_0, f_1 \in \mathcal{A}_{w_0}(W, M)$. If the set $A(f_0)$ is empty, then $[f_0] = e$ and $[f_1] = e$. So we assume that $\operatorname{ind}(f_0, A_j) > 0$ for some j. Let $f_2(\zeta) = f_1(\bar{\zeta})$. By Theorem 6.1 $[f_2]_\rho = [f_1]_\rho^{-1} = [f_0]_\rho$. But $\operatorname{ind}(f_0, A_j) > 0$ while $\operatorname{ind}(f_2, A_j) < 0$ and we came to a contradiction.

To show (3) we take $f \in \mathcal{R}_{w_0}(W, M)$ that is smooth and transverse to A and $[f]_\rho = a$. In this case the set $A(f)$ is finite and consists of points $\zeta_1, \dots, \zeta_k$ in $\mathbb{D}$ such that $\operatorname{ind}_{\zeta_j}(f, A) = \pm 1$. We assume that points ζ_j are enumerated in such a way that the local index is 1 when $1 \le j \le n$ and -1 when $n < j \le k$ and change f slightly near these points so it becomes holomorphic when $1 \le j \le n$ and antiholomorphic when $n < j \le k$.

Then we form a factorization (K, g) of f with arms $K_j = \alpha_j \cup D_j$, where disks D_j are so small that f is either holomorphic or antiholomorphic on them. By Theorem 6.1 $[f_j]_\rho = [h_j]_\rho^{-1}$, where $h_j \in \mathcal{A}_{w_0}(K_j, W, M)$ when $n < j \le k$ and by Theorem 7.1

$$[f]_\rho = \prod_{j=1}^n \delta_1([f_j]) \prod_{j=n+1}^k (\delta_1([h_j]))^{-1}.$$

Part (4) has the same proof. □

By Theorem 6.2 if $\pi_1(M, w_0) = \pi_2(M, w_0) = 0$, then the group ρ_1 in Theorem 7.2 can be replaced by $\pi_1(W, w_0)$.

8. Connected components of $\mathcal{A}(W, M)$ and $\mathcal{R}(W, M)$

There are natural mappings of the set $\eta_1(W, M)$ of connected components of $\mathcal{A}(W, M)$ into the set $\rho_1(W, M)$ of connected components of $\mathcal{R}(W, M)$ and $\pi_1(W)$.

We will denote these mappings also by δ_2 and ι_1, respectively. The mapping ι_1 need not be an injection especially when the group $\pi_2(M)$ is non-trivial. For example, if $M = \mathbb{CP}^n$ and A is an algebraic variety in M such that $\pi_1(W)$ is finite, then ι_1 is not an injection because $\eta_1(W, M)$ is always infinite due to the invariance of index.

There is hope that δ_2 is an injection at least when $M = \mathbb{C}^n$. To advance in this direction we introduce the set $\mathcal{R}^{\pm}_{w_0}(W, M)$ of mappings $f \in \mathcal{R}_{w_0}(W, M)$ such that the set $A(f)$ is finite and the points in $A(f)$ have non-zero local indexes. This set is open and dense in $\mathcal{R}_{w_0}(W, M)$.

Lemma 8.1. *Let M be a complex manifold, let A be an analytic variety in M and let $W = M \setminus A$. Suppose that mappings $f_0, f_1 \in \mathcal{R}^{\pm}_{w_0}(W, M)$ are smooth and transverse to A and can be connected by a continuous curve f_t, $0 \le t \le 1$, in $\mathcal{R}^{\pm}_{w_0}(W, M)$. Then there is a smooth path g_t, $0 \le t \le 1$, connecting f_0 and f_1 in $\mathcal{R}^{\pm}_{w_0}(W, M)$ such that:*

1. *the mapping $G(t, \zeta) = g_t(\zeta)$ of $[0, 1] \times \overline{\mathbb{D}}$ into M is transverse to A;*
2. *the set A_G consists of finitely many disjoint smooth curves $\{\alpha_j(t)\}$, $t \in [0, 1]$, such that $\alpha_j(t) = (t, \zeta_j(t))$.*

Proof. We may assume that the mapping $F : [0, 1] \times \overline{\mathbb{D}} \to M$ defined as $F(t, \zeta) = f_t(\zeta)$ is smooth. Let $A_{\text{sing}} = A^1_{\text{sing}}$ be the set of singular points of A. Define by induction the sets $A^k_{\text{sing}} = (A^{k-1}_{\text{sing}})_{\text{sing}}$. For some $k \le n + 1$ the set A^k_{sing} is empty and therefore the set A^{k-1}_{sing} is a manifold. By the Thom Transversality Theorem we can approximate F by a smooth mapping F_k transverse to A^{k-1}_{sing}. By the definition of transversality $F_k([0, 1] \times \overline{\mathbb{D}})$ never meets the set A^{k-1}_{sing} if $\dim A^{k-1}_{\text{sing}} \le n - 2$. Now we let $M_{k-1} = M \setminus A^{k-1}_{\text{sing}}$ and apply the transversality theorem to M_{k-1} and A^{k-2}_{sing} to find F_{k-2}. By induction we obtain an approximation H of F that never meets the set A_{sing} and is transverse to A. Let $h_t(\zeta) = H(t, \zeta)$. Since M admits a real analytic imbedding into some $\mathbb{R}^N$ we can choose H to be real analytic and since the set $\mathcal{R}^{\pm}_{w_0}(W, M)$ is open we may assume that $h_t \in \mathcal{R}^{\pm}_{w_0}(W, M)$ for all $t \in [0, 1]$.

The set A_H is a compact set in $[0, 1] \times \mathbb{D}$ and a smooth submanifold, i.e., it is a collection Γ of finitely many disjoint smooth curves $\{\gamma_j\}$, $1 \le j \le m$.

Suppose that $(t_0, \zeta_0) \in \gamma_j$ and $H(t_0, \zeta_0) = w_0 \in A$. The point w_0 is a regular point of A and there is a neighborhood U of w_0 such that in appropriate coordinates $(z_1, \dots, z_n)$ the set $A \cap U = \{z_1 = 0\}$. In coordinates $(z_1, \dots, z_n)$ the mapping $H(t, \zeta) = (H_1(t, \zeta), \dots, H_n(t, \zeta))$ and the functions H_k are real analytic. Since the rank of dH_1 is 2 and the curve $\gamma_j = \{H_1(t, \zeta) = 0\}$, by the Implicit Function Theorem the curve γ_j admits a real analytic parametrization $\gamma_j(s) = (t(s), \zeta(s))$ near (t_0, ζ_0) with $t(0) = t_0$. The mapping h_{t_0} is not transverse to A at ζ_0 if and only if either $t(s) = t_0$ near 0 or $t(s) = t_0 + as^p + o(s^p)$, $a \ne 0$ and $p > 1$. But the former case is excluded because $h_{t_0} \in \mathcal{R}^{\pm}_{w_0}(W, M)$ and the set $A(h_{t_0})$ is finite. By real analyticity in the latter case there are only finitely many points where $t'(s) = 0$. Hence the set E of those t where h_t is not transverse to A is finite.

Let $t_0 \in E$. If p is even and $a > 0$, then $t(s)$ has a strict local minimum at 0 and if $a < 0$, then it has a strict local maximum there. In both cases $\operatorname{ind}_{\zeta_0}(h_{t_0}, A) = 0$ and this contradicts the assumption that $h_{t_0} \in \mathcal{R}^{\pm}_{w_0}(W, M)$. If $p > 1$ is odd, then $t(s)$ is either strictly increasing or decreasing near 0 and the set $h_t(\overline{\mathbb{D}}) \cap A$ has only one point in a small neighborhood of ζ_0 for t sufficiently close to ζ_0. Hence we can choose a real analytic parametrization $\gamma_j(s)$ such that $s \in [0,1]$, the function $\tau_j(s) = t(\gamma_j(s))$ is strictly increasing and $\tau_j'(s) = 0$ only at finitely many points.

Finally, we take a smooth diffeomorphism $\Phi(t,\zeta)$ of $[0,1] \times \overline{\mathbb{D}}$ such that $\Phi(t,\zeta) = (t, \phi(t,\zeta))$ and the functions $t(\Phi(\gamma_j(s)))$ have strictly positive derivatives. Let $s_j(t)$ be the inverse of the latter function. The mapping $G = H \circ \Phi^{-1}$ and the curves $\alpha_j(t) = \Phi(\gamma_j(s_j(t)))$ have all the required properties. □

The proof of the lemma below follows the same line of argument as that in the proof of Assertion 2 in the proof of [12, Lemma 2.1].

Lemma 8.2. *Let $\zeta_k(t)$, $1 \le k \le n$, be smooth mappings of $[0,1]$ into $\mathbb{D}$ such that $\zeta_i(t) \ne \zeta_j(t)$ when $i \ne j$ and $0 \le t \le 1$. Then there is a C^∞ mapping $\Phi : \overline{\mathbb{D}} \times [0,1] \to \overline{\mathbb{D}}$ such that $\Phi_t(\zeta) = \Phi(\zeta,t)$ is a diffeomorphism of $\overline{\mathbb{D}}$ onto itself for each t, $\Phi_t(\zeta) = \zeta$ when $|\zeta| = 1$ and $\Phi_t(\zeta_j(0)) = \zeta_j(t)$ for $j = 1,\dots,n$.*

Proof. By the Whitney Extension Theorem (see [6, Theorem 1.5.6]) we can find a complex valued C^∞-function $F(t,\zeta)$ on $[0,1] \times \mathbb{C}$ such that $F(t,\zeta_j(t)) = \partial\zeta_j(t)/\partial t$ for $0 \le t \le 1$, $j = 1,\dots,n$. Replacing F with the product $F\phi$, where ϕ is a C^∞-function with $\phi = 1$ on $\bigcup_{j=1}^n \{(t,\zeta_j(t)) : 0 \le t \le 1\}$ and $\phi = 0$ for $|\zeta| \ge 1$, we can make $F(t,\zeta) = 0$ for $|\zeta| \ge 1$. Then by standard existence and uniqueness theorems for ordinary differential equations, the initial value problem $\partial x/\partial t(t) = F(t,x(t))$, $x(0,\zeta) = \zeta$, $0 \le t \le 1$, has a unique solution $x(t,\zeta)$. Since $F(t,\zeta)$ is smooth, this solution is smooth on $[0,1] \times \mathbb{C}$.

Now define a mapping $\Phi : \mathbb{C} \times [0,1] \to \mathbb{C}$ by $\Phi(\zeta,t) = x(t,\zeta)$. Then for each $0 \le t \le 1$, Φ_t is a diffeomorphism and since the related initial value problem has a unique solution we have $\Phi(\zeta_j(0),t) = \zeta_j(t)$ for $j = 1,\dots,n$. Also note that $\Phi(\zeta,t) = \zeta$ for all $0 \le t \le 1$ when $|\zeta| \ge 1$. So, the restriction of Φ to $\overline{\mathbb{D}} \times [0,1]$ has the desired properties. □

Let $\gamma : [0,1] \to M$ be a continuous curve connecting the base point w_0 with a point w in a regular part of some component A_i and such that $\gamma([0,1)) \subset W$. Given a neighborhood U of w and $\varepsilon > 0$ consider continuous mappings $f_{U,\varepsilon}$ of $[0,1-\varepsilon] \cup \overline{\mathbb{D}}(2-\varepsilon,1)$ such that $f(t) = \gamma(t)$ when $0 \le t \le 1-\varepsilon$ and the restriction of f to $\overline{\mathbb{D}}(2-\varepsilon,1)$ is an analytic disk in U transversal to A_i and whose index with respect to A_i is 1. Rephrasing the definition from [11] we call such mappings *lassos* λ_γ around A_i. Clearly, there are U and ε_0 such that all mappings $f_{V,\varepsilon}$ are equivalent in $\mathcal{H}[\overline{\mathbb{D}},W,M]$ when $V \subset U$ and $\varepsilon < \varepsilon_0$.

Lemma 8.3. *Let λ_{γ_0} and λ_{γ_1} be lassos around A_i. Suppose that there is a continuous mapping $\phi : [0,1]^2 \to M$ such that for all t we have $\phi(0,t) = \gamma_0(t)$, $\phi(1,t) = \gamma_1(t)$, $\phi(t,0) = w_0$, $\phi(t,1) \in A_i^{\mathrm{reg}}$ and $\phi(t,s) \in W$ when $s \ne 1$. Then λ_{γ_0} and λ_{γ_1} are equivalent in $\mathcal{H}_{0,w_0}[\overline{\mathbb{D}},W,M]$.*

Proof. For some small $\varepsilon > 0$ we can construct a continuous family g_t of analytic disks transversal to A_i and of index 1, centered at $\phi(t,1)$ and such that $g_t(1) = \phi(t,1-\varepsilon)$. Then define $K = [0,1-\varepsilon] \cup \overline{\mathbb{D}}(2-\varepsilon,1)$ and $\lambda_{\gamma_t} : K \to M$ as $\phi(t,s)$ on $[0,1-\varepsilon]$ and as $g_t(\zeta - 2 + \varepsilon)$ on $\overline{\mathbb{D}}(2-\varepsilon,1)$. The path λ_{γ_t} is continuous in $\mathcal{T}(W,M)$ and by Theorem 3.5 $[\lambda_{\gamma_0}]_{0,w_0} = [\lambda_{\gamma_1}]_{0,w_0}$. □

Theorem 8.4. *Let M be a complex manifold, let A be an analytic set in M and let $W = M \setminus A$. If f_0 and f_1 in $\mathcal{A}_{1,w_0}(W,M)$ belong to the same connected component of $\mathcal{R}^{\pm}_{w_0}(W,M)$, then $[f_0]_{1,w_0} = [f_1]_{1,w_0}$.*

Proof. We may assume that f_0 and f_1 are transverse to A. Since they belong to the same connected component of $\mathcal{R}^{\pm}_{w_0}(W, M)$ there is a smooth mapping $G(t, \zeta) : [0, 1] \times \overline{\mathbb{D}} \to M$ satisfying the conclusions of Lemma 8.1. Let us apply Lemma 8.2 to the curves $\gamma_j(t)$ in Lemma 8.1 to get the diffeomorphisms Φ_t of $\overline{\mathbb{D}}$.

Let us take a factorization (K, g_0) of f_0 consisting of arms (K_j, g_j), $1 \leq j \leq k$, such that $f_0(K_j)$ are lassos α_j so that

$$[f_0] = \prod_{j=1}^{k} [\alpha_j].$$

Let $L_j = \Phi_1(K_j)$. Then the L_j form a star and the mappings $f_1(L_j)$ are also lassos. Moreover, the lassos α_j and β_j are h-homotopic. The needed h-homotopy is achieved by the family $f_t(\Phi_t(K_j))$. Thus $[\alpha_j] = [\beta_j]$. Since

$$[f_1] = \prod_{j=1}^{k} [\beta_j]$$

the theorem is proved. □

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