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EXOTIC ELLIPTIC ALGEBRAS

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Abstract. The 4-dimensional Sklyanin algebras, over $\mathbb{C}$, $A(E,\tau)$, are constructed from an elliptic curve E and a translation automorphism τ of E. The Klein vierergruppe Γ acts as graded algebra automorphisms of $A(E,\tau)$. There is also an action of Γ as automorphisms of the matrix algebra $M_2(\mathbb{C})$ making it isomorphic to the regular representation. The main object of study in this paper is the invariant subalgebra $\widetilde{A} := \big(A(E,\tau)\otimes M_2(\mathbb{C})\big)^{\Gamma}$. Like $A(E,\tau)$, $\widetilde{A}$ is noetherian, generated by 4 degree-one elements modulo six quadratic relations, Koszul, Artin-Schelter regular of global dimension 4, has the same Hilbert series as the polynomial ring on 4 variables, satisfies the χ condition, and so on. These results are special cases of general results proved for a triple (A,T,H) consisting of a Hopf algebra H, an (often graded) H-comodule algebra A, and an H-torsor T. Those general results involve transferring properties between A, $A\otimes T$, and $(A\otimes T)^{\mathrm{coH}}$. We then investigate $\widetilde{A}$ from the point of view of non-commutative projective geometry. We examine its point modules, line modules, and a certain quotient $\widetilde{B} := \widetilde{A}/(\Theta,\Theta')$ where Θ and Θ' are homogeneous central elements of degree two. In doing this we show that $\widetilde{A}$ differs from A in interesting ways. For example, the point modules for A are parametrized by E and 4 more points, whereas $\widetilde{A}$ has exactly 20 point modules. Although $\widetilde{B}$ is not a twisted homogeneous coordinate ring in the sense of Artin and Van den Bergh, a certain quotient of the category of graded $\widetilde{B}$-modules is equivalent to the category of quasi-coherent sheaves on the curve $E/E[2]$ where $E[2]$ is the 2-torsion subgroup. We construct line modules for $\widetilde{A}$ that are parametrized by the disjoint union $(E/\langle\xi_1\rangle)\sqcup(E/\langle\xi_2\rangle)\sqcup(E/\langle\xi_3\rangle)$ of the quotients of E by its three subgroups of order 2.

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1. Introduction

1.1. The 3- and 4-dimensional Sklyanin algebras are among the most interesting algebras that have appeared in non-commutative algebraic geometry. Such an algebra determines and is determined by an elliptic curve, E, a translation automorphism, τ, of E, and an invertible $\mathcal{O}_E$-module $\mathcal{L}$ of degrees 3 and 4, respectively. The representation theory of the Sklyanin algebra $A(E,\tau,\mathcal{L})$ and, what is almost the same thing, the geometric aspects of the non-commutative projective space $\operatorname{Proj}_{nc}\big(A(E,\tau,\mathcal{L})\big)$, is governed by the geometry of E and τ when E is embedded as a cubic or quartic curve in $\mathbb{P}\big(H^0(E,\mathcal{L})^*\big)$. We refer the reader to [1] and [31] for overviews of the 3- and 4-dimensional Sklyanin algebras. The n in "n-dimensional" refers to the Gelfand-Kirillov dimension of $A(E,\tau)$ or its global dimension or the dimension of $A(E,\tau,\mathcal{L})_1$, which is equal to $H^0(E,\mathcal{L})$.

Odesskii and Feigin have defined generalizations of the 4-dimensional Sklyanin algebras in [24], [25], and [11]. Their algebras depend on a pair (E,τ) and a line bundle $\mathcal{L}$, of degree ≥ 4, that is used to construct $A(E,\tau,\mathcal{L})$. In particular, when $\deg(\mathcal{L})=n^2$, $n\geq 2$, Odesskii and Feigin construct an algebra that they denote by $Q_{n^2}(E,\tau)$.

Following an idea of Odesskii in [23], described in §1.5 below, we construct for every such pair (E,τ) and integer $n\geq 2$ a connected graded algebra $\widetilde{Q}=\widetilde{Q}_{n^2}(E,\tau)$ by a kind of Galois descent procedure applied to $Q_{n^2}(E,\tau)$. We show that the algebras obtained in this manner inherit many of the good properties enjoyed by $Q_{n^2}(E,\tau)$. For example, they are Artin-Schelter regular.

1.2. This paper examines the case $n=2$ and shows that the algebras $\widetilde{Q}$ exhibit a range of novel features. They are still governed very strongly by the geometry of E and τ. For this reason we call them "elliptic algebras", the name Odesskii and Feigin adopted for their algebras, and we append the adjective "exotic" to indicate that they are somewhat novel when compared to the familiar 4-dimensional Sklyanin algebras and other 4-dimensional Artin-Schelter regular algebras.

1.3. The procedure we use to construct the algebras $\widetilde{Q}$ is quite general. Let H be a finite-dimensional Hopf algebra over a field k and let A be an H-comodule algebra. One might also require A to be a graded algebra and that every homogeneous component be a subcomodule. Let T be an H-torsor (see §3.1) and define the algebra $A' := A\otimes T$. If A is graded one places T in degree zero to make A' a graded algebra. Let $\widetilde{A}$ denote the subalgebra of A' consisting of the H-coinvariant elements. In §§3 and 4 we show how various properties pass back and forth between A, A', and $\widetilde{A}$. For example, we consider the noetherian property, that of being finite as a module over its center, and numerous homological properties that play an important role in non-commutative algebraic geometry. When H is commutative, which is the case in the definition of $\widetilde{Q}$, A' is an H-comodule algebra.

In §4 we assume that $\dim_k(H)<\infty$, and (usually) A is a connected graded H-comodule algebra. We show A is Koszul (m-Koszul) if and only if $\widetilde{A}$ is. We show A is Artin-Schelter regular of dimension d if and only if $\widetilde{A}$ is. We show $\widetilde{A}$ satisfies the χ condition, introduced in [6], if A does.

1.4. The construction $A\rightsquigarrow\widetilde{A}$, and our results about properties shared by A and $\widetilde{A}$, should be useful in other situations. It would be sensible to examine the effect of this construction on 2- and 3-dimensional Artin-Schelter regular algebras now that J.J. Zhang and his co-authors have determined (many/all?) the finite-dimensional Hopf algebras for which such algebras can be comodule algebras. Even the case when A is a polynomial ring, or an enveloping algebra, deserves investigation.

1.5. Let $Q = A(E, \tau, \mathcal{L})$ be a 4-dimensional Sklyanin algebra. It was shown in [34] that $\Gamma = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ acts as graded k-algebra automorphisms of Q when $k = \mathbb{C}$. The action there is induced by the translation action of the 2-torsion subgroup, $E[2]$, on E. Here, working over an arbitrary algebraically closed field k of characteristic $\neq 2$, we define an action of Γ as graded k-algebra automorphisms of Q and show that this "corresponds" to the translation action of $E[2]$ on E.

In the language of §1.3, we take H to be the Hopf algebra of k-valued functions on Γ and T to be $M_2(k)$, the ring of 2×2 matrices, with an appropriate H-comodule algebra structure. We then have $\widetilde{Q} = (Q \otimes T)^{\mathrm{coH}} = (Q \otimes T)^{\Gamma}$. The results in §§3 and 4 show that $\widetilde{Q}$ has "all" the good properties Q has. It is a noetherian domain, has global dimension 4, has the same Hilbert series as the polynomial ring on 4 indeterminates, is Artin-Schelter regular, satisfies the χ condition, etc.

1.6. Among the most important results about Sklyanin algebras are classifications of their point and line modules; by definition these are the graded cyclic modules with Hilbert series $(1-t)^{-1}$ and respectively $(1-t)^{-2}$.

The point modules of a 3-dimensional Sklyanin algebra are naturally parametrized by E or, more informatively, by a natural copy of E embedded as a smooth cubic curve in $\mathbb{P}^2 = \mathbb{P}(Q_1^*)$. The point modules for a 4-dimensional Sklyanin are parametrized by a natural copy of E as a smooth quartic curve in $\mathbb{P}^3 = \mathbb{P}(Q_1^*)$ and 4 additional points, the extra points being the vertices of the 4 singular quadrics that contain the copy of E. The line modules are, in both cases, parametrized by the secant lines to E, the lines in $\mathbb{P}(Q_1^*)$ that meet E with multiplicity ≥ 2.

The results for $\widetilde{Q}$ are very different. For example, $\widetilde{Q}$ has only 20 point modules. In a note circulated in 1988 [43], Van den Bergh showed that a generic 4-dimensional AS-regular algebra (with some other properties) has exactly 20 point modules. Since then, there have been a number of examples showing that particular algebras, rather than the ephemeral "generic algebras", have exactly 20 point modules. We believe that ours are the first such examples that turn up "in vivo", so to speak.

1.7. Van den Bergh and Tate [41] showed that the Odesskii-Feigin algebras Q_{n^2} are noetherian, Koszul, Artin-Schelter regular algebras of dimension n^2 with Hilbert series $(1-t)^{-n^2}$. It follows from the relations for Q_{n^2} that $\Gamma = (\mathbb{Z}/n) \times (\mathbb{Z}/n)$, realized as the n-torsion subgroup $E[n] \subset E$, acts as graded algebra automorphisms of Q_{n^2}. It is an easy matter to see that the ring of $n \times n$ matrices $M_n(\mathbb{C})$ is an H-torsor where H is the Hopf algebra of k-valued functions on Γ. In §5 we show that for all $n \geq 2$, $\widetilde{Q_{n^2}} = \big(Q_{n^2} \otimes M_n(k)\big)^{\Gamma}$ has "the same" properties as Q_{n^2}.

1.8. In §6 we begin a detailed examination of the algebra $\widetilde{Q}$ in §1.5. We give explicit generators and relations for $\widetilde{Q}$. It has 4 generators and 6 quadratic relations (Proposition 6.1). Since $\Gamma = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ acts on Q_1 it acts as automorphisms of $\mathbb{P}(Q_1)^* = \mathbb{P}^3$. This $\mathbb{P}^3$ contains a natural copy of E embedded as a quartic curve, and Γ restricts to an action as automorphisms of E.

In §7 we show that this action is the same as the translation action of the 2-torsion subgroup $E[2]$. Each $\gamma \in \Gamma$ acts as an auto-equivalence $M \rightsquigarrow \gamma^* M$ of the graded-module category $\mathsf{Gr}(Q)$. Because Γ acts as $E[2]$ does, if M_p, $p \in E$, is the point module corresponding to $p \in E$, then $\gamma^* M_p \cong M_{p+\omega}$ for a suitable $\omega \in E[2]$. There is a similar result for line modules: $\gamma^* M_{p,q} \cong M_{p+\omega,q+\omega}$.

1.9. By [33], there is a regular sequence in Q consisting of two homogeneous central elements of degree 2, Ω and Ω' say, such that $Q/(\Omega, \Omega')$ is a twisted homogeneous coordinate ring, $B(E, \tau, \mathcal{L})$, in the sense of Artin and Van den Bergh [5]. The main result in [5] tells us that the quotient category $\mathsf{QGr}(B(E, \tau, \mathcal{L}))$ is equivalent to $\mathsf{Qcoh}(E)$, the category of quasi-coherent sheaves on E.

The algebra $\widetilde{Q}$ also has a regular sequence consisting of two homogeneous central elements of degree 2, Θ and Θ' say. Although $\widetilde{B} := \widetilde{Q}/(\Theta, \Theta')$ is not a twisted homogeneous coordinate ring, Theorem 8.1 proves that $\mathsf{QGr}(\widetilde{B})$ is equivalent to $\mathsf{Qcoh}(E/E[2])$.[1] Nevertheless, $\widetilde{B}$ has no point modules. The points on $E/E[2]$ correspond to fat point modules of multiplicity 2 over $\widetilde{B}$. Another new feature is that $\widetilde{B}$ is not a domain although B is. Still, $\widetilde{B}$ is a prime ring.

1.10. In §9 we prove that $\widetilde{Q}$ has exactly 20 point modules. These modules correspond to 20 points in $\mathbb{P}^3 = \mathbb{P}(\widetilde{Q}_1^*)$ that we determine explicitly. The "meaning" of these 20 points eludes us. Let $\mathfrak{P}$ denote that set of 20 points. The degree shift functor $M \rightsquigarrow M(1)$ induces a permutation $\theta : \mathfrak{P} \to \mathfrak{P}$ of order 2. Shelton and Vancliff [29] have shown that the data $(\mathfrak{P}, \theta)$ determines $\widetilde{Q}$ in the sense that the subspace $R \subseteq Q_1 \otimes Q_1$ of bihomogeneous forms vanishing on the graph of θ has the property that $\widetilde{Q}$ is isomorphic to $T(Q_1)/(R)$, the tensor algebra on Q_1 modulo the ideal generated by R.

In §11, we exhibit three families of line modules for $\widetilde{Q}$ parametrized by $\big(E/\langle\xi\rangle\big) \sqcup \big(E/\langle\xi'\rangle\big) \sqcup \big(E/\langle\xi''\rangle\big)$ where $\{\xi, \xi', \xi''\}$ is the set of 2-torsion points on E. These are *not* all the line modules for $\widetilde{Q}$.

1.11. In §§8 and 11 we examine Γ-equivariant objects in $\mathsf{Gr}(Q)$ and other categories of interest. So as not to interrupt the flow of the paper we collect some basic facts about group actions on categories and equivariant objects in an appendix. The material there is known in one form or another in various degrees of generality, but we have not found a suitable reference. The reader might find the appendix useful in filling in the details of some of the proofs in §10.

1.12. In late January 2015, after proving most of the results in this paper, we found an announcement on the web of a seminar talk by Andrew Davies at the University of Manchester in January 2014 that appeared to contain some of the results we prove here. On 1/20/2015, we found a copy of his Ph.D. thesis ([8–10]), which has substantial overlap with this paper. Davies also proves several things we don't. For example, he describes $\widetilde{B}$ (when τ has infinite order) in the manner of Artin and Stafford [2]. Nevertheless, most of what we do is more general, and most of our arguments differ from his. For example, when we deal with the 4-dimensional Sklyanin algebras we make no assumption on the order of τ, we do not restrict our base field to the complex numbers, and we describe some of the line modules for $\widetilde{Q}$. Also, the results in §§3 and 4 for arbitrary H and T are proved by Davies only in the case H is the ring of k-valued functions on a finite abelian group.

[1] Although $E/E[2]$ is isomorphic to E, it is "better" to think of $\mathsf{QGr}(\widetilde{B})$ as equivalent to $\mathsf{Qcoh}(E/E[2])$.

2. Preliminaries

In §§2 to 4, we work over an arbitrary field k. Once we begin discussing the 4-dimensional Sklyanin algebras, k will be an algebraically closed field of characteristic $\neq 2$.

2.1. We will use what is now standard terminology and notation for graded rings and non-commutative projective algebraic geometry. There are several sources for unexplained terminology: the Artin-Tate-Van den Bergh papers ([3], [4]), which started the subject of non-commutative projective algebraic geometry; Stafford and Van den Bergh's survey [37]; papers by Stafford and Smith [33] and Levasseur and Smith [17] on 4-dimensional Sklyanin algebras; the survey [31] on 4-dimensional Sklyanin algebras; Artin and Van den Bergh's paper on twisted homogeneous coordinate rings [5]; and Artin and Zhang's on non-commutative projective schemes [6].

Suppose A is an $\mathbb{N}$-graded k-algebra such that $\dim_k(A_i) < \infty$ for all i. The category of $\mathbb{Z}$-graded left A-modules with degree-preserving A-module homomorphisms is denoted by $\mathsf{Gr}(A)$. The full subcategory of $\mathsf{Gr}(A)$ consisting of modules that are the sum of their finite-dimensional submodules is denoted by $\mathsf{Fdim}(A)$. This is a Serre subcategory, so we can form the quotient category

$$\mathsf{QGr}(A) \;:=\; \frac{\mathsf{Gr}(A)}{\mathsf{Fdim}(A)}.$$

In fact, $\mathsf{Fdim}(A)$ is a localizing subcategory, so the quotient functor $\pi^* : \mathsf{Gr}(A) \to \mathsf{QGr}(A)$ has a right adjoint π_*. The functor π^* is exact. By definition, $\mathsf{QGr}(A)$ has the same objects as $\mathsf{Gr}(A)$. Since $\pi^*\pi_*$ is isomorphic to the identity functor we may view objects in $\mathsf{QGr}(A)$ as objects in $\mathsf{Gr}(A)$.

2.2. We write Vect for the category of vector spaces over k.

2.3. Throughout this paper, H is a Hopf algebra over k with bijective antipode. We write ${}^H\mathcal{M}$ for the category of left H-comodules and $\mathcal{M}^H$ for the category of right H-comodules. Furthermore A denotes a right H-comodule-algebra, i.e., an algebra object in $\mathcal{M}^H$.

Let Υ be an abelian group. We call A an Υ-graded H-comodule algebra or an Υ-graded algebra in $\mathcal{M}^H$ if it is an H-comodule algebra such that each homogeneous component, A_i, is an H-subcomodule. For example, if V is a right H-comodule and $R \subseteq V \otimes V$ is an H-subcomodule, then the tensor algebra, TV, and its quotient $TV/(R)$ are $\mathbb{Z}$-graded algebras in $\mathcal{M}^H$.

We write $\mathsf{Mod}(R)$ for the category of left modules over a ring R. We write ${}_A\mathcal{M}^H$ for the category of A-modules internal to the category of H-comodules, i.e., vector spaces V equipped with an A-module structure and an H-comodule structure such that $A \otimes V \to V$ is an H-comodule map. If A is an Υ-graded algebra in $\mathcal{M}^H$ we write ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$ for the category of Υ-graded A-modules internal to $\mathcal{M}^H$; i.e., each homogeneous component M_i is an H-comodule. Similar conventions apply to right A-modules, with the algebra subscripts appearing on the right in that case.

3. Torsors, twisting, and descent

In this section we prove some general results on the inheritance of various properties for certain rings of (co)invariants, relating various good properties of A to those of the algebra $\widetilde{A}$ defined in (3-5) below. In §§3.1-3.3, the only assumption on H is that it is a Hopf algebra with bijective antipode. In §3.4 we add the hypothesis that H is commutative.

3.1. **Torsors.** A left H-torsor (or just torsor for short) is a left H-comodule-algebra T such that

(1) $T \cong H$ in ${}^H\mathcal{M}$,
(2) the ring of coinvariants, ${}^{\mathrm{coH}}T$, is k, and
(3) the linear map

$$T \otimes T \xrightarrow{\rho \otimes \mathrm{id}} H \otimes T \otimes T \xrightarrow{\mathrm{id} \otimes m} H \otimes T \tag{3-1}$$

is bijective where $\rho : T \to H \otimes T$ is the comodule structure and $m : T \otimes T \to T$ is multiplication.

Throughout §3, T denotes a left H-torsor.

3.1.1. A comodule algebra for which the composition in (3-1) is an isomorphism is sometimes called a left H-Galois object (see e.g. [7, Defn. 1.1]); [7] and the references therein are good sources for background on torsors. Left H-torsors classify exact monoidal functors $\mathcal{M}^H \to \mathrm{V{\scriptsize ECT}}$, the functor corresponding to T being

$$M \;\mapsto\; M \square_H T \;:=\; \{x \in M \otimes T \mid (\rho_M \otimes \mathrm{id})(x) = (\mathrm{id} \otimes \rho)(x)\}, \tag{3-2}$$

where $\rho_M : M \to M \otimes H$ and $\rho : T \to H \otimes T$ are the comodule structure maps. The vector space $M \square_H T$ is called the *cotensor product* of M and T.

3.1.2. *Left versus right comodules.* Since the antipode, $s : H \to H$, is an algebra anti-isomorphism, the categories ${}^H\mathcal{M}$ and $\mathcal{M}^H$ are equivalent: if $\rho : X \to H \otimes X$ is a left H-comodule, then X becomes a right H-comodule with respect to the structure map

$$X \xrightarrow{\rho} H \otimes X \xrightarrow{s \otimes \mathrm{id}} H \otimes X \xrightarrow{\tau} X \otimes H \tag{3-3}$$

where the right-most map is $\tau(h \otimes x) = x \otimes h$. Explicitly, if $x \mapsto x_{-1} \otimes x_0$ is the left-comodule structure, then the right comodule structure is $x \mapsto x_0 \otimes s(x_{-1})$.

3.1.3. *Left versus right comodule algebras.* The operation (3-3) does *not* turn a left H-comodule algebra into a right H-comodule algebra. However, if X is a left H-comodule algebra and X^{op} denotes X with the opposite multiplication, then X^{op} becomes a right H-comodule algebra with respect to the structure map (3-3). To see this, first denote the composition in (3-3) by ρ° and, when $x \in X$, write x° for x viewed as an element in X^{op}. Thus, if $x, y \in X$, then $x^\circ y^\circ = (yx)^\circ$. Therefore if $x, y \in X$ and $\rho(x) = x_{-1} \otimes x_0$, then $\rho^\circ(x^\circ) = x_0^\circ \otimes s(x_{-1})$, so

$$\begin{aligned} \rho^\circ(x^\circ y^\circ) = \rho^\circ\big((yx)^\circ\big) &= \tau(s \otimes \mathrm{id})\rho(yx) = \tau(s \otimes \mathrm{id})(y_{-1}x_{-1} \otimes y_0 x_0) \\ &= y_0 x_0 \otimes s(x_{-1})s(y_{-1}), \end{aligned}$$

which is equal to $\big(x_0^\circ \otimes s(x_{-1})\big)\big(y_0^\circ \otimes s(y_{-1})\big) = \rho^\circ(x^\circ)\rho^\circ(y^\circ)$.

Since T is a left H-torsor, T^{op} with the structure map $\rho^\circ : T^{\mathrm{op}} \to T^{\mathrm{op}} \otimes H$ is a right H-torsor.

3.1.4. *The monoidal functor* $\widetilde{\bullet} : M \mapsto \widetilde{M}$. By [42, Lem. 1.4], the functor $M \mapsto M\square_H T$ in §3.1.1 is a monoidal functor. We denote it by $\widetilde{\bullet} : M \mapsto \widetilde{M}$. By the next result it is naturally equivalent to $M \mapsto (M \otimes T)^{\mathrm{coH}}$.

In the expression $M\square_H T$ we treat T as a left H-comodule. In the expression $(M \otimes T)^{\mathrm{coH}}$ we treat T as a right H-comodule using the new structure map in (3-3). The algebra structure on T in not used in constructing either $M\square_H T$ or $(M \otimes T)^{\mathrm{coH}}$.

Proposition 3.1. *Let* $M\square_H T = (M \otimes T)^{\mathrm{coH}}$.

Proof. We will abuse notation by using simple tensors $m \otimes t$ instead of arbitrary elements of $M \otimes T$.

Let $m \otimes t \in M\square_H T$. Then $m_0 \otimes m_1 \otimes t = m \otimes t_{-1} \otimes t_0$.

Since H is a Hopf algebra there is an internal tensor product on $\mathcal{M}^H$. It is "dual" to the internal tensor product on the category of right H^*-modules. If N and N' are right H^*-modules the latter is defined to be the left-most vertical arrow in the following commutative diagram:

$$\begin{array}{ccc} N \otimes N' \otimes H^* & \longrightarrow & N \otimes N' \otimes H^* \otimes H^* \\ \downarrow & & \downarrow \\ N \otimes N' & \longleftarrow & N \otimes H^* \otimes N' \otimes H^* \end{array}$$

The right H-comodule structure on $M \otimes T$ is defined to be the left-most vertical arrow in the following commutative diagram:

$$\begin{array}{ccc} M \otimes T \otimes H & \longleftarrow & M \otimes T \otimes H \otimes H \\ \uparrow & & \uparrow \\ M \otimes T & \longrightarrow & M \otimes H \otimes T \otimes H \end{array}$$

The right comodule structure on $M \otimes T$ is therefore

$$\begin{aligned} m \otimes t \mapsto (m \otimes t)_0 \otimes (m \otimes t)_1 &= (m_0 \otimes t_0) \otimes m_1 s(t_{-1}) \\ &= m \otimes t_0 \otimes t_{-2} s(t_{-1}) \\ &= m \otimes t_0 \otimes \varepsilon(t_{-1}) 1 \\ &= m \otimes t \otimes 1. \end{aligned}$$

This shows that $m \otimes t \in (M \otimes T)^{\mathrm{coH}}$.

Conversely, suppose $m \otimes t \in (M \otimes T)^{\mathrm{coH}}$. Thus

$$m_0 \otimes t_0 \otimes m_1 s(t_{-1}) = m \otimes t \otimes 1. \tag{3-4}$$

Now apply to both sides of this equation the endomorphism of $M \otimes T \otimes H$ defined by

$$m \otimes t \otimes h \mapsto m \otimes t_0 \otimes h t_{-1}.$$

The left-hand side of (3-4) is sent to

$$m_0 \otimes t_0 \otimes m_1 s(t_{-2}) t_{-1} = m_0 \otimes t_0 \otimes m_1 \varepsilon(t_{-1}) = m_0 \otimes t \otimes m_1,$$

and the right-hand side is sent to $m \otimes t_0 \otimes t_{-1}$. Since the two must be equal, $m \otimes t \in M\square_H T$. □

3.1.5. Since $\widetilde{\bullet}$ is a monoidal functor, it sends algebras in $\mathcal{M}^H$ to algebra objects in Vect, and hence for $A \in \mathcal{M}^H$ as in §2.3, the H-coinvariant subspace

$$\widetilde{A} \;:=\; (A \otimes T)^{\mathrm{coH}} \tag{3-5}$$

has a natural algebra structure. We treat T as a right H-comodule in the expression $(A \otimes T)^{\mathrm{coH}}$.

Although T has two algebra structures, its original one and the opposite one, neither makes $A \otimes T$ into an H-comodule algebra unless additional hypotheses are made (see §3.4). Nevertheless, $\widetilde{A}$ is a subalgebra of $A \otimes T$ (T having its initial algebra structure, not the opposite one). In §3.4 below we specialize to commutative H, in which case $A \otimes T$ is a comodule algebra.

$\widetilde{\bullet}$ lifts to a functor ${}_A\mathcal{M}^H \to \mathsf{Mod}(\widetilde{A})$, and similarly when everything in sight is Υ-graded for some abelian group Υ. We denote all of these functors by the same symbol, relying on context to differentiate between them.

3.1.6. In the definition of a torsor, the condition that $T \cong H$ in ${}^H\mathcal{M}$ makes the Galois object T *cleft*; this condition follows automatically from (3-1) when H is finite dimensional, which is the case we are really interested in here. This is (part of) [7, Thm. 1.9], which cites [16] for a proof.

Cleft objects have an alternative characterization by means of Hopf *cocycles*. Recall (e.g. [7, Ex. 1.3]) that the latter are linear maps $\sigma : H \otimes H \to k$ satisfying certain conditions which we will not spell out here and which are reminiscent of those from group cohomology.

By [7, Thm. 1.8], every left torsor in the sense of §3.1 can be obtained from such a gadget σ by twisting H: T can be identified with H as a vector space but has a new multiplication defined by

$$g \circ h = g_1 h_1 \sigma(g_2 \otimes h_2) \quad \text{for all } \ g, h \in H.$$

Here, $g \mapsto g_1 \otimes g_2$ is the comultiplication in H and juxtaposition on the right-hand side means multiplication in H. Similarly, the algebra $\widetilde{A}$ can be identified with the vector space A endowed with the modified multiplication

$$a \circ b = a_0 b_0 \sigma(a_1 \otimes b_1) \quad \text{for all } \ a, b \in A,$$

where $a \mapsto a_0 \otimes a_1$ is the H-comodule structure ($\widetilde{A}$ is then a *cocycle twist* of A).

We note in passing that in this case the new algebra $\widetilde{A}$ has a right comodule algebra structure with respect to a twisted version H^σ of H, consisting of the same coalgebra underlying H, with the modified multiplication

$$h \circ k = \sigma^{-1}(h_1 \otimes k_1) h_2 k_2 \sigma(h_3 \otimes k_3),$$

where $\sigma^{-1} : H \otimes H \to k$ is the convolution-inverse of σ. There is then a twisting functor implementing an equivalence between H and H^σ-comodules (see e.g. [7, Defn. 3.14] and surrounding discussion), sending A to $\widetilde{A}$ and implementing an equivalence between $\mathcal{M}_A^H$ and $\mathcal{M}_{\widetilde{A}}^{H^\sigma}$. Moreover, H and H^σ play symmetric roles, with A obtainable as a cocycle twist of the H^σ-comodule algebra $\widetilde{A}$.

When H is the function algebra of an abelian group Γ whose order is not divisible by the characteristic of k this construction specializes in the following way.

H can be identified with the group algebra $k\widehat{\Gamma}$ of the character group of Γ; i.e., A is $\widehat{\Gamma}$-graded. Every Hopf cocycle $H \otimes H \to k$ is the linear extension of a normalized group 2-cocycle $\mu : \widehat{\Gamma} \times \widehat{\Gamma} \to k^\times$ in the usual sense. Now, denoting by A_α the

α-homogeneous component of A with respect to the $\widehat{\Gamma}$-grading, the twisted algebra $\widetilde{A}$ can be identified with the vector space A together with the new multiplication

$$a \circ b = \mu(\alpha, \beta)ab \qquad \text{for all } \alpha, \beta \in \widehat{\Gamma},\ a \in A_\alpha,\ b \in A_\beta.$$

3.2. **Generalities.** We prove some auxiliary general results of use below.

Lemma 3.2. *The categories $\mathcal{M}^H_{T^{\mathrm{op}}}$ and* VECT *are equivalent via the mutually quasi-inverse functors*

$$\text{(3-6)} \qquad \mathrm{V{\small ECT}} \underset{\bullet^{\mathrm{coH}}}{\overset{\bullet \otimes T^{\mathrm{op}}}{\rightleftarrows}} \mathcal{M}^H_{T^{\mathrm{op}}}$$

Proof. By [27, Thm. I] applied to the comodule algebra $T^{\mathrm{op}} \in \mathcal{M}^H$ the assertion follows from the torsor condition (3-1) if T^{op} is injective as an H-comodule. It is because $T \cong H$ as a left comodule and every coalgebra is self-injective in the same way that every algebra is self-projective. □

Throughout the rest of this subsection, unadorned Hom denotes hom spaces over the ground field k.

In the following result, T^{op} is regarded as a right comodule-algebra by twisting its original left comodule algebra structure using the antipode, as explained in §3.1.3. When the algebra structure of T^{op} is not relevant, we will on occasion drop the op superscript in order to streamline our notation.

Proposition 3.3. *There is an isomorphism*

$$\text{(3-7)} \qquad \mathrm{Hom}^H(M, N \otimes T) \cong \mathrm{Hom}(\widetilde{M}, \widetilde{N}),$$

functorial in $M, N \in \mathcal{M}^H$. Moreover, it restricts to a functorial isomorphism

$$\text{(3-8)} \qquad \mathrm{Hom}^H_A(M, N \otimes T) \cong \mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$$

for $M, N \in {}_A\mathcal{M}^H$.

Proof. By the adjunction between scalar extension $\bullet \otimes T^{\mathrm{op}} : \mathcal{M}^H \to \mathcal{M}^H_{T^{\mathrm{op}}}$ and scalar restriction (i.e., simply forgetting the T^{op}-action) the left-hand side of (3-7) is naturally isomorphic to the space $\mathrm{Hom}^H_{T^{\mathrm{op}}}(M \otimes T^{\mathrm{op}}, N \otimes T^{\mathrm{op}})$, where T^{op} acts on just the T^{op} tensorands. In turn, this is naturally isomorphic to the right-hand side of (3-7) by Lemma 3.2.

To verify the second assertion note that the left-hand side of (3-8) can be realized as an equalizer,

(3-9)

$$\mathrm{Hom}^H_A(M, N \otimes T) \longrightarrow \mathrm{Hom}^H(M, N \otimes T) \underset{f \mapsto \triangleright \circ (\mathrm{id}_A \otimes f)}{\overset{f \mapsto f \circ \triangleright}{\rightrightarrows}} \mathrm{Hom}^H(A \otimes M, N \otimes T)$$

where the upper and lower $\triangleright$ symbols denote the action $A \otimes M \to M$ and $A \otimes N \to N$ respectively.

Applying the natural isomorphism from the first part of the proposition to the two parallel arrows in (3-9) and keeping in mind the fact that $\widetilde{\bullet}$ is a monoidal functor, we get the arrows

$$\operatorname{Hom}(\widetilde{M}, \widetilde{N}) \underset{f \mapsto \triangleright \circ (\mathrm{id}_A \otimes f)}{\overset{f \mapsto f \circ \triangleright}{\rightrightarrows}} \operatorname{Hom}(\widetilde{A} \otimes \widetilde{M}, \widetilde{N}).$$

Their equalizer is precisely the right hand side of (3-8). □

3.2.1. There is a graded version of Proposition 3.3 with virtually the same proof (M and N are graded comodules, etc.).

The following simple observation turns out to be rather important.

Lemma 3.4. *Suppose H is finite-dimensional. The functors $\widetilde{\bullet} : {}_A\mathcal{M}^H \to \mathsf{Mod}(\widetilde{A})$ and $\widetilde{\bullet} : {}_{\mathsf{Gr}(A)}\mathcal{M}^H \to \mathsf{Gr}(\widetilde{A})$ send projective objects to projective objects.*

Proof. Let $A\sharp H^*$ denote the smash product. The category ${}_A\mathcal{M}^H$ can be identified with $\mathsf{Mod}(A\sharp H^*)$. Under this identification, projectives are direct summands of direct sums of copies of $A\sharp H^*$. It therefore suffices to show that the image of $A\sharp H^*$ under $\widetilde{\bullet}$ is projective over $\widetilde{A}$.

As an A-module $A\sharp H^*$ is simply $A \otimes H^*$ with the A-action on the left tensorand. As an H-comodule $A\sharp H^*$ is the tensor product $A \otimes H^*$, with H coacting on H^* regularly. Since $\widetilde{\bullet}$ is a monoidal functor, it sends $A\sharp H^* \in {}_A\mathcal{M}^H$ to $\widetilde{A} \otimes \widetilde{H^*}$ with the obvious action of $\widetilde{A}$. This is a direct sum of copies of $\widetilde{A}$ in $\mathsf{Mod}(\widetilde{A})$ and hence projective. □

3.3. **The noetherian property and GK-dimension.**

Proposition 3.5. *Let Υ be an abelian group and let A be an Υ-graded H-comodule algebra. Then $\dim_k(A_i) = \dim_k(\widetilde{A}_i)$ for all $i \in \Upsilon$.*

Proof. We are assuming $T \cong H$ in $\mathcal{M}^H$ so $W \otimes T \cong W \otimes H$ in $\mathcal{M}^H$ for all $W \in \mathcal{M}^H$. As in the proof of Proposition 3.10, the map $W \otimes H \to W \otimes H$, $w \otimes h \mapsto w_0 \otimes w_1 h$, is an isomorphism from $W \otimes H$ with the diagonal H-coaction to $W \otimes H$ with the regular H-coaction on the right-hand tensorand. As a consequence, there is a vector space isomorphism $W \cong (W \otimes T)^{\mathrm{coH}}$. Now apply this fact with W equal to each homogeneous component of A. □

Below, we will be referring to the *Gelfand-Kirillov dimension* of a module M over a k-algebra A, denoted (when the module structure is implicit) by $\operatorname{GKdim}(M)$. It is defined as

$$\sup_{V, M_0} \limsup_{n \to \infty} \log_n(\dim V^n M_0),$$

where V and M_0 range over the finite-dimensional subspaces of A and respectively M (see e.g. [35] and the references cited therein, or §4 of the original source for the concept, [12]).

Lemma 3.6 ([15, Lem. 6.1])**.** *Let A be an $\mathbb{N}$-graded k-algebra such that $\dim_k(A_i) < \infty$ for all i, and let M be a finitely generated graded A-module. Then*

$$\operatorname{GKdim}(M) = 1 + \limsup \log_n(\dim_k(M_n)).$$

Proposition 3.7. *If A is a $\mathbb{Z}$-graded comodule algebra such that $\dim_k(A_i) < \infty$ for all i, then A and $\widetilde{A}$ have the same Gelfand-Kirillov dimension.*

Lemma 3.8. *The functor* FORGET : ${}_{\mathsf{Gr}(A)}\mathcal{M}^H \to \mathsf{Gr}(A)$ *preserves projectivity, as does the analogous functor for ungraded modules.*

Proof. This follows from the fact that FORGET is left adjoint to an exact functor, namely $\bullet \otimes H : \mathsf{Gr}(A) \to {}_{\mathsf{Gr}(A)}\mathcal{M}^H$. The same proof works in the ungraded case. □

Proposition 3.9. *Suppose H is finite-dimensional. If A is left or right noetherian, then so is $\widetilde{A}$.*

Proof. Suppose A is left noetherian. (The right noetherian case has a similar proof using the right-handed version of Proposition 3.3.)

Let S be an arbitrary set. The goal is to show that for any $\widetilde{A}$-module map $f : \widetilde{A}^{\oplus S} \to \widetilde{A}$ the images of the restrictions $f_{S'} : \widetilde{A}^{\oplus S'} \to \widetilde{A}$ stabilize as $S' \subseteq S$ ranges over ever larger finite subsets.

By Proposition 3.3, f can be identified with some A-module H-comodule map $\varphi : A^{\oplus S} \to A \otimes T$. By naturality, this identification is compatible with taking restrictions $\varphi_{S'}$ to $A^{\oplus S'}$ for finite subsets $S' \subseteq S$ (in the sense that $f_{S'}$ gets identified with $\varphi_{S'}$).

From the proof of Proposition 3.3 we see that the image of $f_{S'}$ consists of the H-coinvariants of the T^{op}-submodule of $A \otimes T$ generated by the image of $\varphi_{S'}$. Hence, it suffices to show that the images of $\varphi_{S'}$ stabilize as S' increases. This, however, is a consequence of the noetherianness of A and the fact that T is finite-dimensional (so that the A-module $A \otimes T$ is finitely generated). □

3.4. **The case when H is commutative, and the algebra A'.** In this section we assume that H is commutative, i.e., the ring of regular functions on an affine group scheme (not necessarily reductive or reduced).

Because H is commutative, if V and W are right H-comodules, the map $V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$, is an isomorphism of right H-comodules. It follows from this that if T is made into a right H-comodule via the procedure in §3.1.2, then

$$A' := A \otimes T \tag{3-10}$$

becomes a right H-comodule algebra with its usual tensor product algebra structure. We emphasize that the T factor in $A \otimes T$ has its original multiplication and is made into a right H-comodule algebra by the procedure in §3.1.2 and *not* by giving T the opposite multiplication.

As mentioned in §3.1.5, $\widetilde{A}$ is a subalgebra of A'. The following result therefore makes sense.

Proposition 3.10. *Suppose H is commutative. Then, the categories ${}_{A'}\mathcal{M}^H$ and $\mathsf{Mod}(\widetilde{A})$ are equivalent via the mutually quasi-inverse functors*

$$\mathsf{Mod}(\widetilde{A}) \underset{\bullet^{\mathrm{coH}}}{\overset{A' \otimes_{\widetilde{A}} \bullet}{\rightleftarrows}} {}_{A'}\mathcal{M}^H \tag{3-11}$$

Furthermore, the extension $\widetilde{A} \to A'$ is faithfully flat on the right and on the left.

Proof. It will be convenient to phrase the proof in terms of left comodules. Note that since H is commutative its antipode is an automorphism, and therefore the equivalence between $\mathcal{M}^H$ and ${}^H\mathcal{M}$ described in §3.1.2 is a *monoidal* equivalence. In this manner, we think of A and A' as left comodule algebras for the duration of the proof and show that the two functors above implement an equivalence between $\mathsf{Mod}(\widetilde{A})$ and ${}^H_{A'}\mathcal{M}$. We will also freely interchange the order of tensorands, as permitted by the commutativity of H.

By [27, Thm. I], both assertions follow if A' is injective as an H-comodule and the map

$$A' \otimes A' \xrightarrow{\ \rho\otimes \mathrm{id}\ } H \otimes A' \otimes A' \xrightarrow{\ \mathrm{id}\otimes m\ } H \otimes A' \tag{3-12}$$

analogous to (3-1) is onto, where $\rho : A' \to H \otimes A'$ is the left comodule structure mentioned at the beginning of the proof and m is multiplication.

The H-comodule $T \cong H$ is injective in ${}^H\mathcal{M}$ (every coalgebra is self-injective in the same way that every algebra is self-projective). Now, for any left H-comodule M, the map

$$H \otimes M \to H \otimes M, \qquad h \otimes m \mapsto hm_{-1} \otimes m_0$$

is an isomorphism from $M \otimes H \cong H \otimes M$ with the tensor product comodule structure to $M \otimes H$ with the comodule structure coming from the right-hand tensorand alone. In other words $M \otimes H$ is isomorphic in ${}^H\mathcal{M}$ to a direct sum of $\dim_k(M)$ copies of H and in particular is injective. Applying this to $M = A$, it follows that $A' = A \otimes T \cong A \otimes H$ is injective in ${}^H\mathcal{M}$.

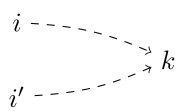
To check the surjectivity of (3-12) note that since (3-1) is an isomorphism so is the composition

$$T \otimes A' = T \otimes T \otimes A \to H \otimes T \otimes A' = H \otimes T \otimes T \otimes A \to H \otimes T \otimes A = H \otimes A';$$

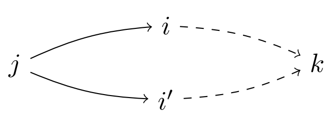
i.e., the restriction of (3-12) to $T \otimes A' \subseteq A' \otimes A'$ already surjects onto $H \otimes A'$. □

Lemma 3.11. *Keeping the notation and conventions of Proposition* 3.10*, if* $N \in {}_{A'}\mathcal{M}^H$ *is finitely generated over* A'*, then* N^{coH} *is finitely generated over* $\widetilde{A}$*.*

Proof. Finite generation can be characterized in category-theoretic terms as follows. Let I be a *filtered* small category in the sense of [18, Sec. IX.1]: Every two objects i, i' fit inside a diagram



and every solid left-hand wedge as in the picture below can be completed to a commutative diagram by a dotted right-hand wedge



For any functor $F : I \to \mathsf{Mod}(A')$ we have a canonical map

$$\varinjlim_{i\in I} \mathrm{Hom}_{A'}(N, F(i)) \to \mathrm{Hom}_{A'}(N, \varinjlim_{i} F(i)). \tag{3-13}$$

We leave it to the reader to check that N is finitely generated if and only if for every filtered I and every functor F such that every arrow $F(i \to i')$ is an embedding the map (3-13) is an isomorphism. Also, the hom spaces on the two sides of the arrow are H-comodules, and the isomorphism respects these comodule structures.

Let $F : I \to {}_{A'}\mathcal{M}^H$ be a functor from a filtered small category such that all $F(i \to i')$ are monomorphisms. Since by Proposition 3.10 the equivalence ${}_{A'}\mathcal{M}^H \equiv \mathsf{Mod}(\widetilde{A})$ is affected by the functor $(\bullet)^{\mathrm{coH}}$ which preserves filtered colimits, the analogue of (3-13) over A^{coH} is obtained by applying this functor to (3-13). Since (3-13) is an isomorphism, so is its image under $(\bullet)^{\mathrm{coH}}$. □

There are analogous graded versions of Lemma 3.11 and Proposition 3.10.

4. Homological properties under twisting

We keep the notation and conventions from the previous section, under the assumption that H is finite-dimensional. We do not assume H is commutative until Theorem 4.12.

4.1. Let A be a (usually connected) graded k-algebra. For $M, N \in \mathsf{Gr}(A)$ we define the graded vector space

$$\underline{\mathrm{Hom}}(M,N) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}(M, N(d)),$$

where $N(d)$ is the degree shift of N by d and Hom here is understood from context to be the space of degree-preserving A-module maps. Just like ordinary Hom, $\underline{\mathrm{Hom}}$ has derived functors $\underline{\mathrm{Ext}}^i$ taking values in the category of graded vector spaces. We denote the degree-j component of $\underline{\mathrm{Ext}}^i(M,N)$ by $\underline{\mathrm{Ext}}^i(M,N)_j$, as usual.

If A is noetherian and M is finitely generated, then $\underline{\mathrm{Ext}}(M,-)$ and $\mathrm{Ext}(M,-)$ agree or, more precisely, $\mathrm{Ext}(M,-)$ is the vector space obtained by forgetting the grading on $\underline{\mathrm{Ext}}(M,-)$. This is not the case in general though.

4.2. Let A be a graded k-algebra in $\mathcal{M}^H$. If we make the smash product $A\sharp H^*$ into a $\mathbb{Z}$-graded k-algebra by placing H^* in degree 0, then ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$ is equivalent to $\mathsf{Gr}(A\sharp H^*)$. Therefore every $M \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$ has a resolution by projective objects in ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$. Let (P_*, d) be such a projective resolution; it is also a projective resolution in $\mathsf{Gr}(A)$ by Lemma 3.8. If $N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$, then the homology of $\underline{\mathrm{Hom}}_A(P_*, N)$ is in $\mathcal{M}^H$. Thus, if $M, N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$, then every $\underline{\mathrm{Ext}}^i_A(M,N)_j$ is in $\mathcal{M}^H$:

Lemma 4.1. *Let A be a graded H-comodule algebra and let $M, N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$. Then the components $\underline{\mathrm{Ext}}^i_A(M,N)_j$ acquire H-comodule structures natural in $M, N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$.*

Similarly, if $M, N \in {}_A\mathcal{M}^H$, then $\mathrm{Ext}^i_A(M,N) \in \mathcal{M}^H$, naturally in M and N.

The following result will be used repeatedly.

Theorem 4.2. *Let A be a graded H-comodule algebra and let $M, N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$. There is a natural isomorphism of bigraded vector spaces*

$$\underline{\mathrm{Ext}}^*_A(M,N)_\bullet \cong \underline{\mathrm{Ext}}^*_{\widetilde{A}}\big(\widetilde{M},\widetilde{N}\big)_\bullet. \tag{4-1}$$

Proof. Let (P_*, d) be a projective resolution of ${}_AM$ in ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$ (and hence also in $\mathsf{Gr}(A)$ by Lemma 3.8). Then $(\widetilde{P}_*, \widetilde{d})$ is a projective resolution of $\widetilde{M}$ in ${}_{\mathsf{Gr}(\widetilde{A})}\mathcal{M}$ by Lemma 3.4, and $\underline{\mathrm{Ext}}^*_{\widetilde{A}}(\widetilde{M},\widetilde{N})_\bullet$ is the cohomology of the complex $\underline{\mathrm{Hom}}_{\widetilde{A}}(\widetilde{P}_*,\widetilde{N})$.

By Proposition 3.3 (or rather its graded version; see §3.2.1), this is the same as the cohomology of the complex

$$\underline{\mathrm{Hom}}^H_A(P_*, N\otimes T) \cong \underline{\mathrm{Hom}}_A(P_*, N\otimes T)^{\mathrm{coH}} \cong (\underline{\mathrm{Hom}}_A(P_*, N)\otimes T)^{\mathrm{coH}}, \tag{4-2}$$

where the second isomorphism uses the finite-dimensionality of T.

The right-most complex is the image of $\underline{\mathrm{Hom}}_A(P_*, N)$ (regarded as a complex of $\mathbb{Z}$-graded H-comodules) under the functor $\widetilde{\bullet}$ to graded vector spaces. Since this functor is exact, it turns the cohomology of $\underline{\mathrm{Hom}}_A(P_*, N)$, i.e., $\underline{\mathrm{Ext}}^*_A(M,N)_\bullet$, into that of (4-2). In other words, $\widetilde{\bullet}$ turns the left-hand side of (4-1) into its right-hand side.

Finally, $\widetilde{\bullet}$ is isomorphic to the forgetful functor $\mathcal{M}^H \to \mathrm{VECT}$ as a linear functor (though not as a monoidal functor) because $T\cong H$ is a comodule; the conclusion follows. □

There is a version of Theorem 4.2 for ungraded modules $M, N \in {}_A\mathcal{M}^H$; the same proof, with obvious modifications, works.

For the next result we specialize to the case when the graded algebra A is connected.

Corollary 4.3. *Let A be a connected graded H-comodule algebra. If $A\cong TV/(R)$, then $A \cong TV/(\widetilde{R})$ where $\widetilde{R}$ and R are isomorphic as graded vector spaces.*

Proof. This follows by applying Theorem 4.2 to $M = N = k$ from the fact that there are isomorphisms $\mathrm{Ext}^1_A(k,k)\cong V^*$ and $\mathrm{Ext}^2_A(k,k)\cong R^*$ of bigraded vector spaces. □

4.3. **The Koszul property.**

Definition 4.4. Let m be an integer ≥ 2. A connected graded algebra A is m-Koszul if $A\cong TV/(R)$ with $\deg(V)=1$, $R\subseteq V^{\otimes m}$, and $\mathrm{Ext}^i_A(k,k)$ is concentrated in just one degree for all i.

Corollary 4.5. *Let m be an integer ≥ 2. A connected graded H-comodule algebra A is m-Koszul if and only if $\widetilde{A}$ is.*

Proof. This follows immediately from Corollary 4.3 and Theorem 4.2 applied to $M = N = k$. □

4.4. **Artin-Schelter regularity.** We begin by recalling the relevant notions.

Definition 4.6. A connected graded k-algebra A is *Artin-Schelter Gorenstein* (AS-Gorenstein for short) of dimension d if the left and right injective dimensions of A as a graded A-module equal d and

$$\underline{\mathrm{Ext}}^i_A(k,A) = \underline{\mathrm{Ext}}^i_{A^\circ}(k,A) \cong \delta_{id}\, k(\ell), \tag{4-3}$$

for some integer ℓ.

If A is AS-Gorenstein we say it is *Artin-Schelter regular* (AS-regular for short) of dimension d if in addition $\mathrm{gldim}(A) = d < \infty$.

Artin and Schelter's original definition of regularity included a restriction on the growth of $\dim_k(A_i)$, but in some situations it is sensible to avoid that restriction. We will show that if A is AS-regular of dimension d, then so is $\widetilde{A}$. Since $\dim_k(A_i) = \dim_k(\widetilde{A}_i)$ for all i (Proposition 3.7), if A is AS-regular with the growth restriction so is $\widetilde{A}$.

Proposition 4.7. *For all noetherian connected graded algebras $A \in \mathcal{M}^H$*, $\mathrm{gldim}(\widetilde{A}) = \mathrm{gldim}(A)$.

Proof. This follows immediately from Proposition 3.9, Theorem 4.2, and the fact that for noetherian connected graded algebras the homological dimension can be computed as the supremum of those i for which $\mathrm{Ext}^i(k,k)$ is non-zero. □

Theorem 4.8. *If a noetherian connected graded algebra $A \in \mathcal{M}^H$ is AS-regular of dimension d so is $\widetilde{A}$.*

Proof. By Proposition 4.7, $\mathrm{gldim}(\widetilde{A}) = d$. Theorem 4.2 and its right-handed version applied to $M = k$ and $N = A$ show that (4-3) holds (or does not hold) simultaneously for A and $\widetilde{A}$. □

Corollary 4.9. *If A is a noetherian twisted Calabi-Yau algebra, so is $\widetilde{A}$.*

Proof. By [26, Lem. 1.2], an algebra is twisted Calabi-Yau if and only if it is AS-regular. □

4.5. **Condition χ.** In this subsection we prove that the finiteness condition χ introduced in [6] is preserved under twisting. Throughout, A will be an $\mathbb{N}$-graded algebra.

Definition 4.10 ([6, Defn. 3.7])**.** We say that A has property χ if for all non-negative integers i, d and all finitely generated graded A-modules N there is an integer n_0 such that $\underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, N)_{\geq d}$ is finitely generated over A for all $n \geq n_0$. (The left A-module structure on $\underline{\mathrm{Ext}}$ comes from the right A-action on $A/A_{\geq n}$.)

The χ condition is crucial in proving Serre-type results on finiteness of cohomology for non-commutative projective schemes (see e.g. [6, Thm. 7.4]).

Theorem 4.11. *If the noetherian connected graded algebra $A \in \mathcal{M}^H$ of finite global dimension has property χ, then so does $\widetilde{A}$.*

Proof. If the finite generation condition from Definition 4.10 holds for all N for a fixed choice of i and d we say that condition χ^i_d holds.

By Propositions 3.5, 3.9, and 4.7, $\widetilde{A}$ is also noetherian connected graded and of finite global dimension. This latter condition means that all sufficiently high $\underline{\mathrm{Ext}}^i$ vanish, so that we can prove that all χ^i_d hold by descending induction on i. We now do this.

Fix i and suppose we have proved that χ^j_d holds for all d and all $j > i$. Fix $N \in \mathsf{Gr}(\widetilde{A})$ and d as in Definition 4.10. Because $\widetilde{A}$ is noetherian, N is the cokernel in a short exact sequence

$$0 \to K \to \widetilde{A}^{\oplus S} \to N \to 0$$

of finitely generated graded modules. Applying the resulting long exact $\underline{\mathrm{Ext}}$ sequence and the induction hypothesis we conclude that it suffices to prove that the graded $\widetilde{A}$-module $\underline{\mathrm{Ext}}^i_{\widetilde{A}}(\widetilde{A}/\widetilde{A}_{\geq n}, \widetilde{A}^{\oplus S})_{\geq d}$ is finitely generated for sufficiently large n.

Just as in the proof of Theorem 4.2, $\underline{\mathrm{Ext}}^i_{\widetilde{A}}(\widetilde{A}/\widetilde{A}_{\geq n}, \widetilde{A}^{\oplus S})_{\geq d}$ is the image of

$$U_n = \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, A^{\oplus S})_{\geq d} \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$$

under the functor $\widetilde{\bullet}$. By hypothesis, U_n is finitely generated over A for sufficiently large n. Since U_n is also an H-comodule, it is finitely generated over $A\sharp H^*$ and hence is a quotient of some finite direct sum of copies of $A\sharp H^*$ in ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$. Applying $\widetilde{\bullet}$ we obtain

$$\widetilde{U_n} = \underline{\mathrm{Ext}}^i_{\widetilde{A}}(\widetilde{A}/\widetilde{A}_{\geq n}, \widetilde{A}^{\oplus S})_{\geq d}$$

as a quotient of a finite direct sum of copies of $\widetilde{A\sharp H^*} \cong \widetilde{A} \otimes \widetilde{H^*} \in \mathsf{Gr}(\widetilde{A})$. □

When H is commutative the noetherian and global dimension hypotheses are not needed.

Theorem 4.12. *If H is commutative and the graded algebra $A \in \mathcal{M}^H$ satisfies condition χ, then so does $\widetilde{A}$.*

Proof. Let N be a finitely generated graded $\widetilde{A}$-module and let i, d be fixed integers. Because A has property χ, there is some n_0 for which the finiteness condition in Definition 4.10 holds for the graded A-module $N' = A' \otimes_{\widetilde{A}} N$ (the A-module structure is obtained by restricting scalars from $A' = A \otimes T^{\mathrm{op}}$ to A). We will show that n_0 satisfies the requirements of Definition 4.10 for N.

Apply the graded analogue of Proposition 3.10 to identify $\mathsf{Gr}(\widetilde{A})$ with ${}_{\mathsf{Gr}(A')}\mathcal{M}^H$. Arguing as in the proof of Theorem 4.2 we see that the $\widetilde{A}$-module $\underline{\mathrm{Ext}}^i_{\widetilde{A}}(\widetilde{A}/\widetilde{A}_{\geq n}, N)_{\geq d}$ that we are interested in is precisely the space of H-coinvariants in

$$\underline{\mathrm{Ext}}^i_{A'}(A'/A'_{\geq n}, N')_{\geq d} \cong \underline{\mathrm{Ext}}^i_{A}(A/A_{\geq n}, N')_{\geq d}. \tag{4-4}$$

To conclude, apply Lemma 3.11 (substituting (4-4) for N in that result). □

5. "Exotic" elliptic algebras

We now apply the above results to Sklyanin algebras.

5.1. Fix an integer $n \geq 3$. Let $k = \mathbb{C}$. Fix a primitive $(n^2)^{\mathrm{th}}$ root of unity $\varepsilon \in k$.

Let $Q = Q_{n^2,1}(E, \tau)$ be the Sklyanin algebra defined in [24].

By [24, §1, Remark 2], the finite Heisenberg group of order n^6, H_{n^2}, acts as automorphisms of Q. There is a basis x_i, $1 \leq i \leq n^2$, for the degree-one component of Q on which the generators of the Heisenberg group act as $x_i \mapsto x_{i+1}$ and $x_i \mapsto \varepsilon^i x_i$ where the indices are labelled modulo n^2. The n^{th} powers of the two generators generate a subgroup $\Gamma \subseteq H_{n^2}$ that is isomorphic to $(\mathbb{Z}/n)^2$. The generators of Γ act by $x_i \mapsto x_{i+n}$ and $x_i \mapsto \zeta^i x_i$ where $\zeta = \varepsilon^n$.

Let $H = k(\Gamma)$ denote the algebra of k-valued functions on Γ and let $M_n(k)$ denote the $n \times n$ matrix algebra. We make Γ act on $M_n(k)$ by having its generators act as conjugation by

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \zeta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{n-1} \end{pmatrix}.$$

By duality, the action of Γ as automorphisms of $M_n(k)$ gives $M_n(k)$ the structure of an H-comodule algebra.

Lemma 5.1. *The above action makes $M_n(k)$ into a left H-torsor in the sense of* §3.1.

Proof. Every character of Γ appears with multiplicity one in $M_n(k)$. In particular, $M_n(k)^{\mathrm{coH}} = M_n(k)^{\Gamma} = k$.

A k-algebra on which Γ acts as automorphisms is the same thing as a k-algebra with a grading by the character group of Γ. Every homogeneous component of $T = M_n(k)$ is the k-span of an invertible matrix. Hence, if χ and χ' are characters of Γ, then $T_\chi T_{\chi'} = T_{\chi\chi'}$. In other words, T is a strongly graded algebra. A result of Ulbrich shows that for every group Υ the Υ-graded algebras that are Galois as comodules over the group algebra $k\Upsilon$ are exactly the strongly graded ones [21, Thm. 8.1.7]. Let Υ be the character group of Γ. Using the natural isomorphism, Pontryagin duality, $k\Upsilon \cong k(\Gamma) = H$, so T is a left H-torsor. □

Let $\widetilde{Q} = (Q \otimes M_n(k))^{\mathrm{coH}}$.

Proposition 5.2. *The algebra $\widetilde{Q}$ is AS-regular of dimension n^2, Koszul, and noetherian, and has Hilbert series $(1-t)^{-n^2}$.*

Proof. By [41, Thm. 1.1, Cor. 1.3], all the hypotheses of Propositions 3.5 and 3.9, Corollary 4.5, and Theorem 4.8 are satisfied. □

Lemma 5.1 and Proposition 5.2 hold when $n = 2$ and k is any algebraically closed field of characteristic $\neq 2$. See §6.

6. Generators and relations for $\tilde{Q}_4$

Let k be an algebraically closed field whose characteristic is not 2.

We now specialize the discussion from §5 to $n = 2$, considering the action of the group $\Gamma = \mathbb{Z}/2 \times \mathbb{Z}/2$ on $Q = Q_{n^2} = Q_4$.

6.1. Let $\alpha_1, \alpha_2, \alpha_3 \in k$ be such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$ and $\{\alpha_1, \alpha_2, \alpha_3\} \cap \{0, \pm 1\} = \varnothing$. Often we write $\alpha = \alpha_1$, $\beta = \alpha_2$, and $\gamma = \alpha_3$.

We fix $a, b, c, i \in k$ such that $a^2 = \alpha$, $b^2 = \beta$, $c^2 = \gamma$, and $i^2 = -1$.

When $k = \mathbb{C}$ and $E = \mathbb{C}/\Lambda$, α, β, and γ are the values at τ of certain elliptic functions with period lattice Λ [30, §2], [33, §2.10]. Thus, when $k = \mathbb{C}$ we can take

$$a = \frac{\theta_{11}(\tau)\theta_{00}(\tau)}{\theta_{01}(\tau)\theta_{10}(\tau)}, \qquad b = i\frac{\theta_{11}(\tau)\theta_{01}(\tau)}{\theta_{10}(\tau)\theta_{11}(\tau)}, \qquad c = i\frac{\theta_{11}(\tau)\theta_{10}(\tau)}{\theta_{11}(\tau)\theta_{01}(\tau)},$$

where $\theta_{11}, \theta_{00}, \theta_{01}, \theta_{10}$ are Jacobi's four theta functions as defined in [46, p. 71].

6.2. Let $Q = k[x_0, x_1, x_2, x_3]$ be the quotient of the free algebra $k\langle x_0, x_1, x_2, x_3\rangle$ by the six relations

$$\text{(6-1)} \qquad x_0x_i - x_ix_0 = \alpha_i(x_jx_k + x_kx_j), \qquad x_0x_i + x_ix_0 = x_jx_k - x_kx_j,$$

where (i, j, k) runs over the cyclic permutations of $(1, 2, 3)$.

6.3. The earlier results will be applied to the Hopf algebra H of k-valued functions on

$$\Gamma = \{1, \gamma_1, \gamma_2, \gamma_3 = \gamma_1\gamma_2\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

and its action as k-algebra automorphisms of Q given by Table 1.

The irreducible characters of Γ are labelled $\chi_0, \chi_1, \chi_2, \chi_3$ in such a way that $\gamma(x_j) = \chi_j(\gamma)x_j$ for all $\gamma \in \Gamma$ and $j = 0, 1, 2, 3$.

TABLE 1. The action of Γ as automorphisms of Q

	x_0	x_1	x_2	x_3
γ_1	x_0	x_1	$-x_2$	$-x_3$
γ_2	x_0	$-x_1$	x_2	$-x_3$
γ_3	x_0	$-x_1$	$-x_2$	x_3

6.4. **A quaternionic basis for $M_2(k)$ and the conjugation action of Γ on $M_2(k)$.** Define

$$q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad q_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad q_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad q_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6-2}$$

Then $q_1^2 = q_2^2 = q_3^2 = -1$, and if (i, j, k) is a cyclic permutation of $(1, 2, 3)$, $q_i q_j = q_k$ and $q_i q_j + q_j q_i = 0$.

Define an action of Γ as automorphisms of $M_2(k)$ by $\gamma_j(a) := q_j a q_j^{-1}$, i.e., $g(q_j) = \chi_j(g) q_j$.

As before, $\widetilde{Q} = (Q \otimes M_2(k))^\Gamma$. If $\gamma \in \Gamma$, then $\gamma(x_i q_j) = \chi_i(\gamma)\chi_j(\gamma) x_i q_j$ so

$$y_0 := x_0, \quad y_1 := x_1 q_1, \quad y_2 := x_2 q_2, \quad y_3 := x_3 q_3$$

are Γ-invariant elements of $Q \otimes M_2(k)$.

Proposition 6.1. *The algebra $\widetilde{Q}$ is generated by y_0, y_1, y_2, y_3 modulo the relations*

$$y_0 y_i - y_i y_0 = \alpha_i (y_j y_k - y_k y_j) \qquad \text{and} \qquad y_0 y_i + y_i y_0 = y_j y_k + y_k y_j, \tag{6-3}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Proof. Because $\widetilde{Q}$ is Koszul with Hilbert series $(1-t)^{-4}$, it is generated by 4 degree-one elements subject to 6 degree-two relations. Since y_0, y_1, y_2, y_3 are Γ-invariant elements of degree one, they generate $\widetilde{Q}$. It follows from the quadratic relations for Q_4 that $(x_0 x_i - x_i x_0) q_i = \alpha_i (x_j x_k + x_k x_j) q_j q_k$ and $(x_0 x_i + x_i x_0) q_i = (x_j x_k - x_k x_j) q_j q_k$. Rewriting these relations in terms of y_0, y_1, y_2, y_3 gives the relations in (6-3). □

Remark 6.2. It follows from the presentation given by the relations (6-3) that the identity and the sign-change maps $\widetilde{Q}_1 \to \widetilde{Q}_1$ both extend to algebra anti-automorphisms of $\widetilde{Q}$.

Since $\widetilde{Q}$ is an Artin-Schelter regular noetherian algebra of global dimension and GK-dimension 4, it is a domain by [4, Thm. 3.9].

Proposition 6.3. *There is an action of Γ as graded k-algebra automorphisms of $\widetilde{Q}$ given by Table 2.*

TABLE 2. The action of Γ as automorphisms of $\widetilde{Q}$

	y_0	y_1	y_2	y_3
γ_1	y_0	y_1	$-y_2$	$-y_3$
γ_2	y_0	$-y_1$	y_2	$-y_3$
γ_3	y_0	$-y_1$	$-y_2$	y_3

Using the conjugation action of Γ *as automorphisms of* $M_2(k)$*, this gives an action of* Γ *as automorphisms of* $\widetilde{Q} \otimes M_2(k)$*. The invariant subalgebra* $(\widetilde{Q} \otimes M_2(k))^\Gamma$ *is generated by*

$$z_0 := y_0, \quad z_1 := y_1 q_1, \quad z_2 := y_2 q_2, \quad z_3 := y_3 q_3$$

and is isomorphic to Q *via* $z_j \mapsto x_j$*.*

Proof. A calculation shows that the action of Γ respects the relations (6-3). Because $(\widetilde{Q} \otimes M_2(k))^\Gamma$ is Koszul with Hilbert series $(1-t)^{-4}$, it is generated by 4 degree-one elements subject to 6 degree-two relations. The elements z_0, z_1, z_2, z_3 are Γ-invariant so generate $(\widetilde{Q} \otimes M_2(k))^\Gamma$. It follows from the quadratic relations for $\widetilde{Q}$ that $(y_0 y_i - y_i y_0) q_i = \alpha_i (y_j y_k - y_k y_j) q_j q_k$ and $(y_0 y_i + y_i y_0) q_i = (y_j y_k + y_k y_j) q_j q_k$. Rewriting these relations in terms of z_0, z_1, z_2, z_3 gives the relations $z_0 z_i - z_i z_0 = \alpha_i (z_j z_k + z_k z_j)$ and $z_0 z_i + z_i z_0 = z_j z_k - z_k z_j$. □

6.5. **Central elements in $\widetilde{Q}$.** In [30, Thm. 2], Sklyanin proved that

$$\text{(6-4)} \quad \Omega := -x_0^2 + x_1^2 + x_2^2 + x_3^2 \qquad \text{and} \qquad \Omega' := x_1^2 + \left(\frac{1+\alpha_1}{1-\alpha_2}\right) x_2^2 + \left(\frac{1-\alpha_1}{1+\alpha_3}\right) x_3^2$$

belong to the center of Q when $k = \mathbb{C}$. By the Principle of Permanence of Algebraic Identities, Ω and Ω' are central for all k.

The elements $x_0^2, x_1^2, x_2^2, x_3^2$ are fixed by the action of Γ. Since $y_j^2 = -x_j^2$ for $j = 1, 2, 3$, the elements

$$\Theta := y_0^2 + y_1^2 + y_2^2 + y_3^2 \qquad \text{and} \qquad \Theta' := y_1^2 + \left(\frac{1+\alpha_1}{1-\alpha_2}\right) y_2^2 + \left(\frac{1-\alpha_1}{1+\alpha_3}\right) y_3^2$$

belong to the center of $\widetilde{Q}$. We note that $\Theta = -\Omega \otimes \mathbb{I}_2$ and $\Theta' = -\Omega' \otimes \mathbb{I}_2$. For simplicity, we will often conflate them with $-\Omega$ and $-\Omega'$ respectively.

7. Γ acts on E as translation by the 2-torsion subgroup

7.1. If we use x_0, x_1, x_2, x_3 as an ordered set of coordinate functions on Q_1^*, then the action of Γ on Q_1^* induced by its action on Q_1 is given by the formulas

$$\text{(7-1)} \qquad \begin{cases} \gamma_1(\delta_0, \delta_1, \delta_2, \delta_3) &=& (\delta_0, \delta_1, -\delta_2, -\delta_3), \\ \gamma_2(\delta_0, \delta_1, \delta_2, \delta_3) &=& (\delta_0, -\delta_1, \delta_2, -\delta_3), \\ \gamma_3(\delta_0, \delta_1, \delta_2, \delta_3) &=& (\delta_0, -\delta_1, -\delta_2, \delta_3). \end{cases}$$

We will write $\mathbb{P}^3$ for $\mathbb{P}(Q_1^*)$, the projective space of lines in Q_1^*. The action of Γ on Q_1^* induces an action of Γ as automorphisms of $\mathbb{P}^3$ given by the formulas in (7-1).

The relations for Q, which are elements of $Q_1 \otimes Q_1$, are bi-homogeneous forms on $\mathbb{P}^3 \times \mathbb{P}^3$. We write $R = \ker(Q_1 \otimes Q_1 \stackrel{\text{mult}}{\longrightarrow} Q_2)$ and define the subscheme

$$V \;:=\; \{(\mathbf{u}, \mathbf{v}) \mid r(\mathbf{u}, \mathbf{v}) = 0 \text{ for all } r \in R\} \;\subseteq\; \mathbb{P}^3 \times \mathbb{P}^3.$$

Let $\mathrm{pr}_i : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$, $i = 1, 2$, be the projections of V onto the left and right copies of $\mathbb{P}^3$.

Proposition 7.1 ([33, Props. 2.4, 2.5])**.** *With the above notation,*

$$\mathrm{pr}_1(V) \;=\; \mathrm{pr}_2(V) \;=\; E \cup \big\{(1,0,0,0),\, (1,0,0,0),\, (1,0,0,0),\, (1,0,0,0)\big\},$$

where E is the intersection of the quadrics

$$\begin{aligned} x_0^2+x_1^2+x_2^2+x_3^2 &= 0, \\ (1-\gamma)x_1^2+(1+\alpha\gamma)x_2^2+(1+\alpha)x_3^2 &= 0. \end{aligned}$$

Furthermore, E is an elliptic curve.

The reader will notice that we use the same notation for elements in Q as for elements in the symmetric algebra $S(Q_1)$. Thus, in Proposition 7.1, $x_0^2+x_1^2+x_2^2+x_3^2$ is an element in $S(Q_1)$, i.e., a degree-two form on $\mathbb{P}^3$, whereas in (6-4), $-x_0^2+x_1^2+x_2^2+x_3^2$ denotes an element in Q.

It is clear that Γ fixes the points in $\{(1,0,0,0),\ (1,0,0,0),\ (1,0,0,0),\ (1,0,0,0)\}$. It is also clear that E is stable under the action of Γ (indeed, that must be so because R is Γ-stable). The map $\Gamma\to \operatorname{Aut}(E)$ is injective, so we will identify Γ with a subgroup of $\operatorname{Aut}(E)$. Once we have fixed a group law $+$ on E we will identify E with the subgroup of $\operatorname{Aut}(E)$ consisting of the translation automorphisms; i.e., $E\to\operatorname{Aut}(E)$ sends a point $\mathbf{v}\in E$ to the automorphism $\mathbf{u}\mapsto\mathbf{u}+\mathbf{v}$.

Once we have defined the group $(E,+)$ we will write o for its identity element and

$$E[2] \ := \ \{\mathbf{v}\in E \mid \mathbf{v}+\mathbf{v}=o\}.$$

The next main result, Theorem 7.6, shows we can define $+$ such that $\Gamma=E[2]$ as subgroups of $\operatorname{Aut}(E)$. We will then identify Γ with $E[2]$. In anticipation of that result we define an involution $-:E\to E$ and a distinguished point $o\in E$ by

$$-(w,x,y,z) \ := \ (-w,x,y,z) \tag{7-2}$$

and

$$o \ := \ \big(0,\sqrt{\nu-1},\ \sqrt{1-\mu},\ \sqrt{\mu-\nu}\,\big),$$

where

$$\mu:=\frac{1-\gamma}{1+\alpha} \qquad\text{and}\qquad \nu:=\frac{1+\gamma}{1-\beta}$$

and $\sqrt{\nu-1}$, $\sqrt{1-\mu}$, and $\sqrt{\mu-\nu}$ are some fixed square roots.[2] The restrictions on the values of α, β, γ imply that $|\{1,\mu,\nu\}|=3$. We use this fact in the proof of Lemma 7.5.

Lemma 7.2.

$$\begin{aligned} E\cap\{x_0=0\} &= \Big\{p\in E \ \Big| \ p=-p\Big\} \\ &= \Big\{\big(0,\pm\sqrt{\nu-1},\pm\sqrt{1-\mu},\pm\sqrt{\mu-\nu}\,\big)\Big\}. \end{aligned}$$

Proof. It follows from the definition of $-$ that $E\cap\{x_0=0\}=\{p\in E\mid p=-p\}$. Computing $E\cap\{x_0=0\}$ reduces to computing the intersection of the plane conics $x_1^2+x_2^2+x_3^2=0$ and $\mu x_1^2+\nu x_2^2+x_3^2=0$. The conics meet at four points, namely $\big(\pm\sqrt{\nu-1},\ \pm\sqrt{1-\mu},\ \pm\sqrt{\mu-\nu}\big)\in\mathbb{P}^2$. The result follows. □

[2]The choice of square root doesn't matter—as one takes the different square roots one obtains 4 different candidates for o. But, as we will see, with the choice of $+$ we eventually make, those 4 points are the points in $E[2]$. The situation is analogous to that of a smooth plane cubic: there are nine inflection points, and if one chooses the group law so that one of those points is the identity, then the inflection points are the points in $E[3]$, the 3-torsion subgroup.

Lemma 7.3. *There is a degree-two morphism* $\pi : E \to \mathbb{P}^1$ *such that* $\pi(p) = \pi(-p)$ *for all* $p \in E$*; i.e., the fibers of* π *are the sets* $\{p, -p\}$, $p \in E$. *In particular, the ramification locus of* π *is* $\{p \in E \mid p = -p\} = \{o, \xi_1, \xi_2, \xi_3\}$ *where*

$$\begin{aligned} o &:= \big(0, \sqrt{\nu-1},\ \sqrt{1-\mu},\ \sqrt{\mu-\nu}\,\big), \\ \xi_1 := \gamma_1(o) &= \big(0,\ \sqrt{\nu-1},\ -\sqrt{1-\mu},\ \sqrt{\mu-\nu}\,\big), \\ \xi_2 := \gamma_2(o) &= \big(0,\ -\sqrt{\nu-1},\ \sqrt{1-\mu},\ \sqrt{\mu-\nu}\,\big), \\ \xi_3 := \gamma_3(o) &= \big(0,\ -\sqrt{\nu-1},\ -\sqrt{1-\mu},\ \sqrt{\mu-\nu}\,\big). \end{aligned}$$

Proof. The conic C, given by $\mu x_1^2 + \nu x_2^2 + x_3^2 = 0$, is smooth so isomorphic to $\mathbb{P}^1$. Define $\pi : E \to C$ by $\pi(w, x, y, z) = (x, y, z)$. The result is now obvious. □

Proposition 7.4. *Let* $E' \subseteq \mathbb{P}^2$ *be the curve* $y^2 z = x(x-z)(x-\lambda z)$ *where*

$$\lambda := \frac{\nu - \mu\nu}{\nu - \mu} = \frac{1}{\gamma}\left(\frac{1+\gamma}{1+\alpha}\right)\left(\frac{\alpha+\gamma}{1-\beta}\right),$$

and consider the group $(E', +)$ *in which* $(0,1,0)$ *is the identity and three points of* E' *sum to zero if and only if they are collinear.*

(1) *There is an isomorphism of varieties* $g : E \to E'$ *such that*

$$g(o) = \infty = (0,1,0), \quad g(\xi_1) = (0,0,1), \quad g(\xi_2) = (1,0,1), \quad g(\xi_3) = (\lambda, 0, 1).$$

(2) *If* $(E, +)$ *is the unique group law such that* $g : (E, +) \to (E', +)$ *is an isomorphism of groups, then* $E[2] = \{p \mid p = -p\} = \{o, \xi_1, \xi_2, \xi_3\}$, *and*
(3) $p + (-p) = o$ *for all* $p \in E$, *and*
(4) *4 points on* E *are coplanar if and only if their sum is zero.*

Proof. (1) Let $\pi : E \to C = \{\mu x_1^2 + \nu x_2^2 + x_3^2 = 0\}$ be the morphism $\pi(x_0, x_1, x_2, x_3) = (x_1, x_2, x_3)$ in Lemma 7.3 and let $f : C \to \mathbb{P}^1$ be the isomorphism

$$f(x_1, x_2, x_3) = (\sqrt{-\nu}x_2 + \sqrt{\mu}x_1,\ x_3) = (x_3,\ \sqrt{-\nu}x_2 - \sqrt{\mu}x_1)$$

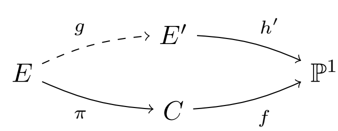
with inverse

$$f^{-1}(s,t) = \Big(\tfrac{1}{\sqrt{\mu}}(s^2 - t^2), \tfrac{1}{\sqrt{-\nu}}(s^2 + t^2), 2st\Big).$$

Let $h = f \circ \pi : E \to \mathbb{P}^1$. The ramification locus of π, and hence of h, is obviously $\{p \in E \mid p = -p\}$. Let E' be the plane cubic $y^2 z = x(x-z)(x-\lambda z)$ and let $h' : E' \to \mathbb{P}^1$ be the morphism $h'(x, y, z) = (x, z)$.

Consider the following diagram:

(7-3)



The following result is implicit in [14, Ch. 4, §4]: If E and E' are elliptic curves and $h : E \to \mathbb{P}^1$ and $h' : E' \to \mathbb{P}^1$ are degree-two morphisms having the same branch points, then there is an isomorphism of varieties $g : E \to E'$ such that $h'g = h$.

The four branch points for h are

$$\Big(\pm\sqrt{\mu\nu - \nu} \pm \sqrt{\mu\nu - \mu},\ \sqrt{\mu - \nu}\Big) = \Big(\sqrt{\mu - \nu},\ \pm\sqrt{\mu\nu - \nu} \mp \sqrt{\mu\nu - \mu}\Big).$$

The cross-ratios of these four points are $\left\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\right\}$ where

$$\lambda := \frac{\nu-\mu\nu}{\nu-\mu} = \frac{1}{\gamma}\left(\frac{1+\gamma}{1+\alpha}\right)\left(\frac{\alpha+\gamma}{1-\beta}\right).$$

The four branch points for $h' : E' \to \mathbb{P}^1$ have the same cross-ratios so $E \cong E'$. In particular, there is an isomorphism of varieties $g : E \to E'$ such that

$$g(o) = \infty = (0,1,0), \quad g(\xi_1) = (0,0,1), \quad g(\xi_2) = (1,0,1), \quad g(\xi_3) = (\lambda,0,1).$$

(2) Let $+$ be the unique group law on E such that $g(p+p') = g(p) + g(p')$ for all $p, p' \in E$. Then g is an isomorphism of algebraic groups. Since $E'[2] = \{(0,1,0), (0,0,1), (1,0,1), (\lambda,0,1)\}$, $E[2] = \{o, \xi_1, \xi_2, \xi_3\} = \{p \in E \mid p = -p\}$.

(3) Since $g : E \to E'$ is a group isomorphism it suffices to show that $g(p) + g(-p) = o$. The fibers of h consist of points that sum to zero, so it suffices to show that $h(g(p)) = h(g(-p))$. However, $hg = f\pi$ and $\pi(p) = \pi(-p)$ so $hg(p) = hg(-p)$.

(4) Let $\Phi : \mathrm{Div}(E) \to E$ be the map

$$\Phi\big((q_1) + \cdots + (q_m) - (r_1) - \cdots - (r_n)\big) := q_1 + \cdots + q_m - r_1 - \cdots - r_n.$$

It is easy to show that if D and D' are divisors of the same degree, then $D \sim D'$ if and only if $\Phi(D) = \Phi(D')$. The points $\{o, \xi_1, \xi_2, \xi_3\}$ are coplanar. Four points $q_0, \ldots, q_3 \in E$ are coplanar if and only if $(o) + (\xi_1) + (\xi_2) + (\xi_3) \sim (q_0) + (q_1) + (q_2) + (q_3)$. Since $o + \xi_1 + \xi_2 + \xi_3 = o$, $q_0, \ldots, q_3 \in E$ are coplanar if and only if $q_0 + q_1 + q_2 + q_3 = o$. □

Lemma 7.5. *There are exactly four singular quadrics that contain E, namely:*

$$\begin{aligned} Q_0 &= \{\mu x_1^2 + \nu x_2^2 + x_3^2 = 0\}, \\ Q_1 &= \{\mu x_0^2 + (\mu-\nu) x_2^2 + (\mu-1) x_3^2 = 0\}, \\ Q_2 &= \{\nu x_0^2 + (\nu-\mu) x_1^2 + (\nu-1) x_3^2 = 0\}, \\ Q_3 &= \{x_0^2 + (1-\mu) x_1^2 + (1-\nu) x_2^2 = 0\}. \end{aligned}$$

Let $p \in E$. For each i, the line through $-p$ and $\gamma_i(p)$ (understood as the tangent to E at $-p$ when $-p = \gamma_i(p)$) lies on Q_i.

Proof. Since the equation defining each Q_i is a linear combination of the equations in Proposition 7.1, Q_i contains E. Each Q_i has a unique singular point, namely e_i, where

$$e_0 := (1,0,0,0), \quad e_1 := (0,1,0,0), \quad e_2 := (0,0,1,0), \quad e_3 := (0,0,0,1).$$

Thus Q_i is a union of lines, and every line on Q_i passes through e_i.

Let f_1, f_2 be quadratic forms such that $E = \{f_1 = f_2 = 0\}$. A quadric contains E if and only if it is the zero locus of $\lambda_1 f_1 + \lambda_2 f_2$ for some $(\lambda_1, \lambda_2) \in \mathbb{P}^1$. Conversely, for all $(\lambda_1, \lambda_2) \in \mathbb{P}^1$ the zero locus of $\lambda_1 f_1 + \lambda_2 f_2$ is a quadric that contains E. Since $|\{1, \mu, \nu\}| = 3$, there are exactly 4 singular quadrics in the pencil of quadrics that contain E; these are the quadrics Q_i (see [17, Prop. 3.4]).

Let $p = (w, x, y, z) \in E$. Let L be a line through $-p$ and e_0. Thus $L = \{(t - sw, sx, sy, sz) \mid (s,t) \in \mathbb{P}^1\}$. The line L lies on Q_0 and meets E when

$$(t-sw)^2 + (sx)^2 + (sy)^2 + (sz)^2 = \mu(sx)^2 + \nu(sy)^2 + (sz)^2 = 0.$$

The second expression is zero for all s. The first expression is zero if and only if $t^2 - 2stw = 0$. One solution to this is $t = 0$, and it corresponds to the point

$-p \in L \cap E$. The other solution occurs when $t - 2sw = 0$ and corresponds to the point $(w, x, y, z) = p$. In other words, the line through e_0 and p intersects E again at $\gamma_0(p)$.

The line through $-p$ and e_1 is $\{(-sw, sx + t, sy, sz) \mid (s,t) \in \mathbb{P}^1\}$. It lies on Q_1 and meets E when

$$(-sw)^2 + (sx+t)^2 + (sy)^2 + (sz)^2 \;=\; \mu(-sw)^2 + \nu(sy)^2 + (sz)^2 \;=\; 0.$$

The second expression is zero for all s and the first is zero if and only if $t^2 + 2stx = 0$. The solution $t = 0$ to this equation corresponds to the point $-p \in L \cap E$. The other solution occurs when $t + 2sx = 0$ and gives the point $(-w, -x, y, z) = \gamma_1(p)$.

Similar calculations show that the line through e_i and $-p$ (which is contained in Q_i) intersects E again at $\gamma_i(p)$ for $i = 2, 3$. This is a rephrasing of the last claim in the statement and hence finishes the proof. □

Theorem 7.6. *There is a group law + on E such that each element in Γ acts as translation by a point in $E[2]$.*

Proof. Let γ_i be the automorphism in Table 1 and let ξ_i be the point in Lemma 7.3. We will show that γ_i is translation by ξ_i, i.e., $\xi_i = \gamma_i(o)$.

Let p and q be arbitrary points of E. The line through $-p$ and $\gamma_i(p)$ lies on Q_i. So does the line through $-q$ and $\gamma_i(q)$. Because these lines are on Q_i they meet at e_i. The lines therefore span a plane; i.e., $-p$, $\gamma_i(p)$, $-q$, and $\gamma_i(q)$ are coplanar. Therefore $(-p) + \gamma_i(p) + (-q) + \gamma_i(q) = o$. Taking $q = o$ and rearranging the equation gives $p = \gamma_i(p) + \gamma_i(o)$ or $\gamma_i(p) = p - \gamma_i(o) = p + \gamma_i(o)$. □

7.2. **Twisting a Q-module by γ_i.** Let $\gamma \in \Gamma$ and let M be a graded left Q-module. We define $\gamma^* M$ to be the graded Q-module which is equal to M as a graded vector space and has the new Q-action

$$r \bullet_\gamma m \;:=\; \gamma^{-1}(r) m$$

for $r \in Q$ and $m \in \gamma^* M = M$. We make γ^* into an auto-equivalence of $\mathsf{Gr}(Q)$ in the obvious way and we note that these auto-equivalences have the property that $\gamma^* \delta^* = (\gamma\delta)^*$.

In preparation for the next result, recall the discussion of point and line modules for Q from §1.6.

Proposition 7.7. *Let $p, q \in E$ and let M_p and $M_{p,q}$ be the associated point and line modules. Then $\gamma_i^* M_p \cong M_{p+\xi_i}$ and $\gamma_i^* M_{p,q} \cong M_{p+\xi_i, q+\xi_i}$.*

Proof. Let $r \in Q_1$ and $p \in \mathbb{P}^3 = \mathbb{P}(Q_1^*)$. The action of γ_i on Q_1 and Q_1^* is such that $\gamma_i(r)(p) = r(\gamma_i^{-1}(p)) = r(\gamma_i(p))$. Thus, $r(p) = 0$ if and only if $\gamma_i(r)$ vanishes at $\gamma_i(p)$. Since $M_p = Q/Qp^\perp$ where $p^\perp$ is the subspace of Q_1 vanishing at p, $\gamma_i^* M_p = Q/Q(p + \xi_i)^\perp$. A similar argument works for line modules. □

8. Properties of $\tilde{B}$

The group law on E is such that the degree-four line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{P}^3}(1)|_E$ is isomorphic to $\mathcal{O}_E(D)$ where D is the divisor $E[2]$. Because Γ acts on E as does translation by $E[2]$, $E[2]$ is stable under Γ and there is therefore a Γ-equivariant structure on $\mathcal{L}$. Hence, since the action of Γ also commutes with translation $p \mapsto p + \tau$, Γ acts on the data $(E, \tau, \mathcal{L})$. It follows that Γ acts as automorphisms of the twisted homogeneous coordinate ring $B(E,\tau,\mathcal{L})$. Accordingly, using the general $\widetilde{\bullet}$ construction described in the first paragraph of §3.1.5, we define

$$\widetilde{B} \;:=\; (B(E,\tau,\mathcal{L}) \otimes M_2(k))^{\Gamma}.$$

8.1. Let $R := Q/(\Omega,\Omega')$. Since Ω and Ω' are fixed by Γ, there is an induced action of Γ on R. It follows that there is an exact sequence

$$(Q\Omega \oplus Q\Omega') \otimes M_2(k) \to Q \otimes M_2(k) \to R \otimes M_2(k) \to 0$$

in which the maps are Γ-equivariant. Since $\Theta = -\Omega \otimes 1$ and $\Theta' = -\Omega' \otimes 1$ are Γ-invariant, $(Q\Omega \otimes M_2(k))^{\Gamma} = \widetilde{Q}\Theta$ and $(Q\Omega' \otimes M_2(k))^{\Gamma} = \widetilde{Q}\Theta'$. Since $Q \otimes M_2(k)$ is a semisimple Γ-module, we obtain an exact sequence

$$\widetilde{Q}\Theta \oplus \widetilde{Q}\Theta' \to \widetilde{Q} \to \widetilde{R} := (R \otimes M_2(k))^{\Gamma} \to 0$$

when we take Γ-invariants in the previous exact sequence. Thus $\widetilde{R} = \widetilde{Q}/(\Theta,\Theta')$. By Proposition 3.5, the Hilbert series of $\widetilde{R}$ is the same as that of R, namely $(1-t^2)^2(1-t)^{-4}$. It follows that Θ,Θ' is a regular sequence on $\widetilde{Q}$.

By [33, §3.9], R is isomorphic to $B(E,\tau,\mathcal{L})$. Let φ denote that isomorphism. Since φ is a Γ-module homomorphism in degree one, it is Γ-equivariant in all degrees. It follows that φ induces an algebra isomorphism $\widetilde{\varphi} : \widetilde{R} \to \widetilde{B}$. Thus, $\widetilde{B} \cong \widetilde{Q}/(\Theta,\Theta')$.

8.2. **The category $\mathsf{QGr}(\widetilde{B})$.** Let $B = B(E,\tau,\mathcal{L})$, $B' = B \otimes M_2(k)$, and $\mathcal{B} = M_2(\mathcal{O}_E)$. The main result in this subsection is

Theorem 8.1. *There is an equivalence of categories* $\mathsf{QGr}(\widetilde{B}) \equiv \mathsf{Qcoh}(E/E[2])$.

Corollary 8.2. *The set of isomorphism classes of simple* $\mathsf{QGr}(\widetilde{B})$*-objects is in natural bijection with* $E/E[2]$.

The plan is to work our way through the chain of equivalences

$$\mathsf{QGr}(\widetilde{B}) \equiv \mathsf{QGr}(B')^{\Gamma} \equiv \mathsf{Qcoh}(\mathcal{B})^{\Gamma} \equiv \mathsf{Qcoh}(\mathcal{B}^{\Gamma}) \equiv \mathsf{Qcoh}(E/E[2]). \tag{8-1}$$

The notation needs some unpacking.

First, Γ acts on the category $\mathsf{QGr}(B')$ (via the diagonal action on the tensor product $B' = B \otimes M_2(k)$) as well as on $\mathsf{Qcoh}(\mathcal{B})$. Such an action comprises an auto-equivalence γ^* of the respective category for each $\gamma \in \Gamma$ together with natural isomorphisms $t_{\gamma,\delta} : \gamma^* \circ \delta^* \cong (\gamma\delta)^*$ for $\gamma,\delta \in \Gamma$ such that

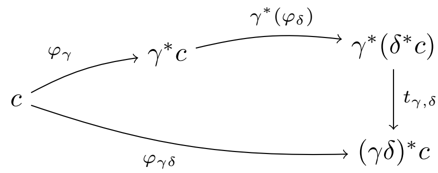
$$\begin{array}{ccc} \gamma^* \circ \delta^* \circ \varepsilon^* & \longrightarrow & (\gamma\delta)^* \circ \varepsilon^* \\ \big\downarrow & & \big\downarrow \\ \gamma^* \circ (\delta\varepsilon)^* & \longrightarrow & (\gamma\delta\varepsilon)^* \end{array} \tag{8-2}$$

commutes for all $\gamma,\delta,\varepsilon \in \Gamma$.

The action of Γ as automorphisms of B' induces an action of Γ on $\mathsf{Gr}(B')$ as described in §7.2. Since the subcategory $\mathsf{Fdim}(B')$ is stable under each γ^*, the Γ-action passes to the quotient category $\mathsf{QGr}(B')$. The action on $\mathsf{Qcoh}(\mathcal{B})$ comes from translation on E by $E[2]$ together with twisting via the Γ-action on the $M_2(k)$ tensorand in $\mathcal{B} = \mathcal{O}_E \otimes M_2(k)$.

If Γ acts on a category $\mathcal{C}$ we can then form the category of Γ-equivariant objects $\mathcal{C}^\Gamma$. The objects of $\mathcal{C}^\Gamma$ are objects $c \in \mathcal{C}$ equipped with isomorphisms $\varphi_\gamma : c \to \gamma^* c$ for $\gamma \in \Gamma$ such that

(8-3)



commutes and the morphisms are those in $\mathcal{C}$ that preserve all the structure. Explicitly, if $(\varphi_\gamma)_{\gamma\in\Gamma}$ and $(\varphi'_\gamma)_{\gamma\in\Gamma}$ are equivariant structures on objects c and c', respectively, a morphism $f : (\varphi_\gamma)_{\gamma\in\Gamma} \to (\varphi'_\gamma)_{\gamma\in\Gamma}$ is a morphism $f : c \to c'$ in $\mathcal{C}$ such that $\alpha^*(f)\varphi_\gamma = \varphi'_\gamma f$ for all $\gamma \in \Gamma$. This elucidates the notation $\mathcal{C}^\Gamma$ in (8-1) for $\mathcal{C} = \mathsf{QGr}(B')$ or $\mathsf{Qcoh}(\mathcal{B})$.

Finally, $\mathcal{B}^\Gamma$ denotes the sheaf of algebras on $E/E[2]$ obtained by descent from $\mathcal{B}$. To make sense of this, let $\rho : E \to E/E[2]$ be the étale quotient morphism and recall the following result.

Proposition 8.3 ([22, Prop. 2, p. 70])**.** *The functors*

$$\mathcal{G} \rightsquigarrow \rho^*\mathcal{G} \qquad \text{and} \qquad \mathcal{F} \rightsquigarrow (\rho_*\mathcal{F})^\Gamma$$

are mutually inverse equivalences between $\mathsf{Qcoh}(E/E[2])$ *and* $\mathsf{Qcoh}(E)^\Gamma$.

The equivalences in Proposition 8.3 are monoidal because ρ^* is. Thus, under these equivalences, Γ-equivariant sheaves of algebras on E (i.e., algebra objects in $\mathsf{Qcoh}(E)^\Gamma$) correspond to sheaves of algebras on $E/E[2]$. Keeping this in mind, we define $\mathcal{B}^\Gamma$ to be the sheaf of $\mathcal{O}_{E/E[2]}$-algebras that corresponds to $\mathcal{B} \in \mathsf{Qcoh}(E)^\Gamma$, i.e., $\mathcal{B}^\Gamma := (\rho_*\mathcal{B})^\Gamma$.

Proof of Theorem 8.1. We go through the equivalences in (8-1) one by one, moving rightward.

First equivalence. The graded version of Proposition 3.10 (applied to B' coacted upon by the function algebra of Γ) provides the equivalence $\mathsf{Gr}(\widetilde{B})$ and $\mathsf{Gr}(B)^\Gamma$. The equivalence restricts to the subcategories $\mathsf{Fdim}(\widetilde{B})$ and $\mathsf{Fdim}(B')^\Gamma$ so descends to the quotient categories QGr.

Second equivalence. By [5, Thm. 3.12], $\mathsf{QGr}(B) \equiv \mathsf{Qcoh}(\mathcal{O}_E)$. Since $\mathcal{B} = \mathcal{O}_E \otimes M_2(k)$, Morita equivalence lifts this to

$$\mathsf{QGr}(B') \equiv \mathsf{Qcoh}(\mathcal{B}). \tag{8-4}$$

Note that Γ acts in the same way on the $M_2(k)$ tensorands in $B' = B \otimes M_2(k)$ and $\mathcal{B} = \mathcal{O}_E \otimes M_2(k)$. This observation together with the precise description of the equivalence $\mathsf{QGr}(B) \equiv \mathsf{Qcoh}(E)$ from [5, Thm. 3.12] shows that (8-4) intertwines the Γ-actions on the two categories. This implies the desired result that it lifts to an equivalence between the respective categories of Γ-equivariant objects.

Third equivalence. Under the equivalences in Proposition 8.3, $\mathcal{B} \in \mathsf{Qcoh}(E)^\Gamma$ corresponds to $\mathcal{B}^\Gamma \in \mathsf{Qcoh}(E/E[2])$. Because those equivalences are monoidal they implement an equivalence between the category of $\mathcal{B}$-modules in $\mathsf{Qcoh}(E)^\Gamma$ and the category of $\mathcal{B}^\Gamma$-modules in $\mathsf{Qcoh}(E/E[2])$.

Fourth equivalence. Because $\rho : E \to E/E[2]$ is étale and $\rho^*(\mathcal{B}^\Gamma) \cong M_2(\mathcal{O}_E)$, $\mathcal{B}^\Gamma$ is a sheaf of Azumaya algebras on $E/E[2]$. The fourth equivalence now follows from Morita equivalence and the fact that $\mathcal{B}^\Gamma$ is Azumaya and hence (because we are working over an algebraically closed field) of the form $\mathcal{E}nd(\mathcal{V})$ for some vector bundle $\mathcal{V}$ on $E/E[2]$. $\square$

Remark 8.4. The proof of Theorem 8.1 also shows that the degree-shift functor on the left-hand side of the equivalence $\mathsf{QGr}(\widetilde{B}) \equiv \mathsf{Qcoh}(E/E[2])$ is identified with (twisting by) translation by the image of τ in $E/E[2]$ on the right-hand side.

We can actually find an explicit vector bundle $\mathcal{V}$ on $E/E[2]$ such that $\mathcal{B}^\Gamma \cong \mathcal{E}nd(\mathcal{V})$.

Proposition 8.5. *Let $\mathcal{V}$ be the unique non-split extension $0 \longrightarrow \mathcal{O}_{E/E[2]} \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_{E/E[2]} \longrightarrow 0$. There is an isomorphism of $\mathcal{O}_{E/E[2]}$-algebras $\mathcal{B}^\Gamma \cong \mathcal{E}nd(\mathcal{V})$.*

Proof. We already know that $\mathcal{B}^\Gamma$ is trivial Azumaya; hence $\mathcal{B}^\Gamma \cong \mathcal{E}nd(\mathcal{U})$ for some rank 2 vector bundle $\mathcal{U}$. By Atiyah's classification of vector bundles on elliptic curves, $\mathcal{U}$ is either decomposable or isomorphic to $\mathcal{V} \otimes \mathcal{L}$ for some $\mathcal{L} \in \operatorname{Pic}(E/E[2])$. If $\mathcal{U}$ is decomposable, then the $\mathcal{O}_{E/E[2]}$-module $\mathcal{B}^\Gamma$ contains two copies of $\mathcal{O}_{E/E[2]}$ as direct summands, whence $\dim H^0(\mathcal{B}^\Gamma) \geqslant 2$. Since $\dim H^0(\mathcal{B}^\Gamma) = \dim H^0(\mathcal{B})^\Gamma = 1$, we must have $\mathcal{B}^\Gamma \cong \mathcal{E}nd(\mathcal{V} \otimes \mathcal{L}) \cong \mathcal{E}nd(\mathcal{V})$. $\square$

8.3. $E/E[2]$ **is a closed subvariety of** $\operatorname{Proj}_{nc}(\widetilde{Q})$**.** The title of this subsection is made precise in the following way. In [44, §3.4], a subcategory $\mathcal{B}$ of an abelian category $\mathcal{D}$ is said to be closed if it is closed under subquotients and the inclusion functor $i_* : \mathcal{B} \to \mathcal{D}$ is fully faithful and has a left adjoint i^* and a right adjoint $i^!$. In [32, Thm. 1.2], which corrects an error in [36], it is shown that if J is a two-sided ideal in an $\mathbb{N}$-graded k-algebra A, then the inclusion functor $\mathsf{Gr}(A/J) \to \mathsf{Gr}(A)$ induces a fully faithful functor $i_* : \mathsf{QGr}(A/J) \to \mathsf{QGr}(A)$ whose essential image is closed in the sense of [44, §3.4]. In particular, since $\widetilde{B}$ is a quotient of $\widetilde{Q}$, this result in conjunction with Theorem 8.1 shows that the essential image of the composition $\mathsf{Qcoh}(E/E[2]) \to \mathsf{QGr}(\widetilde{B}) \to \mathsf{QGr}(\widetilde{Q})$ is closed in the sense of [44, §3.4].

8.4. **Fat point modules for** $\widetilde{B}$**.** Let $p \in E$. Let $p^\perp \subset Q_1$ be the subspace of Q_1 vanishing at p. We call $M_p := Q/Qp^\perp$ the point module associated to p. We view k^2 as a left $M_2(k)$-module in the natural way. Then $M_p \otimes k^2$ is a left $Q \otimes M_2(k)$-module and hence a left $\widetilde{Q}$-module.

Since (Ω, Ω') annihilates M_p, $M_p \otimes k^2$ is a $\widetilde{B}$-module.

Lemma 8.6. *If $p \in E$, then at most one of $\{x_0, x_1, x_2, x_3\}$ vanishes at p.*

Proof. Suppose $x_r(p) = x_s(p) = 0$ and $r \neq s$. Let $t \in \{0,1,2,3\} - \{r,s\}$. There are non-zero scalars λ, μ, ν such that $\lambda x_r^2 + \mu x_s^2 + \nu x_t^2$ vanishes on E so $x_t(p) = 0$ also. But $x_0^2 + x_1^2 + x_2^2 + x_3^2$ vanishes on E, so it would follow that $x_j(p) = 0$ for all j. That is absurd. $\square$

Proposition 8.7. *Let $p \in E$. If $m \otimes v$ is a non-zero element in $(M_p \otimes k^2)_n$, then we have $\widetilde{Q}(m \otimes v) \supseteq (M_p \otimes k^2)_{\geq n+1}$. In particular, every quotient of $M_p \otimes k^2$ by a non-zero graded $\widetilde{Q}$-submodule has finite dimension.*

Proof. Let N be a non-zero graded $\widetilde{Q}$-submodule of $M_p \otimes k^2$. Let $e_n \otimes v$ be a non-zero element in N where $\{e_n\}$ is a basis for the degree-n component of M_p and $v \in k^2 - \{0\}$.

If $p+n\tau=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)$ with respect to the coordinates $x_0,\dots,x_3$, then there is a basis $\{e_{n+1}\}$ for the degree-$(n+1)$ component of M_p such that $x_je_n=\lambda_je_{n+1}$ for $j=0,\dots,3$.

By Lemma 8.6, at least three elements x_j, $j=0,\dots,3$, are non-vanishing at $p+n\tau$.

Suppose, for the sake of argument, that this is the case for $j\geq 1$. Then $\lambda_1^{-1}x_1e_n=\lambda_2^{-1}x_2e_n=\lambda_3^{-1}e_n=e_{n+1}$ or, more precisely,

$$(kx_1\otimes q_1\,+\,kx_2\otimes q_2\,+\,kx_3\otimes q_3)(e_n\otimes v)=e_{n+1}\otimes(kq_1+kq_2+kq_3)v.$$

The set $\{(\lambda,\mu,\nu)\in\mathbb{P}^2\mid \operatorname{rank}(\lambda q_1+\mu q_2+\nu q_3)\leq 1\}$ is the non-degenerate conic $\lambda^2+\mu^2+\nu^2=0$. Since it doesn't contain a line, every 2-dimensional subspace of $kq_1+kq_2+kq_3$ contains an invertible matrix; hence $kq_1+kq_2+kq_3$ does not contain the subspace of $M_2(k)$ that annihilates v; hence $(kq_1+kq_2+kq_3)v=k^2$. Thus, in this case $\widetilde{Q}_1(e_n\otimes v)=e_{n+1}\otimes k^2$. The other cases are similar: the points $(\lambda,\mu,\nu)\in\mathbb{P}^2$ where $\operatorname{rank}(\lambda q_0+\mu q_1+\nu q_2)\leq 1$ form a non-degenerate conic, etc.

Thus, $\widetilde{Q}_1(e_n\otimes v)=e_{n+1}\otimes k^2$ in all cases. It follows by induction on n that $\widetilde{Q}(e_n\otimes v)\supseteq(M_p)_{\geq n+1}\otimes k^2$. This concludes the proof. □

Remark 8.8. Over the complex numbers, the fact that

$$(kx_1\otimes q_1\,+\,kx_2\otimes q_2\,+\,kx_3\otimes q_3)(e_n\otimes v) \tag{8-5}$$

is 2-dimensional can be proved as a consequence of the celebrated Borsuk-Ulam theorem (see e.g. [19, Theorem 2.1.1]).

To see this, let us preserve the notation in the proof and assume as before that x_j, $j\geq 1$, do not annihilate e_n. Now, if (8-5) were 1-dimensional, then as (r_1,r_2,r_3) ranges over the 2-sphere $\mathbb{S}^2$ the map

$$\mathbb{S}^2\ni(r_1,r_2,r_3)\mapsto\sum_{j\geq 1}r_j\lambda_j^{-1}(x_j\otimes q_j)(e_n\otimes v)$$

has its image contained in $\mathbb{C}\cong\mathbb{R}^2$. Since this image does not contain $0\in\mathbb{R}^2$, we have a nowhere-vanishing continuous odd map $\mathbb{S}^2\to\mathbb{R}^2$, in contradiction to [19, Thm. 2.1.1, BU1b].

Corollary 8.9. *Every simple object in* $\mathsf{QGr}(\widetilde{B})$ *is isomorphic to* $\pi^*(M_p\otimes k^2)$ *for some* $p\in E$. *Under the equivalence in Theorem* 8.1, $\pi^*(M_p\otimes k^2)$ *corresponds to* $\mathcal{O}_{p+E[2]}\in\mathsf{Qcoh}(E/E[2])$.

Proof. We first define a set function $r:E\to E/E[2]$. Let $p\in E$. Because $M_p\otimes k^2$ is a 1-critical $\widetilde{B}$-module, $\pi^*(M_p\otimes k^2)$ is a simple object in $\mathsf{QGr}(\widetilde{B})$. The equivalence in Theorem 8.1 sends $\pi^*(M_p\otimes k^2)$ to a simple object in $\textsc{Qcoh}(E/E[2])$; i.e., to a skyscraper sheaf $\mathcal{O}_{r(p)}$ for a unique point $r(p)\in E/E[2]$.

We will show that r is surjective by showing that it agrees with the quotient map $\rho:E\to E/E[2]$ (notation as in the proof of Theorem 8.1) by tracing through the equivalences in (8-1). To do that we must prove that $r(p)=p+E[2]$ or, equivalently, that the support of $\mathcal{O}_{r(p))}$ is $p+E[2]$. Equivalently, this entails showing that the support of $\rho^*\mathcal{O}_{r(p)}\in\mathsf{Qcoh}(E)$ is the $E[2]$-orbit of p. Certainly the support of $\rho^*\mathcal{O}_{r(p)}$ is *some* $E[2]$-orbit, so it suffices to prove that

$$p\in\operatorname{supp}(\rho^*\mathcal{O}_{r(p)}). \tag{8-6}$$

Using the fact that the third equivalence in (8-1) is implemented by ρ^*, note that by the definition of $r(p)$ the object of $\mathsf{QGr}(B')^\Gamma$ corresponding to $\rho^*\mathcal{O}_{r(p)}$ is the image of $B' \otimes_{\widetilde{B}} (M_p \otimes k^2)$ in $\mathsf{QGr}(B')$. The latter surjects onto the image of $M_p \otimes k^2$ in $\mathsf{QGr}(B')$, and hence, identifying

$$\mathsf{QGr}(B') \equiv \mathsf{Qcoh}(E) \quad \text{and} \quad \mathsf{QGr}(B')^\Gamma \equiv \mathsf{Qcoh}(E)^\Gamma,$$

the Γ-equivariant $\mathcal{O}_E$-module $\rho^*\mathcal{O}_{r(p)}$ surjects onto the skyscraper sheaf $\mathcal{O}_p$, meaning that we indeed have (8-6). This finishes the proof. □

The previous result is the reason that $M_p \otimes k^2$ is called a *fat point module* for $\widetilde{Q}$: "point" because in algebraic geometry simple objects in $\mathsf{Qcoh}(X)$ correspond to closed points, "fat" because $\operatorname{Hom}_{\mathsf{QGr}(\widetilde{Q})}(\widetilde{Q}, \pi^*(M_p \otimes k^2)) = 2$, not 1.

Note moreover that in the course of the proof of Corollary 8.9 we have identified the map $E \to E/E[2]$ defined by

$$E \ni p \mapsto M_p \otimes k^2$$

as being precisely the surjection modulo 2-torsion. Consequently, the classes in $\mathsf{QGr}(\widetilde{B})$ (or $\widetilde{Q}$) for $M_p \otimes k^2$ only depend on the class of p modulo $E[2]$. In Proposition 8.10 we lift this observation to $\mathsf{Gr}(\widetilde{Q})$.

Proposition 8.10. *If $\omega \in E[2]$ and $p \in E$, then there is an isomorphism of $\widetilde{Q}$-modules*

$$M_p \otimes k^2 \;\cong\; M_{p+\omega} \otimes k^2.$$

Proof. Write $E[2] = \{o, \xi_1, \xi_2, \xi_3\}$. If $\omega = o$ the identity map is an isomorphism. Fix $i \in \{1, 2, 3\}$.

Let $\{e_n \mid n \geq 0\}$ be a homogeneous basis for M_p with $\deg(e_n) = n$. For each n, let $\xi_{nj} \in k$, $j = 0, 1, 2, 3$, be the unique scalars such that

$$x_j e_n = \xi_{nj} e_{n+1}.$$

Thus, $(\xi_{n0}, \xi_{n1}, \xi_{n2}, \xi_{n3}) = p + n\tau$. Let $\xi'_{n0} = \xi_{n0}$, $\xi'_{ni} = \xi_{ni}$, and $\xi'_{nj} = -\xi_{nj}$ when $j \in \{1, 2, 3\} - \{i\}$. Therefore $p + n\tau + \xi_i = (\xi'_{n0}, \xi'_{n1}, \xi'_{n2}, \xi'_{n3})$. Let $\{f_n \mid n \geq 0\}$ be the unique homogeneous basis for $M_{p+\xi_i}$ with $\deg(f_n) = n$ such that $x_j f_n = \xi'_{nj} f_{n+1}$ for $j = 0, 1, 2, 3$.

Define $\varphi_i : M_p \otimes k^2 \longrightarrow M_{p+\xi_i} \otimes k^2$ by $\varphi_i(e_n \otimes v) := f_n \otimes q_i v$. It follows that

$$\varphi_i\big(y_j \cdot (e_n \otimes v)\big) = \varphi_i(x_j e_n \otimes q_j v) = \varphi_i(\xi_j e_{n+1} \otimes q_j v) = \xi_j f_{n+1} \otimes q_i q_j v$$

and

$$y_j \cdot \varphi_i(e_n \otimes v) = y_j \cdot (f_n \otimes q_i v) = x_j f_{n+1} \otimes q_j q_i v) = \xi'_j f_{n+1} \otimes q_j q_i v.$$

For all j, $\xi_j f_{n+1} \otimes q_i q_j v = \xi'_j f_{n+1} \otimes q_j q_i v$ because

- if $j \in \{0, i\}$, then $\xi_j = \xi'_j$ and $q_i q_j = q_j q_i$;
- if $j \in \{1, 2, 3\} - \{i\}$, then $\xi_j = -\xi'_j$ and $q_i q_j = -q_j q_i$.

Therefore $\varphi_i\big(y_j \cdot (e_n \otimes v)\big) = y_j \cdot \varphi_i(e_n \otimes v)$ for $j = 0, 1, 2, 3$. This proves that φ_i is a homomorphism of graded $\widetilde{Q}$-modules. It is obviously bijective, so the proof is complete. ☐

8.5. $\widetilde{B}$ **is a prime ring.** Davies [9, Cor. 5.3.21] proved that $\widetilde{B}$ is a prime ring when τ has infinite order [9, Hyp. 5.0.2]. We use a different method to prove the result without any restriction on τ.

Proposition 8.11. *Let I_1 and I_2 be graded ideals in an $\mathbb{N}$-graded left and right noetherian k-algebra A. Suppose there is a projective scheme X and an equivalence of categories $\Phi : \mathsf{QGr}(A) \to \mathrm{Qcoh}(X)$. By [32], there are functors α_{1*} and α_{2*} and closed subschemes $Z_1, Z_2 \subseteq X$ such that the essential image of $\Phi\alpha_{i*}$ is equal to $\mathrm{Qcoh}(Z_i)$, and there is a commutative diagram*

(8-7)
$$\begin{array}{ccccc}
\mathsf{Gr}(A/I_1) & \xrightarrow{f_{1*}} & \mathsf{Gr}(A) & \xleftarrow{f_{2*}} & \mathsf{Gr}(A/I_2) \\
\big\downarrow{\scriptstyle \pi_1^*} & & \big\downarrow{\scriptstyle \pi^*} & & \big\downarrow{\scriptstyle \pi_2^*} \\
\mathsf{QGr}(A/I_1) & \xrightarrow[\alpha_{1*}]{} & \mathsf{QGr}(A) & \xleftarrow[a_{2*}]{} & \mathsf{QGr}(A/I_2) \\
& & \big\downarrow{\scriptstyle \Phi} & & \\
& & \mathrm{Qcoh}(X) & &
\end{array}$$

in which $f_{i*} : \mathsf{Gr}(A/I_i) \to \mathsf{Gr}(A)$, $i = 1, 2$, are the natural inclusion functors, and π_1^*, π_2^*, and π^* denote the quotient functors. If $I_1 \cap I_2 = 0$ and X is reduced, then $Z_1 \cup Z_2 = X$.

Proof. Let $\mathcal{O}_x$ be the skyscraper sheaf at a closed point $x \in X$ and let M be an A-module such that $\Phi\pi^*M \cong \mathcal{O}_x$ and $\pi^*(M/N) = 0$ for all non-zero $N \subseteq M$. If $I_2M = 0$, then $\mathcal{O}_x \cong \Phi i_{2*}\pi_2^*M$ so $x \in Z_2$. On the other hand, suppose $I_2M \neq 0$. Then $\pi^*(M/I_2M) = 0$ so $\pi^*(I_2M) \cong \mathcal{O}_x$. Since $I_1I_2M = 0$, $\pi^*(I_2M) = \pi^*f_{1*}(I_2M) = i_{1*}\pi_1^*M$, which implies that $i_{1*}\pi_1^*M \cong \mathcal{O}_x$. Hence $x \in Z_1$.

Thus, every closed point of X belongs to $Z_1 \cup Z_2$. The proposition now follows from the fact that X is reduced. ☐

Theorem 8.12. *Let A be a connected, $\mathbb{N}$-graded, left and right noetherian k-algebra. Suppose there is a projective scheme X and an equivalence of categories $\Phi : \mathsf{QGr}(A) \to \mathrm{Qcoh}(X)$. If A is semiprime and X is reduced and irreducible, then A is a prime ring.*

Proof. Suppose the result is false. Then there are non-zero elements x and y such that $xAy = 0$. If x_m and y_n are the top-degree components of x and y, then $x_mAy_n = 0$. Let $I_1 = Ax_mA$ and $I_2 = Ay_nA$. Then I_1 and I_2 are graded ideals such that $I_1I_2 = 0$. Since $(I_1 \cap I_2)^2 \subseteq I_1I_2$, the fact that A is semiprime implies that $I_1 \cap I_2 = 0$. Hence $Z_1 \cup Z_2 = X$. But X is irreducible, so either $Z_1 = X$ or $Z_2 = X$.

Without loss of generality suppose that $Z_1 = X$. Then the functor i_{1*} : $\mathsf{QGr}(A/I_1) \to \mathsf{QGr}(A)$ is an equivalence. In particular, there is a module $M \in \mathsf{Gr}(A/I_1)$ such that $\pi A \cong i_{1*}\pi_1 M = \pi f_{1*}M$. Hence, if ω is the right adjoint to π constructed by Gabriel, $\omega\pi A \cong \omega\pi f_{1*}M$. By Step 2 in the proof of [32, Thm. 1.2], $\omega\pi f_{1*}M \cong f_{1*}\omega'\pi' M$ where ω' is right adjoint to π'. It follows that I_1 annihilates $\omega\pi A$.

There is an exact sequence $0 \to T \to A \to \omega\pi A$ where T is the largest finite dimensional submodule of A. Since $A_0 = k$, $T \subseteq A_{\geq 1}$. It follows that $T^n = 0$ for $n \gg 0$. But A is semiprime, so $T = 0$. Therefore I_1 annihilates A. Hence $I_1 = 0$. □

Corollary 8.13. *$\widetilde{B}$ is a prime ring.*

Proof. As observed in [8, Cor. 5.1.8], because B is a domain, $B \otimes M_2(k)$ is a prime ring, so [20, Cor. 1.5(1)] shows that $(B \otimes M_2(k))^\Gamma$, which is $\widetilde{B}$, is a semiprime ring. Therefore Theorems 8.1 and 8.12 imply that $\widetilde{B}$ is a prime ring. □

8.5.1. *Remark.* The hypothesis in Theorem 8.12 that the algebra A is connected was needed to show that A does not contain a non-zero left ideal of finite dimension. For $\widetilde{B}$, one can prove that without appealing to the fact that $\widetilde{B}$ is connected. Since $\widetilde{B} = \widetilde{Q}/(\Theta, \Theta')$ where Θ, Θ' is a regular sequence on $\widetilde{Q}$ of length 2, the projective dimension of $\widetilde{B}$ as a left $\widetilde{Q}$-module is 2. Hence, by [17, Prop. 2.1(e)], $\widetilde{B}$ does not contain a non-zero left ideal of finite dimension.

8.5.2. The twisted homogeneous coordinate ring of a reduced and irreducible variety, in particular $B(E, \tau, \mathcal{L})$, is a domain.

Proposition 8.14. *$\widetilde{B}$ is not a domain. In particular, in $\widetilde{B}$,* $0 = y_0^2 + y_1^2 + y_2^2 + y_3^2 =$

$$(y_0 - y_1 - y_2 - y_3)^2 = (y_0 - y_1 + y_2 + y_3)^2 = (y_0 + y_1 - y_2 + y_3)^2 = (y_0 + y_1 + y_2 - y_3)^2.$$

Proof. This is a straightforward calculation: $(y_0 - y_1 - y_2 - y_3)^2$ equals

$$y_0^2 + y_1^2 + y_2^2 + y_3^2 - \sum_{i=1}^{3} (y_0 y_i + y_i y_0 - y_j y_k - y_k y_j),$$

where (i, j, k) is a cyclic permutation of $1, 2, 3$. But $y_0 y_i + y_i y_0 = y_j y_k + y_k y_j$ when (i, j, k) is a cyclic permutation of $1, 2, 3$, and $y_0^2 + y_1^2 + y_2^2 + y_3^2 = -\Omega$, which is zero in $\widetilde{B}$. Similar calculations show that the squares of the other 3 elements are zero in $\widetilde{B}$. Alternatively, one can use the fact that Γ acts as automorphisms of $\widetilde{B}$ and these four elements in $\widetilde{B}_1$ form a Γ-orbit. □

9. Point modules for $\widetilde{Q}$

A point module for a connected graded algebra A is a graded left A-module M such that $M = AM_0$ and $\dim_k(M_i) = 1$ for all $i \geq 0$. The importance of point modules is that they are simple objects in $\mathsf{QGr}(A)$.

9.1. Suppose M is a point module for $\widetilde{Q}$. Its degree-zero component, M_0, is annihilated by a 3-dimensional subspace of $\widetilde{Q}_1$. That 3-dimensional subspace determines and is determined by a point in $\mathbb{P}^3$, its vanishing locus. We will show that the only points in $\mathbb{P}^3$ that arise in this way are those in Table 3, where the coordinates are written with respect to the coordinate system (y_0, y_1, y_2, y_3). We write $\mathfrak{P}$ for this set of points.

Recall that a, b, c, i are fixed square roots of $\alpha, \beta, \gamma, -1$.

Table 3. The points in $\mathfrak{P}$

$\mathfrak{P}_\infty$	$\mathfrak{P}_0$	$\mathfrak{P}_1$	$\mathfrak{P}_2$	$\mathfrak{P}_3$	Γ
$(1,0,0,0)$	$(1,1,1,1)$	$(bc,-i,-ib,-c)$	$(ac,-a,-i,-ic)$	$(ab,-ia,-b,-i)$	
$(0,1,0,0)$	$(1,1,-1,-1)$	$(bc,-i,ib,c)$	$(ac,-a,i,ic)$	$(ab,-ia,b,i)$	γ_1
$(0,0,1,0)$	$(1,-1,1,-1)$	$(bc,i,-ib,c)$	$(ac,a,-i,ic)$	$(ab,ia,-b,i)$	γ_2
$(0,0,0,1)$	$(1,-1,-1,1)$	$(bc,i,ib,-c)$	$(ac,a,i,-ic)$	$(ab,ia,b,-i)$	γ_3

The points in $\mathfrak{P}_\infty$ are fixed by Γ, and every other $\mathfrak{P}_i$ is a Γ-orbit. If $\mathbf{u}$ is the topmost point in one of the columns $\mathfrak{P}_i$, $i = 0, 1, 2, 3$, the other points in that column are $\gamma_1(\mathbf{u})$, $\gamma_2(\mathbf{u})$, and $\gamma_3(\mathbf{u})$, in that order.

We define a permutation θ of $\mathfrak{P}$ with the property $\theta^2 = \mathrm{id}_{\mathfrak{P}}$ by

$$\theta(\mathbf{u}) := \begin{cases} \mathbf{u} & \text{if } \mathbf{u} \in \mathfrak{P}_\infty \cup \mathfrak{P}_0, \\ \gamma_i(\mathbf{u}) & \text{if } \mathbf{u} \in \mathfrak{P}_i,\ i = 1, 2, 3. \end{cases} \tag{9-1}$$

9.2. **The point scheme, $\mathcal{P}$.** Let V denote the linear span of y_0, y_1, y_2, y_3. The defining relations for $\widetilde{Q}$ belong to $V^{\otimes 2}$. Non-zero elements in $V^{\otimes 2}$ are forms of bi-degree $(1, 1)$ on $\mathbb{P}(V^*) \times \mathbb{P}(V^*) = \mathbb{P}^3 \times \mathbb{P}^3$. Let

$\mathcal{P} :=$ the subscheme of $\mathbb{P}^3 \times \mathbb{P}^3$ where the quadratic relations for $\widetilde{Q}$ vanish.

We will show that $\mathcal{P}$ is a reduced scheme consisting of 20 points.

Lemma 9.1. *If $(\mathbf{u}, \mathbf{v}) \in \mathcal{P}$, then $(\mathbf{v}, \mathbf{u}) \in \mathcal{P}$.*

Proof. As noted in Remark 6.2, there is an anti-automorphism of $\widetilde{Q}$ given by $y_i \mapsto -y_i$ for $i = 0, 1, 2, 3$. Thus, if $r = \sum \mu_{ij} y_i \otimes y_j$ is a quadratic relation for $\widetilde{Q}$, so is $r' = \sum \mu_{ij} y_j \otimes y_i$. Obviously, r vanishes at $(\mathbf{u}, \mathbf{v}) \in \mathbb{P}^3 \times \mathbb{P}^3$ if and only if r' vanishes at $(\mathbf{v}, \mathbf{u})$. The lemma now follows from the fact that $\mathcal{P}$ is the zero locus of the set of quadratic relations for $\widetilde{Q}$. □

9.2.1. *From point modules to points in $\mathcal{P}$.* Suppose M is a point module for $\widetilde{Q}$. Let $e_0, e_1, \ldots$ be a homogeneous basis for M with $\deg(e_n) = n$. Define $\lambda_{nj} \in k$ by the requirement that $y_j e_n = \lambda_{nj} e_{n+1}$. Because M is a point module, for each n, some λ_{nj} is non-zero. The point $p_n := (\lambda_{n0}, \lambda_{n1}, \lambda_{n2}, \lambda_{n3}) \in \mathbb{P}^3$ does not depend on the basis $\{e_n\}_{n \geq 0}$. Since $y_j(p_n) = \lambda_{nj}$, the p_n's belong to $\mathbb{P}(V^*)$.

Because M is a $\widetilde{Q}$-module, each quadratic relation $r \in V^{\otimes 2}$ has the property that $r \cdot e_n = 0$ for all n. Thus, r viewed as a (1,1) form on $\mathbb{P}^3 \times \mathbb{P}^3$ vanishes at (p_{n+1}, p_n). Hence $(p_{n+1}, p_n) \in \mathcal{P}$.

9.3. The point modules $M_{\mathbf{u}}$, $\mathbf{u} \in \mathfrak{P}$.

Proposition 9.2. *Let $\mathbf{u} \in \mathfrak{P}$. Let θ be the function defined at (9-1), and for each $n \geq 0$ write $\theta^n(\mathbf{u}) = (\lambda_{n0}, \lambda_{n1}, \lambda_{n2}, \lambda_{n3})$, where the coordinates are written with respect to (y_0, y_1, y_2, y_3). There is a point module, $M_{\mathbf{u}}$, with homogeneous basis $e_0, e_1, \ldots$, $\deg(e_n) = n$, and action*

$$y_j e_n := \lambda_{nj} e_{n+1}. \tag{9-2}$$

These 20 point modules are pairwise non-isomorphic.

Proof. It is clear that $M_{\mathbf{u}}$ is generated by e_0, so it suffices to show that (9-2) really does define a left $\widetilde{Q}$-module. To do this we must show that every relation for $\widetilde{Q}$ annihilates every e_n. In other words, we must show that every quadratic relation for $\widetilde{Q}$, when viewed as a form of bi-degree $(1,1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$, vanishes at $\big((\lambda_{n+1,0}, \lambda_{n+1,1}, \lambda_{n+1,2}, \lambda_{n+1,3}), (\lambda_{n0}, \lambda_{n1}, \lambda_{n2}, \lambda_{n3})\big) \in \mathbb{P}^3 \times \mathbb{P}^3$ for all $n \geq 0$; i.e., it suffices to show that these forms vanish at $(\theta(\mathbf{v}), \mathbf{v})$ for all $\mathbf{v} \in \mathfrak{P}$. Since $\theta^2 = 1$, this is equivalent to showing they vanish at $(\mathbf{v}, \theta(\mathbf{v}))$ for all $\mathbf{v} \in \mathfrak{P}$.

The relations for $\widetilde{Q}$ are the entries in the matrix $\mathsf{M}_1\mathsf{y}$ where

$$\mathsf{M}_1 = \begin{pmatrix} -y_1 & y_0 & \alpha y_3 & -\alpha y_2 \\ -y_2 & -\beta y_3 & y_0 & \beta y_1 \\ -y_3 & \gamma y_2 & -\gamma y_1 & y_0 \\ y_1 & y_0 & -y_3 & -y_2 \\ y_2 & -y_3 & y_0 & -y_1 \\ y_3 & -y_2 & -y_1 & y_0 \end{pmatrix} \quad \text{and} \quad \mathsf{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

We must therefore show that $\mathsf{M}_1(\mathbf{v})\theta(\mathbf{v})^{\mathsf{T}} = 0$ for all $\mathbf{v} \in \mathfrak{P}$. This is a routine calculation. We give one example to illustrate the process.

Let $\mathbf{v} = (\delta_0, \delta_1, \delta_2, \delta_3) \in \mathfrak{P}_1$. Then $\theta(\mathbf{v}) = \gamma_1(\mathbf{v}) = (\delta_0, \delta_1, -\delta_2, -\delta_3)$, so

$$\mathsf{M}_1(\mathbf{v})\theta(\mathbf{v})^{\mathsf{T}} = \begin{pmatrix} -\delta_1 & \delta_0 & \alpha\delta_3 & -\alpha\delta_2 \\ -\delta_2 & -\beta\delta_3 & \delta_0 & \beta\delta_1 \\ -\delta_3 & \gamma\delta_2 & -\gamma\delta_1 & \delta_0 \\ \delta_1 & \delta_0 & -\delta_3 & -\delta_2 \\ \delta_2 & -\delta_3 & \delta_0 & -\delta_1 \\ \delta_3 & -\delta_2 & -\delta_1 & \delta_0 \end{pmatrix} \begin{pmatrix} \delta_0 \\ \delta_1 \\ -\delta_2 \\ -\delta_3 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ -\delta_0\delta_2 - \beta\delta_3\delta_1 \\ -\delta_0\delta_3 + \gamma\delta_1\delta_2 \\ \delta_0\delta_1 + \delta_2\delta_3 \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to check that this 6×1 matrix is 0 for all $\mathbf{v} \in \mathfrak{P}_1$.

The annihilator of e_0 in $\widetilde{Q}_1$ is the subspace that vanishes at $\mathbf{u}$. Hence if $\mathbf{u}$ and $\mathbf{v}$ are different points of $\mathfrak{P}$, $M_{\mathbf{u}} \not\cong M_{\mathbf{v}}$. □

Theorem 9.3. *The 20 point modules $M_{\mathbf{u}}$, $\mathbf{u} \in \mathfrak{P}$, in Proposition 9.2 are all the $\widetilde{Q}$-point modules.*

Proof. Let M be a point module for $\widetilde{Q}$. Let $\{e_n \mid n \geq 0\}$ be a homogeneous basis for M with $\deg(e_n) = n$. Let p_n, $n \geq 0$, be the points in $\mathbb{P}^3$ determined by the procedure described in §9.2.1. Then $(p_{n+1}, p_n) \in \mathcal{P}$ for all $n \geq 0$. By Lemma 9.1, $(p_n, p_{n+1}) \in \mathcal{P}$. Thus, to prove the theorem it suffices to show that $\mathcal{P} = \big\{(\mathbf{u}, \theta(\mathbf{u})) \bigm| \mathbf{u} \in \mathfrak{P}\big\}$. This is what we do in Theorem 9.4 below. □

Theorem 9.4. *Let $\mathcal{P} \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ be the subscheme defined in §9.2. Then*

$$\mathcal{P} \;=\; \{(\mathbf{u},\mathbf{v}) \in \mathbb{P}^3 \times \mathbb{P}^3 \mid \mathsf{M}_1(\mathbf{u})\mathbf{v} = 0\} \;=\; \big\{\big(\mathbf{u},\theta(\mathbf{u})\big) \;\big|\; \mathbf{u} \in \mathfrak{P}\big\}.$$

In particular, $\mathcal{P}$ is the graph of the automorphism θ of $\mathfrak{P}$.

Proof. Let $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{P} \to \mathbb{P}^3$ denote the projections onto the first and second factors of $\mathbb{P}^3 \times \mathbb{P}^3$. We will show that $\mathrm{pr}_1(\mathcal{P}) = \mathfrak{P}$. Let $\mathbf{u} \in \mathrm{pr}_1(\mathcal{P})$. There is a point $\mathbf{v} \in \mathbb{P}^3$ such that $(\mathbf{u},\mathbf{v}) \in \mathcal{P}$, i.e., such that $\mathsf{M}_1(\mathbf{u})\mathbf{v} = 0$. This implies that $\mathrm{rank}(\mathsf{M}_1(\mathbf{u})) \le 3$. Thus the 4×4 minors of M_1 vanish at $\mathbf{u}$. We used SAGE [38] to compute these minors. After removing a common factor of 2, they are:

$$\begin{aligned}
&- b\gamma y_0 y_1^3 - \alpha\gamma y_0 y_1 y_2^2 + \beta\gamma y_1^2 y_2 y_3 + a\gamma y_2^3 y_3 - \alpha\beta y_0 y_1 y_3^2 + \alpha\beta y_2 y_3^3 - y_0^3 y_1 + y_0^2 y_2 y_3\\
&\qquad = (y_2y_3 - y_0y_1)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2),\\
&- \beta\gamma y_0 y_1^2 y_2 - \alpha\gamma y_0 y_2^3 + \beta\gamma y_1^3 y_3 + \alpha\gamma y_1 y_2^2 y_3 - \alpha\beta y_0 y_2 y_3^2 + \alpha\beta y_1 y_3^3 - y_0^3 y_2 + y_0^2 y_1 y_3\\
&\qquad = (y_1y_3 - y_0y_2)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2),\\
&\beta\gamma y_1^3 y_2 + \alpha\gamma y_1 y_2^3 - \beta\gamma y_0 y_1^2 y_3 - \alpha\gamma y_0 y_2^2 y_3 + \alpha\beta y_1 y_2 y_3^2 - \alpha\beta y_0 y_3^3 + y_0^2 y_1 y_2 - y_0^3 y_3\\
&\qquad = (y_1y_2 - y_0y_3)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2),\\
&- \alpha\beta y_1^2 y_3^2 + \alpha\beta y_2^2 y_3^2 - \beta y_0^2 y_1^2 - \alpha y_0^2 y_2^2 + \beta y_1^2 y_3^2 + \alpha y_2^2 y_3^2 - y_0^2 y_1^2 + y_0^2 y_2^2,\\
&- \alpha\beta y_1^2 y_2 y_3 + \alpha\beta y_2 y_3^3 + \beta y_0 y_1^3 - \alpha y_0^2 y_2 y_3 + \alpha y_2^3 y_3 - \beta y_0 y_1 y_3^2 + y_0^3 y_1 - y_0 y_1 y_2^2\\
&\qquad = (y_0y_1 - \alpha y_2y_3)(y_0^2 + \beta y_1^2 - y_2^2 - \beta y_3^2),\\
&- \alpha\beta y_1 y_2^2 y_3 + \alpha\beta y_1 y_3^3 - \alpha y_0 y_2^3 + \beta y_0^2 y_1 y_3 - \beta y_1^3 y_3 + \alpha y_0 y_2 y_3^2 + y_0^3 y_2 - y_0 y_1^2 y_2\\
&\qquad = (y_0y_2 + \beta y_1y_3)(y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2),\\
&\alpha\gamma y_1^2 y_2 y_3 - \alpha\gamma y_2^3 y_3 + \gamma y_0 y_1^3 - \gamma y_0 y_1 y_2^2 - \alpha y_0^2 y_2 y_3 + \alpha y_2 y_3^3 - y_0^3 y_1 + y_0 y_1 y_3^2,\\
&\qquad = (y_0y_1 + \alpha y_2y_3)(-y_0^2 + \gamma y_1^2 - \gamma y_2^2 + y_3^2),\\
&\alpha\gamma y_1^2 y_2^2 - \alpha\gamma y_2^2 y_3^2 - \gamma y_0^2 y_1^2 + \gamma y_1^2 y_2^2 - \alpha y_0^2 y_3^2 + \alpha y_2^2 y_3^2 + y_0^2 y_1^2 - y_0^2 y_3^2,\\
&\alpha\gamma y_1 y_2^3 - \alpha\gamma y_1 y_2 y_3^2 - \gamma y_0^2 y_1 y_2 + \gamma y_1^3 y_2 - \alpha y_0 y_2^2 y_3 + \alpha y_0 y_3^3 + y_0^3 y_3 - y_0 y_1^2 y_3\\
&\qquad = (y_0y_3 - \gamma y_1y_2)(y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2),\\
&\alpha y_0 y_1 y_2^2 + \alpha y_2^3 y_3 - \alpha y_0 y_1 y_3^2 - \alpha y_2 y_3^3 - y_0^3 y_1 + y_0 y_1^3 - y_0^2 y_2 y_3 + y_1^2 y_2 y_3\\
&\qquad = (y_0y_1 + y_2y_3)(-y_0^2 + y_1^2 + \alpha y_2^2 - \alpha y_3^2),\\
&- \beta\gamma y_1^3 y_3 + \beta\gamma y_1 y_2^2 y_3 + \gamma y_0 y_1^2 y_2 - \gamma y_0 y_2^3 + \beta y_0^2 y_1 y_3 - \beta y_1 y_3^3 - y_0^3 y_2 + y_0 y_2 y_3^2\\
&\qquad = (y_0y_2 - \beta y_1y_3)(-y_0^2 + \gamma y_1^2 - \gamma y_2^2 + y_3^2),\\
&- \beta\gamma y_1^3 y_2 + \beta\gamma y_1 y_2 y_3^2 - \gamma y_0^2 y_1 y_2 + \gamma y_1 y_2^3 - \beta y_0 y_1^2 y_3 + \beta y_0 y_3^3 - y_0^3 y_3 + y_0 y_2^2 y_3\\
&\qquad = (y_0y_3 + \gamma y_1y_2)(-y_0^2 - \beta y_1^2 + y_2^2 + \beta y_3^2),\\
&- \beta\gamma y_1^2 y_2^2 + \beta\gamma y_1^2 y_3^2 - \gamma y_0^2 y_2^2 + \gamma y_1^2 y_2^2 - \beta y_0^2 y_3^2 + \beta y_1^2 y_3^2 - y_0^2 y_2^2 + y_0^2 y_3^2,\\
&- \beta y_0 y_1^2 y_{-} \beta y_1^3 y_3 + \beta y_0 y_2 y_3^2 + \beta y_1 y_3^3 - y_0^3 y_2 + y_0 y_2^3 - y_0^2 y_1 y_3 + y_1 y_2^2 y_3\\
&\qquad = (y_0y_2 + y_1y_3)(-y_0^2 - \beta y_1^2 + y_2^2 + \beta y_3^2),\\
&\gamma y_1^3 y_2 - \gamma y_1 y_2^3 + \gamma y_0 y_1^2 y_3 - \gamma y_0 y_2^2 y_3 - y_0^2 y_1 y_2 - y_0^3 y_3 + y_1 y_2 y_3^2 + y_0 y_3^3\\
&\qquad = (y_0y_3 + y_1y_2)(-x_0^2 + \gamma y_1^2 - \gamma y_2^2 + y_3^2).
\end{aligned}$$

Some reorganization and changes of sign show that the linear span of the above 15 polynomials is the same as the linear span of the following 15 polynomials:

$$\begin{aligned}
&(y_2y_3-y_0y_1)(y_0^2+\beta\gamma y_1^2+\alpha\gamma y_2^2+\alpha\beta y_3^2),\\
&(y_1y_3-y_0y_2)(y_0^2+\beta\gamma y_1^2+\alpha\gamma y_2^2+\alpha\beta y_3^2),\\
&(y_1y_2-y_0y_3)(y_0^2+\beta\gamma y_1^2+\alpha\gamma y_2^2+\alpha\beta y_3^2),\\
&(y_0y_1+y_2y_3)(y_0^2-y_1^2-\alpha y_2^2+\alpha y_3^2),\\
&(y_0y_2+\beta y_1y_3)(y_0^2-y_1^2-\alpha y_2^2+\alpha y_3^2),\\
&(y_0y_3-\gamma y_1y_2)(y_0^2-y_1^2-\alpha y_2^2+\alpha y_3^2),\\
&(y_0y_1-\alpha y_2y_3)(y_0^2+\beta y_1^2-y_2^2-\beta y_3^2),\\
&(y_0y_2+y_1y_3)(y_0^2+\beta y_1^2-y_2^2-\beta y_3^2),\\
&(y_0y_3+\gamma y_1y_2)(y_0^2+\beta y_1^2-y_2^2-\beta y_3^2),\\
&(y_0y_1+\alpha y_2y_3)(y_0^2-\gamma y_1^2+\gamma y_2^2-y_3^2),\\
&(y_0y_2-\beta y_1y_3)(y_0^2-\gamma y_1^2+\gamma y_2^2-y_3^2),\\
&(y_0y_3+y_1y_2)(y_0^2-\gamma y_1^2+\gamma y_2^2-y_3^2),\\
&\alpha\beta y_1^2y_3^2-\alpha\beta y_2^2y_3^2+\beta y_0^2y_1^2-\beta y_1^2y_3^2+\alpha y_0^2y_2^2-\alpha y_2^2y_3^2+y_0^2y_1^2-y_0^2y_2^2,\\
&\beta\gamma y_1^2y_2^2-\beta\gamma y_1^2y_3^2+\gamma y_0^2y_2^2-\gamma y_1^2y_2^2+\beta y_0^2y_3^2-\beta y_1^2y_3^2+y_0^2y_2^2-y_0^2y_3^2,\\
&\alpha\gamma y_1^2y_2^2-\alpha\gamma y_2^2y_3^2+\alpha y_2^2y_3^2-\gamma y_0^2y_1^2+\gamma y_1^2y_2^2-\alpha y_0^2y_3^2+y_0^2y_1^2-y_0^2y_3^2.
\end{aligned}$$

The proof of Proposition 9.2 showed that $\mathsf{M}_1(\mathbf{u})\theta(\mathbf{u})^{\mathsf{T}}=0$ for all $\mathbf{u}\in\mathfrak{P}$, so these 15 polynomials vanish at the points in $\mathfrak{P}$. One can also check this directly by evaluating these quartic polynomials at $\mathbf{u}\in\mathfrak{P}$. For example, it is obvious that y_iy_j vanishes on $\mathfrak{P}_\infty$ if $i\neq j$, from which it immediately follows that all 15 polynomials vanish on $\mathfrak{P}_\infty$. As another example, $y_2y_3-y_0y_1$, $y_1y_3-y_0y_2$, and $y_1y_2-y_0y_3$ vanish on $\mathfrak{P}_0$, whence the first 3 of the 15 polynomials vanish on $\mathfrak{P}_0$; the other twelve polynomials belong to the ideal $(y_0^2-y_1^2,y_0^2-y_2^2,y_0^2-y_3^2)$, so they too vanish on $\mathfrak{P}_0$. As a final example, consider $\mathfrak{P}_2$. The first three quartics vanish on $\mathfrak{P}_2$ because $y_0^2+\beta\gamma y_1^2+\alpha\gamma y_2^2+\alpha\beta y_3^2$ does. The second three quartics vanish on $\mathfrak{P}_2$ because $y_0^2-y_1^2-\alpha y_2^2+\alpha y_3^2$ does. The third three quartics vanish on $\mathfrak{P}_2$ because the ideal $(y_0y_1-\alpha y_2y_3,y_0y_2+y_1y_3,y_0y_3+\gamma y_1y_2)$ does. The fourth three quartics vanish on $\mathfrak{P}_2$ because $y_0^2-\gamma y_1^2+\gamma y_2^2-y_3^2$ does. A calculation shows the last three quartics vanish on $\mathfrak{P}_2$.

Suppose these 15 quartics vanish at a point $\mathbf{u}\in\mathbb{P}^3$. To complete the proof we will show that $\mathbf{u}\in\mathfrak{P}$.

The determinant

$$\det\begin{pmatrix}1&\beta\gamma&\alpha\gamma&\alpha\beta\\1&-1&-\alpha&\alpha\\1&\beta&-1&-\beta\\1&-\gamma&\gamma&-1\end{pmatrix}\;=\;-(1+\alpha\beta+\beta\gamma+\gamma\alpha)^2$$

is non-zero: the hypothesis that $\alpha+\beta+\gamma+\alpha\beta\gamma=0$ implies that $1+\alpha\beta+\beta\gamma+\gamma\alpha=(1+\alpha)(1+\beta)(1+\gamma)$, which is non-zero because we are assuming that

$\{\alpha,\beta,\gamma\}\cap\{0,\pm1\}=\varnothing$. Because the determinant is non-zero the polynomials

$$\left\{\begin{array}{l} y_0^2+\beta\gamma y_1^2+\alpha\gamma y_2^2+\alpha\beta y_3^2,\\ y_0^2-y_1^2-\alpha y_2^2+\alpha y_3^2,\\ y_0^2+\beta y_1^2-y_2^2-\beta y_3^2,\\ y_0^2-\gamma y_1^2+\gamma y_2^2-y_3^2 \end{array}\right. \tag{9-3}$$

are linearly independent. Their linear span is therefore the same as that of $\{y_0^2, y_1^2, y_2^2, y_3^2\}$. Hence the common zero locus of the polynomials in (9-3) is empty, and at most three of them vanish at $\mathbf{u}$.

We now do some case-by-case analysis to show that $\mathbf{u}$ belongs to some $\mathfrak{P}_i$.

$\underline{\mathfrak{P}_\infty\cup\mathfrak{P}_0.}$ Suppose $\mathbf{u}$ is not in the zero locus of $y_0^2+\beta\gamma y_1^2+\alpha\gamma y_2^2+\alpha\beta y_3^2$. Then

$$y_0y_1-y_2y_3=y_0y_2-y_1y_3=y_0y_3-y_1y_2=0 \tag{9-4}$$

at $\mathbf{u}$. If one of the coordinate functions y_0,y_1,y_2,y_3 vanishes at $\mathbf{u}$, then three do, so

$$\mathbf{u}\ \in\ \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}\ =\ \mathfrak{P}_\infty. \tag{9-5}$$

If none of y_0,y_1,y_2,y_3 vanish at $\mathbf{u}$, then it follows from (9-4) that

$$\mathbf{u}\ \in\ \{(1,1,1,1),(1,1,-1,-1),(1,-1,1,-1),(1,-1,-1,1)\}\ =\ \mathfrak{P}_0.$$

$\underline{\mathfrak{P}_1.}$ Suppose $\mathbf{u}$ is not in the zero locus of $y_0^2-y_1^2-\alpha y_2^2+\alpha y_3^2$ and not in $\mathfrak{P}_\infty\cup\mathfrak{P}_0$. Then

$$y_0y_1+y_2y_3=y_0y_2+\beta y_1y_3=y_0y_3-\gamma y_1y_2=0 \tag{9-6}$$

at $\mathbf{u}$. If one of y_0,y_1,y_2,y_3 vanishes at $\mathbf{u}$, then three of them do, so $\mathbf{u}\in\mathfrak{P}_\infty$. This is not the case, so none of y_0,y_1,y_2,y_3 vanish at $\mathbf{u}$. Without loss of generality we can, and do, assume that $\mathbf{u}=(bc,y_1,y_2,y_3)$. It follows from (9-6) that $y_0^3(y_1y_2y_3)=\beta\gamma(y_1y_2y_3)^2$. Therefore $bc=y_1y_2y_3$. It also follows from (9-6) that $\beta\gamma y_1^2=\gamma y_2^2=-\beta y_3^2$. Some case-by-case analysis shows that

$$\mathbf{u}\ \in\ \{(bc,-i,ib,c),(bc,-i,-ib,-c),(bc,i,ib,-c),(bc,i,-ib,c)\}\ =\ \mathfrak{P}_1.$$

$\underline{\mathfrak{P}_2.}$ Suppose $\mathbf{u}$ is not in the zero locus of $y_0^2+\beta y_1^2-y_2^2-\beta y_3^2$ and not in $\mathfrak{P}_\infty\cup\mathfrak{P}_0$. Then

$$y_0y_1-\alpha y_2y_3=y_0y_2+y_1y_3=y_0y_3+\gamma y_1y_2=0 \tag{9-7}$$

at $\mathbf{u}$. As in the previous paragraph, $y_0y_1y_2y_3$ does not vanish at $\mathbf{u}$. Without loss of generality we can, and do, assume that $\mathbf{u}=(ac,y_1,y_2,y_3)$. The same sort of analysis as that in the previous paragraph shows that

$$\mathbf{u}\ \in\ \{(ac,a,-i,ic),(ac,a,i,-ic),(ac,-a,-i,-ic),(ac,-a,i,ic)\ =\ \mathfrak{P}_2.$$

$\underline{\mathfrak{P}_3.}$ Suppose $\mathbf{u}$ is not in the zero locus of $y_0^2-\gamma y_1^2+\gamma y_2^2-y_3^2$ and not in $\mathfrak{P}_\infty\cup\mathfrak{P}_0$. Then

$$y_0y_1+\alpha y_2y_3=y_0y_2-\beta y_1y_3=y_0y_3+y_1y_2=0 \tag{9-8}$$

at $\mathbf{u}$. Proceeding as before, we eventually see that

$$\mathbf{u}\ \in\ \{(ab,ia,b,-i),(ab,ia,-b,i),(ab,-ia,b,i),(ab,-ia,-b,-i)\}\ =\ \mathfrak{P}_3.$$

This completes the proof that $\mathrm{pr}_1(\mathcal{P})\subset\mathfrak{P}$. Thus $\mathrm{pr}_2(\mathcal{P})=\mathfrak{P}$.

By Lemma 9.1, $\mathrm{pr}_2(\mathcal{P})=\mathfrak{P}$ also. Since $\mathrm{pr}_2(\mathcal{P})$ does not contain a line, the rank of $\mathsf{M}_1(\mathbf{u})$ is 3 for all $\mathbf{u}\in\mathrm{pr}_1(\mathcal{P})$. Let $\mathbf{u}\in\mathfrak{P}$. Since $\mathsf{M}_1(\mathbf{u})\theta(\mathbf{u})^{\mathsf{T}}=0$, $\theta(\mathbf{u})^{\mathsf{T}}$ is the only $\mathbf{v} \in \mathbb{P}^3$ such that $\mathsf{M}_1(\mathbf{u})\mathbf{v}^\mathsf{T} = 0$. Hence $(\mathbf{u}, \theta(\mathbf{u}))$ is the only point in $\mathrm{pr}_1^{-1}(\mathbf{u})$. It follows that $\mathcal{P} = \{(\mathbf{u}, \theta(\mathbf{u})) \mid \mathbf{u} \in \mathfrak{P}\}$. □

Proposition 9.5. *The central element* $\Theta = y_0^2 + y_1^2 + y_2^2 + y_3^2$ *does not annihilate any point modules for* $\widetilde{Q}$. *Consequently,* $\widetilde{B}$ *has no point modules.*

Proof. Let $\mathbf{u} \in \mathfrak{P}$.

To describe the action of Θ on $M_\mathbf{u}$ we must fix a basis for $M_\mathbf{u}$. We pick a basis for $M_\mathbf{u}$ that is compatible with the entries in Table 3. To do this it is helpful, for a moment, to think of the entries in Table 3 as points in k^4. Suppose $\mathbf{u} = (\delta_0, \delta_1, \delta_2, \delta_3)$. Let e_0 be any non-zero element in $(M_\mathbf{u})_0$. Let e_1 be the unique element in $(M_\mathbf{u})_1$ such that $y_i e_0 = \delta_i e_1$ for $i = 0, 1, 2, 3$. Likewise, if $(\delta'_0, \delta'_1, \delta'_2, \delta'_3)$ is the entry in Table 3 for $\theta(\mathbf{u})$, there is a unique element $e_2 \in (M_\mathbf{u})_2$ such that $y_i e_1 = \delta'_i e_2$ for $i = 0, 1, 2, 3$.

If $\mathbf{u} \in \mathfrak{P}_\infty$, then $\Theta e_0 = e_2$. If $\mathbf{u} \in \mathfrak{P}_0$, then $\Theta e_0 = 4e_2$.

Let $\mathbf{u} = (bc, -i, -ib, -c) \in \mathfrak{P}_1$. Then $\theta(\mathbf{u}) = (bc, -i, ib, c)$. Therefore

$$\begin{aligned}
\Theta e_0 &= (y_0^2 + y_1^2 + y_2^2 + y_3^2)e_0 \\
&= (bcy_0 - iy_1 - iby_2 - cy_3)e_1 \\
&= \big((bc)^2 - 1 + b^2 - c^2\big)e_2 \\
&= (\beta - 1)(\gamma + 1)e_2.
\end{aligned}$$

Likewise, if $\mathbf{u} = (bc, i, -ib, c) \in \mathfrak{P}_1$, then $\theta(\mathbf{u}) = (bc, i, ib, -c)$, and a similar calculation shows that $\Theta e_0 = (\beta - 1)(\gamma + 1)e_2$. Thus, $\Theta e_0 = (\beta - 1)(\gamma + 1)e_2$ for all $\mathbf{u} \in \mathfrak{P}_1$.

Similar calculations show that $\Theta e_0 = (\alpha + 1)(\gamma - 1)e_2$ for all $\mathbf{u} \in \mathfrak{P}_2$. Finally, if $\mathbf{u} \in \mathfrak{P}_3$, then $\Theta e_0 = (\alpha - 1)(\beta + 1)e_2$. □

9.4. Not only do the relations for $\widetilde{Q}$ determine $\mathcal{P}$, but $\mathcal{P}$ determines the defining relations for $\widetilde{Q}$: the quadratic relations for $\widetilde{Q}$ are precisely the elements of $V^{\otimes 2}$ that vanish at $\mathcal{P}$. This is a consequence of the following remarkable result.

Theorem 9.6 (Shelton-Vancliff [29])**.** *Let* V *be a* 4*-dimensional vector space and let* $R \subseteq V^{\otimes 2}$ *be a* 6*-dimensional subspace. Let* TV *denote the tensor algebra on* V *and let* $\mathcal{P} \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ *be the scheme-theoretic zero locus of* R. *If* $\dim(\mathcal{P}) = 0$, *then*

$$R = \{f \in V^{\otimes 2} \mid f|_\mathcal{P} = 0\}.$$

9.5. There has been some interest in Artin-Schelter regular algebras with Hilbert series $(1-t)^{-4}$ that have only finitely many point modules [45], [28], [39], [40]. The interest arises because this phenomenon does not occur for Artin-Schelter regular algebras with Hilbert series $(1-t)^{-3}$; the point modules for the latter algebras are parametrized either by a cubic divisor in $\mathbb{P}^2$ or by $\mathbb{P}^2$. In 1988, M. Van den Bergh circulated a short note showing that a generic 4-dimensional Artin-Schelter regular algebra with Hilbert series $(1-t)^{-4}$ has exactly 20 point modules [43]. Van den Bergh's example is a generic Clifford algebra. In particular, it is a finite module over its center.

Davies [8, §5.1] shows, when the translation automorphism has infinite order, that $\widetilde{Q}$ is not isomorphic to any of the previously found examples of 4-dimensional regular algebras having 20 point modules.

Proposition 9.7. *The point modules $M_{\mathbf{u}}$ for $\mathbf{u} \in \mathfrak{P}_\infty \cup \mathfrak{P}_0$ are quotient rings of $\widetilde{Q}$. If $\mathbf{u} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathfrak{P}_\infty \cup \mathfrak{P}_0$, then*

$$M_{\mathbf{u}} \cong \frac{\widetilde{Q}}{(\lambda_j y_i - \lambda_i y_j \mid 0 \le i, j \le 3)} \cong k[t].$$

Proof. The identification of points $\mathbf{p} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ with point modules $M_{\mathbf{p}}$ is such that the latter is the quotient of $\widetilde{Q}$ by the *left* ideal generated by all elements $y_{ij} = \lambda_j y_i - \lambda_i y_j$ for $0 \le i, j \le 3$. We will show that in the cases under consideration the left ideal in question is a *two-sided* ideal.

The two-sided ideal generated by y_{ij} contains the left ideal generated by the same elements, so in order to check that they are the same it suffices to show that the quotient by the two-sided ideal has Hilbert series $(1-t)^{-1}$. It is enough to exhibit a surjective ring homomorphism $\widetilde{Q} \to k[t]$ that vanishes on all y_{ij}.

For points in $\mathfrak{P}_\infty$, the relations (6-3) make it clear that annihilating three y_js and sending the fourth one to t is such a homomorphism. For the point $(1,1,1,1) \in \mathfrak{P}_0$ there is a similar homomorphism $\widetilde{Q} \to k[t]$ sending all generators y_j to t. The same conclusion for the other points in $\mathfrak{P}_0$ is obtained from the previous sentence by applying the Γ-action on $\widetilde{Q}$. □

Proposition 9.8. *The scheme-theoretic zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$ of the relations for $\widetilde{Q}$ is a reduced scheme with* 20 *points.*

Proof (Van den Bergh [43]). We have already seen that the relations for $\widetilde{Q}$ vanish at 20 points in $\mathbb{P}^3 \times \mathbb{P}^3$. Let X denote the image of the Segre embedding $\mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^{15}$. If we view $\mathbb{P}^{15}$ as the space of 4×4 matrices, then X is the space of rank-one matrices. By [13, §18.15], for example, the degree of X is $\binom{6}{3} = 20$. The 6 defining relations for $\widetilde{Q}$ are linear combinations of terms $x_i x_j$ which, under the Segre embedding, become linear combinations of the coordinate functions x_{ij}. Hence the vanishing locus of the relations in $\mathbb{P}^{15}$ is the vanishing locus of 6 linear forms, hence a linear subspace, L say, of dimension 9. Hence, by Bézout's Theorem, if the scheme-theoretic intersection $L \cap X$ is finite it has degree 20. But, $L \cap X$ consists of 20 different points, so it is reduced. □

10. Secant lines to E and line modules for Q

The relevance of this section will become apparent in §11 when we construct some line modules for $\widetilde{Q}$ that are parametrized by certain lines in $\mathbb{P}(Q_1^*)$. To make the word "parametrized" precise we will show that the parametrizing space is a closed subvariety of the Grassmannian of lines in $\mathbb{P}(Q_1^*)$.

10.1. **Secant lines.** The second symmetric power of E is the quotient variety $S^2E := (E \times E)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by $(p,q) \mapsto (q,p)$. We think of the points in S^2E as effective divisors of degree 2 on E and write $(p)+(q)$ for the image of $(p,q) \in E \times E$.

Because the quartic curve $E \subset \mathbb{P}(Q_1^*) = \mathbb{P}^3$ has no trisecants, there is a well-defined morphism $E \times E \to \mathbb{G}(1,3)$ that sends $(p,q) \in E \times E$ to $\overline{pq}$, the line in $\mathbb{P}(Q_1^*) = \mathbb{P}^3$ whose scheme-theoretic intersection with E is $(p)+(q)$. By the universal property of the quotient $(E \times E)/\mathbb{Z}_2$ this morphism factors through a morphism $\gamma : S^2E \to \mathbb{G}(1,3)$. The image of γ is a closed subscheme of $\mathbb{G}(1,3)$ called the variety of secant lines to E. See [13, Ex. 8.3], for example.

Proposition 10.1. *The map* $\gamma : S^2E \to \mathbb{G}(1,3)$ *defined by* $\gamma\big((p)+(q)\big) := \overline{pq}$ *is a closed immersion.*

Proof. The morphism γ is injective because E has no trisecants, so it suffices to argue that the image of the morphism is smooth. This follows from the standard description of the singular points of a secant variety: a line in the image of γ is singular if and only if it is a trisecant (see e.g. the discussion on page 312 of [13] regarding Exercise 16.11 in that book). □

10.2. **The line modules** $M_{p,q}$**.** A *line module* for Q, or $\widetilde{Q}$, is a cyclic graded module whose Hilbert series is $(1-t)^{-2}$.

Theorem 10.2 ([17, Thm. 4.5])**.** *The function that sends* $(p)+(q) \in S^2E$ *to* $Q/Qx+Qx'$ *where* $\overline{pq} = \{x = x' = 0\}$ *is a bijection from* S^2E *to the set of isomorphism classes of line modules for* Q*.*

If $(p)+(q) \in S^2E$ and $\overline{pq} = \{x = x' = 0\}$ we write $M_{p,q} := Q/Qx+Qx'$.

10.3. In §11 we will show that if $y = y' = 0$ is a line in $\mathbb{P}(\widetilde{Q}_1^*) = \mathbb{P}(\widetilde{Q}_1^*)$ that meets E at $(p)+(p+\xi)$ for some $p \in E$ and $\xi \in E[2]-\{o\}$, then $\widetilde{Q}/\widetilde{Q}y+\widetilde{Q}y'$ is a line module for $\widetilde{Q}$. Such lines will be parametrized by the subscheme of $\mathbb{G}(1,3)$ that is the image of the composition $E/\langle\xi\rangle \to S^2E \to \mathbb{G}(1,3)$.

Lemma 10.3. *The morphism* $\beta : E/\langle\xi\rangle \to S^2E$ *defined by* $\beta(p+\langle\xi\rangle) = (p)+(p+\xi)$ *is a closed immersion.*

Proof. It is clear that β is injective as a set map on the closed points of $E/\langle\xi\rangle$, so it suffices to prove that its derivative is one-to-one on each tangent space or equivalently that the composition of β with the étale map $\pi : E \to E/\langle\xi\rangle$ has this same property.

The composition $\beta\pi$ is

$$E \to E \times E \to S^2E,$$

where the left-hand arrow sends p to $(p, p+\xi)$ and the right-hand arrow is the quotient morphism. Since the latter is étale off the diagonal $\Delta \subset E \times E$ and the former is a closed immersion into $E \times E \setminus \Delta$ the conclusion follows. □

11. Line modules for $\tilde{Q}$

11.1. In this section we exhibit three families of line modules for $\widetilde{Q}$ parametrized by the disjoint union of the three elliptic curves $E/\langle\xi\rangle$ as ξ ranges over the three 2-torsion points of E. The isomorphism classes of the line modules parametrized by $E/\langle\xi\rangle$ are in natural bijection with the lines $\overline{p, p+\xi}$, $p \in E$; the union of these lines is an elliptic scroll in $\mathbb{P}(\widetilde{Q}_1^*)$.

These are *not* all the line modules for $\widetilde{Q}$.

11.2. By Proposition 7.7, the elements in $\Gamma = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ and $E[2] = \{\xi_0, \xi_1, \xi_2, \xi_3\}$ may be labelled in such a way that $\gamma_i^* M_{p,q} \cong M_{p+\xi_i, q+\xi_i}$. Thus, $\gamma^*\big(M_{p,q} \oplus M_{r,s}\big) \cong M_{p,q} \oplus M_{r,s}$ for all $\gamma \in \Gamma$ if and only if $\{p, q, r, s\}$ is an $E[2]$-coset.

11.3. Recall that $Q' = Q \otimes M_2(k)$. The next result follows from Proposition 3.10.

Proposition 11.1. *The function $M \mapsto M^\Gamma$ is a bijection from isomorphism classes of Γ-equivariant Q'-modules with Hilbert series $4(1-t)^{-2}$ to isomorphism classes of $\widetilde{Q}$-line modules.*

By Morita equivalence, a Γ-equivariant Q'-module M with Hilbert series $4(1-t)^{-2}$ is isomorphic to $N \otimes k^2$ for some Q-module N with Hilbert series $2(1-t)^{-2}$ (a "fat line" of multiplicity two over Q). Moreover, by the remark in §11.2, the equivariance ensures/requires that the isomorphism class of M is invariant under translation by the 2-torsion subgroup.

The main ingredient in constructing $\widetilde{Q}$-lines will be Q-modules with Hilbert series $2(1-t)^{-2}$. The obvious such modules are those of the form $M_{p,q} \oplus M_{r,s}$ where the invariance condition requires $\{p, q, r, s\}$ to be an $E[2]$-coset. Theorem 11.6 will provide the examples announced in §11.1.

Lemma 11.2. *Let $E[2] = \{o, \xi, \xi', \xi''\}$, let $x, y \in E/E[2]$, and let ω be a 2-torsion point. Define*

$$M_{x,\xi} := \big(M_{p,p+\xi} \oplus M_{p+\xi',p+\xi''}\big) \otimes k^2, \tag{11-1}$$

where p is any point in E such that $x = p + E[2]$.

1. *The Q'-module $M_{x,\xi}$ does not depend on the choice of p.*
2. *$M_{x,\xi} \cong M_{y,\omega}$ if and only if $(x, \xi) = (y, \omega)$.*
3. *The map $\Phi : k^\times \times k^\times \to \operatorname{Aut}_{Q'}\big(M_{x,\xi}\big)$, $\Phi(\lambda, \lambda')(m, m') := (\lambda m, \lambda' m')$, is an isomorphism.*

Proof. (1) Suppose x is also the image of $q \in E$. Since $\xi' + \xi'' = \xi$,

$$\big\{\{q, q+\xi\}, \{q+\xi', q+\xi''\}\big\} = \big\{\{p, p+\xi\}, \{p+\xi', p+\xi''\}\big\}.$$

Therefore $M_{p,p+\xi} \oplus M_{p+\xi',p+\xi''} = M_{q,q+\xi} \oplus M_{q+\xi',q+\xi''}$. Hence $M_{x,\xi}$ does not depend on the choice of p. In particular, if $(x, \xi) = (y, \omega)$, then $M_{x,\xi} = M_{y,\omega}$.

(2) Suppose that the Q'-modules $M_{x,\xi}$ and $M_{y,\omega}$ are isomorphic. Let $q \in E$ be such that $y = q + E[2]$. By Morita equivalence, there is an isomorphism of Q-modules

$$M_{p,p+\xi} \oplus M_{p+\xi',p+\xi''} \cong M_{q,q+\omega} \oplus M_{q+\omega',q+\omega''}$$

where $E[2] = \{o, \omega, \omega', \omega''\}$. Since isomorphism classes of line modules for Q are in natural bijection with effective divisors of degree 2 on E,

$$\big\{\{q, q+\omega\}, \{q+\omega', q+\omega''\}\big\} = \big\{\{p, p+\xi\}, \{p+\xi', p+\xi''\}\big\}.$$

It follows immediately from this equality that $q + E[2] = p + E[2]$, i.e., $x = y$. Since ω can be recovered from $\big\{\{q, q+\omega\}, \{q+\omega', q+\omega''\}\big\}$ as the difference between the elements in $\{q, q+\omega\}$ and also as the difference between the elements in $\{q+\omega', q+\omega''\}$, it follows that $\omega = \xi$.

(3) Every line module for Q is cyclic, so its graded automorphism group is isomorphic to $k^\times$, each $\lambda \in k^\times$ acting on the line module by scalar multiplication.

By Morita equivalence, $\operatorname{Aut}_{Q'}(M_{x,\xi}) = \operatorname{Aut}_Q(M_{p,p+\xi} \oplus M_{p+\xi',p+\xi''}) \cong k^\times \times k^\times$ where the isomorphism is because $M_{p,p+\xi} \not\cong M_{p+\xi',p+\xi''}$. An automorphism $(\lambda, \lambda') \in (k^\times)^2$ acts on $M_{x,\xi} = (M_{p,p+\xi} \otimes k^2) \oplus (M_{p+\xi',p+\xi''} \otimes k^2)$ as multiplication by λ on the first summand and multiplication by λ' on the second summand. □

Lemma 11.3. *Let $E[2] = \{o, \xi, \xi', \xi''\}$. Let $x \in E/E[2]$ and write $M = M_{x,\xi}$.*

(1) *If $\gamma \in \Gamma$, then $\gamma^* M \cong M$ as Q'-modules.*
(2) *If $\gamma \in \Gamma$ and $a \in \operatorname{Aut}_{Q'}(M)$, then there is a unique element $\gamma \triangleright a \in \operatorname{Aut}_{Q'}(M)$ such that*

$$\begin{array}{ccc} M & \xrightarrow{\varphi_\gamma} & \gamma^* M \\ {\scriptstyle \gamma \triangleright a}\downarrow & & \downarrow{\scriptstyle \gamma^*(a)} \\ M & \xrightarrow[\varphi_\gamma]{} & \gamma^* M \end{array}$$

commutes for all isomorphisms $\varphi_\gamma : M \to \gamma^* M$.
(3) *The map $(\gamma, a) \mapsto \gamma \triangleright a$ defines a left action of Γ on $\operatorname{Aut}_{Q'}(M)$.*
(4) *If we identify $k^\times \times k^\times$ with $\operatorname{Aut}_{Q'}\big(M_{x,\xi}\big)$ via the isomorphism Φ in Lemma 11.2, then the Γ-action on $\operatorname{Aut}_{Q'}(M)$ is*

$$\xi \triangleright (\lambda, \lambda') = (\lambda, \lambda') \qquad \text{and} \qquad \xi' \triangleright (\lambda, \lambda') = \xi'' \triangleright (\lambda, \lambda') = (\lambda', \lambda)$$

for all $(\lambda, \lambda') \in k^\times \times k^\times$.

Proof. Let $p \in E$ be such that $x = p + E[2]$. Thus $M = \big(M_{p,p+\xi} \oplus M_{p+\xi',p+\xi''}\big) \otimes k^2$.

(1) This follows from the remark in §11.2.

(2) Choose an isomorphism $\varphi_\gamma : M \to \gamma^* M$. Define $\gamma \triangleright a := \varphi_\gamma^{-1} \gamma^*(a) \varphi_\gamma$. Certainly the diagram commutes. If $\psi_\gamma : M \to \gamma^* M$ is another isomorphism, then ψ_γ is a multiple of φ_γ by an element in $\operatorname{Aut}_{Q'}(\gamma^* M)$. But $\operatorname{Aut}_{Q'}(\gamma^* M)$ is abelian, so $\psi_\gamma^{-1} \gamma^*(a) \psi_\gamma = \varphi_\gamma^{-1} \gamma^*(a) \varphi_\gamma$.

(3) This is standard. See, for example, Lemma A.1.

(4) By Proposition 7.7, $\xi^* M_{p,p+\xi} \cong M_{p,p+\xi}$ and $\xi^* M_{p+\xi',p+\xi''} \cong M_{p+\xi',p+\xi''}$ so φ_ξ preserves the summands $M_{p,p+\xi} \otimes k^2$ and $M_{p+\xi',p+\xi''} \otimes k^2$. Therefore ξ acts on $(k^\times)^2$ trivially. On the other hand, $(\xi')^* M_{p,p+\xi} \cong (\xi'')^* M_{p,p+\xi} \cong M_{p+\xi',p+\xi''}$ so ξ' and ξ'' act on $(k^\times)^2$ by switching the two components. □

A Γ-equivariant structure on a Q'-module M is the same thing as a left Q'-module M endowed with a left action $\Gamma \times M \to M$, $(\gamma, m) \mapsto m^\gamma$, such that $(xm)^\gamma = \gamma(x) m^\gamma$ for all $x \in Q'$, $m \in M$, and $\gamma \in \Gamma$. We adopt this point of view several times in the rest of this section.

Recall that the action of Γ as automorphisms of Q' is defined in terms of the actions of Γ as automorphisms of Q and $M_2(k)$ (see §6.4).

Lemma 11.4. *Let N be a graded left Q-module that is generated by N_0. The function that sends a Γ-equivariant structure $\{\varphi_\gamma : N \otimes k^2 \longrightarrow \gamma^*(N \otimes k^2) \mid \gamma \in \Gamma\}$ on the Q'-module $N \otimes k^2$ to the Γ-equivariant structure $\{\varphi_\gamma\big|_{N_0 \otimes k^2} : N_0 \otimes k^2 \longrightarrow \gamma^*(N_0 \otimes k^2) \mid \gamma \in \Gamma\}$ on the $M_2(k)$-module $N_0 \otimes k^2$ is injective.*

Proof. Certainly, if the maps $\{\varphi_\gamma : N \otimes k^2 \longrightarrow \gamma^*(N \otimes k^2) \mid \gamma \in \Gamma\}$ give $N \otimes k^2$ the structure of a Γ-equivariant Q'-module, then their restrictions to the degree-zero components give $N_0 \otimes k^2$ the structure of a Γ-equivariant $M_2(k)$-module.

Since Q' is generated as an algebra by Q'_0 and Q'_1, the formula $(xm)^\gamma = \gamma(x) m^\gamma$ implies that the action of γ on $N_{n+1} \otimes k^2$ is completely determined by the action of γ on $N_n \otimes k^2$. Thus, if two Γ-equivariant structures on $N \otimes k^2$ agree on $N_0 \otimes k^2$, then they agree on N. □

11.3.1. *Warning.* The result in Lemma 11.4 does *not* extend to a result saying that two equivariant structures on N are isomorphic if and only if their restrictions to $N_0 \otimes k^2$ are isomorphic. Proposition 11.5 says that all Γ-equivariant structures on $N_0 \otimes k^2$ are isomorphic to each other.

The group Γ acts as k-algebra automorphisms of $M_2(k)$. We fixed a basis for k^2 such that $\omega \in \Gamma$ acts on $M_2(k)$ as conjugation by the quaternionic basis element q_ω defined in §6.4. We use that basis in the next result.

Proposition 11.5. *Fix $\zeta, \eta, \xi \in \Gamma$ such that q_ζ, q_η, q_ξ is a cyclic permutation of q_1, q_2, q_3.*

(1) *Let $\phi_\omega : M_2(k) \to M_2(k)$, $\omega \in \Gamma$, be the linear isomorphisms that take the following values on the basis $1, q_\zeta, q_\eta, q_\xi$ for $M_2(k)$:*

TABLE 4. Action of Γ on $M_2(k)$

q	1	q_ζ	q_η	q_ξ
$\phi_0(q)$	1	q_ζ	q_η	q_ξ
$\phi_\zeta(q)$	1	q_ζ	$-q_\eta$	$-q_\xi$
$\phi_\eta(q)$	1	$-q_\zeta$	q_η	$-q_\xi$
$\phi_\xi(q)$	1	$-q_\zeta$	$-q_\eta$	q_ξ

The action of Γ on $M_2(k)$ given by the maps ϕ_ω, together with the action of $M_2(k)$ on $M_2(k)$ by left multiplication, gives $M_2(k)$ the structure of a Γ-equivariant left $M_2(k)$-module.

(2) *Every Γ-equivariant $M_2(k)$-module is isomorphic to a direct sum of copies of the Γ-equivariant $M_2(k)$-module in* (2).

(3) *Let V be a finite-dimensional Γ-equivariant $M_2(k)$-module. As a Γ-module, V is isomorphic to a direct sum of copies of the regular representation. If $\omega \in \{\zeta, \eta, \xi\}$, then the $(+1)$- and (-1)-eigenspaces for the action of ω on V have dimension $\frac{1}{2}\dim_k(V)$.*

Proof. (1) Whenever a group Γ acts as automorphisms of a ring R, R viewed as a left R-module via multiplication is a Γ-equivariant R-module with respect to the action of Γ as automorphisms of R. The value of $\phi_\omega(q_{\omega'})$ in the table is $q_\omega q_{\omega'} q_\omega^{-1}$, so, by the previous sentence, this action of Γ makes $M_2(k)$ a Γ-equivariant $M_2(k)$-module.

(2) By Lemma 3.2, there is an equivalence from the category of Γ-equivariant $M_2(k)$-modules to the category of vector spaces, the functor implementing the equivalence being $M \rightsquigarrow M^\Gamma$. Since $M_2(k)^\Gamma \cong k$, the result follows.

Alternatively, a Γ-equivariant left $M_2(k)$-module is the same thing as a left module over the 16-dimensional skew group ring $M_2(k) \rtimes \Gamma$; the Γ-equivariant $M_2(k)$-module in (1) is irreducible of dimension 4, so we conclude that $M_2(k) \rtimes \Gamma \cong M_4(k)$. The result follows.

(3) follows from (2) because $M_2(k)$ is isomorphic as a Γ-module to the regular representation. □

Theorem 11.6. *Let $E[2] = \{o, \xi, \xi', \xi''\}$. Let M be the Q'-module $(M_{p,p+\xi} \oplus M_{p+\xi',p+\xi'+\xi}) \otimes k^2$.*

(1) *There are exactly two Γ-equivariant structures on M up to isomorphism.*

(2) *The group* $H^1(\Gamma, \mathrm{Aut}_{Q'}(M))$ *acts simply transitively on this two-element set.*
(3) *Up to isomorphism one equivariant structure is obtained from the other by interchanging the* $(+1)$*- and* (-1)*-eigenspaces for the action of* ξ *on* M *and simultaneously interchanging the* $(+1)$*- and* (-1)*-eigenspaces for the action of* $\xi+\xi'$ *on* M *and leaving the* $(+1)$*- and* (-1)*-eigenspaces for the action of* ξ' *unchanged.*

Proof. If $x = p + E[2]$, then M is the module $M_{x,\xi}$ in Lemmas 11.2 and 11.3.

Step 1. *Existence of an equivariant structure.* Let $\varphi_\gamma : M \to \gamma^* M$, $\gamma \in \Gamma$, be arbitrary Q'-module isomorphisms. An arbitrary choice of such isomorphisms need not give an equivariant structure on M; i.e., there is no reason the diagrams (8-3) should commute. The failure of (8-3) to commute is measured by the elements

$$a_{\gamma,\delta} \;:=\; \varphi_{\gamma\delta}^{-1} \circ t_{\gamma,\delta} \circ \gamma^*(\varphi_\delta) \circ \varphi_\gamma, \quad \gamma, \delta \in \Gamma, \tag{11-2}$$

in $\mathrm{Aut}_{Q'}(M)$ where $t_{\gamma,\delta}$ is as in (8-3) and the right-hand side of (11-2) is the clockwise composition of the automorphisms in (8-3).

A tedious calculation (see Lemma A.2) shows that the function $(\gamma, \delta) \mapsto a_{\gamma,\delta}$ is a 2-cocycle for Γ valued in the Γ-module $\mathrm{Aut}_{Q'}(M) \cong (k^\times)^2$ defined in Lemma 11.3. Let $\xi' \in \Gamma - \langle\xi\rangle$. Since $\Gamma = \langle\xi\rangle \times \langle\xi'\rangle$ it follows from the Hochschild-Serre spectral sequence that

$$E_2^{a,b} = H^a(\langle\xi\rangle, H^b(\langle\xi'\rangle, (k^\times)^2)) \Rightarrow H^{a+b}(\Gamma, (k^\times)^2) \tag{11-3}$$

and from the cohomology of $\mathbb{Z}/2$ that $H^2(\Gamma, (k^\times)^2)$ is trivial. Hence the obstruction cocycle $(a_{\gamma,\delta})$ is cohomologous to zero. Thus $(a_{\gamma,\delta})$ is the coboundary of some function $\Gamma \to \mathrm{Aut}_{Q'}(M)$, $\gamma \mapsto a_\gamma$; the isomorphisms $\varphi_\gamma a_\gamma^{-1}$ now form an equivariant structure on M.

Step 2. *Classification of equivariant structures.* By Step 1, there is at least one Γ-equivariant structure on M. Suppose the maps $\varphi_\gamma : M \to \gamma^* M$, $\gamma \in \Gamma$, provide such an equivariant structure.

Let $(\psi_\gamma)_{\gamma\in\Gamma}$ be another equivariant structure on M. Running through the compatibility conditions comprising equivariance, the maps $a_\gamma = (\varphi_\gamma)^{-1}\psi_\gamma$ can be seen to form a 1-cocycle of Γ valued in the Γ-module $\mathrm{Aut}_{Q'}(M) \cong (k^\times)^2$. We similarly leave it to the reader to check that cocycles (a_γ) and (a'_γ) give rise to isomorphic equivariant structures

$$\psi_\gamma = \varphi_\gamma a_\gamma \quad \text{and} \quad \psi'_\gamma = \varphi_\gamma a'_\gamma$$

if and only if they are cohomologous. In other words, the set of isomorphism classes of equivariant structures on M is acted upon simply and transitively by $H^1(\Gamma, (k^\times)^2)$. Using the Hochschild-Serre spectral sequence once more we get $H^1(\Gamma, (k^\times)^2) \cong \mathbb{Z}/2$ (see the proof of (3) below).

This completes the proof of (1) and (2).

(3) The Hochschild-Serre spectral sequence yields an isomorphism

$$H^1(\Gamma, \mathrm{Aut}(M)) \;\cong\; H^1\big(\langle\xi\rangle, H^0(\langle\xi'\rangle, (k^\times)^2)\big) \;\oplus\; H^0\big(\langle\xi\rangle, H^1(\langle\xi'\rangle, (k^\times)^2)\big). \tag{11-4}$$

Since ξ' interchanges the two copies of $k^\times$, the H^1 term in the second summand vanishes, so we are left with a natural isomorphism

$$H^1(\Gamma, \mathrm{Aut}(M)) \;\cong\; H^1(\langle\xi\rangle, k^\times) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(\langle\xi\rangle, k^\times),$$

where this time $k^\times$ is the diagonal subgroup of $\mathrm{Aut}_{Q'}(M)$.

The function $f : \Gamma \to \mathrm{Aut}_{Q'}(M)$ defined by $f(\xi) = f(\xi' + \xi) = (-1, -1)$ and $f(o) = f(\xi') = (1, 1)$ is a 1-cocycle whose class $[f]$ in $H^1(\Gamma, \mathrm{Aut}(M))$ is non-trivial. If the Q'-module isomorphisms $\{\phi_\gamma : M \to \gamma^* M \mid \gamma \in \Gamma\}$ give M a Γ-equivariant structure, then the Γ-equivariant structure on M associated to the result of $[f]$ acting on the given equivariant structure is given by the isomorphisms $\{\phi_\gamma \circ f(\gamma) : M \to \gamma^* M \mid \gamma \in \Gamma\}$. Recall that $\gamma^* M$ is M as a graded vector space. The $(+1)$-eigenspace for the action of ξ on M with equivariant structure $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is $\{m \in M \mid \phi_\xi(m) = m\}$, which is the (-1)-eigenspace for $\phi_\xi \circ f(\xi)$. Likewise, the (-1)-eigenspace for the action of $\phi_{\xi+\xi'} \circ f(\xi + \xi')$ is the $(+1)$-eigenspaces for the action of $\phi_{\xi+\xi'}$. On the other hand, the eigenspaces for ξ' are the same for both equivariant structures on $M_{x,\xi}$. □

11.3.2. There is a lack of symmetry in part (3) of Theorem 11.6: the eigenspaces for $\xi+\xi'$ are switched, but those for ξ' are not. The explanation is that the equivariant structure obtained by interchanging the eigenspaces for ξ' but not $\xi + \xi'$ (but still exchanging the eigenspaces for ξ) is isomorphic to that obtained by switching the eigenspaces for $\xi + \xi'$ but not those for ξ'.

11.3.3. The proof of Theorem 11.6 illustrates a familiar pattern in obstruction theory. The class of structures we are interested in, isomorphism classes of equivariant structures in this case, is a *pseudotorsor* over a cohomology group. Whether or not it is empty is controlled by an obstruction living in a cohomology group, H^2 for us, as in Step 1 of the proof, and when this obstruction vanishes the cohomology group of one degree lower, H^1 in our case, acts on the class of structures simply transitively.

11.4. **An explicit equivariant structure on $M_{x,\xi}$.** Let $\{\xi_1, \xi_2, \xi_3\}$ denote both the 2-torsion points on E and the corresponding elements in Γ, labelled so that the action of Γ as automorphisms of $M_2(k)$ is such that each ξ_j acts as conjugation by the element q_j in (6-2).

Let $p \in E$ and let $x = p + \langle\xi_1\rangle \in E/\langle\xi_1\rangle$. Let $M = M_{x,\xi_1} = (M_{p,p+\xi_1} \oplus M_{p+\xi_2,p+\xi_3}) \otimes k^2$. Fix a basis e for the degree-zero component of $M_{p,p+\xi_1}$ and a basis e' for the degree-zero component of $M_{p+\xi_2,p+\xi_3}$.

If $u = \binom{0}{1}$ and $v = \binom{1}{0}$, then

$$q_1 u = -iu, \qquad q_2 u = iv, \qquad q_3 u = -v,$$
$$q_1 v = iv, \qquad q_2 v = iu, \qquad q_3 v = u.$$

Lemma 11.7. *Let $\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$ be a linear form that vanishes at p and $p + \xi_1$. Then*

(1) *the line through p and $p + \xi_1$ is $\beta_0 x_0 + \beta_1 x_1 = \beta_2 x_2 + \beta_3 x_3 = 0$,*
(2) *the line through $p + \xi_2$ and $p + \xi_3$ is $\beta_0 x_0 - \beta_1 x_1 = \beta_2 x_2 - \beta_3 x_3 = 0$,*
(3) *$\beta_0 y_0 + i\beta_1 y_1$ and $i\beta_2 y_2 + \beta_3 y_3$ annihilate $e \otimes u + e' \otimes v$ and are linearly independent, and*
(4) *$\beta_0 y_0 - i\beta_1 y_1$ and $i\beta_2 y_2 - \beta_3 y_3$ annihilate $e \otimes v + e' \otimes u$ and are linearly independent.*

Proof. By Lemma 8.6, at least three of the coordinate functions x_0, x_1, x_2, x_3 are non-zero at p. Thus $(\beta_0, \beta_1) \neq (0, 0)$ and $(\beta_2, \beta_3) \neq (0, 0)$. Therefore the equations in (1) and (2) really do define lines in $\mathbb{P}(Q_1^*)$. It also follows that $\beta_0 y_0 + i\beta_1 y_1$ and $i\beta_2 y_2 + \beta_3 y_3$ are linearly independent.

(1) Translation by ξ_1 leaves the set $\{p, p+\xi_1\}$ stable so $\xi_1(\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)$ also vanishes at p and $p+\xi_1$. Since $\xi_1(\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) = \beta_0 x_0 + \beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3$, (1) follows.

(2) Since translation by ξ_2 sends $\{p, p+\xi_1\}$ to $\{p+\xi_2, p+\xi_3\}$, $\xi_2(\beta_0 x_0 + \beta_1 x_1)$ and $\xi_2(\beta_2 x_2 + \beta_3 x_3)$ vanish at $p+\xi_2$ and $p+\xi_3$. Thus (2) is true.

(3) Since

$$\begin{aligned} y_0 \cdot (e\otimes u + e'\otimes v) &= (x_0 \otimes q_0)\cdot(e\otimes u + e'\otimes v) = x_0 e\otimes u + x_0 e'\otimes v, \\ y_1 \cdot (e\otimes u + e'\otimes v) &= (x_1 \otimes q_1)\cdot(e\otimes u + e'\otimes v) = -ix_1 e\otimes u + ix_1 e'\otimes v, \\ y_2 \cdot (e\otimes u + e'\otimes v) &= (x_2 \otimes q_2)\cdot(e\otimes u + e'\otimes v) = ix_2 e\otimes v + ix_2 e'\otimes u, \text{ and} \\ y_3 \cdot (e\otimes u + e'\otimes v) &= (x_3 \otimes q_3)\cdot(e\otimes u + e'\otimes v) = -x_3 e\otimes v + x_3 e'\otimes u, \end{aligned}$$

$(\beta_0 y_0 + i\beta_1 y_1 - i\beta_2 y_2 - \beta_3 y_3)\cdot(e\otimes u + e'\otimes v)$ equals

$$(\beta_0 x_0 + \beta_1 x_1)e\otimes u + (\beta_2 x_2 + \beta_3 x_3)e\otimes v + (\beta_2 x_2 - \beta_3 x_3)e'\otimes u + (\beta_0 x_0 - \beta_1 x_1)e'\otimes v.$$

Since $e \in (M_{p,p+\xi_1})_0$ it follows from (1) that $(\beta_0 x_0 + \beta_1 x_1)e = (\beta_2 x_2 + \beta_3 x_3)e = 0$. Since $e' \in (M_{p+\xi_2,p+\xi_3})_0$ it follows from (2) that $(\beta_0 x_0 - \beta_1 x_1)e' = (\beta_2 x_2 - \beta_3 x_3)e' = 0$. Therefore (3) is true. The proof of (4) is similar. □

Let ϕ_0 be the identity map on M_0 and let $\phi_1, \phi_2 \in \mathrm{GL}(M_0)$ be the linear automorphisms which act on the basis $\{e\otimes u, e\otimes v, e'\otimes u, e'\otimes v\}$ as in Table 5.

TABLE 5. Equivariant structure on M_0

	$e\otimes u$	$e\otimes v$	$e'\otimes u$	$e'\otimes v$
ϕ_1	$e\otimes u$	$-e\otimes v$	$-e'\otimes u$	$e'\otimes v$
ϕ_2	$e'\otimes v$	$e'\otimes u$	$e\otimes v$	$e\otimes u$

Let $\phi_3 = \phi_1\phi_2$.

The following observation is elementary.

Lemma 11.8. *Let a be an element in a ring R such that $a^2 = 1$. There is a group homomorphism $\mathbb{Z}/2 \to \mathrm{Aut}(R)$ given by sending the non-identity element to the automorphism $b \mapsto aba^{-1}$. Let M be a left R-module and define the group homomorphism $\mathbb{Z}/2 \to \mathrm{Aut}_{\mathbb{Z}}(M)$ by sending the non-identity element to the automorphism $m \mapsto am$. This action of $\mathbb{Z}/2$ makes M a $\mathbb{Z}/2$-equivariant R-module.*

Theorem 11.9. *Let each ξ_i act on M_0 as the linear map ϕ_i in Table 5.*

1. *This action of Γ on M_0 extends to an action of Γ on M that makes M a Γ-equivariant Q'-module.*
2. *The $\widetilde{Q}$-line module M^Γ is generated by $e\otimes u + e'\otimes v$.*
3. *If $\beta_0 x_0 + \beta_1 x_1 = \beta_2 x_2 + \beta_3 x_3 = 0$ is the line in $\mathbb{P}(Q_1^*)$ that passes through p and $p+\xi_1$, then the line in $\mathbb{P}(\widetilde{Q}_1^*)$ corresponding to M^Γ is $\beta_0 y_0 + i\beta_1 y_1 = i\beta_2 y_2 + \beta_3 y_3 = 0$.*

Proof. (1) We will use Lemma 11.8 to show that M_0 is a Γ-equivariant $M_2(k)$-module.

First, consider the action of ξ_1 by ϕ_1 on $e \otimes k^2$. With respect to the ordered basis $\{e \otimes u, e \otimes v\}$, ξ_1 acts on $e \otimes k^2$ as multiplication by $1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The action of ξ_1 on $M_2(k)$ is $b \mapsto q_1 b q_1^{-1}$. Since conjugation by q_1 is the same as conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, Lemma 11.8 tells us that $e \otimes k^2$ is a $\langle \xi_1 \rangle$-equivariant $M_2(k)$-module.

Now consider the action of ξ_1 by ϕ_1 on $e' \otimes k^2$. With respect to the ordered basis $\{e \otimes u, e \otimes v\}$, ξ_1 acts on $e' \otimes k^2$ as multiplication by $1 \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Since conjugation by q_1 is the same as conjugation by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, Lemma 11.8 tells us that $e' \otimes k^2$ is a $\langle \xi_1 \rangle$-equivariant $M_2(k)$-module.

Thus, M_0 is a $\langle \xi_1 \rangle$-equivariant $M_2(k)$-module. A similar argument shows that M_0 is a $\langle \xi_j \rangle$-equivariant $M_2(k)$-module for the other j's. Since $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ is a subgroup of $\mathrm{GL}(M_0)$ isomorphic to Γ, these $\mathbb{Z}/2$-equivariant structures fit together to make $M_0 = (e \otimes k^2) \oplus (e' \otimes k^2)$ a Γ-equivariant $M_2(k)$-module.

To extend the equivariant structure to all of M, simply define automorphisms ϕ_i of M by

$$\phi_i(am) = \xi_i(a)\phi_i(m), \quad \forall a \in Q', \ m \in M_0.$$

That this action is well-defined boils down to checking that whenever $a \in Q'$ annihilates $m \in M_0$, $\xi_i(a)$ annihilates $\phi_i(m)$. For this it suffices to assume that m is an eigenvector of ϕ_i (since M_0 breaks up as a direct sum of Γ-eigenspaces) and hence to prove that

$$am = 0 \Rightarrow \xi_i(a)m = 0, \quad \forall a \in Q', \ m \in M_0.$$

The conclusion follows from the fact that all twists $\xi_i^* M$ are isomorphic to M as Q'-modules (because we already know there are equivariant structures on M).

(2) By Proposition 11.1, M^Γ is a line module for $\widetilde{Q}$. One sees from Table 1 that $e \otimes u + e' \otimes v$ is in M_0^Γ so it generates the $\widetilde{Q}$-line module M^Γ.

(3) The correspondence between line modules for $\widetilde{Q}$ and lines in $\mathbb{P}(\widetilde{Q}_1^*)$ is given by sending a line module $\widetilde{Q}/\widetilde{Q}y + \widetilde{Q}y'$ to the line $y = y' = 0$. Thus, (3) follows from Lemma 11.7(3). $\square$

11.5. **3 elliptic curves parametrizing some line modules.** Let $\mathbb{G}(1,3)$ be the Grassmannian of lines in $\mathbb{P}(\widetilde{Q}_1^*)$. There is a bijection

$$\mathbb{G}(1,3) \longleftrightarrow \{\text{isomorphism classes of cyclic graded } \widetilde{Q}\text{-modules with Hilbert series } 1+2t\}$$

given by the function sending a line $y = y' = 0$ to the module $\widetilde{Q}/\widetilde{Q}y + \widetilde{Q}y' + \widetilde{Q}_{\geq 2}$ and its inverse which sends a cyclic graded $\widetilde{Q}$-module N with Hilbert series $1 + 2t$ to the vanishing locus of the subspace of $\widetilde{Q}_1$ that annihilates N_0.

Let L be a line module for $\widetilde{Q}$. The Hilbert series for $L/L_{\geq 2}$ is $1 + 2t$, so L determines a point in $\mathbb{G}(1,3)$. Since $L \cong \widetilde{Q}/\widetilde{Q}y + \widetilde{Q}y'$ for some linearly independent elements $y, y' \in \widetilde{Q}_1$, the isomorphism class of L is determined by the isomorphism class of $L/L_{\geq 2}$. Thus, there is a well-defined map

$$\{\text{isomorphism classes of line modules for } \widetilde{Q}\} \longrightarrow \mathbb{G}(1,3).$$

Proposition 11.10. *Let $g : \mathbb{P}(Q_1^*) \to \mathbb{P}(\widetilde{Q}_1^*)$ be the isomorphism induced by the linear isomorphism $\widetilde{Q}_1 \to Q_1$,*

$$y_0 \mapsto x_0, \qquad y_1 \mapsto -ix_1, \qquad y_2 \mapsto -ix_2, \qquad y_3 \mapsto x_3.$$

The function $f : E/\langle\xi_1\rangle \to \mathbb{G}(1,3)$ *defined by*

$$f\big(p + \langle\xi_1\rangle\big) := g(\text{the line in } \mathbb{P}(Q_1^*) \text{ that passes through } p \text{ and } p+\xi_1)$$

is a closed immersion, and $f\big(E/\langle\xi_1\rangle\big)$ *parametrizes the isomorphism classes of* Γ*-equivariant* Q'*-modules of the form* M_{x,ξ_1}, $x \in E/E[2]$. *If* $x = p + E[2]$, *then the lines* $f\big(p + \langle\xi_1\rangle\big)$ *and* $f\big(p + \xi_2 + \langle\xi_1\rangle\big)$ *correspond to the two non-isomorphic equivariant structures on* M_{x,ξ_1}.

Proof. The map that sends a point $p \in E$ to the line through p and $p + \xi_1$ is a morphism from E to the Grassmanian of lines in $\mathbb{P}(Q_1^*)$. Composing that map with g gives a morphism $h : E \to \mathbb{G}(1,3)$. Since $h(p) = h(p + \xi_1)$, h factors as a composition

$$E \longrightarrow E/\langle\xi_1\rangle \longrightarrow \mathbb{G}(1,3), \tag{11-5}$$

where the first map is the quotient map and the second is f. By the universal property of the quotient map, f is a morphism. In fact, f is the composition $\gamma\beta$ of the two maps from Proposition 10.1 and Lemma 10.3 and hence is a closed immersion.

The line in $\mathbb{P}(Q_1^*)$ through p and $p+\xi_1$ is of the form $\beta_0x_0 + \beta_1x_1 = \beta_2x_2 + \beta_3x_3 = 0$. Therefore $f\big(p + \langle\xi_1\rangle\big)$ is the line $g(\beta_0x_0 + \beta_1x_1) = g(\beta_2x_2 + \beta_3x_3) = 0$, i.e., the line $i\beta_0y_0 - \beta_1y_1 = \beta_2y_2 - i\beta_3y_3 = 0$. Thus, $f(p + \langle\xi_1\rangle)$ is the line in $\mathbb{P}(\widetilde{Q}_1^*)$ that corresponds to the $\widetilde{Q}$-line module, M^Γ, that corresponds to the Γ-equivariant structure on $M = M_{x,\xi_1}$ with the equivariant structure described in Theorem 11.9. □

There are versions of all the results in §11.4 with ξ_2 and ξ_3 in place of ξ_1. In particular, by Proposition 11.10 there are closed immersions $E/\langle\xi_1\rangle \to \mathbb{G}(1,3)$, $E/\langle\xi_2\rangle \to \mathbb{G}(1,3)$, and $E/\langle\xi_3\rangle \to \mathbb{G}(1,3)$. It is clear that the images of these morphisms are disjoint from one another.

Theorem 11.11. *The set of* Γ*-equivariant* Q'*-modules in Theorem* 11.6 *is parametrized by*

$$\big(E/\langle\xi\rangle\big) \sqcup \big(E/\langle\xi'\rangle\big) \sqcup \big(E/\langle\xi''\rangle\big)$$

where $\{\xi, \xi', \xi''\}$ *is the set of* 2*-torsion points on* E.

In fact, we can say more about these three components of the scheme of line modules. We will say that a closed subscheme of a projective space $\mathbb{P}^N$ is *spatial* if its inclusion factors through some linear $\mathbb{P}^3 \subset \mathbb{P}^N$ but not through a linear $\mathbb{P}^2 \subset \mathbb{P}^N$.

Proposition 11.12. *For each* 2*-torsion point* ξ *the elliptic curve* $E/\langle\xi\rangle \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ *is a degree-four curve spanning a linear subspace* $\mathbb{P}^3 \subseteq \mathbb{P}^5$.

Proof. That $E/\langle\xi\rangle$ is contained in a $\mathbb{P}^3 \subseteq \mathbb{P}^5$ follows from its construction in Proposition 11.10. Indeed, suppose in order to fix notation that $\xi = \xi_1$ and denote $\overline{E} = E/\langle\xi\rangle$. If the Plücker coordinates of the line

$$\sum_{j=0}^{3} \lambda_j y_j = \sum_{j=0}^{3} \lambda'_j y_j = 0$$

are the minors M_{ij}, $0 \le i < j \le 3$ of the matrix

$$M = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda'_0 & \lambda'_1 & \lambda'_2 & \lambda'_3 \end{pmatrix}$$

supported on columns i and j, then the two coordinates M_{01} and M_{23} vanish on $\overline{E}$ by part (3) of Theorem 11.9.

The fact that $\overline{E}$ is not contained in a $\mathbb{P}^2$ will follow once we prove that the degree of the embedding into $\mathbb{P}^5$ is four, as claimed in the statement.

To check the degree assertion we will intersect $\overline{E}$ with a hyperplane section of $\mathbb{G}(1,3) \subset \mathbb{P}^5$, judiciously chosen so that it is not tangent to $\overline{E}$ and the number of intersection points is clearly four.

For every line ℓ in $\mathbb{P}^3$ the collection of all lines in $\mathbb{G}(1,3)$ intersecting ℓ is a hyperplane section H_ℓ of $\mathbb{G}(1,3) \subset \mathbb{P}^5$. Let $\ell = \overline{pq}$ be a secant line of E. The points in $\overline{E} \cap H_\ell$ are the classes modulo $\langle \xi \rangle$ of those $u \in E$ for which the secant line $\overline{u(u+\xi)}$ intersects ℓ.

If

$$q \neq p + \xi \quad \text{and} \quad 3p + q + \xi \neq 0,\ p + 3q + \xi \neq 0, \tag{11-6}$$

then there are exactly four such classes modulo $\langle \xi \rangle$, namely those of p, q, u, and $u + \xi'$, where $u + (u+\xi) + p + q = 0$ and $E[2] - \{0\} \ni \xi' \neq \xi$.

It remains to check that $p, q \in E$ can be chosen so that H_ℓ is not tangent to $\overline{E}$ at any of the four points where they intersect, in addition to satisfying (11-6).

Identify, as usual, the tangent space to $\mathbb{G}(1,3)$ at some line m (simultaneously regarded as a 2-plane in the 4-dimensional vector space V) with the space of linear maps $m \to V/m$. Generally, we will conflate linear subspaces of V and their projectivized versions.

For any $u \in E$, the tangent line to $\overline{E} \subset \mathbb{G}(1,3)$ at $\overline{u(u+\xi)}$ can be identified with the space of linear maps $\overline{u(u+\xi)} \to V/\overline{u(u+\xi)}$ that send the lines u and $u+\xi$ in V to the 2-planes T_uE and $T_{u+\xi}E$ in V respectively modulo $\overline{u(u+\xi)}$.

On the other hand, reverting to the notation introduced above for $u \in E$ so that $2u + \xi + p + q = 0$, the tangent space at $\overline{u(u+\xi)} \in \mathbb{G}(1,3)$ to H_ℓ consists of those linear maps $\overline{u(u+\xi)} \to V/\overline{u(u+\xi)}$ that send the intersection $s = \overline{pq} \cap \overline{u(u+\xi)}$ to $\overline{pq}$ modulo $\overline{u(u+\xi)}$ (see e.g. [13, Ex. 16.6]).

Since the line $s \subset V$ is in the span of u and $u+\xi$, we would be certain that the tangent space in the previous paragraph does not contain the tangent line described two paragraphs up if we knew that the tangents to E at u and $u+\xi$ are coplanar. This is indeed the case if $4u = 0$, so simply take $u \in E[4]$ and afterwards select p and q so that (11-6) holds. $\square$

11.5.1. There is another perspective on the Γ-equivariant Q'-modules parametrized by $E/\langle \xi \rangle$. The family of Q'-modules $M_{x,\xi}$ is parametrized by $x \in E/E[2]$. The quotient of the fundamental groups, $\pi_1(E/E[2])/\pi_1(E/\langle \xi \rangle)$, which is naturally isomorphic to $E[2]/\langle \xi \rangle$, acts freely and transitively on each fiber of the natural map $E/\langle \xi \rangle \to E/E[2]$. If we identify the fiber over x with the set of isomorphism classes of equivariant structures on $M_{x,\xi}$, then $H^1(\Gamma, \mathrm{Aut}(M_{x,\xi}))$ also acts on the fiber over x. As the paragraph explains, these actions of $E/\langle \xi \rangle$ and $H^1(\Gamma, \mathrm{Aut}(M_{x,\xi}))$ on the fibers are compatible in a natural way.

The Weil pairing $\langle \cdot, \cdot \rangle : E[2] \times E[2] \to \mu_2 = \{\pm 1\} \subseteq k^\times$ is a non-degenerate skew-symmetric bilinear form on $E[2]$ viewed as a 2-dimensional vector space over $\mathbb{F}_2$. Since $\langle \xi, \xi \rangle = 1$, there is an induced non-degenerate bilinear map $\langle \xi \rangle \times E[2]/\langle \xi \rangle \to \mu_2$ or, what is essentially the same thing, a group isomorphism

$$E[2]/\langle \xi \rangle \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\langle \xi \rangle, \mu_2) = \mathrm{Hom}_{\mathbb{Z}}(\langle \xi \rangle, k^\times) \cong H^1(\Gamma, \mathrm{Aut}(M_{x,\xi})),$$

where the right-most isomorphism was established in the proof of Theorem 11.6(3).

11.6. Under quite general conditions, which $\widetilde{Q}$ satisfies, Shelton and Vancliff prove that every irreducible component of the scheme parametrizing the line modules has dimension ≥ 1 [29, Cor. 2.6] and that every point module is a quotient of a line module [29, Prop. 3.1]. We will investigate this relationship in a subsequent paper. We also show there that the line modules for $\widetilde{Q}$ described above are *not* all the line modules.

Appendix A. Equivariant structures

A.1. **Groups acting on categories.** An action of a group Γ on a category C consists of data $\{\alpha^*, t_{\alpha,\beta} \mid \alpha, \beta \in \Gamma\}$ where each $\alpha^* : \mathsf{C} \to \mathsf{C}$ is an auto-equivalence and each $t_{\alpha,\beta} : \alpha^*\beta^* \to (\alpha\beta)^*$ is a natural isomorphism such that the diagrams

$$\begin{array}{ccc} \alpha^* \circ \beta^* \circ \gamma^* & \xrightarrow{\alpha^* \cdot t_{\beta,\gamma}} & \alpha^* \circ (\beta\gamma)^* \\ {\scriptstyle t_{\alpha,\beta}\cdot\gamma^*}\downarrow & & \downarrow{\scriptstyle t_{\alpha,\beta\gamma}} \\ (\alpha\beta)^* \circ \gamma^* & \xrightarrow[t_{\alpha\beta,\gamma}]{} & (\alpha\beta\gamma)^* \end{array}$$

commute for all $\alpha, \beta, \gamma \in \Gamma$.

Lemma A.1. *Let $x \in \mathsf{Ob}(\mathsf{C})$ and let $\phi = \{\phi_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ be a set of isomorphisms. If $\operatorname{Aut}(x)$ is abelian, then there is an action of Γ on $\operatorname{Aut}(x)$ given by the formula*

$$\begin{aligned} \Gamma \times \operatorname{Aut}(x) &\to \operatorname{Aut}(x) \\ (\alpha, f) &\mapsto \alpha \cdot f := \phi_\alpha^{-1} \alpha^*(f) \phi_\alpha. \end{aligned}$$

This action does not depend on the choice of the ϕ_α's.

Proof. Because $t_{\alpha,\beta} : \alpha^* \circ \beta^* \to (\alpha\beta)^*$ is a natural transformation, the diagram

$$\begin{array}{ccc} \alpha^*(\beta^* x) & \xrightarrow{(t_{\alpha,\beta})_x} & (\alpha\beta)^* x \\ {\scriptstyle \alpha^*\beta^*(f)}\downarrow & & \downarrow{\scriptstyle (\alpha\beta)^*(f)} \\ \alpha^*(\beta^* x) & \xrightarrow[(t_{\alpha,\beta})_x]{} & (\alpha\beta)^* x \end{array}$$

commutes for all $f \in \operatorname{Aut}(x)$ and all $\alpha, \beta \in \Gamma$. In other words,

$$(\alpha\beta)^*(f) = (t_{\alpha,\beta})_x \circ \alpha^*\beta^*(f) \circ (t_{\alpha,\beta})_x^{-1}. \tag{A-1}$$

Since $\operatorname{Aut}(\alpha^*\beta^* x)$ is abelian, $(t_{\alpha,\beta})_x^{-1} \phi_{\alpha\beta} \phi_\alpha^{-1} \alpha^*(\phi_\beta)^{-1}$ commutes with $\alpha^*\beta^*(f)$. This fact can be expressed as

$$\phi_\alpha^{-1} \alpha^*(\phi_\beta^{-1}) \circ \alpha^*\beta^*(f) \circ \alpha^*(\phi_\beta)\phi_\alpha = \phi_{\alpha\beta}^{-1} (t_{\alpha,\beta})_x \circ \alpha^*\beta^*(f) \circ (t_{\alpha,\beta})_x^{-1} \phi_{\alpha\beta},$$

which we rewrite as

$$\phi_\alpha^{-1} \alpha^* \Big(\phi_\beta^{-1} \beta^*(f) \phi_\beta \Big) \phi_\alpha = \phi_{\alpha\beta}^{-1} (t_{\alpha,\beta})_x \circ \alpha^*\beta^*(f) \circ (t_{\alpha,\beta})_x^{-1} \phi_{\alpha\beta}. \tag{A-2}$$

The left-hand side of (A-2) is $\phi_\alpha^{-1}\alpha^*(\beta \cdot f)\phi_\alpha = \alpha \cdot (\beta \cdot f)$, and, by (A-1), the right-hand side of (A-2) is equal to

$$\phi_{\alpha\beta}^{-1}(\alpha\beta)^*(f)\phi_{\alpha\beta},$$

which equals $(\alpha\beta) \cdot f$. Thus $\alpha \cdot (\beta \cdot f) = (\alpha\beta) \cdot f$.

To see that the action does not depend on the choice of the ϕ_α's suppose that $\{\phi'_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ is another collection of isomorphisms. There are automorphisms $\psi_\alpha \in \operatorname{Aut}(\alpha^* x)$ such that $\phi'_\alpha = \psi_\alpha \phi_\alpha$. The action of Γ on $\operatorname{Aut}(x)$ associated to the ϕ'_α, $\alpha \in \Gamma$, is

$$(\alpha, f) \;\mapsto\; (\phi'_\alpha)^{-1}\alpha^*(f)\phi'_\alpha \;=\; \phi_\alpha^{-1}\psi_\alpha^{-1}\alpha^*(f)\psi_\alpha\phi_\alpha,$$

but $\psi_\alpha^{-1}\alpha^*(f)\psi_\alpha = \alpha^*(f)$ because $\operatorname{Aut}(\alpha^* x)$ is abelian, so the right-hand side of the displayed equation is equal to $\alpha \cdot f$. □

A.2. **Equivariant objects.** Suppose Γ acts on C. A Γ-equivariant structure on an object $x \in \mathsf{C}$ is a set of isomorphisms $\{\phi_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ such that the diagrams

(A-3)
$$\begin{array}{ccc} x & \xrightarrow{\ \phi_\alpha\ } & \alpha^* x \\ {\scriptstyle \phi_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \alpha^*(\phi_\beta)} \\ (\alpha\beta)^* x & \xleftarrow[(t_{\alpha,\beta})_x]{} & \alpha^*(\beta^* x) \end{array}$$

commute for all $\alpha, \beta, \gamma \in \Gamma$.

An arbitrary set of isomorphisms $\phi_\alpha : x \to \alpha^* x$, $\alpha \in \Gamma$, will not usually give an equivariant structure on x. Their failure to do so, i.e., the failure of (A-3) to commute, is measured by the automorphisms

(A-4)
$$a_{\alpha,\beta} \;:=\; \phi_{\alpha\beta}^{-1} \circ (t_{\alpha,\beta})_x \circ \alpha^*(\phi_\beta) \circ \phi_\alpha$$

of x.

Lemma A.2. *Let $x \in \mathsf{Ob}(\mathsf{C})$ and let $\{\phi_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ be a set of isomorphisms. If $\operatorname{Aut}(x)$ is abelian, then the function*

$$a : \Gamma \times \Gamma \to \operatorname{Aut}(x), \qquad (\alpha, \beta) \mapsto a_{\alpha,\beta},$$

is a 2-cocycle.

Proof. We must show that $a_{\alpha\beta,\gamma} \circ a_{\alpha,\beta} = a_{\alpha,\beta\gamma} \circ (\alpha \cdot a_{\beta,\gamma})$ for all $\alpha, \beta, \gamma \in \Gamma$.

First, $a_{\alpha\beta,\gamma} \circ a_{\alpha,\beta}$ equals

$$\begin{aligned} &\phi_{\alpha\beta\gamma}^{-1} \circ (t_{\alpha\beta,\gamma})_x \circ (\alpha\beta)^*(\phi_\gamma) \circ \phi_{\alpha\beta} \circ \phi_{\alpha\beta}^{-1} \circ (t_{\alpha,\beta})_x \circ \alpha^*(\phi_\beta) \circ \phi_\alpha \\ &= \phi_{\alpha\beta\gamma}^{-1} \circ (t_{\alpha\beta,\gamma})_x \circ (\alpha\beta)^*(\phi_\gamma) \circ (t_{\alpha,\beta})_x \circ \alpha^*(\phi_\beta) \circ \phi_\alpha \\ &= \phi_{\alpha\beta\gamma}^{-1} \circ (t_{\alpha\beta,\gamma})_x \circ (t_{\alpha,\beta})_{\gamma^* x} \circ \alpha^*\beta^*(\phi_\gamma) \circ \alpha^*(\phi_\beta) \circ \phi_\alpha, \end{aligned}$$

where the last equality follows from the commutative diagram

$$\begin{array}{ccc} \alpha^*\beta^* x & \xrightarrow{\ (t_{\alpha,\beta})_x\ } & (\alpha\beta)^* x \\ {\scriptstyle \alpha^*\beta^*(\phi_\gamma)}\downarrow & & \downarrow{\scriptstyle (\alpha\beta)^*(\phi_\gamma)} \\ \alpha^*\beta^*(\gamma^* x) & \xrightarrow[(t_{\alpha,\beta})_{\gamma^* x}]{} & (\alpha\beta)^*(\gamma^* x) \end{array}$$

which exists by virtue of the fact that $t_{\alpha,\beta}$ is a natural transformation (applied to the isomorphism $\phi_\gamma : x \to \gamma^* x$).

On the other hand, $a_{\alpha,\beta\gamma} \circ (\alpha \cdot a_{\beta,\gamma})$ equals

$$\begin{aligned}
&\phi_{\alpha\beta\gamma}^{-1} \circ (t_{\alpha,\beta\gamma})_x \circ \alpha^*(\phi_{\beta\gamma}) \circ \phi_\alpha \circ \phi_\alpha^{-1} \circ \alpha^*\big(\phi_{\beta\gamma}^{-1} \circ (t_{\beta,\gamma})_x \circ \beta^*(\phi_\gamma) \circ \phi_\beta\big) \circ \phi_\alpha \\
&= \phi_{\alpha\beta\gamma}^{-1} \circ (t_{\alpha,\beta\gamma})_x \circ \alpha^*((t_{\beta,\gamma})_x) \circ \alpha^*\beta^*(\phi_\gamma) \circ \alpha^*(\phi_\beta) \circ \phi_\alpha \\
&= \phi_{\alpha\beta\gamma}^{-1} \circ (t_{\alpha,\beta\gamma})_x \circ (\alpha^* \cdot t_{\beta,\gamma})_x \circ \alpha^*\beta^*(\phi_\gamma) \circ \alpha^*(\phi_\beta) \circ \phi_\alpha \\
&= \phi_{\alpha\beta\gamma}^{-1} \circ (t_{\alpha\beta,\gamma})_x \circ (t_{\alpha,\beta})_{\gamma^* x} \circ \alpha^*\beta^*(\phi_\gamma) \circ \alpha^*(\phi_\beta) \circ \phi_\alpha.
\end{aligned}$$

Thus, $a_{\alpha,\beta\gamma} \circ (\alpha \cdot a_{\beta,\gamma}) = a_{\alpha\beta,\gamma} \circ a_{\alpha,\beta}$. □

Proposition A.3. *Let $x \in \mathsf{Ob}(\mathsf{C})$ and suppose $\operatorname{Aut}(x)$ is abelian. If the 2-cocycle $(\alpha,\beta) \mapsto a_{\alpha,\beta}$ defined in* (A-4) *is the coboundary of the function $f : \Gamma \to \operatorname{Aut}(x)$, $\alpha \mapsto a_\alpha$, then the isomorphisms $\{\phi_\alpha a_\alpha^{-1} : x \to \alpha^* x \mid \alpha \in \Gamma\}$ form an equivariant structure on x.*

Proof. The hypothesis says that

$$\phi_{\alpha\beta}^{-1} \circ (t_{\alpha,\beta})_x \circ \alpha^*(\phi_\beta) \circ \phi_\alpha \;=\; (df)(\alpha,\beta) \;=\; (\alpha \cdot a_\beta) \circ a_{\alpha\beta}^{-1} \circ a_\alpha$$

for all $\alpha, \beta \in \Gamma$. Since $\operatorname{Aut}(x)$ is abelian, we can rewrite this as

$$\begin{aligned}
\phi_{\alpha\beta}^{-1} \circ (t_{\alpha,\beta})_x \circ \alpha^*(\phi_\beta) \circ \phi_\alpha \;&=\; a_{\alpha\beta}^{-1} \circ a_\alpha \circ (\alpha \cdot a_\beta) \\
&=\; a_{\alpha\beta}^{-1} \circ a_\alpha \circ \phi_\alpha^{-1}\alpha^*(a_\beta)\phi_\alpha,
\end{aligned}$$

whence $(t_{\alpha,\beta})_x \circ \alpha^*(\phi_\beta) = \phi_{\alpha\beta} a_{\alpha\beta}^{-1} \circ a_\alpha \phi_\alpha^{-1} \circ \alpha^*(a_\beta)$. In other words, the diagram

$$\begin{array}{ccc}
x & \xrightarrow{\;\phi_\alpha a_\alpha^{-1}\;} & \alpha^* x \\
{\scriptstyle \phi_{\alpha\beta} a_{\alpha\beta}^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha^*(\phi_\beta a_\beta^{-1})} \\
(\alpha\beta)^* x & \xleftarrow[\;(t_{\alpha,\beta})_x\;]{} & \alpha^*(\beta^* x)
\end{array}$$

commutes; i.e., the maps $\{\phi_\alpha a_\alpha^{-1} : x \to \alpha^* x \mid \alpha \in \Gamma\}$ form an equivariant structure on x. □

A.3. **Classification of equivariant structures.** In order to classify equivariant structures we must first say what it means for two equivariant structures to be the "same".

Suppose that Γ acts on C. The objects in the category C^Γ of Γ-*equivariant objects in* C are pairs (x, ϕ) consisting of an object x in C and a set of isomorphisms $\phi = \{\phi_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ that give x the structure of a Γ-equivariant object. A morphism $f : (x, \phi) \to (y, \psi)$ in C^Γ is a morphism $f : x \to y$ in C such that the diagram

$$\begin{array}{ccc}
x & \xrightarrow{\;\phi_\alpha\;} & \alpha^* x \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha^*(f)} \\
y & \xrightarrow[\;\psi_\alpha\;]{} & \alpha^* y
\end{array}$$

commutes for all $\alpha \in \Gamma$.

We will classify equivariant structures on an $x \in \mathsf{Ob}(\mathsf{C})$ up to isomorphism in the special case when $\operatorname{Aut}(x)$ is abelian.

Lemma A.4. *Let $x \in \mathsf{Ob}(\mathsf{C})$. Suppose that $\{\phi_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ and $\{\psi_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ are equivariant structures on x. If $\mathrm{Aut}(x)$ is abelian, then the function $f : \Gamma \to \mathrm{Aut}(x)$, $f(\alpha) := \psi_\alpha^{-1}\phi_\alpha$, is a 1-cocycle.*

Proof. By definition,

$$(df)(\alpha,\beta) = (\alpha \cdot \psi_\beta^{-1}\phi_\beta) \circ \left(\psi_{\alpha\beta}^{-1}\phi_{\alpha\beta}\right)^{-1} \circ \psi_\alpha^{-1}\phi_\alpha. \tag{A-5}$$

Because the ϕ's and ψ's define equivariant structures,

$$\begin{aligned} \psi_{\alpha\beta}^{-1}\phi_{\alpha\beta} &= \Big(t_{\alpha,\beta}\alpha^*(\psi_\beta)\psi_\alpha\Big)^{-1} \circ \Big(t_{\alpha,\beta}\alpha^*(\phi_\beta)\phi_\alpha\Big) \\ &= \psi_\alpha^{-1}\alpha^*(\psi_\beta^{-1}\phi_\beta)\phi_\alpha. \end{aligned}$$

Therefore

$$\begin{aligned} (df)(\alpha,\beta) &= \phi_\alpha^{-1}\alpha^*(\psi_\beta^{-1}\phi_\beta)\phi_\alpha \circ \left(\psi_\alpha^{-1}\alpha^*(\psi_\beta^{-1}\phi_\beta)\phi_\alpha\right)^{-1} \circ \psi_\alpha^{-1}\phi_\alpha \\ &= \mathrm{id}_x. \end{aligned}$$

Thus, f is a 1-cocycle as claimed. □

Let $x \in \mathsf{Ob}(x)$. We write $\Phi(x)$ for the set of equivariant structures on x and $\Phi(x)_{\mathsf{Isom}}$ for the set of isomorphism classes of equivariant structures on x. If $\phi = \{\phi_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\} \in \Phi(x)$ we write $[\phi]$ for the isomorphism class of ϕ; i.e., $\phi \mapsto [\phi]$ denotes the obvious function $\Phi(x) \to \Phi(x)_{\mathsf{Isom}}$.

Proposition A.5. *Let $x \in \mathsf{Ob}(\mathsf{C})$ and suppose $\mathrm{Aut}(x)$ is abelian. If $\phi = \{\phi_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ is an equivariant structure on x and $f : \Gamma \to \mathrm{Aut}(x)$, $\alpha \mapsto f_\alpha$, a 1-cocycle, then*

$$(f \cdot \phi) := \{\phi_\alpha f_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$$

is an equivariant structure on x that depends only on the class of f in $H^1(\Gamma, \mathrm{Aut}(x))$. This gives an action of $H^1(\Gamma, \mathrm{Aut}(x))$ on $\Phi(x)_{\mathsf{Isom}}$. Furthermore, if $\Phi(x) \neq \varnothing$, then $H^1(\Gamma, \mathrm{Aut}(x))$ acts simply transitively on $\Phi(x)_{\mathsf{Isom}}$.

Proof. Let $[f] \in H^1(\Gamma, \mathrm{Aut}(x))$ where f is a 1-cocycle. Let $\phi = \{\phi_\alpha\} \in \Phi(x)$. Because f is a 1-cocycle, $(\alpha \cdot f_\beta) f_{\alpha\beta}^{-1} f_\alpha = \mathrm{id}_x$. Because $\mathrm{Aut}(x)$ is abelian this equality can be rewritten as

$$f_{\alpha\beta} = (\alpha \cdot f_\beta) f_\alpha = \phi_\alpha^{-1}\alpha^*(f_\beta)\phi_\alpha f_\alpha.$$

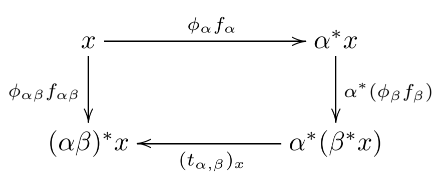
Since the ϕ_α's form an equivariant structure on x,

$$\phi_{\alpha\beta} = (t_{\alpha,\beta})_x \alpha^*(\phi_\beta)\phi_\alpha$$

for all $\alpha, \beta \in \Gamma$. Therefore

$$\phi_{\alpha\beta} f_{\alpha\beta} = \Big((t_{\alpha,\beta})_x \alpha^*(\phi_\beta)\phi_\alpha\Big) \circ \Big(\phi_\alpha^{-1}\alpha^*(f_\beta)\phi_\alpha f_\alpha\Big) = (t_{\alpha,\beta})_x \alpha^*(\phi_\beta f_\beta)\phi_\alpha f_\alpha.$$

In other words, the diagram



commutes; i.e., the maps $\{\phi_\alpha f_\alpha : x \to \alpha^* x \mid \alpha \in \Gamma\}$ form an equivariant structure on x.

We now show that the isomorphism class of $(x, f \cdot \phi)$ depends only on the cohomology class of f. Let $f, f' : \Gamma \to \mathrm{Aut}(x)$ be 1-cocycles. They are cohomologous if and only if $f' f^{-1} = dg$ for some $g \in C^0(\Gamma, \mathrm{Aut}(x)) = \mathrm{Aut}(x)$, i.e., if and only if there is $g \in \mathrm{Aut}(x)$ such that

$$f'_\alpha f_\alpha^{-1} = (dg)(\alpha) = (\alpha \cdot g) g^{-1}$$

for all $\alpha \in \Gamma$. On the other hand, $(x, f \cdot \phi) \cong (x, f' \cdot \phi)$ if and only if there is an isomorphism $g : x \to x$ such that the diagram

$$\begin{array}{ccc} x & \xrightarrow{\phi_\alpha f_\alpha} & \alpha^* x \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \alpha^*(g)} \\ x & \xrightarrow[\phi_\alpha f'_\alpha]{} & \alpha^* x \end{array}$$

commutes for all $\alpha \in \Gamma$, i.e., if and only if $\alpha^*(g)\phi_\alpha f_\alpha = \phi_\alpha f'_\alpha g$ or, equivalently, $\phi_\alpha^{-1}\alpha^*(g)\phi_\alpha f_\alpha = f'_\alpha g$ for all $\alpha \in \Gamma$. Since $\mathrm{Aut}(x)$ is abelian, this is equivalent to the condition that $\phi_\alpha^{-1}\alpha^*(g)\phi_\alpha g^{-1} = f'_\alpha f_\alpha^{-1}$ for all $\alpha \in \Gamma$, i.e., $(\alpha \cdot g)g^{-1} = f'_\alpha f_\alpha^{-1}$. This completes the proof that $(x, f \cdot \phi) \cong (x, f' \cdot \phi)$ if and only if $[f] = [f']$. Thus, once we have shown that $([f], [\phi]) \mapsto [f \cdot \phi]$ really is an action, as we do in the next paragraph, we will have shown that $H^1(\Gamma, \mathrm{Aut}(x))$ acts on $\Phi(x)_{\mathsf{Isom}}$ and all isotropy groups are trivial.

We now check that $([f], \phi) \mapsto (f \cdot \phi)$ is an action of $H^1(\Gamma, \mathrm{Aut}(x))$ on $\Phi(x)$. Let $f, f' : \Gamma \to \mathrm{Aut}(x)$ be 1-cocycles. Then $f \cdot (f' \cdot \phi) = \{\phi_\alpha f'_\alpha f_\alpha \mid \alpha \in \Gamma\}$. Since f_α and f'_α are elements in the abelian group $\mathrm{Aut}(x)$, $f'_\alpha f_\alpha = f_\alpha f'_\alpha$, from which it follows that $f \cdot (f' \cdot \phi) = (ff') \cdot \phi$.

It remains to show that $H^1(\Gamma, \mathrm{Aut}(x))$ acts transitively on $\Phi(x)_{\mathsf{Isom}}$ is transitive. Let $\phi, \phi' \in \Phi(x)$. We will show there is a 1-cocycle f such that $\phi' \cong f \cdot \phi$. By Lemma A.6 below, the function $f : \Gamma \to \mathrm{Aut}(x)$ defined by $f(\alpha) := \phi_\alpha^{-1}\phi'_\alpha$ is a 1-cocycle. But $(f \cdot \phi)_\alpha = \phi_\alpha f_\alpha = \phi'_\alpha$, so $\phi' = f \cdot \phi$. □

Lemma A.6. *Let* $x \in \mathsf{Ob}(\mathsf{C})$ *and suppose that* $\mathrm{Aut}(x)$ *is abelian. If* $\phi, \psi \in \Phi(x)$, *then* $\phi^{-1}\psi := \{\phi_\alpha^{-1}\psi_\alpha \mid \alpha \in \Gamma\}$ *is a* 1*-cocycle for* Γ *with values in* $\mathrm{Aut}(x)$.

Proof. We must show that $d(\phi^{-1}\psi)(\alpha, \beta)$ is the identity for all $\alpha, \beta \in \Gamma$. This is the case because

$$\begin{aligned} d(\phi^{-1}\psi)(\alpha,\beta) &= \alpha \cdot (\phi_\beta^{-1}\psi_\beta) \circ (\phi_{\alpha\beta}^{-1}\psi_{\alpha\beta})^{-1} \circ \phi_\alpha^{-1}\psi_\alpha \\ &= \alpha \cdot (\phi_\beta^{-1}\psi_\beta) \circ \phi_\alpha^{-1}\psi_\alpha \circ (\phi_{\alpha\beta}^{-1}\psi_{\alpha\beta})^{-1} \\ &= \phi_\alpha^{-1}\alpha^*(\phi_\beta^{-1}\psi_\beta)\phi_\alpha \circ \phi_\alpha^{-1}\psi_\alpha \circ \psi_{\alpha\beta}^{-1}\phi_{\alpha\beta} \\ &= \phi_\alpha^{-1}\alpha^*(\phi_\beta^{-1}\psi_\beta)\psi_\alpha \circ \psi_\alpha^{-1}\alpha^*(\psi_\beta)^{-1}(t_{\alpha,\beta})_x^{-1} \circ (t_{\alpha,\beta})_x \alpha^*(\phi_\beta)\phi_\alpha \\ &= \phi_\alpha^{-1}\alpha^*(\phi_\beta^{-1}\psi_\beta)\alpha^*(\psi_\beta)^{-1}\alpha^*(\phi_\beta)\phi_\alpha, \end{aligned}$$

which is certainly equal to id_x. □

A.4. **Equivariant modules.** Let Γ act as k-algebra automorphisms of a k-algebra R. If $\alpha \in \Gamma$ and M is a left R-module we define $\alpha^* M$ to be M as a k-vector space with a new action of R, namely, $x \cdot_\alpha m := \alpha^{-1}(x)m$. If $f : M \to N$ is an R-module homomorphism we define $\alpha^*(f) : \alpha^* M \to \alpha^* N$ to be the function f, now viewed as a homomorphism from $\alpha^* M$ to $\alpha^* N$. In this way, α^* becomes an auto-equivalence of the category of left R-modules, $\mathsf{Mod}(R)$. Since $\alpha^*\beta^* = (\alpha\beta)^*$ this gives an action of Γ on $\mathsf{Mod}(R)$.

Suppose M is a Γ-equivariant left R-module via the isomorphisms $\phi_\alpha : M \to \alpha^* M$, $\alpha \in \Gamma$. Since $\alpha^* M = M$, each ϕ_α is a k-linear map $\phi_\alpha : M \to M$ and it has the property that $\phi_\alpha(xm) = x \cdot_\alpha \phi_\alpha(m) = \alpha^{-1}(x)\phi_\alpha(m)$ or, equivalently, $\phi_\alpha^{-1}(xm) = \alpha(x)\phi_\alpha^{-1}(m)$, for all $x \in R$ and $m \in M$. If we write $m^\alpha := \phi_\alpha^{-1}(m)$, then we obtain a left action of Γ on M with the property that $(xm)^\alpha = \alpha(x)m^\alpha$ for all $x \in R$, $\alpha \in \Gamma$, and $m \in M$.

Conversely, if M is a left R-module with a left action of Γ on M such that $(xm)^\alpha = \alpha(x)m^\alpha$ for all $x \in R$, $\alpha \in \Gamma$, and $m \in M$, then the maps $\phi_\alpha : M \to \alpha^* M$ defined by $\phi_\alpha(m) = m^{\alpha^{-1}}$ gives M the structure of a Γ-equivariant R-module.

Thus, a Γ-equivariant R-module is an R-module, M, say, together with an action of Γ via a group homomorphism $\Gamma \to \mathrm{Aut}_{\mathbb{Z}}(M)$, $\alpha \mapsto (m \mapsto m^\alpha)$, such that $(xm)^\alpha = \alpha(x)m^\alpha$ for all $\alpha \in \Gamma$ and $m \in M$.

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