TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 371, Number 1, 1 January 2019, Pages 1–31
https://doi.org/10.1090/tran/7581
Article electronically published on September 20, 2018

POWER MOMENTS AND VALUE DISTRIBUTION OF FUNCTIONS

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ABSTRACT. In this paper we study various "abscissae" which one can associate to a given function f, or rather to the power moments of f. These are motivated by long-standing open problems in analytic number theory. We show how these abscissae connect to the distribution of values of f in a very elegant way using convex conjugates. This connection allows us to show which abscissae are realizable for both general and more specific arithmetical functions. Further it may give a new approach to, for example, Dirichlet's divisor problem.

INTRODUCTION

A number of major open problems in analytic number theory show similar characteristics. As typical examples, consider two of these — the *Dirichlet divisor problem* (DDP) and the *Lindelöf Hypothesis* (LH). For the DDP, the problem consists of finding the maximal behaviour of the function[1]

$$\Delta(x) = \sum_{n\le x}\Big(d(n) - \log n - 2\gamma\Big),$$

where $d(n)$ is the number of divisors of n and γ is Euler's constant. Dirichlet proved in an elementary way that $\Delta(x) = O(\sqrt{x})$, which has been sharpened to $O(x^{\frac{1}{3}})$ by Voronoi and subsequently by many researchers, the most recent being $O(x^{\frac{517}{1648}+\varepsilon})$ ($\forall\varepsilon > 0$) by Bourgain and Watt [5]. On the other hand, Hardy showed it is not $o(x^{\frac{1}{4}})$. The problem is to find the optimal exponent which is widely believed to be $\frac{1}{4}$; i.e., $\Delta(x) = O(x^{\frac{1}{4}+\varepsilon})$ for all $\varepsilon > 0$. Although this has never been proven, it is true in a mean-square average sense as Cramér [6] proved that $\int_0^x \Delta^2 \sim cx^{3/2}$ for some $c > 0$.

Turning to the second problem, LH is the statement that

$$\zeta(\tfrac{1}{2} + it) = O(t^{\varepsilon})$$

for all $\varepsilon > 0$, where $\zeta(s)$ is Riemann's zeta function and its analytic continuation. It was shown by Hardy and Littlewood that $\zeta(\frac{1}{2} + it) = O(t^{\frac{1}{6}})$, a bound which has subsequently been improved by many authors, the most recent being Bourgain's $O(t^{\frac{13}{84}+\varepsilon})$ ($\forall\varepsilon > 0$) (see [4]). As for Δ, the conjecture is true in a mean-square

Received by the editors July 11, 2016.
2010 *Mathematics Subject Classification.* Primary 11N64, 52A41.
Key words and phrases. Power moments of functions, Dirichlet divisor problem, Lindelőf Hypothesis, convex conjugate, Lindelőf function.

[1]Usually this is defined by $\Delta(x) = \sum_{n\le x} d(n) - (x\log x + (2\gamma - 1)x)$, but this differs from our definition by an insignificant $O(\log x)$.

average sense, for it is known that $\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt \sim T\log T$. Furthermore, it is also widely believed as LH is implied by the Riemann Hypothesis (RH).

In both cases, we see that we have a function $f:(0,\infty)\to\mathbb{C}$ satisfying $f(x)=O(x^\lambda)$ and we need to find the optimal λ for which this holds. We have some further information about the mean-square of the function $\int_0^x |f|^2$ and, in these two instances, also about some higher powers. In the first case, the function has an arithmetical character, namely $f(x)=f([x])$, while in the second case, f is holomorphic.

This motivates the following definition. For a function $f:(0,\infty)\to\mathbb{C}$ on the positive reals which satisfies $f(x)=O(x^\lambda)$ for $x\ge 1$, define the following *abscissae*: let θ be the infimum of such λ, and for each $p>0$ let

$$\theta_p = \inf\left\{\lambda : \Big(\frac{1}{x}\int_x^{2x} |f|^p\Big)^{1/p} = O(x^\lambda)\right\}.$$

Of course, the main motivation for studying these "abscissae" comes from special functions in number theory, but our set up here is very general. When f is "arithmetical" (i.e. $f(x)=f([x])$), then it is a step function of the form

$$f(x)=\sum_{n\le x} a_n$$

for some a_n and θ corresponds to the usual abscissa of convergence of the Dirichlet series $\sum a_n n^{-s}$ (if $\sum a_n$ diverges). Determining these abscissae can be very difficult for "naturally occurring" examples like $a_n = d(n)-\log n - 2\gamma$. For this example it is known that $\theta_p=\frac{1}{4}$ for $p\le 9$ (see [12]) while $\frac{1}{4}\le\theta\le\frac{517}{1648}$ as mentioned before.

It is straightforward to show that θ_p is increasing as a function of p, and further that $p\theta_p$ is convex. Natural questions that now arise are (i) when does $\theta_p\to\theta$ as $p\to\infty$? (ii) when do we have $\theta_p\equiv\theta$?, (iii) what type of functions can θ_p be?

We shall answer these in a very general way by linking these abscissae to a measurement of how often $|f(x)|$ is larger than a given power of x; namely by using the sets

$$U_\lambda(X) := \{x\in[X,2X] : |f(x)|\ge x^\lambda\}.$$

Indeed, let $\sigma(\lambda)$ be defined by $|U_\lambda(X)| \ll X^{1-\sigma(\lambda)+\varepsilon}$ for all $\varepsilon>0$ but no $\varepsilon<0$. Then we shall find that $p\theta_p=\sigma^*(p)$ – the *convex conjugate* or *Legendre-Fenchel transform* of $\sigma(\lambda)$ (Theorem 2.2). In this way we give necessary and sufficient conditions for which $\theta_p\to\theta$ and $\theta_p\equiv\theta$, as well as showing that θ_p can be *any* increasing function such that $p\theta_p$ is convex (Theorem 2.5). By adjusting the proof, we show in §3 that this remains essentially true among the "arithmetical" f. The method also provides a possible new approach to Dirichlet's divisor problem: *if the slope of $\sigma(\lambda)$ at $\theta-$ is finite and <9, then $\theta=\frac{1}{4}$.*

In §4, we consider the abscissae $\mu(\sigma)$ and $\mu_p(\sigma)$ (or "Lindelőf" functions) of the associated Mellin transform. The main result here is that, in a quite general way, $\mu_2(\theta_2)=\frac{1}{2}$.

In §5, we explore a number of examples from diverse areas to highlight the different possible behaviours of θ_p, $\sigma(\lambda)$, and $\mu(\sigma)$.

In §6, we extend the results to the infinite case by considering a class of functions for which θ_p is finite for all $p>0$ but $\theta=\infty$. A typical "arithmetical" example is $d([e^{x-1}])$, where d is the divisor function.

Notation

The symbols $o, O, \ll, \Omega$ have their usual meanings, namely: (i) $f(x) = O(g(x))$ (equivalently $f(x) \ll g(x)$) if $|f(x)| \le Ag(x)$ for some constant A and all $x \ge x_0$ (some x_0); (ii) $f(x) = o(g(x))$ if $f(x)/g(x) \to 0$ as $x \to \infty$; while (iii) $f(x) = \Omega(g(x))$ is its negation; i.e., there exist $x_n \to \infty$ such that $|f(x_n)| \ge c|g(x_n)|$ for some $c > 0$ and all n.

1. Various abcissae: θ, θ_p, and $\overline{\theta}$

Let S denote the class of functions $f : [A, \infty) \to \mathbb{C}$ (some constant $A > 0$) which is locally of bounded variation and satisfies $f(x) = O(x^\lambda)$ for some λ.

Definition 1.1. Let $f \in S$ and let $p > 0$. Define θ_p and θ as follows:

$$\theta_p = \inf\left\{\lambda : \Big(\frac{1}{x}\int_x^{2x} |f|^p\Big)^{1/p} = O(x^\lambda)\right\}, \tag{1.1}$$
$$\theta = \inf\{\lambda : f(x) = O(x^\lambda)\}.$$

We note that θ_p can be characterised in terms of $\int_0^x |f|^p$ for large x as follows:

$$\theta_p = \begin{cases} \inf\{\lambda : (\frac{1}{x}\int_0^x |f|^p)^{1/p} = O(x^\lambda)\} & \text{if } \int_0^\infty |f|^p = \infty, \\ \inf\{\lambda : (\frac{1}{x}\int_x^\infty |f|^p)^{1/p} = O(x^\lambda)\} & \text{if } \int_0^\infty |f|^p < \infty. \end{cases}$$

Remark 1.1. It is quite possible that either infimum does not exist, for example if $f(x)$ is exponentially small. In this case we write $\theta_p = -\infty$ or $\theta = -\infty$. It is also possible to have $\theta_p = -\infty$ for all p but θ finite, for example, if $f(n) = 1$ for $n \in \mathbb{N}$ and zero otherwise.

Proposition 1.1 (Basic properties)**.**

(a) *θ_p increases with p;*
(b) *$\theta_p \le \theta$ for all p;*
(c) *if $\theta_q = -\infty$ for some q, then $\theta_p = -\infty$ for all p;*
(d) *$p\theta_p$ is a convex function of p.*

Proof. (a) Let $q > p > 0$. By Hölder's inequality

$$\int_x^{2x} |f|^p = \int_x^{2x} (|f|^q)^{p/q} \cdot 1^{1-p/q} \le \Big(\int_x^{2x} |f|^q\Big)^{p/q} \Big(\int_x^{2x} 1\Big)^{1-p/q} \tag{1.2}$$
$$\ll x^{(1+q\theta_q+\varepsilon)\frac{p}{q}} \cdot x^{1-\frac{p}{q}} \le x^{1+p\theta_q+\varepsilon}$$

and $\theta_p \le \theta_q$ follows.

(b) We have $f(x) \ll x^{\theta+\varepsilon}$ for every $\varepsilon > 0$, so $\int_x^{2x} |f|^p \ll x^{1+p(\theta+\varepsilon)}$. Thus $\theta_p \le \theta$.

(c) Suppose $\theta_q = -\infty$. If $p < q$, then (1.2) tells us that $\int_x^{2x} |f|^p \ll x^{-A}$ for all A, and so $\theta_p = -\infty$.

Now suppose $p > q$. Since $|f(x)| \le x^c$ for some c, we have

$$\int_x^{2x} |f|^p = \int_x^{2x} |f|^q |f|^{p-q} \ll x^c \int_x^{2x} |f|^q \ll x^{-A}$$

for all A. Thus, again, $\theta_p = -\infty$.

(d) Let $p < q < r$. Then by Hölder's inequality

$$\int_x^{2x} |f|^q = \int_x^{2x} |f|^{p(\frac{r-q}{r-p})} |f|^{r(\frac{q-p}{r-p})} \le \Big(\int_x^{2x} |f|^p\Big)^{\frac{r-q}{r-p}} \Big(\int_x^{2x} |f|^r\Big)^{\frac{q-p}{r-p}}.$$

Thus

$$\frac{1}{x}\int_x^{2x}|f|^q \le \left(\frac{1}{x}\int_x^{2x}|f|^p\right)^{\frac{r-q}{r-p}}\left(\frac{1}{x}\int_x^{2x}|f|^r\right)^{\frac{q-p}{r-p}} = O\Big(x^{\frac{r-q}{r-p}p\theta_p+\frac{q-p}{r-p}r\theta_r+\varepsilon}\Big)$$

for all $\varepsilon>0$. The LHS integral above is $\Omega(x^{1+q\theta_q-\varepsilon})$, so

$$q\theta_q \le \frac{r-q}{r-p}p\theta_p + \frac{q-p}{r-p}r\theta_r, \qquad \text{as required.} \tag{1.3}$$

□

Remarks 1.2.

(a) As a consequence of convexity, we find that (i) θ_p is continuous, and (ii) $\theta_{p_0}=\theta$ for some p_0 implies $\theta_p\equiv\theta$ (also using monotonicity). Furthermore, $p(\theta-\theta_p)$ is increasing. For it is concave, and so for $p<q<r$,

$$(r-p)q(\theta-\theta_q) \ge (q-p)r(\theta-\theta_r)+(r-q)p(\theta-\theta_p) \ge (r-q)p(\theta-\theta_p),$$

which implies $r(p(\theta-\theta_p)-q(\theta-\theta_q)) \le pq(\theta_q-\theta_p)$. If $p(\theta-\theta_p)>q(\theta-\theta_q)$, then the LHS can be made arbitrarily large by increasing r – a contradiction.
Let $\Delta=\lim_{p\to\infty}p(\theta-\theta_p)\in[0,\infty]$. Thus $\Delta=0$ if and only if $\theta_p\equiv\theta$.

(b) Using $\theta_r\le\theta$, letting $r\to\infty$ in (1.3), and putting $p=1$ gives, for $q>1$

$$q\theta_q \le \theta_1+(q-1)\theta.$$

So, for example, $2\theta_2\le\theta_1+\theta$.

The abscissa $\overline{\theta}$. For $f\in S$, let $V_f(x)$ denote the total variation of f on $[0,x]$. Let $\overline{\theta}$ be defined by

$$\overline{\theta}=\inf\{\lambda : V_f(x)=O(x^\lambda)\}.$$

It may happen that there is no λ for which $V_f(x)=O(x^\lambda)$; in this case we say $\overline{\theta}=\infty$. Trivially, $\overline{\theta}\ge\theta$.

Basic facts.

(a) If $f(x)=\sum_{n\le x}a_n$, then $V_f(x)=\sum_{n\le x}|a_n|$ and $\overline{\theta}$ corresponds to the abscissae of absolute convergence of $\sum a_n n^{-s}$ (if $\sum|a_n|$ diverges). Note that in this case $\overline{\theta}\le\theta+1$ (if $\theta\ge-1$). For

$$V_f(x)=\sum_{n\le x}|f(n)-f(n-1)| \le 2\sum_{n\le x}|f(n)| \ll \sum_{n\le x}n^{\theta+\varepsilon} \ll x^{\theta+1+\varepsilon}.$$

(b) If f is absolutely continuous, then $V_f(x)=\int_0^x|f'|$ (see [29, p. 393]).
Unlike in case (a), in general we have no control over the size of $\overline{\theta}-\theta$. For example, let $f(x)=\sin(x^\lambda)$ where $\lambda>0$. Then $\theta=0$ while $\overline{\theta}=\lambda$, since

$$V_f(x)=\int_0^x|\lambda t^{\lambda-1}\cos(t^\lambda)|\,dt \asymp x^\lambda.$$

In fact it may happen that $\overline{\theta}=\infty$ (see example (iv) in §3).

(c) Observe that θ and θ_p depend only on $|f|$ (i.e., f and $|f|$ having the same abscissae) while $\overline{\theta}$ may be different for f and $|f|$.

2. Connecting θ_p to the distribution of f: The function $\sigma(\lambda)$

In order to answer the questions about the possibilities for θ_p, we define the following quantities which measure how often f is "large".

Definition 2.1. Let $f \in S$ with finite abscissa θ. For $\lambda \in \mathbb{R}$ and $x \geq 1$ let

$$U_\lambda(x) = \{x \leq t \leq 2x : |f(t)| \geq t^\lambda\},$$

and denote the Lebesque measure of $U_\lambda(x)$ by $|U_\lambda(x)|$.

Note that for $\lambda < \mu$ we have $U_\lambda(x) \supset U_\mu(x)$, $|U_\lambda(x)| \leq x$, while $U_\lambda(x) = \emptyset$ for $\lambda > \theta$. As $|U_\lambda(\cdot)| \in S$, we may define the function $\sigma(\lambda) \in [0, \infty]$ via

$$|U_\lambda(x)| = O(x^{1-\sigma(\lambda)+\varepsilon}) \quad \text{for all } \varepsilon > 0 \text{ but no } \varepsilon < 0.$$

We see immediately that (i) $\sigma(\lambda)$ is increasing, (ii) $\sigma(\lambda) \geq 0$ for all λ, and (iii) $\sigma(\lambda) = \infty$ for $\lambda > \theta$.

2.1. Connecting θ_p and $\sigma(\lambda)$.

Given the example from Remark 1.1, it is clear that we need some sort of regularity condition on f in order for $\theta_p \to \theta$ as $p \to \infty$ to hold. In Theorem 2.1 below, we find a necessary and sufficient condition for $\theta_p \to \theta$, as well as for $\theta_p \equiv \theta$ using $\sigma(\lambda)$.

In Theorem 2.2, we go further and show how $\sigma(\lambda)$ determines θ_p.

Theorem 2.1. *Let $f \in S$ with abscissae θ and θ_p. Then*

$$\text{(i)} \qquad \theta_p \to \theta \iff \sigma(\lambda) < \infty \quad \textit{for all } \lambda < \theta,$$
$$\text{(ii)} \qquad \theta_p \equiv \theta \iff \sigma(\lambda) = 0 \quad \textit{for all } \lambda < \theta.$$

Proof. (i) ($\Leftarrow$) Let $\varepsilon > 0$. By assumption, there exists a such that $|U_{\theta-\varepsilon}(x)| = \Omega(x^{-a})$. Hence

$$x^{1+p\theta_p+\varepsilon} \gg \int_x^{2x} |f|^p \geq \int_{U_{\theta-\varepsilon}(x)} t^{p(\theta-\varepsilon)}\,dt \asymp x^{p(\theta-\varepsilon)}|U_{\theta-\varepsilon}(x)| = \Omega(x^{p(\theta-\varepsilon)-a}).$$

Hence $1 + p\theta_p + \varepsilon \geq p(\theta - \varepsilon) - a$; i.e.,

$$\theta_p \geq \theta - \varepsilon - \frac{1+a+\varepsilon}{p}. \tag{2.1}$$

As such, $\liminf_{p\to\infty} \theta_p \geq \theta - \varepsilon$. But this holds for every $\varepsilon > 0$, so $\liminf_{p\to\infty} \theta_p \geq \theta$, and $\theta_p \to \theta$ follows.

($\Rightarrow$) Suppose now that $\sigma(\lambda) = \infty$ for some $\lambda < \theta$; i.e., $|U_\lambda(x)| = O(x^{-A})$ for all A. Then, for all $\varepsilon > 0$, we have

$$\int_x^{2x} |f|^p \ll \int_{U_\lambda(x)} t^{p(\theta+\varepsilon)}dt + \int_{[x,2x]\setminus U_\lambda(x)} t^{p\lambda}\,dt \ll x^{p(\theta+\varepsilon)}|U_\lambda(x)| + x^{1+p\lambda} \ll x^{1+p\lambda}$$

for every $\varepsilon > 0$. The LHS is $\Omega(x^{1+p\theta_p-\varepsilon})$, so $\theta_p \leq \lambda < \theta$. Hence $\theta_p \not\to \theta$.

(ii) ($\Rightarrow$) Suppose $\theta_p \equiv \theta$ but that $\sigma(\lambda) > 0$ for some $\lambda < \theta$. Then $|U_\lambda(x)| = O(x^{1-\delta})$ for some $\delta > 0$. Now

$$\int_x^{2x} |f|^p = \int_{U_\lambda(x)} |f|^p + \int_{[x,2x]\setminus U_\lambda(x)} |f|^p \ll x^{p(\theta+\varepsilon)}|U_\lambda(x)| + x^{1+p\lambda}$$
$$\ll x^{1+p(\theta+\varepsilon)-\delta} + x^{1+p\lambda}$$

using the fact that for $t \in [x, 2x] \setminus S_\lambda(x)$, $|f(t)| \le t^\lambda \ll x^\lambda$. The LHS above is $\Omega(x^{1+p(\theta_p-\varepsilon)})$, so

$$\theta_p \le \max\Big\{\theta - \frac{\delta}{p}, \lambda\Big\} < \theta$$

– a contradiction.

($\Leftarrow$) For the converse implication, suppose that $\sigma(\lambda) = 0$ for all $\lambda < \theta$. Thus $|U_\lambda(x)| = \Omega(x^{1-\varepsilon})$ for all $\varepsilon > 0$ and $\lambda < \theta$. Hence

$$\int_x^{2x} |f|^p \ge \int_{U_\lambda(x)} |f|^p \gg x^{p\lambda}|U_\lambda(x)| = \Omega(x^{1+p\lambda-\varepsilon}).$$

Thus $\theta_p \ge \lambda$ for every $\lambda < \theta$. It follows that $\theta_p \ge \theta$, and hence equality must occur. □

Remarks 2.1.

- **(i):** For Theorem 2.1(i), we could equally well work with smaller intervals. For example, put $T_\lambda(x) = \{x \le t \le x + 1 : |f(t)| \ge t^\lambda\}$ with $|T_\lambda(x)|$ its Lebesque measure. Then $\theta_p \to \theta$ if and only if the same condition holds with U replaced by T. This follows immediately from the facts that $|T_\lambda(x)| \le |U_\lambda(x)|$ and $|U_\lambda(x)| \le \sum_{0\le k\le x} |T_\lambda(x + k)|$.
- **(ii):** If in Theorem 2.1(i) we assume that the condition holds *uniformly*; i.e., *there exists* $a(> -1)$ *such that* $\forall\ \varepsilon > 0$, $|U_{\theta-\varepsilon}(x)| = \Omega(x^{-a})$, then the same argument as in the ($\Leftarrow$) direction of the above proof shows that we have the stronger lower bound
$$\theta_p \ge \theta - \frac{1 + a}{p};$$
i.e., $p(\theta - \theta_p)$ remains bounded and so also $\Delta \le a + 1$.
- **(iii):** In the special case that f is a step function of the form $\sum_{n\le x} a_n$, then the condition holds uniformly with $a = 0$. For, given $\varepsilon > 0$, there exists a sequence of positive integers $n_k \nearrow \infty$ for which $|f(n_k)| \ge (n_k+1)^{\theta-\varepsilon}$. Hence for $x \in [n_k, n_k + 1]$, $|f(x)| = |f(x_n)| \ge x^{\theta-\varepsilon}$ and $|U_{\theta-\varepsilon}(n_k)| \ge |T_{\theta-\varepsilon}(n_k)| = 1$. Thus $\theta_p \ge \theta - \frac{1}{p}$.

Theorem 2.2. *Let $f \in S$ for which $\theta_p \to \theta$ as $p \to \infty$. Then*

$$p\theta_p = \sigma^*(p) := \sup_{\lambda\in\mathbb{R}} (p\lambda - \sigma(\lambda)).$$

Proof. We first prove $p\theta_p \ge \sigma^*(p)$.

Let $\varepsilon > 0$. By definition of $\sigma^*(p)$, there exists λ such that $p\lambda - \sigma(\lambda) > \sigma^*(p) - \varepsilon$. Hence

$$x^{1+p\theta_p+\varepsilon} \gg \int_x^{2x} |f|^p \ge \int_{U_\lambda(x)} t^{p\lambda}\, dt \gg x^{p\lambda}|U_\lambda(x)|$$
$$= \Omega(x^{p\lambda+1-\sigma(\lambda)-\varepsilon}) = \Omega(x^{1+\sigma^*(p)-2\varepsilon}).$$

This holds for all $\varepsilon > 0$, so $p\theta_p \ge \sigma^*(p)$.

Now we show the reverse inequality. Let λ be sufficiently small so that

$$\int_{[x,2x]\setminus U_\lambda(x)} |f|^p \ll x^c$$

for some $c < 1 + p\theta_p$. This is possible since the integral is $\ll \int_x^{2x} t^{p\lambda} dt \ll x^{1+p\lambda}$, so it suffices to have $\lambda < \theta_p$. As such, $\int_{U_\lambda(x)} |f|^p = \int_x^{2x} |f|^p + O(x^c)$. Now let $K \in \mathbb{N}$ (large). Choose $\mu > \theta$ and let x be sufficiently large so that $U_\mu(x) = \emptyset$. Then

$$\sum_{k=1}^{K} \int_{U_{\frac{(k-1)\mu}{K}+\lambda}(x) \setminus U_{\frac{k\mu}{K}+\lambda}(x)} |f|^p = \int_{U_\lambda(x)} |f|^p - \int_{U_\mu(x)} |f|^p = \int_{U_\lambda(x)} |f|^p.$$

The kth-term on the left is

$$\ll x^{p\frac{k\mu}{K}+p\lambda} |U_{\frac{(k-1)\mu}{K}+\lambda}(x)| \ll x^{\frac{pk\mu}{K}+p\lambda+1-\sigma(\frac{(k-1)\mu}{K}+\lambda)+\varepsilon} \leq x^{\sigma^*(p)+1+\frac{p\mu}{K}+\varepsilon}$$

for all $\varepsilon > 0$, since $t \notin U_{\frac{k\mu}{K}+\lambda}(x) \Rightarrow |f(t)| < t^{\frac{k\mu}{K}+\lambda} \ll x^{\frac{k\mu}{K}+\lambda}$. Thus

$$\int_{U_\lambda(x)} |f|^p \ll x^{\sigma^*(p)+1+\frac{p\mu}{K}+\varepsilon}.$$

The LHS is $\int_x^{2x} |f|^p + O(x^c) = \Omega(x^{1+p\theta_p-\varepsilon})$ for ε sufficiently small (since $c < 1+p\theta_p$). Thus $p\theta_p \leq \sigma^*(p) + \frac{p\mu}{K}$. But K was arbitrarily chosen. Hence $p\theta_p \leq \sigma^*(p)$ and the result follows. □

Remarks 2.2.

(a) From the theory of convexity, given g, the quantity

$$g^*(y) = \sup_{x\in\mathbb{R}} (xy - g(x))$$

is well-known and has been studied in great detail. It is called the *convex conjugate* of g, or sometimes the *Legendre-Fenchel transform* of g. It is necessarily convex where it is defined (whether or not g is convex). Furthermore, g^{**} is the *convex envelope* of g – the largest convex function h such that $h \leq g$. In particular, if g is convex, then $g^{**} = g$ (see [22] for a detailed discussion).

Letting $\tilde{\sigma}$ denote the *convex envelope* of σ (i.e., the largest convex function h such that $h \leq \sigma$), we find that

$$p\theta_p = \sup_{\lambda\in\mathbb{R}} (p\lambda - \tilde{\sigma}(\lambda)). \tag{2.2}$$

For $\tilde{\sigma}$ is convex, so $\tilde{\sigma}^{**} = \tilde{\sigma} = \sigma^{**}$ (from above). Hence $\tilde{\sigma}^* = \sigma^{***} = \sigma^*$, which is (2.2).

(b) We therefore see that the function $\sigma(\lambda)$ determines θ_p (given θ), but in the other direction θ_p only determines $\tilde{\sigma}(\lambda)$ (and not $\sigma(\lambda)$).

2.2. **Possible θ_p and $\sigma(\lambda)$.** The function $\sigma(\lambda)$ gives a very clear picture of the distribution of values of $|f(t)|$. We can use it to show that any ϕ_p satisfying (a), (b), and (d) of Proposition 1.1 can be realised as some θ_p. Indeed we show that, essentially, any non-negative increasing function on $(-\infty, \theta)$ can be realised as a σ-function. First we need the following lemma, which shows that in order to calculate $\sigma(\lambda)$ it is enough to know the behaviour of $|U_\lambda(2^n)|$.

Lemma 2.3. *Let $\lambda \in \mathbb{R}$. Suppose there exists σ such that $|U_\lambda(2^n)| = O(2^{n(1-\sigma+\varepsilon)})$ for every $\varepsilon > 0$ but for no $\varepsilon < 0$. Then $\sigma = \sigma(\lambda)$.*

Proof. Every $x \geq 1$ lies in some interval $[2^n, 2^{n+1})$. As such $U_\lambda(x) \subset U_\lambda(2^n) \cup U_\lambda(2^{n+1})$ and so

$$|U_\lambda(x)| \leq |U_\lambda(2^n)| + |U_\lambda(2^{n+1})| \ll 2^{n(1-\sigma+\varepsilon)} \ll x^{1-\sigma+\varepsilon}$$

for every $\varepsilon > 0$. That $|U_\lambda(x)| = \Omega(x^{1-\sigma-\varepsilon})$ follows directly from the $x = 2^n$ case. Hence $\sigma = \sigma(\lambda)$. □

Theorem 2.4. *Let $\phi \in \mathbb{R}$ and let $k : (-\infty, \phi) \to [0, \infty)$ be increasing and left-continuous. Then there exists $f \in S$ for which $\theta = \phi$ and $\sigma(\lambda) = k(\lambda)$ for $\lambda < \theta$.*

Proof. Let $k^{\leftarrow}$ denote the generalised inverse[2] of k. Let $f(x) = x^{\tau(x)}$ for $x \geq 1$ and zero otherwise, where τ is defined on intervals $(2^n, 2^{n+1}]$ ($n \in \mathbb{N}_0$) by

$$(t \leq 1) \qquad \tau(2^n + 2^{tn}) = k^{\leftarrow}(1-t) = \sup\{y < \phi : k(y) \leq 1-t\}.$$

Note that $\tau(x) \leq \phi$ always with equality when $x = 2^n+$ (i.e., $t \to -\infty$ above). Thus we see that $\theta = \phi$. We make the following convention: if there is *no* $y < \theta$ such that $k(y) \leq 1-t$ (e.g., when k is bounded below by a positive constant), then we take the sup to be $-\infty$ and $f(2^n + 2^{tn}) = 0$.

Now consider $U_\lambda(2^n)$ for $\lambda < \theta$. We have

$$U_\lambda(2^n) = \{x \in [2^n, 2^{n+1}] : \tau(x) \geq \lambda\} = \{2^n + 2^{tn} : -\infty \leq t \leq 1, k^{\leftarrow}(1-t) \geq \lambda\}.$$

If $t \leq 1 - k(\lambda)$, then $k^{\leftarrow}(1-t) \geq \lambda$, and so $2^n + 2^{tn} \in U_\lambda(2^n)$. Thus $U_\lambda(2^n) \supset (2^n, 2^n + 2^{n(1-k(\lambda))})$ and $|U_\lambda(2^n)| \geq 2^{n(1-k(\lambda))}$.

On the other hand, if $t > 1 - k(\lambda - \varepsilon)$ (where $\varepsilon > 0$), then $k^{\leftarrow}(1-t) \leq \lambda - \varepsilon < \lambda$ and $2^n + 2^{tn} \notin U_\lambda(2^n)$. Thus $U_\lambda(2^n) \subset [2^n, 2^n + 2^{n(1-k(\lambda-\varepsilon))})$. Thus $2^{n(1-k(\lambda))} \leq |U_\lambda(2^n)| \leq 2^{n(1-k(\lambda-\varepsilon))}$. But k is left-continuous so, letting $\varepsilon \to 0+$ gives $|U_\lambda(2^n)| = 2^{n(1-k(\lambda))}$. By Lemma 2.3, it follows that $\sigma(\lambda) = k(\lambda)$ for all $\lambda < \theta$. □

Remark 2.4. In Theorem 2.4, it is clear that $\overline{\theta} = \theta$. However, since θ and θ_p only depend on $|f|$, we could, by taking $f_1 = e^{i\varphi} f$ with φ real, find examples where $\overline{\theta}$ is any given real number greater than θ (by making φ oscillate).

Theorem 2.5. *Let $\phi, \phi_p \in \mathbb{R}$ ($p > 0$) such that $\phi_p \nearrow \phi$ as $p \to \infty$ and $p\phi_p$ is convex. Then there exists $f \in S$ such that $\theta = \phi$ and $\theta_p = \phi_p$.*

Proof. Consider the conjugate function of $p\phi_p$:

$$g(\lambda) = \sup_{p>0} (p\lambda - p\phi_p).$$

By the general theory, g is convex but also we observe that (i) $g(\lambda) = \infty$ for $\lambda > \phi$ while $g(\lambda) < \infty$ for $\lambda < \phi$, (ii) $g \geq 0$, and (iii) g is increasing.

For if $\lambda = \phi + \delta$, with $\delta > 0$, then $p(\lambda - \phi_p) \geq p\delta$ and the supremum can be made arbitrarily large. On the other hand, if $\lambda < \phi$, then, with p sufficiently large, $\phi_p > \lambda$ and so $p(\lambda - \phi_p) < 0$. For (ii), note that $p(\lambda - \phi_p) \geq p(\lambda - \phi)$, which can be made arbitrarily small, and hence $g(\lambda) \geq 0$. (iii) is immediate.

By Theorem 2.4, we can find $f \in S$ for which $\theta = \phi$ and $\sigma \equiv g$. Since $p\phi_p$ is convex, $g^*(p) = p\phi_p$. Thus

$$p\phi_p = \sup_{\lambda<\phi} (p\lambda - g(\lambda)) = \sup_{\lambda \in \mathbb{R}} (p\lambda - \sigma(\lambda)).$$

[2]For $f : (-\infty, X) \to \mathbb{R}$, the *generalised inverse* is defined to be the function $f^{\leftarrow}(x) = \inf\{y < X : f(y) > x\}$. If f is increasing, we automatically have $f^{\leftarrow}(x) = \sup\{y < X : f(y) \leq x\}$.

The RHS is $p\theta_p$ by Theorem 2.2, and the result follows. □

2.3. **The behaviour of θ_p for p large.** The behaviour of $\sigma(\lambda)$ for λ near θ is closely related to the behaviour of θ_p for p large. The function $\sigma(\lambda)$ is increasing, so $\lim_{\lambda\to\theta-}\sigma(\lambda)$ exists or is ∞. Denote this by $\sigma(\theta-)$ in either case. Recall that $\Delta=\lim_{p\to\infty}p(\theta-\theta_p)\in[0,\infty]$.

Proposition 2.6. *$p(\theta-\theta_p)$ is bounded above if and only if $\sigma(\theta-)<\infty$, in which case $\sigma(\theta-)=\Delta$.*

Proof. ($\Leftarrow$) Suppose $\sigma(\theta-)<\infty$. For every $\lambda<\theta$ and $p>0$, $p\theta_p\geq p\lambda-\sigma(\lambda)$, or $\sigma(\lambda)\geq p(\lambda-\theta_p)$. Letting $\lambda\to\theta$ gives $\sigma(\theta-)\geq p(\theta-\theta_p)$ for each $p>0$. Thus $\sigma(\theta-)\geq\Delta$.
($\Rightarrow$) Suppose $\Delta<\infty$. Then $\sup_{\lambda<\theta}(p\lambda-\sigma(\lambda))=p\theta_p\geq p\theta-\Delta$ for all $p>0$. But for every $\delta>0$,

$$\sup_{\lambda\leq\theta-\frac{\Delta+\delta}{p}}(p\lambda-\sigma(\lambda))\leq\sup_{\lambda\leq\theta-\frac{\Delta+\delta}{p}}p\lambda=p\theta-\Delta-\delta<p\theta-\Delta.$$

Thus

$$p\theta-\Delta\leq\sup_{\theta-\frac{\Delta+\delta}{p}<\lambda<\theta}(p\lambda-\sigma(\lambda))\leq p\theta-\sigma\Big(\theta-\frac{\Delta+\delta}{p}\Big).$$

Hence $\sigma(\theta-\frac{\Delta+\delta}{p})\leq\Delta$ for all $p,\delta>0$; i.e., $\sigma(\theta-)\leq\Delta$. □

In many examples (e.g., in the "extremal" case – see §3.3), we find that $p(\theta-\theta_p)$ is constant from some point onwards. We now give a necessary and sufficient condition for this to hold.

In order to state it we mention some facts concerning the smoothness of convex functions (see [22]). For g convex on an open interval, both left and right derivatives exist; that is,

$$g'_-(x)=\lim_{h\to0-}\frac{g(x+h)-g(x)}{h}\quad\text{and}\quad g'_+(x)=\lim_{h\to0+}\frac{g(x+h)-g(x)}{h}$$

exist, and both functions are increasing. By convexity it follows that $g'_-(x)\leq g'_+(x)\leq g'_-(y)$ whenever $x<y$. Furthermore,

$$g(y)-g(x)=\int_x^y g'_-=\int_x^y g'_+.$$

In particular, with $\tilde\sigma$ denoting the convex envelope of σ, we have

$$\tilde\sigma'_-(x)\leq\tilde\sigma'_+(x)\leq\tilde\sigma'_-(y)\leq\tilde\sigma'_+(y)\quad(x<y<\theta).$$

Thus both $\tilde\sigma'_\pm(\theta-)=\lim_{\delta\to0+}\tilde\sigma'_\pm(\theta-\delta)$ exist (in $\mathbb{R}\cup\{+\infty\}$), and furthermore, they must be equal. (Trivially $\tilde\sigma'_-(\theta-)\leq\tilde\sigma'_+(\theta-)$ and for the reverse, note that $\tilde\sigma'_+(\theta-2\delta)\leq\tilde\sigma'_-(\theta-\delta)$ for $\delta>0$. Letting $\delta\to0$ gives $\tilde\sigma'_+(\theta-)\leq\tilde\sigma'_-(\theta-)$.) Denote by $\tilde\sigma'(\theta-)$ this common value, which we may call the *slope of $\tilde\sigma$ at $\theta-$*. Note that this slope may be $+\infty$.

Proposition 2.7. *Let $f\in S$ and suppose $\theta_p\to\theta$ as $p\to\infty$. Then*

$$p(\theta-\theta_p)\text{ is eventually constant}\quad\Longleftrightarrow\quad\tilde\sigma'(\theta-)<\infty$$

in which case $p(\theta-\theta_p)=\Delta(=\sigma(\theta-))$ for $p\geq\tilde\sigma'(\theta-)$.

Proof. We may write

$$\tilde{\sigma}(y) = \tilde{\sigma}(x) + \int_x^y g \tag{2.4}$$

for $x < y < \theta$, where g is increasing and locally bounded on $(-\infty, \theta)$. We may take g to be $\tilde{\sigma}'_+$ or $\tilde{\sigma}'_-$. Observe that $g(x) \to 0$ as $x \to -\infty$, for otherwise $\tilde{\sigma}(y)$ is unbounded as $y \to -\infty$.

Let $k(\lambda) = p\lambda - \tilde{\sigma}(\lambda)$, where p is fixed but arbitrary. Thus for $\lambda < \mu < \theta$,

$$k(\mu) - k(\lambda) = \int_\lambda^\mu p - g(t)\,dt.$$

Now suppose g remains bounded near θ. Then, for $p \geq \sup g$, $g(t) - p \geq 0$ for all $t > 0$, so that k is increasing on $(-\infty, \theta)$. As such

$$p\theta_p = \sup_{\lambda<\theta} k(\lambda) = k(\theta-) = p\theta - \tilde{\sigma}(\theta-).$$

That is, $p(\theta - \theta_p)$ is constant $(= \Delta)$ for $p \geq g(\theta-) = \tilde{\sigma}'(\theta-)$.

Conversely, suppose g is unbounded at θ. Then, no matter how large p, there exists $\lambda' < \theta$ such that $g(\lambda') > p$, while $g(\lambda) - p < 0$ for λ sufficiently small. Hence

$$\lambda_p \stackrel{\text{def}}{=} \inf\{\lambda < \theta : g(\lambda) \geq p\} \quad \text{exists, and } \lambda_p < \theta.$$

As such $k(\lambda) - k(\lambda_p) = \int_{\lambda_p}^\lambda p - g(t)\,dt \leq 0$ for $\lambda \geq \lambda_p$ and also ≤ 0 for $\lambda \leq \lambda_p$; i.e.,

$$p\theta_p = \sup_{\lambda<\theta} h(\lambda) = h(\lambda_p) = p\lambda_p - \tilde{\sigma}(\lambda_p).$$

Thus for $q > p$,

$$\begin{aligned} q(\theta - -\theta_q) = q\theta - q\lambda_q + \tilde{\sigma}(\lambda_q) &= p\theta + (q-p)(\theta - \lambda_q) - [p\lambda_q - \tilde{\sigma}(\lambda_q)] \\ &\geq p(\theta - \theta_p) + (q-p)(\theta - \lambda_q) > p(\theta - \theta_p), \end{aligned}$$

showing that $p(\theta - \theta_p)$ is strictly increasing. □

An approach to Dirichlet's divisor problem? Proposition 2.7 may be an approach to proving $\theta_p \equiv \theta$ in particular cases, where we may be able to judge whether $\tilde{\sigma}'(\theta-) < \infty$. For example, if for Dirichlet's divisor problem, we have $\tilde{\sigma}'(\theta-) < 9$, then

$$\theta_p = \theta - \frac{c}{p} \quad \text{for } p \geq 9 - \eta \text{ (some constants } c, \eta \text{ with } c \geq 0 \text{ and } \eta > 0).$$

But $\theta_p = \frac{1}{4}$ for $p \leq 9$, so this would force $c = 0$ and hence $\theta = \frac{1}{4}$. In other words, it is enough to prove $\tilde{\sigma}'(\theta-) < 9$ to settle DDP.

2.4. **The behaviour of θ_p for p small.** The behaviour of $\sigma(\lambda)$ for λ large and negative is closely related to the behaviour of θ_p for p small. Since $\sigma(\lambda)$ is increasing and non-negative, so $\lim_{\lambda\to-\infty} \sigma(\lambda)$ exists, which we denote by $\sigma(-\infty)$. Recall that both θ_p and $p(\theta - \theta_p)$ decrease as p decreases to zero. Thus we may define

$$\theta_0 = \lim_{p\to 0+} \theta_p \in [-\infty, \theta] \quad \text{and} \quad \Delta_0 = \lim_{p\to 0+} p(\theta - \theta_p).$$

Note that $\lim_{p\to 0+} p\theta_p = -\Delta_0 (\leq 0)$.

Proposition 2.8.

(a) *We have* $\sigma(-\infty) = \Delta_0$.

(b) $\sigma(\lambda) = 0$ *on* $(-\infty, d)$ *where* d *is optimal (i.e.,* $\sigma(\lambda) > 0$ *for* $\lambda > d$*) if and only if* $\theta_0 = d$.

Proof.

(a), Note that $p\theta_p \geq p\lambda - \sigma(\lambda)$ for every $\lambda < \theta$. Let $p \to 0$ to give $\sigma(\lambda) \geq \Delta_0$ for all $\lambda < \theta$. Thus $\sigma(-\infty) \geq \Delta_0$. On the other hand,

$$p\theta_p = \sup_{\lambda<\theta}(p\lambda - \sigma(\lambda)) \leq p\theta + \sup_{\lambda<\theta}(-\sigma(\lambda)) = p\theta - \sigma(-\infty).$$

Thus $\sigma(-\infty) \leq p(\theta - \theta_p)$ and so $\sigma(-\infty) \leq \Delta_0$.

(b) ($\Rightarrow$) If $d = \theta$ the result is trivially true as this is equivalent to $\theta_p \equiv \theta$. So we may suppose $d < \theta$. We have

$$p\theta_p = \max\{\sup_{\lambda<d}(p\lambda - \sigma(\lambda)), \sup_{d\leq\lambda<\theta}(p\lambda - \sigma(\lambda))\}$$

and so, on the assumption that $\sigma(\lambda) = 0$ on $(-\infty, d)$,

$$\theta_p = \max\{d, \sup_{d\leq\lambda<\theta}(\lambda - \frac{\sigma(\lambda)}{p})\}.$$

For each $\delta > 0$, $\sup_{d+\delta\leq\lambda<\theta}(\lambda - \frac{\sigma(\lambda)}{p}) \leq \theta - \frac{\sigma(d+\delta)}{p} \to -\infty$ as $p \to 0$, so $\theta_0 = d$ follows.

($\Leftarrow$) For the converse implication, suppose $\theta_0 = d$. Then by monotonicity, $\theta_p \geq d$ for all p. But $\sup_{\lambda\leq d-\delta}(p\lambda - \sigma(\lambda)) < pd$ for every $\delta > 0$. Hence

$$\theta_p = \sup_{d-\leq\lambda<\theta}\Big(\lambda - \frac{\sigma(\lambda)}{p}\Big).$$

If $\sigma(d-) > 0$, then $\theta_p \leq \theta - \frac{\sigma(d-)}{p}$ which is $< d$ for all p sufficiently small, contradicting $\theta_p \geq d$. Thus $\sigma(d-) = 0$. By the ($\Rightarrow$) part we cannot have $\sigma(\lambda) = 0$ for any $\lambda > d$. ☐

3. Some special classes of functions — I: The case where $f(x) = f([x])$

Let S_{disc} denote the subset of S of functions f for which $f(x) = f([x])$; i.e., f is of the form

$$f(x) = \sum_{n\leq x} a_n$$

for some a_n; indeed $a_n = f(n) - f(n-1)$. We can generally say more about f with this extra condition. For example, from Remarks 2.1(iii) we see that $\theta_p \geq \theta - \frac{1}{p}$, while in general $p(\theta - \theta_p)$ need not be bounded. In many examples of interest (e.g., the general Dirichlet divisor problem) one knows more about the a_n. Here we investigate what can be deduced under the further condition that $a_n \ll n^\varepsilon$ for every $\varepsilon > 0$. The condition of course implies that $\theta \leq 1$. Indeed $\overline{\theta} \leq 1$ since $\sum_{n\leq x} |a_n| \ll x^{1+\varepsilon}$.

Note that $\sigma(\theta-) = \Delta \leq 1$ in this case.

Proposition 3.1. *Let* $f \in S_{\text{disc}}$ *for which* $f(n) - f(n-1) \ll n^\varepsilon$ *for every* $\varepsilon > 0$ *and* $\theta > 0$. *Then* $\sigma(\theta-) = \Delta \leq 1 - \theta$.

Proof. Let $\lambda < \theta$. There exists $n_k \in \mathbb{N}$ such that $n_k \nearrow \infty$ and

$$|f(n_k)| \geq 2(2n_k)^\lambda$$

since $\frac{|f(x)|}{x^{\theta-\varepsilon}}$ is unbounded for every $\varepsilon > 0$. Let $\delta > 0$. Then, for k sufficiently large,

$$|a_r| \leq r^\delta \quad \text{ for } r \geq n_k.$$

As such, with $n_k \leq t \leq 2n_k$,

$$|f(t)| \geq |f(n_k)| - |f(t) - f(n_k)| \geq 2(2n_k)^\lambda - \sum_{n_k < r \leq t} |a_r|$$
$$\geq 2(2n_k)^\lambda - (2n_k)^\delta (t - n_k)$$

for k sufficiently large. Thus for $t \leq n_k + (2n_k)^{\lambda-\delta}$ we have

$$|f(t)| \geq 2(2n_k)^\lambda - (2n_k)^\lambda = (2n_k)^\lambda \geq t^\lambda$$

if $\theta \geq 0$. Hence $t \in U_\lambda(n_k)$ for $n_k \leq t \leq n_k + (2n_k)^{\lambda-\delta}$; i.e., $|U_\lambda(n_k)| \geq (2n_k)^{\lambda-\delta}$. It follows that $1 - \sigma(\lambda) \geq \lambda - \delta$ for $\lambda \in (0, \theta)$. This is true for every $\delta > 0$, so $1 - \sigma(\lambda) \geq \lambda$ for $\lambda \in (0, \theta)$. Hence $\sigma(\theta-) \leq 1 - \theta$. □

Proposition 3.1 also holds for $f \in S$ if $f(x + y) - f(x) \ll x^\varepsilon$ uniformly for $0 \leq y \leq 1$ and any $\varepsilon > 0$. For $f \in S_{\text{disc}}$, the condition simplifies to $f(n) - f(n-1) \ll n^\varepsilon$.

It is interesting to see to what extent Theorem 2.5 can be proved for the special class of functions in Proposition 3.1. We prove that the only extra condition required is $\sigma(\theta-) \leq 1 - \theta$. Furthermore, we show that we may even take $f(n) - f(n - 1) \ll \log n$. Note that to prove the equivalent of Theorem 2.5, we only need Theorem 2.4 for $k(\cdot)$ concave, increasing, and bounded by 1. This in turn makes k strictly increasing (once it is positive) and continuous. We just consider case (b) of Proposition 2.8 as it is of greater interest. The construction resembles that of Theorem 2.4 but now we choose $\tau(x)$ more carefully in order to avoid the large jump at 2^n.

Theorem 3.2. *Let $\phi \in (0, 1)$ and $k : [d, \phi] \to \mathbb{R}$ be strictly increasing and continuous such that $k(d) = 0$ and $k(\phi) \leq 1 - \phi$. Then there exists $f \in S_{\text{disc}}$ such that $\theta = \phi$, $f(n) - f(n - 1) \ll \log n$, and*

$$\sigma(\lambda) = \begin{cases} k(\lambda) & \text{if } d < \lambda \leq \phi, \\ 0 & \text{if } \lambda \leq d. \end{cases}$$

Proof. First we find $f_1 \in S$ satisfying the conditions. Let $f_1(x) = x^{\tau(x)}$ where τ is defined on each interval $[2^n, 2^{n+1}]$ by

$$\tau\Big(\tfrac{3}{2}2^n \pm \tfrac{\phi-\lambda}{2(\phi-d)} 2^{n(1-k(\lambda))}\Big) = \lambda \qquad (d \leq \lambda \leq \phi, n \in \mathbb{N}_0).$$

Observe τ is strictly increasing on $[2^n, \frac{3}{2}2^n]$ and strictly decreasing on $[\frac{3}{2}2^n, 2^{n+1}]$, with minimum and maximum given by $\tau(2^n) = d$ and $\tau(\frac{3}{2}2^n) = \phi$. Note that τ is continuous on $(1, \infty)$.

Clearly, $f_1 \in S$ with $\theta = \phi$ (so we now replace ϕ by θ). Also, for $\lambda \in [d, \theta]$, $t \in U_\lambda(2^n) \Leftrightarrow \tau(t) \geq \lambda \Leftrightarrow t \in [\frac{3}{2}2^n - \frac{\theta-\lambda}{2(\theta-d)} 2^{n(1-k(\lambda))}, \frac{3}{2}2^n + \frac{\theta-\lambda}{2(\theta-d)} 2^{n(1-k(\lambda))}]$. Hence

$$|U_\lambda(2^n)| = \tfrac{\theta-\lambda}{\theta-d} 2^{n(1-k(\lambda))}$$

and so $\sigma(\lambda) = 1 - k(\lambda)$ in this range. As $k(d) = 0$ it follows that $\sigma(\lambda) = 0$ for $\lambda < d$. It remains to find bounds on $f_1(n) - f_1(n - 1)$. To this end, it is enough to consider $f_1(m) - f_1(m-1)$ for integers m with $2^n < m \le 2^{n+1}$, $n \in \mathbb{N}_0$. Consider first $m > \frac{3}{2}2^n$, the case when $m \le \frac{3}{2}2^n$ can be treated similarly. We can write

$$\begin{aligned} m &= \tfrac{3}{2}2^n + \tfrac{\theta-\lambda_1}{2(\theta-d)}2^{n(1-k(\lambda_1))}, \\ m-1 &= \tfrac{3}{2}2^n + \tfrac{\theta-\lambda_2}{2(\theta-d)}2^{n(1-k(\lambda_2))} \end{aligned}$$

for some λ_1, λ_2 such that $d \le \lambda_1 < \lambda_2 \le \theta$. As such $\tau(m-1) - \tau(m) = \lambda_2 - \lambda_1$. Subtracting the above two equations gives

$$(\theta - \lambda_1)2^{n(1-k(\lambda_1))} - (\theta - \lambda_2)2^{n(1-k(\lambda_2))} = 2(\theta - d). \tag{3.1}$$

Then, since $k(\lambda_2) > k(\lambda_1)$,

$$\begin{aligned} \tau(m-1) - \tau(m) = (\theta - \lambda_1) - (\theta - \lambda_2) &= \frac{(\theta - \lambda_1)2^{n(1-k(\lambda_1))} - (\theta - \lambda_2)2^{n(1-k(\lambda_1))}}{2^{n(1-k(\lambda_1))}} \\ &\le \frac{(\theta - \lambda_1)2^{n(1-k(\lambda_1))} - (\theta - \lambda_2)2^{n(1-k(\lambda_2))}}{2^{n(1-k(\lambda_1))}} = \frac{2(\theta-d)}{2^{n(1-k(\lambda_1))}} \end{aligned}$$

by (3.1). But $2^n \ge \frac{m}{2}$ and $1 - k(\lambda_1) \ge \theta$, so for $m \ge 2$,

$$|\tau(m) - \tau(m-1)| \le \frac{2(\theta-d)}{(\frac{m}{2})^\theta} \le \frac{4\theta}{m^\theta}. \tag{3.2}$$

A similar argument gives the above inequality for $m \le \frac{3}{2}2^n$ and so (3.2) holds for $m \ge 2$. Thus, for $m \ge 2$,

$$\begin{aligned} f_1(m) - f_1(m-1) &= m^{\tau(m)} - (m-1)^{\tau(m-1)} \\ &= m^{\tau(m)}\Big\{1 - \Big(1 - \frac{1}{m}\Big)(m-1)^{\tau(m-1)-\tau(m)}\Big\} \\ &= m^{\tau(m)}\Big(1 - \Big(1 + O\Big(\frac{1}{m}\Big)\Big)\Big(1 + O\Big(\frac{\log m}{m^\theta}\Big)\Big)\Big) \\ &= O\Big(\frac{m^{\tau(m)}\log m}{m^\theta}\Big) = O(\log m). \end{aligned}$$

Thus f_1 satisfies the conditions.

Now let $f(x) = f_1([x])$. Then $f \in S_{\text{disc}}$ with the same θ and $f(n) - f(n-1) \ll \log n$ follow immediately. It remains to prove that f has the same $\sigma-$function as f_1.

First note that for $m \in \mathbb{N}$, we have $m \in U_{\lambda,f}(2^n)$ if and only if $m \in U_{\lambda,f_1}(2^n)$, which in turn holds if and only if

$$m \in [\tfrac{3}{2}2^n - \tfrac{\theta-\lambda}{2(\theta-d)}2^{n(1-k(\lambda))}, \tfrac{3}{2}2^n + \tfrac{\theta-\lambda}{2(\theta-d)}2^{n(1-k(\lambda))}] \qquad (\text{for } d \le \lambda \le \theta).$$

Let $\varepsilon > 0$ and $d < \lambda < \theta$. Then for all sufficiently large m, $m \in U_{\lambda,f}(2^n)$ implies $[m, m+1) \subset U_{\lambda-\varepsilon,f}(2^n)$. For this involves $f(t) \ge t^{\lambda-\varepsilon}$ for every $t \in [m, m+1)$; i.e., $m^{\tau(m)} \ge t^{\lambda-\varepsilon}$. Since $\tau(m) \ge \lambda$, this certainly holds if $m^\lambda \ge t^{\lambda-\varepsilon}$. But $t = m + O(1)$ and the RHS is $m^{\lambda-\varepsilon}(1 + O(\frac{1}{m}))$, and so it clearly holds for m sufficiently large. Hence

$$|U_{\lambda-\varepsilon,f}(2^n)| \ge \sum_{|m - \frac{3}{2}2^n| \le \frac{\theta-\lambda}{2(\theta-d)}2^{n(1-k(\lambda))}} 1 \ge c\,2^{n(1-k(\lambda))}$$

for some $c > 0$, and so $\sigma_f(\lambda - \varepsilon) \le k(\lambda)$. This holds for all $\varepsilon > 0$, so actually $\sigma_f(\lambda-) \le k(\lambda)$ for $d < \lambda < \theta$.

For the reverse inequality, we shall need (3.2) for non-integral m. In fact using the monotonicity of τ on both halves of the interval $[2^n, 2^{n+1}]$, one has

$$\tau(x+y)-\tau(x) \ll \frac{1}{x^\theta} \quad \text{uniformly for } y \in [0,1].$$

For if $2^n \le x < \frac{3}{2}2^n - 1$, then

$$\tau(x+y)-\tau(x) \le \tau([x+y]+1)-\tau([x+y])+\tau([x+y])-\tau([x]) \ll x^{-\theta}.$$

This is similar for $x \in [\frac{3}{2}2^n, 2^{n+1}-1)$. In the remaining ranges one has (by monotonicity)

$$|\tau(x+y)-\tau(x)|$$
$$\le \begin{cases} 2(\tau(\frac{3}{2}2^n)-\tau(\frac{3}{2}2^n-1)) & \text{if } \frac{3}{2}2^n - 1 \le x < \frac{3}{2}2^n, \\ \max\{\tau(2^{n+1}+1)-\tau(2^{n+1}), \tau(2^{n+1}-1)-\tau(2^{n+1})\} & \text{if } 2^{n+1}-1 \le x < 2^{n+1}. \end{cases}$$

In either case one has $\tau(x+y)-\tau(x) \ll x^{-\theta}$ uniformly.

Let $\varepsilon > 0$ and $d < \lambda < \theta$ as before. Then $U_{\lambda,f}(2^n) \subset U_{\lambda-\varepsilon,f_1}(2^n)$ for n sufficiently large. For $t \in S_{\lambda,f}(2^n) \Leftrightarrow m^{\tau(m)} \ge t^\lambda$ (where $m=[t]$) and $\tau(t)=\tau(m)+O(m^{-\theta})$. Thus

$$t^{\tau(t)} \sim t^{\tau(m)} = (m+O(1))^{\tau(m)} \sim m^{\tau(m)} \ge t^\lambda,$$

and so $\tau(t) \ge \lambda - \varepsilon$ for all t (i.e., n) sufficiently large, and $t \in U_{\lambda-\varepsilon,f_1}(2^n)$. Hence

$$|U_{\lambda,f}(2^n)| \le |U_{\lambda-\varepsilon,f_1}(2^n)| \ll 2^{n(1-k(\lambda-\varepsilon))}$$

and $\sigma_f(\lambda) \ge k(\lambda-\varepsilon)$. By continuity of k, it follows that $\sigma_f(\lambda) \ge k(\lambda)$ for $d < \lambda < \theta$ and hence we must have equality. □

Theorem 3.3. *Let $\phi, \phi_p \in \mathbb{R}$ $(p>0)$ such that ϕ_p is increasing and tends to ϕ, $\phi \le 1$, $p\phi_p$ is convex, $\phi_p \ge \phi - \frac{1-\phi}{p}$ for all p, and $\phi_p \to \phi_0 > -\infty$ as $p \to 0+$. Then there exists $f \in S_{\text{disc}}$ with $f(n)-f(n-1) \ll \log n$ such that $\theta = \phi$ and $\theta_p = \phi_p$.*

Proof. The proof follows that of Theorem 2.5 and note that the condition $\lim_{p\to 0+} \phi_p > -\infty$ implies that only case (b) of Proposition 2.8 occurs. Thus Theorem 3.2 applies in precisely the same way that Theorem 2.4 applies to Theorem 2.5. The extra condition $\phi_p \ge \phi - \frac{1-\phi}{p}$ ensures that $\Delta_0 \le 1-\phi$. □

Example. By Theorem 3.2, we can find an $f \in S_{\text{disc}}$ with $\theta \in (0,1)$ satisfying $f(n)-f(n-1) \ll n^\varepsilon$ with

$$\sigma(\lambda) = \begin{cases} \sqrt{2P(\theta-\lambda)} & \text{if } \theta - \frac{1}{2P} < \lambda < \theta, \\ 0 & \text{if } \lambda \le \theta - \frac{1}{2P}, \end{cases}$$

(some $P>0$). This leads to, via Theorem 2.2,

$$\theta_p = \begin{cases} \theta - \frac{1}{2P} & \text{for } p < P, \\ \theta - \frac{1}{p} + \frac{P}{2p^2} & \text{for } p \ge P, \end{cases}$$

after some calculation. The example shows quite explicitly that θ_p may be constant on an interval $(0,P)$ of arbitrary length before increasing.

II: Analytic functions. Let S_{holo} denote the set of functions f which are holomorphic in a half-strip

$$\{x+iy : x \geq 1, a \leq y \leq b\}$$

such that for each $y \in [a,b]$, the function f_y defined by $f_y(x) = f(x+iy)$ lies in S. As such, $\theta_p^{(y)}$ and $\theta^{(y)}$ (the "θ_p" and "θ" for f_y, respectively) are defined for $a \leq y \leq b$. We note that $\theta^{(y)}$ and $\theta_p^{(y)}$ are convex functions of y (for every p). The convexity of $\theta^{(y)}$ follows from a general result about functions of finite order in a strip (see [29, p. 180]), while the convexity of $\theta_p^{(y)}$ follows from the results in [8]. Just as for the "discrete" case we find the same inequality connecting these. This is a consequence of results in [8] (see Theorem 2 therein).

Proposition 3.4. *With $f \in S_{\text{holo}}$ as above,*

$$\theta^{(y)} \leq \theta_p^{(y)} + \frac{1}{p}$$

for every $y \in (a,b)$.

Proof. It suffices the prove the result for $p \in \mathbb{N}$ since $p(\theta^{(y)} - \theta_p^{(y)})$ increases with p so the result for p integral implies the same for every $p > 0$.

Let $a < y < b$ and $h > 0$ chosen sufficiently small so that $a < y-h < y+h < b$. By Cauchy's integral formula

$$f(x+iy)^p = \frac{1}{2\pi}\int_0^{2\pi} f(x+iy+re^{i\theta})^p \, d\theta$$

for every $r \in [0,h]$. In particular,

$$|f(x+iy)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(x+iy+re^{i\theta})|^p \, d\theta.$$

Now apply $\int_0^h \ldots r\,dr$ to both sides. Thus

$$\begin{aligned}\frac{h^2}{2}|f(x+iy)|^p &\leq \frac{1}{2\pi}\int_0^h r \int_0^{2\pi} |f(x+iy+re^{i\theta})|^p \, d\theta\, dr \\ &\leq \frac{1}{2\pi}\int_{x-h}^{x+h}\int_{y-h}^{y+h} |f(u+iv)|^p \, du\, dv.\end{aligned}$$

In particular for $x \geq \frac{h}{2}$,

$$\begin{aligned}h^2|f(x+iy)|^p &\leq \frac{1}{\pi}\int_{y-h}^{y+h}\int_{x/2}^{2x} |f(u+iv)|^p \, du\, dv \ll_\varepsilon \int_{y-h}^{y+h} x^{1+p\theta_p^{(v)}+\varepsilon}\, dv \\ &\ll x^{1+p\theta_p^{(y+h')}+\varepsilon},\end{aligned}$$

where $|h'| \leq h$. Thus $p\theta^{(y)} \leq 1 + p\theta_p^{(y+h')} + \varepsilon$. This holds for every $\varepsilon > 0$ and h sufficiently small. Since $\theta_p^{(y)}$ is continuous (w.r.t. y), it follows that $p\theta^{(y)} \leq 1 + p\theta_p^{(y)}$. ☐

Example. The inequality given in Proposition 2.12 is sharp. Let

$$f(z) = \frac{1}{1 - z^i + z^{-1}},$$

where z^i is the usual principle value: $(re^{i\theta})^i = e^{-\theta}r^i$ for $r > 0$ and $|\theta| < \pi$. Thus f is well defined and holomorphic in the cut-plane $\mathbb{C} \setminus (-\infty, 0]$ except at the zeros of $1 - z^i + \frac{1}{z}$. These zeros are bounded away from 0 since at the zeros we require $\frac{1}{|z|} = |1 - z^i| \le 1 + e^\pi$. For large $|z|$ the zeros lie roughly on the line $\Im z = -1$. To see this, take absolute values of $z^i = 1 + \frac{1}{z}$ with $z = re^{i\theta}$ to get

$$e^{-2\theta} = 1 + \frac{2\cos\theta}{r} + \frac{1}{r^2}.$$

Thus $\theta = -\frac{\cos\theta}{r} + O(\frac{1}{r^2})$, and so $z = r - i + O(\frac{1}{r})$. It follows that for every $y \ne -1$, $f(x+iy)$ is well defined for x sufficiently large. Fix $y > -1$. A straightforward estimation shows that, for large x,

$$\frac{1}{f(x+iy)} = 1 - x^i e^{-\frac{y}{x}} + \frac{1}{x} - \frac{iy}{x^2} + O\Big(\frac{1}{x^3}\Big).$$

Now

$$(3.3)\quad \Big|1 - x^i e^{-\frac{y}{x}} + \frac{1}{x} - \frac{iy}{x^2}\Big|^2 = \Big(1 - e^{-\frac{y}{x}}\cos\log x + \frac{1}{x}\Big)^2 + \Big(e^{-\frac{y}{x}}\sin\log x + \frac{y}{x^2}\Big)^2$$

and $1 - e^{-\frac{y}{x}}\cos\log x + \frac{1}{x} \ge \frac{1+y\cos\log x}{x} + O(x^{-2}) \ge \frac{a}{x}$ for some $a > 0$ (depending on y). Thus $|f(x+iy)| \ll x$. On the other hand, with $x = e^{2\pi k}$ ($k \in \mathbb{N}$), the RHS of (3.3) $= (1 - e^{-\frac{y}{x}} + \frac{1}{x})^2 + \frac{y^2}{x^4} = \frac{(1+y)^2}{x^2} + O(x^{-3})$. Hence $|f(x+iy)| \sim \frac{x}{1+y}$ for such x. Thus $\theta^{(y)} = 1$ for every $y > -1$. (Similarly we can discuss $y < -1$.)

In order to calculate $\theta_p^{(y)}$, we first calculate $\sigma(\lambda)$ and use Lemma 2.3. Fix $y > -1$ and $\lambda < 1$. Then $|f_y(t)| \ge t^\lambda$ is satisfied if

$$(3.4)\qquad \Big|1 - (t+iy)^i + \frac{1}{t+iy}\Big|^2 \le t^{-2\lambda}.$$

The LHS above is bounded so, for $\lambda < 0$, (3.4) is always satisfied for large enough t. Thus $|U_\lambda(x)| = x$ (for x large) and $\sigma(\lambda) = 0$ for $\lambda < 0$.

Now consider $0 < \lambda < 1$. Now $(t+iy)^i = t^i(1 - \frac{y}{t} + O(\frac{1}{t^2})) = t^i + o(1)$, so (3.4) can only hold if $t = e^{2\pi n + o(1)}$ ($n \in \mathbb{N}$). Write $t = e^{2\pi(n+h)}$. Then $t^i = e^{2\pi i h}$ and (after some calculations)

$$\Big|1 - (t+iy)^i + \frac{1}{t+iy}\Big|^2 = (1-\cos 2\pi h)^2 + (\sin 2\pi h)^2\Big(1 - \frac{y}{t}\Big)$$
$$+ \frac{2}{t}(1-\cos 2\pi h)(1 + y\cos 2\pi h) + O\Big(\frac{1}{t^2}\Big).$$

This is $\sim 4\pi^2 h^2$ (whenever $h \succ \frac{1}{t}$). Thus (3.4) is satisfied for $h \le c_1 t^{-\lambda}$ and fails for $h \ge c_2 t^{-\lambda}$ (suitable $c_1, c_2 > 0$). Hence (3.4) holds in the range $[e^{2\pi(n+c_1e^{-2\pi n})}, e^{2\pi(n+c_2e^{-2\pi n\lambda})}] \approx [e^{2\pi n} + O(1), e^{2\pi n} + c' e^{2\pi(1-\lambda)n}]$ for some $c' > 0$. Hence $|U_\lambda(e^{2\pi n})| \asymp e^{2\pi(1-\lambda)n}$, so that $\sigma(\lambda) = \lambda$. That is,

$$\sigma(\lambda) = \begin{cases} \lambda & \text{if } 0 < \lambda < 1, \\ 0 & \text{if } \lambda \le 0. \end{cases}$$

Now Theorem 2.2 gives

$$\theta_p = \begin{cases} 0 & \text{if } 0 < p < 1, \\ 1 - \frac{1}{p} & \text{if } p \ge 1. \end{cases}$$

Thus $\theta^{(y)} = \theta_p^{(y)} + \frac{1}{p}$ for the range $p \ge 1$ and every $y > -1$.

The problem whether Theorems 2.4 and 2.5 extend to the holomorphic case remains open.

4. The abscissae of the associated Mellin transform

4.1. **Lindelöf functions.** Formally the Mellin transform of a function $f : (0, \infty) \to \mathbb{C}$ is defined by

$$\hat{f}(s) = \int_0^\infty x^{-s}\, df(x).$$

We shall assume that $f \in S$ and that $f(x) = 0$ for $x < 1$, as this covers most cases of interest. As such, the Mellin transform is well defined for $\sigma = \Re s > \theta$. For

$$\int_0^X x^{-s}\, df(x) = \frac{f(X)}{X^s} + s \int_1^X \frac{f(x)}{x^{s+1}}\, dx.$$

The first term on the right tends to 0 as $X \to \infty$ whenever $\sigma > \theta$ while the integral converges absolutely for $\sigma > \theta_1$. In particular, for $\sigma > \theta$

$$\hat{f}(s) = s \int_1^\infty \frac{f(x)}{x^{s+1}}\, dx \tag{4.1}$$

and by standard results of complex analysis, $\hat{f}$ is holomorphic in this region. However the RHS of (4.1) converges absolutely for $\sigma > \theta_1$ and so $\hat{f}$ has an automatic analytic continuation to the half-plane H_{θ_1}[3] and (4.1) holds for $\sigma > \theta_1$. We shall denote any analytic continuation of $\hat{f}$ beyond H_{θ_1} by $\hat{f}$.

We see that, for $\sigma > \theta_1$ at least, $\hat{f}$ is of *finite order*: i.e., $\hat{f}(\sigma + it) = O(t^A)$ for some A (indeed we may take $A = 1$ here). As such we can define the usual 'Lindelöf' functions (wherever they make sense):

$$\mu(\sigma) = \inf\{\lambda : \hat{f}(\sigma + it) = O(t^\lambda) \text{ for } t \geq 1\},$$
$$\mu_p(\sigma) = \inf\left\{\lambda : \left(\frac{1}{T}\int_T^{2T} |\hat{f}(\sigma + it)|^p\, dt\right)^{1/p} = O(T^\lambda)\right\}.$$

Occasionally, we may write $\mu_{\hat{f}}(\sigma)$ and $\mu_{\hat{f},p}(\sigma)$ for these functions to show the dependence on $\hat{f}$. These are examples of the "analytic case" from §3II.

From above we automatically have $\mu(\sigma) \leq 1$ for $\sigma > \theta_1$.

Proposition 4.1. *Let $f \in S$. Then $\mu(\cdot)$ and $\mu_p(\cdot)$ are convex for every p. Furthermore, if $\overline{\theta} < \infty$ and $f(1) \neq 0$, then both functions are zero for $\sigma > \overline{\theta}$. Consequently, both are decreasing functions of σ.*

Proof. The convexity of $\mu(\sigma)$ follows from a general result about functions of finite order in a vertical strip (see [29, p. 180]), while the convexity of $\mu_p(\sigma)$ follows from the results in [8].

For the second part, we have for $\sigma > \overline{\theta}$

$$|\hat{f}(s)| \leq \int_{1-}^\infty x^{-\sigma}\, dV_f(x) = \hat{V}_f(\sigma)$$

so that $\mu(\sigma) \leq 0$ for $\sigma > \overline{\theta}$. But for σ sufficiently large we have

$$|\hat{f}(\sigma + it)| = \left| f(1) + \int_1^\infty x^{-s}\, df(x) \right| \geq |f(1)| - \int_1^\infty x^{-\sigma}\, dV_f(x).$$

The final integral tends to zero as $\sigma \to \infty$, hence if $f(1) \neq 0$, then $|\hat{f}(\sigma + it)| \geq \frac{1}{2}|f(1)|$ for σ sufficiently large and all $t \in \mathbb{R}$. Thus $\mu(\sigma), \mu_p(\sigma) \geq 0$ for σ sufficiently

[3] $H_\alpha = \{z \in \mathbb{C} : \Re z > \alpha\}$.

large (all p). By convexity, we must have $\mu(\sigma) = \mu_p(\sigma) = 0$ for $\sigma > \overline{\theta}$, and both functions are decreasing. □

Slightly more generally, if f is zero on $(0, c)$ but $f(c) \neq 0$ for some $c > 1$, then the same conclusions hold. In other cases[4] it may be that $\mu(\sigma)$ is negative for all σ. It follows immediately from the convexity that (under the conditions of Proposition 4.1)

$$\mu(\sigma) \leq \frac{\overline{\theta} - \sigma}{\overline{\theta} - \theta_1} \quad \text{for } \theta_1 < \sigma \leq \overline{\theta}. \tag{4.2}$$

Proposition 3.4 can be applied to $\hat{f}$ to give:

$$\mu(\sigma) \leq \mu_p(\sigma) + \frac{1}{p}$$

for every $p > 0$ and $\sigma > \theta_1$. Furthermore, the result remains true whenever $\hat{f}$ has an analytic continuation to H_α for some $\alpha < \theta_1$ of finite order.

This inequality is sharp (at least for the case $p = 2$) as an example of Kahane shows (see [20, Theorem 2.2]). For his example, $\mu(\sigma) = \mu_2(\sigma) + \frac{1}{2}$ for a range of values of σ. See §5 (example (vii)) for more details.

4.2. **Connection between θ_2 and $\mu_2(\sigma)$.** As for the abscissae in §1, determining the Lindelöf functions is often exceedingly difficult. We aim to see to what extent the abscissae of a function are connected to the abscissae of its Mellin transform. First we prove a connection between θ_2 and μ_2. At the heart of this connection lies Parseval's equality.

Theorem 4.2. *Let $f \in S$ with θ_2, $\hat{f}$ and $\mu_2(\sigma)$ defined as before, and suppose $\theta_2 > -\infty$. Then $\mu_2(\theta_2+) \leq \frac{1}{2}$.*

Furthermore, suppose that $\hat{f}$ has an analytic continuation of finite order to H_α for some $\alpha < \theta_2$. Then $\mu_2(\theta_2) = \frac{1}{2}$.

Note: in particular, if $\theta_1 < \theta_2$, then the extra analyticity assumption is satisfied automatically with $\alpha = \theta_1$.

Proof. Put $g(s) = \int_1^\infty \frac{f(x)}{x^{s+1}} dx$. Thus $g(s) = \hat{f}(s)/s$ for $\sigma > \theta_1$. Let ν denote the infimum of σ for which $\int_{-\infty}^\infty |g_\sigma|^2$ converges. By an argument identical to that given in Theorem 12.5 of [30], we have $\nu = \theta_2$ and

$$\frac{1}{2\pi} \int_{-\infty}^\infty |g(\sigma + it)|^2 \, dt = \int_0^\infty \frac{|f(x)|^2}{x^{2\sigma+1}} \, dx \tag{4.3}$$

for $\sigma > \theta_2$ (see also Theorem 71 of [31]). As such, for $\sigma > \theta_2$,

$$\int_T^{2T} |\hat{f}(\sigma + it)|^2 \, dt \asymp T^2 \int_T^{2T} \left| \frac{\hat{f}(\sigma + it)}{\sigma + it} \right|^2 dt = o(T^2)$$

and so $\mu_2(\sigma) \leq \frac{1}{2}$ for every $\sigma > \theta_2$; i.e., $\mu_2(\theta_2+) \leq \frac{1}{2}$ as required.

Now assume that $\hat{f}$ has an analytic continuation of finite order to H_α for some $\alpha < \theta_2$.

[4] For example, if $f(x) = x - 1$ for $x \geq 1$, then $\hat{f}(s) = \frac{1}{s-1}$ and $\mu(\sigma) \equiv -1$.

Suppose, for a contradiction, that $\mu_2(\theta_2) < \frac{1}{2}$. Then, by continuity of μ_2, there exists $\sigma < \theta_2$ such that

$$\int_T^{2T} |\hat{f}(\sigma+it)|^2\,dt \ll T^{2-\delta}$$

for some $\delta > 0$. But then

$$\int_T^{2T} |g(\sigma+it)|^2\,dt \asymp \frac{1}{T^2}\int_T^{2T} |\hat{f}(\sigma+it)|^2\,dt = O(T^{-\delta}).$$

By telescoping it follows that the LHS of (4.3) is finite; i.e., $\sigma \geq \nu$. This contradicts the fact that $\sigma < \nu(=\theta_2)$. □

Remarks 4.1.

(i): The analyticity condition of Theorem 4.2 is necessary if we want $\mu_2(\theta_2) = \frac{1}{2}$ to hold. For example, taking $f(x) = [x]$, then $\theta_2 = 1$. In this case $\hat{f}(s) = \zeta(s)$, $\mu(\sigma)$ is the usual Lindelöf function for $\zeta(s)$ and μ_2 is given by

$$\mu_2(\sigma) = \begin{cases} 0 & \text{if } \sigma \geq \frac{1}{2}, \\ \frac{1}{2}-\sigma & \text{if } \sigma \leq \frac{1}{2}. \end{cases}$$

Theorem 4.2 just says $\mu_2(1) \leq \frac{1}{2}$, whereas we know $\mu_2(1) = 0$.

As we only have one pole it is easy to adjust the example to make the Mellin transform holomorphic to the left of θ_1 whilst keeping μ and μ_2 the same. In the above example, one could take $f_1(x) = [x] - x$. As such, $\hat{f}_1(s) = \zeta(s) - \frac{s}{s-1}$ which is entire. Now $\theta_2 = 0$, while μ, μ_2 remain the same as above. Theorem 4.2 now says $\mu_2(0) = \frac{1}{2}$.

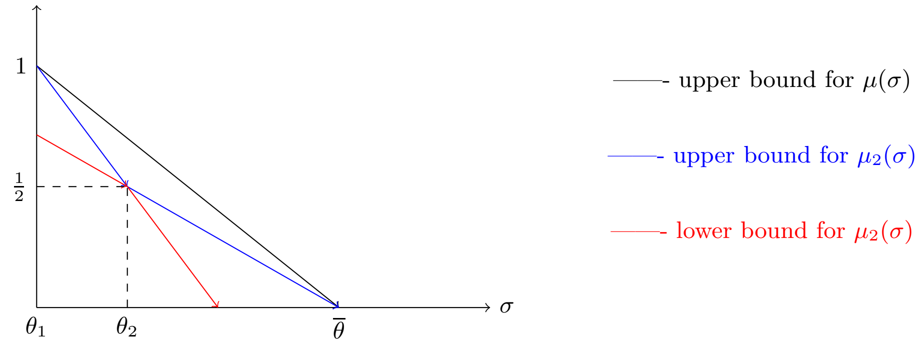
The same procedure can be used if we have a finite number of poles. However, with an infinite number of poles we run into difficulty. For example, take $f(x) = M(x) = \sum_{n\leq x}\mu(n)$, where $\mu(n)$ is the Möbius function. It is well known that

$$\theta = \Theta := \sup\{\Re\rho : \zeta(\rho) = 0\}.$$

Further, if $\int_0^x |M| = O(x^{1+\alpha})$ for some $\alpha < 1$, then $\int_1^\infty \frac{M(x)}{x^{s+1}}dx$ converges (absolutely) for $\sigma > \alpha$ and $\frac{1}{\zeta(s)}$ is holomorphic and $\zeta(s) \neq 0$ for $\sigma > \alpha$; i.e., we must have $\theta_1 \geq \Theta$. This implies $\theta_p \equiv \Theta$.

(ii): The Babenko-Beckner inequality (Haussdorff-Young for Fourier transforms – see [2]) gives $\mu_q(\theta_p+) \leq \frac{1}{p}$ for $1 < p \leq 2$, where $\frac{1}{p} + \frac{1}{q} = 1$. This interpolates the inequalities $\mu(\theta_1+) \leq 1$ and $\mu_2(\theta_2+) \leq \frac{1}{2}$. However, unlike the $p = 2$ case, in general one doesn't have equality.

4.3. **Further inequalities for $\mu(\sigma)$ and $\mu_2(\sigma)$ and the extremal case.** The convexity of μ and μ_2 together with Theorem 4.2 and (4.2) give further inequalities for these functions. This is perhaps best seen with the aid of a diagram.



From Theorem 4.2 and (4.2) we must have (given the conditions)

$$\frac{1}{2} = \mu_2(\theta_2) \le \mu(\theta_2) \le \frac{\overline{\theta} - \theta_2}{\overline{\theta} - \theta_1} \qquad (\text{provided } \theta_1 < \overline{\theta} < \infty).$$

In particular, if the left and right hand sides are equal (i.e., $2\theta_2 = \theta_1 + \overline{\theta}$) we must have equality throughout. But $2\theta_2 \le \theta_1 + \theta$, so this is only possible if $\overline{\theta} = \theta$. As such, convexity forces $\mu(\sigma)$ equal to the upper bound on $(\theta_1, \overline{\theta})$ in (4.2); i.e., the blue, red, and black lines coincide. This leads to the following:

Corollary 4.3. *Suppose that $\theta_1 < \theta = \overline{\theta}$ and $2\theta_2 = \theta_1 + \theta$. Then $\mu_p \equiv \mu$ for every $p > 0$ on (θ_1, ∞) and*

$$\mu(\sigma) = \begin{cases} \frac{\theta-\sigma}{\theta-\theta_1} & \text{if } \theta_1 < \sigma < \theta, \\ 0 & \text{for } \sigma \ge \theta. \end{cases}$$

We shall call this case *extremal*. The case where $\theta_p \equiv \theta = \overline{\theta}$ can be thought of as a degenerate extremal case, since now the interval $(\theta_1, \overline{\theta})$ is empty and Corollary 4.3 says nothing.

Note that in the extremal case, convexity of $p\theta_p$ forces $p(\theta - \theta_p)$ to be constant for $p \ge 1$. Furthermore, the condition $2\theta_2 = \theta_1 + \theta$ is equivalent to $\theta - \theta_1 = 2(\theta - \theta_2)$, which in turn is easily seen to be equivalent to $\sigma(\theta - \lambda) \ge \sigma(\theta-) - \lambda$ and forces $\sigma(\theta-) = \theta - \theta_1$. This case shows that it is possible for $\mu_p \equiv \mu$ even though $\theta_p \not\equiv \theta$.

5. Examples

(i): Let $f(x) = \{x\}$ (for $x \ge 1$), where $\{x\}$ denotes the fractional part of x. For this function $\theta_p \equiv \theta = 0$. Also $V_f(x) \sim 2x$ since the total variation is 2 for each interval $(n, n+1]$. Hence $\overline{\theta} = 1$. In this case $\sigma(\lambda) \equiv 0$ trivially as $|U_\lambda(x)| \sim x$ for all $\lambda < 0$.

The Mellin transform is

$$\hat{f}(s) = \frac{s}{s-1} - \zeta(s)$$

which is entire of finite order. The mean values of ζ imply that

$$\mu_2(\sigma) = \begin{cases} 0 & \text{if } \sigma \ge \frac{1}{2}, \\ \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2}, \end{cases}$$

and by the functional equation it is also known that $\mu(\sigma) = \mu_2(\sigma)$ for $\sigma \ge 1$ and $\sigma \le 0$. What happens inside the critical strip $(0 < \sigma < 1)$ is still unknown. The Lindelöf Hypothesis (LH) is that $\mu(\sigma) = \mu_2(\sigma)$ here. This is equivalent to $\mu_p \equiv \mu$ on $\mathbb{R}$. The fourth power moment of ζ is known and gives $\mu_4 \equiv \mu_2$. So $\mu_p \equiv \mu_2$ for $0 < p \leq 4$ follows.

(ii): (Dirichlet's divisor problem). Let $\Delta(x)$ be defined by

$$\Delta(x) = \sum_{n \leq x} \Big(d(n) - \log n - 2\gamma \Big),$$

where $d(n)$ is the number of divisors of n and γ is Euler's constant. The best bounds on θ are $\frac{1}{4} \leq \theta \leq \frac{517}{1648}$. It is conjectured that $\theta = \frac{1}{4}$.

Regarding θ_p, it is known that $\theta_p = \frac{1}{4}$ for $0 < p \leq 9$ (at least); see [16], [12]. As for $\sigma(\lambda)$, by Proposition 2.8(b), $\sigma(\lambda) = 0$ for $\lambda < \frac{1}{4}$, while $\sigma(\theta-) \leq 1 - \theta$.

More generally, take $f(x) = \Delta_k(x)$ where

$$(k \geq 2) \qquad \Delta_k(x) = \sum_{n \leq x} \Big(d_k(n) - P_k(\log n) \Big).$$

Here $d_k(n)$ is the coefficient of n^{-s} in the Dirichlet series for $\zeta(s)^k$ and P_k is a polynomial of degree $k-1$, suitably chosen so that the associated Mellin transform is entire. Then $\theta = \alpha_k$ and $\theta_2 = \beta_k$ (using the notation of Titchmarsh [30]). The conjecture is that (for every k)

$$\alpha_k \overset{(i)}{=} \beta_k \overset{(ii)}{=} \frac{1}{2} - \frac{1}{2k}.$$

Note that (ii) (for all k) is equivalent to the Lindelöf Hypothesis, while (i) and (ii) together are not known to follow from LH or even RH. Equality (i) (without (ii)) would just be saying that $\theta_p \equiv \theta$.

As for the corresponding Lindelöf functions, we have $\mu(\sigma) = k\mu_\zeta(\sigma)$ while $\mu_p(\sigma) = \mu_{\zeta^k,p}(\sigma)$. These functions are not known for any value inside the critical strip except when $k = 1, 2$ and $p \leq 2$. On LH, all these functions are equal; namely

$$\mu_p(\sigma) \equiv \mu(\sigma) = \begin{cases} 0 & \text{if } \sigma \geq \frac{1}{2}, \\ k(\frac{1}{2} - \sigma) & \text{if } \sigma < \frac{1}{2}. \end{cases}$$

(iii): Let

$$f(x) = \begin{cases} |\zeta(\frac{1}{2} + ix)|^2 & \text{for } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For this function, $\theta = 2\mu_\zeta(\frac{1}{2})$ where μ_ζ is the Lindelöf function for $\zeta(s)$. It is known that $\mu_\zeta(\frac{1}{2}) \leq \frac{13}{84}$ (see [5]) and, on LH, $\theta = 0$. Here, $\theta_1 = \theta_2 = 0$ on account of the second and fourth power moments of $\zeta(\frac{1}{2} + ix)$.

Notice that $\zeta(\frac{1}{2} + ix)^2 \in S_{\text{holo}}$ since $\zeta(y + ix)^2 \in S$ for each fixed y. By Proposition 3.4, $\theta \leq \theta_p + \frac{1}{p}$ and so $\theta_p \to \theta$. The Lindelöf Hypothesis is the statement that θ_p is constant.

Also for this function, $\overline{\theta} \leq 1$. For $f'(x) = -2\Im(\zeta'(\frac{1}{2} + ix)\zeta(\frac{1}{2} - ix))$ and so, by Cauchy-Schwarz,

$$V_f(x) \leq 2\sqrt{\int_1^x |\zeta'(\tfrac{1}{2} + it)|^2 \int_1^x |\zeta(\tfrac{1}{2} + it)|^2} \sim \frac{2}{\sqrt{3}} x (\log x)^2.$$

Presumably $\overline{\theta} = 1$. The Mellin transform of this function has been studied by a number of authors (starting with [17]) and holomorphic properties and various bounds on the associated Lindelöf functions are known. Thus, using the notation of [17],

$$\hat{f}(s) = s\int_1^\infty \frac{|\zeta(\frac{1}{2}+ix)|^2}{x^{s+1}}\,dx = sZ_1(s+1).$$

It is known (see [18]) that Z_1 has an analytic continuation to the whole plane except for a double pole at $s = 1$ and singularities which are at most simple poles at $-1, -3, -5, \ldots$. Thus $\hat{f}$ has an analytic continuation to the whole plane except for a simple pole at $s = 0$ with residue 1, and at most simple poles at $-2, -4, -6, \ldots$.

Regarding the Lindelöf functions, the relation $\mu(\sigma) = 1+\mu_{Z_1}(\sigma+1)$ (and similarly for $\mu_2(\sigma)$) and using known bounds (see [18] and [19]) we have

$$\mu(\sigma) \le \begin{cases} \frac{2-4\sigma}{3} & \text{for } -1 \le \sigma \le -\frac{1}{2}, \\ \frac{5}{6}-\sigma & \text{for } -\frac{1}{2} \le \sigma \le 0, \end{cases}$$

and $\mu_2(\sigma) \le -2\sigma$ for $-1 \le \sigma \le -\frac{1}{2}$ while

$$\mu_2(\sigma) = \frac{1}{2} - \sigma \text{ for } -\tfrac{1}{2} \le \sigma \le 0.$$

Note that this last equality contains the value $\mu_2(0) = \frac{1}{2}$ from[5] Theorem 4.2. Ivić [18] suggested that $\mu(\sigma) = \mu_2(\sigma)$ for $-\frac{1}{2} \le \sigma \le 0$ (at least). Perhaps $\mu_2 \equiv \mu$ here.

(iv): Another example comes from Bernoulli convolutions. For $\rho > 1$, let

$$C(x) = \#\{\varepsilon_1 + \varepsilon_2\rho + \cdots + \varepsilon_n\rho^{n-1} \le x : \varepsilon_i \in \{0,1\}, n \in \mathbb{N}\}.$$

One finds that $C(x) \asymp x^\tau$ where $\tau = \frac{\log 2}{\log \rho}$. Furthermore $C(\rho^n x) \sim 2^n\mu[0,x]$, where μ is the unique probability measure satisfying $\mu(E) = \frac{1}{2}\mu(\rho E) + \frac{1}{2}\mu(\rho E - 1)$ (see [13]). It is known that μ is either absolutely continuous or purely singular, and the problem is to decide which. It is known to be singular for ρ a Pisot number (as shown by Erdős) and also for $\rho > 2$. A major breakthrough came in 1995 when Solomyak [27] proved that μ is absolutely continuous with an L^2-density for almost all $\rho \in (1,2]$.

Now let $f(x) = C(x) - C(x-1/\rho)$. In [13], it was shown that $\int^x f \asymp x^\tau$, so $\theta_1 = \tau - 1$, while $\int^x f^2 \ll x^{2\tau-1}$ was shown to be equivalent to μ being absolutely continuous and having an L^2-density (this was based on a criterion by Kahane and Salem [21]). Thus by Solomyak's result, $\theta_2 = \tau - 1$ for almost all $\rho \in (1,2]$. Furthermore, $\overline{\theta} = \tau$ (since for every number of the form $\varepsilon_1 + \varepsilon_2\rho + \cdots + \varepsilon_n\rho^{n-1}$ there is a jump of one at least) while

$$\theta \le \tau(1 - \tfrac{1}{n}) \quad \text{for } n-1 < \tau \le n \ \text{ where } n \in \mathbb{N}$$

(Theroem 1.5 [13]). In particular $\theta_p \equiv \theta$ if $\tau \in \mathbb{N}$ (i.e., ρ is an nth-root of 2). But for $\rho = \frac{1+\sqrt{5}}{2}$ (a Pisot number) it was shown that $\theta \ge \frac{\tau}{3} > \tau - 0.9603$, so in this case $\theta_p < \theta$ for all p.

An open problem is then: for which ρ do we have $\theta_p \equiv \theta$?

[5] Actually Theorem 4.2 would only give $\le$ here. To get equality we need to cancel off the simple pole at 0. This can be done by considering instead $f(x) = |\zeta(\frac{1}{2}+ix)|^2 - \log x$, which keeps all the abscissa the same.

(v): Let $f(x) = \sin(e^x)$ for $x \geq 1$ and zero otherwise. In this case $\theta_p \equiv \theta = 0$ while $\overline{\theta} = \infty$. For $1 \geq |f(x)| \not\to 0$ while $\int_0^x f^2 = \int_1^x (\sin(e^t))^2 dt = \int_e^{e^x} \frac{1-\cos 2y}{2} dy = \frac{x}{2} + O(1)$. These imply $\theta = 0 = \theta_2$. Noting that $\int_0^\infty |f|^p = \infty$ gives $\theta_p \equiv 0$. On the other hand,

$$V_f(x) = \int_0^x e^t |\cos(e^t)|\, dt = \int_1^{e^x} |\cos u|\, du \asymp e^x,$$

which is exponentially large. Thus $\overline{\theta} = \infty$.

The Mellin transform is

$$\hat{f}(s) = s \int_1^\infty \frac{\sin(e^x)}{x^{s+1}}\, dx = s \int_e^\infty \frac{\sin y}{y(\log y)^{s+1}}\, dy. \tag{5.1}$$

This integral converges absolutely for $\sigma > 0$. Integrating by parts gives (for $\sigma > 0$)

$$\hat{f}(s) = s\frac{\cos e}{e} - s(s+1) \int_e^\infty \frac{\cos y}{y^2 (\log y)^{s+2}}\, dy - s \int_e^\infty \frac{\cos y}{y^2 (\log y)^{s+1}}\, dy,$$

showing that $\hat{f}$ is entire, with $\hat{f}(\sigma + it) = O(t^2)$. Thus $\mu_2(\sigma)$ and $\mu(\sigma)$ are constant (being convex and bounded). Since $\mu_2(\theta_2) = \mu_2(0) = \frac{1}{2}$, it follows that $\mu_2(\sigma) \equiv \frac{1}{2}$.

Further, from (5.1) $\mu(\sigma) \equiv$ constant $\in [\frac{1}{2}, 1]$. Which constant?

(vi): Let $f \in S_{\text{disc}}$ be defined by $f(x) = f_1([x])$, $f_1(x) = x^{\tau(x)}$ and $\tau(\cdot)$ is given on each interval $[2^n, 2^{n+1}]$ by

$$\tau\Big(\tfrac{3}{2}2^n \pm \lambda 2^{(\lambda+\frac{1}{2})n}\Big) = \frac{1}{2} - \lambda \qquad (0 \leq \lambda \leq \tfrac{1}{2},\, n \in \mathbb{N}_0).$$

After Theorem 3.2, f satisfies $f(n) - f(n-1) \ll \log n$, $\overline{\theta} = \theta = \frac{1}{2}$ and $\nu(\lambda) = \lambda + \frac{1}{2}$ for $0 < \lambda < \frac{1}{2}$ while $\nu(\lambda) = 1$ for $\lambda \geq \frac{1}{2}$. This leads easily to

$$\theta_p = \begin{cases} \frac{1}{2} - \frac{1}{2p} & \text{if } p > 1, \\ 0 & \text{if } p \leq 1. \end{cases}$$

This is an "extremal" case as in Corollary 4.3. As such we have

$$\mu_p(\sigma) \equiv \mu(\sigma) = \begin{cases} 1 - 2\sigma & \text{if } 0 < \sigma < \frac{1}{2}, \\ 0 & \text{if } \sigma \geq \frac{1}{2}. \end{cases}$$

This is an example where $f(x) = \sum_{n \leq x} a_n$ where $a_n \ll \log n$, and $\theta_p, \theta, \mu_p(\sigma)$, $\mu(\sigma)$ are explicitly known and $\theta_p < \theta$ while $\mu_p \equiv \mu$.

(vii): (Kahane's example). In [20] (Theorem 2.2), Kahane proves the existence of a Dirichlet series of the form $\sum_{n=1}^\infty \varepsilon_n ((2n-1)^{-s} - (2n)^{-s})$ with $\varepsilon_n = \pm 1$ for which

$$\mu(\sigma) = \begin{cases} 1 - \sigma & \text{for } 0 < \sigma < 1, \\ 0 & \text{for } \sigma \geq 1, \end{cases} \qquad \text{while} \qquad \mu_2(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{for } 0 < \sigma < \frac{1}{2}, \\ 0 & \text{for } \sigma \geq \frac{1}{2}. \end{cases}$$

(Actually, Kahane proved the formula for $\mu(\sigma)$, but the formula for $\mu_2(\sigma)$ is easily proven using methods from, say, [24].) Let $f(x) = \sum_{n \leq x} a_n$ where $a_{2k-1} = -a_{2k} = \varepsilon_k$. Then $|f(x)| = 1$ on intervals $[2k-1, 2k)$ and zero on $[2k, 2k+1)$. It readily follows that $\theta_p \equiv \theta = 0$, while $\overline{\theta} = 1$. This is an example of an $f \in S_{\text{disc}}$ where $\theta_p \equiv \theta$ while $\mu_p < \mu$.

Final comments. Many of these examples point to $\mu_2 \equiv \mu$ (and hence $\mu_p \equiv \mu$), through proofs or conjectures. It suggests that $\mu_2 \equiv \mu$ in quite general circumstances, though not in all as shown by Kahane's example and the example at the end of §3. Deciding what mechanism makes $\mu_2 \equiv \mu$ is likely to be extremely difficult.

6. The case $\theta = \infty$

There is an interesting class of functions which are not of polynomial growth yet θ_p is finite for all p. Examples are $d([e^{x-1}])$ (where d is the divisor function) and, conjecturally, $\zeta(\frac{1}{2} + ie^x)$. These will be discussed later. We shall see that much of the previous theory extends to this case. Some results extend without trouble but extra care is needed when earlier we relied on θ being finite.

Let S_∞ denote the space of functions $f : (0, \infty) \to \mathbb{C}$ for which (i) $f(x) = 0$ for $x < 1$ and (ii) f has bounded variation on any bounded interval, but for which $\int_0^x |f|^p = O(x^A)$ for every $p > 0$ (for some A, dependent on p) but $f(x) = \Omega(x^A)$ for every A.

The proof of Proposition 1.1 shows that both (a), (c), and (d) of Proposition 1.1 remain true for $f \in S_\infty$; namely, θ_p increases with p and $p\theta_p$ is a convex function of p.

Further, we may again define $U_\lambda(x)$ and $\sigma(\lambda)$ as before, this time for all $\lambda \in \mathbb{R}$. As before, $\sigma(\lambda)$ is increasing and non-negative.

6.1. **Connecting θ_p and $\sigma(\lambda)$.** We can characterise those functions for which $\theta_p \to \infty$ in terms of $\sigma(\lambda)$.

Theorem 6.1. *Let $f \in S_\infty$. Then $\theta_p \to \infty$ if and only if $\sigma(\lambda) < \infty$ for all $\lambda \in \mathbb{R}$.*

Proof. ($\Leftarrow$) Suppose $\sigma(\lambda) < \infty$ for all λ. Then for every λ,

$$\int_x^{2x} |f|^p \geq \int_{U_\lambda(x)} |f|^p \geq \int_{U_\lambda(x)} t^{p\lambda}\, dt \geq cx^{p\lambda}|U_\lambda(x)| = \Omega(x^{p\lambda+1-\sigma(\lambda)-\varepsilon})$$

for every $\varepsilon > 0$. The LHS is $\ll x^{p\theta_p+1+\varepsilon}$. Thus

$$p\theta_p \geq p\lambda - \sigma(\lambda) \quad \text{for all } \lambda. \tag{6.1}$$

Hence $\theta_p \geq \lambda - \frac{\sigma(\lambda)}{p}$ and letting $p \to \infty$ gives

$$\liminf_{p\to\infty} \theta_p \geq \lambda.$$

This is true for every λ, so $\theta_p \to \infty$ as $p \to \infty$.

For the converse, suppose $\theta_p \to \infty$. Then, given $\lambda \in \mathbb{R}$, there exists p such that $\theta_p > \lambda$. As such

$$\int_{[x,2x]\setminus U_\lambda(x)} |f|^p \leq cx^{p\lambda+1} = o(x^{p\theta_p+1-\eta})$$

for some $\eta > 0$. It follows that

$$\int_{U_\lambda(x)} |f|^p = \int_x^{2x} |f|^p - \int_{[x,2x]\setminus U_\lambda(x)} |f|^p = \Omega(x^{p\theta_p+1-\varepsilon}) \quad \text{for every } \varepsilon > 0. \tag{6.2}$$

Now suppose $w(\lambda) = -\infty$. By Cauchy-Schwarz,

$$\Big(\int_{U_\lambda(x)} |f|^p\Big)^2 \leq \int_{U_\lambda(x)} dt \cdot \int_{U_\lambda(x)} |f|^{2p} \leq |U_\lambda(x)| \int_x^{2x} |f|^{2p} \ll x^{-A}$$

for all $A > 0$, since $\sigma(\lambda) = \infty$. This contradicts (6.2). Hence $\sigma(\lambda) < \infty$ for $\lambda < \theta_p$; i.e., $\sigma(\lambda) < \infty$ for all λ. □

Remarks 6.1.

(a) The proof of Theorem 6.1 actually shows that, on writing $\theta_\infty = \lim_{p\to\infty} \theta_p \in [\theta_1, \infty]$, we have $\sigma(\theta_\infty-) < \infty$ while $\sigma(\theta_\infty+) = \infty$.

(b) We see from (6.1), with say $p = 1$, that $\sigma(\lambda) \geq \lambda - \theta_1 \to \infty$ as $\lambda \to \infty$. In fact, it is even true that $\frac{\sigma(\lambda)}{\lambda} \to \infty$. For

$$\liminf_{\lambda\to\infty} \frac{\sigma(\lambda)}{\lambda} \geq \liminf_{\lambda\to\infty} \Big(p - \frac{p\theta_p}{\lambda}\Big) = p$$

for every $p > 0$.

Now we show that Theorem 2.2 extends to this setting.

Theorem 6.2. *We have*

$$p\theta_p = \sigma^*(p) = \sup_{\lambda\in\mathbb{R}}(p\lambda - \sigma(\lambda)).$$

Proof. From (6.1) we see already that $p\theta_p \geq \sigma^*(p)$. It remains to show the reverse inequality.

Fix $\lambda < \theta_p$ and let $\mu > \lambda$ (fixed but arbitrary). Let $K \in \mathbb{N}$ (large) and put $\mu_k = \lambda + \frac{k}{K}(\mu - \lambda)$ for $k = 0, 1, \ldots, K$. Thus $\lambda = \mu_0 < \ldots < \mu_K = \mu$. As such

$$\int_{U_\lambda(x)} |f|^p = \int_{U_\mu(x)} |f|^p + \sum_{k=1}^{K} \int_{U_{\mu_{k-1}}(x)\setminus U_{\mu_k}(x)} |f|^p. \tag{6.3}$$

For $t \in U_{\mu_{k-1}}(x) \setminus U_{\mu_k}(x)$, we have $|f(t)| \leq t^{\mu_k} \leq cx^{\mu_k}$. Hence

$$\int_{U_{\mu_{k-1}}(x)\setminus U_{\mu_k}(x)} |f|^p \leq cx^{p\mu_k}|U_{\mu_{k-1}}(x)| \ll x^{p\mu_k+1-\sigma(\mu_{k-1})+\varepsilon} \leq x^{\sigma^*(p)+1+\frac{p(\mu-\lambda)}{K}+\varepsilon}.$$

Hence the RHS sum of (6.3) is $O(x^{\sigma^*(p)+1+\frac{p(\mu-\lambda)}{K}+\varepsilon})$. Suppose for a contradiction that $p\theta_p > \sigma^*(p)$ for some p. Then we can choose K so large that $\sigma^*(p) + \frac{p(\mu-\lambda)}{K} < p\theta_p$. As such, the RHS sum in (6.3) is $o(x^{p\theta_p+1-\eta})$ for some $\eta > 0$ and so, for some $c > 0$,

$$\int_{U_\mu(x)} |f|^p \geq \int_{U_\lambda(x)} |f|^p - cx^{p\theta_p+1-\eta} = \Omega(x^{p\theta_p+1-\varepsilon}) \quad \text{for all } \varepsilon > 0 \text{ for every } \mu. \tag{6.4}$$

But, using Cauchy-Schwarz again,

$$\int_{U_\mu(x)} |f|^p \leq \sqrt{|U_\mu(x)| \int_x^{2x} |f|^{2p}} \ll x^{\frac{1}{2}-\frac{1}{2}\sigma(\mu)+\frac{1}{2}+p\theta_{2p}+\varepsilon},$$

so (6.4) implies $\sigma(\mu) \leq 2p(\theta_{2p} - \theta_p)$. This is true for every $\mu > \lambda$ as μ was chosen arbitrarily. Thus $\sigma(\cdot)$ is bounded above. This contradicts the fact that $\sigma(\mu) \geq p\mu - p\theta_p \to \infty$ as $\mu \to \infty$. □

As before we have $\sigma(\lambda) \geq p\lambda - p\theta_p$, so

$$\sigma(\lambda) \geq \sup_{p>0} p\lambda - p\theta_p = \tilde{\sigma}(\lambda). \tag{6.5}$$

Proposition 2.8(b) also generalises to this setting. As θ_p is increasing, $\theta_0 = \lim_{p\to 0+} \theta_p$ exists in $\mathbb{R} \cup \{-\infty\}$.

Proposition 6.3. *Suppose $\theta_0 > -\infty$. Then $\sigma(\lambda) = 0$ for $\lambda < \theta_0$ and $\sigma(\lambda) > 0$ for $\lambda > \theta_0$.*

Proof. First observe that for $\lambda < \theta_0$ and $p > 0$, $p\lambda - \sigma(\lambda) \le p\lambda < p\theta_p$. Thus

$$p\theta_p = \sup_{\lambda \ge \theta_0 -} (p\lambda - \sigma(\lambda)).$$

After Remarks 6.1, there exists $\lambda_0 > 0$ such that $\sigma(\lambda) \ge \lambda$ for $\lambda \ge \lambda_0$. As such, $p\lambda - \sigma(\lambda) \le 0$ for $p \in (0, 1]$. But $p\theta_p \to 0$ as $p \to 0+$, so by continuity, $p\_p > 0$ for p sufficiently small. Thus for such p,

$$p\theta_p = \sup_{\theta_0 - \le \lambda \le \lambda_0} (p\lambda - \sigma(\lambda)) \le p\lambda_0 - \sigma(\theta_0 -).$$

Letting $p \to 0+$, gives $\sigma(\theta_0-) \le 0$. Hence $\sigma(\theta_0-) = 0$ as required.

For the second part, if $\lambda > \theta_0$, then $\exists p > 0$ such that $\lambda > \theta_p$. As such, $\sigma(\lambda) \ge p(\lambda - \theta_p) > 0$. □

Theorem 6.4. *Let $k : [\alpha, \infty) \to \mathbb{R}$ denote an arbitrary continuous and strictly increasing function for which $k(\alpha) = 0$ and $k(\lambda)/\lambda \to \infty$ as $\lambda \to \infty$. Then there exists $f \in S_\infty$ such that*

$$\sigma(\lambda) = \begin{cases} k(\lambda) & \textit{for } \lambda \ge \alpha, \\ 0 & \textit{for } \lambda < \alpha. \end{cases}$$

Proof. Let $f(t) = t^{\tau(t)}$ for $t \ge 1$ where $\tau(t)$ is defined on intervals $[2^n, 2^{n+1}]$ ($n \in \mathbb{N}_0$) as follows:

$$\tau(\tfrac{3}{2}2^n \pm \tfrac{1}{2}2^{n(1-k(\lambda))}) = \begin{cases} \lambda & \text{if } \alpha \le \lambda \le \lambda_n, \\ \lambda_n & \text{if } \lambda > \lambda_n. \end{cases}$$

Here λ_n is any strictly increasing sequence tending to infinity such that $\lambda_n \ge \alpha$ and $k(\lambda_n) \ll 2^{bn}$ for some b. As such, $\tau(2^n) = \alpha$, and, since $\lambda_n \to \infty$, $\theta = \infty$.

Now given $\lambda \in [\alpha, \lambda_n]$ and $t \in [2^n, 2^{n+1}]$, we have $f(t) \ge t^\lambda$ if and only if $\tau(t) \ge \lambda$. Thus

$$U_\lambda(2^n) = [\tfrac{3}{2}2^n - \tfrac{1}{2}2^{n(1-k(\lambda))}, \tfrac{3}{2}2^n + \tfrac{1}{2}2^{n(1-k(\lambda))}]$$

and so $|U_\lambda(2^n)| = 2^{n(1-k(\lambda))}$. Thus, taking $x \in [2^n, 2^{n+1}]$,

$$|U_\lambda(x)| \le |U_\lambda(2^n)| + |U_\lambda(2^{n+1})| \le (1 + 2^{1-k(\lambda)})2^{n(1-k(\lambda))} \ll x^{1-k(\lambda)},$$

and so $\sigma(\lambda) = k(\lambda)$ for $\lambda \ge \alpha$. Since $k(\alpha) = 0$ and σ is increasing, it follows that $\sigma(\lambda) = 0$ for $\lambda < \alpha$.

Now it remains to show that θ_p exists for this f. It suffices to show that $\int_x^{2x} |f|^p \ll x^A$ for some A (depending on p) and moreover it is enough to consider $x = 2^n$. As such, we have

$$\int_{2^n}^{2^{n+1}} |f|^p \asymp \int_{2^n}^{2^{n+1}} 2^{np\tau(t)}\,dt = 2\int_0^{2^{n-1}} 2^{np\tau(\frac{3}{2}2^n + y)}\,dy$$

$$= 2^{n(p\lambda_n + 1 - k(\lambda_n))} + n\log 2 \int_\alpha^{\lambda_n} 2^{n(p\lambda + 1 - k(\lambda))}\,dk(\lambda).$$

But $p\lambda + 1 - k(\lambda) \to -\infty$ as $\lambda \to \infty$, so the first term tends to zero while the integrand is bounded by 2^{an} with $a = \sup_{\lambda \geq \alpha}(p\lambda + 1 - k(\lambda))$. Hence

$$\int_{2^n}^{2^{n+1}} |f|^p \ll n2^{an} \int_{\alpha}^{\lambda_n} dk(\lambda) \ll n2^{an}k(\lambda_n) \ll 2^{cn}$$

for some c. □

Theorem 6.5. *Let $\phi_p \in \mathbb{R}$ $(p > 0)$ be increasing without bound such that $p\phi_p$ is convex. Suppose further that $\phi_p \to \phi_0 > -\infty$ as $p \to 0+$. Then there exists $f \in S_\infty$ such that $\theta_p = \phi_p$.*

Proof. As in the proof of Theorem 2.5, let $g(\lambda) = \sup_{p>0}(p\lambda - p\phi_p)$, which is convex and increasing. By taking arbitrarily close to 0 and using the condition $\lim_{p\to 0+} \phi_p > -\infty$ implies that $g \geq 0$. Further, as in Proposition 2.8, $g(\lambda) = 0$ for $\lambda < \phi_0$ while $g(\lambda) > 0$ for $\lambda > \phi_0$. By convexity, g is therefore strictly increasing on $[\phi_0, \infty)$. By Theorem 6.4, there exists $f \in S_\infty$ with $\sigma(\lambda) = g(\lambda)$ for all λ. Hence

$$p\phi_p = \sup_\lambda(p\lambda - g(\lambda)) = \sup_\lambda(p\lambda - \sigma(\lambda)) = p\theta_p$$

as required. □

6.2. **A special class of arithmetical functions.** Now we consider f of the form

$$f(x) = F([e^{x-1}]), \tag{6.6}$$

where F is an arithmetical function. In fact we shall restrict further to non-negative multiplicative functions for which $F(n) = O(n^\varepsilon)$. As such the size of θ_p is largely determined by the behaviour of F at the primes. To ensure $\theta = \infty$, we need $F(n) = \Omega((\log n)^A)$ for all A.

Theorem 6.6. *Let f be of the form* (6.6) *where F is multiplicative and non-negative. Further assume that $F(n) \ll n^\varepsilon$ for all $\varepsilon > 0$ and $F(p) \to \alpha > 0$ as $p \to \infty$ through primes. Then $f \in S_\infty$ with $\theta_q = \frac{\alpha^q - 1}{q}$.*[6]

Proof. Let $g(n) = \frac{F(n)^q}{n}$, which is multiplicative and non-negative. Further $g(n) \ll n^{-1+\varepsilon}$ for all $\varepsilon > 0$ and

$$\sum_{e^x < p \leq e^{\lambda x}} g(p) = \sum_{e^x < p \leq e^{\lambda x}} \frac{F(p)^q}{p} \sim \alpha^q \log \lambda$$

as $x \to \infty$ for every $\lambda > 0$. Thus g satisfies the conditions of Theorem 6.3.2 of [3]; namely,

$$\sum_{p,k\geq 2} g(p^k), \sum_p g(p)^2 < \infty$$

and

$$\sum_{x < \log p \leq \lambda x} g(p) \to b \log \lambda$$

as $x \to \infty$ for every $\lambda > 0$ for some $b \geq 0$. As such, Theorem 6.3.2 of [3] says that $\sum_{n \leq e^x} g(n)$ is regularly-varying of index α^q; in particular this means

$$\sum_{n \leq e^x} \frac{F(n)^q}{n} = x^{\alpha^q + o(1)}. \tag{6.7}$$

[6]We use q here, as primes will be denoted by p.

But, on writing $F(x)=F([x])$,

$$\begin{aligned}\int_0^x |f|^q &= \int_1^x F(e^{t-1})^q\,dt = \int_1^{e^{x-1}} \frac{F(y)^q}{y}\,dy \\ &= \sum_{n\le e^{x-1}} \int_n^{n+1} \frac{F(y)^q}{y}\,dy + \int_{[e^{x-1}]}^{e^{x-1}} \frac{F(y)^q}{y}\,dy \\ &= \sum_{n\le e^{x-1}} F(n)^q \log\Big(1+\frac{1}{n}\Big) + O\Big(\frac{F(e^{x-1})^q}{e^{x-1}}\Big).\end{aligned}$$

The O-term is $o(1)$ since $F(x) \ll x^{\varepsilon}$ and the first sum is $x^{\alpha^q+o(1)}$ by (6.7). Hence the result follows. □

Remarks 6.2. (i) Note that $\theta_p \to 0$ as $p\to\infty$ if $\alpha \le 1$. (ii) Using (6.5) we have, in case $\alpha > 1$,

$$\tilde{\sigma}(\lambda\log\alpha) = \begin{cases} \lambda\log\lambda - \lambda + 1 & \text{if } \lambda > 1, \\ 0 & \text{if } \lambda \le 1, \end{cases}$$

while if $\alpha \le 1$

$$\tilde{\sigma}(\lambda) = \begin{cases} \infty & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda \le 0. \end{cases}$$

(iii) The result easily generalises to the case where $F(p)$ oscillates asymptotically between different values along different arithmetic progressions. *If f satisfies the conditions of Theorem* 6.6, *but now $F(p) \to \alpha_k$ as $p\to\infty$ through primes $p \equiv k (\mathrm{mod}\ m)$, with $k \le m$ coprime to m, then $f \in S_\infty$ with*

$$q\theta_q = \frac{1}{\varphi(m)} \sum_{\substack{k\le m\\(k,m)=1}} (\alpha_k^q - 1).$$

For this time

$$\sum_{e^x<p\le e^{\lambda x}} \frac{F(p)^q}{p} \sim \sum_{\substack{k\le m\\(k,m)=1}} \alpha_k^q \sum_{\substack{e^x<p\le e^{\lambda x}\\ p\equiv k(\mathrm{mod} m)}} \frac{1}{p} \to \frac{1}{\varphi(m)}\Big(\sum_{\substack{k\le m\\(k,m)=1}} \alpha_k^q\Big)\log\lambda$$

as $x\to\infty$. Again, the conditions of Theorem 6.3.2 of [3] are satisfied and the result follows.

Examples.

(a) Take $f(x) = d([e^{x-1}])$ where $d(n)$ is the divisor function. By Theorem 6.6, $f \in S_\infty$ with $\theta_p = \frac{2^p-1}{p}$. Hence $\theta_0 = \log 2$ and by Proposition 6.3, $\sigma(\lambda) = 0$ for $\lambda < \log 2$ and positive for $\lambda > \log 2$. Indeed $\sigma(\lambda\log 2) \ge \tilde{\sigma}(\lambda\log 2) = \int_1^\lambda \log t\,dt$ for $\lambda > 1$.

(b) Take $f(x) = F([e^{x-1}])$ where $F(n) = \sum_{d|n}\chi(d)$, where $\chi(n) = (-1)^{\frac{n-1}{2}}$ for n odd and zero otherwise. Thus $F(n) = \frac{1}{4}r(n)$, where $r(n)$ is the number of ways of writing n as a sum of two squares. In this case, for p an odd prime

$$F(p) = \begin{cases} 2 & \text{if } p \equiv 1 (\mathrm{mod}\ 4), \\ 0 & \text{if } p\equiv 3(\mathrm{mod}\ 4). \end{cases}$$

By Remarks 6.2(iii), $f \in S_\infty$ we have

$$\theta_p = \frac{2^{p-1}-1}{p}.$$

Note that here $\theta_0 = -\infty$ and $\sigma(\lambda) > 0$ for every λ.

Indeed, (6.5) shows after a little calculation that

$$\tilde{\sigma}(\lambda \log 2) = \begin{cases} \lambda \log 2\lambda - \lambda + 1 & \text{if } \lambda > \frac{1}{2}, \\ \frac{1}{2} & \text{if } \lambda \leq \frac{1}{2}. \end{cases}$$

(c) Let $f(x) = |\zeta(\frac{1}{2} + ie^x)|^2$ for $x \geq 1$ and zero otherwise. On the Riemann Hypothesis, it was shown that

$$I_p(T) = \int_0^T |zeta(\tfrac{1}{2} + it)|^{2p}\, dt \asymp T(\log T)^{p^2}$$

for every $p > 0$. The lower bound (which holds unconditionally for p rational) is by Ramachandra [23] and Heath-Brown [11], while the upper bound is due to Harper [10], building on the work of Soundararajan [28]. Hence

$$\int_x^{2x} |f|^p = \int_x^{2x} |\zeta(\tfrac{1}{2} + ie^y)|^{2p}\, dy = \Big[\frac{I_p(t)}{t}\Big]_{e^x}^{e^{2x}} + \int_{e^x}^{e^{2x}} \frac{I_p(t)}{t^2}\, dt.$$

Assuming RH, the first term is $\ll x^{p^2+\varepsilon}$ while the integral is $x^{p^2+1+o(1)}$. Thus $f \in S_\infty$ with $\theta_p = p$.

Now $\theta_0 = 0$, so by Proposition 6.3, $\sigma(\lambda) = 0$ for $\lambda < 0$. For $\lambda \geq 0$, (6.5) gives $\tilde{\sigma}(\lambda) = \frac{\lambda^2}{4}$. Thus, on RH, we find that for every $a \geq 0$ and every $\varepsilon > 0$,

$$|\{t \in [x, 2x] : |\zeta(\tfrac{1}{2} + ie^t)| \geq t^a\}| \ll x^{1-a^2+\varepsilon}.$$

(d) Let $f(x) = |S(e^{e^x})|$, where $S(t) = \frac{1}{\pi}\arg\zeta(\frac{1}{2} + it)$. Selberg [25] showed that $\int_0^T S(t)^{2k} dt \sim \frac{(2k)!}{k!(2\pi)^{2k}} T((\log\log T))^k$ for each positive integer k. This easily leads to $\int_x^{2x} |f|^{2k} \sim a_k x^{k+1}$ for some constant $a_k > 0$. Thus $\theta_{2k} = \frac{1}{2}$ for $k \in \mathbb{N}$ while $\theta = \infty$ (since $S(t)$ is sometimes as large as a power of $\log t$). It follows that $\theta_p = \frac{1}{2}$ for all $p > 0$. Thus also $\sigma(\lambda) = \tilde{\sigma}(\lambda) = \infty$ for $\lambda > \frac{1}{2}$ and zero otherwise.

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