

PULL-BACK COMPONENTS OF THE SPACE OF FOLIATIONS OF CODIMENSION ≥ 2

W. COSTA E SILVA AND A. LINS NETO

ABSTRACT. We present a new list of irreducible components for the space of k -dimensional holomorphic foliations on \mathbb{P}^n , $n \geq 3$, $k \geq 2$. They are associated to pull-back of dimension one foliations on \mathbb{P}^{n-k+1} by non-linear rational maps.

CONTENTS

1.	Introduction	949
2.	Preliminaries	951
3.	Proof of Theorem A	957
4.	Related problems	968
	Acknowledgments	968
	References	968

1. INTRODUCTION

A codimension q singular holomorphic foliation \mathcal{F} on a complex manifold M , $\dim_{\mathbb{C}} M \geq q + 1$, can be defined locally by the following data:

- (1) a covering $\mathcal{U} = (U_{\alpha})_{\alpha \in A}$ of M by open sets.
- (2) A collection $(\eta_{\alpha})_{\alpha \in A}$ of q -forms, $\eta_{\alpha} \in \Omega_{U_{\alpha}}^q$, having the following properties:
 - (a) Local decomposability: given $p \in U_{\alpha}$ such that $\eta_{\alpha}(p) \neq 0$ there exist a neighborhood U of p , $U \subset U_{\alpha}$, and holomorphic 1-forms $\omega_1, \dots, \omega_q$ such that $\eta_{\alpha}|_U = \omega_1 \wedge \dots \wedge \omega_q$.
 - (b) The decomposition of η_{α} satisfies the Frobenius integrability condition $d\omega_i \wedge \eta_{\alpha} = 0$ for every $i = 1, \dots, q$.
- (3) A multiplicative cocycle $G := (g_{\alpha\beta})_{U_{\alpha} \cap U_{\beta} \neq \emptyset}$ such that $\eta_{\alpha} = g_{\alpha\beta} \eta_{\beta}$.

When $\dim_{\mathbb{C}} M = n$ we say also that the foliation is $(n - q)$ -dimensional.

The line bundle induced by the cocycle G is denoted by $N_{\mathcal{F}}$ and called the normal bundle of \mathcal{F} . The family $(\eta_{\alpha})_{\alpha \in A}$, defines a global section $\eta \in H^0(M, \Omega_M^q \otimes N_{\mathcal{F}})$. The analytic subset, $Sing(\mathcal{F}) := \bigcup_{\alpha} \{p \in M \mid \eta_{\alpha}(p) = 0\}$ is the singular set of \mathcal{F} . We can always assume that $Sing(\mathcal{F})$ has codimension greater than or equal to two.

If $M = \mathbb{P}^n$, the n -dimensional complex projective space, then a codimension q singular foliation \mathcal{F} is given by a global section of

$$H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q \otimes \mathcal{O}_{\mathbb{P}^n}(\Theta + q + 1)),$$

Received by the editors August 24, 2016, and, in revised form, February 20, 2017, March 14, 2017, and March 28, 2017.

2010 *Mathematics Subject Classification*. Primary 37F75; Secondary 32G34, 32S65.

Key words and phrases. Holomorphic foliations, irreducible components.

where Θ (called the degree of \mathcal{F}) is the degree of the divisor of tangencies of the foliation with a generic \mathbb{P}^q linearly embedded in \mathbb{P}^n . On the other hand, a global section of $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q \otimes \mathcal{O}_{\mathbb{P}^n}(\Theta + q + 1))$ can be represented, in homogeneous coordinates, by a polynomial q -form η on \mathbb{C}^{n+1} with homogeneous coefficients of degree $\Theta + 1$ and satisfying $i_R\eta = 0$, where $R = \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$ is the radial vector field. In fact, the q -form η defines the foliation $\Pi^*(\mathcal{F})$, where $\Pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ is the canonical projection. For more details see [10] and [11].

The projectivisation of the set of integrable q -forms as above, defining in homogeneous coordinates k -dimensional foliations on \mathbb{P}^n of degree Θ , will be denoted by $\text{Fol}(\Theta; k, n)$. Due to the integrability condition, $\text{Fol}(\Theta; k, n)$ is a quasi-projective algebraic subset of $\mathbb{P}H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q \otimes \mathcal{O}_{\mathbb{P}^n}(\Theta + n - k + 1))$. A natural question that arises is the following.

Problem. Identify and classify the irreducible components of $\text{Fol}(\Theta; k, n)$ on \mathbb{P}^n , such that $\Theta \geq 0$ and $n \geq k + 1$.

The classification of the irreducible components of $\text{Fol}(0; k, n)$ was given in [2, Th. 3.8 p. 46] (a k -dimensional foliation of degree zero on \mathbb{P}^n is a rational fibration defined by a linear projection from \mathbb{P}^n to \mathbb{P}^{n-k}). The classification of the irreducible components of $\text{Fol}(1; k, n)$, which require more details to be explained here, was given in [14, Th. 6.2 and Cor. 6.3 pp. 935-936]. The situation for $\Theta \geq 2$ remains wide open, unless in the case of codimension one and degree $\Theta = 2$ (see [3]).

Usually, the space of codimension one foliations on \mathbb{P}^n of degree k is denoted by $\text{Fol}(k, n)$. The study of irreducible components of these spaces has been initiated by Jouanolou in [10], where the irreducible components of $\text{Fol}(k, n)$ for $k = 0$ and $k = 1$ are described. The case of degree two was studied in the paper [3], where the authors proved that $\text{Fol}(2, n)$ has six irreducible components, which can be described by geometric and dynamic properties of a generic element. In the general case, degree ≥ 3 , one can exhibit some kind of list of irreducible components in every degree, but it is not known if this list is complete.

When we study the components of the space $\text{Fol}(k, n)$, $n \geq 3$ we perceive that there are families of irreducible components in which the typical element is a pull-back of a foliation on \mathbb{P}^2 by a rational map. More precisely, the situation is as follows: given a generic rational map $f : \mathbb{P}^n \rightarrow \mathbb{P}^2$ of degree $\nu \geq 1$, and a degree d foliation \mathcal{G} of \mathbb{P}^2 , then it can be associated to the pair (f, \mathcal{G}) the pull-back foliation $f^*\mathcal{G}$ on \mathbb{P}^n . If f and \mathcal{G} are generic, then the degree of $f^*\mathcal{G}$ is $\nu(d + 2) - 2$, as proved in [4]. Denote by $PB(d, \nu, n)$ the closure in $\text{Fol}(\nu(d + 2) - 2, n)$, $n \geq 3$ of the set of foliations $f^*\mathcal{G}$ as above. The main result of [4] is the following.

Theorem 1.1 ([4]). *The set $PB(d, \nu, n)$ is a unirational irreducible component of $\text{Fol}(\nu(d + 2) - 2, n)$ for all $n \geq 3$, $\nu \geq 1$, and $d \geq 2$.*

The case $\nu = 1$, of linear pull-backs, was proven in [1], whereas the case $\nu > 1$, of non-linear pull-backs, was proved in [4]. In the paper [6] the authors were able to prove the existence of irreducible components of linear pull-back foliations in arbitrary codimension. This was obtained by using the techniques of stability of the tangent sheaf of the foliation. Linear pull-back foliations have tangent sheaf which is locally free and therefore stable. On the other hand, irreducible components of the space of foliations where a typical element is a non-linear pull-back was known only for the codimension one situation. The main purpose of this work is to show that at least in some cases there exist non-linear pull-back type components of

$\mathbb{F}ol(\Theta; k, n)$, for several values of Θ . Let us describe, briefly, the type of pull-back foliation that we shall consider.

From now on we will always assume $n \geq 3$. Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^{q+1}$ be a rational map, represented in the homogeneous coordinates $z \in \mathbb{C}^{n+1}$ and $x \in \mathbb{C}^{q+2}$ by $\tilde{f} = (F_0, F_1, \dots, F_{q+1})$, where the F_i 's are homogeneous polynomials of degree ν , without common factors. Let \mathcal{G} be a one dimensional foliation on \mathbb{P}^{q+1} of degree d and let $\Pi_{q+1} : \mathbb{C}^{q+2} \setminus \{0\} \rightarrow \mathbb{P}^{q+1}$ be the canonical projection. Then $\Pi_{q+1}^* \mathcal{G}$ can be defined by a q -form $\Omega = i_R i_X dV$, where R is the radial vector field, $X = \sum_{i=0}^{q+1} P_i(x) \frac{\partial}{\partial x_i}$ is a homogeneous polynomial vector field of degree d and $dV = dx_0 \wedge \dots \wedge dx_{q+1}$. If f and \mathcal{G} are generic, in a sense that will be explained later, then the form $f^* \Omega$ represents in homogeneous coordinates the foliation $f^* \mathcal{G} := \mathcal{F}$, of codimension q on \mathbb{P}^n . It can be checked that the degree of \mathcal{F} is $\Theta(\nu, d, q) = (d + q + 1)\nu - q - 1$. Let $PB(\nu, d, k, n)$, $k = n - q$, be the closure in $\mathbb{F}ol(\Theta; k, n)$, $\Theta = \Theta(\nu, d, q)$, of the set of this kind of foliations. In this paper we are able to prove the following.

Theorem A. *The set $PB(\nu, d, k, n)$ is a unirational irreducible component of $\mathbb{F}ol(\Theta; k, n)$ for all $\nu \geq 2$, $d \geq 2$, $k \geq 2$, and $n \geq 3$.*

Let us observe that a generic element in the set $PB(\nu, d, k, n)$ has no algebraic invariant leaf because a generic foliation by curves of degree ≥ 2 also does not have [13]. It is worth pointing out that we recover Theorem 1.1 in the case $n \geq 3$ and $k = n - 1$. On the other hand, for $n \geq 4$ we present new families of irreducible components that were not known.

The proof of Theorem A will be done first in the case $k = 2$, that is, for two dimensional foliations on \mathbb{P}^n that are pull-back of one dimensional foliations on \mathbb{P}^{n-1} . The general case, $k \geq 3$, will be done in §3.4.

2. PRELIMINARIES

2.1. One dimensional foliations on \mathbb{P}^m , $m \geq 2$. Let us recall some definitions and basic facts about one dimensional foliations that we will use. Let \mathcal{X} be a germ at $0 \in \mathbb{C}^m$ of holomorphic vector field, $0 \in Sing(\mathcal{X})$ and denote by $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ the eigenvalues of the linear part of $D\mathcal{X}(0)$. The germ of foliation defined by \mathcal{X} will be denoted by $\mathcal{G}_{\mathcal{X}}$. We say that 0 is a non-degenerate singularity of Kupka-type of \mathcal{X} if $\lambda_j \neq 0$ for all $j = 1, \dots, m$ and $tr(D\mathcal{X}(0)) = \lambda_1 + \dots + \lambda_m \neq 0$. In this case, 0 is an isolated singularity of \mathcal{X} . The singularity is hyperbolic if it is non-degenerate and all the quotients λ_i/λ_j , $i \neq j$, are not real. When 0 is a hyperbolic singularity of \mathcal{X} then there are exactly m germs of analytic $\mathcal{G}_{\mathcal{X}}$ -invariant curves, say $\Gamma_1, \dots, \Gamma_m$, through $0 \in \mathbb{C}^m$, called the separatrices of $\mathcal{G}_{\mathcal{X}}$ through 0 . Moreover, Γ_j is smooth and tangent to the eigenspace associated to λ_j , $1 \leq j \leq m$ (see [13]).

Let us denote by $\mathbb{F}ol(d; 1, m)$ the space of one dimensional foliations of degree d on \mathbb{P}^m . From [12] and [13] we know that, given $m \geq 2$ and $d \geq 2$, there is an open and dense subset $\mathcal{M}(d) \subset \mathbb{F}ol(d; 1, m)$ such that any $\mathcal{G} \in \mathcal{M}(d)$ satisfies:

- (1) \mathcal{G} has exactly $N = \frac{d^{m+1}-1}{d-1}$ singularities, all of them hyperbolic,
- (2) \mathcal{G} has no algebraic invariant curve,
- (3) all singularities are of Kupka-type.

When $m = 2$ the property (2) implies that the Zariski closure of any one dimensional leaf of \mathcal{G} is \mathbb{P}^2 . When $m \geq 3$ it is not known if there exists an open

and dense subset of $\mathbb{F}ol(d; 1, m)$ with a similar property. However, it is known that there exists a Zariski generic subset (complement of a countable union of hypersurfaces) $\mathcal{M}_g(d) \subset \mathcal{M}(d) \subset \mathbb{F}ol(d; 1, m)$ such that any foliation $\mathcal{G} \in \mathcal{M}_g(d)$ has no proper algebraic invariant subset of positive dimension (see [5]). In particular, if $\mathcal{G} \in \mathcal{M}_g(d)$, then the Zariski closure of any leaf of \mathcal{G} is \mathbb{P}^m .

2.2. Rational maps. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, and let $\tilde{f} : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m+1}$ be its natural lifting in homogeneous coordinates.

Definition 2.1. We denote by $RM(n, m, \nu)$ the set of maps $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ of degree $\nu \geq 2$ given by $\tilde{f} = (F_0, F_1, \dots, F_m)$ where the F_{j_s} , are homogeneous polynomials of degree ν without common factors.

The *indeterminacy locus* of f is, by definition, the set $I(f) = \Pi_n(\tilde{f}^{-1}(0))$, where $\Pi_n : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ is the canonical projection. Observe that the restriction $f|_{\mathbb{P}^n \setminus I(f)}$ is holomorphic.

Definition 2.2. We say that $f \in RM(n, m, \nu)$ is *generic* if for all $p \in \tilde{f}^{-1}(0) \setminus \{0\}$ we have $dF_0(p) \wedge dF_1(p) \wedge \dots \wedge dF_m(p) \neq 0$.

This is equivalent to saying that $f \in RM(n, m, \nu)$ is *generic* if $I(f)$ is the transverse intersection of the $m + 1$ hypersurfaces $\Pi_n(F_i = 0)$ for $i = 0, \dots, m$. If f is generic, $n = m + 1$ and $deg(f) = \nu$, then $I(f)$ consists of ν^{m+1} distinct points, by Bezout’s theorem. On the other hand, if $n > m + 1$, then $I(f)$ is a connected smooth complete intersection of degree ν^{m+1} .

Let $U(f) = \mathbb{P}^n \setminus I(f)$, $P(f)$ be the set of critical points of f in $U(f)$ and let $C(f) = f(P(f))$ be the set of the critical values of f . If f is generic, then $\overline{P(f)} \cap I(f) = \emptyset$, so that $\overline{P(f)} = P(f) \subset U(f)$ (where \overline{A} denotes the closure of $A \subset \mathbb{P}^n$ in the usual topology). Since $P(f) = \{p \in U(f); rank(df(p)) \leq m - 1\}$, it follows that $P(f)$ is a proper algebraic subset of \mathbb{P}^n and $C(f)$ is a proper algebraic subset of \mathbb{P}^m . The set of generic maps of degree ν will be denoted by $Gen(n, m, \nu)$.

Proposition 2.3. *Gen(n, m, ν) is a Zariski open and dense subset of RM(n, m, ν).*

2.3. Generic pairs.

Definition 2.4. Let f be an element of $Gen(n, m, \nu)$. We say that $\mathcal{G} \in \mathcal{M}(d) \subset \mathbb{F}ol(d; 1, m)$ is in generic position with respect to f if $Sing(\mathcal{G}) \cap C(f) = \emptyset$. In this case we will say that (f, \mathcal{G}) is a generic pair.

Set $\mathcal{W} = \{\mathcal{F}; \mathcal{F} = f^*\mathcal{G} \text{ and } (f, \mathcal{G}) \in Gen(n, m, \nu) \times \mathcal{M}(d) \text{ is a generic pair}\}$. We remark that \mathcal{W} is an open and dense subset of $PB(\nu, d, k, n)$, where $k = n - m + 1$. In fact it is a real Zariski open set.

As we have seen before, $f^*\mathcal{G}$ can be represented in homogeneous coordinates by $\tilde{f}^*\Omega$, where

$$\tilde{f} = (F_0, \dots, F_m)$$

represents f and

$$\Omega = i_{R^i X} dV = \sum_{i \neq k} (-1)^{i+k+1} x_k P_i dx_0 \wedge \dots \wedge \widehat{dx}_i \wedge \dots \wedge \widehat{dx}_k \wedge \dots \wedge dx_m$$

represents \mathcal{G} . The pull-back foliation, $f^*(\mathcal{G})$, is then defined in homogeneous coordinates by the $(m - 1)$ -form

$$\eta_{[f,\mathcal{G}]} = \left[(-1)^{i+k+1} \sum_{i \neq k} F_k \cdot (P_i \circ \tilde{f}) dF_0 \wedge \dots \wedge \widehat{dF_i} \wedge \dots \wedge \widehat{dF_k} \wedge \dots \wedge dF_m \right].$$

Since the $F_{k's}$ are homogenous of degree ν and $P_{i's}$ homogeneous of degree d , the coefficients of $\eta_{[f,\mathcal{G}]}$ are homogeneous of degree $(d + m)\nu - m + 1$. From these considerations we get the following.

Proposition 2.5. *If $\mathcal{F} = f^*\mathcal{G}$ where (f, \mathcal{G}) is a generic pair, then the degree of \mathcal{F} is $\Theta(\nu, d, n) = (d + m)\nu - m$.*

2.4. The Kupka set. The purpose of this section is to expose the main facts about the Kupka set of two dimensional foliations that will be used in the first part of the proof of Theorem A (see §3.3).

Let η be a germ at (\mathbb{C}^n, p) of holomorphic $(n - 2)$ -form. Since $d\eta$ is an $(n - 1)$ -form there exists a germ at (\mathbb{C}^n, p) of vector field Z such that $d\eta = i_Z\nu$, where $\nu = dz_1 \wedge \dots \wedge dz_n$. The vector field Z is called the *rotational* of η with respect to ν . We will denote $Z := rot_\nu(\eta)$. If $\tilde{\nu}$ is another non-vanishing n -form, then there is a vector field $\tilde{Z} = rot_{\tilde{\nu}}(\eta)$ such that $d\eta = i_{\tilde{Z}}\nu'$. Since $\nu' = u \cdot \nu$, where $u(p) \neq 0$, we have $rot_{\tilde{\nu}}(\eta) = u \cdot rot_\nu(\eta)$. In particular, $rot_\nu(\eta)$ and $rot_{\tilde{\nu}}(\eta)$ define the same germ of one dimensional foliation. We say that p is a singularity of *Kupka-type* of η if $\eta(p) = 0$ and $rot_\nu(\eta)(p) \neq 0$. This condition is equivalent to $\eta(p) = 0$ and $d\eta(p) \neq 0$. Therefore, if $\tilde{\eta} = u \cdot \eta$, where $u(p) \neq 0$, then

$$\eta(p) = 0 \text{ and } d\eta(p) \neq 0 \iff \tilde{\eta}(p) = 0 \text{ and } d\tilde{\eta}(p) \neq 0.$$

In other words, the concept depends only on the foliation defined by η , \mathcal{F}_η . In particular, it can be extended to foliations on complex manifolds.

Definition 2.6. Let \mathcal{F} be a two dimensional foliation on a complex manifold M . We say that $p \in M$ is a singularity of Kupka-type of \mathcal{F} if \mathcal{F} is represented in a neighborhood of p by an $(n - 2)$ -form with a Kupka singularity at p . The set of singularities of Kupka-type of \mathcal{F} will be denoted by $K(\mathcal{F})$.

The following result is a special case of a more general one proved in [15] (see also [12]).

Proposition 2.7 (Local product structure). *Let η be a germ at (\mathbb{C}^n, p) of integrable $(n - 2)$ -form and $Z = rot(\eta)$. Assume that $Z(p) \neq 0$. Then there exists a coordinate system around p , say $\phi = (y, t) : U \rightarrow \mathbb{C}^{n-1} \times \mathbb{C}$, $y : U \rightarrow \mathbb{C}^{n-1}$, $t : U \rightarrow \mathbb{C}$ such that $\phi(p) = (0, 0)$ and when expressed in this coordinate system η depends only on y . In other words, we can write $\eta = i_Y dy_1 \wedge \dots \wedge dy_{n-1}$ where Y is a vector field of the form $Y = \sum_{j=1}^{n-1} Y_j(y) \frac{\partial}{\partial y_j}$ and $Y(0) = 0$. Moreover, $tr(DY(0)) \neq 0$, that is, 0 is a singularity of Kupka-type of Y .*

Remark 2.8. It follows from Proposition 2.7 that \mathcal{F}_η is equivalent to a foliation which is a product of a one dimensional non-singular foliation by the foliation induced by Y . By this reason, the vector field Y is called the normal type of the Kupka set of \mathcal{F}_η at p .

If the normal type Y has an isolated singularity at 0 , $det(DY(0)) \neq 0$, then in the coordinate system (y, t) of Proposition 2.7, the germ of $K(\eta)$ at p is the smooth

curve ($y = 0$). Moreover, the normal type is constant along this curve. An example is when Y has a non-degenerate singularity at p . In this case, we say that p is a non-degenerate Kupka singularity of \mathcal{F}_η .

In the global case, if a foliation \mathcal{F} on a complex compact manifold M has a non-degenerate Kupka singularity p , then the irreducible component of $Sing(\mathcal{F})$ that contains p is a compact complex curve Γ such that $\Gamma \setminus K(\mathcal{F})$ is finite, say $\Gamma \setminus K(\mathcal{F}) = \{p_1, \dots, p_r\}$, and the normal type of \mathcal{F} is constant along $\Gamma \setminus \{p_1, \dots, p_r\}$.

The Kupka set is locally stable under deformations, as explained below.

Let $(\eta_t)_{t \in D_r}$ be a holomorphic one parameter family of integrable $(n - 2)$ -forms defined in a neighborhood of a closed ball $\overline{B} \subset \mathbb{C}^n$, $D_r = \{\tau \in \mathbb{C} \mid |\tau| < r\}$. Assume that $K(\eta_0)$ contains a holomorphic curve Γ with the following properties:

- (i). $\Gamma' := \Gamma \cap \overline{B}$ is biholomorphic to a closed disc, $\overline{\delta} \subset \mathbb{C}$, and cuts transversely the boundary ∂B of \overline{B} .
- (ii). The normal type of η_0 along Γ' , say Y_0 , is non-degenerate.

The following result is a consequence of [15].

Proposition 2.9. *In the above situation there exists $r' < r$, a C^∞ isotopy $\Phi: \Gamma' \times D_{r'} \rightarrow \overline{B}$, and a holomorphic family of germs at $0 \in \mathbb{C}^{n-1}$ of holomorphic vector fields $(Y_\tau)_{\tau \in D_{r'}}$, with the following properties:*

- (a). $\Phi_0(\Gamma') = \Gamma'$ and for any $\tau \in D_{r'}$ then $\Phi_\tau(\Gamma') \subset K(\eta_\tau)$, where $\Phi_\tau(z) = \Phi(z, \tau)$.
- (b). Y_τ is the normal type of η_τ along $\Phi_\tau(\Gamma')$ and has a non-degenerate singularity at $0 \in \mathbb{C}^n \forall \tau \in D_{r'}$.

Next we will describe the Kupka set of a foliation $\mathcal{F} = f^*\mathcal{G}$ on \mathbb{P}^n , where $f: \mathbb{P}^n \rightarrow \mathbb{P}^{n-1}$, $\mathcal{G} \in \mathcal{M}(d)$ and (f, \mathcal{G}) is a generic pair. Let $Sing(\mathcal{G}) = \{q_1, \dots, q_N\}$ and recall that, by definition, q_1, \dots, q_N are all singularities of Kupka-type of \mathcal{G} . Set $V_{q_i} = \overline{f^{-1}(q_i)}$, $1 \leq i \leq N$. Observe first that V_{q_i} is a smooth algebraic curve of \mathbb{P}^n that contains $I(f)$. In fact, it is the transverse intersection of $n - 1$ hypersurfaces because the pair (f, \mathcal{G}) is generic. For fixed $i \in \{1, \dots, N\}$ let Y_i be a holomorphic vector field representing \mathcal{G} in a neighborhood of q_i . Since $\mathcal{G} \in \mathcal{M}(d)$, q_i is a non-degenerate singularity of Y_i of Kupka-type; $tr(DY_i(q_i)) \neq 0$.

Lemma 2.10. *In the above situation we have:*

- $V_{q_i} \setminus I(f) \subset K(f^*(\mathcal{G})) \forall i$.
- *The normal type of $f^*(\mathcal{G})$ along $V_{q_i} \setminus I(f)$ is equivalent to $Y_i \forall i$.*

Proof. Fix $i \in \{1, \dots, N\}$ and $p \in V_{q_i} \setminus I(f)$. Since q_i is a regular value of $f|_{U(f)}$, then f is a submersion in a neighborhood of p . Hence, there exist local analytic coordinate systems (U, y, t) , $y: U \rightarrow \mathbb{C}^{n-1}$, $t: U \rightarrow \mathbb{C}$, and (V, u) , $u: V \rightarrow \mathbb{C}^{n-1}$, at p and $q_i = f(p)$, respectively, such that $f(U) \subset V$, $u(q_i) = 0 \in \mathbb{C}^{n-1}$, $f(y_1, y_2, \dots, y_{n-1}, t) = (y_1, y_2, \dots, y_{n-1})$ and \mathcal{G} is represented in V by a vector field Z holomorphically equivalent near $0 \in \mathbb{C}^{n-1}$ to Y_i near q_i . In particular, $det(DZ(0)) \neq 0$ and $tr(DZ(0)) \neq 0$. Therefore, if $Z = \sum_{j=1}^{n-1} Z_j(u) \frac{\partial}{\partial u_j}$, then \mathcal{G} is represented on V by the $(n - 1)$ -form $\omega = i_Z du_1 \wedge \dots \wedge du_{n-1}$. Since $f(y, t) = y$ the form $\eta := f^*(\omega)$ has essentially the same expression as ω :

$$\eta = f^*\omega = i_{Z(y)} dy_1 \wedge \dots \wedge dy_{n-1} .$$

In particular, we get $dn(p) = tr(DZ(0)) dy_1 \wedge \dots \wedge dy_{n-1}$ and $rot(\eta) = tr(DZ(0)) \frac{\partial}{\partial t} \neq 0$. Therefore, $p \in K(f^*(\mathcal{G}))$ and the normal type of $f^*(\mathcal{G})$ is equivalent to the germ of Z at 0, which proves Lemma 2.10. \square

2.5. Conic singularities. The purpose of this section is to describe the so-called *conic* singularities in the case of two dimensional foliations. We begin by proving that a pull-back foliation $f^*(\mathcal{G})$, where $f: \mathbb{P}^n \rightarrow \mathbb{P}^{n-1}$ and (f, \mathcal{G}) is a generic pair, has a conic structure near a point $p \in I(f)$.

Fix a point $p \in I(f)$ and let $\tilde{p} \in \Pi_n^{-1}(p)$. Without loss of generality we can assume that $\tilde{p} = (1, 0, \dots, 0) \in \mathbb{C}^{n+1}$. Let $\tilde{f} = (F_0, \dots, F_{n-1}): \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ be the homogeneous lifting of f . Since f is generic, \tilde{f} is a submersion at \tilde{p} ; there exists a local coordinate system $(U, x = (x_0, \dots, x_n) \in \mathbb{C}^n)$ around \tilde{p} such that

$$\tilde{f}(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n) \implies f[1 : x_1 : \dots : x_n] = [x_1 : \dots : x_n] \in \mathbb{P}^{n-1} .$$

In other words, in the affine chart $[1 : x] \simeq x \in \mathbb{C}^n \subset \mathbb{P}^n$, the map f is the canonical projection $x \in \mathbb{C}^n \setminus \{0\} \mapsto [x] \in \mathbb{P}^{n-1}$. In particular, the pull-back foliation $f^*(\mathcal{G})$ is defined in these coordinates by an integrable $(n - 2)$ -form η with homogeneous coefficients of degree $d + 1$ and such that $i_R\eta = 0$, R the radial vector field on \mathbb{C}^n . In fact, the form η defines \mathcal{G} in homogeneous coordinates.

In the next lemma we study the rotational of a form η defining in homogeneous coordinates a foliation on \mathbb{P}^{n-1} . Let \mathcal{G} be a foliation of degree $d \geq 2$ on \mathbb{P}^{n-1} and let η be an $(n - 2)$ -form on \mathbb{C}^n , with homogeneous coordinates of degree $d + 1$, defining $\Pi_{n-1}^*(\mathcal{G})$. Let $X := rot(\eta)$; $d\eta = i_X\nu$, $\nu = dx_1 \wedge \dots \wedge dx_n$.

Lemma 2.11. *Assume that all singularities of \mathcal{G} are non-degenerate. Then $0 \in \mathbb{C}^n$ is an isolated singularity of X if and only if all singularities of \mathcal{G} are of Kupka-type.*

Proof. We will use the identity:

$$(2.1) \quad (n + d - 1)\eta = i_R d\eta .$$

Let us prove (2.1). Since the coefficients of η are homogeneous of degree $d + 1$ we have $L_R\eta = (n + d - 1)\eta$. On the other hand, from $i_R\eta = 0$ we get

$$(n + d - 1)\eta = L_R(\eta) = i_R d\eta + d(i_R\eta) = i_R d\eta .$$

Note that relation (2.1) implies $Sing(X) = Sing(d\eta) \subset Sing(\eta)$. Since η represents \mathcal{G} in homogeneous coordinates, we have

$$Sing(\eta) = \{0\} \cup \Pi_{n-1}^{-1}(Sing(\mathcal{G})) .$$

On the other hand, if $q \in Sing(\mathcal{G})$, then $\Pi_{n-1}^{-1}(q)$ is a line ℓ through $0 \in \mathbb{C}^n$ and, as we have seen in the proof of Lemma 2.10 the normal type of η along ℓ coincides with the analytic type of a germ vector field representing \mathcal{G} at q . Therefore, if $\mathcal{G} \in \mathcal{M}(d)$, then $\ell \setminus \{0\} \subset K(\eta)$ and $d\eta(p) \neq 0 \forall p \in \ell \setminus \{0\}$, so that 0 is an isolated singularity of $d\eta$. Conversely, if 0 is an isolated singularity of $d\eta$, then $\ell \setminus \{0\} \subset K(\eta)$ and q is a singularity of Kupka-type of \mathcal{G} . \square

The next definition can be found in [12].

Definition 2.12. Let η be a germ at $0 \in \mathbb{C}^n$ of integrable $(n - 2)$ -form with a singularity at 0, $\eta(0) = 0$. We say that 0 is a *generalized Kupka singularity* (briefly: GK singularity) if it is an isolated singularity of $d\eta$, or equivalently, an isolated singularity of the rotational, $X = rot(\eta)$. We say that 0 is an NGK (nilpotent generalized Kupka) singularity if $DX(0)$ is nilpotent.

In [12] it is proved that if η has an NGK singularity at $0 \in \mathbb{C}^n$, then there is a holomorphic germ of vector field Y at $0 \in \mathbb{C}^n$ such that $\eta = i_Y i_X \nu$, where $X = \text{rot}(\eta)$ and $\nu = dz_1 \wedge \dots \wedge dz_n$. Moreover, the eigenvalues of $DY(0)$ are all rational positive and $\text{tr}(DY(0)) < 1$ and there exists a holomorphic coordinate system in which we can take $Y = S + N$, where S is linear semi-simple, N and X are polynomial vector fields, $DN(0)$ is nilpotent and they satisfy $[S, N] = 0$, $[N, X] = 0$ and $[S, X] = \lambda X$, $\lambda = 1 - \text{tr}(S) > 0$.

Remark 2.13. Since the eigenvalues of S are rational positive they can be written as $\frac{p_1}{q}, \dots, \frac{p_n}{q}$, where $p_1, \dots, p_n, q \in \mathbb{N}$ and p_1, \dots, p_n are relatively prime. Note that the number $\tilde{\lambda} := q\lambda = q - q \cdot \text{tr}(S)$ is a positive integer. If we set $\tilde{S} = qS$, then $[\tilde{S}, X] = \tilde{\lambda}X$. This relation says that the vector field X is quasi-homogeneous with weights p_1, \dots, p_n . In this case, we will say that the singularity is NGK of type $(p_1, \dots, p_n : \tilde{\lambda})$.

Observe that when the weights are $p_1 = \dots = p_n = 1$, then $\tilde{S} = R$, the radial vector field, and X has homogeneous coefficients of degree $d = \tilde{\lambda} + 1$. A consequence of Lemma 2.11 is that if $\mathcal{G} \in \mathcal{M}(d)$, then the form η that represents \mathcal{G} in homogeneous coordinates has an NGK singularity of type $(1, \dots, 1; d - 1)$ at $0 \in \mathbb{C}^n$. A nilpotent singularity of this type will be called an NGK *conic* singularity of degree d .

Remark 2.14. The vector field Y such that $\eta = i_Y i_X \nu$ can be decomposed as $Y = S + N$, where S is semi-simple and $[S, N] = 0$, $[N, X] = 0$ and $[S, X] = \lambda X$, $\lambda = 1 - \text{tr}(S)$. In fact, the vector field N vanishes if we assume that X satisfies a Zariski open condition (proposition 3 of [12]).

In the case of a conic NGK singularity of degree $d \geq 2$, N is necessarily linear nilpotent and this condition is Zariski open and dense. In fact, in [12] it is proved that there exists a Zariski open and dense subset $\mathcal{U}(d) \subset \text{Fol}(d; 1, n - 1)$ such that for any $\mathcal{G} \in \mathcal{U}(d)$, if η represents \mathcal{G} in homogeneous coordinates, $X = \text{rot}(\eta)$, N is linear nilpotent and $[X, N] = 0$, then $N = 0$. We will use the notation $\mathcal{M}'(d) := \mathcal{M}(d) \cap \mathcal{U}(d)$ and $\mathcal{M}'_g(d) = \mathcal{M}_g(d) \cap \mathcal{U}(d)$.

Another result proved in [12] is the persistence of nilpotent singularities under deformation. Let $(\eta_t)_{t \in B_r}$ be a holomorphic family of integrable $(n - 2)$ -forms on an open set of $U \subset \mathbb{C}^n$, where $B_r = \{t \in \mathbb{C}^m; \|t\| < r\}$. Assume that η_0 has a nilpotent singularity of type $(p_1, \dots, p_n; \lambda)$ at some point $q \in U$.

Theorem 2.15. *In the above situation, there exist $0 < r' < r$ and a holomorphic map $Q: B_{r'} \rightarrow U$ with the following properties:*

- $Q(0) = q$ and $Q(t)$ is a nilpotent singularity of type $(p_1, \dots, p_n; \lambda)$ of η_t for all $t \in B_{r'}$.

Next, let us consider a holomorphic family $(\eta_t)_{t \in B_r}$ of $(n - 2)$ -forms on a neighborhood of a closed ball $0 \in \overline{B} \subset \mathbb{C}^n$ with boundary ∂B . Let us assume:

- η_0 has homogeneous coefficients of degree $d + 1$ and represents a foliation $\mathcal{G}_0 \in \mathcal{M}'(d)$.

In this case, we can write $\eta_0 = \rho i_R i_{X_0} \nu$, where R is the radial vector field on \mathbb{C}^n , $X_0 = \text{rot}(\eta_0)$, $\rho = 1/(n + d - 1)$ and $\nu = dz_1 \wedge \dots \wedge dz_n$. As we have seen in Lemma 2.11 the vector field X_0 has an isolated singularity and η_0 a conic NGK singularity of degree d at $0 \in \mathbb{C}^n$.

Note also that the singular set of η_0 contains exactly $N = (d^n - 1)/(d - 1)$ straight lines through $0 \in \mathbb{C}^n$, each line corresponding to a singularity of \mathcal{G}_0 .

Let $X_t = \text{rot}(\eta_t)$. Since $X_0|_{\partial B}$ doesn't vanish, by taking a smaller r if necessary, we can assume that $X_t|_{\partial B}$ also doesn't vanish. In particular, $\text{Sing}(X_t) \cap \overline{B}$ is necessarily finite and so $\text{cod}(\text{Sing}(X_t)) \geq 3$. By de Rham's division theorem (cf. [16]) there exists a holomorphic family of vector fields $(Y_t)_{t \in D_r}$, defined in a neighborhood of \overline{B} , such that $Y_0 = \rho R$ and

$$(2.2) \quad \eta_t = i_{Y_t} i_{X_t} \nu \quad \forall t \in D_r .$$

Given $q \in \overline{B}$ and $k \geq 0$ we will denote by $j_q^k(\eta_t)$ the k^{th} -jet of η_t at q . As a consequence of Theorem 2.15 and from the compactness of \overline{B} we can state the following.

Corollary 2.16. *In the above conditions, there exists $0 < r' \leq r$ and a holomorphic map $Q: B_{r'} \rightarrow B$, with $Q(0) = 0$, and such that:*

- (a). $\text{Sing}(X_t) \cap \overline{B} = \text{Sing}(Y_t) \cap \overline{B} = \{Q(t)\} \quad \forall t \in B_{r'}$.
- (b). $Q(t)$ is a conic NGK singularity of degree d of η_t for all $t \in B_{r'}$. In particular, $j_{Q(t)}^d(\eta_t) = 0$ and $j_{Q(t)}^{d+1}(\eta_t)$, viewed as an $(n - 2)$ -form with homogeneous coefficients, represents a foliation $\mathcal{G}_t \in \mathcal{M}'(d)$. In particular, the correspondence $t \in B_{r'} \mapsto \mathcal{G}_t \in \mathcal{M}'(d)$ is holomorphic.
- (c). $DY_t(Q(t)) = \rho R$ for all $t \in B_{r'}$, where R is the radial vector field on \mathbb{C}^n . In particular, by Poincaré's linearization theorem $\rho^{-1} Y_t$ is holomorphically equivalent at $Q(t)$ to R .

Remark 2.17. Let Y_t and X_t be as in (2.2), so that $\eta_t = i_{Y_t} i_{X_t} \nu$ and $d\eta_t = i_{X_t} \nu$. From these relations we get

$$L_{Y_t} \eta_t = i_{Y_t} d\eta_t + d(i_{Y_t} \eta_t) = \eta_t .$$

As a consequence of the above relation, the singular set of η_t is Y_t -invariant. In fact, $\text{Sing}(\eta_t)$ is the closure of N orbits of Y_t . Since Y_t is conjugated to a multiple of the radial vector field in a neighborhood of $Q(t)$, the closure of each orbit is a smooth curve containing $Q(t)$. On the other hand, since we are supposing that $Y_0 = \rho R$, by continuity of the solutions of the flow of Y_t with $t \in B_{r'}$, if r' is small enough, then the orbits of Y_t are transverse to ∂B and have as a unique adherent point the point $Q(t)$. Hence, we can conclude that $\text{Sing}(\eta_t) \cap \overline{B} = \bigcup_{j=1}^N \ell_j(t)$, where:

- (a). For all $j \in \{1, \dots, N\}$, $\ell_j(t)$ is the union of an orbit of Y_t in \overline{B} with $Q(t)$. In particular, $\ell_j(t)$ is transverse to ∂B and biholomorphic to a closed disc.
- (b). For all $i \neq j$, $\ell_i(t) \cap \ell_j(t) = \{Q(t)\}$.
- (c). For all j , there exists a C^∞ isotopy $\Phi^j: B_{r'} \times \overline{\delta} \rightarrow \overline{B}$ such that $\Phi_0^j(\overline{\delta}) = \ell_j(0)$ and $\Phi_t^j(\overline{\delta}) = \ell_j(t)$, $\Phi^j(t, z) = \Phi_t^j(z)$.

3. PROOF OF THEOREM A

3.1. Plan of the proof. We begin by proving the theorem in the case of two dimensional foliations. In §3.3 we will prove that $PB(\nu, d, 2, n)$ is an irreducible component of $\text{Fol}(\Theta; 2, n)$, $\Theta = (d + n - 1)\nu - n + 1$. Let us give an idea of the proof.

First of all, $PB(\nu, d, 2, n)$ is a unirational irreducible algebraic subset of $\text{Fol}(\Theta; 2, n)$, because it is the closure in $\text{Fol}(\Theta; 2, n)$ of the set $\{f^*(\mathcal{G}) \mid f \in RM(n, n - 1, \nu), \mathcal{G} \in \text{Fol}(d, 1, n - 1)\}$. Let Z be the (unique) irreducible component of

$\mathbb{F}ol(\Theta; 2, n)$ containing $PB(\nu, d, 2, n)$. Since $PB(\nu, d, 2, n)$ and Z are irreducible it is sufficient to prove that there exists $\mathcal{F} = f^*(\mathcal{G}) \in PB(\nu, d, 2, n)$ such that for any germ of holomorphic one parameter family $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ of foliations $\mathcal{F}_t \in Z$ with $\mathcal{F}_0 = \mathcal{F}$ then $\mathcal{F}_t \in PB(\nu, d, 2, n) \forall t \in (\mathbb{C}, 0)$.

We choose $\mathcal{F} = f^*(\mathcal{G})$, where (f, \mathcal{G}) is a generic pair (see §2.3), and $\mathcal{G} \in \mathcal{M}'_g(d)$ (see §2.1 and Remark 2.14 of §2.5). Given the one parameter family $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ with $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$, we will construct in §3.2 two holomorphic one parameter families $(f_t)_{t \in (\mathbb{C}, 0)}$ and $(\mathcal{G}_t)_{t \in (\mathbb{C}, 0)}$ of generic maps and foliations, such that $f_0 = f$, $\mathcal{G}_0 = \mathcal{G}$ and $f_t^*(\mathcal{G}_t) = \mathcal{F}_t$ for all $t \in (\mathbb{C}, 0)$, so that $\mathcal{F}_t \in PB(\nu, d, 2, n)$ for all $t \in (\mathbb{C}, 0)$. In the next section we will describe briefly how to find these families.

A problem with the families $(f_t)_{t \in (\mathbb{C}, 0)}$ and $(\mathcal{G}_t)_{t \in (\mathbb{C}, 0)}$ that we will construct in §3.2 is that we cannot assert a priori that $\mathcal{F}_t = F_t^*(\mathcal{G}_t) \forall t \in (\mathbb{C}, 0)$. This fact will be proved in §3.3 in the case of two dimensional foliations. In §3.4 we will see how to reduce the case of foliations of dimension ≥ 3 to the case of two dimensional foliations.

3.2. Construction of the families $(\mathcal{G}_t)_{t \in (\mathbb{C}, 0)}$ and $(f_t)_{t \in (\mathbb{C}, 0)}$. Recall that $Sing(\mathcal{F}_0)$ contains N smooth complete intersection curves $V_{q_1, \dots, V_{q_N}}$, such that:

- $V_{q_j} = \overline{f^{-1}(q_j)}$, $1 \leq j \leq N$, where $\{q_1, \dots, q_N\} = Sing(\mathcal{G}_0)$.
- $V_{q_i} \cap V_{q_j} = I(f_0) \forall i \neq j$. In particular, $I(f_0) \subset V_{q_j} \forall j$.
- $V_{q_j} \setminus I(f_0) \subset K(\mathcal{F}_0) \forall j$.

In fact, it can be proved that $K(\mathcal{F}_0) = \bigcup_j V_{q_j} \setminus I(f_0)$, but we will not use this essentially. Using Corollary 2.16, about deformations of conic NGK points we can state the following.

Corollary 3.1. *Set $I(f_0) = \{p_1, \dots, p_\rho\}$, $\rho = \nu^n$. Then there are holomorphic germs of curves $P_j: (\mathbb{C}, 0) \rightarrow \mathbb{P}^n$, $1 \leq j \leq \rho$, such that $P_j(0) = p_j$ and $P_j(t)$ is a conic NGK singularity of degree d . Furthermore, for each $j \in \{1, \dots, \rho\}$ there exists a germ of a holomorphic one parameter family of foliations $t \in (\mathbb{C}, 0) \mapsto \mathcal{G}_t^j \in \mathcal{M}'(d)$ such that $\mathcal{G}_0^j = \mathcal{G}_0$ and \mathcal{F}_t represents \mathcal{G}_t^j near $P_j(t)$.*

Notation. We will use the notation: $Con(\mathcal{F}_t) = \{P_1(t), \dots, P_\rho(t)\}$.

Remark 3.2. We would like to observe that when we blow-up at the point $P_j(t)$ the foliation $\mathcal{G}_t^j \in \mathcal{M}'(d)$ of Corollary 3.1 appears as a foliation on the exceptional divisor of the blow-up. Indeed, if we denote the blow-up by $\pi_t: (\widetilde{\mathbb{P}^n}, E_t) \rightarrow (\mathbb{P}^n, P_j(t))$, then the divisor E_t is biholomorphic \mathbb{P}^{n-1} and $\pi_t^*(\mathcal{F}_t)$ extends to \mathbb{P}^n , the complex manifold obtained after the blow-up. On the other hand, by (c) of Corollary 2.16, the radial vector field is tangent to \mathcal{F}_t , and its strict transform by π_t is transverse to E_t . In fact, it can be verified that $\pi_t^*(\mathcal{F}_t)|_E \simeq \mathcal{G}_t^j$.

Notation. In the situation of Remark 3.2 we will say that \mathcal{F}_t represents \mathcal{G}_t^j near $P_j(t)$.

Remark 3.3. Since, in principle, we have $\rho = \nu^n$ different one parameter families of one dimensional foliations, $(\mathcal{G}_t^j)_{t \in (\mathbb{C}, 0)}$, $1 \leq j \leq \rho$, we cannot assert a priori that, if $i \neq j$, then \mathcal{G}_t^i is equivalent to \mathcal{G}_t^j for any $t \in (\mathbb{C}, 0)$. Indeed, this fact is true, but it will be a consequence of the final result.

Notation. We will choose the family $(\mathcal{G}_t)_{t \in (\mathbb{C}, 0)}$ of §3.1 as the family $(\mathcal{G}_t^1)_{t \in (\mathbb{C}, 0)}$.

Since $\mathcal{G}_0 \in \mathcal{M}_g(d)$ and $\mathcal{M}_g(d)$ is Zariski generic, there exists a countable subset $C \subset (\mathbb{C}, 0)$ such that $\mathcal{G}_t \in \mathcal{M}_g(d)$ for all $t \notin C$.

Next we will describe briefly how to obtain the family of maps $(f_t)_{t \in (\mathbb{C}, 0)}$ of §3.1. The construction of this family will be done by using the deformation of the curves V_{q_j} , $1 \leq j \leq N$, and a theorem of Sernesi [17]. First of all, an easy consequence of Proposition 2.9 and Corollary 2.16 is the following.

Lemma 3.4. *There exist N germs of C^∞ isotopies $\phi^j : (\mathbb{C}, 0) \times V_{q_j} \rightarrow \mathbb{P}^n$, $1 \leq j \leq N$, such that if we denote $V_j(t) := \phi_{q_j}(\{t\} \times V_{q_j})$, then:*

- (a) $V_j(0) = V_{q_j}$ and $V_j(t) \setminus \text{Con}(\mathcal{F}_t)$ is contained in the Kupka set of $\mathcal{F}_t \forall t \in (\mathbb{C}, 0)$ and $\forall 1 \leq j \leq N$.
- (b) $\text{Con}(\mathcal{F}_t) \subset V_j(t) \forall 1 \leq j \leq N$ and $\forall t \in (\mathbb{C}, 0)$. Moreover, if $i \neq j$, then $V_i(t) \cap V_j(t) = \text{Con}(\mathcal{F}_t) \forall t \in (\mathbb{C}, 0)$.

In particular, $V_j(t)$ is an algebraic smooth curve of $\mathbb{P}^n \forall 1 \leq j \leq N$ and $\forall t \in (\mathbb{C}, 0)$.

Proof. The argument is similar to [11, lemma 2.3.3, p. 83] and uses essentially the local stability under deformations of the Kupka set and of $\text{Con}(\mathcal{F}_0)$ (see also [4]). \square

The map $f_t : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$, $f_t \in \text{Gen}(n, n-1, \nu)$ will be constructed in such a way that the curves $V_j(t)$, $1 \leq j \leq N$, will be fibers of $f_t \forall t \in (\mathbb{C}, 0)$. Since $d \geq 2$ we have $\# \text{Sing}(\mathcal{G}_0) = N = (d^n - 1)/(d - 1) > n$, and we can assume that $\{q_1, \dots, q_n\} \subset \text{Sing}(\mathcal{G}_0)$, where $q_1 = [1 : 0 : \dots : 0], \dots, q_n = [0 : 0 : \dots : 1]$.

Lemma 3.5. *Let $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ be as before; $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$. Then there exists a holomorphic germ of deformation $(f_t)_{t \in (\mathbb{C}, 0)}$ of f_0 in $\text{Gen}(n, n-1, \nu)$ such that:*

- (i) $\text{Con}(\mathcal{F}_t) = I(f_t) \forall t \in (\mathbb{C}, 0)$.
- (ii) $V_j(t)$ is a fiber of $f_t \forall 1 \leq j \leq N \forall t \in (\mathbb{C}, 0)$.
- (iii) The pair (f_t, \mathcal{G}_t) is generic for all $t \in (\mathbb{C}, 0)$.

Proof. Let $\tilde{f}_0 = (F_0, \dots, F_{n-1}) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ be the homogeneous lifting of f_0 . By the choice of q_1, \dots, q_n , the first n curves $V_{q_1}, V_{q_2}, \dots,$ and V_{q_n} appear as the complete intersections $V_{q_i} = \Pi_n(F_0 = F_1 = \dots = \widehat{F_{i-1}} = \dots = F_{n-1} = 0)$, where the symbol $\widehat{F_j}$ means omission of F_j in the sequence. The remaining curves $V_{q_i}, i > n$ are also defined by $n - 1$ polynomials in the ideal $\mathcal{I}_0 := \langle F_0, \dots, F_{n-1} \rangle$.

Now, we use Sernesi's stability criteria [17, sec. 4.6 pp. 235-236]. It follows from Lemma 3.4 and Sernesi's criteria that for each $j \in \{1, \dots, N\}$ the curve $V_j(t)$ is also a complete intersection. Moreover, it is defined by an ideal of homogeneous polynomials of degree ν , $\mathcal{I}_t^j = \langle G_{1t}^j, \dots, G_{n-1t}^j \rangle$ such that each germ $t \in (\mathbb{C}, 0) \mapsto G_{it}^j \in \mathbb{C}[X_0, \dots, X_n]$, $1 \leq i \leq n - 1$, is holomorphic and moreover $\mathcal{I}_0^j \subset \mathcal{I}_0$. For instance, in the case $j = 1$ we have $V_{q_1} = \Pi_n(F_1 = \dots = F_{n-1} = 0)$ and there are holomorphic families of homogeneous polynomials $t \in (\mathbb{C}, 0) \mapsto F_{it} \in \mathbb{C}[X_0, \dots, X_n]$, $1 \leq i \leq n - 1$, such that $F_{i0} = F_i$ and

$$V_1(t) = \Pi_n(F_{1t} = \dots = F_{n-1t} = 0) \quad \forall t \in (\mathbb{C}, 0) .$$

Similarly, there are holomorphic germs $t \in (\mathbb{C}, 0) \mapsto F_{0t} \in \mathbb{C}[X_0, \dots, X_n]$, $t \in (\mathbb{C}, 0) \mapsto \tilde{F}_{2t} \in \mathbb{C}[X_0, \dots, X_n]$, \dots , $t \in (\mathbb{C}, 0) \mapsto \tilde{F}_{n-1t} \in \mathbb{C}[X_0, \dots, X_n]$ such that $F_{00} = F_0, \tilde{F}_{20} = F_2, \dots, \tilde{F}_{n-10} = F_{n-1}$ and

$$V_2(t) = \Pi_n(F_{0t} = \tilde{F}_{2t} = \dots = \tilde{F}_{n-1t} = 0) \quad \forall t \in (\mathbb{C}, 0) .$$

Notation. We choose the family $(f_t)_{t \in (\mathbb{C}, 0)}$ in such a way that $\tilde{f}_t = (F_{0,t}, \dots, F_{n-1,t})$, where \tilde{f}_t is the homogeneous lifting of f_t for all $t \in (\mathbb{C}, 0)$. Note that f_t is a generic map for any $t \in (\mathbb{C}, 0)$.

Let us prove that $Con(\mathcal{F}_t) = I(f_t) \forall t \in (\mathbb{C}, 0)$. First of all, $\Pi_n(F_{0,t} = 0), \dots, \Pi_n(F_{n-1,t} = 0)$ intersect multitransversally at ν^n points, because f_t is generic for all $t \in (\mathbb{C}, 0)$. Since $I(f_t) = \Pi_n(F_{0,t} = \dots = F_{n-1,t} = 0)$ we get $\#I(f_t) = \nu^n$. Moreover, $Con(\mathcal{F}_t) \subset I(f_t)$, because

$$Con(\mathcal{F}_t) = V_1(t) \cap V_2(t) = \Pi_n(F_{1,t} = \dots = F_{n-1,t} = 0) \cap \Pi_n(F_{0,t} = \tilde{F}_{2,t} = \dots = \tilde{F}_{n-1,t} = 0) \subset I(f_t).$$

Finally $Con(\mathcal{F}_t) = I(f_t)$ because $\#Con(\mathcal{F}_t) = \nu^n = \#I(f_t)$.

It remains to prove that $V_j(t)$ is a fiber of $f_t \forall t \in (\mathbb{C}, 0)$. Here we use Noether's theorem about multitransversal intersections. The fact that $Con(\mathcal{F}_t) = I(f_t)$ is a multitransversal intersection and $Con(\mathcal{F}_t) \subset V_j(t)$ imply that the ideal $\mathcal{I}_j(t)$ that defines $V_j(t)$ in homogeneous coordinates is contained in the ideal $\mathcal{I}_t := \langle F_{0,t}, \dots, F_{n-1,t} \rangle$. This of course implies that $V_j(t)$ is a fiber of f_t .

Finally, the pair (f_t, \mathcal{G}_t) is generic for all $t \in (\mathbb{C}, 0)$, because (f_0, \mathcal{G}_0) is generic and the set of generic pairs is open. □

3.3. End of the proof of Theorem A in the case of two dimensional foliations. Let $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ be the germ of deformation of $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$ of §3.1 and $(f_t, \mathcal{G}_t)_{t \in (\mathbb{C}, 0)}$ be the germ of deformation of (f_0, \mathcal{G}_0) obtained in §3.2. Consider the holomorphic family of foliations $(\tilde{\mathcal{F}}_t)_{t \in (\mathbb{C}, 0)}$ defined by $\tilde{\mathcal{F}}_t = f_t^*(\mathcal{G}_t) \forall t \in (\mathbb{C}, 0)$. Of course $\tilde{\mathcal{F}}_0 = \mathcal{F}_0$ and $\tilde{\mathcal{F}}_t \in PB(\nu, d, 2, n) \forall t \in (\mathbb{C}, 0)$.

Lemma 3.6. $\tilde{\mathcal{F}}_t = \mathcal{F}_t$ for all $t \in (\mathbb{C}, 0)$. In particular, $\mathcal{F}_t \in PB(\nu, d, 2, n) \forall t \in (\mathbb{C}, 0)$.

Proof. The idea is to prove that $\tilde{\mathcal{F}}_t$ and \mathcal{F}_t have a common leaf $L_t \forall t \in (\mathbb{C}, 0)$. In particular, the foliations $\tilde{\mathcal{F}}_t$ and \mathcal{F}_t coincide in the Zarisk closure \bar{L}_t^Z of L_t . Recall that there exists a germ of countable set $C \subset (\mathbb{C}, 0)$ such that $\mathcal{G}_t \in \mathcal{M}_g(d)$ for all $t \notin C$. The fact that $\mathcal{G}_t \in \mathcal{M}_g(d)$ implies that the Zariski closure of any leaf of \mathcal{G}_t is the whole \mathbb{P}^{n-1} . As we will see, this will imply that \bar{L}_t^Z is the whole $\mathbb{P}^n \forall t \notin C$. Since C is countable this will finish the proof of Lemma 3.6.

We begin by blow-up once at the $\rho = \nu^n$ points $P_1(t), \dots, P_\rho(t)$ of $Con(\mathcal{F}_t)$. Let us denote by $M(t)$ the complex manifold obtained from this blow-up procedure, by $\pi_t: M(t) \rightarrow \mathbb{P}^n$ the blow-up map and by $E_1(t), \dots, E_\rho(t)$ the exceptional divisors obtained, where $\pi(E_j(t)) = P_j(t), 1 \leq j \leq \rho$. Denote by $V'_j(t)$ the strict transform of $V_j(t)$ by π_t .

Remark 3.7. Since the pair (f_t, \mathcal{G}_t) is generic and $I(f_t) = Con(\mathcal{F}_t) = \{P_1(t), \dots, P_\rho(t)\}$, we can assert the following facts:

- (I). The map $f_t \circ \pi_t$ extends to a holomorphic map $f'_t: M(t) \rightarrow \mathbb{P}^{n-1}$. Moreover, for any $1 \leq j \leq \rho$, there exists a neighborhood U_j of $P_j(t)$ such that $f'_t|_{U_j}: U_j \rightarrow \mathbb{P}^{n-1}$ is a submersion.
- (II). The fiber $f'^{-1}_t(q), q \in \mathbb{P}^{n-1}$ is the strict transform of $f_t^{-1}(q)$ by π_t . It is smooth near $E_j(t)$ and cuts $E_j(t)$ transversely in just one point $\forall 1 \leq j \leq \rho$.
- (III). $V'_j(t)$ is a smooth curve and f'_t is a submersion in some neighborhood of $V'_j(t) \forall 1 \leq j \leq \rho$.

Assertion (I) follows from the fact that f_t is equivalent to the canonical projection $\Pi_{n-1}: \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ near $P_j(t)$, $1 \leq j \leq \rho$. Of course, (I) \implies (II) \implies (III).

Let us denote by \mathcal{F}'_t and $\tilde{\mathcal{F}}'_t$ the strict transforms by π of the foliations \mathcal{F}_t and $\tilde{\mathcal{F}}_t$, respectively. Note that, for each $1 \leq j \leq \rho$, the foliation $\tilde{\mathcal{F}}'_t|_{E_j(t)}$ is a foliation by curves on $E_j(t) \simeq \mathbb{P}^{n-1}$ equivalent to \mathcal{G}_t . Similarly, $\mathcal{F}'_t|_{E_j(t)}$ is a foliation by curves on $E_j(t)$ and by construction \mathcal{G}_t is equivalent to $\mathcal{F}'_t|_{E_1(t)}$.

Now, by using (II) we can define a holomorphic map $\Phi_t: M(t) \rightarrow E_1(t)$ by

$$\Phi_t(q) := f_t^{-1}(f'_t(q)) \cap E_1(t) \quad \forall q \in \mathbb{P}^n .$$

Note that the fibers of Φ_t coincide with the fibers of f'_t . In fact, the maps Φ_t and f'_t are equivalent, in the sense that there exists a biholomorphism $h: \mathbb{P}^{n-1} \rightarrow E_1(t)$ such that $\Phi_t = h \circ f'_t$. In particular, identifying \mathcal{G}_t with $\mathcal{F}'_t|_{E_1(t)}$ we can assert that

$$\tilde{\mathcal{F}}'_t = \Phi_t^*(\mathcal{G}_t) .$$

Now, we fix a singularity of \mathcal{G}_t , say $q(t) = q_1(t)$, with $V'_1(t) = \Phi_t^{-1}(q_1(t))$. Since $\mathcal{G}_t \in \mathcal{M}(d)$ it has $n - 1$ analytic separatrices through $q_1(t)$, all smooth, say $\gamma_1(t), \dots, \gamma_{n-1}(t)$, and no other local analytic separatrix. Each separatrix $\gamma_j(t)$ is a germ of complex curve through $q(t)$ such that $\overline{\gamma_j(t) \setminus \{q(t)\}}$ is contained in some leaf of \mathcal{G}_t . If $\mathcal{G}_t \in \mathcal{M}_g(d)$, then its Zariski closure $\overline{\gamma_j(t)}^Z$ is $E_1(t) = \mathbb{P}^{n-1}$, because \mathcal{G}_t has no proper algebraic invariant subset of positive dimension. We fix one of these separatrices, say $\gamma_1(t)$. By construction the set $\Phi_t^{-1}(\gamma_1(t))$ satisfies the following property:

1. It is $\tilde{\mathcal{F}}'_t$ -invariant. In other words, $V'_1(t) \subset \Phi_t^{-1}(\gamma_1(t))$ and $\Phi_t^{-1}(\gamma_1(t)) \setminus V'_1(t)$ is an open subset of some leaf of $\tilde{\mathcal{F}}'_t$.

We can assert also that:

2. If $\mathcal{G}_t \in \mathcal{M}_g(d)$, then the Zariski closure $\overline{\Phi_t^{-1}(\gamma_1(t))}^Z$ is \mathbb{P}^n . This follows from the relation

$$\Phi_t^{-1} \left(\overline{\gamma_1(t)}^Z \right) = \overline{\Phi_t^{-1}(\gamma_1(t))}^Z .$$

Notation. An ℓ -dimensional *strip* along $V'_1(t)$, $\ell \geq 2$, is a germ of a smooth complex manifold of dimension ℓ along $V'_1(t)$, containing $V'_1(t)$ and transverse to $E_1(t)$. We say that the strip Θ along $V'_1(t)$ is a separatrix of \mathcal{F}'_t (resp., of $\tilde{\mathcal{F}}'_t$) along $V'_1(t)$ if it is two dimensional and $\Theta \setminus V'_1(t)$ is contained in some leaf of \mathcal{F}'_t (resp., of $\tilde{\mathcal{F}}'_t$). If Θ is a separatrix of \mathcal{F}'_t (or of $\tilde{\mathcal{F}}'_t$) along $V'_1(t)$, then $\Theta \cap E_1(t)$ is one of the separatrices $\gamma_j(t)$, $1 \leq j \leq n - 1$, of \mathcal{G}_t .

We say that the strip is Φ_t -invariant if it is a union of fibers of Φ_t , or equivalently $\Phi_t^{-1}(\Theta \cap E_1(t)) = \Theta$.

Claim 3.8. Let Θ be a two dimensional strip along $V'_1(t)$. Assume that there exists an $(n - 1)$ -dimensional strip Γ along $V'_1(t)$ such that:

- (a). Γ is Φ_t -invariant.
- (b). Θ and Γ are transverse and $\Theta \cap \Gamma = V'_1(t)$.

Then Θ is Φ_t -invariant. In particular, if Θ is a separatrix of \mathcal{F}'_t , then it is also a separatrix of $\tilde{\mathcal{F}}'_t$.

Proof. Consider representatives of Θ and Γ transverse to $E_1(t)$, denoted by the same symbols. Since Φ_t is a submersion at the points of $V'_1(t) = \Phi_t^{-1}(q(t))$ and by

(b), there exists a holomorphic coordinate system around $q(t) \in E_1(t) \subset M(t)$, say $(x, y): U \rightarrow \mathbb{C}^{n-1} \times \mathbb{C}$, $x = (x_1, \dots, x_{n-1})$, such that

- (i). $x(q(t)) = 0 \in \mathbb{C}^{n-1}$ and $y(q(t)) = 0 \in \mathbb{C}$.
- (ii). $E_1(t) \cap U = (y = 0)$ and $V'_1(t) \cap U = (x = 0)$.
- (iii). $\Phi_t(x, y) = (x, 0)$.
- (iv). $\Theta \cap E_1(t) \cap U \subset (x_2 = \dots = x_n = y = 0)$.
- (v). $\Gamma \cap E_1(t) \cap U \subset (x_1 = y = 0)$.

Fix a polydisc $Q = \{(x, 0) \mid |x_j| < \epsilon, 1 \leq j \leq n - 1\} \subset U \subset E_1(t)$. We can take the representatives Θ and Γ in such a way that

- (vi). $\Theta \cup \Gamma \subset \Phi_t^{-1}(Q)$.

Let us denote $p(s) := (s, 0, \dots, 0) \in \Theta \cap E_1(t) \subset Q$. We assert that there exists $0 < \delta < \epsilon$ such that if $|s| < \delta$ then $\Phi_t^{-1}(p(s)) \subset \Theta$. Note that this implies Claim 3.8 because $\Phi_t^{-1}(\{p(s) \mid |s| < \delta\})$ is an open subset of Θ .

Let us prove the assertion. Given $s \in \mathbb{C}$ with $|s| < \epsilon$ set $\alpha_s = \{(x, 0) \in Q \mid x_1 = s\}$ and $\Gamma_s := \Phi_t^{-1}(\alpha_s)$, so that $\Gamma = \Gamma_0$. Note that α_s is a hypersurface of $E_1(t)$ transverse to $\Theta \cap E_1(t)$ at the point $p(s) \in \Theta \cap E_1(t)$. Since Θ intersects Γ along $V'_1(t)$ and $V'_1(t)$ is compact, if ϵ is small, by standard arguments there exists a C^∞ isotopy $\psi: \Gamma \times D_\epsilon \rightarrow M(t)$ with $\psi(\Gamma \times \{s\}) = \Gamma_s$, $D_\epsilon = \{z \mid |z| < \epsilon\}$. In particular, the compactness of $V'_1(t)$ implies, via transversality theory, that Γ_s intersects Θ in a compact complex curve, say γ_s . Finally, $p(s) \in \Phi_t(\gamma_s) \subset Q$ and since $\Phi_t(\gamma_s)$ is a compact analytic subset of Q we must have $\Phi_t(\gamma_s) = \{p(s)\}$, so that $\Phi_t^{-1}(p(s)) = \gamma_s \subset \Theta$, as asserted. \square

Now, the idea is to prove that there are two strips Θ and Γ as in Claim 3.8 such that Θ is a separatrix of \mathcal{F}'_t along $V'_1(t)$. By Claim 3.8, Θ will be also separatrix of $\tilde{\mathcal{F}}'_t$ along $V'_1(t)$ and the two foliations will have a common leaf. This will conclude the proof of Lemma 3.6 and of Theorem A.

In the construction of Θ and Γ as above, we will work with the deformation $t \in (\mathbb{C}, 0) \mapsto M(t)$. We can assume that there exists $\epsilon > 0$ such that (I), (II), and (III) of Remark 3.7 are true if $|t| < \epsilon$. Consider the complex manifold $\widehat{M} = \{(z, t) \mid |t| < \epsilon \text{ and } z \in M(t)\}$. Note that $\{(z, t) \in \widehat{M} \mid t = t_o\} = M(t_o) \times \{t_o\}$, so that it will be denoted by $M(t_o)$. In \widehat{M} consider the following objects:

- (A). The two dimensional holomorphic foliation $\widehat{\mathcal{F}}'$ such that $\widehat{\mathcal{F}}'|_{M(t)} = \mathcal{F}'_t$. Note that, by construction, the projection $(z, t) \in \widehat{M} \mapsto t \in \mathbb{C}$ is a first integral of $\widehat{\mathcal{F}}'$.
- (B). The n -dimensional submanifold \widehat{E}_1 of \widehat{M} defined by $\widehat{E}_1 \cap M(t) = E_1(t)$.
- (C). The one dimensional foliation $\widehat{\mathcal{G}}$ of \widehat{E}_1 defined by $\widehat{\mathcal{G}}|_{E_1(t)} = \mathcal{G}_t$. Note that the projection $(z, t) \in \widehat{E}_1 \mapsto t \in \mathbb{C}$ is a first integral of $\widehat{\mathcal{G}}$.
- (D). The map $\widehat{\Phi}: \widehat{M} \rightarrow \widehat{E}_1$ defined by $\widehat{\Phi}(z, t) = \Phi_t(z) \in E_1(t) \subset \widehat{E}_1$.
- (E). The two dimensional submanifold \widehat{V}'_1 defined by $\widehat{V}'_1 \cap M(t) = V'_1(t)$.

The idea is to construct two germs of complex submanifolds along \widehat{V}'_1 , say $\widehat{\Theta}$ and $\widehat{\Gamma}$, such that:

- (a). $\widehat{\Gamma}$ is $\widehat{\Phi}$ -invariant.
- (b). $\widehat{\Theta}$ and $\widehat{\Gamma}$ are transverse and $\widehat{\Theta} \cap \widehat{\Gamma} = \widehat{V}'_1$.
- (c). $\widehat{\Theta} \cap M(t)$ is a separatrix of \mathcal{F}'_t along $V'_1(t)$, if $|t| < \delta$, where $0 < \delta \leq \epsilon$. In particular, $\dim_{\mathbb{C}}(\widehat{\Theta}) = 3$ and $\dim_{\mathbb{C}}(\widehat{\Gamma}) = n$.

If $\widehat{\Theta}$ and $\widehat{\Gamma}$ are as in (a), (b), and (c), then $\Theta_t := \widehat{\Theta} \cap M(t)$ and $\Gamma_t := \widehat{\Gamma} \cap M(t)$ satisfy the hypothesis of Claim 3.8, if $|t| < \delta$, as the reader can check.

In the construction of $\widehat{\Theta}$ we will use that \widehat{V}'_1 is contained in the Kupka set of $\widehat{\mathcal{F}}'$. Let us prove this fact.

First of all, $V'_1(t)$ is contained in the Kupka set of \mathcal{F}'_t . Indeed, $V'_1(t) \setminus \bigcup_j E_j(t) \subset K(\mathcal{F}'_t)$ by Lemma 3.4, because $\mathcal{F}'_t = \pi_t^*(\mathcal{F}_t)$. Moreover, for each $j = 1, \dots, \nu^n$, $V'_1(t) \cap E_j(t)$ consists of one point, which is also in the Kupka set because $P_j(t)$ is a conic NGK singularity of \mathcal{F}_t (see §2.5). We leave the details to the reader. Given $w_o = (z_o, t_o) \in \widehat{V}'_1$, if ω is an $(n - 2)$ -form representing \mathcal{F}'_{t_o} in a neighborhood of z_o , then, by construction, the form $\widehat{\omega} := \omega \wedge dt$ represents $\widehat{\mathcal{F}}'$ in a neighborhood of w_o . Finally, $d\widehat{\omega}(w_o) = d\omega(z_o) \wedge dt \neq 0$, because z_o is in the Kupka set of \mathcal{F}'_{t_o} , which proves the assertion.

Next we will find the normal type of $\widehat{\mathcal{F}}'$ at the point $(q_1(0), 0) \in V'_1(0) \cap E_1(0) \subset \widehat{E}_1$. Note that, by construction, this normal type coincides with the germ $\widehat{\mathcal{G}}_0$ of $\widehat{\mathcal{G}}$ at $(q_1(0), 0)$ (see (A), (B) and (C)). Since $(z, t) \mapsto t$ is a first integral of $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}|_{E_1(t)} = \mathcal{G}_t$, the normal type is done essentially by a holomorphic one parameter family of vector fields $(Y_t)_{t \in (\mathbb{C}, 0)}$, where Y_t is a germ of the vector field at $q_1(t)$ representing \mathcal{G}_t at $q_1(t)$. For each fixed $t \in D_\epsilon$ the vector field Y_t has $n - 1$ one dimensional separatrices. It follows from the theory of invariant manifolds for hyperbolic vector fields (see [8]) that it is possible to find a germ of holomorphic function $\gamma: (\mathbb{C}^2, 0) \rightarrow \widehat{E}_1$ such that

- (i). $\gamma(0, t) = q_1(t) \ \forall t \in (\mathbb{C}, 0)$.
- (ii). $s \mapsto \gamma(s, t)$ is a holomorphic parametrization of the separatrix $\gamma_1(t)$ of $\mathcal{G}_t \ \forall t \in (\mathbb{C}, 0)$.

Note that $\{\gamma(s, t) \mid s \in (\mathbb{C}, 0), t = t_o\} \subset E_1(t_o) \ \forall t_o$. Moreover, $\widehat{\gamma} := \gamma(\mathbb{C}^2, 0)$ is a germ at $q_1(0)$ of a smooth surface in \widehat{E}_1 and $\widehat{\gamma} \cap E_1(t)$ is a separatrix of $\mathcal{G}_t \ \forall t \in (\mathbb{C}, 0)$. Let us finish the construction of $\widehat{\Theta}$ using the property of local product along the Kupka set.

Since \widehat{V}'_1 is contained in the Kupka set of $\widehat{\mathcal{F}}'$, there exist a finite covering $(U_\alpha)_{\alpha \in A}$ of $V'_1(0) \subset \widehat{V}'_1$ by open sets of \widehat{M} and a family of submersions $(\varphi_\alpha)_{\alpha \in A}$, where $\varphi_\alpha: U_\alpha \rightarrow W$ (W a neighborhood of $(q_1(0), 0)$ in \widehat{E}_1) such that

$$(3.1) \quad \widehat{\mathcal{F}}'|_{U_\alpha} = \varphi_\alpha^* \left(\widehat{\mathcal{G}}|_W \right) \quad \forall \alpha \in A .$$

Fix $0 < \delta \leq \epsilon$ such that γ has a representative defined in $D_\delta \times D_\delta$, denoted by the same symbol, such that $\widehat{\gamma}_\delta := \gamma(D_\delta \times D_\delta) \subset W$. Let Θ_α be the germ of a $\varphi_\alpha^{-1}(\widehat{\gamma}_\delta) \subset U_\alpha$ along $V'_1(0) \cap U_\alpha$. Note that for each $\alpha \in A$, Θ_α is a germ of a smooth three dimensional manifold. Moreover, Θ_α is $\widehat{\mathcal{F}}'$ -invariant, by (3.1), so that $\Theta_\alpha \cap M(t)$ is \mathcal{F}'_t -invariant $\forall \alpha \in A$. Using again (3.1), if $U_\alpha \cap U_\beta \neq \emptyset$, then $\Theta_\alpha \cap U_\alpha \cap U_\beta = \Theta_\beta \cap U_\alpha \cap U_\beta$. If we set

$$\widehat{\Theta} = \bigcup_{\alpha} \Theta_\alpha,$$

then $\widehat{\Theta}$ satisfies (c): $\widehat{\Theta} \cap M(t)$ is a separatrix of \mathcal{F}'_t along $V'_1(t)$.

The construction of $\widehat{\Gamma}$ as above is easier. First of all, we fix any germ at $q_1(0)$ of a smooth complex submanifold of \widehat{E}_1 , say \widehat{C} , with the property that is transverse to $\widehat{\gamma}_\delta$ and $\widehat{C} \cap \widehat{\gamma}_\delta = q_1(0)$. Note that $\dim_{\mathbb{C}}(\widehat{C}) = n - 1$. Set $\widehat{\Gamma} = \widehat{\Phi}^{-1}(\widehat{C})$. Now,

we use that for $t = 0$ we have $\mathcal{F}'_0 = \widetilde{\mathcal{F}}'_0 = \Phi_0^*(\mathcal{G}_0)$. This implies that $\widehat{\Gamma} \cap M(0)$ is transverse to $\widehat{\Theta} \cap M(0)$ along $\widehat{V}' \cap M(0) = V'_1(0)$. Therefore, by transversality theory, $\widehat{\Gamma} \cap M(t)$ and $\widehat{\Theta} \cap M(t)$ are transverse and $\widehat{\Theta} \cap \widehat{\Gamma} \cap M(t) = V'_1(t)$, as wished. This finishes the proof of Theorem A in the case of two dimensional foliations. \square

3.4. Proof of Theorem A in the case $k \geq 3$. As in the proof in §3.3 we will consider a germ of a one parameter family of foliations $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ on \mathbb{P}^n such that $\mathcal{F}_0 = f^*(\mathcal{G}) \in PB(\nu, d, k, n)$, where the pair (f, \mathcal{G}) is generic, and we will prove that $\mathcal{F}_t \in PB(\nu, d, k, n)$ for all $t \in (\mathbb{C}, 0)$. We will assume also that $\mathcal{G} \in \mathcal{M}_g(d) \subset \mathbb{F}ol(d, 1, m)$, as in the preceding proof. Since the dimension of \mathcal{F} is $k \geq 3$, its codimension is $q = n - k$, the same codimension of \mathcal{G} in \mathbb{P}^m , so that $m = q + 1$.

Now we describe briefly the foliation \mathcal{F} near the indeterminacy locus $I(f)$. Fix $p \in I(f)$. With an argument similar to the argument of §2.2 there exists a holomorphic coordinate system $(U, (x, y))$, where $x = (x_1, \dots, x_{m+1}): U \rightarrow \mathbb{C}^{m+1}$, $y: U \rightarrow \mathbb{C}^{n-m-1}$, such that

$$f(x_1, \dots, x_{m+1}, y) = [x_1 : \dots : x_{m+1}] \quad \forall (x, y) \in U .$$

If \mathcal{G} is represented in homogeneous coordinates by the integrable q -form η on \mathbb{C}^{m+1} , with homogeneous coefficients, then $\mathcal{F} = f^*(\mathcal{G})$ is represented in U by $f^*(\eta)$, which has the same expression of η . In particular, $\mathcal{F}|_U$ is equivalent to the product of a two dimensional homogeneous foliation on \mathbb{C}^{m+1} by the regular foliation of dimension $n - m - 1 = k - 2$ whose leaves are the levels $x = c, c \in \mathbb{C}^{m+1}$. Observe that the restriction $\mathcal{F}|_{y=c'}$, $c' \in \mathbb{C}^{n-m-1}$, has a conic singularity at $(0, c')$.

Definition 3.9. Let \mathcal{H} be a germ of k -dimensional foliation on $(\mathbb{C}^n, 0)$, $k \geq 2$. We say that \mathcal{H} has a *conic* singularity at $0 \in \mathbb{C}^n$ if there exists a decomposition $\mathbb{C}^n = \mathbb{C}^{n-k+2} \times \mathbb{C}^{k-2}$ for which the restriction $\mathcal{H}|_{\mathbb{C}^{n-k+2} \times \{0\}}$ has a conic singularity at $(0, 0)$. We say that the singularity is NGK if the conic singularity is NGK (see §2.5).

Let \mathcal{H} be as in Definition 3.9 and let η be a germ of integrable $(n - k)$ -form defining \mathcal{H} . Clearly, the condition in Definition 3.9 is equivalent to the fact that $\widetilde{\eta} := \eta|_{\mathbb{C}^{n-k+2} \times \{0\}}$ has the first non-zero jet of conic type. The singularity is NGK if and only if $\widetilde{\eta}$ is conic and $(0, 0)$ is an isolated singularity of $d\widetilde{\eta}$ in the plane $\mathbb{C}^{n-k+2} \times \{0\}$.

If $\mathcal{F} = f^*(\mathcal{G})$, where the pair (f, \mathcal{G}) is generic, then all points of $I(f)$ are conic NGK singularities of \mathcal{F} . This is a consequence of the structure of local product described above, the fact that $\mathcal{G} \in \mathcal{M}(d)$, and of Lemma 2.11.

Let $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ be a germ of deformation of $\mathcal{F}_0 = \mathcal{F} = f^*(\mathcal{G})$. We will denote the set of conic points of \mathcal{F}_t by $Con(\mathcal{F}_t)$. The proof of Theorem A will be finished with two auxiliary results.

Lemma 3.10. *Let $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ be a germ of one parameter deformation of $\mathcal{F}_0 = f^*(\mathcal{G})$, where the pair (f, \mathcal{G}) is generic. Then there exists a germ of C^∞ -isotopy $\Phi: I(f) \times (\mathbb{C}, 0) \rightarrow \mathbb{P}^n$ such that $\Phi_0(I(f)) = I(f)$ and $\Phi_t(I(f)) = Con(\mathcal{F}_t) \quad \forall t \in (\mathbb{C}, 0)$, where $\Phi_t(z) = \Phi(z, t)$.*

Lemma 3.11. *Let η be a germ at $0 \in \mathbb{C}^n$ of integrable q -form, where $n \geq q + 3$. Assume that η has a conic NGK singularity of degree $d \geq 2$ at $0 \in \mathbb{C}^n$. Then there exists a germ of biholomorphism $\phi = (x, y): (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{q+2} \times \mathbb{C}^{n-q-2}, (0, 0))$ such*

that $\phi_*(\eta)$ is dicritical, has homogeneous coefficients of degree $d + 1$, and depends only on $x = (x_1, \dots, x_{q+2})$:

$$\phi_*(\eta) = \sum_{\sigma} A_{\sigma}(x) dx_{\sigma(1)} \wedge \dots \wedge dx_{\sigma(q)} .$$

In other words, the foliation \mathcal{F}_{η} is equivalent to a product of a singular foliation of dimension two on \mathbb{C}^{q+2} by a regular foliation of dimension $n - q - 2$.

The proof of Lemmas 3.10 and 3.11 will be done at the end.

Notation. The form $\phi_*(\eta)$ represents a one dimensional foliation \mathcal{G} on \mathbb{P}^{q+1} . The foliation \mathcal{G} will be called the normal type of the foliation at the conic singularity.

Remark 3.12. When the normal type is *NGK*, then Lemma 3.11 implies that it is locally constant along the set of conic points.

Let us finish the proof of Theorem A. First of all, $Con(\mathcal{F}_0) = I(f)$ is the complete intersection of $m + 1$ hypersurfaces of degree ν , so that $dim_{\mathbb{C}}(Con(\mathcal{F}_0)) = n - m - 1 = k - 2 \geq 1$. On the other hand, by Lemma 3.10, $Con(\mathcal{F}_t)$ is a deformation of $Con(\mathcal{F}_0)$, and so we can use Sernesi’s theorem [17]: there are holomorphic families of homogeneous polynomials of degree ν , F_{0t}, \dots, F_{mt} , that define $Con(\mathcal{F}_t)$ in homogeneous coordinates, for all $t \in (\mathbb{C}, 0)$. Each $F_t = (F_{0t}, \dots, F_{mt}) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m+1}$ defines a rational map $f_t : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ such that $I(f_t) = Con(\mathcal{F}_t)$.

Now, the normal type of \mathcal{F}_t along $Con(\mathcal{F}_t)$ is locally constant, by Remark 3.12. Since $Con(\mathcal{F}_t)$ is a complete intersection, it is connected, so that the normal type is constant along $Con(\mathcal{F}_t)$ for all $t \in (\mathbb{C}, 0)$. Let \mathcal{G}_t be the normal type of \mathcal{F}_t along $Con(\mathcal{F}_t)$. The idea is to prove that \mathcal{F}_t is equivalent to $f_t^*(\mathcal{G}_t)$ for all $t \in (\mathbb{C}, 0)$.

Let us define f_t in such a way that $f_t^*(\mathcal{G}_t) = \mathcal{F}_t$ for all $t \in (\mathbb{C}, 0)$. Fix a $q + 2$ plane H_o transverse to $Con(\mathcal{F}_0)$ and fix a point $p \in H_o \cap Con(\mathcal{F}_0)$. Since p is a conic NGK point of $\mathcal{F}_0|_{H_o}$, by Theorem 2.15, there exists a holomorphic germ $Q : (\mathbb{C}, 0) \rightarrow H_o$ such that $Q(0) = p$ and $Q(t)$ is a conic NGK singularity of $\mathcal{F}_t|_{H_o}$. Since $t \mapsto Q(t)$ is holomorphic, after an automorphism of \mathbb{P}^n that preserves H_o , we can assume that $Q(t) = p$ for all $t \in (\mathbb{C}, 0)$. Now, we blow up H_o at p obtaining a divisor $E \simeq \mathbb{P}^{q+1}$ and a holomorphic one parameter family of foliations on E that we can assume to be the family $(\mathcal{G}_t)_{t \in (\mathbb{C}, 0)}$. Fix a ball U around p in H_o such that $I(f_t) \cap H_o \cap U$ contains only the point p . If U is small, then we can assume that all fibers of f_t cut $H_o \cap U$ in a smooth curve passing through p . The strict transform of this curve after the blow-up cuts E in a unique point and this defines a rational map $\mathbb{P}^n \dashrightarrow E \simeq \mathbb{P}^{q+1}$. This map is equivalent to f_t , so that we can assume that f_t is constructed in this way. Now, if we apply the argument of §3.3 we see that $f_t^*(\mathcal{G}_t)|_{H_o} = \mathcal{F}_t|_{H_o}$ for all $t \in (\mathbb{C}, 0)$. The same argument can be applied to any $q + 2$ plane H transverse to $Con(\mathcal{F}_0)$ to show that $\mathcal{F}_t|_H = f_t^*(\mathcal{G}_t)|_H$ for all $t \in (\mathbb{C}, 0)$. This implies that $\mathcal{F}_t = f_t^*(\mathcal{G}_t) \forall t \in (\mathbb{C}, 0)$, as the reader can check. This finishes the proof of Theorem A.

Proof of Lemma 3.10. Lemma 3.10 is a consequence of the stability of conic singularities, Theorem 2.15. First of all, let us prove the local stability. Given $z \in Con(\mathcal{F}_0)$ fix coordinates $(U, (x, y))$, $\phi = (x, y) : U \rightarrow Q_1 \times Q_2 \subset \mathbb{C}^{n-q-2} \times \mathbb{C}^{q+2}$ with $z \in U$ and $\phi(z) = (0, 0)$, such that $U \cap Con(\mathcal{F}_0) = (y = 0)$. Given $(x_o, 0) \in Con(\mathcal{F}_0)$ set $L_{x_o} := \{(x, y) \in U \mid x = x_o\} \simeq Q_2$. Each L_x is transverse to $Con(\mathcal{F}_0)$, so that $\mathcal{F}_0|_{L_x}$ has a conic NGK singularity at $(x, 0)$. Applying Theorem 2.15 to the family of foliations $(\mathcal{F}_t|_{L_x})_{(t,x)}$, viewed as a family of foliations

on the open set Q_2 of \mathbb{C}^{q+2} , we obtain a holomorphic map $\psi: Q_1 \times (\mathbb{C}, 0) \rightarrow Q_2$ such that $\psi(x, 0) = (x, 0)$ and $\psi(x, t)$ is the unique conic singularity of $\mathcal{F}_t|_{L_x}$ in L_x . This holomorphic local version implies that, locally “ $Con(\mathcal{F}_t)$ is a graph over $Con(\mathcal{F}_0)$ ”, and this implies the C^∞ global version. We leave the details to the reader. \square

Proof of Lemma 3.11. According to Definition 3.9, there exists a decomposition $\mathbb{C}^n = \mathbb{C}^{q+2} \times \mathbb{C}^{k-2}$ such that $\eta|_{(\mathbb{C}^{q+2}, 0) \times \{0\}}$ has a conic NGK singularity at $(0, 0)$. Let us fix some notation. We denote the coordinates in $\mathbb{C}^{q+2} \times \mathbb{C}^{k-2}$ as (x, y) , where $x = (x_1, \dots, x_{q+2})$ and $y = (y_1, \dots, y_{k-2})$. Given sequences $\alpha = (1 \leq \alpha_1 < \dots < \alpha_s \leq q + 2)$ and $\beta = (1 \leq \beta_1 < \dots < \beta_r \leq k - 2)$ we set $\#\alpha = s$, $\#\beta = r$ and $dx^\alpha \wedge dy^\beta = dx_{\alpha_1} \wedge \dots \wedge dx_{\alpha_s} \wedge dy_{\beta_1} \wedge \dots \wedge dy_{\beta_r}$. With this notation a germ Θ of holomorphic q -form can be written as

$$(3.2) \quad \Theta = \sum_{\#\alpha + \#\beta = q} A_{\alpha, \beta}(x, y) dx^\alpha \wedge dy^\beta, \quad A_{\alpha, \beta} \in \mathcal{O}_n.$$

We will say that the q -form Θ depends on r variables y , where $0 \leq r \leq k - 2$, if it can be written as

$$\Theta = \sum_{\substack{\#\alpha + \#\beta = q \\ 1 \leq \beta_j \leq r}} A_{\alpha, \beta}(x, y_1, \dots, y_r) dx^\alpha \wedge dy^\beta \text{ if } r > 0 \text{ or as } \Theta = \sum_{\#\alpha = q} A_\alpha(x) dx^\alpha$$

if $r = 0$. In other words, Θ depends only on $x_1, \dots, x_{q+2}, y_1, \dots, y_r$ and on $dx_1, \dots, dx_{q+2}, dy_1, \dots, dy_r$.

The idea is to prove that if η depends on r variables y , where $1 \leq r \leq k - 2$, then there exists a germ of biholomorphism $\varphi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$, of the form $\varphi(x, y) = (\varphi_1(x, y), y)$, such that $\varphi^*(\eta)$ depends on $r - 1$ variables y . Of course, this implies Lemma 3.11. The induction step will be reduced to the following.

Claim 3.13. There exists a germ of vector field Y of the form

$$(3.3) \quad Y = \frac{\partial}{\partial y_r} + \sum_{j=1}^{q+2} B_j(x, y_1, \dots, y_r) \frac{\partial}{\partial x_j}$$

such that $i_Y \eta = 0$ and $i_Y d\eta = 0$.

We will prove Claim 3.13 at the end. Let us finish the proof of the induction step by using it.

Let $\phi: (\mathbb{C} \times \mathbb{C}^{q+2+r}, (0, 0)) \rightarrow (\mathbb{C}^{q+2+r}, 0)$ be the local flow of Y . The reader can check using (3.3) that ϕ is of the form

$$\phi(t, x, y) = (\phi_1(t, x, y), y_1, \dots, y_{r-1}, y_r + t).$$

Define $\varphi: (\mathbb{C} \times \mathbb{C}^{q+2+r-1}, (0, 0)) \rightarrow (\mathbb{C}^{q+2+r}, 0)$ as

$$\varphi(t, x, y_1, \dots, y_{r-1}) = \phi(t, x, y_1, \dots, y_{r-1}, 0) = (\phi_1(t, x, y_1, \dots, y_{r-1}, 0), y_1, \dots, y_{r-1}, t).$$

It can be verified that φ is a germ of biholomorphism and that $\varphi^*(Y) = \frac{\partial}{\partial t}$. In particular, if we set $\tilde{\eta} = \varphi^*(\eta)$, then

$$i_{\frac{\partial}{\partial t}} d\tilde{\eta} = 0 \text{ and } i_{\frac{\partial}{\partial t}} \tilde{\eta} = 0 \implies L_{\frac{\partial}{\partial t}} \tilde{\eta} = 0.$$

Since $i_{\frac{\partial}{\partial t}} \tilde{\eta} = 0$ the form $\tilde{\eta}$ does not contain terms with dt . Since $L_{\frac{\partial}{\partial t}} \tilde{\eta} = 0$ the coefficients of $\tilde{\eta}$ do not depend on t . Therefore, $\tilde{\eta}$ depends only on $r - 1$ variables y , as wished. \square

Proof of Claim 3.13. Fix coordinates $(x, y) \in (\mathbb{C}^{q+2} \times \mathbb{C}^{k-2}, (0, 0))$ such that $\eta|_{(y=0)}$ has a conic NGK singularity. We begin proving that we can assume that the set of conic NGK singularities of η , $Con(\eta)$, is $\{(x, y) \mid x = 0\}$.

Indeed, fix coordinates $(x, y) = (x_1, \dots, x_{q+2}, y_1, \dots, y_{k-2}) \in \mathbb{C}^{q+2} \times \mathbb{C}^{k-2}$ such that $\eta|_{(y=0)}$ has a conic NGK singularity at $(0, 0)$. Given $y_o \in (\mathbb{C}^{k-2}, 0)$ set $\eta_{y_o} := \eta|_{(y=y_o)}$. If η is written as in (3.2), then

$$\eta_y = \sum_{\#\alpha=q} A_\alpha(x, y) dx^\alpha \quad \forall y \in (\mathbb{C}^{k-2}, 0) .$$

According to the definition, η_0 has a conic NGK singularity at $0 \in \mathbb{C}^{q+2}$. We can consider $y \mapsto \eta_y$ as a holomorphic family of integrable q -forms. In this case, Theorem 2.15 implies that there exists a holomorphic germ $Q: (\mathbb{C}^{k-2}, 0) \rightarrow (\mathbb{C}^{q+2}, 0)$ such that $Q(0) = 0$ and $Q(y)$ is the unique conic NGK singularity of η_y . In particular, $Con(\eta) = \{(Q(y), y) \mid y \in (\mathbb{C}^{k-2}, 0)\}$. Let ϕ be the germ of biholomorphism defined by $\phi(x, y) = (x + Q(y), y)$. Since $\phi^{-1}(Con(\eta)) = (x = 0)$, we get

$$Con(\phi^*(\eta)) = \phi^{-1}(Con(\eta)) = (x = 0),$$

which proves the assertion. From now on we will assume that $Con(\eta) = (x = 0)$.

Let us assume that η depends on r variables y , where $1 \leq r \leq k - 2$. Given $y_o \in (\mathbb{C}^r, 0)$ let η_{y_o} be as before:

$$\eta_{y_o} = \eta|_{(y=y_o)} = \sum_{\#\alpha=q} A_\alpha(x, y_o) dx^\alpha .$$

We can consider $d\eta_{y_o}$ as a $(q + 1)$ -form on $(\mathbb{C}^{q+2}, 0)$. In particular, there exists a holomorphic vector field

$$(3.4) \quad X = \sum_{j=1}^{q+2} X_j(x, y_o) \frac{\partial}{\partial x_j}$$

such that $d\eta_{y_o} = i_X \nu_o$, where $\nu_o = dx_1 \wedge \dots \wedge dx_{q+2}$. We consider X as a germ of vector field on $(\mathbb{C}^n, 0)$. Since $d\eta$ is integrable, the fact that $i_X d\eta_y = 0$ for all $y \in (\mathbb{C}^r, 0)$ implies that $i_X d\eta = 0$. The integrability of η implies that $i_X \eta = 0$.

Remark 3.14. Note that $Sing(X) = Con(\eta) = (x = 0)$.

From now on, we will write $y = (\tilde{y}, y_r)$, where $\tilde{y} = (y_1, \dots, y_{r-1})$. Given $\tilde{y}_o = (y_{1o}, \dots, y_{r-1o}) \in (\mathbb{C}^{r-1}, 0)$ fixed, set $\Sigma_{\tilde{y}_o} = \{(x, \tilde{y}, y_r) \mid \tilde{y} = \tilde{y}_o\}$ and $\eta_{\tilde{y}} := \eta|_{\Sigma_{\tilde{y}}}$, so that $d\eta_{\tilde{y}} = d\eta|_{\Sigma_{\tilde{y}}}$. It follows from the above notation that there exists a germ of q -form θ , of the type

$$\theta = \sum_{\#\alpha=q} C_\alpha(x, y) dx^\alpha$$

such that

$$d\eta_{\tilde{y}} = i_X \nu_o + \theta \wedge dy_r .$$

From $i_X d\eta = 0$ we obtain $i_X d\eta_{\tilde{y}} = 0$ for all $\tilde{y} \in (\mathbb{C}^{r-1}, 0)$. In particular,

$$i_X d\eta_{\tilde{y}} = i_X(\theta \wedge dy_r) = i_X \theta \wedge dy_r = 0 \implies i_X \theta = 0 .$$

Now, we use de Rham's division theorem in the parametric form, considering θ and X depending on the parameter $y \in (\mathbb{C}^r, 0)$. Since $Sing(X) = (x = 0)$ for $y \in (\mathbb{C}^r, 0)$

fixed then X has an isolated singularity at $0 \in \mathbb{C}^{q+2}$. De Rham's theorem implies that there exists a $q+1$ -form μ such that $\theta = i_X \mu$, where

$$\mu = \sum_{\#\alpha=q+1} D_\alpha(x, y) dx^\alpha .$$

Hence, we can write

$$d\eta_{\tilde{y}} = i_X(\nu_o + \mu \wedge dy_r) .$$

The reader can verify that the $(q+2)$ -form $\nu_o + \mu \wedge dy_r$ can be written as

$$\nu_o + \mu \wedge dy_r = i_Y (dy_r \wedge dx_1 \wedge \dots \wedge dx_{q+2}) ,$$

where Y is as in (3.3). On the other hand, $i_Y(\nu_o + \theta \wedge dy_r) = 0$ implies that $i_Y(d\eta_{\tilde{y}}) = 0$ for all $\tilde{y} \in (\mathbb{C}^{r-1}, 0)$. Since η and $d\eta$ are integrable, this implies that $i_Y d\eta = 0$ and $i_Y \eta = 0$, which proves Claim 3.13 and Lemma 3.11. \square

4. RELATED PROBLEMS

Motivated by Theorem A, we would like to state some problems:

Problem 1. Is there a generalization of Theorem A in the case of pull-backs by branched maps (not generic) like in [18]?

Problem 2. Let \mathcal{F} be a k -dimensional foliation on \mathbb{P}^n , where $2 \leq k < n$. Assume that \mathcal{F} has a conic NGK singularity. Is it true that \mathcal{F} is a pull-back foliation?

In fact, we don't know any example of foliation of dimension ≥ 2 having a conic singularity which is not a pull-back. Another problem, motivated by Problem 2, is the following:

Problem 3. Is there a holomorphic foliation of dimension ≥ 2 , on a complex connected manifold, having two or more conic NGK singularities of different normal types?

ACKNOWLEDGMENTS

We are deeply grateful to D. Cerveau and T. Fassarella for the discussions, suggestions, and comments. We also thank the referee for helpful comments and corrections. This work was developed at IRMAR(Rennes, France) and was supported by Capes-Brasil, process number (9814-13-2).

REFERENCES

- [1] César Camacho and Alcides Lins Neto, *The topology of integrable differential forms near a singularity*, Inst. Hautes Études Sci. Publ. Math. **55** (1982), 5–35. MR672180
- [2] Julie Déserti and Dominique Cerveau, *Feuilletages et actions de groupes sur les espaces projectifs* (French, with English and French summaries), Mém. Soc. Math. Fr. (N.S.) **103** (2005), vi+124 pp. (2006). MR2200857
- [3] D. Cerveau and A. Lins Neto, *Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{C}P(n)$, $n \geq 3$* , Ann. of Math. (2) **143** (1996), no. 3, 577–612. MR1394970
- [4] D. Cerveau, A. Lins Neto, and S. J. Edixhoven, *Pull-back components of the space of holomorphic foliations on $\mathbb{C}P(n)$, $n \geq 3$* , J. Algebraic Geom. **10** (2001), no. 4, 695–711. MR1838975
- [5] S. C. Coutinho and J. V. Pereira, *On the density of algebraic foliations without algebraic invariant sets*, J. Reine Angew. Math. **594** (2006), 117–135. MR2248154
- [6] Fernando Cukierman and Jorge Vitório Pereira, *Stability of holomorphic foliations with split tangent sheaf*, Amer. J. Math. **130** (2008), no. 2, 413–439. MR2405162

- [7] F. Cukierman, J. V. Pereira, and I. Vainsencher, *Stability of foliations induced by rational maps* (English, with English and French summaries), *Ann. Fac. Sci. Toulouse Math.* (6) **18** (2009), no. 4, 685–715. MR2590385
- [8] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant manifolds*, *Lecture Notes in Mathematics*, Vol. 583, Springer-Verlag, Berlin-New York, 1977. MR0501173
- [9] Ivan Kupka, *The singularities of integrable structurally stable Pfaffian forms*, *Proc. Nat. Acad. Sci. U.S.A.* **52** (1964), 1431–1432. MR0173214
- [10] J. P. Jouanolou, *Équations de Pfaff algébriques* (French), *Lecture Notes in Mathematics*, vol. 708, Springer, Berlin, 1979. MR537038
- [11] Alcides Lins Neto, *Componentes irredutíveis dos espaços de folheações* (Portuguese), *Publicações Matemáticas do IMPA*. [IMPA Mathematical Publications], Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2007. 26^o Colóquio Brasileiro de Matemática. [26th Brazilian Mathematics Colloquium]. MR2370063
- [12] A. Lins Neto, *Germens of complex two dimensional foliations*, *Bull. Braz. Math. Soc. (N.S.)* **46** (2015), no. 4, 645–680. MR3436562
- [13] A. Lins Neto and M. G. Soares, *Algebraic solutions of one-dimensional foliations*, *J. Differential Geom.* **43** (1996), no. 3, 652–673. MR1412680
- [14] Frank Loray, Jorge Vitério Pereira, and Frédéric Touzet, *Foliations with trivial canonical bundle on Fano 3-folds*, *Math. Nachr.* **286** (2013), no. 8-9, 921–940. MR3066408
- [15] Airton S. de Medeiros, *Structural stability of integrable differential forms*, *Geometry and topology* (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Springer, Berlin, 1977, pp. 395–428. *Lecture Notes in Math.*, Vol. 597. MR0451274
- [16] Georges de Rham, *Sur la division de formes et de courants par une forme linéaire* (French), *Comment. Math. Helv.* **28** (1954), 346–352. MR0065241
- [17] Edoardo Sernesi, *Deformations of algebraic schemes*, *Grundlehren der Mathematischen Wissenschaften* [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006. MR2247603
- [18] W. Costa e Silva, *Stability of branched pull-back projective foliations*, *Bull. Braz. Math. Soc. (N.S.)* **48** (2017), no. 1, 29–44. MR3623760

IMPA, EST. D. CASTORINA, 110, 22460-320, RIO DE JANEIRO, RJ, BRAZIL
Email address: wancossil@gmail.com

IMPA, EST. D. CASTORINA, 110, 22460-320, RIO DE JANEIRO, RJ, BRAZIL
Email address: alcides@impa.br