

INTEGRAL POINTS AND ORBITS OF ENDOMORPHISMS ON THE PROJECTIVE PLANE

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ABSTRACT. We analyze when integral points on the complement of a finite union of curves in \mathbb{P}^2 are potentially dense. When the logarithmic Kodaira dimension $\bar{\kappa}$ is $-\infty$, we completely characterize the potential density of integral points in terms of the number of irreducible components at infinity and the number of multiple members in a pencil naturally associated to the surface. When $\bar{\kappa} = 0$, we prove that integral points are always potentially dense. The bulk of our analysis concerns the subtle case of $\bar{\kappa} = 1$. We determine the potential density of integral points in a number of cases by incorporating the structure theory of affine surfaces and developing an arithmetic framework for studying integral points on surfaces fibered over curves.

We also prove, assuming Lang–Vojta’s conjecture, that an orbit under an endomorphism ϕ of \mathbb{P}^2 can contain a Zariski-dense set of integral points only if there is a nontrivial completely invariant proper Zariski-closed subset of \mathbb{P}^2 under ϕ . This may be viewed as a generalization of a result of Silverman on \mathbb{P}^1 .

1. INTRODUCTION

In this paper we will study integral points on affine open subsets of \mathbb{P}^2 and the more specific problem of integral points that lie in an orbit of an endomorphism of \mathbb{P}^2 . Our analyses are based on structure theorems for affine surfaces classified by their logarithmic Kodaira dimension. To our knowledge, these results in affine algebraic geometry have not been previously used for arithmetic applications.

For curves, there are two fundamental results characterizing the finiteness of integral and rational points. Siegel’s theorem asserts that an affine curve $C \subset \mathbb{A}^N$ over a number field k has only finitely many integral points if either the curve C has positive genus or the curve C is rational and has more than two points at infinity. For curves of genus $g \geq 2$, Siegel’s theorem is superseded by Faltings’s theorem that the set of rational points $C(k)$ is finite. Both theorems may be unified into the single statement that a curve of log general type has only finitely many integral points (note that for a projective curve, integral points are the same as rational points). In higher dimensions, the Lang–Vojta conjecture predicts that this unified statement continues to hold, with finiteness replaced by Zariski nondensity.

Conjecture 1 (Lang–Vojta). *Let V be a variety defined over a number field k and let S be a finite set of places of k containing the archimedean places. If V is of log general type, then any set of S -integral points on V is not Zariski dense in V .*

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This conjecture is a consequence of a much more general height inequality conjectured by Vojta [31, Conjecture 3.4.3]. When V is a projective surface this was posed as a question by Bombieri.

In higher dimensions, outside of some important special cases (e.g., subvarieties of semiabelian varieties [12, 13, 32, 34]), not much is known towards Conjecture 1. For instance, if $V = \mathbb{P}^2 \setminus D$, where D is a nonsingular plane curve, then V is of log general type if and only if $\deg D \geq 4$. However, there is not a single such V for which Conjecture 1 is known for all applicable k and S . In view of this, in studying integral points on affine subsets of \mathbb{P}^2 we will often work under the assumption of Conjecture 1.

Suppose now that V is an affine surface given as the complement in \mathbb{P}^2 of a (possibly reducible) curve D defined over a number field k . A basic invariant of V is the log Kodaira dimension $\bar{\kappa}(V)$, whose definition will be recalled in Section 2. Here we just note that $\bar{\kappa}(V) \in \{-\infty, 0, 1, 2\}$ and $\bar{\kappa}(V) = 2$ if and only if V is of log general type. If $V = \mathbb{P}^2 \setminus D$, where D is a normal crossings divisor, then

$$\bar{\kappa}(V) = \begin{cases} -\infty & \text{if } \deg D = 1, 2, \\ 0 & \text{if } \deg D = 3, \\ 2 & \text{if } \deg D \geq 4. \end{cases}$$

However, if D is not a normal crossings divisor, then computing $\bar{\kappa}(V)$ is more subtle. A first goal of this paper is to understand integral points on V via the invariant $\bar{\kappa}(V)$. If $\bar{\kappa}(V) = 2$, then, as discussed, this is accomplished (conjecturally) by the Lang–Vojta conjecture. To avoid situations where the lack of integral points is caused by special arithmetic properties of certain number fields or certain sets of primes, we analyze *potential density* of integral points, namely whether there exist a number field $L \supset k$ and a finite set of places S of L for which $V(\mathcal{O}_{L,S})$ is Zariski dense.

When $\bar{\kappa}(V) = -\infty$, we prove the following result using structure theory for affine surfaces.

Theorem 2. *Let D be an effective divisor on \mathbb{P}^2 defined over $\overline{\mathbb{Q}}$, and let $V = \mathbb{P}^2 \setminus D$. Suppose that $\bar{\kappa}(V) = -\infty$. Let Λ be the associated pencil of Miyanishi and Sugie (Theorem 27). Let D' be the union of D and the multiple members of Λ . Let r be the number of irreducible components of D' . Then integral points on V are potentially dense if and only if $r \leq 2$. Moreover, if $r \geq 3$, then for any finite set of places S of k containing the archimedean places, $V(\mathcal{O}_{k,S})$ is contained in the union of finitely many effectively computable curves.*

When $\bar{\kappa}(V) = 0$ it turns out that integral points on V are always potentially dense.

Theorem 3. *Let D be an effective divisor on \mathbb{P}^2 defined over $\overline{\mathbb{Q}}$, and let $V = \mathbb{P}^2 \setminus D$. Suppose that $\bar{\kappa}(V) = 0$. Then integral points on V are potentially dense.*

This agrees with a very general conjecture of Campana [8, Th. 5.1, Conj. 9.20] that if V is a projective variety and $\bar{\kappa}(V) = 0$, then rational points on V are potentially dense.

When $\bar{\kappa}(V) = 1$, the situation is more subtle. In this case, we know from Kawamata [21] that there is a pencil Λ of curves in \mathbb{P}^2 whose restriction to V yields a \mathbb{G}_m -fibration. Kojima [22] has proved a structure theorem for this case,

describing a certain open subset $\mathbb{P}^2 \setminus D' \subset V$, but unlike the $\bar{\kappa} = -\infty$ case, additional components in D' come from *singular* fibers, rather than multiple fibers. This makes it harder to retrieve Diophantine information about D from that of D' .

In Section 3.3, we will define two weights associated to (D, Λ) , the gcd weight and the Campana weight. These weights measure the multiplicities of the members of Λ , using either the gcd of the multiplicities appearing in a member of the pencil or the minimum (following Campana [8]). Our definitions of the weights also take into account the divisor D with respect to which integrality is defined. We prove the following result relating these weights to integral points.

Theorem 4 (cf. Theorem 23). *Let D be an effective divisor on \mathbb{P}^2 defined over a number field k , and let $V = \mathbb{P}^2 \setminus D$. Let S be a finite set of places of k containing the archimedean places. Let Λ be a pencil of curves on \mathbb{P}^2 such that the base points of Λ are contained in the support of D .*

- (i) *If the gcd weight of (D, Λ) is greater than 2, then the S -integral points of $\mathbb{P}^2 \setminus D$ are contained in the union of finitely many members of Λ . Furthermore, if the support of D contains the support of a member of Λ , then this finite union is effectively computable.*
- (ii) *If the Campana weight of (D, Λ) is greater than 2, then the abc conjecture implies that the S -integral points of $\mathbb{P}^2 \setminus D$ are contained in the union of finitely many members of Λ .*

Theorem 4(i) will be used in the proof of Theorem 2. In the $\bar{\kappa} = 1$ case, we will use both parts of Theorem 4, but mainly (ii). Among our results in this case, we just mention the following result here.

Theorem 5 (cf. Theorems 35 and 37). *Let D be a rational curve defined over $\overline{\mathbb{Q}}$ having just cusps as singularities, and let $V = \mathbb{P}^2 \setminus D$. Suppose that $\bar{\kappa}(V) = 1$. Then assuming the abc conjecture, integral points on V are potentially dense only if D is projectively equivalent to one of the five types of curves listed in Theorems 35(i) and 37.*

We also analyze potential density of integral points for other varieties V with $\bar{\kappa}(V) = 1$, such as when $\deg D$ is small or when D consists of a line and an irreducible curve. Furthermore, in many cases of Theorem 5, we can actually construct Zariski-dense sets of integral points, providing a partial converse to the theorem. We will defer the precise statements to Theorems 35–40 and the examples in Section 7.

Our proofs of Theorems 2–5 combine results from affine algebraic geometry with results from Diophantine analysis. From geometry, key ingredients in our proofs include the structure theory of surfaces $V = \mathbb{P}^2 \setminus D$ with $\bar{\kappa}(V) = -\infty$ (Miyanishi-Sugie [23]) and $\bar{\kappa}(V) \leq 1$ (Kojima [22]). For Theorem 5, another key result is the classification of cuspidal plane curves whose complements have $\bar{\kappa} = 1$, due to Tono [27, 28]. From Diophantine analysis, we use various results including Darmon-Granville’s results [11], unit equations, and Baker’s theory to prove Theorem 4.

We now discuss integral points in an orbit under an endomorphism ϕ of \mathbb{P}^2 . Here we denote by ϕ^n the n th iterate of ϕ , and by $\mathcal{O}_\phi(\alpha)$ the orbit

$$\{\alpha, \phi(\alpha), \phi^2(\alpha), \dots\}$$

of α . Our starting point is Silverman’s theorem on integral points in orbits of rational functions.

Theorem 6 (Silverman [25]). *Let $\phi(z) \in k(z)$ be a rational function of degree $d \geq 2$, over a number field k , with the property that $\phi^2(z)$ is not a polynomial. Let S be a finite set of places of k containing the archimedean places. Then for any $\alpha \in k$, the orbit $\mathcal{O}_\phi(\alpha)$ contains only finitely many S -integral points.*

If $\phi^2(z) \in k[z]$, then for an appropriate choice of S and α , $\mathcal{O}_\phi(\alpha)$ will contain infinitely many S -integral points. Note also that if $\phi^2(z)$ is a polynomial, then $E = \{\infty, \phi(\infty)\}$ is a *completely invariant* set for ϕ , that is, $\phi^{-1}(E) = E = \phi(E)$. Thus Silverman's theorem easily implies the following.

Corollary 7. *Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be an endomorphism defined over a number field k . Let S be a finite set of places of k containing the archimedean places. Let D be a nontrivial effective divisor on \mathbb{P}^1 . If $\mathcal{O}_\phi(P)$ contains infinitely many S -integral points in $(\mathbb{P}^1 \setminus D)(\mathcal{O}_{k,S})$ for some point P , then there exists a nonempty completely invariant finite subset of \mathbb{P}^1 under ϕ .*

In view of Corollary 7, the second author has asked the following higher-dimensional analogue.

Question 8 (cf. [37, Question 2]). *Let $\phi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be an endomorphism defined over a number field k . Let S be a finite set of places of k containing the archimedean places. Let D be a nontrivial effective divisor on \mathbb{P}^n . If $\mathcal{O}_\phi(P) \cap (\mathbb{P}^n \setminus D)(\mathcal{O}_{k,S})$ is Zariski dense in \mathbb{P}^n for some point P , then does there exist a nonempty completely invariant proper Zariski-closed subset of \mathbb{P}^n under ϕ ?*

Assuming the Lang–Vojta conjecture, we will use our analysis of integral points on open affine subsets of \mathbb{P}^2 to answer this question positively in the case $n = 2$.

Theorem 9 (cf. Theorem 42). *Assume the Lang–Vojta conjecture. Let $\phi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be an endomorphism of degree > 1 defined over a number field k . Let S be a finite set of places of k containing the archimedean places. Let D be a nontrivial effective divisor on \mathbb{P}^2 defined over k . If $\mathcal{O}_\phi(P) \cap (\mathbb{P}^2 \setminus D)(\mathcal{O}_{k,S})$ is Zariski dense in \mathbb{P}^2 for some point P , then there exists a nonempty completely invariant proper Zariski-closed subset of \mathbb{P}^2 under ϕ .*

The completely invariant set constructed in the proof of Theorem 9 may consist of a finite set of points. It is plausible, however, that there always exists a codimension one completely invariant set in the conclusion of Theorem 9 (or in Question 8), and in this case one can explicitly characterize the possible pairs (ϕ, D) (see the discussion following Theorem 42).

Finally, we remark that our results should admit, via Vojta's dictionary [31], corresponding results in Nevanlinna theory and for holomorphic maps $f : \mathbb{C} \rightarrow \mathbb{P}^2 \setminus D$. We will not pursue this here but content ourselves with a few remarks. Recall that via Vojta's dictionary, an infinite (or Zariski-dense) set of (D, S) -integral points on \mathbb{P}^2 corresponds to a holomorphic map $f : \mathbb{C} \rightarrow \mathbb{P}^2 \setminus D$ that is nonconstant (or with Zariski-dense image). Since our proofs combine geometric results with arguments from Diophantine approximation involving (local) height functions, all of our proofs and results on nondensity of integral points on $\mathbb{P}^2 \setminus D$ should translate, in a straightforward manner, to corresponding proofs involving Nevanlinna theory and results on holomorphic maps $f : \mathbb{C} \rightarrow \mathbb{P}^2 \setminus D$. One notable difference is that in Nevanlinna theory the analogue of the *abc* conjecture is known (Nevanlinna's Second Main Theorem with truncated counting functions). Thus, our results on

integral points that are conditional on the *abc* conjecture will yield unconditional corresponding results for holomorphic curves.

The paper is organized as follows. In Section 2, we recall some basic definitions from algebraic geometry and Diophantine geometry. In Section 3, we collect together theoretical results on integral points, including a geometric lemma (Lemma 25) used to produce Zariski-dense sets of integral points. In this section, we also define the two notions of weights of pencils and discuss their relation to integral points (Theorem 23). In Section 4, we quote various structure theorems for affine subsets of \mathbb{P}^2 . Section 5 is the heart of the paper and we work towards classifying divisors D for which $\mathbb{P}^2 \setminus D$ has a potentially dense set of integral points, using results of Sections 3 and 4. In Section 6, we prove a more precise version of Theorem 9 (Theorem 42). In Section 7, we discuss a series of specific examples. We end with several open problems for further study.

2. DEFINITIONS AND NOTATION

In this section, we recall some definitions and notation from algebraic geometry and Diophantine geometry. See [7, 18, 19] for further details. We begin by defining the Kodaira–Iitaka dimension κ of a divisor.

Definition 10. Let D be a divisor on a smooth projective variety X over an algebraically closed field of characteristic 0. If $\dim H^0(X, \mathcal{O}(nD)) = 0$ for all $n > 0$, then we define $\kappa(X, D) = -\infty$. Otherwise, define $\kappa = \kappa(X, D)$ to be the integer for which

$$\limsup_{n \rightarrow \infty} \frac{\dim H^0(X, \mathcal{O}(nD))}{n^\kappa}$$

exists and is nonzero.

It is well known that $\kappa(X, D) \in \{-\infty, 0, 1, \dots, \dim X\}$. We now define the logarithmic Kodaira dimension of a quasi-projective variety.

Definition 11. Let V be a smooth quasi-projective variety over an algebraically closed field of characteristic 0. Let $V = X \setminus D$, where X is a smooth projective variety and D is a normal crossings divisor on X . Then the *log Kodaira dimension* $\bar{\kappa}(V)$ of V is defined to be the Kodaira–Iitaka dimension of the divisor $K_X + D$, where K_X is the canonical divisor of X .

Note that there always exists a choice of X and D as in the definition (by Hironaka’s results), and it is known that $\kappa(K_X + D)$ is independent of the choice of X and D (so that $\bar{\kappa}(V)$ is well defined). We say that V is of *log general type* if $\bar{\kappa}(V)$ is as large as possible, that is, if $\bar{\kappa}(V) = \dim V$.

Let k be a number field, and let us now assume that the smooth projective variety X and the effective divisor D are both defined over k . There are several essentially equivalent ways of defining a set of integral points on $X \setminus D$, including the natural scheme-theoretic definition coming from a choice of integral model of $X \setminus D$. Here we will follow Vojta [31] and define sets of integral points via local height functions. Let M_k be the canonical set of places of k consisting of one place for each prime ideal \mathfrak{p} of \mathcal{O}_k , one place for each real embedding $\sigma : k \rightarrow \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \bar{\sigma} : k \rightarrow \mathbb{C}$. For each $v \in M_k$, let $|\cdot|_v$ denote the corresponding absolute value, normalized so that $|p|_v = p^{-[k_v:\mathbb{Q}_v]/[k:\mathbb{Q}]}$ if v corresponds to a prime \mathfrak{p} and \mathfrak{p} lies above a rational prime

p , and $|x|_v = |\sigma(x)|^{[k_v:\mathbb{Q}_v]/[k:\mathbb{Q}]}$ if v corresponds to an embedding σ . With this normalization, the product formula

$$\prod_{v \in M_k} |x|_v = 1$$

holds for all $x \in k^*$. Let $\text{Supp } D$ denote the support of D . From the theory of heights, for each place $v \in M_k$ we can associate to D a *local height function* $\lambda_v(D, -) : X(k) \setminus \text{Supp } D \rightarrow \mathbb{R}$, unique up to a bounded function, such that

$$\sum_{v \in M_k} \lambda_v(D, P) = h(D, P) + O(1)$$

for all $P \in X(k) \setminus \text{Supp } D$, where $h(D, -)$ is a Weil height with respect to D . We will not give a precise definition here, but $\lambda_v(D, P)$ is, up to a bounded function, $-\log |f(P)|_v$, where f is a local equation of D around P . We recall that both global and local heights are functorial with respect to pullbacks by morphisms, in the following sense: if $\phi : Y \rightarrow X$ is a morphism of smooth projective varieties with $\phi(Y) \not\subset \text{Supp } D$, then

$$\begin{aligned} \lambda_v(\phi^* D, P) &= \lambda_v(D, \phi(P)) + O(1), \\ h(\phi^* D, P) &= h(D, \phi(P)) + O(1). \end{aligned}$$

In fact, since the global (Weil) height depends only on the linear equivalence class of the divisor D , if one works with divisor classes (or line bundles) the condition $\phi(Y) \not\subset \text{Supp } D$ may be avoided for the global height.

Now, let S be a finite subset of M_k containing all the archimedean places. Then a *set of (k -rational) (D, S) -integral points* on X is defined to be any subset of a set of the form

$$\left\{ P \in X(k) \setminus \text{Supp } D : \sum_{v \in M_k \setminus S} \lambda_v(D, P) \leq C \right\}$$

for some choice of local height functions and for some constant C . This notion depends only on the support of D . In fact, it depends only on the variety $V = X \setminus D$. Thus, for a smooth quasi-projective variety V over k we will call a subset $R \subset V(k)$ a set of S -integral points on V if there exist a smooth projective variety X and an effective divisor D on X such that $V = X \setminus D$ and R is a set of (D, S) -integral points on X . We say that integral points on V are *potentially dense* if there exists a Zariski-dense set of S -integral points on V for some number field k and some finite subset $S \subset M_k$.

On \mathbb{P}^n , one can choose the local height functions in a canonical way, and thus define sets of integral points unambiguously. For a polynomial $f \in k[x_0, \dots, x_n]$ and $v \in M_k$, we let $|f|_v$ denote the maximum of the v -adic absolute values of the coefficients of f . Now let D be a hypersurface in \mathbb{P}^n defined by a homogeneous polynomial $f \in k[x_0, \dots, x_n]$ of degree d . For $v \in M_k$ and $P = (x_0, \dots, x_n) \in \mathbb{P}^n(k) \setminus \text{Supp } D$, $x_0, \dots, x_n \in k$, we define the local height function

$$(1) \quad \lambda_v(D, P) = \log \frac{|f|_v \max_i |x_i|_v^d}{|f(P)|_v}.$$

This definition is independent of the choice of the defining polynomial f and the choice of the coordinates for P . Note also that for a nonarchimedean v , $\lambda_v(D, P) \geq 0$ for all $P \in \mathbb{P}^n(k) \setminus \text{Supp } D$. Then we can define *the* set of S -integral points $(\mathbb{P}^n \setminus D)(\mathcal{O}_{k,S})$ to be the set of points

$$\left\{ P \in \mathbb{P}^n(k) \setminus \text{Supp } D : \sum_{v \in M_k \setminus S} \lambda_v(D, P) = 0 \right\}.$$

Finally, we recall the notion of local heights with truncation. This is an analog of the truncated counting function in Nevanlinna theory. For each nonarchimedean $v \in M_k$, we define the minimum positive valuation ν_v to be

$$(2) \quad \nu_v = \min\{-\log |x|_v : x \in k^*, -\log |x|_v > 0\}.$$

We can then define the *truncated local height* for a nonarchimedean v by

$$\lambda_v^{(1)}(D, P) = \min(\lambda_v(D, P), \nu_v).$$

This captures the nontrivial contribution to the v -adic local height, but only counted with the smallest possible contribution.

3. INTEGRAL POINTS

In this section, we collect together theoretical results on integral points that we will use. These results will be combined with the geometric structure theorems of the next section to yield a study of integral points on affine subsets of \mathbb{P}^2 . Some of our results are also conditional on well-known conjectures, whose statements we now recall.

3.1. Conjectures. We discuss several variants of the *abc* conjecture.[†] The following is the most standard formulation.

Conjecture 12 (Masser–Oesterlé *abc* conjecture). *For all $\epsilon > 0$, there is a constant $C > 0$ such that for all $a, b, c \in \mathbb{Z}$ with $a + b + c = 0$ and $\gcd(a, b, c) = 1$, we have*

$$\max\{|a|, |b|, |c|\} \leq C \prod_{p|abc} p^{1+\epsilon}.$$

Translating the conjecture into the language of heights (see [33] for the details) and allowing for arbitrary number fields and (finite) sets of places S , we obtain the following formulation of the *abc* conjecture for number fields.

Conjecture 13 (*abc* conjecture for number fields). *Let k be a number field, and let S be a finite set of places of k containing the archimedean places. Let $\epsilon > 0$. There exists a constant C such that for all $x \in \mathbb{P}^1(k) \setminus \{0, 1, \infty\}$,*

$$(1 - \epsilon)h(x) \leq \sum_{v \in M_k \setminus S} \left(\lambda_v^{(1)}((0), x) + \lambda_v^{(1)}((1), x) + \lambda_v^{(1)}((\infty), x) \right) + C.$$

The special case $k = \mathbb{Q}$ and $S = \infty$ is precisely the conjecture of Masser and Oesterlé. Using Belyi maps, Conjecture 13 is equivalent to the following conjecture, which replaces the three points $\{0, 1, \infty\}$ by an arbitrary finite set of points on the projective line (see [30] for a precise relationship between the two conjectures).

[†]Mochizuki [24] has claimed a proof of this conjecture.

Conjecture 14 (General *abc* conjecture). *Let k be a number field, and let S be a finite set of places of k containing the archimedean places. Let $\alpha_1, \dots, \alpha_q \in \mathbb{P}^1(k)$ be distinct points, and let $\epsilon > 0$. There exists a constant C such that for all $x \in \mathbb{P}^1(k) \setminus \{\alpha_1, \dots, \alpha_q\}$,*

$$(3) \quad (q - 2 - \epsilon)h(x) \leq \sum_{\ell=1}^q \sum_{v \in M_k \setminus S} \lambda_v^{(1)}((\alpha_\ell), x) + C.$$

It is this form of the *abc* conjecture that will be the most convenient for our applications. We note that Conjecture 14 is the special case for \mathbb{P}^1 of a general conjecture of Vojta valid for all smooth complete varieties [33, Conjecture 2.3].

3.2. Campana and gcd multiplicities and weights. In this section, we define two weights associated to a fibration over \mathbb{P}^1 and discuss their importance relative to integral points. In a general setting, Campana [8] associates to a fibration $f : V \rightarrow W$ a certain \mathbb{Q} -divisor on W reflecting the multiple fibers of the fibration, and uses this divisor to give the base of the fibration the structure of an orbifold (in the sense of Campana), called the orbifold base of the fibration. One can then define the canonical bundle and Kodaira dimension of this orbifold base. As in the Lang–Vojta conjecture (Conjecture 1), when the Kodaira dimension of the orbifold base is maximal, one expects integral points on the orbifold base to be sparse, and then the same conclusion holds for the variety V . We now make this more precise in the special case of a fibration over \mathbb{P}^1 , where all of the relevant information is captured in the definition of certain weights associated to the fibration. We first define a weight using the minimum of the multiplicities in a fiber, following Campana [8].

Definition 15. Let D be an effective divisor on a nonsingular projective variety V over a field k of characteristic 0. Let $\phi : V \rightarrow \mathbb{P}^1$ be a nonconstant morphism over k . For each point $P \in \mathbb{P}^1(\bar{k})$, we define the *Campana multiplicity* $m_{D,\phi}(P)$ to be the infimum of the multiplicities of the irreducible components of the fiber over P , excluding any irreducible components of D . The infimum of the empty set is defined as usual to be ∞ , so if all components of a fiber $\phi^{-1}(P)$ are inside D , then $m_{D,\phi}(P)$ is defined to be ∞ . Then the *Campana weight* of (D, ϕ) is

$$\sum_{P \in \mathbb{P}^1(\bar{k})} \left(1 - \frac{1}{m_{D,\phi}(P)} \right).$$

Remark 16. Campana dealt with the case $D = 0$ in the above definition, where the infimum is taken over all irreducible components of the fibers. If we use the convention that every component of D has multiplicity ∞ in any fiber, then we can still define $m_{D,\phi}(P)$ to be the infimum of the multiplicities of *all* the irreducible components of the fiber over P , including those inside D .

The significance of the Campana weight of (D, ϕ) is contained in the next result, conditional on the *abc* conjecture (see also [26, Theorem 5.3] for a related result in the case V is a curve).

Theorem 17. *Assume the *abc* conjecture (Conjecture 14). Let D be an effective divisor on a nonsingular projective variety V , both defined over a number field k . Let $\phi : V \rightarrow \mathbb{P}^1$ be a nonconstant morphism over k . Let S be a finite set of places of k containing the archimedean places. If the Campana weight of (D, ϕ) is greater*

than 2, then any set of S -integral points on $V \setminus D$ is contained in finitely many fibers of ϕ . In particular, any set of S -integral points on $V \setminus D$ is Zariski nondense.

Proof. Let $\alpha_1, \dots, \alpha_q$ be the points of $\mathbb{P}^1(\bar{k})$ over which ϕ has multiple fibers or contains an irreducible component of D . Let

$$\phi^*((\alpha_\ell)) = \sum_i m_{\ell i} D_{\ell i} + \sum_j n_{\ell j} F_{\ell j}$$

be the decomposition into irreducible components, where the $D_{\ell i}$'s are the components contained in D and the $F_{\ell j}$'s are the remaining components.

Now let R be a (D, S) -integral set of points. For $P \in R$, we derive the following from Conjecture 14 and functoriality of heights with respect to pullbacks by morphisms:

$$\begin{aligned} (q - 2 - \epsilon)h(\phi(P)) &\leq \sum_{\ell=1}^q \sum_{v \in M_k \setminus S} \lambda_v^{(1)}((\alpha_\ell), \phi(P)) + O(1) \quad (\because (3)) \\ &= \sum_{\ell=1}^q \sum_{v \in M_k \setminus S} \lambda_v^{(1)} \left(\sum_i m_{\ell i} D_{\ell i} + \sum_j n_{\ell j} F_{\ell j}, P \right) + O(1) \\ &= \sum_{\ell=1}^q \sum_{v \in M_k \setminus S} \lambda_v^{(1)} \left(\sum_j n_{\ell j} F_{\ell j}, P \right) + O(1) \quad (\because P \text{ } S\text{-integral}) \\ &= \sum_{\ell=1}^q \sum_{v \in M_k \setminus S} \lambda_v^{(1)} \left(\sum_j F_{\ell j}, P \right) + O(1) \\ &\leq \sum_{\ell=1}^q h \left(\sum_j F_{\ell j}, P \right) + O(1) \\ (4) \quad &\leq \sum_{\ell=1}^q \frac{1}{\inf_j n_{\ell j}} h \left(\sum_j n_{\ell j} F_{\ell j}, P \right) + O(1) \\ &\leq \sum_{\ell=1}^q \frac{1}{\inf_j n_{\ell j}} h \left(\sum_i m_{\ell i} D_{\ell i} + \sum_j n_{\ell j} F_{\ell j}, P \right) + O(1) \\ &= \sum_{\ell=1}^q \frac{1}{\inf_j n_{\ell j}} h((\alpha_\ell), \phi(P)) + O(1) \\ &= \left(\sum_{\ell=1}^q \frac{1}{m_{D, \phi}(\alpha_\ell)} \right) h(\phi(P)) + O(1). \end{aligned}$$

Note that the inequality (4) holds even when $\{n_{\ell j}\}_j$ is the empty set for some ℓ . Since the Campana weight of (D, ϕ) is assumed to be greater than 2, $\sum_{\ell=1}^q \frac{1}{m_{D, \phi}(\alpha_\ell)} < q - 2$, and by choosing ϵ sufficiently small, we conclude that $h(\phi(P))$ is bounded for $P \in R$. Hence, P must be contained in the union of finitely many fibers of ϕ . \square

To obtain unconditional (and even effective) results, we consider another version of multiplicities and weights, replacing inf by gcd. This construction is in fact more classical in algebraic geometry.

Definition 18. Let D be an effective divisor on a nonsingular projective variety V over a field k of characteristic 0. Let $\phi : V \rightarrow \mathbb{P}^1$ be a nonconstant morphism over k . For each point $P \in \mathbb{P}^1(\bar{k})$, we define the *classical multiplicity* $m_{D,\phi}^-(P)$ to be the greatest common divisor of the multiplicities of the irreducible components of the fiber over P , excluding any irreducible components of D . By convention, we define the gcd of the empty set to be ∞ , so if all components of a fiber $\phi^{-1}(P)$ are inside D , then $m_{D,\phi}^-(P)$ is defined to be ∞ . Then the *gcd weight* of (D, ϕ) is

$$\sum_{P \in \mathbb{P}^1(\bar{k})} \left(1 - \frac{1}{m_{D,\phi}^-(P)} \right).$$

One can give an unconditional version of Theorem 17 using the gcd weight of (D, ϕ) . This is essentially due to Darmon and Granville [11] (see also [10]).

Theorem 19. *Let D be an effective divisor on a nonsingular projective variety V , both defined over a number field k . Let $\phi : V \rightarrow \mathbb{P}^1$ be a nonconstant morphism over k . Let S be a finite set of places of k containing the archimedean places. If the gcd weight of (D, ϕ) is greater than 2, then any set of S -integral points on $V \setminus D$ is contained in finitely many fibers of ϕ . In particular, any set of S -integral points on $V \setminus D$ is Zariski nondense.*

Proof. Let R be a set of S -integral points on $V \setminus D$, and let $P \in R$. Let

$$\{P_1, \dots, P_n\} = \{P \in \mathbb{P}^1(\bar{k}) \mid m_{D,\phi}^-(P) > 1\}$$

and replace k by the finite extension $k(P_1, \dots, P_n)$. Set $m_i = m_{D,\phi}^-(P_i)$. Then for each i , we have $\text{Supp } \phi^*P_i \subset \text{Supp } D$ if $m_i = \infty$, and otherwise $\phi^*P_i = D_i + m_iF_i$ for some effective divisors D_i and F_i on V with $\text{Supp } D_i \subset \text{Supp } D$. It follows from functoriality of local heights that

$$(5) \quad \lambda_v(P_i, \phi(P)) = m_i \lambda_v(F_i, P) + O_v(1), \quad m_i < \infty,$$

$$(6) \quad \lambda_v(P_i, \phi(P)) = O_v(1), \quad m_i = \infty,$$

for all $v \in M_k \setminus S$, where $O_v(1) = 0$ for all but finitely many v . Since $\lambda_v(P_i, \phi(P))$ can be taken as the product of a v -adic intersection pairing $(P_i, \phi(P))_v$ and ν_v of (2), it follows that for some finite set of places S' of k containing the archimedean places, we can choose an integral model on \mathbb{P}^1 so that the set $\phi(R)$ is contained in a set of S' -integral points on the M -curve $(\mathbb{P}^1; P_1, m_1; \dots; P_n, m_n)$, in the language of Darmon [10]. Our assumption on the gcd weight of (D, ϕ) implies that this M -curve has negative Euler characteristic, and so by [10, Theorem (Faltings plus epsilon)], $\phi(R)$ is a finite set. Thus, R lies in finitely many fibers of ϕ . \square

Under the additional assumption that one of the multiplicities is infinite, one obtains an effective result. This is essentially due to Bilu [6, Th. 1.2], who proved an explicit quantitative result when V is a curve.

Theorem 20. *Let D be an effective divisor on a nonsingular projective variety V , both defined over a number field k . Let $\phi : V \rightarrow \mathbb{P}^1$ be a nonconstant morphism over k . Let S be a finite set of places of k containing the archimedean places. If the gcd weight of (D, ϕ) is greater than 2 and $m_{D,\phi}^-(P) = \infty$ for some point $P \in \mathbb{P}^1(\bar{k})$, then any set of S -integral points on $V \setminus D$ is contained in finitely many effectively computable fibers of ϕ .*

Implicit in Theorem 20, we assume that a set of S -integral points on $V \setminus D$ is given in an explicit fashion such that the constants in (5) and (6) in the proof of Theorem 19 are effectively computable.

Proof. As before, let R be a set of S -integral points on $V \setminus D$, and let

$$\{P_1, \dots, P_n\} = \{P \in \mathbb{P}^1(\bar{k}) \mid m_{D,\phi}^-(P) > 1\}.$$

Replacing k by the finite extension $k(P_1, \dots, P_n)$, we may assume that P_1, \dots, P_n are k -rational. We consider four cases.

Case I: There are at least three distinct points P_i with $m_{D,\phi}^-(P_i) = \infty$. After an automorphism of \mathbb{P}^1 , we can assume that $m_{D,\phi}^-(0) = m_{D,\phi}^-(1) = m_{D,\phi}^-(\infty) = \infty$. Then we can enlarge S so that $\phi(P)$ and $1 - \phi(P)$ are S -units for all $P \in R$. Thus, setting $u = \phi(P)$ and $v = 1 - \phi(P)$, we obtain a solution to the S -unit equation

$$u + v = 1, \quad u, v \in \mathcal{O}_{k,S}^*.$$

As the S -unit equation has finitely many solutions, which are effectively computable, it follows that $\phi(R)$ is a finite set and R is contained in finitely many effectively computable fibers of ϕ .

Case II: There are exactly two distinct points P_i with $m_{D,\phi}^-(P_i) = \infty$. After an automorphism of \mathbb{P}^1 , we can assume that $m_{D,\phi}^-(0) = m_{D,\phi}^-(\infty) = \infty$. Since the gcd weight of (D, ϕ) is greater than 2, there is at least one other point P_i with $m_{D,\phi}^-(P_i) = m$, $1 < m < \infty$. After an automorphism of \mathbb{P}^1 , we may assume that $m_{D,\phi}^-(1) = m$. Then we can enlarge S so that $\phi(P)$ is an S -unit and the ideal generated by $\phi(P) - 1$ is an m th power of an ideal in $\mathcal{O}_{k,S}$ for all $P \in R$. In fact, after enlarging S so that $\mathcal{O}_{k,S}$ is a principal ideal domain and adjoining the m th roots of the finitely many generators of $\mathcal{O}_{k,S}^*$, we obtain a number field L and a finite set of places T of L such that $\phi(P)$ is a T -unit and $\phi(P) - 1$ an m th power in $\mathcal{O}_{L,T}$ for all $P \in R$. Let $u = \phi(P)$. Then u satisfies the equation

$$(7) \quad u - 1 = z^m, \quad u \in \mathcal{O}_{L,T}^*, z \in \mathcal{O}_{L,T}.$$

Rewriting this as $u = z^m + 1$ and noting that the right-hand side has at least two distinct roots (over \bar{k}), equation (7) easily reduces to unit equations. It follows that $\phi(R)$ is a finite set and R is contained in finitely many effectively computable fibers of ϕ .

Case III: There is exactly one point P_i with $m_{D,\phi}^-(P_i) = \infty$ and there is at least one point P_j with finite multiplicity $m_{D,\phi}^-(P_j) > 2$. Note that by our assumptions there must exist a third point P_k , distinct from P_i and P_j , with finite multiplicity $m_{D,\phi}^-(P_k) > 1$. After an automorphism of \mathbb{P}^1 , we can assume that

$$\begin{aligned} m_{D,\phi}^-(\infty) &= \infty, \\ m_{D,\phi}^-(0) &= m \geq 2, \\ m_{D,\phi}^-(1) &= n \geq 3. \end{aligned}$$

Then arguing as before, there exist a number field L and a finite set of places T of L such that $\phi(P) = x^m$ for some $x \in \mathcal{O}_{L,T}$ and $x^m - 1 = y^n$ for some $y \in \mathcal{O}_{L,T}$ for

all $P \in R$. Since $m, n > 1$ and $(m, n) \neq (2, 2)$, the superelliptic equation

$$x^m - y^n = 1, \quad x, y \in \mathcal{O}_{L,T}$$

has only finitely many effectively computable solutions. It follows that $\phi(R)$ is a finite set and R is contained in finitely many effectively computable fibers of ϕ .

Case IV: There is exactly one point P_i with $m_{D,\phi}^-(P_i) = \infty$ and there are at least three points P_j with multiplicity $m_{D,\phi}^-(P_j) = 2$. After an automorphism of \mathbb{P}^1 , we can assume that $m_{D,\phi}^-(\infty) = \infty$ and $m_{D,\phi}^-(0) = m_{D,\phi}^-(a) = m_{D,\phi}^-(b) = 2$ for some distinct $a, b \in k^*$. Then arguing as before, there exist a number field L and a finite set of places T of L such that for all $P \in R$, $\phi(P) = x^2$ for some $x \in \mathcal{O}_{L,T}$ and

$$(8) \quad x^2 - a = y^2,$$

$$(9) \quad x^2 - b = z^2$$

for some $y, z \in \mathcal{O}_{L,T}$. As is well known, equations (8) and (9) yield an affine model of an elliptic curve. Since Siegel’s theorem is effective for elliptic curves, we again obtain that $\phi(R)$ is a finite set and R is contained in finitely many effectively computable fibers of ϕ .

It’s clear that every possible case is covered by Cases I–IV, finishing the proof. □

3.3. Pencils of plane curves and integral points. In this section we reformulate the definitions and results of the last section in the context of pencils of plane curves.

Definition 21. Let Λ be a pencil of curves on \mathbb{P}^2 , and let D be an effective divisor on \mathbb{P}^2 containing the base points of Λ . For each member $C \in \Lambda$, let $m_{D,\Lambda}(C)$ (resp., $m^-(D, \Lambda)(C)$) be the infimum (resp., gcd) of the multiplicities of the irreducible components of C , excluding irreducible components which are also components of D . Then the *Campana weight* of (D, Λ) is

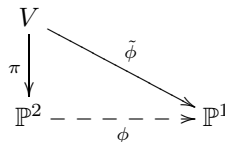
$$\sum_{C \in \Lambda} \left(1 - \frac{1}{m_{D,\Lambda}(C)} \right)$$

and the *gcd weight* of (D, Λ) is

$$\sum_{C \in \Lambda} \left(1 - \frac{1}{m_{D,\Lambda}^-(C)} \right).$$

We connect these definitions with the results of the last section through the following lemma.

Lemma 22. *Let Λ be a pencil of curves on \mathbb{P}^2 , and let D be an effective divisor on \mathbb{P}^2 containing the base points of Λ . Let $\phi = \phi_\Lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the corresponding rational map. Let $\tilde{\phi} : V \rightarrow \mathbb{P}^1$ be a morphism resolving the indeterminacy locus of ϕ , that is, a map for which*



commutes where $\pi : V \rightarrow \mathbb{P}^2$ is a birational morphism of nonsingular varieties. The Campana weight of (D, Λ) is the same as the Campana weight of $(\pi^*D, \tilde{\phi})$. The gcd weight of (D, Λ) is the same as the gcd weight of $(\pi^*D, \tilde{\phi})$.

Proof. Let $P \in \mathbb{P}^1(\bar{k})$. Let $C_P \in \Lambda$ be the corresponding member of the pencil. Since D contains the base points of Λ , when computing $m_{\pi^*D, \tilde{\phi}}(P)$ we ignore any exceptional divisors of V lying over base points of Λ . It's clear then that $m_{\pi^*D, \tilde{\phi}}(P) = m_{D, \Lambda}(C_P)$ and $m_{\pi^*D, \tilde{\phi}}^-(P) = m_{D, \Lambda}^-(C_P)$. \square

Using Lemma 22 and the results of the last section, we immediately obtain the following result which will be used repeatedly to analyze integral points on affine subsets of \mathbb{P}^2 .

Theorem 23. *Let Λ be a pencil of curves on \mathbb{P}^2 , and let D be an effective divisor on \mathbb{P}^2 containing the base points of Λ , with D and Λ both defined over a number field k . Let S be a finite set of places of k containing the archimedean places.*

- (a) *Assume the abc conjecture. If the Campana weight of (D, Λ) is greater than 2, then the S -integral points of $\mathbb{P}^2 \setminus D$ are contained in the union of finitely many members of Λ .*
- (b) *If the gcd weight of (D, Λ) is greater than 2, then the S -integral points of $\mathbb{P}^2 \setminus D$ are contained in the union of finitely many members of Λ .*
- (c) *If the gcd weight of (D, Λ) is greater than 2 and the support of D contains the support of a member of Λ , then the S -integral points of $\mathbb{P}^2 \setminus D$ are contained in the union of finitely many effectively computable members of Λ .*

Remark 24. We note that Theorem 23 holds regardless of the log Kodaira dimension of $\mathbb{P}^2 \setminus D$. Further, we note that the pencil does not necessarily have to be the one coming from the structure theory of Kawamata [21] and Gurjar–Miyanishi [15]. That is, as long as a pencil satisfies the hypotheses, it does not have to be the one created from the so-called “peeling” theory.

3.4. Density of integral points. For some rational pencils, we will use the following general lemma to construct Zariski-dense sets of integral points.

Lemma 25. *Let D be an effective divisor on \mathbb{P}^2 defined over $\overline{\mathbb{Q}}$. Let Λ be a pencil of plane curves with the property that for a general member $C \in \Lambda$, either $C \setminus D \cong \mathbb{A}^1$ or $C \setminus D \cong \mathbb{G}_m$ (over $\overline{\mathbb{Q}}$). Suppose that there is a plane curve C_0 defined over $\overline{\mathbb{Q}}$ such that C_0 is not a component of any member of Λ and either $C_0 \setminus D \cong \mathbb{A}^1$ or $C_0 \setminus D \cong \mathbb{G}_m$ (over $\overline{\mathbb{Q}}$). Then integral points on $\mathbb{P}^2 \setminus D$ are potentially dense.*

Proof. Let k be a number field with the following properties:

- The divisor D is defined over k .
- Generators for the pencil Λ are defined over k .
- $C_0 \setminus D$ is isomorphic over k to either \mathbb{A}^1 or \mathbb{G}_m .
- k has at least one complex archimedean place.

Since D is very ample, we may embed $\mathbb{P}^2 \setminus D$ over k into some affine space \mathbb{A}^N . By our assumption on C_0 , for some finite set of places S of k containing the archimedean places, $(C_0 \setminus D)(\mathcal{O}_{k,S}) = (C_0 \setminus D)(k) \cap \mathbb{A}^N(\mathcal{O}_{k,S})$ is infinite. We may further assume that $|S| \geq 2$. For every point $P \in C_0$, let C_P denote a member of Λ containing P . For all but finitely many $P \in C_0(\mathcal{O}_{k,S})$, we have $C_P \setminus D \cong \mathbb{A}^1$ or $C_P \setminus D \cong \mathbb{G}_m$ (over \bar{k}). Let P be such a point and assume further that P is not a base point

of Λ . We first note that C_P is defined over k . Indeed, since Λ has generators defined over k and C_P contains the k -rational point P , any conjugate of C_P over k would also be a member of Λ containing P , and since P is not a base point, it must coincide with C_P . Let $C'_P = C_P \setminus D$. Since $P \in C'_P(\mathcal{O}_{k,S})$ is a nonsingular point of C'_P , C'_P is rational and has at most two points at infinity, k has at least one complex archimedean place, and $|S| \geq 2$, it follows from [1, Theorem 1.1] that $C'_P(\mathcal{O}_{k,S}) = C'_P(k) \cap \mathbb{A}^N(\mathcal{O}_{k,S})$ is infinite. Since C_0 is not a component of any member of Λ , C_P varies as P varies, and it is clear that there exists a Zariski-dense set of (D, S) -integral points on \mathbb{P}^2 . \square

Finally, we recall a result of Beukers [5] on integral points on the complement of a nonsingular plane cubic.

Theorem 26 (Beukers). *Let C be a nonsingular projective cubic plane curve defined over a number field k . Then integral points on $\mathbb{P}^2 \setminus C$ are potentially dense.*

More precisely, Beukers [5, Th. 3.3] proves that S -integral points on $\mathbb{P}^2 \setminus C$ are Zariski dense if C has a k -rational flex F and the tangent line to C through F is not a component of C modulo any prime outside S . Since this condition is clearly satisfied for large enough k and S , we obtain the potential density of Theorem 26.

4. STRUCTURE THEOREMS FOR AFFINE SUBSETS OF \mathbb{P}^2

In this section, we recall various results classifying affine subsets of \mathbb{P}^2 via the logarithmic Kodaira dimension. Since we will use the Lang–Vojta conjecture (Conjecture 1) to handle surfaces of log general type ($\bar{\kappa} = 2$), we will only be interested in the remaining three possibilities for a surface: $\bar{\kappa} = -\infty, 0, 1$. Throughout this section, we work over the complex numbers (or more generally, an algebraically closed field of characteristic 0).

4.1. $\bar{\kappa} = -\infty$.

The following structure theorem of Miyanishi and Sugie [23, Theorem] (see also Kojima [22, Theorem 1.1 (i)]) will be the key ingredient in analyzing integral points in this case.

Theorem 27 (Miyanishi–Sugie, Kojima). *Let $V = \mathbb{P}^2 \setminus D$, where D is a reduced effective divisor on \mathbb{P}^2 . Suppose that $\bar{\kappa}(V) = -\infty$. Then there exists a pencil Λ on \mathbb{P}^2 such that:*

- (a) *Every member of Λ is irreducible (but possibly nonreduced).*
- (b) *The pencil Λ has at most two multiple members. If it has two distinct multiple members aF and bG , then $\gcd(a, b) = 1$.*
- (c) *The divisor D is a union of supports of members of Λ .*
- (d) *The pencil Λ has a unique base point P_0 , and for a general member $C \in \Lambda$, $C \setminus \{P_0\} \cong \mathbb{A}^1$.*
- (e) *Let D' be the union of D and the multiple members of Λ . Let r be the number of irreducible components of D' . Then*

$$\mathbb{P}^2 \setminus D' \cong \mathbb{P}^2 \setminus \{r \text{ lines through a single point } P\}.$$

4.2. $\bar{\kappa} = 0$.

A structure theorem of Kojima is the key ingredient in this case.

Theorem 28 (Kojima [22, Theorem 1.1 (ii)]). *Let $V = \mathbb{P}^2 \setminus D$, where D is a reduced effective divisor on \mathbb{P}^2 . If $\bar{\kappa}(V) = 0$, then either D is a nonsingular cubic curve or V contains an open subset isomorphic to $\mathbb{G}_m \times \mathbb{G}_m \cong \mathbb{P}^2 \setminus \{3 \text{ lines in general position}\}$.*

4.3. $\bar{\kappa} = 1$.

In this case, while we know from Kawamata [21] and Gurjar–Miyanishi [15] that there is a \mathbb{G}_m -fibration over \mathbb{P}^1 , there is not a complete classification of the affine surfaces $V \subset \mathbb{P}^2$ with $\bar{\kappa}(V) = 1$ that is sufficient for our purposes. Instead, we give a variety of classification results under various hypotheses. In each such case, in the next section we will prove results on integral points on the classified affine surfaces.

We begin by recalling a result of Wakabayashi which greatly restricts the possibilities of an irreducible plane curve C with $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$.

Theorem 29 (Wakabayashi [35]). *Let C be an irreducible curve in \mathbb{P}^2 and suppose that $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$. Then one of the following is true:*

- (a) C is a nonsingular cubic curve.
- (b) C is a rational curve with at most two singular points. If C has two singular points, then both singularities are cuspidal.

In particular, if $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$, then C is a rational curve with exactly one or two singular points. In the latter case, both singularities are cuspidal.

It is elementary that in case (a), $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 0$, and if C is a nonsingular rational curve, then $\bar{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$. Tsunoda [29] showed that if C has two cuspidal singularities, then $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$ or 2 , and that $\bar{\kappa}(\mathbb{P}^2 \setminus C) \neq 0$ if C is a rational curve with a single cuspidal point.

We now describe in more detail classifications of $V = \mathbb{P}^2 \setminus D$ with $\bar{\kappa}(V) = 1$ in the following cases:

- (a) D is a rational curve with two cusps and no other singularities.
- (b) D is a rational curve with a single cusp and no other singularities.
- (c) D is an irreducible curve with $\deg D \leq 5$ with exactly one singularity.
- (d) D is a union of a line and an irreducible curve.

Remark 30. From Theorem 29, we see that the cases not treated by (a)–(d) above are when D is irreducible of degree at least 6 having exactly one singularity which is not a cusp, or when D is reducible and either the number of components is at least 3 or none of the components is a line.

4.3.1. *Complement of a rational bicuspidal curve.* In this case, Tono [27, Theorem 4.1.2] has a complete classification up to projective equivalence.

Theorem 31 (Tono). *Let D be a rational bicuspidal curve such that $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$.*

Given $\vec{v} = (v_1, \dots, v_{n+1}) \in \mathbb{C}^{n+1}$, denote by $J_{\vec{v}}(x, y, z)$ the polynomial $x^n z + \sum_{j=1}^{n+1} v_j x^{n+1-j} y^j$. Then there exists a sequence D_0, \dots, D_s of curves satisfying:

- (i) D_0 is projectively equivalent to D .
- (ii) D_s is defined by $F_1 + F_2 = 0$, where F_1 and F_2 are one of the following three possibilities:

$$(10) \quad F_1 = y^{\alpha_1}, \quad F_2 = x^{\alpha_1 - \alpha_0} (z + ay)^{\alpha_0},$$

$$(11) \quad F_1 = (J_{\vec{a}}(x, y, z))^{\alpha_0}, \quad F_2 = x^{(n+1)\alpha_0 - \alpha_1} y^{\alpha_1}, \quad \alpha_1 < (n+1)\alpha_0,$$

$$(12) \quad F_1 = y^{\alpha_1}, \quad F_2 = x^{\alpha_1 - (n+1)\alpha_0} (J_{\vec{a}}(x, y, z))^{\alpha_0}, \quad (n+1)\alpha_0 < \alpha_1,$$

where $1 < \alpha_0 < \alpha_1$ with $\gcd(\alpha_0, \alpha_1) = 1$, $a \in \mathbb{C}$, and $\vec{a} \in \mathbb{C}^n \times \mathbb{C}^*$ with $n \geq 1$.

- (iii) For each $i = 1, \dots, s$, $D_{i-1} = \tau_{\vec{a}_i}^{-1}(D_i)$, where $\tau_{\vec{a}_i}$ is a De Jonquières transformation $(x, y, z) \mapsto (x^{m_i+1}, J_{\vec{a}_i}(x, y, z), x^{m_i}y)$ for $\vec{a}_i \in \mathbb{C}^{m_i} \times \mathbb{C}^*$ with $m_i \geq 1$.

Conversely, any curve D satisfying the above conditions defines a rational bicuspidal curve with $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$.

4.3.2. *Complement of a rational unicuspidal curve.* Here we quote a different result of Tono. In [28, Theorem 2], Tono classifies, into three different cases, unicuspidal plane curves whose complements have log Kodaira dimension 1.

Theorem 32 (Tono). *Let C be a rational unicuspidal curve with $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$. Then one of the following holds:*

- (i) *There exist $n, s \geq 2$ and $a_2, \dots, a_s \in \mathbb{C}$ with $a_s \neq 0$ such that C is projectively equivalent to the curve*

$$\left(\left(f^{s-1}y + \sum_{i=2}^s a_i f^{s-i} x^{(n+1)i-n} \right)^{\mu_A} - f^{\mu_G} \right) / x^n = 0,$$

where $f = x^n z + y^{n+1}$, $\mu_A = n + 1$, and $\mu_G = (n + 1)(s - 1) + 1$.

- (ii) *There exists $n \geq 2$ such that the curve C is projectively equivalent to the curve*

$$\left((g^n y + x^{2n+1})^{\mu_A} - (g^{2n} z + 2x^{2n} y g^n + x^{4n+1})^{\mu_G} \right) / g^n = 0,$$

where $g = xz - y^2$, $\mu_A = 4n + 1$, and $\mu_G = 2n + 1$.

- (iii) *There exist a positive integer $n \geq 2$, a positive integer s , and $a_1, \dots, a_s \in \mathbb{C}$ with $a_s \neq 0$ such that C is projectively equivalent to the curve*

$$\left(\left(h^{2s-1} (g^n y + x^{2n+1}) + \sum_{i=1}^s a_i h^{2(s-i)} g^{mi-n} \right)^{\mu_A} - h^{\mu_G} \right) / g^n = 0,$$

where $m = \mu_A = 4n + 1$, $g = xz - y^2$, $h = g^{2n} z + 2x^{2n} y g^n + x^m$, and $\mu_G = 2((4n + 1)s - n)$.

Conversely, any curve C defined in (i)–(iii) is a rational unicuspidal curve satisfying $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$.

4.3.3. *Complement of a rational curve of degree ≤ 5 with a unique singularity.* From Wakabayashi’s theorem (Theorem 29), when D is irreducible and $\bar{\kappa} = 1$, D has to be a rational curve having two cusps or having one singular point. The former case was already treated in Section 4.3.1. The latter case with $\deg D \leq 5$ has been classified by Yoshihara [38].

Theorem 33. *Let C be a rational plane curve with a unique singularity, let $\deg C \leq 5$, and let $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$. Let e be the multiplicity at the singular point, and let N be the number of points in the normalization above the singular point. Then $(e, N) \in \{(3, 3), (4, 3), (4, 4)\}$.*

4.3.4. *Complement of a line and a curve.* Suppose now that D is the union of a line and an irreducible curve. After an automorphism of \mathbb{P}^2 , we may assume that the line is given by $Z = 0$. Let $f(x, y)$ be the (dehomogenized) irreducible polynomial defining the component of D that is not the line ($Z = 0$). In this case, Aoki [4, Theorem 3.7, Lemmas 3.8–3.12] has determined $f(x, y)$ for which $\bar{\kappa}(\mathbb{A}^2 \setminus (f(x, y) = 0)) = 1$, up to changes of coordinates in x, y .

Theorem 34 (Aoki). *Let D be the union of the line L defined by $Z = 0$ and an irreducible curve C defined by a homogeneous polynomial $\tilde{f}(X, Y, Z)$. Let $f(x, y) = \tilde{f}(x, y, 1)$. Suppose that $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$. Then after a suitable change of coordinates on $\mathbb{A}^2 = \mathbb{P}^2 \setminus L$, $f(x, y)$ is one of the following:*

- (i) $f(x, y) = x^a y^b + 1$, where $\gcd(a, b) = 1$ and $a, b > 1$.
- (ii) $f(x, y) = x^a (x^l y + p(x))^b + 1$, where $a > 0, b > 1, l > 0, \gcd(a, b) = 1, \deg p(x) < l$, and $p(0) \neq 0$.
- (iii) $f(x, y) = a_0(x)y + a_1(x)$, where $a_0(x)$ and $a_1(x)$ have no common factors, $\deg a_1 < \deg a_0$, and $a_0(x)$ has at least two distinct roots over \mathbb{C} .
- (iv) $f(x, y) = x^a - y^b$, where $a, b > 1$ and $\gcd(a, b) = 1$.

Conversely, the complement in \mathbb{A}^2 of any of the curves defined in (i)–(iv) above satisfies $\bar{\kappa} = 1$.

5. INTEGRAL POINTS ON \mathbb{P}^2

In this section we prove results on integral points corresponding to the geometric classification results of the last section.

5.1. $\bar{\kappa} = -\infty$.

Using Theorems 23 and 27, we completely classify integral points in the case $\bar{\kappa}(\mathbb{P}^2 \setminus D) = -\infty$, proving Theorem 2 from the introduction.

Proof of Theorem 2. We may assume that the divisor D is reduced. We first give a proof ignoring the claim of effectivity. Let r and D' be as in Theorem 27. When $r \leq 2$, by Theorem 27, $\mathbb{P}^2 \setminus D'$ is isomorphic to $\mathbb{P}^2 \setminus \{\text{one line}\} \cong \mathbb{A}^2$ or $\mathbb{P}^2 \setminus \{\text{two lines}\} \cong \mathbb{G}_m \times \mathbb{A}^1$. In both cases, it is evident that D' -integral points are potentially dense, hence so are D -integral points.

Now suppose that $r \geq 3$. Since every member of Λ is irreducible, we have the following three possibilities:

- (a) D has one irreducible component, and there are two distinct multiple members aF and bG of Λ , with F and G distinct from D .

- (b) D has two irreducible components, and there is at least one multiple member aF of Λ such that $F \not\subset \text{Supp } D$.
- (c) D has at least three irreducible components.

Recall that in the first case, $\gcd(a, b) = 1$ by Theorem 27(b).

The gcd weight of (D, Λ) is at least

$$1 + (1 - 1/2) + (1 - 1/3) > 2$$

in case (a), at least

$$1 + 1 + (1 - 1/2) > 2$$

in case (b), and at least

$$1 + 1 + 1 > 2$$

in case (c). Thus, in all cases, the gcd weight of (D, Λ) is greater than 2. Since D is a union of supports of members of Λ , by Theorem 23(c), $(\mathbb{P}^2 \setminus D)(\mathcal{O}_{k,S})$ is contained in the union of finitely many curves (in fact, the union of finitely many members of Λ).

We now prove the claim about effectivity. In view of the effective nature of Theorem 23(c), it suffices to show that one can effectively determine whether or not $r \geq 3$, and (possibly of independent interest) if so effectively compute the Miyanishi–Sugie pencil Λ and its multiple members. We will obtain as a by-product of the proof that Λ is unique when $r \geq 3$.

Let f be a homogeneous polynomial defining D . By factoring f (over \bar{k}), we can determine the number of irreducible components of D . Suppose that D has at least three irreducible components (in which case $r \geq 3$). It follows from Theorem 27 that any irreducible component of D of maximal degree must be a member of Λ , and for any other component of D , some appropriate multiple of it is a member of Λ . Thus, we obtain explicit generators for Λ (and Λ is uniquely determined by D). In this case, we do not need to compute any other multiple members (if any) of Λ to effectively apply Theorem 23c, but we can effectively compute them by solving a suitable associated system of equations (e.g., using Gröbner bases).

Suppose next that D has exactly two irreducible components. Then Λ is generated by a component of D of maximal degree and a suitable multiple of the remaining component. Using Gröbner bases, we can again effectively solve for other multiple members of Λ in \bar{k} . We can find a solution if and only if $r \geq 3$.

Suppose now that D has exactly one irreducible component. We first claim that one can effectively determine if there exists a pencil Λ' containing D as a member, and containing multiple members aF and bG with $\gcd(a, b) = 1$. Indeed, if $a, b > 1$ are coprime divisors of $d = \deg D$, then we consider the equation

$$f(x, y, z) = \left(\sum_{\substack{i,j,k \\ i+j+k=\frac{d}{a}}} a_{i,j,k} x^i y^j z^k \right)^a + \left(\sum_{\substack{i,j,k \\ i+j+k=\frac{d}{b}}} b_{i,j,k} x^i y^j z^k \right)^b$$

in the indeterminates $a_{i,j,k}, b_{i,j,k}$. Equating monomial coefficients yields a system of equations in $a_{i,j,k}$ and $b_{i,j,k}$, and we can (in principle) determine if the system

of equations has a solution in \bar{k} using Gröbner bases. Running over all possible integers a and b , we can thus determine if there exists a pencil Λ' as above. If no such Λ' exists, then obviously $r \leq 2$.

Otherwise, by taking some (effectively computable) solution to the system of equations, we obtain an (effectively computable) pencil Λ' containing D as a member, and containing (effectively computable) multiple members aF and bG with $\gcd(a, b) = 1$. As before, the gcd weight of (D, Λ') is greater than 2, and so integral points on V are not potentially dense. Then we must have $r \geq 3$ (for the Miyanishi–Sugie pencil Λ) and be in case (a) above. Let $C \in \Lambda$ be a general member of Λ . Then $C \setminus D \cong \mathbb{A}^1$, and it follows that C will contain an infinite number of integral points in $V(\mathcal{O}_{L,S})$ for some number field L and some finite set of places S of L . But then by Theorem 23(c) (applied with Λ'), C must be contained in the support of a member of Λ' . Since the general member of Λ' is reduced and irreducible (as the member D is), it follows that a general member $C \in \Lambda$ must be a member of Λ' . Then we have $\Lambda' = \Lambda$, Λ is effectively computable, and Λ is unique. \square

5.2. $\bar{\kappa} = 0$.

We now prove Theorem 3, showing potential density of integral points when $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 0$.

Proof of Theorem 3. By Theorem 28, D is a nonsingular cubic curve or $\mathbb{P}^2 \setminus D$ contains an open subset isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$. If D is a nonsingular cubic, then the result follows from Theorem 26. Otherwise, potential density of (D, S) -integral points follows from potential density of integral points on $\mathbb{G}_m \times \mathbb{G}_m$. \square

5.3. $\bar{\kappa} = 1$.

In this section, we analyze the arithmetic of the surfaces that were classified in Section 4.3. As mentioned previously, this does not cover all the possible surfaces $V \subset \mathbb{P}^2$ of interest with $\bar{\kappa}(V) = 1$. In contrast to previous sections, we are also only able to give a partial analysis of the arithmetic of the surfaces in Section 4.3. We give several examples in Section 7 which go beyond the results of this section.

The general strategy of our analysis is as follows. Using the explicit equations of Section 4.3, we construct a corresponding pencil Λ of curves on \mathbb{P}^2 . Most of the time, the Campana weight of (D, Λ) is greater than 2, and so Theorem 23 gives us Zariski nondensity of integral points. For the rest, we attempt to explicitly construct a Zariski-dense set of integral points. Since there is always a \mathbb{G}_m -fibration to \mathbb{P}^1 for such surfaces, this amounts to constructing a horizontal curve C_0 as in Lemma 25.

5.3.1. *Complement of a rational bicuspidal curve.* We begin by analyzing integral points on the complement of a rational bicuspidal curve.

Theorem 35. *Suppose that D is a rational curve defined over $\bar{\mathbb{Q}}$ having exactly two cusps and no other singularities and that $V = \mathbb{P}^2 \setminus D$ satisfies $\bar{\kappa}(V) = 1$.*

- (i) Assume the abc conjecture. If integral points on V are potentially dense, then D is projectively equivalent to one of the following:

$$\left[X^{2n+1}Y + \sum_{j=1}^{n+1} b_j X^{2(n+1-j)}(XZ + cY^2)^j \right]^2 + X^2(XZ + cY^2)^{2n+1} = 0,$$

$$Y^{\alpha+1} + X(Z + aY)^\alpha = 0,$$

$$XY^{(n+1)\alpha-1} + \left(X^n Z + \sum_{j=1}^{n+1} a_j X^{n+1-j} Y^j \right)^\alpha = 0,$$

$$Y^{(n+1)\alpha+1} + X \left(X^n Z + \sum_{j=1}^{n+1} a_j X^{n+1-j} Y^j \right)^\alpha = 0,$$

where $n \geq 1$, $\alpha \geq 2$, $a, a_j, b_j \in \overline{\mathbb{Q}}$, and $a_{n+1}, b_{n+1}, c \in (\overline{\mathbb{Q}})^*$.

- (ii) Suppose that D is defined by one of the following:

$$Y^{\alpha+1} + X(Z + aY)^\alpha = 0,$$

$$XY^{(n+1)\alpha-1} + (X^n Z + a_1 X^n Y + a_{n+1} Y^{n+1})^\alpha = 0,$$

$$Y^{(n+1)\alpha+1} + X(X^n Z + a_1 X^n Y + a_{n+1} Y^{n+1})^\alpha = 0,$$

where $n \geq 1$, $\alpha \geq 2$, $a, a_1 \in \overline{\mathbb{Q}}$, and $a_{n+1} \in (\overline{\mathbb{Q}})^*$. Then integral points on V are potentially dense.

Proof. We first prove part (i). Using the notation of Theorem 31, let $\tau = \tau_{\overline{a}_s} \circ \dots \circ \tau_{\overline{a}_1}$ when $s \geq 1$ and $\tau = \text{id}$ when $s = 0$. Let Λ be the pencil generated by $F_1(\tau(x, y, z))$ and $F_2(\tau(x, y, z))$, where F_i are chosen to be one of the three possibilities stated in Theorem 31(ii). By construction, the base points of Λ are on D . Suppose at first that $s \geq 1$, i.e., some De Jonquières transformation is necessary. Since the x -coordinate of $\tau(x, y, z)$ is a pure power of x with power at least 2 and since $\alpha_1 > \alpha_0 > 1$, straightforward consideration of cases gives that the Campana weight of (D, Λ) is at least $1 + (1 - \frac{1}{2}) + (1 - \frac{1}{3}) > 2$, unless $(s, \alpha_0, \alpha_1, n, m_1) = (1, 2, 2n + 1, n, 1)$ and D_1 is defined by (11). So Theorem 23 shows Zariski nondensity of integral points except for this case. In this special case, the Campana weight is exactly 2. Let $\tau = (x^2, xz + a_1xy + a_2y^2, xy)$, and let D_1 be defined by $(x^n z + \sum_{j=1}^{n+1} b_j x^{n+1-j} y^j)^2 + xy^{2n+1}$, where $a_2, b_{n+1} \in (\overline{\mathbb{Q}})^*$. Then D is defined by

$$\left[x^{2n+1}y + \sum_{j=1}^{n+1} b_j x^{2(n+1-j)}(xz + a_1xy + a_2y^2)^j \right]^2 + x^2(xz + a_1xy + a_2y^2)^{2n+1} = 0.$$

By replacing $z + a_1y$ by z , the equation simplifies to

$$\left[x^{2n+1}y + \sum_{j=1}^{n+1} b_j x^{2(n+1-j)}(xz + a_2y^2)^j \right]^2 + x^2(xz + a_2y^2)^{2n+1},$$

yielding the first equation in (i).

We are now left with the case $s = 0$. For (10) and (12), when the power of x in F_2 is at least 2, as $\alpha_1 \geq 3$, the Campana weight is greater than 2 and we may

apply Theorem 23. For (11), when $\alpha_0 = 2$, the gcd condition forces the power of x in F_2 to be odd, so the minimum power in F_2 cannot be 2. Therefore, in the case (11), the Campana weight is greater than 2 unless, again, the power of x in F_2 is exactly equal to 1. Thus, the cases where the exponent of x in F_2 is equal to 1 yield the other three possibilities in (i). Note that in each of these cases the Campana weight of (D, Λ) is less than 2.

We now prove (ii). For the first case, we can use Lemma 25 with the line defined by $z = by$ for some $b \neq -a$. For the last two cases, we can use Lemma 25 with the line defined by $z = -a_1y$. □

Remark 36. Note that we need to use the Campana weight, rather than the gcd weight, to eliminate many cases in the proof of Theorem 35(i). Since the member of Λ corresponding to F_2 in (10)–(12) has at least two components, the gcd multiplicity of this member may be significantly smaller than the Campana multiplicity, resulting in many cases where the the Campana weight is greater than two, but the gcd weight is not.

5.3.2. *Complement of a rational unicuspidal curve.* In contrast to the previous case, we prove an unconditional result for rational unicuspidal curves.

Theorem 37. *Suppose that D is a rational curve defined over $\overline{\mathbb{Q}}$ having exactly one cusp and no other singularities and that $V = \mathbb{P}^2 \setminus D$ satisfies $\bar{\kappa}(V) = 1$. If integral points on V are potentially dense, then D is projectively equivalent to*

$$\frac{((X^2Z + Y^3)Y + aX^4)^3 - (X^2Z + Y^3)^4}{X^2} = 0$$

for some $a \in \overline{\mathbb{Q}}^*$.

Proof. We use Theorem 32. We begin with the first case of that theorem. Let D be defined by

$$\left(\left(f^{s-1}y + \sum_{i=2}^s a_i f^{s-i} x^{(n+1)i-n} \right)^{\mu_A} - f^{\mu_G} \right) / x^n = 0,$$

where $f = x^n z + y^{n+1}$, $\mu_A = n + 1$, $\mu_G = (n + 1)(s - 1) + 1$, and $n, s \geq 2$. Let Λ be the pencil generated by $(f^{s-1}y + \sum_{i=2}^s a_i f^{s-i} x^{(n+1)i-n})^{\mu_A}$ and f^{μ_G} . Since the divisor D occurs in the same member of Λ as $x^n = 0$, this member contributes a gcd multiplicity of n . So the gcd weight of (D, Λ) is at least $(1 - \frac{1}{\mu_A}) + (1 - \frac{1}{\mu_G}) + (1 - \frac{1}{n})$. When $n \geq 3$, we have $\mu_A \geq 4$ and $\mu_G \geq 5$, so the gcd weight of (D, Λ) is at least $\frac{3}{4} + \frac{4}{5} + \frac{2}{3} > 2$. In addition, if $n = 2$ and $s \geq 3$, then $\mu_A = 3$ and $\mu_G \geq 7$, so the gcd weight of (D, Λ) is at least $\frac{2}{3} + \frac{6}{7} + \frac{1}{2} = \frac{85}{42} > 2$. Therefore, the only situation where (D, Λ) possibly has gcd weight ≤ 2 is when $n = s = 2$. In this situation, the divisor is projectively equivalent to

$$\frac{((x^2z + y^3)y + ax^4)^3 - (x^2z + y^3)^4}{x^2} = 0$$

for some $a \in (\overline{\mathbb{Q}})^*$. Since every member other than the two multiple fibers and the reducible fiber containing D is irreducible and reduced [28, Theorem 1], the gcd weight (as well as the Campana weight) of this (D, Λ) is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12} < 2$.

We now work with the second and third cases of Theorem 32. For the second case, let D be defined by

$$((g^n y + x^{2n+1})^{\mu_A} - (g^{2n} z + 2x^{2n} y g^n + x^{4n+1})^{\mu_G})/g^n = 0,$$

where $g = xz - y^2$, $\mu_A = 4n + 1 \geq 9$, and $\mu_G = 2n + 1 \geq 5$. Let Λ be the pencil generated by $(g^n y + x^{2n+1})^{\mu_A}$ and $(g^{2n} z + 2x^{2n} y g^n + x^{4n+1})^{\mu_G}$. Since the divisor D occurs in the same member of Λ as $g^n = 0$, this member has gcd multiplicity n . Therefore, the gcd weight of (D, Λ) is at least $\frac{8}{9} + \frac{4}{5} + \frac{1}{2} > 2$. In the third case, $\mu_A = 4n + 1 \geq 9$ and $\mu_G = 2((4n + 1)s - n) \geq 2(3n + 1) \geq 14$ as $s \geq 1$. By the same argument (with the natural choice of Λ), the gcd weight of (D, Λ) is at least $\frac{8}{9} + \frac{13}{14} + \frac{1}{2} > 2$. Applying Theorem 23(b) finishes the proof. \square

5.3.3. *Complement of a rational curve of degree ≤ 5 with a unique singularity.* Under certain conditions on the singularities, we prove potential density of integral points on the complements of singular quartic and quintic curves.

Theorem 38. *Suppose that D is a quartic plane curve defined over $\overline{\mathbb{Q}}$ with a triple point, i.e., a singularity of multiplicity three. Then integral points on $\mathbb{P}^2 \setminus D$ are potentially dense.*

Proof. Let Λ be the pencil of lines in \mathbb{P}^2 passing through the singular point of D . Then for a general member $C \in \Lambda$, $C \setminus D \cong \mathbb{G}_m$ over $\overline{\mathbb{Q}}$. By [16, Th. 3.5], D has four bitangent lines whose eight points of contact lie on a conic. Then taking C_0 to be one of the bitangents and applying Lemma 25 to Λ and C_0 , we conclude that integral points on $\mathbb{P}^2 \setminus D$ are potentially dense. \square

Theorem 39. *Suppose that D is a quintic rational plane curve defined over $\overline{\mathbb{Q}}$ with a unique singular point, which has multiplicity 4 with 3 points above it in the normalization. Then integral points on $\mathbb{P}^2 \setminus D$ are potentially dense.*

Proof. We may assume that the singular point is $[0 : 0 : 1]$. By the genus-multiplicity formula (cf. [17, Example V.3.9.2]), the singularity of D is resolved after one blowup. Since the preimage of the singular point in the normalization consists of 3 points, the defining equation of D is

$$F(X, Y, Z) = F_5(X, Y) + L_1(X, Y)L_2(X, Y)L_3(X, Y)^2Z,$$

where F_5 is a homogeneous polynomial of degree 5 in two variables and L_1, L_2, L_3 are distinct linear polynomials. Without loss of generality, we may assume that $L_1 = X$ and $L_2 = Y$. Then for a suitable choice of a and b in $\overline{\mathbb{Q}}$, $F_5(X, Y) - (aX + bY)^5$ is divisible by XY . We let G be the cubic polynomial

$$G(X, Y, Z) = \frac{F(X, Y, Z) - (aX + bY)^5}{XY} = \frac{F_5(X, Y) - (aX + bY)^5}{XY} + L_3(X, Y)^2Z.$$

The curve C defined by $G = 0$ has a cusp, and $D \cap C = \{aX + bY = 0\} \cap C$, so it follows that $C \setminus D$ is isomorphic to \mathbb{G}_m . We also note that a general line through $[0 : 0 : 1]$ meets D in two points, and so we conclude from Lemma 25 that integral points on $\mathbb{P}^2 \setminus D$ are potentially dense. \square

5.3.4. *Complement of a line and a curve.* Under the *abc* conjecture, we partially classify integral points on the complement of a line and a plane curve when the complement has logarithmic Kodaira dimension one.

Theorem 40. *Suppose that D is a union of a line and an irreducible curve defined over $\overline{\mathbb{Q}}$ and let $V = \mathbb{P}^2 \setminus D$. Assume the abc conjecture. If integral points on V are potentially dense and $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$, then $V \cong \mathbb{P}^2 \setminus D'$, where D' is defined by one of the following two equations:*

- (i) $Z(Z^{1+b(l+1)} + X(X^l Y + p(X, Z)Z^{l+1-\deg p})^b) = 0$, where $b > 1$, $l > 0$, $\deg p < l$, and $p(0, 1) \neq 0$.
- (ii) $Z(a_0(X, Z)Y + a_1(X, Z)Z^{1+\deg a_0-\deg a_1}) = 0$, where a_0 and a_1 do not have a common factor, $\deg a_1 < \deg a_0$, and a_0 has at least two distinct factors.

Proof. Let L be the line defined by $Z = 0$ and identify $\mathbb{A}^2 = \mathbb{P}^2 \setminus L$. If D is a sum of L and a curve C and $\phi : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ is an automorphism, then integral points on $\mathbb{P}^2 \setminus D = \mathbb{A}^2 \setminus C$ will be transformed via ϕ to integral points on $\mathbb{A}^2 \setminus \phi(C \cap \mathbb{A}^2)$. Thus, we may reduce to analyzing the four cases classified by Aoki in Theorem 34. Note that since $Z = 0$ is a part of the divisor, any multiplicity of Z is ignored in computing the gcd/Campana weights.

Suppose first that $f(x, y) = x^a y^b + 1$, where $\gcd(a, b) = 1$ and $a > 1$ and $b > 1$. Letting Λ be the pencil defined by $X^a Y^b$ and Z^{a+b} , the Campana weight of (D, Λ) is at least $\left(1 - \frac{1}{\min(a,b)}\right) + 1 + 1 > 2$. So according to Theorem 23 (assuming the abc conjecture), integral points are always Zariski nondense.

Suppose now that $f(x, y) = x^a(x^l y + p(x))^b + 1$, where $a > 0$, $b > 1$, $l > 0$, $\gcd(a, b) = 1$, $\deg p(x) < l$, and $p(0) \neq 0$. Letting Λ be the pencil generated by $Z^{a+b(l+1)}$ and the homogenization of $x^a(x^l y + p(x))^b$, the Campana weight of (D, Λ) is at least $\left(1 - \frac{1}{\min(a,b)}\right) + 1 + 1$. This is greater than 2 if and only if $a > 1$. Then by Theorem 23, integral points are Zariski nondense unless $a = 1$. The case $a = 1$ yields the first possibility in the theorem.

The second possibility in the theorem is the third case in Theorem 34, to which we have nothing to add.

Finally, suppose that $f(x, y) = x^a - y^b$ with $a > 1$, $b > 1$, and $\gcd(a, b) = 1$. Without loss of generality, let us assume that $a < b$, and let Λ be the pencil generated by $X^a Z^{b-a}$ and Y^b . Then the gcd/Campana weight of (D, Λ) is at least

$$\left(1 - \frac{1}{a}\right) + \left(1 - \frac{1}{b}\right) + 1 \geq \frac{1}{2} + \frac{2}{3} + 1 > 2,$$

since $\gcd(a, b) = 1$ and $a, b > 1$. Then by Theorem 23, integral points are Zariski nondense (without assuming the abc conjecture). □

Remark 41. Here we list the cases of $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$ for which we have been unable to classify integral points (even under the abc conjecture). As in Remark 30, a part of the problem is the lack of a classification theory in certain cases. As we will see in Lemma 43, integral points on $\mathbb{P}^2 \setminus D$ are never Zariski dense if D has at least four components. Thus, the unresolved cases due to a lack of an appropriate classification are when D is irreducible of degree at least 6 having exactly one singularity which is not a cusp, or when D is reducible and either the number of components is exactly 3 or none of the components is a line.

For the rest of the unresolved cases, we have been unable to construct a Zariski-dense set of integral points (or prove nondensity), even though at least some classification results are known:

- (i) the bicuspidal case which is projectively equivalent to one of

$$\left[X^{2n+1}Y + \sum_{j=1}^{n+1} b_j X^{2(n+1-j)}(XZ + cY^2)^j \right]^2 + X^2(XZ + cY^2)^{2n+1} = 0,$$

$$XY^{(n+1)\alpha-1} + \left(X^n Z + \sum_{j=1}^{n+1} a_j X^{n+1-j} Y^j \right)^\alpha = 0,$$

$$Y^{(n+1)\alpha+1} + X \left(X^n Z + \sum_{j=1}^{n+1} a_j X^{n+1-j} Y^j \right)^\alpha = 0,$$

where $n \geq 1$, $\alpha \geq 2$, $a_j, b_j \in \overline{\mathbb{Q}}$, at least one of a_2, \dots, a_n is nonzero, and $a_{n+1}, b_{n+1}, c \in (\overline{\mathbb{Q}})^*$.

- (ii) the unicuspidal case which is projectively equivalent to

$$\frac{((X^2Z + Y^3)Y + aX^4)^3 - (X^2Z + Y^3)^4}{X^2} = 0$$

for some $a \in \overline{\mathbb{Q}}^*$.

- (iii) a rational plane curve of degree 5 with a unique singularity which has multiplicity e and e points above it in the normalization, $e = 3, 4$.
- (iv) a line and a geometrically irreducible curve which is projectively equivalent to either

$$Z(Z^{1+b(l+1)} + X(X^l Y + p(X, Z)Z^{l+1-\deg p})^b) = 0,$$

$$Z(a_0(X, Z)Y + a_1(X, Z)Z^{1+\deg a_0-\deg a_1}) = 0,$$

where $b > 1$, $l > 0$, $\deg p < l$, and $p(0, 1) \neq 0$ for the first type, and where a_0 and a_1 do not have a common factor, $\deg a_1 < \deg a_0$, and a_0 has at least two distinct factors for the second type.

The examples in Section 7 will show that there are some special examples of the cases listed above for which we can still conclude potential density of integral points.

6. INTEGRAL POINTS IN ORBITS

To prove Theorem 9, we will actually prove the following refinement, which describes in more detail the situation when integral points in an orbit are Zariski dense in \mathbb{P}^2 .

Theorem 42. *Let $\phi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a morphism of degree $d > 1$ defined over a number field k . Let D be a nontrivial effective divisor on \mathbb{P}^2 defined over k . Let S be a finite set of places of k containing the archimedean places. Let r be the (possibly infinite) quantity*

$$r = \sup\{\# \text{ of distinct irreducible components of } \text{Supp}(\phi^n)^*D \text{ over } \bar{k} \mid n \in \mathbb{N}\}.$$

- (i) *Suppose that there exists $P \in \mathbb{P}^2(k)$ such that $\mathcal{O}_\phi(P)$ contains a Zariski-dense set of (D, S) -integral points on \mathbb{P}^2 . Then $r \leq 3$.*

- (ii) *In the situation of (i), fix n such that $\text{Supp}(\phi^n)^*D$ has r irreducible components over \bar{k} , and let C be an irreducible component of $\text{Supp}(\phi^n)^*D$ over \bar{k} . Then for $i \in \mathbb{N}$, $C_i = \phi^{-i}(C)$ is a geometrically irreducible curve, and assuming the Lang-Vojta conjecture (Conjecture 1) for \mathbb{P}^2 , one of the following two conditions is satisfied:*
 - (a) *C is a line and $\phi^{-m}(C) = C$ (as a set) for some positive integer m .*
 - (b) *each C_i is a rational curve, and there exist a positive integer i_0 and a sequence of points $P_{i_0}, P_{i_0+1}, P_{i_0+2}, \dots$ such that for all $i \geq i_0$, $\deg C_{i+1} = d \deg C_i$, P_i is a singular point of C_i , and $\phi^{-1}(P_i) = \{P_{i+1}\}$ (as a set). Further, the set $\{P_i : i \geq i_0\}$ is a finite set.*

We note that as an immediate corollary, if r as above is at least 4, a set of (D, S) -integral points in any orbit is Zariski nondense in \mathbb{P}^2 .

If case (a) of (ii) holds for every irreducible component of $\text{Supp}(\phi^n)^*D$, then after replacing k by a finite extension, D consists of r distinct lines L_1, \dots, L_r , $1 \leq r \leq 3$, over k , and there is an integer m such that $\phi^{-m}(L_i) = L_i$ for $i = 1, \dots, r$. In fact, since $\bigcup_{i=1}^r \bigcup_{j=1}^m \phi^{-j}(L_i)$ is a completely invariant set, it is clear that we can take $m \in \{1, 2, 3\}$ by Lemma 46 below. If $r = 3$, then the lines L_1, L_2 , and L_3 must be in general position (otherwise, one can project from $\mathbb{P}^2 \setminus D$ to $\mathbb{P}^1 \setminus \{\text{three points}\}$ and apply Siegel’s theorem to deduce that integral points on $\mathbb{P}^2 \setminus D$ are not Zariski dense). Then after possibly replacing k by a finite extension again and up to an automorphism of \mathbb{P}^2 over k , ϕ^6 and D have one of the following forms:

- (1) D is the line $z = 0$ and

$$\phi^6 = [F(X, Y, Z) : G(X, Y, Z) : Z^{d^6}]$$

for some homogeneous polynomials $F(X, Y, Z), G(X, Y, Z) \in k[X, Y, Z]$ of degree d^6 .

- (2) D is defined by $yz = 0$ and

$$\phi^6 = [F(X, Y, Z) : Y^{d^6} : Z^{d^6}]$$

for some homogeneous polynomial $F(X, Y, Z) \in k[X, Y, Z]$ of degree d^6 .

- (3) D is defined by $xyz = 0$ and

$$\phi^6 = [X^{d^6} : Y^{d^6} : Z^{d^6}].$$

In case (3), each point in the orbit of $[3 : 2 : 1]$ under ϕ^6 is (D, S) -integral for $S = \{\infty, 2, 3\}$, and it is easy to see from the 2-adic and the 3-adic valuations that an algebraic curve can only contain finitely many points in this orbit. For cases (1) and (2), it follows from a recent work of Xie [36, Th. 1.1] that there exists an algebraic point P whose orbit under ϕ^6 is Zariski dense. In order to apply Xie’s theorem we only need to verify the hypothesis that there is no nonconstant rational function g on \mathbb{P}^2 such that $g \circ \phi^6 = g$. However, since ϕ^6 is an endomorphism of \mathbb{P}^2 of degree at least 2 (Xie considers more general maps), this is immediate, as otherwise each member of the pencil associated to g contains a curve in the ramification locus of ϕ^6 , yielding a contradiction. We can also invoke [2, Corollary 2.7] instead to obtain a Zariski-dense orbit if there exists $\ell \geq 1$ such that the two eigenvalues of the tangent map at a fixed point of $\phi^{6\ell}$ are multiplicatively independent. By enlarging k and S so that all coordinates of the point P are S -units and such that all the coefficients of F and G lie in $\mathcal{O}_{k,S}$, we see that each point in the orbit of P is (D, S) -integral. Therefore, if case (b) of Theorem 42 never occurs, we have proved,

under the Lang–Vojta conjecture, a full generalization of Silverman’s theorem to \mathbb{P}^2 , completely characterizing endomorphisms of \mathbb{P}^2 and divisors D for which there exists an orbit of ϕ with a Zariski-dense set of (D, S) -integral points.

We also note that a general endomorphism does not have any totally ramified points. Even for an endomorphism with a totally ramified point, a general divisor D will not satisfy (a) or (b).

We will first discuss several lemmas which will be used in the proof. We begin by recalling a special case of a result of Vojta [31, Corollary 2.4.3].

Lemma 43. *Let S be a finite set of places of a number field k . Let D be an effective divisor on \mathbb{P}^n defined over k . If D has at least $n + 2$ irreducible components (over \bar{k}), then any set of (D, S) -integral points is not Zariski dense in \mathbb{P}^n .*

More generally, using a reduction to unit equations, Vojta [31, Theorem 2.4.1] proved a result for an arbitrary nonsingular projective variety X , replacing the quantity $n + 2$ in the lemma by $\dim X + \rho + r + 1$, where ρ is the Picard number of X and r is the rank of $\text{Pic}^0(X)(k)$. Using results on integral points on closed subvarieties of semiabelian varieties, this quantity was subsequently improved by Vojta [32, Corollary 0.3] to $\dim X - h^1(X, \mathcal{O}_X) + \rho + 1$.

The next lemma is an immediate corollary of known results for the logarithmic Kodaira dimension of the complement of an irreducible plane curve.

Lemma 44. *Assume the Lang–Vojta conjecture for open subvarieties of \mathbb{P}^2 . Let $D = C$ be a geometrically irreducible curve in \mathbb{P}^2 defined over a number field k , and let S be a finite set of places of k containing the archimedean places. If there exists a Zariski-dense set of (D, S) -integral points, then either C is a nonsingular cubic curve or C is a rational curve with at most two singular points.*

Proof. By the Lang–Vojta conjecture, integral points are never Zariski dense when $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$. Therefore, the lemma follows from the result of Wakabayashi (Theorem 29). □

Lemma 45. *Let ϕ be an endomorphism of a projective variety X , both defined over a number field k . Let D be a nontrivial effective divisor on X defined over k . Let S be a finite set of places of k containing the archimedean places. Suppose that $P \in X(k)$ and $\mathcal{O}_\phi(P)$ contains a Zariski-dense set of (D, S) -integral points on X . Then for any positive integer n , $\mathcal{O}_\phi(P)$ contains a Zariski-dense set of $((\phi^n)^*D, S)$ -integral points on X .*

Proof. By functoriality of local and global height functions with respect to pullbacks by morphisms, we have

$$\begin{aligned} \sum_{v \in M_k \setminus S} \lambda_v((\phi^n)^*D, Q) &= h((\phi^n)^*D, Q) - \sum_{v \in S} \lambda_v((\phi^n)^*D, Q) + O(1) \\ &= h(D, \phi^n(Q)) - \sum_{v \in S} \lambda_v(D, \phi^n(Q)) + O(1) \\ &= \sum_{v \in M_k \setminus S} \lambda_v(D, \phi^n(Q)) + O(1) \end{aligned}$$

for all $Q \in X(k)$. Let R be a set of (D, S) -integral points in $\mathcal{O}_\phi(P)$. Then by the

above calculation, the set

$$R_n = \{Q \in \mathcal{O}_\phi(P) \mid \phi^n(Q) \in R\}$$

is a set of $((\phi^n)^*D, S)$ -integral points in $\mathcal{O}_\phi(P)$, and the result follows. □

The following is a theorem in holomorphic dynamics on $\mathbb{C}\mathbb{P}^2$.

Lemma 46. *Let $\phi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a morphism (defined over \mathbb{C}). If $C \subset \mathbb{P}^2$ is a finite union of algebraic curves such that $\phi^{-1}(C) = C$ as sets, then C is a union of three or fewer lines.*

Proof. Fornæss and Sibony [14, Proposition 4.2] show that such a curve must have degree ≤ 3 , and then remove all nonlinear possibilities, except for a conic. Cerveau and Lins Neto [9, Théorèm 2] remove the conic possibility. □

The final lemma we will use is a simple consequence of the chain rule.

Lemma 47. *Let $\phi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a morphism. If $C \subset \mathbb{P}^2$ is a curve and P is a singular point of C , then $\phi^{-1}(C)$ is singular at every point in $\phi^{-1}(P)$.*

Proof of Theorem 42. We first note that the bound $r \leq 3$ in (i) follows immediately from combining Lemmas 43 and 45. From the definition of r , it also follows immediately that each C_i is geometrically irreducible. If $\deg C_{i+1} \neq d \deg C_i$, then it must be that C_{i+1} is in the ramification locus of ϕ . If this happens for infinitely many i , then since the ramification locus contains only finitely many curves, we must have $\phi^{-m}(C_j) = C_j$ for some j and some m (by taking multiples, we may assume that $m > j$). It follows that $C = \phi^j(C_j) \subseteq \phi^{-(m-j)}(C_j)$, and since $\phi^{-(m-j)}(C_j)$ is irreducible, in fact we have equality. Therefore, $\phi^{-m}(C) = C$, and C is a line by Lemma 46. So we are in case (a). Hence we may now assume that $\deg C_{i+1} = d \deg C_i$ for all sufficiently large i . By Lemmas 44 and 45, each such C_i must be rational and have 1 or 2 singular points. Choose i_0 so that C_{i_0} has the maximum number of singular points in the family of curves C_i . Let P_{i_0} be a singular point of C_{i_0} . From the definition of C_{i_0} and Lemma 47, $\phi^{-1}(P_{i_0})$ must consist of a single point P_{i_0+1} , which is a singular point of C_{i_0+1} . Continuing inductively, we define points $P_i \in C_i$, $i \geq i_0$, and we are in case (b). Finally, [3, Theorem 1] shows that there are at most 9 points in \mathbb{P}^2 over which the preimage set is a singleton. So $\{P_i : i \geq i_0\}$ must be a finite set. □

Proof of Theorem 9. If (a) of Theorem 42 occurs, then C is a completely invariant proper Zariski-closed subset of \mathbb{P}^2 , while if (b) occurs, then the finite set $\{P_{i_0}, P_{i_0+1}, \dots\}$ is a completely invariant proper Zariski-closed subset of \mathbb{P}^2 . □

7. EXAMPLES

Here we collect together a series of examples, some of which demonstrate our theorems, while others extend our results in certain special cases.

We first discuss an example where $\bar{\kappa}(\mathbb{P}^2 \setminus D) = -\infty$.

Example 48. Let D be Yoshihara’s quintic: the zero locus of $F(X, Y, Z) = (YZ - X^2)(YZ^2 - X^2Z - 2XY^2) + Y^5$. Then $\bar{\kappa}(\mathbb{P}^2 \setminus D) = -\infty$ and, as in [23], the associated pencil Λ of Theorem 2 has two multiple fibers: $2D$ and $5E$, where E is the zero locus of $G(X, Y, Z) = YZ - X^2$. Then in the notation of Theorem 2, $D' = (FG = 0)$, $r = 2$, and integral points on $\mathbb{P}^2 \setminus D$ are potentially dense. On the other hand, if C is another member of the pencil, say defined by

$F^2 + G^5$ (which is irreducible over $\overline{\mathbb{Q}}$), then the corresponding divisor D' is given by $(FG(F^2 + G^5) = 0)$. So $r = 3$, and by Theorem 2, any set of integral points on $\mathbb{P}^2 \setminus C$ is contained in an effectively computable (possibly reducible) curve.

Next, we provide an example of Theorem 23(c).

Example 49. Let D be defined by $Z(Y^2Z - X^3) = 0$. As was proved in [20], $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$, and so we are in the situation of Theorem 40 (but we won't need the *abc* conjecture). The pencil Λ generated by Y^2Z and X^3 has two base points, and for a general member C , $C \setminus D$ is isomorphic to \mathbb{G}_m . Since $Z = 0$ is contained in D , the gcd/Campana weight of (D, Λ) is at least $1 + \frac{2}{3} + \frac{1}{2} > 2$. Therefore, Theorem 23(c) shows unconditionally that the set of S -integral points on $\mathbb{P}^2 \setminus D$ is Zariski nondense and effectively computable. In this particular case, we can also prove this directly. An S -integral point in this case is asking for $[x : y : 1]$ with $x, y \in \mathcal{O}_{k,S}$ such that $y^2 - x^3 \in \mathcal{O}_{k,S}^*$. Choosing representatives u_1, \dots, u_ℓ of $\mathcal{O}_{k,S}^*/(\mathcal{O}_{k,S}^*)^6$, we see that such a point induces an integral point on one of the finitely many elliptic curves $Y^2 = X^3 + u_j$. Therefore, any such pair (x, y) lies in one of finitely many effectively computable curves.

An example that sits somewhere between Theorem 23(a) and Theorem 23(b) is the following.

Example 50. Let D be the divisor defined by $Y^2Z^3 - X^5 = 0$. This is a bicuspidal curve with $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$, so this also serves as an example of Theorem 35(i) (note that this is listed as (10) in Theorem 31, but it is not one of the curves listed in Theorem 35(i)). The Campana weight of the pencil generated by Y^2Z^3 and X^5 is $(1 - \frac{1}{\min(2,3)}) + \frac{4}{5} + 1 > 2$, while the gcd weight is $(1 - \frac{1}{\gcd(2,3)}) + \frac{4}{5} + 1 < 2$. Therefore, upon assuming the *abc* conjecture, Theorem 23(a) implies the arithmetically interesting statement that for any finite set of primes S in \mathbb{Z} , the set

$$\{(a, b) \in \mathbb{Z}^2 : a \text{ a powerful number, } b \text{ a fifth power, } a - b \text{ not divisible by } p \notin S\}$$

lies in a finite union of curves (in fact, in a finite union of lines passing through the origin). This example illustrates clearly the role of the *abc* conjecture in analyzing integral points.

We now give an example of Theorem 35(ii).

Example 51. Let D be defined by $F(X, Y, Z) = Y^{2\alpha+1} + X(XZ + Y^2)^\alpha = 0$, where $\alpha \geq 2$. Given a unit $u \in \mathcal{O}_{k,S}^*$ and a natural number m , we have

$$\left(\frac{u-1}{u^{\alpha m}}, 1, \frac{u^m-1}{u-1}u^{\alpha m}\right) \in \mathcal{O}_{k,S}^3 \quad \text{and} \quad F\left(\frac{u-1}{u^{\alpha m}}, 1, \frac{u^m-1}{u-1}u^{\alpha m}\right) = u.$$

Assuming that $\mathcal{O}_{k,S}^*$ is infinite, as u and m vary this yields a Zariski-dense set of S -integral points on the complement of D . Note that D belongs to the $n = 1$ case of the third type listed in Theorem 35(ii). When $n \geq 2$, the explicit description of integral points becomes more complicated, since our proof relies on Lemma 25, which in turn uses the results of [1].

We now discuss an example of a bicuspidal curve for which potential density of integral points can be shown unconditionally, but which is not covered in Theorem 35(ii).

Example 52. Let D be defined by the polynomial

$$F(X, Y, Z) = Y^{3b+1} + X(X^2Z + aXY^2 + Y^3)^b,$$

where $b \geq 2$. This belongs to the fourth type in Theorem 35(i). The pencil Λ generated by Y^{3b+1} and $X(X^2Z + aXY^2 + Y^3)^b$ induces a \mathbb{G}_m -fibration on $\mathbb{P}^2 \setminus D$ and (D, Λ) has gcd/Campana weight $1 + \frac{3b}{3b+1} + (1 - \frac{1}{1}) < 2$. Now, assume that a is a natural number. Then for each $u \in \mathcal{O}_{k,S}^*$, there exists $t \in \mathcal{O}_{k,S}^*$ such that

$$(13) \quad t^{b+1} - t^b - a(u-1) \equiv 0 \pmod{(u-1)^2},$$

namely $t = u^a = ((u-1) + 1)^a$. Therefore, all the coordinates of the point

$$P_u = \left(\frac{u-1}{t^b}, 1, \frac{t^b(t^{b+1} - t^b - a(u-1))}{(u-1)^2} \right)$$

are in $\mathcal{O}_{k,S}$ and $F(P_u) = u$. It follows that $P_u \in (\mathbb{P}^2 \setminus D)(\mathcal{O}_{k,S})$, viewing P_u in $\mathbb{P}^2(k)$, and similar to the proof of Lemma 25, Theorem 1.1 of [1] tells us that for all but finitely many u , the member of the pencil containing P_u contains infinitely many integral points with respect to D . Assuming that $\mathcal{O}_{k,S}^*$ is infinite, by varying u , we conclude that $(\mathbb{P}^2 \setminus D)(\mathcal{O}_{k,S})$ is Zariski dense.

Similarly, a congruence relation can help us with some cases of the union of a line and an irreducible curve (cf. Theorem 40).

Example 53. Let $l = 2$ and let $p(X, Z) = aX + Z$ with $a \in \mathbb{N}$ in the first equation of Theorem 40, so that the divisor D is defined by

$$Z(Z^{1+3b} + X(X^2Y + (aX + Z)Z^2)^b) = 0.$$

For each $u \in \mathcal{O}_{k,S}^*$, we choose $t \in \mathcal{O}_{k,S}^*$ satisfying (13). Then as before, the point $\left[\frac{u-1}{t^b}, \frac{t^b(t^{b+1} - t^b - a(u-1))}{(u-1)^2}, 1 \right]$ is an S -integral point with respect to D . Therefore, assuming that $\mathcal{O}_{k,S}^*$ is infinite, the same argument as in Example 52 shows that $(\mathbb{P}^2 \setminus D)(\mathcal{O}_{k,S})$ is Zariski dense.

Another example of a union of a line and an irreducible curve for which potential density of integral points can be shown unconditionally is the following.

Example 54. Let D be the union of a nodal cubic and a nontangent line going through the singular point, say $D = (Y(Y^2Z - X^3 - X^2Z) = 0)$. After one blowup, the boundary divisor becomes normal-crossings, and it is easy to see that $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$. Interchanging the roles of Y and Z , the affine equation on \mathbb{A}^2 is $(1 - x^2)(y + x) - x = 0$, so this is case (iii) of Theorem 34 and the second possibility in Theorem 40. Going back to the original notation, any line through the singular point $[0 : 0 : 1]$ other than $Y = 0$ and $Y = \pm X$ meets D at exactly one other point. So these lines form a pencil Λ such that for a general member $C \in \Lambda$, we have $C \setminus D \cong \mathbb{G}_m$ over \mathbb{Q} . We also have the line $X = -Z$ which does not pass through the singular point, but meets D at only $[-1 : 0 : 1]$ and $[0 : 1 : 0]$. Therefore, this line may serve as the C_0 of Lemma 25.

We end with an example for which we can unconditionally conclude Zariski density of integral points, but which is not a part of Theorems 35–40.

Example 55. Let D be the sum of a conic and two nontangent lines meeting at a common point P on the conic. For example, we can take the divisor D defined by $(X^2 - Y^2)(YZ - X^2) = 0$. By blowing up once at P , the boundary divisor D

becomes a normal-crossings divisor, and it is easy to see that $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$. Any line through $[0 : 0 : 1]$ meets D in just two points, yielding a pencil as in Lemma 25. Moreover, the tangent line to the conic at $[1 : 1 : 1]$ also meets D in just two points, and this serves as the C_0 of Lemma 25.

8. FURTHER QUESTIONS

For integral points on $\mathbb{P}^2 \setminus D$, the obvious question left to study is the remainder of the cases when $\bar{\kappa}(\mathbb{P}^2 \setminus D) = 1$. For the cases listed in Remark 41, we have not determined whether or not integral points are potentially dense. We believe that in these cases, the Campana weight of the pencil constructed from the structure theory of affine surfaces will be less than or equal to 2. In some specific cases, such as the examples mentioned in Section 7, we can prove potential density of integral points, but it seems one needs a more thorough classification theory of affine surfaces and more involved arithmetic and Diophantine analyses to proceed further.

One particular avenue of interest involves extending and generalizing the congruence method of Examples 52 and 53. In these examples, when a is a natural number we were able to reduce the construction of a Zariski-dense set of integral points to a congruence condition (13). For a general $a \in \mathbb{Q}$, it is possible that one may again be able to construct integral points using a congruence condition, rather than using a geometric method as in Lemma 25. For example, assume that k is a number field with a finite set of places S such that there are infinitely many $u \in \mathcal{O}_{k,S}^*$ for which $(u-1)\mathcal{O}_{k,S}$ is a product of primes lying over distinct primes of \mathbb{Z}_S that split completely in $\mathcal{O}_{k,S}$. Then for each such u , there exists an N such that $\mathcal{O}_{k,S}/(u-1)\mathcal{O}_{k,S} \cong \mathbb{Z}/N\mathbb{Z}$. Let $a \in k$. Enlarging S , we can assume that $a \in \mathcal{O}_{k,S}$. Then there exist a natural number m and $\alpha \in \mathcal{O}_{k,S}$ such that $a = m + \alpha(u-1)$. Therefore,

$$t^{b+1} - t^b - a(u-1) \equiv t^{b+1} - t^b - m(u-1) \equiv 0 \pmod{(u-1)^2},$$

and we can argue as in Example 52 to construct a Zariski-dense set of integral points. As a case in point, if $p = 2^n - 1$, $n \geq 3$, is a (rational) prime, then p splits completely in $\mathcal{O}_{\mathbb{Q}(\sqrt{2})}$ and $(\sqrt{2})^n - 1$ generates a prime ideal in $\mathcal{O}_{\mathbb{Q}(\sqrt{2})}$ lying over p . So the assumption made above is satisfied for $k = \mathbb{Q}(\sqrt{2})$ if there are infinitely many Mersenne primes (taking S to consist of the archimedean places and the place above the 2-adic place).

As for integral points in orbits, we would like to further refine Question 8. From [37, Proposition 15], one can have an endomorphism of \mathbb{P}^2 with a completely invariant singleton set and yet integral points in every orbit are Zariski nondense in \mathbb{P}^2 . Therefore, it is natural to ask: for endomorphisms on \mathbb{P}^2 , can one conclude that there must be a one-dimensional completely invariant Zariski-closed subset instead of just a nonempty completely invariant proper Zariski-closed subset in Question 8? This is related to determining whether (b) of Theorem 42(ii) can actually occur.

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REFERENCES

- [1] Paraskevas Alvanos, Yuri Bilu, and Dimitrios Poulakis, *Characterizing algebraic curves with infinitely many integral points*, Int. J. Number Theory **5** (2009), no. 4, 585–590. MR2532274
- [2] E. Amerik, F. Bogomolov, and M. Rovinsky, *Remarks on endomorphisms and rational points*, Compos. Math. **147** (2011), no. 6, 1819–1842. MR2862064
- [3] E. Amerik and F. Campana, *Exceptional points of an endomorphism of the projective plane*, Math. Z. **249** (2005), no. 4, 741–754. MR2126212
- [4] Hisayo Aoki, *Étale endomorphisms of smooth affine surfaces*, J. Algebra **226** (2000), no. 1, 15–52. MR1749875
- [5] F. Beukers, *Ternary form equations*, J. Number Theory **54** (1995), no. 1, 113–133. MR1352640
- [6] Yuri F. Bilu, *Quantitative Siegel’s theorem for Galois coverings*, Compositio Math. **106** (1997), no. 2, 125–158. MR1457336
- [7] Enrico Bombieri and Walter Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs, vol. 4, Cambridge University Press, Cambridge, 2006. MR2216774
- [8] Frédéric Campana, *Orbifolds, special varieties and classification theory* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **54** (2004), no. 3, 499–630. MR2097416
- [9] D. Cerveau and A. Lins Neto, *Hypersurfaces exceptionnelles des endomorphismes de $\mathbf{CP}(n)$* (French, with English and French summaries), Bol. Soc. Brasil. Mat. (N.S.) **31** (2000), no. 2, 155–161. MR1785266
- [10] H. Darmon, *Faltings plus epsilon, Wiles plus epsilon, and the generalized Fermat equation*, C. R. Math. Rep. Acad. Sci. Canada **19** (1997), no. 1, 3–14. MR1479291
- [11] Henri Darmon and Andrew Granville, *On the equations $z^m = F(x, y)$ and $Ax^p + By^q = Cz^r$* , Bull. London Math. Soc. **27** (1995), no. 6, 513–543. MR1348707
- [12] Gerd Faltings, *Diophantine approximation on abelian varieties*, Ann. of Math. (2) **133** (1991), no. 3, 549–576. MR1109353
- [13] Gerd Faltings, *The general case of S. Lang’s conjecture*, Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspect. Math., vol. 15, Academic Press, San Diego, CA, 1994, pp. 175–182. MR1307396
- [14] John Erik Fornæss and Nessim Sibony, *Complex dynamics in higher dimension. I*, Complex analytic methods in dynamical systems (Rio de Janeiro, 1992), Astérisque **222** (1994), 5, 201–231. MR1285389
- [15] R. V. Gurjar and M. Miyanishi, *Affine lines on logarithmic \mathbf{Q} -homology planes*, Math. Ann. **294** (1992), no. 3, 463–482. MR1188132
- [16] Joe Harris, *Theta-characteristics on algebraic curves*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 611–638. MR654853
- [17] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157
- [18] Marc Hindry and Joseph H. Silverman, *Diophantine geometry*, An introduction, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000. MR1745599
- [19] Shigeru Iitaka, *Algebraic geometry*, An introduction to birational geometry of algebraic varieties; Graduate Texts in Mathematics, vol. 76, North-Holland Mathematical Library, 24, Springer-Verlag, New York-Berlin, 1982. MR637060
- [20] Shigeru Iitaka, *Daisū kikagaku. I–III* (Japanese), 2nd ed., Iwanami Shoten Kiso Sūgaku [Iwanami Lectures on Fundamental Mathematics], vol. 23, Iwanami Shoten, Tokyo, 1984. Daisū [Algebra], vii. MR845274
- [21] Yujiro Kawamata, *On the classification of noncomplete algebraic surfaces*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, pp. 215–232. MR555700
- [22] Hideo Kojima, *Structure of affine surfaces $\mathbf{P}^2 - B$ with $\bar{\kappa} \leq 1$* , J. Algebra **253** (2002), no. 1, 100–111. MR1925009
- [23] Masayoshi Miyanishi and Tohru Sugie, *On a projective plane curve whose complement has logarithmic Kodaira dimension $-\infty$* , Osaka J. Math. **18** (1981), no. 1, 1–11. MR609973
- [24] Shinichi Mochizuki, *Inter-universal Teichmüller theory IV: Log-volume computations and set-theoretic foundations*, preprint available on his webpage, 2012.
- [25] Joseph H. Silverman, *Integer points, Diophantine approximation, and iteration of rational maps*, Duke Math. J. **71** (1993), no. 3, 793–829. MR1240603

- [26] Arne Smeets, *Insufficiency of the étale Brauer-Manin obstruction: towards a simply connected example*, Amer. J. Math. **139** (2017), no. 2, 417–431. MR3636635
- [27] Keita Tono, *Defining equations of certain rational cuspidal curves*, 2000, Thesis (Ph.D.)–Saitama University.
- [28] Keita Tono, *Rational unicuspidal plane curves with $\bar{\kappa} = 1$* , Newton polyhedra and singularities (Japanese) (Kyoto, 2001), Sūrikaiseikikenkyūsho Kōkyūroku **1233** (2001), 82–89. MR1905271
- [29] Shuichiro Tsunoda, *The complements of projective plane curves*, Sūrikaiseikikenkyūsho Kōkyūroku **446** (1981), 48–55.
- [30] Machiel van Frankenhuysen, *ABC implies the radicalized Vojta height inequality for curves*, J. Number Theory **127** (2007), no. 2, 292–300. MR2362438
- [31] Paul Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Mathematics, vol. 1239, Springer-Verlag, Berlin, 1987. MR883451
- [32] Paul Vojta, *Integral points on subvarieties of semiabelian varieties. I*, Invent. Math. **126** (1996), no. 1, 133–181. MR1408559
- [33] Paul Vojta, *A more general abc conjecture*, Internat. Math. Res. Notices **21** (1998), 1103–1116. MR1663215
- [34] Paul Vojta, *Integral points on subvarieties of semiabelian varieties. II*, Amer. J. Math. **121** (1999), no. 2, 283–313. MR1680329
- [35] Isao Wakabayashi, *On the logarithmic Kodaira dimension of the complement of a curve in P^2* , Proc. Japan Acad. Ser. A Math. Sci. **54** (1978), no. 6, 157–162. MR0498590
- [36] Junyi Xie, *The existence of Zariski dense orbits for polynomial endomorphisms of the affine plane*, Compos. Math. **153** (2017), no. 8, 1658–1672. MR3705271
- [37] Yu Yasufuku, *Deviations from S -integrality in orbits on \mathbb{P}^N* , Bull. Inst. Math. Acad. Sin. (N.S.) **9** (2014), no. 4, 603–631. MR3309943
- [38] Hisao Yoshihara, *On plane rational curves*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 4, 152–155. MR533711

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