

SIMPLICES FOR NUMERAL SYSTEMS

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ABSTRACT. The family of lattice simplices in \mathbb{R}^n formed by the convex hull of the standard basis vectors together with a weakly decreasing vector of negative integers include simplices that play a central role in problems in enumerative algebraic geometry and mirror symmetry. From this perspective, it is useful to have formulae for their discrete volumes via Ehrhart h^* -polynomials. Here we show, via an association with numeral systems, that such simplices yield h^* -polynomials with properties that are also desirable from a combinatorial perspective. First, we identify n -simplices in this family that associate via their normalized volume to the n th place value of a positional numeral system. We then observe that their h^* -polynomials admit combinatorial formula via descent-like statistics on the numeral strings encoding the nonnegative integers within the system. With these methods, we recover ubiquitous h^* -polynomials including the Eulerian polynomials and the binomial coefficients arising from the factoradic and binary numeral systems, respectively. We generalize the binary case to base- r numeral systems for all $r \geq 2$, and prove that the associated h^* -polynomials are real-rooted and unimodal for $r \geq 2$ and $n \geq 1$.

1. INTRODUCTION

We consider the family of simplices defined by the convex hull

$$\Delta_{(1,q)} := \text{conv}(e_1, \dots, e_n, -q) \subset \mathbb{R}^n,$$

where e_1, \dots, e_n denote the standard basis vectors in \mathbb{R}^n , and $q = (q_1, q_2, \dots, q_n)$ is any weakly increasing sequence of n positive integers. A convex polytope $P \subset \mathbb{R}^n$ with vertices in \mathbb{Z}^n (i.e., a *lattice polytope*) is called *reflexive* if its polar body

$$P^\circ := \{y \in \mathbb{R}^n : y^T x \leq 1 \text{ for all } x \in P\} \subset \mathbb{R}^n$$

is also a lattice polytope. In the case of $\Delta_{(1,q)}$, the reflexivity condition is equivalent to the arithmetic condition

$$q_i \mid 1 + \sum_{j \neq i} q_j \quad \text{for all } j \in [n],$$

which was used to classify all reflexive n -simplices in [7]. Reflexive simplices of the form $\Delta_{(1,q)}$ have been studied extensively from an algebro-geometric perspective since they exhibit connections with string theory that yield important results in enumerative geometry [8]. These connections result in an explicit formula for all Gromov-Witten invariants which, in turn, encode the number of rational curves of degree d on the quintic threefold [6]. This observation led to the development

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of the mathematical theory of *mirror symmetry* in which reflexive polytopes and their polar bodies play a central role. In particular, the results of [6] are intimately tied to the fact that $\Delta_{(1,q)}$ for $q = (1, 1, 1, 1) \in \mathbb{R}^4$ contains significantly less lattice points (i.e., points in \mathbb{Z}^n) than its polar body.

As a result of this connection, the lattice combinatorics and volume of the simplices $\Delta_{(1,q)}$ have been studied from the perspective of geometric combinatorics in terms of their (Ehrhart) h^* -polynomials. The *Ehrhart function* of a d -dimensional lattice polytope P is the function $i(P; t) := |tP \cap \mathbb{Z}^n|$, where $tP := \{tp : p \in P\}$ denotes the t th dilate of the polytope P for $t \in \mathbb{Z}_{\geq 0}$. It is well known [9] that $i(P; t)$ is a polynomial in t of degree d , and the *Ehrhart series* of P is the rational function

$$\sum_{t \geq 0} i(P; t)z^t = \frac{h_0^* + h_1^*z + \cdots + h_d^*z^d}{(1-z)^{d+1}},$$

where the coefficients $h_0^*, h_1^*, \dots, h_d^*$ are all nonnegative integers [19]. The polynomial $h^*(P; z) := h_0^* + h_1^*z + \cdots + h_d^*z^d$ is called the h^* -polynomial of P . The h^* -polynomial of P encodes the typical Euclidean volume of P in the sense that $d! \operatorname{vol}(P) = h^*(P; 1)$. It also encodes the number of lattice points in P since $h_1^* = |P \cap \mathbb{Z}^n| - d - 1$ (see, for instance, [1]).

In addition to combinatorial interpretations of the coefficients $h_0^*, \dots, h_d^* \in \mathbb{Z}_{\geq 0}$, distributional properties of these coefficients is a popular research topic. Let $p := a_0 + a_1z + \cdots + a_dz^d$ be a polynomial with nonnegative integer coefficients. The polynomial p is called *symmetric* if $a_i = a_{d-i}$ for all $i \in [d]$, it is called *unimodal* if there exists an index j such that $a_i \leq a_{i+1}$ for all $i < j$ and $a_i \geq a_{i+1}$ for all $i \geq j$, it is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i \in [d]$, and it is called *real-rooted* if all of its roots are real numbers. An important result in Ehrhart theory states that $P \subset \mathbb{R}^n$ is reflexive if and only if $h^*(P; x)$ is symmetric of degree n [12]. It is well known that if p is real-rooted, then it is log-concave, and if all $a_i > 0$, then it is also unimodal. Unlike symmetry, there is no general characterization of any one of these properties for h^* -polynomials.

The distributional (and related algebraic) properties of the h^* -polynomials for the simplices $\Delta_{(1,q)}$ were recently studied in terms of the arithmetic structure of q [3, 4]. In particular, [4, Theorem 2.5] provides an arithmetic formula for $h^*(\Delta_{(1,q)}; z)$ in terms of q which the authors use to prove unimodality of the h^* -polynomial in some special cases. On the other hand, the literature lacks examples of $\Delta_{(1,q)}$ admitting combinatorial formula for their h^* -polynomials. Thus, while the simplices $\Delta_{(1,q)}$ constitute a class of convex polytopes fundamental in algebraic geometry, we would like to observe that they are of interest from a combinatorial perspective as well. To demonstrate that this is in fact the case, we will identify simplices $\Delta_{(1,q)}$ whose h^* -polynomials are well known in combinatorics and admit the desirable distributional properties mentioned above. We observe that simplices within this family can be associated in a natural way to *numeral systems* and the h^* -polynomials of such simplices admit a combinatorial interpretation in terms of descent-type statistics on the numeral representations of nonnegative integers within these systems. Most notably, we find such ubiquitous generating polynomials as the *Eulerian polynomials* and the *binomial coefficients* arising from the *factoradic* and *binary* numeral systems, respectively. For these examples, we also find that the combinatorial interpretation of $h^*(\Delta_{(1,q)}; z)$ is closely tied to that of q . We then generalize the simplices arising from the binary system to simplices

associated to any base- r numeral system for $r \geq 2$, all of whose h^* -polynomials admit a combinatorial interpretation in terms of the base- r representations of the nonnegative integers. Finally, we show that, even though these simplices are not all reflexive, their h^* -polynomials are always real-rooted, log-concave, and unimodal.

The remainder of the paper is outlined as follows. In Section 2, we review the basics of positional numeral systems and outline our notation. In Section 3, we describe when a numeral system associates to a family of reflexive simplices of the form $\Delta_{(1,q)}$, one in each dimension $n \geq 1$. We then show that the n -simplex for the factoradic numeral system has the $(n + 1)^{st}$ Eulerian polynomial as its h^* -polynomial. We also prove the n -simplex for the binary numeral system has h^* -polynomial $(1 + x)^n$. In Section 4, we generalize the binary case to base- r numeral systems for $r \geq 2$. We provide a combinatorial formula for these h^* -polynomials and prove they are real-rooted, log-concave, and unimodal for $r \geq 2$ and $n \geq 1$.

2. NUMERAL SYSTEMS

The typical (positional) numeral system is a method for expressing numbers that can be described as follows. A *numeral* is a sequence of nonnegative integers $\eta := \eta_{m-1}\eta_{m-2} \dots \eta_1\eta_0$, and the location of η_i in the string is called *place i* . The *digits* are the numbers allowable in each place, and the *i th radix (or base)* is the number of digits allowed in place i . A *numeral system* is a sequence of positive integers $a = (a_n)_{n=0}^\infty$ satisfying $a_0 := 1 < a_1 < a_2 < \dots$, which we call *place values*. To see why a yields a system for expressing nonnegative integers, let $b \in \mathbb{Z}_{\geq 0}$ and let n be the smallest integer so that $a_n > b$. Dividing b by a_{n-1} and iterating gives

$$\begin{array}{ll}
 b = q_{n-1}a_{n-1} + r_{n-1} & 0 \leq r_{n-1} < a_{n-1} \\
 r_{n-1} = q_{n-2}a_{n-2} + r_{n-2} & 0 \leq r_{n-2} < a_{n-2} \\
 \vdots & \vdots \\
 r_{i+1} = q_i a_i + r_i & 0 \leq r_i < a_i \\
 \vdots & \vdots \\
 r_2 = q_1 a_1 + r_1 & 0 \leq r_2 < a_2 \\
 r_1 = q_0 a_0. &
 \end{array}$$

Denoting $b_a(i) := q_i$ for all i , it follows that

$$(1) \quad b = b_a(n - 1)a_{n-1} + b_a(n - 2)a_{n-2} + \dots + b_a(1)a_1 + b_a(0)a_0.$$

On the other hand, if $b = \sum_{i=0}^{n-1} \beta_i a_i$ where $\sum_{j=0}^i \beta_j a_j < a_{i+1}$ for every $i \geq 0$, then $b_a(i) = \beta_i$ for every i . In particular, the representation of b in equation (1) is unique (see, for instance, [11, Theorem 1]). Thus, the representation of the nonnegative integer b in the numeral system a is the numeral

$$b_a := b_a(n - 1)b_a(n - 2) \dots b_a(1)b_a(0).$$

An important family of numeral systems we will consider are the mixed radix systems. A numeral system $a = (a_n)_{n=0}^\infty$ is *mixed radix* if there exists a sequence of integers $(c_n)_{n=0}^\infty$ with $c_0 := 1$ and $c_n > 1$ for $n > 1$ such that the product $c_0 c_1 \dots c_n = a_n$ for all $n \geq 0$. In this case, c_{n+1} is the n th radix of the system a .

Example 2.1 (Numeral systems). The following are examples of numeral systems:

- (1) The binary numbers are the numeral system $a = (2^n)_{n=0}^\infty$. The radix of place i is 2 for every i since each place assumes only digits 0 or 1. The binary system is mixed radix with sequence of radices $c = (1, 2, 2, 2, \dots)$. The numeral representation of the number 102 in this system is 1100110.
- (2) The ternary (base-3) numeral system is the system $a = (3^n)_{n=0}^\infty$. The radix of place i is 3 for every i since each place assumes only digits 0, 1, or 2. The ternary numeral system is also mixed radix, and it has sequence of radices $c = (1, 3, 3, 3, 3, \dots)$. The numeral representation of the number 102 in the ternary system is 10210.
- (3) An example of a numeral system that is not mixed radix is the system $a = (F_{n+1})_{n=0}^\infty$, where $a_n = F_{n+1}$ is the $(n + 1)^{st}$ Fibonacci number. This sequence is not mixed radix since there exist prime Fibonacci numbers other than 2. The radix of every place is 2 since $2F_{n+1} = F_{n+1} + F_n > F_{n+1}$ for all n , and so each place assumes digits 0 or 1. The numeral representation of the number 102 in this system is 1000100000.

3. REFLEXIVE NUMERAL SYSTEMS

In this section, we demonstrate a method for attaching an n -simplex of the form $\Delta_{(1,q)}$ to the n th place value a_n of a numeral system a such that $a_n = n! \text{vol}(\Delta_{(1,q)})$, i.e., the *normalized volume of P* . For certain families of numeral systems, which we call *reflexive numeral systems*, these simplices can be chosen so that they are all reflexive. We show that the factoradic and binary numeral systems are reflexive, and recover the h^* -polynomials of their associated simplices. We also discuss the relationship between reflexive numeral systems and the mixed radix numeral systems, and the geometric relationship between the factoradic n -simplex and the s -lecture hall n -simplex with the same h^* -polynomial, i.e., the n th Eulerian polynomial.

Definition 3.1. A numeral system $a = (a_n)_{n=0}^\infty$ is called a *reflexive (numeral) system* if there exists an increasing sequence of positive integers $d = (d_n)_{n=0}^\infty$ satisfying the following properties:

- (1) $d_i \mid a_n$ for all $0 \leq i \leq n - 1$, $n \geq 1$, and
- (2) $1 + \sum_{i=0}^{n-1} \frac{a_n}{d_i} = a_n$ for all $n \geq 1$.

Such a sequence d is called a *divisor system (for a)*.

Example 3.1 (A reflexive numeral system). Recall from Example 2.1 that the binary numeral system is given by $a = (a_n)_{n=0}^\infty := (2^n)_{n=0}^\infty$. The binary system a admits the divisor system $d = (d_n)_{n=0}^\infty := (2^{n+1})_{n=0}^\infty$. This is because for all $n \geq 1$ we have that $2^{i+1} \mid 2^n$ for all $0 \leq i \leq n - 1$, and

$$1 + \sum_{i=0}^{n-1} \frac{a_n}{d_i} = 1 + \sum_{i=0}^{n-1} \frac{2^n}{2^{i+1}} = 1 + \sum_{i=0}^{n-1} 2^i = 2^n = a_n.$$

Thus, the binary numeral system a is a reflexive system with divisor system d . Furthermore, in Theorem 3.6 we will prove that $h^*(\Delta_{(1,q)}; z) = (1 + z)^n$.

Recall that we would like to associate an n -simplex $\Delta_{(1,q)}$ to the n th place value a_n of a numeral system a by the relation $a_n = n! \text{vol}(\Delta_{(1,q)})$. In [4, 17] it is shown that the normalized volume of $\Delta_{(1,q)}$ (i.e., the value $n! \text{vol}(\Delta_{(1,q)})$) is $1 + q_1 + \dots + q_n$. When a is a reflexive system with divisor system d , condition (1) tells us that $\frac{a_n}{d_i}$

is a positive integer for all $n \geq 1$ and $0 \leq i \leq n - 1$, and condition (2) tells us that the n -simplex $\Delta_{(1,q)}$ with

$$q := \left(\frac{a_n}{d_{n-1}}, \frac{a_n}{d_{n-2}}, \dots, \frac{a_n}{d_0} \right) \in \mathbb{Z}_{>0}^n$$

will have the desired normalized volume. Moreover, it turns out the h^* -polynomial of this n -simplex can be computed in a recursive fashion in terms of the numeral representations of the first a_n nonnegative integers. This recursive formula is presented in Proposition 3.2, but it is essentially due to the following theorem of [4].

Theorem 3.1 ([4, Theorem 2.5]). *The h^* -polynomial for the n -simplex $\Delta_{(1,q)}$ for $q = (q_1, \dots, q_n)$ is*

$$h^*(\Delta_{(1,q)}; z) = \sum_{b=0}^{q_1+q_2+\dots+q_n} z^{\omega(b)},$$

where

$$\omega(b) = b - \sum_{i=1}^n \left\lfloor \frac{q_i b}{1 + q_1 + q_2 + \dots + q_n} \right\rfloor.$$

Proposition 3.2. *Let $a = (a_n)_{n=0}^\infty$ be a reflexive system that admits a divisor system $d = (d_n)_{n=0}^\infty$. Then for*

$$q := \left(\frac{a_n}{d_{n-1}}, \frac{a_n}{d_{n-2}}, \dots, \frac{a_n}{d_0} \right)$$

the reflexive n -simplex $\Delta_{(1,q)} \subset \mathbb{R}^n$ has h^* -polynomial $h^*(\Delta_{(1,q)}; z) = \sum_{b=0}^{a_n-1} z^{\omega(b)}$, where

$$\omega(b) = \omega(b') + b_a(n - 1) - \left\lfloor \frac{b}{d_{n-1}} \right\rfloor,$$

and $b' := b - b_a(n - 1)a_{n-1}$.

Proof. Applying Theorem 3.1 to our particular case, we can simplify the formula for $\omega(b)$ as follows:

$$\begin{aligned} \omega(b) &= b - \sum_{i=0}^{n-1} \left\lfloor \frac{b}{d_i} \right\rfloor \\ &= b_a(n - 1)a_{n-1} + b' - \sum_{i=0}^{n-2} \left\lfloor \frac{b_a(n - 1)a_{n-1} + b'}{d_i} \right\rfloor - \left\lfloor \frac{b}{d_{n-1}} \right\rfloor \\ &= b_a(n - 1)a_{n-1} + b' - b_a(n - 1) \left(\sum_{i=0}^{n-2} \frac{a_{n-1}}{d_i} \right) - \sum_{i=0}^{n-2} \left\lfloor \frac{b'}{d_i} \right\rfloor - \left\lfloor \frac{b}{d_{n-1}} \right\rfloor \\ &= \omega(b') + b_a(n - 1) - \left\lfloor \frac{b}{d_{n-1}} \right\rfloor. \quad \square \end{aligned}$$

We note that not every mixed radix system is a reflexive system. However, if a mixed radix system is reflexive, then its corresponding divisor system is unique. If a is mixed radix, then $a_n = c_0 \cdots c_n$ where $c = (c_n)_{n=0}^\infty$ is its sequence of radices. Using this fact and part (2) of Definition 3.1, we deduce the following proposition.

Proposition 3.3. *If $a = (a_n)_{n=0}^\infty$ is a reflexive mixed radix system with sequence of radices $c = (c_n)_{n=0}^\infty$, then its corresponding divisor system d is unique and*

$$d = (d_n)_{n=0}^\infty = \left(\frac{a_{n+1}}{c_{n+1} - 1} \right)_{n=0}^\infty.$$

Example 3.2 (Reflexive and nonreflexive mixed radix systems). Recall from Example 3.1 that the binary numeral system $a = (2^n)_{n=0}^\infty$ is a reflexive system with divisor system $d = (2^{n+1})_{n=0}^\infty$. Since a is also a mixed radix system with sequence of radices $c = (1, 2, 2, 2, 2, \dots)$, then we can apply Proposition 3.3 to recover the divisor system d as the unique divisor system for a .

It is also important to notice that there are mixed radix systems that are not reflexive. For example, consider the numeral system $a = (2^n \cdot n!)_{n=0}^\infty$ whose n th place value is the order of the hyperoctahedral group. Then a is reflexive with sequence of radices $c = (1, 2, 4, \dots, 2n, \dots)$. However, it follows from Proposition 3.3 that a has no divisor system since there are infinitely many times when $\frac{2^{n+1}(n+1)!}{2(n+1)-1}$ is not an integer. For instance, every prime is expressible as $2(n+1) - 1$ for some n , and this prime will never be a factor of $2^{n+1}(n+1)!$.

Although identifying a divisor system for a given numeral system is generally a nontrivial problem, Proposition 3.3 makes it easier in the case of mixed radix systems, and it provides a quick check to deduce if the system is reflexive. Moreover, when a mixed radix system is reflexive, the resulting simplices $\Delta_{(1,q)}$ appear to have well-known h^* -polynomials with a combinatorial interpretation closely related to a combinatorial interpretation of q . We now give two examples of this phenomenon.

3.1. The factoradics and the Eulerian polynomials. In this subsection, we study the numeral system $a = (a_n)_{n=0}^\infty := ((n+1)!)_{n=0}^\infty$, which is commonly called the *factoradics*. By Proposition 3.3, we see that the factoradics are reflexive with divisor system $d = (d_n)_{n=0}^\infty := ((n+1)! + n!)_{n=0}^\infty$. We will see that the q -vectors given by d admit a combinatorial interpretation in terms of descents, and the resulting simplices $\Delta_{(1,q)}$ have h^* -polynomials the Eulerian polynomials. For $\pi \in S_n$, we let

$$\begin{aligned} \text{Des}(\pi) &:= \{i \in [n-1] : \pi_{i+1} > \pi_i\}, \\ \text{des}(\pi) &:= |\text{Des}(\pi)|, \\ \text{maxDes}(\pi) &:= \begin{cases} \max\{i \in [n-1] : i \in \text{Des}(\pi)\} & \text{for } \pi \neq 12 \cdots n, \\ 0 & \text{for } \pi = 12 \cdots n. \end{cases} \end{aligned}$$

We then consider the pair of polynomial generating functions

$$A_n(z) := \sum_{\pi \in S_n} z^{\text{des}(\pi)} \quad \text{and} \quad B_n(z) := \sum_{\pi \in S_n} z^{\text{maxDes}(\pi)},$$

where $A_n(z)$ is the well-known n th *Eulerian polynomial*. The polynomial $B_n(z)$ is a different generating function for permutations that satisfies a recursion described in Lemma 3.4. In the following, we let $A(n, k)$ and $B(n, k)$ denote the coefficient of z^k in $A_n(z)$ and $B_n(z)$, respectively.

Lemma 3.4. *For each $n \in \mathbb{Z}_{>0}$ we have that*

$$B(n, 0) = 1, \quad B(n, 1) = n - 1,$$

and for $k > 1$

$$B(n, k) = (n)_{k-1}(n - k) = nB(n - 1, k - 1).$$

Moreover, for the polynomials $B_n(z)$, we have the recursive expression

$$B_n(z) = 1 - z + nzB_{n-1}(z), \quad (n > 1),$$

and the closed-form expression

$$B_n(z) = 1 + \sum_{k=1}^{n-1} \frac{n!}{(n-k)! + (n-k-1)!} z^k.$$

Proof. By the definition of $B_n(z)$ we can see that $B(n, 0) = 1$ and $B(n, 1) = n - 1$ for all $n \in \mathbb{Z}_{>0}$. To see that $B(n, k) = (n)_{k-1}(n - k)$ for $k > 1$, notice first that the falling factorial $(n)_{k-1}$ counts the number of ways to pick the first $k - 1$ letters of $\pi = \pi_1\pi_2 \cdots \pi_n$. Multiplying this value by $(n - k)$ accounts for the fact that there are $(n - k)$ remaining choices for the k th letter such that $\pi_k > \pi_{k+1}$. The remaining $(n - k)$ letters of the permutation are then arranged in increasing order, and so $\text{maxDes}(\pi) = k$. The recursion for $B(n, k)$ for $k > 1$ and $B_n(z)$ for $n > 1$ now follow readily from these observations. The closed form expression for $B_n(z)$ is immediate from the closed forms presented for the coefficients $B(n, k)$ for $k \geq 1$ and the identity $n! = (n - 1)((n - 1)! + (n - 2)!)$. \square

We now show that the sequence of coefficients for the nonconstant terms of $B_n(z)$ is a vector q for which the n -simplex $\Delta_{(1,q)}$ has h^* -polynomial $A_n(z)$; thereby offering, in a sense, a geometric transformation between the two generating polynomials.

Recall that for two integer strings $\eta := \eta_1\eta_2 \cdots \eta_n$ and $\mu := \mu_1\mu_2 \cdots \mu_n$ the string η is *lexicographically larger* than the string μ if and only if the leftmost nonzero number in the string $(\eta_1 - \mu_1)(\eta_2 - \mu_2) \cdots (\eta_n - \mu_n)$ is positive. For $0 \leq b < n!$, the factoradic representation of b , denoted $b_i := b_i(n - 1)b_i(n - 2) \cdots b_i(1)b_i(0)$, is known as the *Lehmer code* of the b th largest permutation of $[n]$ under the lexicographic ordering [15]. In particular, if we let $\pi^{(b)}$ denote the b th largest permutation of $[n]$ under the lexicographic ordering, then for all $0 \leq i < n$

$$b_i(i) = \left| \{0 \leq j < i : \pi_{n-i}^{(b)} > \pi_{n-j}^{(b)}\} \right|.$$

It is straightforward to check that $b_i(i) > b_i(i + 1)$ if and only if $n - i \in \text{Des}(\pi^{(b)})$. Thus, counting descents in $\pi^{(b)}$ is equivalent to counting descents in the factoradic representation of the integer b . This fact allows for the following theorem.

Theorem 3.5. *The factoradic numeral system $a = (a_n)_{n=0}^\infty = ((n + 1)!)_{n=0}^\infty$ admits the divisor system $d = (d_n)_{n=0}^\infty = ((n + 1)! + n!)_{n=0}^\infty$ for which the reflexive simplex $\Delta_{(1,q)} \subset \mathbb{R}^n$ with*

$$q := (B(n + 1, 1), B(n + 1, 2), \dots, B(n + 1, n))$$

has h^* -polynomial

$$h^*(\Delta_{(1,q)}; z) = A_{n+1}(z).$$

Proof. First notice that $d = ((n + 1)! + n!)_{n=0}^\infty$ is in fact a divisor system for $a = ((n + 1)!)_{n=0}^\infty$ by Lemma 3.4. We now show, via induction on n , that for all integers $b = 0, 1, \dots, n! - 1$ $\omega(b) = \text{des}(\pi^{(b)})$, where $\pi^{(b)}$ denotes the b th largest permutation of $[n]$. Since the base case ($n = 1$) is clear, we assume the result holds for $n - 1$. For convenience, we reindex $a = (n!)_{n=1}^\infty$ and $d = (n! + (n - 1)!)_{n=1}^\infty$.

Then by Proposition 3.2 we know that for $b = 0, 1, \dots, n!$

$$\begin{aligned}
 \omega(b) &= \omega(b') + b_l(n-1) - \left\lfloor \frac{b}{d_{n-1}} \right\rfloor \\
 &= \omega(b') + b_l(n-1) - \left\lfloor \frac{(n-1)b_l(n-1)(n-1)! + b'}{n!} \right\rfloor \\
 &= \omega(b') + b_l(n-1) - \left\lfloor b_l(n-1) - \left(\frac{b_l(n-1)(n-1)! - (n-1)b'}{n!} \right) \right\rfloor \\
 (2) \quad &= \omega(b') + \left\lfloor \frac{b_l(n-1)(n-1)! - (n-1)b'}{n!} \right\rfloor.
 \end{aligned}$$

With equation (2) in hand, we now consider a few cases. First, notice that if $b_l(n-1) = 0$, then $\omega(b) = \omega(b')$, and the result follows from the inductive hypothesis. This is because $0 \leq b_{n-1} \leq (n-1)! - 1$. So whenever $b_l(n-1) = 0$ then

$$\left\lfloor \frac{b_l(n-1)(n-1)! - (n-1)b'}{n!} \right\rfloor = \left\lfloor -\frac{(n-1)b'}{n!} \right\rfloor = 0.$$

Suppose now that $0 < b_l(n-1) < n$. Then $\pi^{(b)}$ satisfies $\pi_1^{(b)} = b_l(n-1) + 1$, and we consider the following three cases.

First, if $b' = 0$, then since $0 < b_l(n-1) < n$, we have that $\omega(b) = \omega(b') + 1$, and the result follows from the inductive hypothesis. Second, suppose that $b' \neq 0$ and that $0 < b_l(n-1)(n-1)! - (n-1)b'$. Then since $0 < (n-1)b' \leq b_l(n-1)(n-1)! < n!$, we know that

$$\left\lfloor \frac{b_l(n-1)(n-1)! - (n-1)b'}{n!} \right\rfloor = 1.$$

Thus, $\omega(b) = \omega(b') + 1$. Since the first $b_l(n-1)(n-2)!$ permutations of $[n]$ with $\pi_1 = b_l(n-1) + 1$ satisfy $\pi_1 > \pi_2$, the result follows from the inductive hypothesis. Finally, if $b' \neq 0$ and $0 \leq (n-1)b' - b_l(n-1)(n-1)!$, since $0 < (n-1)b' \leq (n-1)((n-1)! - 1) < n!$, we have that

$$\left\lfloor \frac{b_l(n-1)(n-1)! - (n-1)b'}{n!} \right\rfloor = - \left\lfloor \frac{(n-1)b' - b_l(n-1)(n-1)!}{n!} \right\rfloor = 0.$$

Thus, $\omega(b) = \omega(b')$. Since the last $(n-1)! - b_l(n-1)(n-2)!$ permutations of $[n]$ satisfying $\pi_1 = b_l(n-1) + 1$ satisfy $\pi_1 < \pi_2$, the result follows from the inductive hypothesis, completing the proof. \square

Remark 3.1 (Relationship with lecture hall simplices). Let \mathcal{F}_n denote the n -simplex described in Theorem 3.5. To the best of the author’s knowledge, the only reflexive n -simplex Δ with $h^*(\Delta; z) = A_{n+1}(z)$, other than \mathcal{F}_n , is the s -lecture hall simplex

$$P_n^{(2,3,\dots,n+1)} := \left\{ x \in \mathbb{R}^n : 0 \leq \frac{x_1}{2} \leq \frac{x_2}{3} \leq \dots \leq \frac{x_n}{n+1} \leq 1 \right\}.$$

Using the classification of [7], we can deduce that \mathcal{F}_n and $P_n^{(2,3,\dots,n+1)}$ define distinct toric varieties in the following sense: For a lattice n -simplex $\Delta \subset \mathbb{R}^n$ containing the origin in its interior, [7, Definition 2.3] assigns a *weight* $q := (q_0, \dots, q_n) \in \mathbb{Z}_{>0}^{n+1}$ and a *factor* $\lambda := \gcd(q_0, \dots, q_n)$. We say that Δ is of *type* $(q_{\text{red}}, \lambda)$, where $q_{\text{red}} := \frac{1}{\lambda}q$. When $\lambda = 1$, the toric variety of Δ is the weighted projective space $\mathbb{P}(q_{\text{red}})$, and when $\lambda > 1$, it is a quotient of $\mathbb{P}(q_{\text{red}})$ by the action of a finite group of index λ .

It follows from our construction that \mathcal{F}_n has factor 1, and so its toric variety is the weighted projective space $\mathbb{P}(B(n))$, where $B(n) := (1, B(n+1, 1), \dots, B(n+1, n))$. On the other hand, for $n > 2$, $P_n^{(2,3,\dots,n+1)}$ empirically exhibits factor

$$\lambda = \frac{n!}{\text{lcm}(1, 2, \dots, n)},$$

(see sequence [18, A025527]), and $q_{\text{red}} \neq B(n)$. Thus, \mathcal{F}_n and $P_n^{(2,3,\dots,n+1)}$ define distinct toric varieties in terms of the classification of [7]. Moreover, \mathcal{F}_n appears to be the only known weighted projective space with Eulerian h^* -polynomial.

A second way to see that \mathcal{F}_n and $P_n^{(2,3,\dots,n+1)}$ define distinct toric varieties is to note that \mathcal{F}_n is not *self-dual* for $n \geq 3$; that is, for $n \geq 3$, the polar body of \mathcal{F}_n is not \mathcal{F}_n itself up to a translation and/or a rotation. On the other hand, [13] proves that $P_n^{(2,3,\dots,n+1)}$ is a self-dual polytope. Thus, from a discrete geometric perspective, this allows us to see that \mathcal{F}_n and $P_n^{(2,3,\dots,n+1)}$ define distinct toric varieties.

Remark 3.2 (Type B Eulerian polynomials). We also note that the *Type B Eulerian polynomials* cannot arise from a numeral system since the sequence $a := (2^n \cdot n!)_{n=0}^\infty$ is a mixed radix system that is not reflexive (see Example 3.2).

3.2. The binary numbers and binomial coefficients. Using Proposition 3.3, we can see that the binary numeral system $a = (a_n)_{n=0}^\infty = (2^n)_{n=0}^\infty$ is also reflexive (see Example 3.2). Here, the h^* -polynomial of the resulting n -simplices are given by counting the number of 1's in the binary representation of the first 2^n nonnegative integers. In the following, we let $\text{supp}_2(b)$ denote the number of nonzero digits in the binary representation $b_2 := b_2(n-1)b_2(n-1) \cdots b_2(0)$ of the integer b .

Theorem 3.6. *The binary numeral system $a = (a_n)_{n=0}^\infty = (2^n)_{n=0}^\infty$ admits the divisor system $d = (d_n)_{n=0}^\infty = (2^{n+1})_{n=0}^\infty$ for which the reflexive simplex $\Delta_{(1,q)} \subset \mathbb{R}^n$ with*

$$q := (1, 2, 4, 8, \dots, 2^{n-1})$$

has h^* -polynomial

$$h^*(\Delta_{(1,q)}; z) = \sum_{b=0}^{2^n-1} z^{\text{supp}_2(b)} = (1+z)^n.$$

Proof. To prove the result we show that $\omega(b) = \text{supp}_2(b)$ for all $b = 0, 1, 2, \dots, 2^n - 1$ via induction on n . For the base case, we take $n = 1$. By [4, Theorem 2.5] we have that

$$h^*(\Delta_{(1,q)}; z) = z^{\omega(0)} + z^{\omega(1)},$$

where $\omega(0) = 0 = \text{supp}_2(0)$, and $\omega(1) = 1 = \text{supp}_2(1)$. For the inductive step, suppose that $\omega(b) = \text{supp}_2(b)$ for all $b = 0, 1, 2, \dots, 2^{n-1}$. Then by Proposition 3.2 we have that

$$\omega(b) = \omega(b') + b_2(n-1) - \left\lfloor \frac{b}{d_{n-1}} \right\rfloor = \omega(b') + b_2(n-1),$$

where the last equality holds since $0 \leq b < 2^n$. Since

$$\text{supp}_2(b) = \text{supp}_2(b') + b_2(n-1),$$

the result follows by the inductive hypothesis. Finally, the fact that $h^*(\Delta_{(1,q)}) = (1+z)^n$ follows from [11, Theorem 1] and the fact that any $(0, 1)$ -string of length n is a valid binary representation of a nonnegative integer less than 2^n . \square

Similar to the factoradics, the q -vector in Theorem 3.6 can be viewed as the coefficients of a “max-descent” polynomial for binary strings of length n . Namely, q_i is the number of binary strings $\eta_{n-1}\eta_{n-2}\cdots\eta_0$ with rightmost nonzero digit η_i . Analogously, Theorem 3.6 provides a geometric transformation between these two generating polynomials for counting binary strings in terms of their nonzero entries.

Proposition 3.3 implies that the only reflexive base- r numeral system for $r \geq 2$ is the binary system. Thus, there does not exist a base- r generalization of Theorem 3.6 that results in simplices with symmetric h^* -polynomials. On the other hand, there is a generalization that preserves other desirable properties of the h^* -polynomial $(1+x)^n$, including real-rootedness and unimodality. This is the focus of the next section.

4. THE POSITIONAL BASE- r NUMERAL SYSTEMS

For $r \geq 2$, consider the base- r numeral system, $a := (r^n)_{n=0}^\infty$. Just as in the base-2 case, we denote the base- r representation of an integer $b \in \mathbb{Z}_{\geq 0}$ by

$$b_r := b_r(n-1)b_r(n-2)\cdots b_r(0).$$

We will now study a generalization of the n -simplices from Theorem 3.6 for $r \geq 2$ whose h^* -polynomials preserve many of the nice properties of the $r = 2$ case, including real-rootedness and unimodality. Our generalization is motivated as follows:

In order to simplify the formula for $\omega(b)$ to the desired state in the proofs of Theorem 3.5 and Theorem 3.6, respectively, we required the identities

$$1 + \sum_{k=0}^{n-1} k \cdot k! = n! \quad \text{and} \quad 1 + \sum_{k=0}^{n-1} 2^k = 2^n.$$

Notice that these identities are different from the one requested in Definition 3.1 (2) to certify reflexivity of a numeral system. In fact, it follows from [11, Theorem 2] that any mixed radix system $a = (a_n)_{n=0}^\infty$ with sequence of radices $c = (c_n)_{n=1}^\infty$ satisfies the identity

$$1 + \sum_{k=0}^{n-1} (c_{k+1} - 1)a_k = a_n.$$

In the case of base- r numeral systems, this identity yields a natural generalization of Theorem 3.6. For two integers $r \geq 2$ and $n \geq 1$, we define the *base- r n -simplex* to be the n -simplex $\mathcal{B}_{(r,n)} := \Delta_{(1,q)} \subset \mathbb{R}^n$ for

$$q := ((r-1), (r-1)r, (r-1)r^2, \dots, (r-1)r^{n-1}).$$

In the following, we show that, while symmetry of $h^*(\mathcal{B}_{(r,n)}; z)$ does not hold for $r > 2$, many of the nice properties of $h^*(\mathcal{B}_{(2,n)}; z)$ carry over to this more general family. In Subsection 4.1 we prove that $h^*(\mathcal{B}_{(r,n)}; z)$ admits a combinatorial interpretation in terms of a descent-like statistic applied to the nonzero digits of the base- r representations of the nonnegative integers. Then, in Subsection 4.2, we prove that $h^*(\mathcal{B}_{(r,n)}; z)$ is real-rooted and unimodal for all $r \geq 2$ and $n \geq 1$.

4.1. A descent-like statistic. We now define a descent-like statistic on the base- r representations of nonnegative integers that we then use to give a combinatorial interpretation of the h^* -polynomial of $\mathcal{B}_{(r,n)}$ for $r \geq 2$ and $n \geq 1$. Given two indices $i \geq j$ and an integer $b \in \mathbb{Z}_{\geq 0}$, we can think of the integer quantity $b_r(i) - b_r(j)$ as the *height* of the index i “above” the index j in the string b_r . Of course, a negative

TABLE 1. The statistic $\text{nasc}_r(b)$ for some integers b with respect to the base-4 numeral system $a = (4^n)_{n=0}^\infty$ and $n = 3$.

b	base-4 numeral	$\text{Supp}_4(b)$	awheight of numerals in $\text{Supp}_4(b)$	$\text{nasc}_4(b)$
19	103	$\{0, 2\}$	awheight(0) = 1 awheight(2) = 1	2
22	112	$\{0, 1, 2\}$	awheight(0) = 1 awheight(1) = -1 awheight(2) = $-\frac{1}{2}$	1
31	133	$\{0, 1, 2\}$	awheight(0) = 1 awheight(1) = 0 awheight(2) = -5	2

height simply means we think of i as “below” j in b_r . We define the (*average weighted*) height of an index $i > 0$ in b_r to be

$$\text{awheight}(i) := \frac{1}{i} \sum_{j=0}^{i-1} (b_r(i) - b_r(j))r^j \quad \text{and} \quad \text{awheight}(0) := \begin{cases} 0 & \text{if } b_r(0) = 0, \\ 1 & \text{if } b_r(0) \neq 0. \end{cases}$$

In a sense, this statistic measures the height of i above the remaining substring of b_r where the value of the height of an index closer to i (in absolute value) is higher. When $\text{awheight}(i)$ is nonnegative, we can think of i as being at least as high as the remaining portion of the string, and so we say that i is a *nonascent* of b_r if $0 \leq \text{awheight}(i)$. Define the *support* of b to be the set

$$\text{Supp}_r(b) := \{i \in \mathbb{Z}_{\geq 0} : b_r(i) \neq 0\} \quad \text{and let} \quad \text{supp}_r(b) := |\text{Supp}_r(b)|.$$

We then consider the collection of indices

$$\text{Nasc}_r(b) := \{i \in \text{Supp}_r(b) : 0 \leq \text{awheight}(i)\},$$

and we let $\text{nasc}_r(b) := |\text{Nasc}_r(b)|$.

Example 4.1 (Computing $\text{nasc}_r(b)$ for base-4 numerals). The base-4 numeral system is $a = (4^n)_{n=0}^\infty$. The numeral representations for the first 64 nonnegative integers are the strings of length three $b_4(2)b_4(1)b_4(0)$ in which each term $b_4(i)$ can assume values 0, 1, 2, or 3. For example, the number $b = 19$ has base-4 representation 103, and so the support of b is $\text{Supp}_4(b) = \{0, 2\}$. To compute $\text{nasc}_4(b)$ we must compute the (averaged weighted) height of the indices 0 and 2. Since $b_4(0) = 3$, then $\text{awheight}(0) = 1$, and

$$\text{awheight}(2) = \frac{1}{2} ((1 - 0) \cdot 4^1 + (1 - 3) \cdot 4^0) = 1.$$

Thus, $\text{nasc}_4(b) = 2$. Table 1 presents this statistic for a couple more integers. Given the statistic $\text{nasc}_4(b)$ for all integers $0 \leq b < 64$ we can compute

$$h^*(\mathcal{B}_{(4,3)}; z) = \sum_{b=0}^{63} z^{\text{nasc}_4(b)} = 1 + 19z + 34z^2 + 10z^3.$$

The following theorem shows that this formula generalizes to the base- r n -simplices for all $n \geq 1$ and $r \geq 2$.

Theorem 4.1. *For two integers $r \geq 2$ and $n \geq 1$ the base- r n -simplex $\mathcal{B}_{(r,n)}$ has h^* -polynomial*

$$h^*(\mathcal{B}_{(r,n)}; z) = \sum_{b=0}^{r^n-1} z^{\text{nasc}_r(b)}.$$

Proof. By [4, Theorem 2.5], it suffices to show for $0 \leq b < r^n$ that $\omega(b) = \text{nasc}_r(b)$. We prove this fact via induction on n . Notice first that

$$\omega(b) = b - \sum_{k=1}^n \left\lfloor \frac{(r-1)b}{r^k} \right\rfloor.$$

For the base, take $n = 1$, and notice that $\omega(b) = \left\lfloor \frac{b}{r} \right\rfloor$, and so $\omega(b) = 0$ where $b = 0$, and $\omega(b) = 1$ for $1 \leq b \leq r-1$. Suppose now that the result holds for $n-1$. Letting $b' := b - b_r(n-1)r^{n-1}$, we then observe that

$$\begin{aligned} \omega(b) &= b - \sum_{k=1}^n \left\lfloor \frac{(r-1)b}{r^k} \right\rfloor \\ &= b - \sum_{k=1}^n \left\lfloor \frac{(r-1)(b_r(n-1)r^{n-1} + b')}{r^k} \right\rfloor \\ &= b - \sum_{k=1}^n \left\lfloor \frac{(r-1)b_r(n-1)r^{n-1}}{r^k} \right\rfloor - \sum_{k=1}^n \left\lfloor \frac{(r-1)b'}{r^k} \right\rfloor - \left\lfloor \frac{(r-1)b}{r^n} \right\rfloor \\ &= \omega(b') + b_r(n-1)r^{n-1} - b_r(n-1) \left(\sum_{k=0}^{n-2} r^k \right) - \left\lfloor \frac{(r-1)b}{r^n} \right\rfloor \\ &= \omega(b') + \left\lfloor \frac{rb_r(n-1)r^{n-1} - (r-1)b'}{r^n} \right\rfloor. \end{aligned}$$

Notice that $rb_r(n-1)r^{n-1} - (r-1)b' > 0$ if and only if $b_r(n-1) \neq 0$ and

$$\begin{aligned} b' &< b_r(n-1) \left(\frac{r^{n-1}}{r-1} \right), \\ \sum_{j=0}^{n-2} b_r(j)r^j &< \sum_{j=0}^{n-2} b_r(n-1)r^j + \frac{b_r(n-1)}{r-1} \\ - \frac{b_r(n-1)}{(r-1)(n-1)} &< \frac{1}{n-1} \sum_{j=0}^{n-2} (b_r(n-1) - b_r(j))r^j. \end{aligned}$$

Since $1 \leq b_r(n-1) \leq r-1$, this last inequality is equivalent to $n-1$ being a nonascent of b_r , which completes the proof. \square

4.2. Real-rootedness and unimodality. To prove real-rootedness of the h^* -polynomial of $\mathcal{B}_{(r,n)}$, we will use the well-developed theory of interlacing polynomials. Let $f, g \in \mathbb{R}[z]$ be nonzero, real-rooted polynomials, and let $d := \deg(f)$, and $c := \deg(g)$ denote the degree of f and g , respectively. Suppose $\alpha_d \leq \dots \leq \alpha_2 \leq \alpha_1$ and $\beta_c \leq \dots \leq \beta_2 \leq \beta_1$ are the roots of f and g , respectively. We say that g *interlaces* f , written $g \preceq f$, if either $d = c$ and

$$\beta_d \leq \alpha_d \leq \dots \leq \beta_2 \leq \alpha_2 \leq \beta_1 \leq \alpha_1,$$

or $d = c + 1$ and

$$\alpha_{d+1} \leq \beta_d \leq \alpha_d \leq \dots \leq \beta_2 \leq \alpha_2 \leq \beta_1 \leq \alpha_1.$$

If all inequalities are strict, we say that g *strictly interlaces* f and we write $g \prec f$.

A sequence $F_m := (f_i)_{i=1}^m$ of real-rooted polynomials is called (*strictly*) *interlacing* if f_i (strictly) interlaces f_j for all $1 \leq i < j \leq m$. Let \mathcal{F}_m^+ denote the space of all interlacing sequences F_m for which f_i has only nonnegative coefficients for all $1 \leq i \leq m$. In [5], Brändén characterized when a matrix $G = (G_{i,j}(z))_{i,j=1}^m$ of polynomials maps \mathcal{F}_m^+ to \mathcal{F}_m^+ . We say that such a map *preserves interlacing*. It preserves *strict* interlacing if it further maps strictly interlacing sequences to strictly interlacing sequences. In the following we use such polynomial maps to prove the real-rootedness of $h^*(\mathcal{B}_{(r,n)}; z)$ for $r \geq 2, n \geq 1$.

For $r \geq 2$ and $n \geq 1$ define the univariate polynomial

$$f_{(r,n)} := (1 + z + z^2 + \dots + z^{r-1})^n.$$

As noted in [14], for every $r \geq 1$ and $f \in \mathbb{R}[z]$ there are uniquely determined $f^{(0)}, \dots, f^{(r-2)} \in \mathbb{R}[z]$ such that

$$f(z) = f^{(0)}(z^{r-1}) + zf^{(1)}(z^{r-1}) + \dots + z^{r-2}f^{(r-2)}(z^{r-1}).$$

So, for $\ell = 0, 1, \dots, r - 2$, we consider the operator

$$\langle r-1, \ell \rangle : \mathbb{R}[z] \longrightarrow \mathbb{R}[z] \quad \text{where} \quad \langle r-1, \ell \rangle : f \longrightarrow f^{(\ell)}.$$

The following theorem gives a second interpretation of the h^* -polynomial of $\mathcal{B}_{(r,n)}$, now in terms of the polynomials $f_{(r,n)}^{(r-1,\ell)}$.

Theorem 4.2. *For two integers $r \geq 2$ and $n \geq 1$ the base- r n -simplex $\mathcal{B}_{(r,n)}$ has h^* -polynomial*

$$h^*(\mathcal{B}_{(r,n)}; z) = f_{(r,n)}^{(r-1,0)} + z \sum_{\ell=1}^{r-2} f_{(r,n)}^{(r-1,\ell)}.$$

Proof. To prove this result, we prove a slightly stronger statement. We will show, via induction on n , that

$$\sum_{b=0}^{1+r+r^2+\dots+r^{n-1}} z^{\omega(b)} = f_{(r,n)}^{(r-1,0)},$$

and for each $i \in [r - 2]$,

$$\sum_{b=1+i(1+r+r^2+\dots+r^{n-1})}^{(i+1)(1+r+r^2+\dots+r^{n-1})} z^{\omega(b)} = z f_{(r,n)}^{(r-1,r-\ell-1)}.$$

For the base case, we let $n = 1$, and so we must verify that

$$z^{\omega(0)} + z^{\omega(1)} = f_{(r,1)}^{(r-1,0)},$$

and for each $i \in [r - 2]$,

$$z^{\omega(i+1)} = z f_{(r,1)}^{(r-1,r-\ell-1)}.$$

Notice first that since $n = 1$, then $\omega(b) = \lceil \frac{b}{r} \rceil$ for all $b = 0, 1, \dots, r - 1$, and so

$$\omega(b) = \begin{cases} 0 & \text{if } b = 0, \\ 1 & \text{if } b \in [r - 1]. \end{cases}$$

The base case follows immediately from the fact that $f_{(r,1)} = 1 + r + r^2 + \dots + r^{n-1}$.

As for the inductive step, we begin by partitioning the sequence of numbers

$$\mathbb{B} = (\mathbb{B}_j)_{j=0}^{r^n-1} := (0, 1, 2, \dots, r^n - 1)$$

into r consecutive sequences

$$B_i = (B_{i,j})_{j=0}^{r^{n-1}-1} := (\mathbb{B}_j)_{j=ir^{n-1}}^{(i+1)r^{n-1}-1}$$

for $i = 0, 1, \dots, r - 1$. Notice that if $b \in B_i$ then $b_r(n - 1) = i$. Even more, the number b has the base- r representation $b_r(n - 1)b_r(n - 2) \dots b_r(1)b_r(0)$ being the b th sequence of n digits $0, 1, \dots, r - 1$ in lexicographic ordering. It then follows that for all $i = 1, 2, \dots, r - 1$

$$(3) \quad \omega(B_{i,j}) = \begin{cases} \omega(B_{0,j}) + 1 & \text{if } j \leq 1 + r + r^2 + \dots + r^{n-1}, \\ \omega(B_{0,j}) & \text{if } j > 1 + r + r^2 + \dots + r^{n-1}. \end{cases}$$

This is because the base- r representation of $b = i(1+r+r^2+\dots+r^{n-1})$ is $b_r = ii \dots i$. Combining the observation in equation (3) with the inductive hypothesis, we then have that

$$\begin{aligned} \sum_{b=0}^{1+r+r^2+\dots+r^{n-1}} z^{\omega(b)} &= \sum_{b=0}^{r^{n-1}-1} z^{\omega(b)} + \sum_{b=r^{n-1}}^{1+r+r^2+\dots+r^{n-1}} z^{\omega(b)} \\ &= h^*(\mathcal{B}_{(r,n-1)}; z) + z f_{(r,n-1)}^{\langle r-1,0 \rangle} \\ &= f_{(r,n-1)}^{\langle r-1,0 \rangle} + z \sum_{\ell=1}^{r-2} f_{(r,n-1)}^{\langle r-1,\ell \rangle} + z f_{(r,n-1)}^{\langle r-1,0 \rangle} \\ &= f_{(r,n)}^{\langle r-1,0 \rangle}, \end{aligned}$$

which proves the first part of the claim.

For the second part of the claim, we just want to see that for $i = 1, 2, \dots, r - 2$

$$\sum_{b=B_{i,2+r+r^2+\dots+r^{n-2}}}^{B_{i,r^{n-1}-1}} z^{\omega(b)} + \sum_{b=B_{i+1,0}}^{B_{i+1,1+r+r^2+\dots+r^{n-2}}} z^{\omega(b)} = z f_{(r,n)}^{\langle r-1,r-\ell-1 \rangle}.$$

However, by the inductive hypothesis and equation (3) it follows that

$$\begin{aligned} \sum_{b=B_{i,2+r+r^2+\dots+r^{n-2}}}^{B_{i,r^{n-1}-1}} z^{\omega(b)} + \sum_{b=B_{i+1,0}}^{B_{i+1,1+r+r^2+\dots+r^{n-2}}} z^{\omega(b)} &= \sum_{\ell=i}^{r-2} f_{(r,n-1)}^{\langle r-1,\ell \rangle} + z \sum_{\ell=0}^i f_{(r,n-1)}^{\langle r-1,\ell \rangle} \\ &= f_{(r,n)}^{\langle r-1,r-\ell-1 \rangle}. \end{aligned}$$

Thus, the claim holds for all $r \geq 2$ and $n \geq 1$. The desired expression for $h^*(\mathcal{B}_{(r,n)}; z)$ then follows immediately from [4, Theorem 2.5]. □

Example 4.2 (The h^* -polynomial of a base-4 simplex). Example 4.1 presents the h^* -polynomial of the base-4 3-simplex $\mathcal{B}_{(4,3)}$ in terms of the average weighted height statistic. We can recompute this polynomial using the formula proved in

Theorem 4.2. If we expand the polynomial $f_{(4,3)}$ and write it diagrammatically as

$$\begin{aligned} f_{(4,3)} &= (1 + z + z^2 + z^3)^3 \\ &= 1 + 3z + 6z^2 + 10z^3 + 12z^4 + 12z^5 + 10z^6 + 6z^7 + 3z^8 + z^9 \\ &= 1 \qquad \qquad + 10z^{1 \cdot 3} \qquad \qquad + 10z^{2 \cdot 3} \qquad \qquad + z^{3 \cdot 3} \\ &\quad + 3z^{0 \cdot 3+1} \qquad \quad + 12z^{1 \cdot 3+1} \qquad \quad + 6z^{2 \cdot 3+1} \\ &\quad \quad + 6z^{0 \cdot 3+2} \qquad \quad + 12z^{1 \cdot 3+2} \qquad \quad + 3z^{2 \cdot 3+2}, \end{aligned}$$

then we see by the decomposed presentation of $f_{(4,3)}$ in the third equality that

$$\begin{aligned} f_{(4,3)}^{(3,0)} &= 1 + 10z + 10z^2 + z^3, \\ f_{(4,3)}^{(3,1)} &= 3 + 12z + 6z^2, \text{ and} \\ f_{(4,3)}^{(3,2)} &= 6 + 12z + 3z^2. \end{aligned}$$

Then, by Theorem 4.2, we know that

$$(4) \quad \begin{aligned} h^*(\mathcal{B}_{(4,3)}; z) &= (1 + 10z + 10z^2 + z^3) + z(9 + 24z + 9z^2) \\ &= 1 + 19z + 34z^2 + 10z^3, \end{aligned}$$

and thus we recover the h^* -polynomial originally computed in Example 4.1.

Remark 4.1 (The symmetric decomposition of an h^* -polynomial). The first line of equation (4) in Example 4.2 highlights a more general phenomenon. It is a well-known result that if P is a lattice polytope containing an interior lattice point, then there is a unique decomposition of $h^*(P; z)$ as

$$h^*(P; z) = a(z) + zb(z),$$

where $a(z) = z^d a(\frac{1}{z})$ and $b(z) = z^{d-1} b(\frac{1}{z})$. Moreover, these polynomials admit a nice combinatorial interpretation as

$$\begin{aligned} a(z) &= \sum_{\Delta \in T} h(\text{link}_T(\Delta); z) B_{\Delta}(z), \text{ and} \\ b(z) &= \frac{1}{z} \sum_{\Delta \in T} h(\text{link}(\Delta); z) B_{\text{conv}(\Delta, 0)}(z), \end{aligned}$$

where T is any triangulation of the boundary of P , $h(\text{link}_T(\Delta); z)$ is the h -polynomial of the link of the simplex Δ in T , and $B_S(z)$ is the box polynomial of a simplex S . A summary of these various definitions and a proof of this decomposition of $h^*(P; z)$ is provided in [1, Chapter 10, Theorem 10.5].

For the base- r n -simplex $\mathcal{B}_{(r,n)}$, it follows from Theorem 4.2 that

$$a(z) = f_{(r,n)}^{\langle r-1, 0 \rangle} \quad \text{and} \quad b(z) = \sum_{\ell=1}^{r-2} f_{(r,n)}^{\langle r-1, \ell \rangle}.$$

In Theorem 4.5, we will use the formulation of $h^*(\mathcal{B}_{(r,n)}; z)$ given in Theorem 4.2 to prove that $h^*(\mathcal{B}_{(r,n)}; z)$ is real-rooted and unimodal. In fact, it follows along the way, that the symmetric polynomials $a(z)$ and $b(z)$ for $\mathcal{B}_{(r,n)}$ have these properties as well. To the best of the author’s knowledge, this is the first known proof of real-rootedness of a (nonsymmetric) h^* -polynomial that comes by way of proving its symmetric decomposition consists of real-rooted polynomials as well.

We also note that Theorem 4.2 provides us with a second combinatorial interpretation of the coefficients of $h^*(\mathcal{B}_{(r,n)}; z)$. Given a subset $S \subset \mathbb{Z}_{>0}$ and two integers $t, m \in \mathbb{Z}_{>0}$, we let $\text{comp}_t(m; S)$ denote the number of compositions of m of length t with parts in S . Since for all $k \in \mathbb{Z}_{\geq 0}$ the coefficient of z^k in $f_{(r,n)}$ is

$$[z^k] \cdot f_{(r,n)} = \text{comp}_n(n + k; [r]),$$

then we have the following corollary to Theorem 4.2.

Corollary 4.3. *For integers $r \geq 2$ and $n \geq 1$*

$$[z^k] \cdot h^*(\mathcal{B}_{(r,n)}; z) = \text{comp}_n(n + k; [r]) + \sum_{\ell=1}^{r-2} \text{comp}_n(n + (k - 1)(r - 1) + \ell; [r])$$

for each $k = 0, 1, \dots, n$.

We now use Theorem 4.2 to verify that $h^*(\mathcal{B}_{(r,n)}; z)$ is real-rooted. To do so, we first prove that a useful polynomial map G preserves (strict) interlacing.

Lemma 4.4. *The polynomial map*

$$G := \begin{pmatrix} z + 1 & 1 & 1 & \cdots & 1 \\ z & z + 1 & 1 & & \vdots \\ z & z & z + 1 & \ddots & 1 \\ \vdots & & \ddots & \ddots & 1 \\ z & z & \cdots & z & z + 1 \end{pmatrix} \in \mathbb{R}[z]^{(r-1) \times (r-1)}$$

preserves strict interlacing.

Proof. In [5, Theorem 7.8.5] Brändén gives a complete characterization of all such matrices. Applying this characterization, it suffices to prove that each of the five 2×2 matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} z & z \\ z & z \end{pmatrix}, \begin{pmatrix} z + 1 & 1 \\ z & z + 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ z + 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} z & z + 1 \\ z & z \end{pmatrix}$$

preserve interlacing and nonnegativity. This follows from a series of results in [10, Section 3.11] proven by Fisk. In particular, the result follows for the first two matrices by applying [10, Lemma 3.71], for the third matrix by [10, Lemma 3.79], and for the fourth matrix by [10, Lemma 3.83(1)]. Finally, the fifth matrix is seen to preserve interlacing and nonnegativity by factoring it as

$$\begin{pmatrix} z & z + 1 \\ z & z \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ z & z \end{pmatrix},$$

and applying [10, Lemma 3.71] to each of these factors. □

Using Lemma 4.4, we can now prove our main result of this subsection.

Theorem 4.5. *For two integers $r \geq 2$ and $n \geq 1$, the h^* -polynomial of the base- r n -simplex $\mathcal{B}_{(r,n)}$ is real-rooted and thus unimodal.*

Proof. By Theorem 4.2 we know that the h^* -polynomial of $\mathcal{B}_{(r,n)}$ is expressible as

$$h^*(\mathcal{B}_{(r,n)}; z) = f_{(r,n)}^{\langle r-1,0 \rangle} + z \sum_{\ell=1}^{r-2} f_{(r,n)}^{\langle r-1,\ell \rangle}.$$

Notice next that

$$(5) \quad f_{(r,n)} = (1 + z + z^2 + \cdots + z^{r-1})f_{(r,n-1)}.$$

For each index k , let $a_k := [z^k].f_{(r,n-1)}$ and $b_k := [z^k].f_{(r,n)}$. Then recall that we can write each k uniquely as $k = i(r-1) + j$ for integers i and $0 \leq j < r-1$. It follows that

$$[z^i].f_{(r,n-1)}^{(r-1,j)} = a_{i(r-1)+j}, \quad \text{and similarly} \quad [z^i].f_{(r,n)}^{(r-1,j)} = b_{i(r-1)+j}.$$

Therefore, for each $\ell = 0, 1, \dots, r-2$, it follows from equation (5) that

$$[z^i].f_{(r,n)}^{(r-1,\ell)} = b_{i(r-1)+\ell} = \sum_{j=0}^{\ell} a_{i(r-1)+(\ell-j)} + \sum_{j=\ell}^{r-2} a_{(i-1)(r-1)+(r-2+\ell-j)}.$$

For an example of this computation, we refer the reader to Example 4.2. More concisely, this expression for $[z^i].f_{(r,n)}^{(r-1,\ell)}$ for each index i is equivalent to saying that the vector of polynomials

$$\left(f_{(r,n)}^{(r-1,r-2)} \quad \cdots \quad f_{(r,n)}^{(r-1,1)} \quad f_{(r,n)}^{(r-1,0)} \right)^T$$

is produced by multiplying the vector of polynomials

$$\left(f_{(r,n-1)}^{(r-1,r-2)} \quad \cdots \quad f_{(r,n-1)}^{(r-1,1)} \quad f_{(r,n-1)}^{(r-1,0)} \right)^T$$

on the left by the $(r-1) \times (r-1)$ matrix of polynomials G in Lemma 4.4. Thus, by way of induction, Lemma 4.4 implies that $f_{(r,n)}^{(r-1,r-2)} \prec \cdots \prec f_{(r,n)}^{(r-1,1)} \prec f_{(r,n)}^{(r-1,0)}$ is a sequence of strictly mutually interlacing and nonnegative polynomials. Moreover, it is well known that the polynomial map

$$H := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ z & 1 & 1 & & \vdots \\ z & z & 1 & \ddots & 1 \\ \vdots & & \ddots & \ddots & 1 \\ z & z & \cdots & z & 1 \end{pmatrix} \in \mathbb{R}[z]^{(r-2) \times (r-2)}$$

also preserves interlacing. For instance, proofs of this fact can be found in [10, Example 3.73], [5, Corollary 7.8.7], and [14, Proposition 2.2]. Applying this map once to the vector of polynomials

$$\left(f_{(r,n)}^{(r-1,r-2)} \quad \cdots \quad f_{(r,n)}^{(r-1,1)} \quad f_{(r,n)}^{(r-1,0)} \right)^T$$

produces a vector of polynomials

$$(g_{r-2} \quad \cdots \quad g_1 \quad g_0)^T$$

and it follows that $g_{r-2} \prec \cdots \prec g_1 \prec g_0$ is a sequence of strictly mutually interlacing polynomials with the property that $g_0 = h^*(\mathcal{B}_{(r,n)}; z)$. The result then follows. \square

Remark 4.2. In order to prove Theorem 4.5 we used Lemma 4.4 to first show that

$$\left(f_{(r,n)}^{(r-1,r-\ell-1)} \right)_{\ell=1}^{r-1}$$

is a strictly interlacing sequence. Other important h -polynomials have been shown to be real-rooted using a closely related construction. In particular, in order to verify a conjecture of [2], Jochemko shows in [14] that the sequence

$$\left(f_{(r,n)}^{\langle r, r-\ell \rangle} \right)_{\ell=1}^r$$

is strictly interlacing. Similarly, in [16] and [20] Leander and Zhang independently showed that

$$\left(f_{(r,n)}^{\langle r+1, r-\ell+1 \rangle} \right)_{\ell=1}^{r+1}$$

is a strictly interlacing sequence in order to prove that the r th edgewise and cluster subdivisions of the simplex have real-rooted local h -polynomials. Each of these strictly interlacing sequences constitutes a distinct family of real-rooted polynomials, and collectively they represent the growing prevalence of decompositions of the polynomial $f_{(r,n)}$ in unimodality questions for h -polynomials.

5. A CLOSING REMARK

To conclude our discussions, we remark that two natural classes of simplices associated to numeral systems have been introduced and analyzed in this note. In Section 3, we searched for numeral systems admitting divisor systems that allowed us to construct a q -vector yielding a reflexive n -simplex with normalized volume the n th place value for all $n \geq 0$. In the identified examples, we saw that these numeral systems yield combinatorial interpretations of the associated h^* -polynomials that are closely related to interpretations for the associated q -vectors. Furthermore, these examples all exhibited the desirable distributional properties implied by real-rootedness. To produce more examples of this nature, the (seemingly difficult) problem is to identify a divisor system for some numeral system a , and then study the geometry of the simplex whose q -vector is given by identity (2) in Definition 3.1.

On the other hand, in Section 4, we used a natural choice of q -vector for any mixed radix numeral system to generalize the geometry associated to the binary numeral system in Theorem 3.6. The result is a family of nonreflexive simplices associated to the base- r numeral systems for $r \geq 2$. We observed that while symmetry of the h^* -polynomial is lost in this generalization, real-rootedness is preserved. This suggests a possible extension to a larger family of simplices with real-rooted h^* -polynomials, namely, the simplices associated to mixed radix systems via an analogous choice of q -vector. Amongst such mixed radix simplices, the simplices $\mathcal{B}_{(r,n)}$ correspond exactly to mixed radix systems in which all radices are taken to be equal, and our proof of real-rootedness relies heavily on this fact. Thus, new techniques may be necessary to address real-rootedness for this larger family. On a related note, computations suggest that the simplices $\mathcal{B}_{(r,n)}$ are also *Ehrhart positive*. This curious observation further serves to promote the study of simplices for numeral systems from the combinatorial perspective.

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