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ELEMENTARY EQUIVALENCE VS. COMMENSURABILITY FOR HYPERBOLIC GROUPS

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Abstract. We study to what extent torsion-free (Gromov)-hyperbolic groups are elementarily equivalent to their finite index subgroups. In particular, we prove that a hyperbolic limit group either is a free product of cyclic groups and surface groups or admits infinitely many subgroups of finite index which are pairwise non-elementarily equivalent.

1. Introduction

There are many ways in which one may try to classify finitely generated (or finitely presented) groups. The most obvious one is up to isomorphism, or to commensurability (having isomorphic subgroups of finite index). More generally, Gromov suggested the looser notions of quasi-isometry and measure equivalence (commensurable groups are quasi-isometric and measure equivalent, though the converse is far from true).

First-order logic provides us with a coarser relation than the relation of isomorphism, where one tries to classify groups up to elementary equivalence. Two groups are elementarily equivalent if and only if they satisfy the same first-order sentences. As usual, one must restrict the class of groups under consideration. A representative example is the work of Szmielew [Szm55] that characterized all abelian groups (possibly infinitely generated) up to elementary equivalence.

Our concern here is to investigate the relation between elementary equivalence and commensurability for the special class of torsion-free hyperbolic groups, a class of groups almost orthogonal to the class of abelian groups (hyperbolic will always mean Gromov-hyperbolic, also called word-hyperbolic or δ-hyperbolic).

The solution of the Tarski problem by Kharlampovich-Myasnikov [KM06] and Sela [Sel06] says that all non-abelian free groups are elementarily equivalent. Even more, both works characterized the finitely generated groups that are elementarily free (i.e., elementarily equivalent to a non-abelian free group). Specifically, there exist finitely generated groups which are elementarily free but not free. The simplest examples are fundamental groups of closed surfaces with Euler characteristic

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at most -2, and more generally non-abelian free products of such surface groups and cyclic groups. This particular family is closed under taking finite index subgroups; in particular, these groups are elementarily equivalent to their finite index subgroups.

On the other hand, if G, G' are two torsion-free hyperbolic groups with no non-trivial cyclic splitting (fundamental groups of closed hyperbolic 3-manifolds, for instance), the situation is rigid: if G and G' are elementarily equivalent (or just have the same universal theory), then they are isomorphic (see Proposition 7.1 of [Sel09]). In particular, being co-Hopfian [Sel97], the group G is not elementarily equivalent to any of its proper finite index subgroups.

All finitely generated elementarily free groups are torsion-free and hyperbolic (see [Sel06, p. 713] and [Sel01]). It is therefore natural to ask whether all finitely generated elementarily free groups are elementarily equivalent to their finite index subgroups, and more generally which torsion-free hyperbolic groups are elementarily equivalent to their finite index subgroups.

We shall see that the answer to the first question is very much negative. Since it is conceivable that there exist hyperbolic groups with no proper finite index subgroups, we focus on classes of hyperbolic groups which have many finite index subgroups: limit groups and cubulable groups.

Limit groups were first introduced as finitely generated ω-residually free groups. They may be viewed as limits of free groups in the space of marked groups [CG05]. In model theoretic terms, non-abelian limit groups may be characterized as finitely generated groups having the same universal theory as non-abelian free groups [Rem89]. Some of them, for instance free products of cyclic groups and surface groups as above, are elementarily free, but most of them are not. A group is cubulable (or cubulated) if it acts properly discontinuously and cocompactly on a CAT(0) cubical complex. Cubulable hyperbolic groups played a prominent role in the proof by Agol and Wise of the virtually Haken conjecture about 3-dimensional manifolds [Ago13].

Our main results are the following:

Theorem 1.1. *Let G be a torsion-free hyperbolic group. If G is a limit group or is cubulable, then either G is a free product of surface groups and cyclic groups or it has infinitely many normal subgroups of finite index which are all different up to elementary equivalence.*

Corollary 1.2. *A finitely generated elementarily free group is elementarily equivalent to all its finite index subgroups if and only if it is a free product of surface groups and cyclic groups.*

We also prove:

Theorem 1.3. *There exists a one-ended hyperbolic limit group G which is not elementarily free but has an elementarily free finite index subgroup.*

Among torsion-free hyperbolic groups, Sela [Sel09] singled out a special family, which he called elementary prototypes, with the property that two elementary prototypes are elementarily equivalent if and only if they are isomorphic. Our way of proving Theorem 1.1 is to construct non-isomorphic finite index subgroups of G which are elementary prototypes. In Theorem 1.3 we construct G as the fundamental group of a surface with a socket.

Let us now describe the contents of the paper in a more detailed way.

In order to make our main results as independent as possible of the deep theory needed to solve Tarski's problem and establish the results of [Sel09], we view one-ended prototypes as groups having no non-injective preretraction (in the sense of [Per11]), and in Section 5 we give a direct argument to show that for such groups elementary equivalence is the same as isomorphism. As the name suggests, a non-injective preretraction yields a retraction and thus a hyperbolic floor [Per11].

A one-ended torsion-free hyperbolic group has a canonical splitting over cyclic groups, its JSJ splitting [Sel97, Bow98, GL]. A special role is played by the quadratically hanging (QH) vertex groups: they are isomorphic to the fundamental groups of compact surfaces, and incident edge groups are boundary subgroups.

Using a key technical lemma proved in Section 3, we show (Proposition 4.7) that the existence of a non-injective preretraction is equivalent to that of a surface Σ associated to a QH vertex group of the JSJ decomposition and a map $p : \pi_1(\Sigma) \to G$ with the following properties: p is a conjugation on every boundary subgroup, is not an isomorphism of $\pi_1(\Sigma)$ onto a conjugate, and has non-abelian image. We also show that p cannot exist if the genus of Σ is too small (Corollary 3.5).

The main new results are contained in the last two sections. We first consider fundamental groups of surfaces with sockets, obtained by enlarging the fundamental group of a surface with boundary by adding roots of generators of boundary subgroups. We determine which of these groups are limit groups, which of them are elementarily free, and which of them are elementary prototypes (i.e., do not have non-injective preretractions). In particular, we prove Theorem 1.3 by showing that the fundamental group of an orientable surface of genus 2 with a socket of order 3 is a limit group which is an elementary prototype but has an elementarily free subgroup of index 3. We also study a related family of groups, for which all sockets are identified.

In Section 7 we prove Theorem 1.1, using the fact that a one-ended torsion-free hyperbolic group G is an elementary prototype if its cyclic JSJ decomposition satisfies the following conditions: all surfaces are orientable, the girth of the graph of groups is large compared to the complexity of the surfaces which appear, and edge groups map injectively to the abelianization of G. This condition about edge groups is the main reason why we restrict to limit groups and cubulable groups.

2. Preliminaries

Graphs of groups. Graphs of groups are a generalization of free products with amalgamation and HNN-extensions. Recall that a graph of groups $\mathcal{G}$ consists of a graph Γ, the assignment of a group G_v or G_e to each vertex v or non-oriented edge e, and injections $G_e \to G_v$ whenever v is the origin of an oriented edge (see [Ser77] for precise definitions); the image of G_e is called an incident edge group of G_v. This data yields a group $G(\mathcal{G})$, the fundamental group of $\mathcal{G}$, with an action of $G(\mathcal{G})$ on a simplicial tree T (the Bass-Serre tree of $\mathcal{G}$). We sometimes call the graph of groups Γ rather than $\mathcal{G}$.

A graph of groups decomposition (or splitting) of a group G is an isomorphism of G with the fundamental group of a graph of groups. The vertex and edge groups G_v and G_e are then naturally viewed as subgroups of G. We always assume that G is finitely generated and that the graph of groups is minimal (no proper connected subgraph of Γ carries the whole of G); it follows that Γ is a finite graph. We allow the trivial splitting, with Γ a single vertex v with $G_v = G$.

Killing all vertex groups defines an epimorphism from G to the topological fundamental group of the graph Γ (a free group, possibly cyclic or trivial).

The (closed) star of a vertex v of Γ is the union of all edges containing v. The open star of v is obtained by removing all vertices $w \neq v$ from the closed star.

Grushko decompositions. A finitely generated group has a Grushko decomposition: it is isomorphic to a free product $A_1 * \cdots * A_n * \mathbb{F}_m$, where each A_i for $i \leq n$ is non-trivial, freely indecomposable, and not infinite cyclic, and $\mathbb{F}_m$ is a free group of rank m. Moreover, the numbers n and m are unique, as well as the A_i's up to conjugacy (in G) and permutation. We call the A_i's the Grushko factors of G. When G is torsion-free, they are precisely the one-ended free factors of G by a classical theorem of Stallings.

Decompositions as free products correspond to graph of groups decompositions with trivial edge groups. A one-ended group does not split over a trivial group, but it may split over $\mathbb{Z}$. We discuss this when G is a torsion-free hyperbolic group.

Hyperbolic groups and their cyclic JSJ decompositions. Recall that a finitely generated group is hyperbolic if its Cayley graph (with respect to some, hence any, finite generating set) is a hyperbolic metric space: there exists $\delta > 0$ such that any point on one side of any geodesic triangle is δ-close to one of the other two sides. Small cancellation groups and fundamental groups of negatively curved closed manifolds are hyperbolic. In particular, free groups, fundamental groups of closed surfaces with negative Euler characteristic, and free products of such groups are hyperbolic. A one-ended torsion-free hyperbolic group G is co-Hopfian [Sel97]: it cannot be isomorphic to a proper subgroup.

A one-ended torsion-free hyperbolic group G has a canonical JSJ decomposition Γ_{can} over cyclic groups [Sel97, Bow98, GL]. We mention the properties that will be important for this paper. The graph Γ_{can} is bipartite, with every edge joining a vertex carrying a cyclic group to a vertex carrying a non-cyclic group. The action of G on the associated Bass-Serre tree T is invariant under automorphisms of G and acylindrical in the following strong sense: if a non-trivial element $g \in G$ fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its midpoint has cyclic stabilizer.

Surface-type vertices. There are two kinds of vertices of Γ_{can} carrying a non-cyclic group: rigid ones and quadratically hanging (QH) ones. We will be concerned mostly with QH vertices.

If v is a QH vertex of Γ_{can}, the group G_v is isomorphic to the fundamental group of a compact (possibly non-orientable) surface Σ. Incident edge groups G_e are boundary subgroups of $\pi_1(\Sigma)$: there exists a component C of $\partial\Sigma$ such that the image of G_e is equal to $\pi_1(C)$. We do not specify base points, so we should really say that G_e is conjugate to $\pi_1(C)$ in $\pi_1(\Sigma)$. Similarly, any non-null-homotopic simple closed curve in Σ determines an infinite cyclic subgroup of $\pi_1(\Sigma)$, well-defined up to conjugacy.

Moreover, given any component C of $\partial\Sigma$, there is a unique incident edge e such that G_e equals $\pi_1(C)$. These properties are not true for general QH vertices (see [GL]), so as in [Per11] we recall them by calling QH vertices v of Γ_{can} *surface-type vertices* (see Definition 3.1).

Surfaces. The genus $g = g(\Sigma)$ of a compact surface Σ is the largest number of non-intersecting (possibly one-sided) simple closed curves (other than components of $\partial\Sigma$) that can be drawn on the surface without disconnecting it. If $\Sigma_1, \dots, \Sigma_k$ are disjoint compact subsurfaces of Σ, one clearly has $\sum_k g(\Sigma_i) \leq g(\Sigma)$. Surfaces of genus 0 are obtained by removing open discs from a sphere, and they are called planar.

If Σ has b boundary components, its Euler characteristic is $\chi(\Sigma) = 2 - 2g - b$ if Σ is orientable; $\chi(\Sigma) = 2 - g - b$ otherwise. Surfaces appearing in surface-type vertices will always have non-abelian fundamental group; equivalently, $\chi(\Sigma)$ will be negative. Two surfaces are homeomorphic if and only if they are both orientable or non-orientable, and g and χ are the same.

Given a compact surface Σ, there is an upper bound for the cardinality of a family of disjoint non-parallel simple closed curves which are not null-homotopic (a curve is null-homotopic if it bounds a disc; two curves are parallel if they bound an annulus); for instance, the bound is $3g - 3$ if Σ is a closed orientable surface of genus $g \geq 2$.

There are 5 surfaces with $\chi(\Sigma) = -1$. The orientable ones are the pair of pants ($g = 0$, $b = 3$) and the once-punctured torus ($g = 1$, $b = 1$). The non-orientable ones are the twice-punctured projective plane ($g = 1$, $b = 2$), the once-punctured Klein bottle ($g = 2$, $b = 1$), and the closed surface of genus 3. With the exception of the punctured torus, these surfaces do not carry pseudo-Anosov diffeomorphisms and are not allowed to appear in hyperbolic towers (in the sense of [Sel01, Per11]). They should be considered as exceptional (see Definition 4.3).

Actions of surface groups on trees. Let Σ be a compact surface. We describe a standard construction associating a finite family $\mathcal{C}$ of disjoint simple closed curves on Σ to an action of $\pi_1(\Sigma)$ on a tree T such that boundary subgroups are elliptic (they fix a vertex in T).

The group $\pi_1(\Sigma)$ acts freely on the universal covering $\tilde{\Sigma}$. It also acts on T, and we construct an equivariant continuous map $\tilde{f} : \tilde{\Sigma} \to T$.

First suppose that Σ is closed. Fix a triangulation of Σ and lift it to the universal covering $\tilde{\Sigma}$. We define $\tilde{f}$ on the set of vertices, making sure that vertices $\tilde{\Sigma}$ are mapped to vertices of T. We then extend it to edges in a linear way and to triangles (2-simplices). There is a lot of freedom in this construction, but we make sure that $\tilde{f}$ is in general position with respect to midpoints of edges: the preimage $\tilde{\mathcal{C}}$ of the set of midpoints of edges of T intersects each triangle in a finite collection of disjoint arcs joining one side of the triangle to another side. The construction is the same if Σ has a boundary, but we require that each line in $\partial\tilde{\Sigma}$ be mapped to a single vertex of T (this is possible because boundary subgroups act elliptically on T).

The subset $\tilde{\mathcal{C}} \subset \tilde{\Sigma}$ is $\pi_1(\Sigma)$-invariant, and its projection to Σ is a finite family $\mathcal{C}$ of disjoint simple closed curves.

Limit groups. A group G is residually free if, for every non-trivial element $g \in G$, there exists a morphism $h : G \to \mathbb{F}$ for some free group $\mathbb{F}$ such that $h(g)$ is non-trivial.

A group G is ω-residually free if, for every finite set $\{g_1, \dots, g_n\} \subset G \setminus \{1\}$, there exists a morphism $h : G \to \mathbb{F}$ for some free group $\mathbb{F}$ such that $h(g_i)$ is non-trivial for all $i \leq n$.

Remeslennikov [Rem89] proved that the class of ω-residually free groups coincides with the class of $\forall$-free groups, i.e., the class of groups that have the same universal theory as a free group. In his work on the Tarski problem, Sela viewed finitely generated ω-residually free groups in a more geometric way and called them limit groups because they arise from limiting processes (see [Sel01, CG05]). Limit groups have the same universal theory as a free group, but they are not necessarily elementarily free.

Free abelian groups and free groups are limit groups. Fundamental groups of orientable closed surfaces and of non-orientable surfaces with $g \geq 4$ are limit groups (they are even elementarily free). A finitely generated subgroup of a limit group and a free product of limit groups are limit groups. A limit group is hyperbolic if and only if it does not contain $\mathbb{Z}^2$ ([Sel01]).

3. Pinched curves on surfaces

Definition 3.1 (Surface-type vertex)**.** Let Γ be a graph of groups decomposition of a group G. A vertex v of Γ is called a *surface-type vertex* if the following conditions hold:

- the group G_v carried by v is the fundamental group of a compact surface Σ (usually with boundary), with $\pi_1(\Sigma)$ non-abelian;
- incident edge groups are maximal boundary subgroups of $\pi_1(\Sigma)$, and this induces a bijection between the set of boundary components of Σ and the set of incident edges.

We say that G_v is a surface-type vertex group. If u is any lift of v to the Bass-Serre tree T of Γ, we say that u is a surface-type vertex, and its stabilizer G_u (which is conjugate to G_v) is a surface-type vertex stabilizer.

Surface-type vertices are QH (quadratically hanging) vertices in the sense of [GL].

Definition 3.2 (Pinching)**.** Let Σ be a compact surface. Given a homomorphism $p : \pi_1(\Sigma) \to G$, a *family of pinched curves* is a collection $\mathcal{C}$ of disjoint, non-parallel, two-sided simple closed curves $C_i \subset \Sigma$, none of which are null-homotopic, such that the fundamental group of each C_i is contained in $\ker p$ (the curves may be parallel to a boundary component).

The map p is *non-pinching* if there is no pinched curve.

Let S be a component of the surface $\hat{\Sigma}$ obtained by cutting Σ along $\mathcal{C}$. Its fundamental group is naturally identified with a subgroup of $\pi_1(\Sigma)$, so p restricts to a map from $\pi_1(S)$ to G, which we also denote by p.

Lemma 3.3. *Let Γ be a graph of groups decomposition of a group G, with Bass-Serre tree T. Assume that Γ has a single surface-type vertex v with $G_v = \pi_1(\Sigma)$, together with vertices $v_1, \dots, v_n$ (with $n \geq 1$), and every edge of Γ joins v to some v_i (in particular, edge groups of Γ are infinite cyclic).*

Let $p : \pi_1(\Sigma) \to G$ be a homomorphism such that the image of every boundary subgroup of $\pi_1(\Sigma)$ fixes an edge of T, and p is not an isomorphism onto some subgroup of G conjugate to $\pi_1(\Sigma)$.

Let $\mathcal{C}$ be a maximal family of pinched curves on Σ, and let S be a component of the surface obtained by cutting Σ along $\mathcal{C}$.

Then the image of $\pi_1(S)$ *by* p *is contained in a conjugate of some* G_{v_i} *(i.e., it fixes a vertex of* T *which is not a lift of* v*).*

Proof. The boundary of S consists of components of $\partial\Sigma$ and curves coming from $\mathcal{C}$. Define S_0 by gluing a disc to S along each curve coming from $\mathcal{C}$. There is an induced map $\pi : \pi_1(S_0) \to G$, hence an action of $\pi_1(S_0)$ on the Bass-Serre tree T, and we must show that $\pi_1(S_0)$ fixes a lift of some v_i. We may assume that the image of π is non-trivial.

As in the preliminaries, we consider the universal covering $\tilde{S}_0$ and $\pi_1(S_0)$-equivariant maps $\tilde{f} : \tilde{S}_0 \to T$ such that every line in $\partial\tilde{S}_0$ is mapped to a vertex (such maps exist because boundary subgroups of $\pi_1(\Sigma)$ fix a point in T). Assuming that $\tilde{f}$ is in general position, we consider preimages of midpoints of edges and we project to S_0. This yields a finite family $\mathcal{C}_0$ of simple closed curves on S_0, and we choose $\tilde{f}$ so as to minimize the number of these curves (in particular, no curve in $\mathcal{C}_0$ is null-homotopic).

The map $\tilde{f}$ induces a map $f : S_0 \to \Gamma$ sending each component Y of $S_0 \setminus \mathcal{C}_0$ into the open star of some vertex v_Y of Γ. We shall show that *no* v_Y *may be equal to* v. Assuming this, we deduce that $\mathcal{C}_0$ has to be empty, since every curve in $\mathcal{C}_0$ separates a component mapped to the star of v from a component mapped to the star of some v_i, and the lemma follows.

Therefore let Y be a component with $v_Y = v$ and Z its closure. The components of ∂Z come from either $\mathcal{C}_0$ or $\partial\Sigma$. The restriction π_Z of π to $\pi_1(Z)$ has an image contained in a conjugate of $\pi_1(\Sigma)$, which we may assume to be $\pi_1(\Sigma)$ itself. It is non-pinching (its kernel cannot contain the fundamental group of an essential simple closed curve) by maximality of $\mathcal{C}$, and it sends any boundary subgroup H of $\pi_1(Z)$ into a boundary subgroup of $\pi_1(\Sigma)$ because $\pi_Z(H)$ fixes an edge of T. By Lemma 6.2 of [Per11], there are three possibilities: either $\partial Z = \emptyset$ or the image of π_Z is contained in a boundary subgroup of $\pi_1(\Sigma)$ or it has finite index in $\pi_1(\Sigma)$. We study each possibility.

If ∂Z is empty (so, in particular, $\mathcal{C}_0 = \emptyset$ and $Z = S_0$), we note that $\pi_1(\Sigma)$ is a free group because $n \geq 1$, and we represent $\pi_Z : \pi_1(Z) \to \pi_1(\Sigma)$ by a map from Z to a graph. Maximality of $\mathcal{C}$ implies that $Z = S_0$ is a sphere or a projective plane, so the image of π is trivial, hence contained in every G_{v_i}.

Now suppose that the image of π_Z is contained in a boundary subgroup H of $\pi_1(\Sigma)$. Denote by $\tilde{v}$ the vertex of T fixed by $\pi_1(\Sigma)$ and by $\tilde{v}\tilde{v}_i$ the unique edge incident on $\tilde{v}$ fixed by H (a lift of some edge vv_i of Γ). Consider the component $\tilde{Y}$ of the preimage of Y in $\tilde{S}_0$ mapping to the star of $\tilde{v}$. The union of all regions adjacent to $\tilde{Y}$ is mapped by $\tilde{f}$ into the open star of $\tilde{v}_i$ and we may redefine $\tilde{f}$ so as to remove $\mathcal{C}_0 \cap \partial Z$ from $\mathcal{C}_0$, thus contradicting the original choice of $\tilde{f}$ (if $\mathcal{C}_0 \cap \partial Z = \emptyset$, then $Z = S_0$, and we may replace $\tilde{f}$ by the constant map equal to $\tilde{v}_i$).

If the image of π_Z has finite index d in $\pi_1(\Sigma)$, then d has to be 1 because $\chi(\Sigma) \geq \chi(S_0) \geq \chi(Z) = d\chi(\Sigma)$, and we conclude from Lemma 3.12 of [Per11] that $\Sigma = S_0 = Z$ and p is an isomorphism from $\pi_1(\Sigma)$ to itself, contradicting our hypothesis. □

Remark 3.4. The proof shows the following more general result, which will be useful in [GLS]: Let Γ and Σ be as in Lemma 3.3. Let Σ' be a compact surface, and let $p : \pi_1(\Sigma') \to G$ be a homomorphism such that the image of every boundary subgroup of $\pi_1(\Sigma')$ is contained in a conjugate of some G_{v_i}. Let $\mathcal{C}$ be a maximal family of pinched curves on Σ', and let S be a component of the surface obtained by cutting Σ' along $\mathcal{C}$. Then, up to conjugacy in G, the image of $\pi_1(S)$ by p is contained in some G_{v_i}, or there is an incompressible subsurface $Z \subset S$ such that p maps boundary subgroups of $\pi_1(Z)$ into boundary subgroups of $\pi_1(\Sigma)$, and $p(\pi_1(Z))$ is a finite index subgroup of $\pi_1(\Sigma)$.

Corollary 3.5. *Let Γ be as in Lemma* 3.3*. Denote by n the total number of vertices v_i of Γ, by n_1 the number of v_i's which have valence* 1*, and by b the number of boundary components of Σ.*

If $p : \pi_1(\Sigma) \to G$ maps every boundary subgroup of $\pi_1(\Sigma)$ injectively into a G-conjugate and p is not an isomorphism onto a G-conjugate of $\pi_1(\Sigma)$, then the genus g of Σ satisfies $g \geq n_1$ and $g + b \geq 2n$.

Recall that $g = \frac{1}{2}(2-\chi(\Sigma)-b)$ if Σ is orientable, and $g = 2-\chi(\Sigma)-b$ otherwise. By a G-conjugate of H, we mean a subgroup of G conjugate to H.

Proof. We fix a maximal family of pinched curves $\mathcal{C} \subset \Sigma$, and we let $\hat{\Sigma}$ be the (possibly disconnected) surface obtained by cutting Σ along $\mathcal{C}$.

Consider a vertex v_i of Γ of valence 1. Let C be the boundary component of Σ associated to the edge vv_i, and let S be the component of $\hat{\Sigma}$ containing C. By Lemma 3.3, the image of $\pi_1(S)$ by the restriction of p fixes a vertex of T which is not a lift of v. Because of our assumption on $p(\pi_1(C))$, this vertex must be in the orbit of v_i. It follows that C is the only boundary component of Σ contained in S: since v_i has valence 1, the group G_{v_i} cannot intersect non-trivially a conjugate of $\pi_1(C')$ if $C' \neq C$ is another component of $\partial\Sigma$. The boundary of S thus consists of C and curves from $\mathcal{C}$. Since $\pi_1(C)$ is not contained in $\ker p$ but curves in $\mathcal{C}$ are pinched, S cannot be planar: it cannot be a sphere with holes since a generator of $\pi_1(C)$ would then be a product of elements representing pinched curves, contradicting injectivity of p on $\pi_1(C)$.

We conclude that $\hat{\Sigma}$ has at least n_1 non-planar components. This implies that $g \geq n_1$. The second inequality follows since $n_1 + b \geq 2n$. ☐

4. Preretractions

Definition 4.1 (JSJ-like decomposition, [Per11, Definition 5.8])**.** Let Γ be a graph of groups decomposition of a group G. It is *JSJ-like* if:

- edge groups are infinite cyclic;
- at most one vertex of any given edge is a surface-type vertex (in the sense of Definition 3.1), and at most one carries a cyclic group;
- (acylindricity) if a non-trivial element of G fixes two distinct edges of T, then they are adjacent and their common vertex has cyclic stabilizer.

The basic example of a JSJ-like decomposition is the canonical cyclic JSJ decomposition of a torsion-free one-ended hyperbolic group G. More generally, the tree of cylinders (in the sense of [GL11]) of any splitting of a torsion-free CSA group G over infinite cyclic groups is JSJ-like.

Proposition 4.2 ([Per11, Proposition 6.1])**.** *Let Γ be a JSJ-like graph of groups decomposition of a group G. If $r : G \to G$ sends each vertex group and each edge group isomorphically to a conjugate of itself, then r is an automorphism.*

This statement is stronger than Proposition 6.1 of [Per11], but the proof is identical. If G is torsion-free hyperbolic, Γ may be any graph of groups decomposition with infinite cyclic edge groups (one applies the proposition to its tree of cylinders).

Definition 4.3 (Exceptional surfaces)**.** Pairs of pants, once-punctured Klein bottles, twice-punctured projective planes, and closed non-orientable surfaces of genus 3 are surfaces with Euler characteristic -1 which do not carry pseudo-Anosov diffeomorphisms. We call them, as well as surface-type vertices associated to them, *exceptional.*

Definition 4.4 (Preretraction, [Per11, Definition 5.9])**.** Let Γ be a JSJ-like decomposition of a group G. A *preretraction associated to* Γ is a homomorphism $r : G \to G$ such that, for each vertex group G_v:

(1) if G_v is not surface-type, the restriction of r to G_v is conjugation by some $g_v \in G$;
(2) if G_v is an exceptional surface-type vertex group, the restriction of r to G_v is conjugation by some $g_v \in G$;
(3) if G_v is a non-exceptional surface-type vertex group, then the image of G_v is non-abelian.

Note that this implies that the restriction of r to any edge group of Γ is a conjugation.

Example 4.5. Let G be the fundamental group of a non-exceptional closed surface. Any map from G onto a non-abelian free subgroup is a non-injective preretraction (with Γ the trivial splitting).

Conditions 2 and 3 are important in order to draw conclusions from the existence of a non-injective preretraction (as in [Per11]). In the present paper, however, we often do not need these assumptions.

Definition 4.6 (Weak preretraction)**.** A map $r : G \to G$ satisfying condition 1 of Definition 4.4 will be called a *weak preretraction.*

Proposition 4.2 implies that a preretraction r is an isomorphism if there is no non-exceptional surface-type vertex. More generally:

Proposition 4.7. *Let Γ be a JSJ-like decomposition of a group G. The following are equivalent:*

- *there is a non-injective preretraction r associated to Γ;*
- *Γ has a non-exceptional surface-type vertex group $G_v = \pi_1(\Sigma)$, and there is a map $p : \pi_1(\Sigma) \to G$ which is a conjugation on each boundary subgroup, has non-abelian image, and is not an isomorphism of $\pi_1(\Sigma)$ onto a conjugate.*

Similarly, there exists a non-injective weak *preretraction r if and only if there is an arbitrary surface-type G_v with a map p which is a conjugation on each boundary subgroup and is not an isomorphism of $\pi_1(\Sigma)$ onto a conjugate.*

Proof. The existence of r implies that of p by Proposition 4.2. To prove the converse direction, we may collapse the edges of Γ not containing the surface-type vertex v associated to Σ and thus assume that Γ is as in Lemma 3.3 (unless G is a non-exceptional closed surface group, in which case p itself is a preretraction which is non-injective because G is co-Hopfian). As in Lemma 3.3, we denote by n the number of vertices adjacent to v.

Given p which is a conjugation on each boundary subgroup, we may extend it "by the identity" to a (weak) preretraction r. We first illustrate this when Σ has two boundary components (so $n \leq 2$).

If $n = 2$, we have $G = A_1 *_{C_1} \pi_1(\Sigma) *_{C_2} A_2$, and p equals conjugation by some g_i on the cyclic group C_i; we then define r as being conjugation by g_i on A_i. If $n = 1$, then G has presentation $\langle \pi_1(\Sigma), A, t \mid c_1 = a_1, tc_2t^{-1} = a_2 \rangle$, with $a_i \in A$ and c_i generators of boundary subgroups C_i of $\pi_1(\Sigma)$. If p_i is conjugation by g_i on C_i, we define r as being conjugation by g_1 on A and mapping t to $g_1 t {g_2}^{-1}$.

The general case is similar; if Γ is not a tree, one chooses a maximal subtree in order to get a presentation with stable letters as in the case $n = 1$ above.

There remains to check that r is non-injective. We may assume that p is injective. Then there is no pinching, so by Lemma 3.3 the image of p is contained in a conjugate of some G_{v_i}. Let $\tilde{e} = \tilde{v}\tilde{v}_i \subset T$ be a lift of the edge vv_i. We may assume that $\pi_1(\Sigma)$ is the stabilizer of $\tilde{v}$, and p is the identity on the stabilizer of $\tilde{e}$. The extension r then preserves the stabilizer of $\tilde{v}_i$, and up to postcomposing r with the conjugation by some element of $G_{\tilde{e}}$, we may assume that r is the identity on the stabilizer of $\tilde{v}_i$. The image of p fixes a vertex in the orbit of $\tilde{v}_i$, and by acylindricity this vertex must be $\tilde{v}_i$. It follows that r cannot be injective. $\square$

Remark 4.8. In particular, if there is a non-injective preretraction associated to an arbitrary non-trivial JSJ-like decomposition, there is one associated with a Γ as in Lemma 3.3.

5. Preretractions and elementary equivalence

Let G be a finitely generated group. It may be written as $A_1 * \cdots * A_n * F$, with A_i non-trivial, not isomorphic to $\mathbb{Z}$, freely indecomposable, and F free. We call the A_i's the Grushko factors of G.

The following is folklore, but we give a proof for completeness.

Proposition 5.1. *Let G and G' be torsion-free hyperbolic groups. Assume that G and G' are elementarily equivalent. If A is a Grushko factor of G and A has no non-injective preretraction associated to its cyclic JSJ decomposition Γ_{can}, then A embeds into G'.*

Such an A is an example of an elementary prototype [Sel09, Definition 7.3].

Proof. We assume that A does not embed into G', and we construct a non-injective preretraction. By Sela's shortening argument (see [Per11, Proposition 4.3]), there exist finitely many non-trivial elements $r_i \in A$ such that, given any homomorphism $f : A \to G'$ (necessarily non-injective), there is a modular automorphism σ of A such that the kernel of $f' = f \circ \sigma$ contains one of the r_i's.

Let Γ_{can} be the canonical cyclic JSJ decomposition of A. By definition, modular automorphisms act on non-surface vertex groups G_v of Γ_{can} by conjugation, so f and f' differ by an inner automorphism on such vertex groups G_v. If Γ has an exceptional surface-type vertex group G_u that has a Dehn twist of infinite order (i.e., if Σ is a once-punctured Klein bottle or a closed non-orientable surface of genus 3), this property does not hold for G_u (this detail was overlooked in [Per11, Lemma 5.6]).

To remedy this, one considers Γ'_{can} obtained from Γ_{can} by splitting G_u along the fundamental group of a suitable 2-sided simple closed curve which is invariant under the mapping class group (see Remark 9.32 in [GL]); in Γ'_{can}, all exceptional surfaces have finite mapping class group, and modular automorphisms of A act by conjugation on all exceptional surface groups of Γ'_{can}.

Thus the following statement, which is expressible in first-order logic, holds (compare Section 5.4 of [Per11]): *Given any $f : A \to G'$ such that non-exceptional surface-type vertex groups G_v of Γ'_{can} have non-abelian image, there is $f' : A \to G'$ such that* $\ker f'$ *contains one of the r_i's, non-exceptional surface-type vertex groups G_v have non-abelian image by f', and for each other vertex group G_v of Γ'_{can} there is $g \in G'$ such that f' agrees with $i_g \circ f$ on G_v* (with i_g denoting conjugation by g).

Since G and G' are elementarily equivalent, this statement holds with f the inclusion from A to G and yields a non-injective $f' : A \to G$. Composing with a projection from G onto A yields a non-injective weak preretraction from A to A associated to Γ'_{can}. In order to get a preretraction (i.e., to ensure that non-exceptional surface-type vertex groups have non-abelian image), we apply Proposition 5.12 of [Per11]. By Proposition 4.7, this preretraction also gives a non-injective preretraction associated to Γ_{can}. □

Corollary 5.2. *If a finitely generated group G is elementarily free, all its Grushko factors have non-injective preretractions associated to their cyclic JSJ decompositions.* □

Recall that every finitely generated elementarily free group is torsion-free and hyperbolic ([Sel01], [Sel06]).

Corollary 5.3. *Let G and G' be one-ended torsion-free hyperbolic groups. Suppose that G, G' do not have non-injective preretractions associated to their cyclic JSJ decompositions. If G and G' are elementarily equivalent, they are isomorphic.*

This follows from the proposition and the fact that G and G' are co-Hopfian [Sel97].

6. Surfaces with sockets

It is proved in [KM06] and [Sel06] that the fundamental group $G = \pi_1(\Sigma)$ of a closed hyperbolic surface Σ is a limit group which is elementarily equivalent to a non-abelian free group, with one exception: if Σ is the non-orientable surface of genus 3, then G is not even a limit group.

In this section we consider fundamental groups of surfaces with sockets (which we simply call socket groups). They are obtained from $\pi_1(\Sigma)$, with Σ a compact hyperbolic surface of genus g with $b \geq 1$ boundary components B_i, by adding a root to the element h_i representing B_i (see Figure 1). More precisely, G has presentation

$$\langle h_1, \dots, h_b, z_1, \dots, z_b, a_1, b_1, \dots, a_g, b_g \mid h_1 \cdots h_b = [a_1, b_1] \cdots [a_g, b_g], z_i^{n_i} = h_i \rangle$$

or

$$\langle h_1, \dots, h_b, z_1, \dots, z_b, a_1, \dots, a_g \mid h_1 \cdots h_b = a_1^2 \cdots a_g^2, z_i^{n_i} = h_i \rangle,$$

depending on whether Σ is orientable or not. The exponent n_i, the *order* of the i-th socket, is a positive integer, and we always assume $n_i \geq 3$: if $n_i = 2$, we can remove the socket by attaching a Möbius band to B_i.

Socket groups are torsion-free one-ended hyperbolic groups. They were introduced in [Sel97] to study splittings over maximal cyclic subgroups.

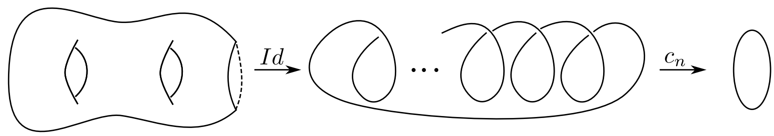

Figure 1. A socket group can be seen topologically as the fundamental group of a surface for which every boundary component is glued onto a circle under an n-cover map for some n.

We shall show:

Proposition 6.1. *The group G is a limit group if and only if one of the following holds:*

- *the surface Σ is orientable with at least four boundary components (i.e., $b \geq 4$);*
- *the surface Σ is orientable with two or three boundary components and genus $g \geq 1$;*
- *the surface Σ is orientable with one boundary component and genus $g \geq \frac{n+1}{2}$, where n is the order of the unique socket;*
- *the surface Σ is non-orientable and $b + g \geq 4$.*

Proposition 6.2. *The following assertions are equivalent:*

1. *the group G is elementarily equivalent to a free group;*
2. *the group G has a non-injective preretraction associated to its cyclic JSJ decomposition;*
3. *the surface Σ is non-orientable, $b + g \geq 4$, every n_i is even, and $g \geq b$.*

The group G is naturally the fundamental group of a graph of groups Γ with a central vertex carrying $\pi_1(\Sigma)$ and b terminal vertices carrying $\langle z_i \rangle$ (the edge groups are generated by the h_i's). It is a torsion-free one-ended hyperbolic group, and the assumption $n_i \geq 3$ ensures that Γ is its canonical cyclic JSJ decomposition (see Proposition 4.24 in [DG11]).

Proof of Proposition 6.1. By Proposition 4.21 of [CG05], G is a limit group if and only if the equation

$$z_1^{n_1} \cdots z_b^{n_b} = [a_1, b_1] \cdots [a_g, b_g] \qquad \text{or} \qquad z_1^{n_1} \cdots z_b^{n_b} = a_1^2 \cdots a_g^2,$$

with unknowns z_i, a_i, b_i in a free group $F(x, y)$, has a solution which is not contained in a cyclic subgroup and with every z_i non-trivial.

Recall that, given integers $k_i \geq 2$, a relation $x_1^{k_1} \cdots x_p^{k_p} = 1$ between elements of $F(x, y)$ implies that $\langle x_1, \ldots, x_p \rangle$ is cyclic when $p \leq 3$ [LS62]. Thus G is not a limit group if Σ is non-orientable and $b+g \leq 3$. On the other hand, if Σ is non-orientable and $b+g \geq 4$, it is easy to find a suitable solution of the equation above by setting all unknowns equal to powers of x or y.

Now suppose that Σ is orientable and there is a single socket ($b = 1$). If $g \geq \frac{n+1}{2}$, it follows from [Cul81] that $[x, y]^n$ is a product of g commutators, so G is a limit group. If $g < \frac{n+1}{2}$, no non-trivial n-th power is a product of g commutators by [CE95], so G is not a limit group.

Finally, assume that Σ is orientable and $b \geq 2$. As suggested by Jim Howie, the equation $z_1^{n_1} z_2^{n_2} = [a_1, b_1]$ is solved by setting $z_1 = x^{n_2}$, $z_2 = yx^{-n_1}y^{-1}$, $a_1 = x^{n_1 n_2}$,

$b_1 = y$. It follows that G is a limit group when $g \geq 1$. If $g = 0$, the fact recalled above implies that G is a limit group if and only if $b \geq 4$. □

Proof of Proposition 6.2. We first prove (2) $\implies$ (3). Assuming that there is a preretraction, we apply Lemma 3.3 to the map p provided by Proposition 4.7. Cut Σ open along $\mathcal{C}$, and call S_i the component of the surface thus obtained which contains B_i. For each i, up to conjugation, we get a map from $\pi_1(S_i)$ to $\langle z_i \rangle$ sending h_i to itself and killing the fundamental group of every other boundary component of S_i. This is possible only if S_i is non-orientable and n_i is even.

The inequality $g \geq b$ follows from Corollary 3.5. By Proposition 4.2, the existence of a non-injective preretraction forces that of a non-exceptional surface. This rules out the once-punctured Klein bottle ($g = 2$, $b = 1$) and thus implies $b + g \geq 4$.

Conversely, we assume that Σ and the n_i's are as in (3), and we construct a non-injective preretraction r fixing every z_i (hence every h_i). Write $n_i = 2m_i$. If $b = 1$, then $g \geq 3$. We choose $z \in G$ not commuting with z_1, and we map a_1 to $z_1^{m_1}$, a_2 to z, a_3 to z^{-1}, and a_i to 1 for $i > 3$. If $2 \leq b \leq g$, we map a_i to $z_i^{m_i}$ for $i \leq b$, to 1 for $i > b$. This proves (3) $\implies$ (2).

(1) $\implies$ (2) follows from Corollary 5.2. Conversely, the map r constructed above expresses G as an extended hyperbolic tower over a free group, showing that G is elementarily free [Sel06]. □

Remark 6.3. The proof shows that G has a non-injective weak preretraction (in the sense of Definition 4.6) associated to its cyclic JSJ decomposition if and only if Σ is non-orientable, every n_i is even, and $g \geq b$.

Corollary 6.4. *Let G be a socket group, with Σ a once-punctured orientable surface of genus* 2, *and a single socket of order* 3 (*see Figure* 2). *Then G is a one-ended hyperbolic limit group G which is not elementarily free but contains an elementarily free subgroup of finite index.*

Remark 6.5. In fact, G has no non-injective weak preretraction by Remark 6.3.

Proof. By Propositions 6.1 and 6.2, G is a limit group that is not elementarily free. It has a subgroup G_0 of index 3, obtained as the kernel of a map from G to $\mathbb{Z}/3\mathbb{Z}$ killing a_1, b_1, a_2, b_2, which is the fundamental group of the space obtained by gluing three once-punctured surfaces of genus 2 along their boundary. Identifying two of these surfaces yields a retraction from G_0 onto the fundamental group of the closed orientable surface of genus 4, so G_0 is elementarily free by [Sel06]. □

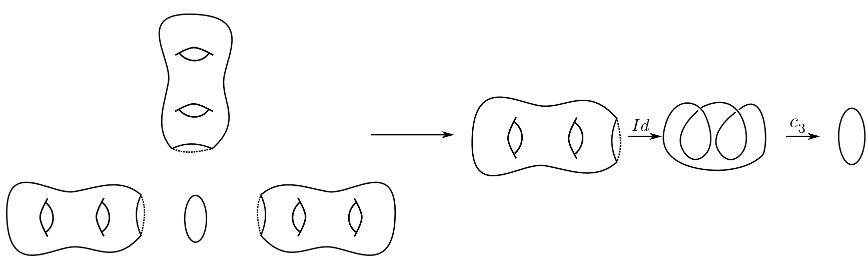

FIGURE 2. A covering of degree 3.

We now consider a slightly different family of groups, where all sockets are identified. More precisely, G is the fundamental group of a graph of groups with one surface-type vertex group $G_v = \pi_1(\Sigma)$, where Σ has at least two boundary components and one vertex carrying a cyclic group $\langle z \rangle$. It is presented as

$$\langle h_1, \ldots, h_b, z, t_1, \ldots, t_b, a_1, b_1, \ldots, a_g, b_g \mid h_1 \cdots h_b = [a_1, b_1] \cdots [a_g, b_g], t_i z^{n_i} {t_i}^{-1} = h_i, t_1 = 1 \rangle$$

or

$$\langle h_1, \ldots, h_b, z, t_1, \ldots, t_b, a_1, \ldots, a_g \mid h_1 \cdots h_b = a_1^2 \cdots a_g^2, t_i z^{n_i} {t_i}^{-1} = h_i, t_1 = 1 \rangle,$$

depending on whether Σ is orientable or not (see Figure 3).

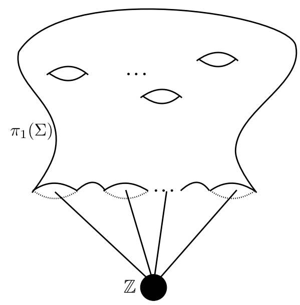

FIGURE 3. A socket group with identified sockets.

When Σ is orientable one may orient the components of $\partial\Sigma$ in a consistent way, and the sign of the n_i's matters. In the non-orientable case the components may be oriented separately, and only $|n_i|$ has a meaning.

Proposition 6.6. *Let G be a socket group with identified sockets as above. There is a non-injective weak preretraction associated to the cyclic JSJ decomposition of G if and only if the sum $\sum_{i=1}^{b} n_i$ is 0 when Σ is orientable; the sum $\sum_{i=1}^{b} n_i$ is even when Σ is non-orientable.*

Proof. Assume that there is $p : \pi_1(\Sigma) \to G$ which is a conjugation on each boundary subgroup and is not an isomorphism of $\pi_1(\Sigma)$ onto a conjugate (see Proposition 4.7). We apply Lemma 3.3. First suppose there is no pinching. All h_i's are mapped into the same conjugate of $\langle z \rangle$, so we may assume that h_i is mapped to a power of z, which must be z^{n_i} (note that z^p conjugate to z^q implies $p = q$). We deduce that $z^{\sum n_i}$ is a product of squares or commutators in $\langle z \rangle$. If there is pinching, apply the argument to each pinched surface.

Conversely, we define a non-injective preretraction r as follows: we map z to z, every h_i to z^{n_i}, every t_i, a_i, b_i to 1, except that we map a_1 to $z^{(\sum n_i)/2}$ in the non-orientable case. ☐

Deciding whether there is a non-injective (true) preretraction r (with Σ non-exceptional and the image of $\pi_1(\Sigma)$ non-abelian) is more subtle.

First suppose that Σ is orientable. If its genus is ≥ 1, one may make $r(\pi_1(\Sigma))$ non-abelian by sending a_1 and b_1 to some $x \in G$ which does not commute with z.

If Σ is a 4-punctured sphere and r exists, Lemma 3.3 implies that there is a pinched curve. It must divide Σ into two pairs of pants, each containing two boundary components of Σ, say B_1 and B_2 on one side, B_3 and B_4 on the other. One must then have $n_1 + n_2 = n_3 + n_4 = 0$. Conversely, if this holds, one defines r on the generating set $\{z, t_2, t_3, t_4\}$ by sending z to itself, t_2 to the trivial element, and t_3 and t_4 to some $x \in G$ such that z and xzx^{-1} do not commute. This argument shows:

Proposition 6.7. *Let G be a socket group with identified sockets as above, with Σ an orientable surface of genus g. There is a non-injective preretraction (i.e., G is elementarily free) if and only if the sum* $\sum_{i=1}^{b} n_i$ *is 0 and either:*

(i) *$g \geq 1$ or*
(ii) *$g = 0$ and there is a proper non-empty subset $I \subset \{1, \ldots, b\}$ with $\sum_{i\in I} n_i = 0$.* □

When Σ is non-orientable, the easy case is when $g \geq 3$ since one may use a_2 and a_3 to make $r(\pi_1(\Sigma))$ non-abelian. When g is 1 or 2, one has to consider the ways in which a maximal family of pinched curves may divide Σ. A case by case analysis yields:

Proposition 6.8. *Let G be a socket group with identified sockets as above, with Σ a non-orientable surface of genus g. Then there is a non-injective preretraction (i.e., G is elementarily free) if and only if the sum* $\sum_{i=1}^{b} n_i$ *is even and one of the following holds:*

(i) *$g \geq 3$;*
(ii) *$g = 2$ and Σ has at least three boundary components (i.e., $b \geq 3$);*
(iii) *$g = 2$, the surface Σ has two boundary components (i.e., Σ is a twice punctured Klein bottle), and $|n_1| = |n_2|$ or both n_1, n_2 are even;*
(iv) *$g = 1$, the surface Σ has at least three boundary components (i.e., $b \geq 3$), and there exist a proper subset $I \subset \{1, \ldots, b\}$ and signs $\varepsilon_i = \pm 1$ with $\sum_{i\in I} \varepsilon_i n_i = 0$.* □

7. Finite index subgroups

The most obvious examples of limit groups, namely free abelian groups, free groups, and surface groups, are elementarily equivalent to their finite index subgroups. The goal of this section is to show that these are basically the only examples of limit groups with this property, at least among hyperbolic limit groups.

Theorem 7.1. *Let G be a hyperbolic limit group or a torsion-free cubulable hyperbolic group. If G is not a free product of cyclic groups and of surface groups, it has infinitely many normal finite index subgroups which are all different up to elementary equivalence.*

Corollary 7.2. *Let G be a finitely generated elementarily free group. If G is not a free product of cyclic groups and surface groups, it has infinitely many finite index subgroups which are not elementarily free.*

On the other hand, if G is a free product of cyclic groups and fundamental groups of closed surfaces with $\chi(\Sigma) \leq -2$, it is elementarily free by [Sel06], and so are all its non-cyclic finitely generated subgroups.

The corollary is clear, because a finitely generated elementarily free group is a hyperbolic limit group by [Sel01] and [Sel06]. We shall deduce the theorem from a more technical statement, which requires a definition.

Definition 7.3. We say that G has *enough abelian virtual quotients* if, given any infinite cyclic subgroup C, there is a finite index subgroup G_C of G such that $C \cap G_C$ has infinite image in the abelianization of G_C. Equivalently, C has a finite index subgroup which is a retract of a finite index subgroup of G.

Note that having enough virtual abelian quotients is inherited by subgroups. This property is slightly weaker than G being LR over cyclic groups in the sense of [LR08] (they require that C itself be a retract of a finite index subgroup). It implies the existence of infinitely many finite index subgroups.

Theorem 7.4. *Let G be a one-ended torsion-free hyperbolic group. If G has enough virtual abelian quotients and is not a surface group, then G has infinitely many characteristic subgroups of finite index which have no non-injective weak preretraction associated to their cyclic JSJ decomposition.*

The theorem readily extends to groups which are only virtually torsion-free and are not virtual surface groups.

Having enough virtual abelian quotients is a technical assumption (which holds for limit groups and cubulable hyperbolic groups, as we shall see). It is not optimal, but note that some assumption is needed: if there is a hyperbolic group which is not residually finite, there is one which has no proper subgroup of finite index [KW00], and such a group obviously does not satisfy Theorem 7.4. Also note that groups with property (T) do not have enough virtual abelian quotients, but they cannot have non-injective weak preretractions because they have no splittings.

We first explain how to deduce Theorem 7.1 from Theorem 7.4.

Lemma 7.5. *Residually free groups* (*in particular limit groups*) *and cubulable hyperbolic groups have enough abelian virtual quotients.*

Proof. If G is residually free, there is a map p from G to a free group which is injective on C. By Hall's theorem, the image of C under p is a free factor of a finite index subgroup, and we let G_C be its preimage.

If G is hyperbolic and cubulable, then by [Ago13] it is virtually special: there is some finite index subgroup $G_1 < G$ acting freely and cocompactly on a CAT(0) cube complex X such that X/G_1 is special. Since quasi-convex subgroups are separable by [HW08, Corollary 7.4], up to passing to a further finite index subgroup, we can assume that the 2-neighbourhoods of hyperplanes are embedded in X/G_1 so that X/G_1 is fully clean. Denote $C_1 = C \cap G_1$.

By [HW08, Proposition 7.2], there is a convex cube complex $Y \subset X$ which is C_1-invariant and C_1-cocompact and defines a local isometry $f : Y/C_1 \to X/G_1$ which is fully special by [HW08, Lemma 6.3]. By [HW08, Proposition 6.5], there is a finite cover X/G_2 of X/G_1 and an embedding $Y/C_1 \to X/G_2$ with a retraction $X/G_2 \to Y/C_1$. Group-theoretically, we get a retraction from the finite index subgroup G_2 of G to C_1. This proves the lemma. ☐

Note that any residually free hyperbolic group is a limit group [Bau67].

Proof of Theorem 7.1. Given G as in the theorem, we write its Grushko decomposition as $G = A_1 * \cdots * A_p * F * S_1 * \cdots * S_q$ where F is free, each S_i is a closed surface group, and each A_i is a one-ended group which is not a surface group (the A_i's are not necessarily distinct up to isomorphism). By assumption, we have $p \geq 1$.

By Theorem 7.4 and Lemma 7.5, we can find in each A_i a decreasing sequence $A_i(1) \supsetneqq A_i(2) \supsetneqq A_i(3) \supsetneqq \dots$ of characteristic finite index subgroups which have no non-injective preretraction associated to their cyclic JSJ decomposition. Let G_n be the kernel of the natural projection from G onto $\prod_{i=1}^{p} A_i/A_i(n)$. It is a finite index subgroup of G whose non-surface Grushko factors are all isomorphic to some $A_i(n)$.

We claim that there are infinitely many distinct G_n's up to elementary equivalence. If not, we may assume that they are all equivalent. Applying Proposition 5.1 to G_1 and G_2, we find that $A_1(1)$ embeds into G_2, hence into some Grushko factor $A_j(2)$. We cannot have $j = 1$ because $A_1(1)$ properly contains $A_1(2)$ and torsion-free one-ended hyperbolic groups are co-Hopfian [Sel97]. We may therefore assume $j = 2$. We then find that $A_2(2)$ embeds in some $A_3(k)$ and $k > 2$ by co-Hopfianity. Iterating this argument leads to a contradiction. □

The remainder of this section is devoted to the proof of Theorem 7.4.

We consider the canonical cyclic JSJ decomposition $\Gamma_{can}(G)$, or simply Γ_{can}, and the associated Bass-Serre tree T_{can}. Its non-rigid vertices are of surface-type.

Recall that the girth of a graph is the smallest length of an embedded circle (∞ if the graph is a tree). We say that Γ_{can} has *large girth* if its girth N is large compared to the complexity of the surfaces which appear in Γ_{can}; precisely, $(N-2)/2$ should be larger than the maximal cardinality of a family of non-parallel disjoint simple closed curves on a given surface (the curves may be boundary parallel, but should not be null-homotopic).

If G_0 is a finite index subgroup, its JSJ tree is obtained by restricting the action of G on T_{can} to G_0. This is because by [Bow98] one may construct T_{can} purely from the topology of the boundary of G, and $\partial G_0 = \partial G$. If v is a surface-type vertex for the action of G on T_{can}, it is one also for the action of G_0, but in general the surface is replaced by a finite cover.

The proof requires two lemmas.

Lemma 7.6. *There is a characteristic subgroup of finite index $G_0 \subset G$ whose JSJ decomposition has the following properties:*

(1) *the surfaces which appear in $\Gamma_{can}(G_0)$ are all orientable;*
(2) *each edge group maps injectively to the abelianization of G_0;*
(3) *the graph $\Gamma_{can}(G_0)$ has large girth* (*as defined above*).

Note that the first two properties are inherited by finite index subgroups.

Lemma 7.7. *Let G be a torsion-free one-ended hyperbolic group whose JSJ decomposition satisfies the properties of the previous lemma. If G is not a surface group, then G has no non-injective weak preretraction associated to its cyclic JSJ decomposition.*

Before proving these lemmas, let us explain how they imply Theorem 7.4.

First suppose that there is G_0 as in Lemma 7.6 such that $\Gamma_{can}(G_0)$ is not a tree. Then choose a decreasing sequence of characteristic finite index subgroups $H_1 \supset H_2 \supset \cdots$ of the topological fundamental group of the graph $\Gamma_{can}(G_0)$, and define G_n as the preimage of H_n under the natural epimorphism from G_0 to the fundamental group of the graph $\Gamma_{can}(G_0)$.

Since $\Gamma_{can}(G_0)$ is invariant under automorphisms of G_0, each G_n is characteristic in G. Its JSJ decomposition is the covering of $\Gamma_{can}(G_0)$ associated to H_n, with the lifted graph of groups structure. The group G_n satisfies the first two properties of Lemma 7.6 and also the third one because the surfaces appearing in $\Gamma_{can}(G_n)$ are the same as in $\Gamma_{can}(G_0)$, so Lemma 7.7 applies.

If $\Gamma_{can}(G_0)$ is a tree for every subgroup of finite index $G_0 < G$ as in Lemma 7.6, we fix any such G_0 and we let G_n be any sequence of distinct characteristic subgroups of finite index (it exists because there are enough abelian virtual quotients). We cannot control the complexity of surfaces, but Lemma 7.7 applies to G_n because the girth is infinite.

We now prove the lemmas.

Proof of Lemma 7.6. Since the first two properties are inherited by finite index subgroups, it suffices to construct G_0 having one given property.

To achieve (1), view the JSJ decomposition Γ_{can} as expressing G as the fundamental group of a graph of spaces X, with a surface Σ for each surface-type vertex. If there is a non-orientable surface, consider a 2-sheeted covering of X which is trivial (a product) above the complement of the non-orientable surfaces and is the orientation covering $\hat{\Sigma}$ over each non-orientable surface (note that a boundary component of Σ lifts to two curves in $\hat{\Sigma}$). Though Γ_{can} is canonical, the 2-sheeted covering is not, so we define G_0 as the intersection of all subgroups of G of index 2.

For (2), let C_i be the edge groups of Γ_{can}, and let G_0 be any characteristic subgroup of finite index contained in every group G_{C_i} provided by Definition 7.3. Each edge group of $\Gamma_{can}(G_0)$ is the image of some $C_i \cap G_0$ by an automorphism of G_0, so maps injectively to the abelianization.

If the girth of Γ_{can} is too small, we choose a characteristic finite index subgroup H_0 of the topological fundamental group $\pi_1(\Gamma_{can})$ such that the associated finite cover of Γ_{can} has large girth, and we define G_0 as the preimage of H_0 under the epimorphism from G to the topological fundamental group of Γ_{can}. As in the proof of the theorem given above, invariance of Γ_{can} under automorphisms implies that G_0 is characteristic, and the complexity of surfaces does not change. ☐

Proof of Lemma 7.7. We consider the canonical cyclic JSJ decomposition $\Gamma = \Gamma_{can}$. Recall that it is bipartite: each edge joins a vertex carrying an infinite cyclic group to a vertex carrying a non-cyclic group.

We argue by way of contradiction, assuming that there is a non-injective weak preretraction. By Proposition 4.7, there is a surface-type vertex group $G_v = \pi_1(\Sigma)$ and a map $p : \pi_1(\Sigma) \to G$ which is a conjugation on each boundary subgroup and is not an isomorphism of $\pi_1(\Sigma)$ onto a conjugate.

Let $\mathcal{C}$ be a maximal family of pinched curves on Σ (see Definition 3.2), and let S be a component of the compact surface obtained by cutting Σ along $\mathcal{C}$. Since G is not a surface group, we may assume that S contains a boundary component C of Σ. It must contain another boundary component C' of Σ: otherwise, since Σ is orientable and curves in $\mathcal{C}$ are pinched, the image of a generator of $\pi_1(C)$ would be a product of commutators, contradicting the second item in Lemma 7.6.

We apply Lemma 3.3 to the graph of groups obtained from Γ by collapsing all edges which do not contain v. We find that the image of $\pi_1(S)$ by p is contained (up to conjugacy) in the fundamental group R_i of a subgraph of groups Γ_i of Γ, which is a component of the complement of the open star of v. The image of the fundamental groups of C and C' are contained in the groups carried by distinct vertices w and w' of Γ_i which are adjacent to v in Γ but far away from each other in Γ_i because the girth of Γ is large.

As in the proof of Lemma 3.3, we construct a surface S_0 by attaching discs to boundary curves of S coming from $\mathcal{C}$, and we represent the induced map $\pi : \pi_1(S_0) \to R_i$ by an equivariant map from the universal covering $\tilde{S}_0$ to the Bass-Serre tree of Γ_i. We consider preimages of midpoints of edges, and we project to S_0. We obtain a finite family $\mathcal{C}_0$ of disjoint simple closed curves on S_0.

View Γ_i as covered by stars of vertices u carrying a cyclic group. If C_0 is a curve in $\mathcal{C}_0$, the image of $\pi_1(C_0)$ by π is contained (up to conjugacy) in a unique G_u, and curves associated to different vertices cannot be parallel. Since any path joining w to w' in Γ_i must go through at least $(N-2)/2$ stars, with N the girth of Γ, the family $\mathcal{C}_0$ must contain at least $(N-2)/2$ non-parallel curves. We get a contradiction if N is large enough. □

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