

GENERIC INJECTIVITY OF THE PRYM MAP FOR DOUBLE RAMIFIED COVERINGS

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WITH AN APPENDIX BY ALESSANDRO VERRA

ABSTRACT. We consider the Prym map for double coverings of curves of genus g ramified at $r > 0$ points; that is, the map associated with a double ramified covering its Prym variety. The generic Torelli theorem states that the Prym map is generically injective as soon as the dimension of the space of coverings is less than or equal to the dimension of the space of polarized abelian varieties. We prove the generic injectivity of the Prym map in the cases of double coverings of curves with (a) $g = 2$, $r = 6$, and (b) $g = 5$, $r = 2$. In the first case the proof is constructive and can be extended to the range $r \geq \max\{6, \frac{2}{3}(g+2)\}$. For (b) we study the fiber along the locus of the intermediate Jacobians of cubic threefolds to conclude the generic injectivity. This completes the work of Marucci and Pirola, who proved this theorem for all the other cases except for the bielliptic case $g = 1$ (solved later by Marucci and the first author); the case $g = 3$, $r = 4$ considered previously by Nagaraj and Ramanan; and by Bardelli, Ciliberto, and Verra, where the degree of the map is 3.

We close with an appendix by Alessandro Verra with an independent result, the rationality of the moduli space of coverings with $g = 2$, $r = 6$, whose proof is self-contained.

It is well known that a general Prym variety of dimension at least 6 is the Prym variety of a unique étale covering. This is the so-called generic Torelli theorem for Prym varieties. In a modular way this result can be reformulated as the generic injectivity of the Prym map

$$\mathcal{R}_g \longrightarrow \mathcal{A}_{g-1},$$

sending an étale covering in \mathcal{R}_g to its Prym variety, which turns out to be a principally polarized abelian variety (ppav). In recent years, starting with the seminal work by Marucci and Pirola (see [18]), the ramified case has attracted attention and the corresponding generic Torelli problem has been shown to be a natural problem with rich geometry. In the mentioned paper generic Torelli is proved for a large number of cases using degeneration techniques, and only a few special situations have remained open. Our purpose is to prove the generic Torelli theorem in these last cases.

In order to establish our results more precisely, let us define the basic objects that take part in the main statements. Let C be an irreducible smooth complex projective curve of genus g , and let $\pi : D \rightarrow C$ be a smooth double covering ramified in $r > 0$ points. The associated Prym variety to π is defined as

$$P(\pi) := \text{Ker Nm}_\pi \subset JD.$$

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This is an abelian subvariety of dimension $g - 1 + \frac{r}{2}$ with induced polarization Ξ of type

$$\delta = (1, \dots, 1, \underbrace{2, \dots, 2}_{g \text{ times}}).$$

Given a divisor B in C of even degree $r > 0$ with no multiple points and a line bundle $\eta \in \text{Pic}^{r/2}(C)$ with $B \in |\eta^{\otimes 2}|$, the projection

$$\pi : D = \mathbf{Spec}(\mathcal{O}_C \oplus \eta^{-1}) \rightarrow \mathbf{Spec} \mathcal{O}_C = C$$

defines a double covering branched over B . Conversely, every double ramified covering of C arises in this way. Hence, these coverings are parametrized by the moduli space

$$\mathcal{R}_{g,r} := \{(C, \eta, B) \mid \eta \in \text{Pic}^{\frac{r}{2}}(C), B \text{ reduced divisor in } |\eta^{\otimes 2}|\}.$$

Let $\mathcal{A}_{g-1+\frac{r}{2}}^\delta$ denote the moduli space of abelian varieties of dimension $g - 1 + \frac{r}{2}$ and polarization of type δ . For any $g \geq 1$ and $r > 0$, we define the Prym map by

$$\begin{aligned} \mathcal{P}_{g,r} : \mathcal{R}_{g,r} &\rightarrow \mathcal{A}_{g-1+\frac{r}{2}}^\delta \\ (\pi : C \rightarrow D) &\mapsto (P(\pi), \Xi). \end{aligned}$$

The codifferential of $\mathcal{P}_{g,r}$ at a generic point $[(C, \eta, B)]$ is given by the multiplication map

$$d\mathcal{P}_{g,r}^* : \text{Sym}^2 H^0(C, \omega_C \otimes \eta) \rightarrow H^0(C, \omega_C^2 \otimes \mathcal{O}(B)),$$

which is known to be surjective [17]; therefore, \mathcal{P} is generically finite if and only if

$$\dim \mathcal{R}_{g,r} \leq \dim \mathcal{A}_{g-1+\frac{r}{2}}^\delta.$$

So, by an elementary count of parameter, we get that the Prym map is generically finite as soon as one of the following conditions holds: $r \geq 6$ and $g \geq 1$, or $r = 4$ and $g \geq 3$, or $r = 2$ and $g \geq 5$.

Marcucci and Pirola [18], and later Marcucci and Naranjo [19], proved the generic injectivity in all the cases except three:

- (a) $r = 4, g = 3$;
- (b) $r = 6, g = 2$;
- (c) $r = 2, g = 5$.

Case (a) was considered previously by Nagaraj and Ramanan (see [24]), also by Bardelli, Ciliberto, and Verra (see [1]). They proved (in different ways) that the degree of the Prym map is 3 and is the only case where the map is generically finite but the generic Torelli theorem fails.

This paper has two goals: to prove the generic Torelli theorem for the remaining cases (b) and (c), and to give a constructive proof of the generic Torelli theorem for $r \geq \max\{6, \frac{2}{3}(g + 2)\}$. More precisely we prove the following (see Theorem 1.1).

Theorem 0.1. *Let (C, η, B) be a generic element in $\mathcal{R}_{g,r}$ and let (P, Ξ) be its Prym variety. Assume that $r \geq \max\{6, \frac{2}{3}(g + 2)\}$. Then the element (C, η, B) is uniquely determined by the base locus B_s of the linear system $|\Xi|$.*

In particular, this establishes the generic injectivity of the Prym map $\mathcal{P}_{2,6} : \mathcal{R}_{2,6} \rightarrow \mathcal{A}_4^{(1,1,2,2)}$ (see Corollary 1.2). The image of the Prym map gives a divisor in $\mathcal{A}_4^{(1,1,2,2)}$, which is invariant under the natural involution in $\mathcal{A}_4^{(1,1,2,2)}$ (see [25]). This divisor could be useful in understanding the nature of the birational geometry of $\mathcal{A}_4^{(1,1,2,2)}$; for instance, in computing its Kodaira dimension, which seems to be unknown.

Recall that in the case of étale double coverings there are two different constructive proofs of the generic injectivity of the Prym map: one due to Welters (see [29]), which uses degeneration methods and works for genus $g \geq 16$; and another one by Debarre (see [8]), where the covering is reconstructed from the tangent cones to the stable singularities of the theta divisor of P , this holds for curves of genus $g \geq 7$. In the same spirit but with a different method, we present in Theorem 0.1 a procedure to reconstruct the covering for $r \geq \max\{6, \frac{2}{3}(g + 2)\}$. This gives another proof of Marcucci and Pirola results in this range. It also provides a different proof for the case $r \geq 6, g = 1$, which was proven in [19]. The idea of the proof of Theorem 0.1 is to show that the base locus of the map $P \dashrightarrow |\Xi|^\vee$ is symmetric and invariant by the action of the kernel of the isogeny $\lambda_\Xi : P \rightarrow P^\vee$. The quotient by this action turns out to be, under some hypothesis of genericity, birational to the symmetric product $D^{(\frac{r}{2}-1)}$. Moreover, the symmetry of the base locus induces an involution on $D^{(\frac{r}{2}-1)}$. Then an extended Torelli theorem proved by Martens allows us to conclude the proof.

The second part of the paper is devoted to proving generic injectivity in case (c). The main result of section 2 is the following (see also Theorem 2.10):

Theorem 0.2. *The Prym map $\mathcal{P} : \mathcal{R}_{5,2} \rightarrow \mathcal{A}_5$ is generically injective.*

The moduli space $\mathcal{R}_{5,2}$ can be embedded into the Beauville partial compactification $\overline{\mathcal{R}}_6$ of admissible étale double coverings of genus 6 curves, by identifying the two branch points on the covering and on the base curve. The closure of the image of $\mathcal{R}_{5,2}$ is an irreducible divisor Δ^n in the boundary of $\overline{\mathcal{R}}_6$ (denoted by Δ_6^{ram} in [11]). Using the definition of the Prym map for admissible coverings, one deduces from the main theorem in [10] that the degree of the Prym map $\mathcal{P}_6 : \mathcal{R}_6 \rightarrow \mathcal{A}_5$ restricted to Δ^n is 1, although the total degree of the map is 27. The class of the image of Δ^n under the Prym map in \mathcal{A}_5 has been computed in [11, Theorem 0.8], but it does not give the degree of the restriction map to Δ^n .

For the proof of Theorem 0.2 we use the description of the fibers of the Prym map \mathcal{P}_6 over the locus of intermediate Jacobians of cubic threefolds (after a suitable blowup) intersected with the boundary divisor Δ^n , and we exploit the geometry of the cubic threefolds.

As a summary, the previous results together with the main theorems in [18], [19], and [24] give the following result.

Theorem 0.3. *Let $\mathcal{P}_{g,r}$ be a Prym map with $r > 0$. We assume that*

$$\dim \mathcal{R}_{g,r} \leq \dim \mathcal{A}_{g-1+\frac{r}{2}}^\delta.$$

Then $\mathcal{P}_{g,r}$ is generically injective except for $g = 3, r = 4$, in which case the degree is 3.

The paper closes with an appendix by Alessandro Verra with an independent result, namely the rationality of the moduli space $\mathcal{R}_{2,6}$, whose proof is self-contained.

1. A CONSTRUCTIVE GENERIC TORELLI THEOREM
 FOR $r \geq \max\{6, \frac{2}{3}(g + 2)\}$

The goal of the section is to prove the following theorem.

Theorem 1.1. *Let (C, η, B) be a generic element in $\mathcal{R}_{g,r}$, and let (P, Ξ) be its Prym variety. Assume that $r \geq \max\{6, \frac{2}{3}(g + 2)\}$. Then the element (C, η, B) is uniquely determined by the base locus B_s of the linear system $|\Xi|$.*

We first explain in a few words how B_s determines explicitly the covering $\pi : D \rightarrow C$ attached to (C, η, B) . The kernel of the polarization map $\lambda_\Xi : P \rightarrow P^\vee$ acts on B_s , which moreover is a symmetric subvariety of P . It turns out that the quotient $B_s/\text{Ker}(\lambda_\Xi)$ is birational to the symmetric product $D^{\binom{g}{2}-2}$ and inherits a natural involution σ' . By a theorem of Martens (see [20]) σ' is induced by an involution σ on D . Since $C = D/\langle\sigma\rangle$ the covering is determined. It is convenient to remark that the Martens proof is also constructible since it relies on the Boolean calculus of special subvarieties of Jacobians. An elegant cohomological proof of the same theorem was given by Ran in [26].

This theorem reproves the main theorem in [18] under the hypothesis $r \geq \max\{6, \frac{2}{3}(g + 2)\}$ and recovers completely the result in [19]. Observe that our proof is constructive: we provide a procedure to recover the covering $\pi : D \rightarrow C$ from its Prym variety. Instead, in both quoted papers, degeneration methods are used. It would be interesting to find also a constructive proof for the cases where the map is generically injective and $r < \frac{2}{3}(g + 2)$.

Corollary 1.2. *The Prym map $\mathcal{R}_{2,6} \rightarrow \mathcal{A}_4^{(1,1,2,2)}$ is generically injective.*

The rest of the section is devoted to proving Theorem 1.1. As above we denote by $\pi : D \rightarrow C$ the ramified double covering attached to (C, η, B) . Remember that the Prym variety (P, Ξ) associated with the covering π has polarization of type $(1, \dots, 2, \dots, 2)$, where 2 appears g times.

By the Riemann–Roch theorem we have that $\dim H^0(P, \mathcal{O}_P(\Xi)) = 2g$. Let $\varphi_\Xi : P \dashrightarrow \mathbb{P}H^0(P, \mathcal{O}_P(\Xi))^\vee$ be the map defined by the polarization. We fix a translation $\Theta_D \subset \text{Pic}^0(D)$ of the theta divisor of JD and a translation of Θ_C such that $\Theta_D \cap JC = 2\Theta_C$. Then we define the map

$$\delta : P \dashrightarrow |2\Theta_C|, \quad z \mapsto (\Theta_{D,z})|_{\pi^*JC},$$

where $\Theta_{D,z} = \Theta_D + z$. This is well-defined: indeed this is equivalent to the statement that

$$(\Theta_{D,z} - \Theta_D)|_{\pi^*JC}$$

is linearly equivalent to the trivial divisor. In other words,

$$(1.1) \quad (\pi^*)^\vee \circ \lambda_{\Theta_D} \circ i = 0,$$

where i is the embedding of P in JD , and λ_{Θ_D} is the isomorphism

$$\lambda_{\Theta_D} : JD \rightarrow \text{Pic}^0(JD)$$

sending z to $\mathcal{O}_{JD}(\Theta_{D,z} - \Theta_D)$. To see that (1.1) holds, we remind the reader of the following standard formula for coverings:

$$\text{Nm}_\pi = \lambda_{\Theta_C}^{-1} \circ (\pi^*)^\vee \circ \lambda_{\Theta_D}.$$

Then $(\pi^*)^\vee \circ \lambda_{\Theta_D} \circ i$ is 0 if and only if

$$\lambda_{\Theta_C}^{-1} \circ (\pi^*)^\vee \circ \lambda_{\Theta_D} \circ i = \text{Nm}_\pi \circ i = 0,$$

which is obvious.

Notice that the base locus of δ is

$$Bs := Bs(\delta) = \{z \in P \mid \pi^*JC \subset \Theta_{D,z}\}.$$

Our aim is to compute Bs and to prove that under some genericity condition the covering π can be recovered from Bs . First we notice that δ is in fact another representation of the map φ_Ξ ; hence, Bs is completely determined by the polarized Prym variety.

Proposition 1.3. *There is a canonical isomorphism $i : \mathbb{P}H^0(P, \mathcal{O}_P(\Xi))^\vee \rightarrow |2\Theta_C|$ making*

$$\begin{array}{ccc} & & \mathbb{P}H^0(P, \mathcal{O}_P(\Xi))^\vee \\ & \nearrow \varphi_\Xi & \downarrow \simeq \\ P \setminus Bs & & |2\Theta_C| \\ & \searrow \delta & \end{array}$$

commutative. In particular, the base locus of φ_Ξ coincides with the base locus of δ .

Proof. The proof in [21, proposition, p. 334] applies verbatim. □

In order to compute explicitly the base locus Bs , it is convenient to work with a suitable translate of P and of Ξ . We consider

$$P^{\text{can}} = \text{Nm}_\pi^{-1}(\omega_C \otimes \eta) \subset \text{Pic}^{2g-2+\frac{r}{2}}(D).$$

Observe that $2g - 2 + \frac{r}{2} = g(D) - 1$. Then there is a canonical representative of the theta divisor

$$\Xi^{\text{can}} = \Theta_D^{\text{can}} \cap P^{\text{can}},$$

where $\Theta_D^{\text{can}} = W_{2g-2+\frac{r}{2}}^0(D) \subset \text{Pic}^{g(D)-1}(D)$ is the canonical theta divisor.

Let $\kappa \in \text{Pic}^{2g-2+\frac{r}{2}}(D)$ be such that $\text{Nm}_\pi \kappa = \omega_C \otimes \eta$ and let $\Theta_D = \Theta_{D,-\kappa}^{\text{can}}$ be the translated theta divisor. Consider the translation $t_{-\kappa} : P^{\text{can}} \xrightarrow{\simeq} P, L \mapsto L \otimes \kappa^{-1}$. Let $\widetilde{Bs} = t_\kappa(Bs)$ be the translation of the base locus in P^{can} , so we obtain the description

$$\widetilde{Bs} = t_\kappa(Bs) = \{L \in P^{\text{can}} \mid \pi^*(JC) \subset \Theta_{D,-L}^{\text{can}}\}.$$

From now on we suppose that $g \geq 2$. The case $g = 1$ will be considered at the end of the proof. We denote by $SU_C(2, \omega_C)$ the moduli space of S -equivalence classes of rank 2 semistable vector bundles on C with determinant ω_C , which is a normal projective variety. Given a line bundle $L \in P^{\text{can}} \subset \text{Pic}^{2g-2+\frac{r}{2}}(D)$, π_*L is a rank 2 vector bundle with

$$\det \pi_*L = \text{Nm}(L) \otimes \det \pi_*(\mathcal{O}_D) = \omega_C \otimes \eta \otimes \det(\mathcal{O}_C \oplus \eta^{-1}) = \omega_C.$$

According to [4, Lemma 1.2] there is an open set $P^{ss} \subset P^{\text{can}}$ such that π_*L is semistable for all $L \in P^{ss}$, so the map $\pi_* : P^{ss} \rightarrow SU_C(2, \omega_C)$ is well-defined. On the other hand, there is a *theta map* defined by

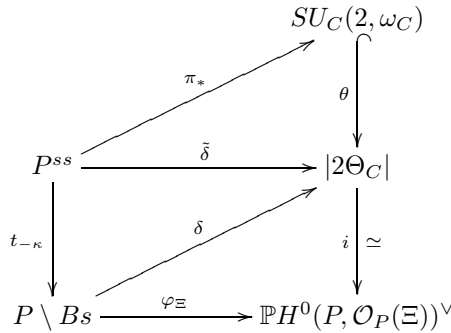
$$\theta : SU_C(2, \omega_C) \rightarrow |2\Theta_C|, \quad E \mapsto \theta(E),$$

where $\theta(E)$ is a divisor whose support is $\{\alpha \in \text{Pic}^0(C) \mid H^0(C, E \otimes \alpha) \neq 0\}$. It is known that the theta map is well-defined [27] (that is, $\theta(E)$ is a divisor linearly equivalent to $2\Theta_C$) for every semistable rank 2 vector bundle with a canonical determinant. The following proposition is a consequence of several results in the literature (see, for example, [13] and the references therein).

Proposition 1.4. *The theta map is an embedding $SU_C(2, \omega_C) \hookrightarrow |2\Theta_C|$ for a generic curve of genus $g \geq 2$.*

In particular, every semistable rank 2 vector bundle with canonical determinant admits a theta divisor in JC [27, Proposition 1.6.2]. Now we set $\tilde{\delta} = \delta \circ t_{-\kappa}$. With this notation we have the following.

Lemma 1.5. *Then*



is commutative.

Proof. The commutativity of the lower triangle is shown in Proposition 1.3. Let $L \in P^{ss}$. Then π_*L is a semistable rank 2 vector bundle with a canonical determinant, and the support of the divisor $\theta(\pi_*L)$ is given by

$$\begin{aligned}
 \text{Supp } \theta(\pi_*L) &= \{\alpha \in \text{Pic}^0(C) \mid H^0(C, \alpha \otimes \pi_*L) \neq 0\} \\
 &= \{\alpha \in \text{Pic}^0(C) \mid H^0(D, \pi^*\alpha \otimes L) \neq 0\} \\
 &= (\Theta_{-L}^{\text{can}})_{|\pi^*JC} \\
 &= \tilde{\delta}(L).
 \end{aligned}$$

This proves the commutativity of the upper triangle. □

Observe that Lemma 1.5 also shows that

$$\widetilde{Bs} \subset B_{nss} := \{L \in P^{\text{can}} \mid \pi_*L \text{ is not semistable}\}.$$

On the other hand, if π_*L is not semistable, then there exists a line subbundle $N \subset \pi_*L$ of slope $> g - 1$. By the Riemann–Roch theorem, we have $0 \neq H^0(C, N \otimes \alpha) \subseteq H^0(C, \pi_*L \otimes \alpha)$ for all $\alpha \in JC$, so

$$H^0(C, \pi_*L \otimes \alpha) = H^0(D, L \otimes \pi^*\alpha) \neq 0 \quad \forall \alpha \in JC;$$

that is, $\pi^*(JC) \subset \Theta_{D, -L}^{\text{can}}$ and $L \in \widetilde{Bs}$. This gives $\widetilde{Bs} = B_{nss}$.

We consider now the subset of P^{can} :

$$B_0 := \{L = \pi^*(A)(p_1 + \dots + p_{\frac{g}{2}-2}) \mid A \in \text{Pic}^g(C), p_i \in D, \text{Nm } L \cong \omega_C \otimes \eta\}.$$

Proposition 1.6. *The equality $B_0 = \widetilde{Bs}$ holds.*

Proof. Let $L = \pi^*(A')(\Sigma p_i) \in B_0$, where A' is effective of maximal degree greater than or equal to g . In particular, putting $\bar{p}_i = \pi(p_i)$ we have that $\bar{p}_i \neq \bar{p}_j$ for $i \neq j$. In order to compute $H^0(D, \pi^*\alpha \otimes L)$ with $\alpha \in JC$, we use a short exact sequence on C given by Mumford in [21, proof of proposition, p. 338] which in our case reads as follows:

$$(1.2) \quad 0 \rightarrow A' \rightarrow \pi_*(L) \rightarrow A'(\Sigma \bar{p}_i) \otimes \eta^{-1} \rightarrow 0,$$

where $\pi_*(L) = \pi_*(\pi^*(A')(\Sigma p_i))$. Tensoring with α , we get

$$H^0(C, A' \otimes \alpha) \subset H^0(C, \pi_*L \otimes \alpha) = H^0(C, \pi_*(L \otimes \pi^*(\alpha))) = H^0(D, L \otimes \pi^*(\alpha)).$$

Since the degree of $A' \otimes \alpha$ is at least $g = g(C)$, then $H^0(C, A' \otimes \alpha) \neq 0$. Therefore, $\pi^*(JC) \subset \Theta_{D,-L}^{\text{can}}$, and this proves $B_0 \subset \widetilde{Bs}$.

Let $L \in B_{n_{ss}}$. Then there exists a line bundle $A \hookrightarrow \pi_*L$ with

$$\text{deg } A > g - 1 = \mu(\pi_*L) := \frac{\text{deg } \pi_*L}{\text{rk } \pi_*L}.$$

Since

$$\text{Hom}(A, \pi_*L) \cong \text{Hom}(\pi^*(A), L),$$

there is a nontrivial map $\pi^*(A) \rightarrow L$ which is necessarily injective. We also obtain that $h^0(D, L \otimes \pi^*(A^{-1})) > 0$, and hence L is of the form $L = \pi^*A(\Sigma p_i)$, $p_i \in D$. Thus, we have

$$\widetilde{Bs} = B_{n_{ss}} \subset B_0,$$

and we are done. □

We use the description of the base locus to recover the covering $\pi : D \rightarrow C$ from the Prym variety (P, Ξ) . Define the variety

$$W := \left\{ (A, \sum_i p_i) \in \text{Pic}^g(C) \times D^{(\frac{r}{2}-2)} \mid A^{\otimes 2}(\sum_i \bar{p}_i) \simeq \omega_C \otimes \eta \right\},$$

where $\bar{p}_i := \pi(p_i)$.

Proposition 1.7. *Assume that (C, η) is general, and assume also that $r \geq \max\{6, \frac{2}{3}(g+2)\}$. Then the natural morphism $W \rightarrow B_0$ sending*

$$(A, \sum_{i=1}^{\frac{r}{2}-2} p_i) \mapsto \pi^*(A)(\sum_{i=1}^{\frac{r}{2}-2} p_i)$$

is birational.

Proof. The main difficulty of the proof is to show that for such a generic element the equality $h^0(C, A) = 1$ holds. To prove this, we denote it by

$$W_\eta = \{A \in \text{Pic}^g(C) \mid h^0(C, \omega_C \otimes \eta \otimes A^{-\otimes 2}) > 0\},$$

which by definition is the image of the first projection $W \rightarrow \text{Pic}^g(C)$. Obviously, if

$$\text{deg}(\omega_C \otimes \eta \otimes A^{-\otimes 2}) = 2g - 2 + \frac{r}{2} - 2g = \frac{r}{2} - 2 \geq g,$$

then the condition on the cohomology is empty and $W_\eta = \text{Pic}^g(C)$. In this case $h^0(C, A) = 1$ generically for any (C, η) . This covers the cases $r \geq 2g + 4$. Assume now that $r = 2g + 2$. Since the second projection $W \rightarrow D^{(\frac{r}{2}-2)}$ is an étale covering, $\dim W_\eta \leq \dim W = \frac{r}{2} - 2 = g - 1$; hence, W_η is a proper subvariety of $\text{Pic}^g(C)$. We claim that it is not contained in the Brill–Noether locus $W_g^1(C)$. If we are

able to prove that $\dim W_\eta = g - 1$, then $W_\eta \not\subset W_g^1(C) \cong W_{g-2}^0(C)$. Since the cohomological condition that defines W_η refers to $A^{\otimes 2}$, we consider \overline{W}_η the image of the étale map:

$$W_\eta \longrightarrow \overline{W}_\eta \subset \text{Pic}^{2g}(C), \quad A \mapsto A^{\otimes 2}.$$

Then, an element $L \in \overline{W}_\eta$ satisfies

$$0 < h^0(C, \omega_C \otimes \eta \otimes L^{-1}) = h^0(C, L \otimes \eta^{-1})$$

by the Riemann–Roch theorem. Then \overline{W}_η is isomorphic to the translated $\eta + W_{g-1}^0(C)$ of the theta divisor on $\text{Pic}^{g-1}(C)$. Thus, for any $\eta \in T = \{\eta \mid \eta^{\otimes 2} \in W_r^0(C)\}$, $\dim W_\eta = g - 1$, which proves the claim.

In the previous argument there is no restriction on η . In order to prove the statement for $r < 2g + 2$, we shall show that for a generic η in T we have $W_\eta \not\subset W_g^1(C)$. We claim that the union of the subsets W_η , as η varies in T , covers all $\text{Pic}^g(C)$. In other words, if we define

$$I := \{(A, \eta) \in \text{Pic}^g(C) \times T \mid A \in W_\eta\},$$

this is equivalent to showing that the first projection $I \rightarrow \text{Pic}^g(C)$ is surjective. Let $T = \{\eta \mid \eta^{\otimes 2} \in W_r^0(C)\}$ and $T_0 = \{\eta \in T \mid |\eta^{\otimes 2}| \text{ has a reduced divisor}\}$. We fix an arbitrary element $A \in \text{Pic}^g(C)$. We ask for the existence of an $\eta \in T$ such that $h^0(C, \omega_C \otimes \eta \otimes A^{-\otimes 2}) > 0$. Then η has to be chosen in the intersection of the Brill–Noether locus

$$T_A := W_{\frac{r}{2}-2}^0 + (A^{\otimes 2} \otimes \omega_C^{-1})$$

with T . If $r \geq g$, then $T = \text{Pic}^{\frac{r}{2}}(C)$ and the existence of such an η is obvious as far as $r \geq 6$. Assume from now on that $r \leq g$, in particular, $\dim T = \dim W_r^0(C) = r$. Since the cohomological class of T_A is a fraction of a power of the theta divisor, the intersection $T_A \cap T$ is nonempty if

$$\dim T_A + \dim T - g = \frac{r}{2} - 2 + r - g \geq 0.$$

Hence, for $r \geq \max\{6, \frac{2}{3}(g + 2)\}$, we have the claimed surjectivity. Thus, for a generic $\eta \in T$, and therefore for a generic $\eta \in T_0$, the generic element in W_η satisfies $h^0(C, A) = 1$.

To finish the proof of the proposition, we observe that the second projection $W \rightarrow D^{(\frac{r}{2}-2)}$ is an étale covering of degree 2^{2g} , hence for a generic element $(A, \sum_i p_i)$ in W we can simultaneously assume that $\sum_i p_i$ is π simple (i.e., it does not contain fibers of π) and $h^0(C, A) = 1$. Let us consider again the exact sequence (1.2)

$$0 \longrightarrow A \longrightarrow \pi_*(\pi^*(A)(\sum_i p_i)) \longrightarrow A(\sum_i \bar{p}_i) \otimes \eta^{-1} \longrightarrow 0.$$

Since $A^{\otimes 2}(\sum_i \bar{p}_i) \cong \omega_C \otimes \eta$ the exact sequence becomes

$$0 \longrightarrow A \longrightarrow \pi_*(\pi^*(A)(\sum_i p_i)) \longrightarrow \omega_C \otimes A^{-1} \longrightarrow 0.$$

Notice that $H^0(C, \omega_C \otimes A^{-1}) \cong H^1(C, A)$. By the Riemann–Roch theorem this group is trivial since $\deg(A) = g$ and $h^0(C, A) = 1$. Therefore

$$H^0(C, A) \cong H^0(C, \pi_*(\pi^*(A)(\sum_i p_i))).$$

In other words, there is only one divisor in the linear series $|\pi^*(A)(\sum_i p_i)|$. This immediately implies the generic injectivity of the map $W \rightarrow B_0$. Since it is obviously surjective, we are done.

Now we consider the case $g = 1$ and $r \geq 6$. We simply remark that the only place in the proof where we use that $g \geq 2$ is in Proposition 1.6, where we use rank 2 vector bundles on curves with a genus of at least 2 to show the inclusion $\widetilde{Bs} \subset B_0$. Let us prove it directly: Assume by contradiction that $L \in \widetilde{Bs}$ does not belong to B_0 , which in this case means that L (which satisfies $h^0(C, L) > 0$ by definition) is represented by an effective π -simple divisor. Then the exact sequence (1.2) reads

$$0 \rightarrow \mathcal{O}_C \rightarrow \pi_*(L) \rightarrow \mathcal{O}_C \rightarrow 0;$$

hence, for any $\alpha \in JC \setminus \{0\}$, we obtain

$$0 = h^0(C, \pi_*(L) \otimes \alpha) = h^0(C, \pi_*(L \otimes \pi^*\alpha)) = h^0(D, L \otimes \pi^*(\alpha)),$$

which is a contradiction since $L \in \widetilde{Bs}$. □

Now we are in disposition of finishing the proof of the main theorem of this section. According to the last proposition, the quotient of the action of $J\mathcal{C}_2 = \text{Ker}(\lambda_\Xi)$ on the base locus Bs of the linear system $|\Xi|$ is birational to $W/J\mathcal{C}_2 = D^{(\frac{r}{2}-2)}$. Since $g(D) - 1 = 2g - 2 + \frac{r}{2} > \frac{r}{2} - 2$, we can apply the main theorem in [20], which gives a constructive method to recover the curve D from a variety birational to $D^{(\frac{r}{2}-2)}$. Moreover, the base locus Bs is symmetric in P and this symmetry corresponds to the involution $\sigma^{(\frac{r}{2}-2)}$ on the symmetric product of the curve. Applying again the result in [20], we recover also the involution σ on D and therefore the whole covering $D \rightarrow D/\langle\sigma\rangle$.

Remark 1.8. The case of Corollary 1.2 ($r = 6$ and $g = 2$) is particularly simple. It is not hard to see that in this case, for a generic η , W and $B_0 \cong Bs$ are isomorphic irreducible curves of genus 81. In this case the condition on η can be determined precisely: we need the vanishing $h^0(C, \eta \otimes \omega_C^{-1}) = 0$. The quotient of the action of $\text{Ker}(\lambda_\Xi)$ on W gives directly the curve D , and the symmetry on W induces the involution σ on D . Finally $D/\langle\sigma\rangle = C$.

2. CASE $g = 5$ AND $r = 2$

In this section we take care of the map $\mathcal{P}_{5,2} : \mathcal{R}_{5,2} \rightarrow \mathcal{A}_5$ which by [18, Corollary (2.3)] is generically finite. This is the last open case of the generic Torelli problem stated in the introduction. We will prove at the end of this section that $\mathcal{P}_{5,2}$ is generically injective.

In order to study the degree of $\mathcal{P}_{5,2}$, we use the classical Prym map $\mathcal{P}_6 : \mathcal{R}_6 \rightarrow \mathcal{A}_5$ defined on irreducible unramified double coverings of curves of genus 6. To do so we first recall how this map was extended by Beauville in [2] to a proper map:

$$\overline{\mathcal{P}}_6 : \overline{\mathcal{R}}_6 \rightarrow \mathcal{A}_5.$$

We give the basic definitions for any g . To start with, we borrow from [2] the description of the elements belonging to $\overline{\mathcal{R}}_g$, called admissible coverings (see [2, (**) and Lemma (5.1)]).

Definition 2.1. Let \tilde{C} be a connected curve with only ordinary double points and arithmetic genus $p_a(\tilde{C}) = 2g - 1$, and let σ be an involution on \tilde{C} . Then $\tilde{C} \rightarrow \tilde{C}/\langle\sigma\rangle$ is an admissible covering if and only if the following conditions are fulfilled:

- (1) There are not fixed nonsingular points of σ .
- (2) At the nodes fixed by σ , the two branches are not exchanged.
- (3) The number of nodes exchanged under σ equals the number of irreducible components of \tilde{C} exchanged under σ .

In particular, σ is not the identity on any component. Under these conditions the arithmetical genus of $\tilde{C}/\langle\sigma\rangle$ is g and the Prym variety attached to the covering can be defined in a similar way to the standard Prym construction and it is a ppav.

An instance of admissible covering is the following: Consider two copies of a smooth curve C of genus $g - 1$, with each one having two marked points. Call p_1, q_1 the marked points in the first copy and p_2, q_2 the same points in the second copy. Then the curve

$$\tilde{C} = C \cup C / (p_1 \sim q_2, p_2 \sim q_1)$$

has a natural involution σ exchanging the components. Observe that $\tilde{C}/\langle\sigma\rangle$ is the irreducible nodal curve $C/p_1 \sim p_2$. This is an admissible covering whose associated Prym variety is the Jacobian of C . These elements (\tilde{C}, σ) are called Wirtinger coverings. Observe that the closure of the set parametrizing these objects describe a divisor in $\overline{\mathcal{R}}_g$ which is birational to the moduli space $\mathcal{M}_{g-1,2}$ of curves of genus $g - 1$ with two marked points. Hence, this divisor is irreducible. We denote it by Δ^W .

Now we come back to our particular situation; hence, $g = 6$. Let $\pi : D \rightarrow C$ be a covering in $\mathcal{R}_{5,2}$. By gluing in C the two branch points and in D the two ramification points, we get an admissible covering. These curves form a divisor in the boundary of $\overline{\mathcal{R}}_6$, birational to $\mathcal{R}_{5,2}$ (and hence irreducible) that we denote by Δ^n . A generic element in this divisor is an irreducible admissible covering of a curve with exactly one node. Moreover, the composition

$$\mathcal{R}_{5,2} \hookrightarrow \Delta^n \hookrightarrow \overline{\mathcal{R}}_6 \xrightarrow{\overline{\mathcal{P}}_6} \mathcal{A}_5$$

is the Prym map $\mathcal{P}_{5,2}$ we are considering.

Remark 2.2. Let us consider the forgetful map $\pi : \mathcal{R}_6 \rightarrow \mathcal{M}_6$ sending $[\tilde{C} \rightarrow C]$ to the class of the curve C . Let $\overline{\mathcal{M}}_6$ be the Deligne–Mumford compactification of the moduli space of curves. Denote by $\overline{\mathcal{R}}'_6$ the normalization of $\overline{\mathcal{M}}_6$ in the function field of \mathcal{R}_6 . Then there is a map $\pi' : \overline{\mathcal{R}}'_6 \rightarrow \overline{\mathcal{M}}_6$, which has been studied in [12]. The preimage of the divisor of nodal curves $\Delta_0 \subset \overline{\mathcal{M}}_6$ under π' is

$$\pi'^*(\Delta_0) = \Delta^{na} + \Delta^W + 2\Delta^n,$$

where the general element $\tilde{C} \rightarrow C$ of Δ^{na} is not admissible (in Beauville’s sense) since \tilde{C} is irreducible with two nodes interchanged by σ . From this, one sees that π' is ramified along Δ^n (which explains the notation Δ^{ram} in [12]). Of course, there are other components of codimension 1 in the boundary of $\overline{\mathcal{R}}'_6$ (and of $\overline{\mathcal{R}}_6$) that are in the preimage of $\Delta_{i,6-i} \subset \overline{\mathcal{M}}_6$. We refer to [12] and the references therein for more details on this map for any g .

In order to study the degree of $\mathcal{P}_{5,2}$, our strategy is to look at the preimage of the locus \mathcal{C} , the set of the smooth cubic threefolds embedded in \mathcal{A}_5 via the intermediate

Jacobian map (see [7]). We start by recalling some basic facts on the geometry of cubic threefolds and the representation of its intermediate Jacobian map as Prym variety. All the results are very classical, and they can be found in [7], [10], [9], and [23].

Let V be a smooth cubic threefold in \mathbb{P}^4 . The intermediate Jacobian JV of V is isomorphic to the Prym variety of many elements in $\overline{\mathcal{R}}_6$. This is done as follows: Given a line $l \subset V$ and a 2-plane $\Pi \subset \mathbb{P}^4$ disjoint with l , the projection $V \setminus l \rightarrow \Pi$ extends to a morphism on the blowup V_l of V on l . The fibers of this map are conics and the discriminant curve on Π parametrizing the points where these conics degenerate is a quintic Q_l . The pairs of lines on the degenerate fibers give a curve \tilde{Q}_l in the Grassmannian of lines in \mathbb{P}^4 . Then $\tilde{Q}_l \rightarrow Q_l$ is an admissible covering and its Prym variety is isomorphic as ppav to JV . Hence $\overline{\mathcal{P}}_6^{-1}(V)$ contains the set of all these coverings of plane quintics, which is in bijection with the Fano surface $F(V)$ of the lines contained in V . In [10, Part V, section 1] it is proved that in fact the fiber is exactly $F(V)$ for any V . Since we are interested in the fiber restricted to the divisor Δ^n , we need to know in which cases the quintic Q_l is not smooth; in other words, the first step is to compute

$$F(V) \cap \Delta^n$$

for any cubic threefold V .

This is related to the geometry of the cubic threefolds due to the following result of Beauville.

Proposition 2.3 ([3, Proposition 1.2]). *Let $V_l \rightarrow \Pi$ be a conic bundle as above and let Q_l be the discriminant curve. Then*

- (1) *The curve Q_l has at most ordinary double points as singularities.*
- (2) *If s is a regular point in Q_l , then the corresponding conic has only one singular point; i.e., it is formed by two different lines.*
- (3) *If s is a node of Q_l , then the corresponding conic is a double line.*

Hence, Q_l is singular if and only if there is a plane in \mathbb{P}^4 intersecting V in $l + 2r$, where $r \in F(V)$. So we are interested in the following two lines:

$$\Gamma = \{l \in F(V) \mid \exists \text{ a plane } L \text{ and a line } r \in F(V) \text{ with } V \cdot L = l + 2r\},$$

$$\Gamma' = \{r \in F(V) \mid \exists \text{ a plane } L \text{ and a line } l \in F(V) \text{ with } V \cdot L = l + 2r\}.$$

According to Proposition 2.3, the set Γ parametrizes the lines l for which Q_l is singular.

The curve Γ' has deserved more attention in the literature. It appears in [7, section 6] as the set of lines of “second type”. In [7, Proposition 10.21] it is shown that Γ' has pure dimension 1 and, as divisor on the Fano surface, it is linearly equivalent to twice the canonical divisor. Moreover, Murre proved that Γ' (\mathcal{F}_0 in Murre’s notation) is smooth and not necessarily connected, and Γ (\mathcal{F}'_0 with his notation) is Zariski closed of dimension at most 1 (see [23, Corollary (1.9), Lemma (1.11)]).

It is easy to go further with the techniques of [23] to show that in fact Γ is also a curve. To this end, let us consider the incidence variety

$$I = \{(l, r) \in \Gamma \times \Gamma' \mid \text{there is a plane } L \text{ with } L \cdot V = l + 2r\}.$$

By Proposition 2.3 the discriminant curve Q_l has at most a finite number of singularities; hence, we have that $I \rightarrow \Gamma$ is finite to one. The following lemma implies that $I \rightarrow \Gamma'$ is bijective.

Lemma 2.4. *Given a line $r \in \Gamma'$ there is only one line l such that there is a plane L with $L \cdot V = l + 2r$.*

Proof. We fix r, l , and L as in the statement. We can choose a reference system in such a way that $(1 : 0 : 0 : 0 : 0), (0 : 1 : 0 : 0 : 0) \in r$ and $(1 : 0 : 0 : 0 : 0), (0 : 0 : 1 : 0 : 0) \in l$. Hence, using coordinates x, y, z, u, v , the equations of r and l are $z = u = v = 0$ and $y = z = v = 0$, respectively. Moreover, L has equations $u = v = 0$. Our hypothesis implies that the cubic polynomial F defining V is of the form

$$F = cyz^2 + uG_2 + vH_2,$$

where G_2, H_2 are the equations of two quadrics Q_1, Q_2 in \mathbb{P}^4 . Observe that if $c = 0$, then, by an easy computation, the points satisfying $G_2 = H_2 = u = v = 0$ are singularities of V , which is a contradiction. So we can assume $c = 1$.

By convenience, in a later computation we decompose the quadratic forms G_2, H_2 separating the part in the variables x, y :

$$\begin{aligned} G_2 &= A(x, y) + zL_1 + uL_2 + vL_3, \\ H_2 &= B(x, y) + zM_1 + uM_2 + vM_3, \end{aligned}$$

where L_i, M_i are linear forms. Hence, we have

$$(2.1) \quad \begin{aligned} F &= yz^2 + u(A(x, y) + zL_1 + uL_2 + vL_3) \\ &\quad + v(B(x, y) + zM_1 + uM_2 + vM_3). \end{aligned}$$

The planes through l are parametrized by the points of the plane $x = y = 0$. We want to know for which points $p = (0 : 0 : \alpha : \beta : \gamma)$ the plane $r \vee p$ intersects V in $2r + l'$ for some line l' . A generic point of $r \vee p$ is of the form $(r : s : t\alpha : t\beta : t\gamma)$ for some parameters r, s, t . Replacing in (2.1) we have to impose that t (the equation of r in the plane) appears with multiplicity 2:

$$\alpha^2 st^2 + t\beta(A(r, s) + t(\alpha L_1 + \dots)) + t\gamma(B(r, s) + t(\alpha M_1 + \dots)).$$

So the condition is $\beta A(r, s) + \gamma B(r, s) = 0$ for any r, s . The next claim implies that $\beta = \gamma = 0$, and therefore the only solution corresponds to the plane L and hence l is unique.

Claim. If the quadratic forms $A(x, y), B(x, y)$ are not linearly independent, then V is singular.

Indeed, assume that $B(x, y) = \lambda A(x, y)$ for some constant λ . Then F is of the form

$$yz^2 + u(A(x, y) + zL_1 + uL_2 + vL_3) + v(\lambda A(x, y) + zM_1 + uM_2 + vM_3).$$

One easily checks that the points which satisfy $A(x, y) = z = u = v = 0$ are nontrivial solutions of the system of the partial derivatives of F ; hence, V is singular. □

Corollary 2.5. *For any smooth cubic threefold, Γ is a curve dominated by Γ' .*

In the case of a general cubic threefold we can go further in the description of the properties of the curves Γ and Γ' .

Proposition 2.6. *For a general cubic threefold V the curves Γ and Γ' are irreducible, smooth, and isomorphic.*

Proof. We first prove the irreducibility. This is a consequence of the fact that for a general V the Fano surface $F(V)$ has Picard number 1 (see [28, p. 382]). Indeed, recall from [7, section 10] that Γ' is bicanonical in $F(V)$ and that canonical divisor is very ample, giving the Plücker embedding $F(V) \hookrightarrow \text{Grass}(1, 4) \hookrightarrow \mathbb{P}^9$. Therefore the positive generator of the Neron–Severi group is ample. Thus, if $\Gamma' = T_1 \cup T_2$, then $T_1 \cdot T_2 > 0$, contradicting the smoothness proved by Murre. Since Γ' dominates Γ , both are irreducible. To prove the smoothness of Γ we will show that there is only one node in the quintic Q_l for any $l \in \Gamma$; this implies that $\Gamma \cong I \cong \Gamma'$.

We finally show the statement on the number of nodes; assume that there is a line l in V such that there are two different lines r, r' and two different planes L, L' with $V \cdot L = l + 2r$ and $V \cdot L' = l + 2r'$. We can choose a reference system such that $(1 : 0 : 0 : 0 : 0) \in l \cap r$, $(0 : 1 : 0 : 0 : 0) \in l \cap r'$, $(0 : 0 : 1 : 0 : 0) \in r$ and $(0 : 0 : 0 : 1 : 0) \in r'$. We use the variables x, y, z, u, v for the coordinates. Then the equation of V is of the form

$$\lambda y^2 z + \mu u x^2 + v Q_2,$$

where Q_2 is the equation of a quadric in \mathbb{P}^4 . The set of these equations represents a linear variety of dimension 16 in the parameter space \mathbb{P}^{34} of the equations (up to constant) of degree 3 in five variables. The projectivities that leave invariant the four fixed points depend on eight projective parameters. Hence, the family of those V 's for which there exist a line $l \in F(V)$ with Q_l with more than one node is a codimension 2 subvariety in \mathcal{C} . □

We come back to our computation of the degree of the map $\mathcal{P}_{5,2}$ identified with $\overline{\mathcal{P}}_{6|\Delta^n}$. Observe that, by the previous results, any $JV \in \mathcal{C}$ can be represented as $P(\tilde{Q}_l, Q_l)$ for a quintic plane curve with only one node. Notice that \tilde{Q}_l must be irreducible; otherwise, $\tilde{Q}_l \rightarrow Q_l$ would be a Wirtinger cover and JV would be a Jacobian contradicting the main result in [7]. Summarizing, we have the following:

- (a) The locus \mathcal{C} is contained in $\overline{\mathcal{P}}_6(\Delta^n)$.
- (b) The fiber over V of $\overline{\mathcal{P}}_{6|\Delta^n}$ is $\Gamma = \Delta^n \cap F(V)$.
- (c) The locus $\mathcal{S}' = (\overline{\mathcal{P}}_{6|\Delta^n})^{-1}(\mathcal{C})$ has dimension 11, all the fibers of $\mathcal{S}' \rightarrow \mathcal{C}$ are one dimensional, and the generic fiber is smooth and irreducible.

In particular, the irreducible component $\mathcal{S} \subset \mathcal{S}'$ containing the generic smooth irreducible fibers is the only component of \mathcal{S}' of dimension 11. Notice that the other possible components of \mathcal{S} have a dimension of at most 9.

Although the map $\overline{\mathcal{P}}_{6|\Delta^n}$ is generically finite, the fibers over \mathcal{C} are one dimensional, as we have already seen. Nevertheless, the degree of $\overline{\mathcal{P}}_{6|\Delta^n}$ can be computed at this locus by using the concept of local degree explored in [10, Part I, section 3]. The local degree d of $\overline{\mathcal{P}}_{6|\Delta^n}$ along \mathcal{S} is the degree of the map obtained from $\overline{\mathcal{P}}_{6|\Delta^n}$ by localizing at \mathcal{S} in the source and at \mathcal{C} in the target:

$$(\Delta^n)_{\mathcal{S}} \longrightarrow (\mathcal{A}_5)_{\mathcal{C}}.$$

We have $\text{deg}(\overline{\mathcal{P}}_{6|\Delta^n}) = d$.

Assume that $\overline{\mathcal{P}}_6|_{\Delta^n}$ lifts to a regular map from the blowup $\widetilde{\Delta}^n$ of Δ^n along \mathcal{S} to the blowup of $\widetilde{\mathcal{A}}_5$ of \mathcal{A}_5 along \mathcal{C} , sending the exceptional divisor $\widetilde{\mathcal{S}}$ to the exceptional divisor $\widetilde{\mathcal{C}}$:

$$\widetilde{\overline{\mathcal{P}}_6}|_{\Delta^n} : \widetilde{\mathcal{S}} \rightarrow \widetilde{\mathcal{C}}.$$

Then we have the following.

Lemma 2.7 ([10, Lemma 3.2]). *If the natural linear map $N_{\mathcal{S}|\Delta^n, s} \rightarrow N_{\mathcal{C}|\mathcal{A}_5, \overline{\mathcal{P}}_6(s)}$ induced by*

$$d(\overline{\mathcal{P}}_6|_{\Delta^n}) : T_{\Delta^n} \rightarrow T_{\mathcal{A}_5}$$

is injective at any $s \in \mathcal{S}$, then the local degree at \mathcal{S} equals the degree of $\widetilde{\overline{\mathcal{P}}_6}|_{\Delta^n} : \widetilde{\mathcal{S}} \rightarrow \widetilde{\mathcal{C}}$.

In other words, under some conditions on the behavior of the normal bundle, we can compute the degree by looking at the degree of the map between exceptional divisors. In [10, Part V] the authors proved that the conditions of the lemma are satisfied by the usual Prym map $\overline{\mathcal{P}}_6 : \overline{\mathcal{R}}_6 \rightarrow \mathcal{A}_5$ with respect to \mathcal{C} . Moreover, they give a nice geometrical description of the blowups involved in the picture. In this way they are able to confirm once again that the degree of P is 27 (see also [9, section 4]). The next proposition is a summary of the results in sections 1 and 2 in [10, Part V]:

Proposition 2.8. *Let \mathcal{F} be the closure of the union of all the Fano surfaces $F(V)$ of smooth cubic threefolds V . Denote by $\pi_1 : \widetilde{\overline{\mathcal{R}}_6} \rightarrow \overline{\mathcal{R}}_6$ the blowup of $\overline{\mathcal{R}}_6$ along \mathcal{F} and by $\pi_2 : \widetilde{\mathcal{A}}_5 \rightarrow \mathcal{A}_5$ the blowup of \mathcal{A}_5 along \mathcal{C} . Then*

- (1) $\overline{\mathcal{P}}_6^{-1}(\mathcal{C}) = \mathcal{F}$.
- (2) *For a cubic threefold V the fiber $\pi_2^{-1}(V)$ can be identified with the dual of the ambient space \mathbb{P}^4 of V .*
- (3) *For an admissible covering $(\widetilde{Q}_l, Q_l) \in \mathcal{F}$ the fiber $\pi_1^{-1}(\widetilde{Q}_l, Q_l)$ can be identified with the dual of the ambient space \mathbb{P}^2 of Q_l .*
- (4) *The map between exceptional divisors $\widetilde{\mathcal{F}} \rightarrow \widetilde{\mathcal{C}}$ sends $(\widetilde{Q}_l, Q_l, m)$ to $(V, l \vee m)$. It is particularly well-defined and injective everywhere, and the conditions of Lemma 2.7 are satisfied.*

Corollary 2.9. *Let \mathcal{S} be the fiber of \mathcal{C} by the map $\overline{\mathcal{P}}_6|_{\Delta^n} : \Delta^n \rightarrow \mathcal{A}_5$, and let $\widetilde{\mathcal{S}}$ be the exceptional divisor of the blowup of Δ^n along \mathcal{S} . Then the degree of $\overline{\mathcal{P}}_6|_{\Delta^n}$ equals the degree of the map $\widetilde{\mathcal{S}} \rightarrow \widetilde{\mathcal{C}}$ sending $(\widetilde{Q}_l, Q_l, m)$ to $(V, l \vee m)$.*

Proof. Since $\Delta^n \subset \overline{\mathcal{R}}_6$ is a closed immersion, the universal property of blowups (see [14, Corollary 7.15]) implies that Proposition 2.5 is valid for the restriction to Δ^n . In particular, the conditions of Lemma 2.7 are satisfied, and the local degree (and hence the degree) can be computed looking at the map between exceptional divisors. □

Now we can prove the main result of this section.

Theorem 2.10. *The Prym map $\mathcal{P}_{5,2} : \mathcal{R}_{5,2} \rightarrow \mathcal{A}_5$ is generically injective.*

Proof. Due to the last corollary, the theorem reduces to check the following fact: Let V be a generic smooth cubic threefold, and let $l \subset V$ be a generic line on Γ

(hence, Q_l is a quintic nodal curve). Given a generic hyperplane H in \mathbb{P}^4 containing l , there is no other line in Γ contained in H .

To prove this, we consider

$$\mathcal{H}_0 = \{(l, H) \in \Gamma \times \mathbb{P}^{4*} \mid l \subset H\}.$$

This a three-dimensional closed subvariety, since all the fibers of $\mathcal{H}_0 \rightarrow \Gamma$ are isomorphic to \mathbb{P}^2 . The variety we are interested in is $\mathcal{H} = p_2(\mathcal{H}_0)$. The image of the rational map

$$\Gamma \times \Gamma \dashrightarrow \mathcal{H}$$

sending (l_1, l_2) to the hyperplane $l_1 \vee l_2$ is a two-dimensional family \mathcal{H}_2 of hyperplanes contained in \mathcal{H} . To finish the proof of the theorem it is enough to show that $\dim \mathcal{H} = 3$. Indeed, otherwise, for a generic $H \in \mathcal{H}$ there are infinitely many lines of Γ contained in H ; thus, the whole Γ gives lines in H . Taking three different linearly independent hyperplanes $H_1, H_2, H_3 \in \mathcal{H}$ such that all the lines of Γ are in the three hyperplanes, we get a contradiction. \square

APPENDIX A. GEOMETRY OF $\mathcal{R}_{2,6}$ VIA KUMMER QUARTIC SURFACES

In this appendix we describe the moduli space $\mathcal{R}_{2,6}$ via the geometry of the sextic models in \mathbf{P}^4 of a smooth, integral genus 2 curve C and of its naturally associated Kummer quartic surface. Our goal is to derive from this some recreational geometry and to prove the following result.

Theorem A.1. *$\mathcal{R}_{2,6}$ is rational.*

To perform the proof, let us introduce some preliminaries. Recall that a point of $\mathcal{R}_{2,6}$ is defined by a triple (C, η, b) , where C is a curve as above, $\eta \in \text{Pic}^3 C$, and b is a smooth element of $|\eta^{\otimes 2}|$. Since $\eta^{\otimes 2}$ has degree 6, we have $\dim |\eta^{\otimes 2}| = 4$. Furthermore, a well-known theorem of Mumford [22] implies that $\eta^{\otimes 2}$ defines a projectively normal embedding

$$C \subset \mathbf{P}^4,$$

with ideal sheaf \mathcal{I}_C generated by quadrics. The linear system $|\mathcal{I}_C(2)|$ is three dimensional and reflects the properties of the Kummer surface of $\text{Pic}^0 C$. We recollect from [5] some of these properties. First observe that C is contained in the scroll

$$R = \bigcup \langle x + \sigma(x) \rangle, \quad x \in C,$$

where σ is the involution on the covering exchanging the sheets over C . So R is either a cone over a rational normal cubic or a smooth cubic scroll, biregular to \mathbf{P}^2 blown up at one point. Notice that R is a cone if and only if C is 3-canonically embedded. In particular, the moduli points of triples (C, η, b) , such that R is not a cone, fill up a dense open set of $\mathcal{R}_{2,6}$. Therefore, it will be not restrictive to assume that R is smooth.

The ideal sheaf \mathcal{I}_R of R is generated by quadrics and $\dim |\mathcal{I}_R(2)| = 2$. This implies that $C = R \cdot Q$, where Q is a quadric smooth along C . Hence, by the Bertini theorem, it follows that a general $Q \in |\mathcal{I}_C(2)|$ is smooth. Then the discriminant locus of $|\mathcal{I}_C(2)|$ is a surface, namely the quintic surface

$$D := \{Q \in |\mathcal{I}_C(2)| \mid \text{Sing } Q \neq \emptyset\}.$$

Notice that D is not an integral. Indeed, by the Lefschetz hyperplane theorem, no quadric containing R is smooth. Hence, the net $|\mathcal{I}_R(2)|$ is a plane contained in D . From now on we will use the notation

$$\mathbf{P}^3 := |\mathcal{I}_C(2)| \quad \text{and} \quad \mathbf{P}^2 := |\mathcal{I}_R(2)|.$$

We describe \mathbf{P}^2 explicitly. The surface R contains a unique exceptional line that we will denote by E . We can associate with each point $v \in R$ a unique quadric Q_v containing R in the following way. Let $p_v : R \rightarrow \mathbf{P}^3$ be the projection from v . Then $p_v(R)$ is a quadric surface and the cone of vertex v over $p_v(R)$ is a quadric Q_v through R . If Q'_v is a second quadric through R singular at v , then $Q_v \cap Q'_v$ is a cone of vertex v over a curve, which implies that R is a cone, so we get a contradiction. Therefore, the assignment $v \rightarrow Q_v$ defines a morphism

$$\beta : R \rightarrow \mathbf{P}^2.$$

Proposition A.2. *The map β is the contraction of E . Moreover, either $v \notin E$ and $\beta(v)$ has rank 4 or $\beta(E)$ has rank 3. In the latter case $E = \text{Sing } \beta(E)$.*

Proof. Since the arguments are standard we sketch only the proof. Let $v \in R$. Then $p_v(R)$ is obtained via the elementary transformation of center v of the ruled surface R [15]. Hence, $p_v(R)$ is a smooth quadric surface if $v \notin E$. If $v \in E$, then $p_v(R)$ has rank 3, $p_v(R)$ is singular at $p_v(E)$, and $\text{Sing } Q_v = E$. Now it is easy to see that there is a unique rank 3 quadric Q_E containing R . Hence, we have $Q_v = Q_E$ and β is constant on E . Then the statement follows. \square

To continue the study of D , we consider the natural involution

$$s : \text{Pic}^3 C \rightarrow \text{Pic}^3 C$$

defined by $s(L) := \eta^{\otimes 2}(-L)$, so $\text{Pic}^3 C / \langle s \rangle$ is biregular to the Kummer surface of $\text{Pic}^0 C$. Let $L \in \text{Pic}^3 C$. Then $\dim |L| = 1$ and each $d \in |L|$ satisfies $\dim \langle d \rangle = 2$. Otherwise, we would have $\dim |\eta^{\otimes 2}(-d)| \geq 2$, which is impossible. This defines a quadric through C , namely

$$Q_L := \bigcup_{d \in |L|} \langle d \rangle,$$

the union of the planes generated by the divisors in $|L|$. Let

$$u : \text{Pic}^3 C \rightarrow \mathbf{P}^3, \quad L \mapsto Q_L,$$

and $S := u(\text{Pic}^3 C)$. Since Q_L is ruled by planes, Q_L is singular and hence $u(\text{Pic}^3 C) \subset D$.

Lemma A.3. *At a general Q_L the fiber of $u : \text{Pic}^3 C \rightarrow S$ is $\{L, s(L)\}$.*

Proof. From $L \otimes s(L) \cong \mathcal{O}_C(1)$ it easily follows that $Q_L = Q_{s(L)}$. Let $|L|$ be a basepoint-free pencil. Then L is uniquely reconstructed from the ruling of planes $\langle d \rangle$ of Q_L . Since for a general $L \in \text{Pic}^3 C$, $|L|$ and $|s(L)|$ are basepoint-free distinct pencils, the statement follows. \square

To complete the description of S , we consider the natural theta divisor

$$\Theta := \{M \in \text{Pic}^3 C \mid M \cong \omega_C(x), x \in C\}$$

and its conjugate by the involution s

$$\Theta_\eta := s(\Theta) = \{\eta^{\otimes 2}(-M), M \in \Theta\}.$$

Observe that $M \in \Theta$ if and only if $|M|$ is basepoint free.

Theorem A.4. *It holds that*

- (1) $u^*\mathbf{P}^2 = \Theta + \Theta_\eta$,
- (2) the linear system $|\Theta + \Theta_\eta|$ defines the map $u : \text{Pic}^3 C \rightarrow \mathbf{P}^3$,
- (3) the surface S is the Kummer quartic model of $\text{Pic}^3 C / \langle s \rangle$.

Proof. (1) Let $Q \in \mathbf{P}^2$ be of rank 4. Then $\text{Sing } Q$ is a point $o \in R$. Consider the projection $p_o : R \rightarrow \mathbf{P}^3$ from o . Then $p_o(R)$ is a smooth quadric and one of its rulings is the image of the ruling of lines of R . If $o \notin C$, then $p_o(C)$ is a curve of type (2,4) in $p_o(R)$ and the two rulings of planes of Q cut $|\omega_C|$ and a basepoint-free pencil of degree 4 on C . Hence, Q is not in S . If $o \in C$, then $Q = u(L)$, where L is $\omega_C(o)$, and then $L \in \Theta$ and $s(L) \in \Theta_\eta$. Secondly, let Q be of rank 3. Then $\text{Sing } Q \cap C = \{o_1, o_2\}$ and one has $Q = u(L)$, where $\{L, s(L)\} = \{\omega_C(o_1), \omega_C(o_2)\} \subset \Theta$. The discussion implies $u^{-1}(\mathbf{P}^2) = \Theta \cup \Theta_\eta$. Now $u : \text{Pic}^3 C \rightarrow S$ is a morphism of degree 2, and $\text{deg } S \leq 4$ since D has degree 5. This easily implies $\text{deg } S = 4$ and $u^*\mathbf{P}^2 = \Theta + \Theta_\eta$. This shows (1), and (1) implies (2) and (3). \square

It is well known that $\text{Sing } S$ consists of 16 nodes. In our construction these are the images of the fixed points of s . In other words,

$$\text{Sing } S = \{Q_L \mid L^{\otimes 2} \cong \mathcal{O}_C(1)\}.$$

Since $L = s(L)$ the quadric Q_L has a unique ruling of planes and rank 3. Hence, $Q_\eta \in \text{Sing } S$ since $s(\eta) = \eta$, and the rank 3 quadric Q_η defines a node of S .

For a general triple, $(C, \eta, b) \in \mathcal{R}_{2,6}$ is not restricted to assuming $\eta \notin \Theta$. Then $|\eta|$ is basepoint free and $\text{Sing } Q_\eta \cap C = \emptyset$, where $\text{Sing } Q_\eta$ is a line. By Proposition A.2 this implies that

$$C = Q_\eta \cdot R.$$

Since R is unique in \mathbf{P}^4 up to projective automorphisms, we fix it. Then a general pair (C, η) is defined via the above construction by a rank 3 quadric not containing R .

Proof of Theorem A.1. Let $G(1, 4)$ be the Grassmannian of lines of \mathbf{P}^4 ; we use the same notation for a line $\ell \subset \mathbf{P}^4$ and its parameter in $G(1, 4)$. Let \mathcal{I}_ℓ be the ideal sheaf of ℓ . Then $|\mathcal{I}_\ell^2(2)|$ is five dimensional. We set

$$\mathbf{P}_\ell^5 := |\mathcal{I}_\ell^2(2)|.$$

This linear system parametrizes all the quadrics singular along ℓ , and its general member has rank 3. Let $\ell \in G(1, 4)$ and $Q \in \mathbf{P}_\ell^5$ be general. Then Q defines a smooth curve $C := Q \cdot R$. By the previous remarks C is a general curve of genus 2. Moreover, the ruling of planes of Q cuts on C a pencil $|\eta|$, where $\eta \in \text{Pic}^3 C$ and $\eta^{\otimes 2} \cong \mathcal{O}_C(1)$. Let $\mathbf{P}^{4*} = |\mathcal{O}_{\mathbf{P}^4}(1)|$. Then a general $(Q, H) \in \mathbf{P}_\ell^5 \times \mathbf{P}^{4*}$ defines a triple (C, η, b) where $b := H \cdot C$, with (C, η) constructed as above. This defines a rational map

$$f_\ell : \mathbf{P}_\ell^5 \times \mathbf{P}^{4*} \rightarrow \mathcal{R}_{2,6}$$

sending (Q, H) to the moduli point of (C, η, b) . Notice that f_ℓ is a map of varieties of the same dimension, depending of course on ℓ .

Let $G_\ell \subset \text{Aut } \mathbf{P}^4$ be the stabilizer of $R \cup \ell$. It is clear that G_ℓ acts linearly on the two factors of $\mathbf{P}_\ell^5 \times \mathbf{P}^{4*}$. Then we consider its action on this product, sending (Q, H) to $a(Q, H) := (a(Q), a(H))$. First we show that f_ℓ is birationally the quotient map of this action.

Assume $f_\ell(p_1) = f_\ell(p_2)$ for two general points $p_i = (Q_i, H_i)$, $i = 1, 2$. Let $C_i := Q_i \cdot R$, and let $\eta_i \in \text{Pic } C_i^3$ be the line bundle defined by the ruling of Q_i . Then C_1, C_2 are biregular to the same general curve C and η_1, η_2 are isomorphic. This implies that there exists $a \in \text{Aut } \mathbf{P}^4$ such that $a(C_1) = C_2$ and $a(Q_1) = Q_2$. We can assume that $\mathcal{O}_{C_i}(1)$ is general in $\text{Pic}^6 C_i$ so that the stabilizer of C_i in $\text{Aut } \mathbf{P}^4$ is trivial. Let $b_i = C_i \cdot H_i$. Then it follows that $a(b_1) = b_2$ and $a(H_1) = H_2$. Moreover, notice that the ruling of lines of R is invariant by a since $a(C_1) = C_2$. Hence, $a(R) = R$. Since $a(Q_1) = a(Q_2)$, then $a(\ell) = \ell$. This implies that $a \in G_\ell$ and that $(Q_2, H_2) = a(Q_1, H_1)$. Since $\mathbf{P}^5 \times \mathbf{P}^{4*}$ and $\mathcal{R}_{2,6}$ are irreducible of the same dimension, it follows that $\mathbf{P}^5 \times \mathbf{P}^{4*}/G_\ell$ is birational to $\mathcal{R}_{2,6}$.

Finally, we claim that G_ℓ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for a general ℓ . Therefore, $\mathbf{P}^5 \times \mathbf{P}^{4*}$ descends to a \mathbf{P}^5 -bundle over the quotient space \mathbf{P}^4/G_ℓ and the latter is rational. □

Proof of the claim. The stabilizer of R in $\text{Aut } \mathbf{P}^4$ is isomorphic to $\text{Aut } R$. Since R is the blowup of \mathbf{P}^2 at a point o , $\text{Aut } R$ is isomorphic to the stabilizer of o in $\text{Aut } \mathbf{P}^2$. Notice that $\text{Aut } R$ is six dimensional and acts on the six-dimensional variety $G(1, 4)$. Recall that we have a map

$$\beta : R \rightarrow |\mathcal{I}_R(2)|$$

such that $\beta(v)$ is the unique $Q_v \in |\mathcal{I}_R(2)|$ singular at v . By Proposition A.2 β is the contraction of the exceptional line $E \subset R$ and $o := \sigma(E)$ is the parameter point of the unique rank 3 quadric Q_E containing R . Let

$$r_\ell : |\mathcal{I}_R(2)| \rightarrow |\mathcal{O}_\ell(2)|$$

be the restriction map. We define the nonempty open sets,

- (1) $U_1 := \{\ell \in G(1, 4) \mid r_\ell \text{ is an isomorphism}\}$,
- (2) $U_2 := \{\ell \in G(1, 4) \mid \ell \text{ is transversal to } Q_E\}$,

and prove our claim for $\ell \in U_1 \cap U_2$. Let $\Delta_\ell := r_\ell^* \Delta$ be the pullback of the diagonal conic $\Delta := \{2x \mid x \in \ell\}$. Then Δ_ℓ is a conic in $|\mathcal{I}_R(2)|$; more precisely, it is the family of quadrics through R which are tangent to ℓ . Let $a \in G_\ell$. Then $a(Q_E \cap \ell) = Q_E \cap \ell$. Since Q_E is transversal to ℓ , it follows that

$$Q_E \cap \ell = \{x_1, x_2\},$$

with $x_1 \neq x_2$. Therefore, the parameter point o of Q_E is not in Δ_ℓ . It is also clear that the induced automorphism $a_* : |\mathcal{I}_R(2)| \rightarrow |\mathcal{I}_R(2)|$ leaves Δ_ℓ invariant. Let $Q_v \in |\mathcal{I}_R(2)|$ be singular at v . Then $Q_{a(v)} = a(Q_v)$, and this equality implies that $a|_R : R \rightarrow R$ is uniquely defined by a_* . Since $a|_R$ is the identity if and only if a is, it follows that a is uniquely reconstructed from a_* .

Now $a_*(o) = o$ implies $a_*(L) = L$ and $a_*(T_1 \cup T_2) = T_1 \cup T_2$, where L is the polar line of o with respect to Δ_ℓ , and T_1, T_2 are the tangent lines to Δ_ℓ at the two points of $L \cap \Delta_\ell$. Let P be the pencil of conics which are tangent to Δ_ℓ at $L \cap \Delta_\ell$. Then a_* acts on P , fixing each of the elements $\Delta_\ell, T_1 + T_2, 2L$. Hence, a_* acts as the identity on P . To conclude, just recall that the group of planar automorphisms acting on a pencil like P as the identity is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Indeed, it is generated by the involution leaving each conic of P invariant, with o and the points of L fixed. Hence, G_ℓ is generated by a , where a_* is such an involution. □

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