TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 371, Number 7, 1 April 2019, Pages 4793–4814
https://doi.org/10.1090/tran/7572
Article electronically published on September 24, 2018

BOUNDEDNESS OF MULTIDIMENSIONAL HAUSDORFF OPERATORS IN H^p SPACES, $0 < p < 1$

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Abstract. Sufficient conditions, of both smoothness and algebraic type, for the boundedness of multidimensional Hausdorff operators in Hardy spaces $H^p(\mathbb{R}^n)$, $n \geq 2$, $0 < p < 1$, are given. Positive and negative examples illustrate all aspects of the behavior of such operators and reveal the special nature of the case $p < 1$.

1. Introduction

For the theory of Hardy spaces H^p, $0 < p < 1$, the Hausdorff operators turn out to be a very effective testing area, in dimension 1 and especially in several dimensions. After the publication of [LM], Hausdorff operators have attracted much attention. A general idea can be had of the subject from the surveys [Lsr] and [CFW]. There are plenty of references therein, and some will be given in the sequel. In contrast to the study of the Hausdorff operators in L^p, $1 \leq p \leq \infty$, and in the Hardy space H^1, the study of these operators in the Hardy spaces H^p with $p < 1$ holds a specific place, and there are very few results on this topic. For the case of one dimension, after [K], [M2], more or less final results were given in [LiMi]. The results differ from those for L^p, $1 \leq p \leq \infty$, and H^1, since they involve smoothness conditions on the averaging function, which seem unusual but unavoidable.

The purpose of this paper is to consider multidimensional Hausdorff operators in the Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p < 1$. We shall establish, in particular, two basic results. First, we show that, in the case $n \geq 2$, mere smoothness condition on the averaging function is not sufficient to imply the boundedness of Hausdorff operators in the Hardy spaces. Second, we introduce an algebraic condition on the functions involved in Hausdorff operators and prove the boundedness of these operators in the Hardy spaces. We also give several examples of multidimensional Hausdorff operators. Special atomic decomposition, special algebraic and smoothness conditions, etc.—all of these devices will work to the full extent.

To explain the issue in more detail, we begin with recalling some results in dimension 1. Given a function ϕ on the half line $(0, \infty)$, the Hausdorff operator $\mathcal{H}_\phi$ is defined by

$$(\mathcal{H}_\phi f)(x) = \int_0^\infty \phi(t) t^{-1} f(t^{-1}x)\, dt, \quad x \in \mathbb{R}.$$

Received by the editors May 5, 2017, and, in revised form, December 3, 2017.
2010 *Mathematics Subject Classification.* Primary 42B30; Secondary 42B10, 42B08, 42B15.
Key words and phrases. Hausdorff operator, Hardy space, atomic decomposition, Fourier transform, smoothness.

If $1 \le p \le \infty$, an application of Minkowski's inequality gives

$$\|\mathcal{H}_\phi f\|_{L^p(\mathbb{R})} \le \int_0^\infty |\phi(t)| \|t^{-1} f(t^{-1}\cdot)\|_{L^p(\mathbb{R})}\, dt = A_p(\phi)\|f\|_{L^p(\mathbb{R})},$$

where

$$A_p(\phi) = \int_0^\infty |\phi(t)| t^{-1+1/p}\, dt.$$

Thus, $\mathcal{H}_\phi$ is bounded in $L^p(\mathbb{R})$, $1 \le p \le \infty$, provided that $A_p(\phi) < \infty$. In [LM], this type of argument was extended to the case of the Hardy space $H^1(\mathbb{R})$.

The study of $\mathcal{H}_\phi$ in the Hardy space $H^p(\mathbb{R})$, $0 < p < 1$, was initiated by Kanjin [K] and continued in [LiMi]. Notice that the above simple argument for using Minkowski's inequality cannot be applied to $H^p(\mathbb{R})$ with $p < 1$. To explain the results of [K], [LiMi], we shall simply say that $\mathcal{H}_\phi$ is bounded in $H^p(\mathbb{R})$ if $\mathcal{H}_\phi$ is well-defined in a dense subspace of $H^p(\mathbb{R})$ and if it is extended to a bounded operator in $H^p(\mathbb{R})$. The following theorems, Theorems A and B, can be found in [K] and [LiMi], respectively.

Theorem A ([K]). *Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. Suppose that $A_1(\varphi) < \infty$, $A_2(\varphi) < \infty$, and suppose that $\widehat{\varphi}$ (the Fourier transform of the function φ extended to the whole real line by setting $\varphi(t) = 0$ for $t \le 0$) is a function of class C^{2M} on $\mathbb{R}$ with $\sup_{\xi\in\mathbb{R}} |\xi|^M |\widehat{\varphi}^{(M)}(\xi)| < \infty$ and $\sup_{\xi\in\mathbb{R}} |\xi|^M |\widehat{\varphi}^{(2M)}(\xi)| < \infty$. Then $\mathcal{H}_\varphi$ is bounded in $H^p(\mathbb{R})$.*

Theorem B ([LiMi]). *Let $0 < p < 1$, $M = [1/p - 1/2] + 1$, and let ϵ be a positive real number. Suppose that φ is a function of class C^M on $(0, \infty)$ such that*

$$|\varphi^{(k)}(t)| \le \min\{t^\epsilon, t^{-\epsilon}\} t^{-1/p-k} \quad \text{for } k = 0, 1, \dots, M.$$

Then $\mathcal{H}_\varphi$ is bounded in $H^p(\mathbb{R})$.

An immediate corollary of Theorems A and B is the following.

Theorem C. *Let $0 < p < 1$, and let $M = [1/p - 1/2] + 1$. If ϕ is a function on $(0, \infty)$ of class C^M and $\operatorname{supp}\phi$ is a compact subset of $(0, \infty)$, then $\mathcal{H}_\phi$ is bounded in $H^p(\mathbb{R})$.*

It is noteworthy that the above theorems impose a certain smoothness assumption on ϕ. In fact, this smoothness assumption cannot be removed since we have the next theorem.

Theorem D ([LiMi]). *There exists a function ϕ on $(0, \infty)$ such that ϕ is bounded, $\operatorname{supp}\phi$ is a compact subset of $(0, \infty)$, and, for every $p \in (0, 1)$, the operator $\mathcal{H}_\phi$ is not bounded in $H^p(\mathbb{R})$.*

Now, considering multidimensional Hausdorff operators is in order. As a multidimensional analogue of $\mathcal{H}_\phi$, we shall consider the operator $\mathcal{H}_{\Phi,A}$ defined as follows. Let $N, n \in \mathbb{N}$, and let $\Phi : \mathbb{R}^N \to \mathbb{C}$ and $A : \mathbb{R}^N \to M_n(\mathbb{R})$ be given, where $M_n(\mathbb{R})$ denotes the class of all $n \times n$ real matrices. Assuming the matrix $A(u)$ to be nonsingular for almost every u with $\Phi(u) \ne 0$, we define $\mathcal{H}_{\Phi,A}$ by

$$(\mathcal{H}_{\Phi,A} f)(x) = \int_{\mathbb{R}^N} \Phi(u) |\det A(u)|^{-1} f(x\, {}^t\!A(u)^{-1})\, du, \quad x \in \mathbb{R}^n,$$

where ${}^t\!A(u)^{-1}$ denotes the inverse of the transpose of the matrix $A(u)$, and $x\, {}^t\!A(u)^{-1}$ denotes the row n-vector obtained by multiplying the row n-vector x by the $n \times n$

matrix ${}^tA(u)^{-1}$. The Fourier transform of $\mathcal{H}_{\Phi,A}f$ is (formally) calculated from the definition as

$$(\mathcal{H}_{\Phi,A}f)^\wedge(\xi) = \int_{\mathbb{R}^N} \Phi(u)\widehat{f}(\xi A(u))\,du, \quad \xi \in \mathbb{R}^n. \tag{1.1}$$

To be precise, we have to put some conditions on Φ, A, and f so that $\mathcal{H}_{\Phi,A}f$ is well-defined and that the formula (1.1) holds; for this, see Section 2, where we give the accurate definition of $\mathcal{H}_{\Phi,A}$. With this definition in hand, formula (1.1) is confirmed to be valid.

On account of Theorems C and D, one may suppose that the multidimensional operator $\mathcal{H}_{\Phi,A}$ is bounded in $H^p(\mathbb{R}^n)$, $0 < p < 1$, if one merely assumes Φ and A to be sufficiently smooth, and Φ to be with compact support. However, certain attempts, like [CFL], [CFLR], show that the situation is more complicated than one may expect. We shall show that this naive generalization of Theorem C is false. We give examples of smooth Φ with compact support and smooth A for which $\mathcal{H}_{\Phi,A}$ is not bounded in the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < 1$, $n \geq 2$ (see Examples 3.1 and 3.2). This leads to the conclusion that A or Φ or both should be subject to additional assumptions. The nature and type of such assumptions is, in a sense, the main issue, or, say, spirit of this work. Indeed, for positive results, we introduce an algebraic condition on A and prove the Hardy space boundedness of $\mathcal{H}_{\Phi,A}$ (see condition (4.1) and Theorem 4.1). This is a generalization of Theorem C to the multidimensional case. We also give some examples of $\mathcal{H}_{\Phi,A}$ that are bounded in $H^p(\mathbb{R}^n)$, $0 < p < 1$ (see Examples 5.2 and 5.3).

The matter of the above described three sections can be treated as "local", first of all since the averaging function Φ is of compact support. The general case is much more difficult and, as a whole, will be considered in our next work. However, in this paper we already make two steps toward that general case. In Section 6, we present a still local but quantitative version of the above result (see Theorem 6.1). Using it and a certain partition of unity, we are able to prove, in Section 7, a global version of the boundedness of Hausdorff operators in $H^p(\mathbb{R}^n)$, $0 < p < 1$ (see Theorem 7.1). It is a generalization of Theorem B rather than a complete extension of our local results; however, it might be considered a bridge to the full generality.

In the last section, Section 8, we give some remarks concerning the algebraic condition introduced in Theorem 4.1.

It is also worth mentioning two issues that are, in a sense, on the sidelines of the main road of our study but are important for the topic in general. First, we bring the attention of the reader to the second, alternative proof of Example 3.2. Though the theory of Hausdorff operators is in a certain sense opposed to that of Fourier multipliers (averaging versus multipliers; see, e.g., [Lsr]), that proof shows that there are deep relations between the two approaches to summability. Second, although this problem is not considered in the present paper, we mention that boundedness of $\mathcal{H}_{\Phi,A}$ in $H^1(\mathbb{R}^n)$ is another interesting problem. This is motivated by the fact that the formula

$$\big\| |\det A|^{-1} f(\cdot\, {}^tA^{-1}) \big\|_{H^1(\mathbb{R}^n)} = \|f\|_{H^1(\mathbb{R}^n)}$$

does not hold for general $A \in GL(n, \mathbb{R})$. For this problem, see [LL], [Li].

Notation. We write

$$B(x,r) = \{y \in \mathbb{R}^k : |y-x| < r\}, \quad x \in \mathbb{R}^k, \quad r > 0,$$
$$\Sigma = \Sigma^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\},$$
$$B_\Sigma(x,r) = \{y \in \Sigma : |y-x| < r\}, \quad x \in \Sigma, \quad r > 0.$$

For $A = (a_{ij}) \in M_n(\mathbb{R})$, we define

$$\|A\| = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{1/2}.$$

The Fourier transform and the inverse Fourier transform are defined by

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi\rangle}\, dx, \quad \xi \in \mathbb{R}^n,$$
$$(f)^\vee(y) = \widehat{f}(-y),$$

where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$.

2. Definition of Hausdorff operators

In this section, we give a preliminary argument concerning the definition of $\mathcal{H}_{\Phi,A}$ and formula (1.1).

For functions $\Phi : \mathbb{R}^N \to \mathbb{C}$, $A : \mathbb{R}^N \to M_n(\mathbb{R})$, and $f : \mathbb{R}^n \to \mathbb{C}$, consider

$$(\mathcal{H}_{\Phi,A} f)(x) = \int_{\mathbb{R}^N} \Phi(u) |\det A(u)|^{-1} f(x\, {}^t\!A(u)^{-1})\, du, \quad x \in \mathbb{R}^n, \tag{2.1}$$

and

$$(\widetilde{\mathcal{H}}_{\Phi,A} f)(x) = \int_{\mathbb{R}^N} \Phi(u) f(xA(u))\, du, \quad x \in \mathbb{R}^n. \tag{2.2}$$

We always assume that Φ, A, and f are Borel measurable functions. Defining

$$L_A(\Phi) = \int_{\mathbb{R}^N} |\Phi(u)|\, |\det A(u)|^{-1/2}\, du,$$

we have the following.

Proposition 2.1. *If $L_A(\Phi) < \infty$, then for all $f \in L^2(\mathbb{R}^n)$ the functions $\mathcal{H}_{\Phi,A} f$ and $\widetilde{\mathcal{H}}_{\Phi,A} f$ are well-defined almost everywhere on $\mathbb{R}^n$, and the inequalities*

$$\|\mathcal{H}_{\Phi,A} f\|_{L^2(\mathbb{R}^n)} \le L_A(\Phi) \|f\|_{L^2(\mathbb{R}^n)} \tag{2.3}$$

and

$$\|\widetilde{\mathcal{H}}_{\Phi,A} f\|_{L^2(\mathbb{R}^n)} \le L_A(\Phi) \|f\|_{L^2(\mathbb{R}^n)}$$

hold. Thus, $\mathcal{H}_{\Phi,A}$ and $\widetilde{\mathcal{H}}_{\Phi,A}$ are well-defined bounded operators in $L^2(\mathbb{R}^n)$ if $L_A(\Phi) < \infty$.

Proof. Suppose that $L_A(\Phi) < \infty$. Then the matrix $A(u)$ is nonsingular for almost every u satisfying $\Phi(u) \ne 0$, and the integrand in (2.1) is defined for almost every $(x,u) \in \mathbb{R}^n \times \mathbb{R}^N$. If $A(u)$ is nonsingular, we have

$$\big\| |\det A(u)|^{-1} f(x\, {}^t\!A(u)^{-1}) \big\|_{L^2_x} = |\det A(u)|^{-1/2} \|f\|_{L^2}.$$

This equality and Minkowski's inequality give

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^N} |\Phi(u)|\,|\det A(u)|^{-1}|f(x\,{}^tA(u)^{-1})|\,du \right\|_{L^2_x} \\
&\quad\le \int_{\mathbb{R}^N} \left\| |\Phi(u)|\,|\det A(u)|^{-1}|f(x\,{}^tA(u)^{-1})| \right\|_{L^2_x} du \\
&\quad= \int_{\mathbb{R}^N} |\Phi(u)|\,|\det A(u)|^{-1/2}\|f\|_{L^2}\,du = L_A(\Phi)\|f\|_{L^2},
\end{aligned}$$

from which we see that $(\mathcal{H}_{\Phi,A}f)(x)$ is well-defined almost everywhere and that inequality (2.3) holds. Similarly, the assertions for $\widetilde{\mathcal{H}}_{\Phi,A}f$ follow from Minkowski's inequality and the equality

$$\|f(xA(u))\|_{L^2_x} = |\det A(u)|^{-1/2}\|f\|_{L^2}.$$

The proof is complete. □

The next proposition gives formula (1.1).

Proposition 2.2. *If $L_A(\Phi) < \infty$, then $(\mathcal{H}_{\Phi,A}f)^\wedge = \widetilde{\mathcal{H}}_{\Phi,A}\widehat{f}$ for all $f \in L^2(\mathbb{R}^n)$.*

Proof. If both Φ and f are in L^1 and if the matrix $A(u)$ is nonsingular for almost every u satisfying $\Phi(u) \neq 0$, then a simple application of Fubini's theorem gives the equality $(\mathcal{H}_{\Phi,A}f)^\wedge = \widetilde{\mathcal{H}}_{\Phi,A}\widehat{f}$. Using the inequalities of Proposition 2.1 and using a limiting argument, we see that the same equality holds for all Φ with $L_A(\Phi) < \infty$, and for all $f \in L^2(\mathbb{R}^n)$. □

In the rest of this paper, we shall always consider Φ and A that satisfy $L_A(\Phi) < \infty$. We shall say that $\mathcal{H}_{\Phi,A}$ is bounded in $H^p(\mathbb{R}^n)$ if there exists a constant C such that the inequality

$$\|\mathcal{H}_{\Phi,A}f\|_{H^p(\mathbb{R}^n)} \le C\|f\|_{H^p(\mathbb{R}^n)}$$

holds for all $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$. The same notion for the operator $\widetilde{\mathcal{H}}_{\Phi,A}$ will be used, with the same meaning.

3. Examples of unbounded Hausdorff operators

One of the main challenges for the positive results for $\mathcal{H}_{\Phi,A}$ is the search for conditions on the averaging ingredients. In this section, we shall give two examples of operators of the form $\mathcal{H}_{\Phi,A}$, with $\Phi \in C_0^\infty(\mathbb{R}^N)$ and $A \in C^\infty$ that are not bounded in $H^p(\mathbb{R}^n)$, $0 < p < 1$, $n \ge 2$. These examples show that in addition to smoothness conditions, one should approach carefully the choice of the set over which averaging is fulfilled.

Example 3.1. Let Φ be a nonnegative smooth function on $(0,\infty)$ with compact support. Assume that $\Phi(s) > 1$ for $1 < s < 2$. Then, for $n \ge 2$ and $0 < p < 1$, the operator

$$(Hf)(x) = \int_0^\infty \Phi(s)f(sx)\,ds, \quad x \in \mathbb{R}^n,$$

is not bounded in $H^p(\mathbb{R}^n)$.

Proof. We write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Take an $\omega_0 \in \mathbb{R}^{n-1}$ with $|\omega_0| = 1$. For $0 < \epsilon < 1/2$, consider the function f on $\mathbb{R}^n$ such that

$$\begin{aligned}
&\{x \in \mathbb{R}^n : f(x) \neq 0\} \subset \{(x', x_n) \in \mathbb{R}^n : |x' - \omega_0| < \epsilon,\ \ -\epsilon < x_n < \epsilon\},\\
&f(x) = \epsilon^{-n/p} \quad \text{if } \ |x' - \omega_0| < \epsilon \ \text{ and } \ 0 < x_n < \epsilon,\\
&\|f\|_{L^\infty} \lesssim \epsilon^{-n/p},\\
&\int f(x) x^\alpha \, dx = 0 \quad \text{for } \ |\alpha| \leq [n/p - n].
\end{aligned}$$

Then f is a constant multiple of an $H^p(\mathbb{R}^n)$ atom, and hence $\|f\|_{H^p(\mathbb{R}^n)} \lesssim 1$. We shall show that $\|Hf\|_{L^p(\mathbb{R}^n)} \to \infty$ as $\epsilon \to 0$, which will a fortiori imply that H is not bounded in $H^p(\mathbb{R}^n)$. Consider the tube type domain B defined by

$$B = \left\{ (x', x_n) \in \mathbb{R}^n \ : \ \frac{4}{7} < |x'| < \frac{4}{5}, \ \ \left| \frac{x'}{|x'|} - \omega_0 \right| < \frac{\epsilon}{2}, \ \ 0 < x_n < \frac{\epsilon}{2} \right\}.$$

We have $|B| \approx \epsilon^{n-1}$. Observe that

$$\begin{aligned}
&x \in B, \ \ \left| s - \frac{1}{|x'|} \right| < \frac{\epsilon}{2}\\
&\Rightarrow 1 < s < 2, \ \ |sx' - \omega_0| < \epsilon, \ \ 0 < sx_n < \epsilon\\
&\Rightarrow \Phi(s) f(sx) > \epsilon^{-n/p}.
\end{aligned}$$

Notice also that if $x \in B$, then $\Phi(s) f(sx) \geq 0$ for all $0 < s < \infty$. Hence, for $x \in B$, we have

$$(Hf)(x) = \int_0^\infty \Phi(s) f(sx) \, ds \geq \int_{|s - 1/|x'|| < \epsilon/2} \epsilon^{-n/p} \, ds = \epsilon^{-n/p+1}.$$

Thus,

$$\|Hf\|_{L^p(\mathbb{R}^n)} \geq \epsilon^{-n/p+1} |B|^{1/p} \approx \epsilon^{1-1/p}$$

and $\|Hf\|_{L^p(\mathbb{R}^n)} \to \infty$ as $\epsilon \to 0$. □

It is worth comparing this example with the results in [An]. In both cases, the averaging is one dimensional. However, whereas in the latter such an averaging simplifies the situation and allows one to obtain positive results for H^1, BMO and some other spaces, the considered example shows that a certain high dimension should also contribute to the boundedness in Hardy spaces $H^p(\mathbb{R}^n)$, $n \geq 2$, $0 < p < 1$. This will be confirmed in the next two sections.

Let $SO(n, \mathbb{R})$ be the Lie group of real $n \times n$ orthogonal matrices with determinant 1, and let μ be the Haar measure on $SO(n, \mathbb{R})$.

Example 3.2. For $n \geq 2$ and $0 < p < 1$, the operator

$$(Hf)(x) = \int_{SO(n,\mathbb{R})} f(xP) \, d\mu(P), \quad x \in \mathbb{R}^n,$$

is not bounded in $H^p(\mathbb{R}^n)$.

Proof. Take an $x_0 \in \Sigma^{n-1}$. For $0 < \epsilon < 1$, consider the function f on $\mathbb{R}^n$ such that

$$\begin{aligned}
&\{x \in \mathbb{R}^n : f(x) \neq 0\} \subset B(x_0, \epsilon),\\
&f(x) = \epsilon^{-n/p} \quad \text{for } \ x \in B(x_0,\epsilon) \cap B(0,1),\\
&\|f\|_{L^\infty} \lesssim \epsilon^{-n/p},\\
&\int f(x) x^\alpha \, dx = 0 \quad \text{for } \ |\alpha| \leq [n/p - n].
\end{aligned}$$

Then f is a constant multiple of an $H^p(\mathbb{R}^n)$ atom, and hence $\|f\|_{H^p(\mathbb{R}^n)} \lesssim 1$. We will show that $\|Hf\|_{L^p(\mathbb{R}^n)} \to \infty$ as $\epsilon \to 0$, which a fortiori implies that H is not bounded in $H^p(\mathbb{R}^n)$.

The integral defining $(Hf)(x)$ can be written as

$$(Hf)(x) = \int_{SO(n,\mathbb{R})} f(xP)\, d\mu(P) = C_n \int_{\Sigma^{n-1}} f(|x|\omega) d\sigma(\omega),$$

where σ denotes the $(n-1)$-dimensional surface measure on Σ^{n-1} and C_n is a positive constant depending only on n. If $1 - \epsilon/2 < |x| < 1$, then

$$\begin{aligned}
(Hf)(x) &= C_n \int_{\Sigma^{n-1}} f(|x|\omega) d\sigma(\omega)\\
&= C_n \int_{\Sigma^{n-1}} \epsilon^{-n/p}\, \mathbf{1}\{|x|\omega \in B(x_0,\epsilon)\}\, d\sigma(\omega) \approx \epsilon^{-n/p+n-1}
\end{aligned}$$

(where $\mathbf{1}\{|x|\omega \in B(x_0,\epsilon)\}$ denotes 1 or 0 according to $|x|\omega \in B(x_0,\epsilon)$ or $|x|\omega \notin B(x_0,\epsilon)$). Thus,

$$\|Hf\|_{L^p(\mathbb{R}^n)} \gtrsim \epsilon^{-n/p+n-1} \left(\int_{1-\epsilon/2<|x|<1} dx \right)^{1/p} \approx \epsilon^{(n-1)(1-1/p)}$$

and $\|Hf\|_{L^p(\mathbb{R}^n)} \to \infty$ as $\epsilon \to 0$. □

Remark 3.3. Here is an alternative proof of Example 3.2. The operator of Example 3.2 is related to a certain oscillating Fourier multiplier. Let g be a radial function on $\mathbb{R}^n$, and let $f = g(\cdot - y)$ with $y \in \Sigma^{n-1}$. Then $\|f\|_{H^p(\mathbb{R}^n)} = \|g\|_{H^p(\mathbb{R}^n)}$. Since $\widehat{g}$ is also a radial function, the Fourier transform of Hf is given by

$$\begin{aligned}
(Hf)^\wedge(\xi) &= \int_{SO(n,\mathbb{R})} \widehat{f}(\xi\, {}^tP^{-1})\, d\mu(P) = \int_{SO(n,\mathbb{R})} \widehat{f}(\xi P)\, d\mu(P)\\
&= \int_{SO(n,\mathbb{R})} e^{-i\langle y, \xi P\rangle}\, \widehat{g}(\xi P)\, d\mu(P) = \left(\int_{SO(n,\mathbb{R})} e^{-i\langle y,\xi P\rangle}\, d\mu(P) \right) \widehat{g}(\xi)\\
&= m(\xi)\widehat{g}(\xi),
\end{aligned}$$

with

$$m(\xi) = \int_{SO(n,\mathbb{R})} e^{-i\langle y,\xi P\rangle}\, d\mu(P) = C_n |\xi|^{-(n-2)/2} J_{(n-2)/2}(|\xi|),$$

where $J_{(n-2)/2}$ is the Bessel function, and C_n is a positive constant depending on n (see [SW, p. 154]). It is known that $m(\xi)$ is a Fourier multiplier for $H^p(\mathbb{R}^n)$ if and only if $1 \leq p < \infty$ (see [M3]). In fact, even if we restrict g to radial functions, $g \mapsto (m\widehat{g})^\vee$ is not bounded in $H^p(\mathbb{R}^n)$ if $0 < p < 1$. To see this, let ψ be a smooth function on $\mathbb{R}$ such that $\psi(t) = 0$ for $t < 1$ and $\psi(t) = 1$ for $t > 2$. Let $\epsilon > 0$ and

$$g = \left(\psi(|\xi|) |\xi|^{n/p-n-\epsilon} \right)^\vee .$$

Then $g \in H^p(\mathbb{R}^n)$. It is known that $Hf = (m\widehat{g})^\vee$ is a smooth function on $\mathbb{R}^n \setminus \Sigma^{n-1}$ and, if $n - 1 - n/p + \epsilon < 0$, it has the following asymptotic behavior:

$$\begin{aligned} Hf(x) =& C_n' \Gamma(-n+1+n/p-\epsilon) \cos\left(\frac{\pi}{2}\left(-n+1+\frac{n}{p}-\epsilon\right)\right) \\ & \times (1-|x|)^{n-1-n/p+\epsilon} + o((1-|x|)^{n-1-n/p+\epsilon}) \quad \text{as } |x| \uparrow 1, \end{aligned}$$

where C_n' is a positive constant depending on n (see, e.g., [M3, Section 3]). Thus, if $0 < p < 1$ and if we choose $\epsilon > 0$ so that $\cos\left(\frac{\pi}{2}\left(-n+1+\frac{n}{p}-\epsilon\right)\right) \neq 0$ and $n - 1 - n/p + \epsilon < -1/p$, then $Hf \notin L^p(\mathbb{R}^n \setminus \Sigma^{n-1})$, a fortiori, $Hf \notin H^p(\mathbb{R}^n)$.

4. A local theorem

In this section, we shall prove the following theorem.

Theorem 4.1. *Let $n \in \mathbb{N}$, let $n \geq 2$, let $0 < p < 1$, and let $M = [n/p - n/2] + 1$. Let $N \in \mathbb{N}$, $\Phi : \mathbb{R}^N \to \mathbb{C}$ be a function of class C^M with compact support, and let $A : \mathbb{R}^N \to M_n(\mathbb{R})$ be a mapping of class C^{M+1}. Assume that the matrix $A(u)$ is nonsingular for all $u \in \operatorname{supp} \Phi$. Also assume that Φ and A satisfy the following condition:*

$$(4.1) \qquad \begin{cases} \text{for all } (u, y, \xi) \in \operatorname{supp} \Phi \times \Sigma^{n-1} \times \Sigma^{n-1}, \\ \text{there exists a } j = j(u, y, \xi) \in \{1, \dots, N\} \text{ such that} \\ \left\langle y, \xi \dfrac{\partial A(u)}{\partial u_j} \right\rangle \neq 0. \end{cases}$$

Then the operator $\mathcal{H}_{\Phi,A}$ is bounded in $H^p(\mathbb{R}^n)$.

To prove Theorem 4.1, we use the same method as in [LiMi], which is based on the modified atomic decomposition for H^p given in [M1], [M2]. We recall the results for the modified atoms.

Definition 4.2 ([M1])**.** Let $n, M \in \mathbb{N}$, and let $0 < p \leq 1$. For $0 < s < \infty$, we define $\mathcal{A}_{p,M}(s, \mathbb{R}^n)$ as the set of all those $f \in L^2(\mathbb{R}^n)$ for which $\widehat{f}(\xi) = 0$ for $|\xi| \leq s^{-1}$, and $\|\partial^\alpha \widehat{f}\|_{L^2(\mathbb{R}^n)} \leq s^{|\alpha| - n/p + n/2}$ for $|\alpha| \leq M$. We define $\mathcal{A}_{p,M}(\mathbb{R}^n)$ as the union of $\mathcal{A}_{p,M}(s, \mathbb{R}^n)$ over all $0 < s < \infty$.

Lemma 4.3 ([M1], [M2])**.** *Let $n \in \mathbb{N}$, let $0 < p \leq 1$, and let $M \in \mathbb{N}$ satisfy $M > n/p - n/2$. Then there exists a constant c depending only on n, p, and M such that the following hold.*

(1) $\|f(\cdot - x_0)\|_{H^p(\mathbb{R}^n)} \leq c$ *for all* $f \in \mathcal{A}_{p,M}(\mathbb{R}^n)$ *and all* $x_0 \in \mathbb{R}^n$.
(2) *Every* $f \in H^p(\mathbb{R}^n)$ *can be decomposed as*

$$(4.2) \qquad f = \sum_{j=1}^{\infty} \lambda_j f_j(\cdot - x_j),$$

where $f_j \in \mathcal{A}_{p,M}(\mathbb{R}^n)$, $x_j \in \mathbb{R}^n$, $0 \leq \lambda_j < \infty$, and

$$\left(\sum_{j=1}^{\infty} \lambda_j^p \right)^{1/p} \leq c \|f\|_{H^p(\mathbb{R}^n)},$$

and where the series in (4.2) converges unconditionally in $H^p(\mathbb{R}^n)$. If $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then this decomposition can be made so that the series in (4.2) converges unconditionally in $L^2(\mathbb{R}^n)$ as well.

This lemma is given in [M1, Lemma 2], except for the assertion on the L^2 convergence. A complete proof of part (2) of the lemma can be found in [M2, Section 3].

Proof of Theorem 4.1. By condition (4.1) and by the compactness of $\operatorname{supp}\Phi\times\Sigma\times\Sigma$, there exists a constant $\eta>0$ such that

$$\max_{j\in\{1,\ldots,N\}}|\langle y,\xi\partial_j A(u)\rangle|>\eta$$

for all $(u,y,\xi)\in\operatorname{supp}\Phi\times\Sigma\times\Sigma$. Hence, by uniform continuity, there exist $\delta_1>0$ and $\delta_2>0$ such that

$$\begin{cases}\forall(u,y,\xi)\in\operatorname{supp}\Phi\times\Sigma\times\Sigma,\ \ \exists j=j(u,y,\xi)\in\{1,\ldots,N\},\\ \forall(\overline{u},\overline{y},\overline{\xi})\in B(u,\delta_1)\times B_\Sigma(y,\delta_2)\times B_\Sigma(\xi,\delta_2),\ |\langle\overline{y},\overline{\xi}\partial_j A(\overline{u})\rangle|>\eta/2.\end{cases}\tag{4.3}$$

By using an appropriate partition of unity, we may assume that $\operatorname{supp}\Phi$ is included in a ball of radius δ_1. By (4.3), this implies the following:

$$\begin{cases}\forall(y,\xi)\in\Sigma\times\Sigma,\ \ \exists j=j(y,\xi)\in\{1,\ldots,N\},\\ \forall(u,\overline{y},\overline{\xi})\in\operatorname{supp}\Phi\times B_\Sigma(y,\delta_2)\times B_\Sigma(\xi,\delta_2),\\ |\langle\overline{y},\overline{\xi}\partial_j A(u)\rangle|>\eta/2.\end{cases}\tag{4.4}$$

By virtue of Lemma 4.3(2), in order to prove the boundedness of $\mathcal{H}_{\Phi,A}$ in $H^p(\mathbb{R}^n)$, it is sufficient to show the estimate

$$\|\mathcal{H}_{\Phi,A}(f(\cdot-x_0))\|_{H^p(\mathbb{R}^n)}\lesssim 1$$

for all $f\in\mathcal{A}_{p,M}(s,\mathbb{R}^n)$, $0<s<\infty$, and $x_0\in\mathbb{R}^n$. If f, s, and x_0 satisfy the last conditions, and if we set $g=s^{n/p}f(s\cdot)$ and $y_0=s^{-1}x_0$, then $g\in\mathcal{A}_{p,M}(1,\mathbb{R}^n)$,

$$\mathcal{H}_{\Phi,A}(f(\cdot-x_0))(x)=s^{-n/p}\mathcal{H}_{\Phi,A}(g(\cdot-y_0))(s^{-1}x),$$

and

$$\|\mathcal{H}_{\Phi,A}(f(\cdot-x_0))\|_{H^p(\mathbb{R}^n)}=\|\mathcal{H}_{\Phi,A}(g(\cdot-y_0))\|_{H^p(\mathbb{R}^n)}.$$

Thus, it is sufficient to show the estimate

$$\|\mathcal{H}_{\Phi,A}(g(\cdot-y_0))\|_{H^p(\mathbb{R}^n)}\lesssim 1\tag{4.5}$$

for all $g\in\mathcal{A}_{p,M}(1,\mathbb{R}^n)$ and $y_0\in\mathbb{R}^n$. To prove this estimate, we shall show that

$$c^{-1}\mathcal{H}_{\Phi,A}(g(\cdot-y_0))\in\mathcal{A}_{p,M}(c_0,\mathbb{R}^n),\tag{4.6}$$

where c and c_0 are positive constants depending only on N, n, p, Φ, and A. Estimate (4.5) follows from (4.6) by the use of Lemma 4.3(1).

Now let $g\in\mathcal{A}_{p,M}(1,\mathbb{R}^n)$, $y_0\in\mathbb{R}^n$, and

$$F(\xi)=(\mathcal{H}_{\Phi,A}(g(\cdot-y_0)))^\wedge(\xi)=\int_{\mathbb{R}^N}\Phi(u)e^{-i\langle y_0,\xi A(u)\rangle}\widehat{g}(\xi A(u))\,du.$$

To prove (4.6), what we have to prove are the following:

$$F(\xi)=0\quad\text{for } |\xi|\le c_0^{-1}\tag{4.7}$$

and

$$\|\partial^\alpha F\|_{L^2(\mathbb{R}^n)}\lesssim 1\quad\text{for } |\alpha|\le M,\tag{4.8}$$

where the constants in $\lesssim$ of (4.8) should depend only on N, n, p, Φ, and A. The assertion (4.7) is obvious since $\widehat{g}(\xi)=0$ for $|\xi|\le 1$, and since there exists $c_0>0$ such that $|\xi A(u)|\le c_0|\xi|$ for all $u\in\operatorname{supp}\Phi$. In the rest of the proof, we shall prove (4.8).

By Leibniz' rule of differentiation, $\partial^\alpha F(\xi)$, $|\alpha| \le M$, can be written as a finite linear combination of the following functions:

$$G(\xi) = \int_{\mathbb{R}^N} \Phi(u)(y_0\, {}^t\!A(u))^{\alpha'} e^{-i\langle y_0, \xi A(u)\rangle} P_\beta(A(u))\, \widehat{g}^{(\beta)}(\xi A(u))\, du,$$

where α' and β are multi-indices with $|\alpha'|+|\beta| = |\alpha| \le M$, $P_\beta(A)$ is a homogeneous polynomial of degree $|\beta|$ of the components of the matrix A, and $\widehat{g}^{(\beta)} = \partial^\beta \widehat{g}$. We shall prove $\|G\|_{L^2(\mathbb{R}^n)} \lesssim 1$ for each such G.

If $|y_0| \le 1$ or $\alpha' = 0$, then the estimate $\|G\|_{L^2(\mathbb{R}^n)} \lesssim 1$ is easy. In fact, in this case,

$$|G(\xi)| \lesssim \int_{\operatorname{supp}\Phi} |\widehat{g}^{(\beta)}(\xi A(u))|\, du,$$

and hence Minkowski's inequality and the estimate $\|\widehat{g}^{(\beta)}\|_{L^2(\mathbb{R}^n)} \le 1$ yield

$$\begin{aligned}\|G\|_{L^2(\mathbb{R}^n)} &\lesssim \int_{\operatorname{supp}\Phi} \|\widehat{g}^{(\beta)}(\cdot A(u))\|_{L^2(\mathbb{R}^n)}\, du \\ &= \int_{\operatorname{supp}\Phi} |\det A(u)|^{-1/2} \|\widehat{g}^{(\beta)}\|_{L^2(\mathbb{R}^n)}\, du \lesssim 1.\end{aligned}$$

In the rest of the proof, we assume that $|y_0| > 1$ and $\alpha' \ne 0$. We take a finite number of balls $B_\Sigma(\xi^{(i)}, \delta_2)$ that cover Σ and set

$$\Gamma_i = \{\xi \in \mathbb{R}^n \setminus \{0\} : \xi/|\xi| \in B_\Sigma(\xi^{(i)}, \delta_2)\}.$$

Thus, $\mathbb{R}^n \setminus \{0\} = \bigcup_i \Gamma_i$. It is sufficient to prove $\|G\|_{L^2(\Gamma_i)} \lesssim 1$ for each i. Hereafter we fix an i and simply write $\xi_0 = \xi^{(i)}$ and $\Gamma = \Gamma_i$. We shall consider $G(\xi)$ for $\xi \in \Gamma$.

By virtue of (4.4), there exists $j = j(y_0/|y_0|, \xi_0) \in \{1, \dots, N\}$ such that

$$\forall (u, \xi) \in \operatorname{supp}\Phi \times \Gamma, \quad |\langle y_0/|y_0|, (\xi/|\xi|)\partial_j A(u)\rangle| > \frac{\eta}{2}. \tag{4.9}$$

With this j, we have

$$(-i\langle y_0, \xi \partial_j A(u)\rangle)^{-1} \partial_j e^{-i\langle y_0, \xi A(u)\rangle} = e^{-i\langle y_0, \xi A(u)\rangle}$$

for $u \in \operatorname{supp}\Phi$ and $\xi \in \Gamma$. Hence, integration by parts gives

$$\begin{aligned} G(\xi) = i^{-|\alpha'|} \int_{\operatorname{supp}\Phi} & e^{-i\langle y_0, \xi A(u)\rangle} \left[\partial_j \circ \langle y_0, \xi \partial_j A(u)\rangle^{-1}\right]^{|\alpha'|} \\ & \left\{\Phi(u)(y_0\, {}^t\!A(u))^{\alpha'} P_\beta(A(u))\, \widehat{g}^{(\beta)}(\xi A(u))\right\} du, \end{aligned} \tag{4.10}$$

where $\partial_j \circ \langle y_0, \xi \partial_j A(u)\rangle^{-1}$ denotes the composite of the differential operator ∂_j and the operator of multiplication by the function $\langle y_0, \xi \partial_j A(u)\rangle^{-1}$. The right-hand side of (4.10) can be written as a finite linear combination of the following:

$$\begin{aligned} \int_{\operatorname{supp}\Phi} & e^{-i\langle y_0, \xi A(u)\rangle} \left(\prod_{\nu=1}^{|\alpha'|} \partial_j^{m_\nu} \langle y_0, \xi \partial_j A(u)\rangle^{-1}\right) \left(\partial_j^{\ell_1}\{\Phi(u) P_\beta(A(u))\}\right) \\ & \times \left(\partial_j^{\ell_2} (y_0\, {}^t\!A(u))^{\alpha'}\right) \left(\partial_j^{\ell_3} \widehat{g}^{(\beta)}(\xi A(u))\right) du, \end{aligned} \tag{4.11}$$

where $m_1, \dots, m_{|\alpha'|}, \ell_1, \ell_2, \ell_3$ are nonnegative integers satisfying

$$m_1 + \dots + m_{|\alpha'|} + \ell_1 + \ell_2 + \ell_3 = |\alpha'|.$$

It suffices to estimate the $L^2(\Gamma)$-norm of (4.11).

By (4.9), we have

$$|\partial_j^{m_\nu}\langle y_0, \xi\partial_j A(u)\rangle^{-1}| \lesssim (|y_0||\xi|)^{-1} \tag{4.12}$$

for $u \in \operatorname{supp}\Phi$ and $\xi \in \Gamma$ (here we used the assumption that A is of class C^{M+1}). We also have

$$|\partial_j^{\ell_1}\{\Phi(u)P_\beta(A(u))\}| \lesssim 1 \tag{4.13}$$

and

$$|\partial_j^{\ell_2}(y_0\, {}^t\!A(u))^{\alpha'}| \lesssim |y_0|^{|\alpha'|} \tag{4.14}$$

for $u \in \operatorname{supp}\Phi$. The derivative $\partial_j^{\ell_3}\widehat{g}^{(\beta)}(\xi A(u))$ can be written as

$$\begin{aligned}\partial_j^{\ell_3}\widehat{g}^{(\beta)}(\xi A(u)) = \sum &(\text{const})\, \widehat{g}^{(\beta+\gamma)}(\xi A(u))\\ &\times \Bigg(\prod_{\nu=1}^{\gamma_1}\partial_j^{k_{1\nu}}(\xi A(u))_1\Bigg)\cdots\Bigg(\prod_{\nu=1}^{\gamma_n}\partial_j^{k_{n\nu}}(\xi A(u))_n\Bigg),\end{aligned}$$

where $(\xi A(u))_i$ denotes the ith component of $\xi A(u)$ and the sum is taken over the multi-indices $\gamma = (\gamma_1, \dots, \gamma_n)$ and positive integers $k_{i\nu}$ satisfying $|\gamma| \le \ell_3$ and $\sum_{i=1}^n \sum_{\nu=1}^{\gamma_i} k_{i\nu} = \ell_3$. Thus,

$$|\partial_j^{\ell_3}\widehat{g}^{(\beta)}(\xi A(u))| \lesssim \sum_{|\gamma|\le\ell_3} |\widehat{g}^{(\beta+\gamma)}(\xi A(u))|\, |\xi|^{|\gamma|} \tag{4.15}$$

for $u \in \operatorname{supp}\Phi$. Combining estimates (4.12)–(4.15), we see that, for $u \in \operatorname{supp}\Phi$ and $\xi \in \Gamma$, the absolute value of the integrand of (4.11) is bounded by

$$(|y_0||\xi|)^{-|\alpha'|}|y_0|^{|\alpha'|}\sum_{|\gamma|\le\ell_3}|\widehat{g}^{(\beta+\gamma)}(\xi A(u))|\,|\xi|^{|\gamma|} \lesssim \sum_{|\gamma|\le\ell_3}|\widehat{g}^{(\beta+\gamma)}(\xi A(u))|$$

(notice that $|\xi|^{-|\alpha'|+|\gamma|} \lesssim 1$ since $|\gamma| \le \ell_3 \le |\alpha'|$ and $|\xi| \ge c_0^{-1}$ on the support of $\widehat{g}^{(\beta+\gamma)}(\xi A(u))$). Thus, by Minkowski's inequality,

$$\begin{aligned}\|(4.11)\|_{L^2(\Gamma)} &\lesssim \sum_{|\gamma|\le\ell_3}\int_{\operatorname{supp}\Phi}\|\widehat{g}^{(\beta+\gamma)}(\cdot A(u))\|_{L^2}\,du\\ &= \sum_{|\gamma|\le\ell_3}\int_{\operatorname{supp}\Phi}|\det A(u)|^{-1/2}\|\widehat{g}^{(\beta+\gamma)}\|_{L^2}\,du \lesssim 1\end{aligned}$$

since $g \in \mathcal{A}_{p,M}(1,\mathbb{R}^n)$ and $|\beta| + |\gamma| \le |\beta| + |\alpha'| = |\alpha| \le M$. This completes the proof of Theorem 4.1. □

We shall observe that Theorem 4.1 immediately implies the boundedness of the operator $\widetilde{\mathcal{H}}_{\Phi,A}$. For this, notice that condition (4.1) does not change if we replace $\forall(u,y,\xi) \in \operatorname{supp}\Phi \times \Sigma^{n-1} \times \Sigma^{n-1}$ by $\forall(u,y,\xi) \in \operatorname{supp}\Phi \times (\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$. Notice also that

$$\frac{\partial\, {}^t\!A(u)^{-1}}{\partial u_i} = -\,{}^t\!A(u)^{-1}\frac{\partial\, {}^t\!A(u)}{\partial u_j}\,{}^t\!A(u)^{-1}$$

and

$$\left\langle y, \xi\frac{\partial\, {}^t\!A(u)^{-1}}{\partial u_j}\right\rangle = -\left\langle yA(u)^{-1}\frac{\partial A(u)}{\partial u_j}, \xi\, {}^t\!A(u)^{-1}\right\rangle.$$

Hence, condition (4.1) is equivalent to the following:

$$\begin{cases} \text{for all } (u,y,\xi) \in \operatorname{supp}\Phi \times (\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\}), \\ \text{there exists a } j = j(u,y,\xi) \in \{1,\dots,N\} \text{ such that} \\ \left\langle y, \xi \dfrac{\partial\, {}^t\!A(u)^{-1}}{\partial u_j} \right\rangle \neq 0. \end{cases}$$

Thus, rewriting ${}^tA(u)^{-1}$ and $\Phi(u)|\det A(u)|^{-1}$ in Theorem 4.1 as $A(u)$ and $\Phi(u)$, respectively, we obtain the following corollary.

Corollary 4.4. *Under the same assumptions as Theorem* 4.1*, the operator* $\widetilde{\mathcal{H}}_{\Phi,A}$ *is bounded in* $H^p(\mathbb{R}^n)$*.*

Remark 4.5. Condition (4.1) is invariant under C^1 change of variables. In fact, if $(u_1,\dots,u_N) \leftrightarrow (v_1,\dots,v_N)$ is a C^1 diffeomorphism, then since

$$\left\langle y, \xi \frac{\partial A}{\partial u_j} \right\rangle = \sum_{k=1}^{N} \frac{\partial v_k}{\partial u_j} \left\langle y, \xi \frac{\partial A}{\partial v_k} \right\rangle,$$

$\langle y, \xi \frac{\partial A}{\partial u_j} \rangle \neq 0$ implies that $\langle y, \xi \frac{\partial A}{\partial v_k} \rangle \neq 0$ for at least one k, and vice versa.

5. Examples of bounded Hausdorff operators

In this section, we shall give examples of operators that satisfy the conditions of Theorem 4.1 or Corollary 4.4.

Prior to this, we give a lemma that will be used in this section. We write $\mathfrak{so}(n,\mathbb{R})$ to denote the set of all real $n \times n$ skew symmetric matrices. For linear subspace $\mathcal{L} \subset \mathfrak{so}(n,\mathbb{R})$, we consider the following condition:

$$\text{(5.1)} \qquad \begin{cases} \text{for any two linearly independent vectors } y, z \in \mathbb{R}^n, \\ \text{there exists a } B \in \mathcal{L} \text{ such that } \langle y, zB \rangle \neq 0. \end{cases}$$

Lemma 5.1.

(1) *For* $n \geq 2$*,* $\mathcal{L} = \mathfrak{so}(n,\mathbb{R})$ *satisfies condition* (5.1)*.*
(2) *If* $n = 2$ *or* 3*, then* $\mathcal{L} = \mathfrak{so}(n,\mathbb{R})$ *is the only linear subspace of* $\mathfrak{so}(n,\mathbb{R})$ *that satisfies condition* (5.1)*.*
(3) *If* $n \geq 4$*, then there exists a proper linear subspace* $\mathcal{L}$ *of* $\mathfrak{so}(n,\mathbb{R})$ *that satisfies condition* (5.1)*.*

Proof. To prove this lemma, we introduce some notations. For two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathfrak{so}(n,\mathbb{R})$, we define the inner product (A,B) by

$$\text{(5.2)} \qquad (A,B) = \frac{1}{2} \sum_{i,\,j} a_{ij} b_{ij} = \sum_{i<j} a_{ij} b_{ij}.$$

For $y = (y_i), z = (z_i) \in \mathbb{R}^n$, we define

$$\Delta_{ij}(y,z) = y_i z_j - y_j z_i = \begin{vmatrix} y_i & y_j \\ z_i & z_j \end{vmatrix}, \quad 1 \leq i,\, j \leq n,$$

and

$$T(y,z) = (\Delta_{ij}(y,z)) \in \mathfrak{so}(n,\mathbb{R}).$$

If $y, z \in \mathbb{R}^n$ and $B \in \mathfrak{so}(n, \mathbb{R})$, then

$$\langle y, zB\rangle = \sum_{i,\,j} y_j z_i b_{ij} = \sum_{i<j} b_{ij}(y_j z_i - y_i z_j) = -(B, T(y,z)).$$

Let $\mathcal{L}^\perp$ denote the orthogonal complement of $\mathcal{L} \subset \mathfrak{so}(n, \mathbb{R})$ with respect to the inner product (5.2). Then condition (5.1) is equivalent to the following:

$$\text{for any two linearly independent vectors } y, z \in \mathbb{R}^n,\ T(y,z) \notin \mathcal{L}^\perp. \tag{5.3}$$

Now we prove assertions (1), (2), (3) of the lemma.

(1) If y and z are linearly independent vectors in $\mathbb{R}^n$, then there exist $1 \le i < j \le n$ such that $\Delta_{ij}(y,z) \neq 0$, and hence $T(y,z) \neq 0$. Thus, $\mathcal{L} = \mathfrak{so}(n, \mathbb{R})$, for which $\mathcal{L}^\perp = \{0\}$, obviously satisfies condition (5.3).

(2) If $n = 2$ or 3, then every nonzero matrix $B \in \mathfrak{so}(n, \mathbb{R}^n)$ can be written as $B = T(y,z)$ with linearly independent vectors $y, z \in \mathbb{R}^n$. In fact, this is obvious when $n = 2$. If $n = 3$, then $\Delta_{ij}(y,z)$ for $(i,j) = (2,3)$, $(3,1)$, $(1,2)$ are the three components of the vector product of y and z, from which, by elementary geometry, we see that every nonzero $B \in \mathfrak{so}(n, \mathbb{R}^n)$ can be written as $B = T(y,z)$ with linearly independent vectors $y, z \in \mathbb{R}^3$. Hence, when $n = 2$ or 3, (5.3) holds only if $\mathcal{L}^\perp = \{0\}$ or, equivalently, $\mathcal{L} = \mathfrak{so}(n, \mathbb{R})$.

(3) Let $n \geq 4$. Observe the following fact:

$$\Delta_{34}(y,z) \neq 0,\ \ \Delta_{14}(y,z) = \Delta_{24}(y,z) = 0 \ \Rightarrow\ \Delta_{12}(y,z) = 0. \tag{5.4}$$

In fact, if the conditions on the left-hand side of $\Rightarrow$ holds, then, for the vectors $v_i = \binom{y_i}{z_i}$, it follows that $v_4 \neq 0$ and that both v_1 and v_2 are parallel to v_4, and hence that $\Delta_{12}(y,z) = 0$. Take a $B^\circ = (b^\circ_{ij}) \in \mathfrak{so}(n, \mathbb{R})$ such that $b^\circ_{34} \neq 0$, $b^\circ_{14} = b^\circ_{24} = 0$, and $b^\circ_{12} \neq 0$, and let $\mathcal{L}$ be the orthogonal complement of $\{tB^\circ : t \in \mathbb{R}\}$ in $\mathfrak{so}(n, \mathbb{R})$. If $y, z \in \mathbb{R}^n$ are linearly independent, from assertion (5.4) we see that $T(y,z) \notin \{tB^\circ : t \in \mathbb{R}\} = \mathcal{L}^\perp$. Thus, $\mathcal{L}$ satisfies condition (5.3), and obviously $\mathcal{L} \neq \mathfrak{so}(n, \mathbb{R})$. ☐

Now we shall give the examples of Hausdorff operators bounded in $H^p(\mathbb{R}^n)$, $0 < p < 1$, $n \geq 2$. Example 5.2 below should be compared with Examples 3.1 and 3.2; the difference is only more dimensions but the result is quite opposite.

Example 5.2. Let $n \in \mathbb{N}$, $n \geq 2$, $0 < p < 1$, and $M = [n/p - n/2] + 1$. Let $\Phi : (0,\infty) \times SO(n, \mathbb{R}) \to \mathbb{C}$ be a function of class C^M with compact support. Then the operator

$$(Hf)(x) = \int_{(0,\infty)\times SO(n,\mathbb{R})} \Phi(s,P) f(sxP)\, ds d\mu(P), \quad x \in \mathbb{R}^n,$$

is bounded in $H^p(\mathbb{R}^n)$.

Proof. We write $L = \dim SO(n, \mathbb{R}) = n(n-1)/2$. Recall that a *coordinate neighborhood* of $SO(n, \mathbb{R})$ is a pair (V, φ) of an open subset V of $SO(n, \mathbb{R})$ and a C^∞ diffeomorphism $\varphi : V \to \varphi(V)$, with $\varphi(V)$ being an open subset of $\mathbb{R}^L$. If (V, φ) is a coordinate neighborhood of $SO(n, \mathbb{R})$ and if α is a smooth function on $SO(n, \mathbb{R})$ supported in V, then by writing

$$P = \varphi^{-1}(u_1, \dots, u_L), \quad P \in V, \quad (u_1, \dots, u_L) \in \varphi(V), \tag{5.5}$$

we can write the operator

$$(H'f)(x) = \int_{(0,\infty)\times SO(n,\mathbb{R})} \Phi(s,P)\alpha(P)f(sxP)\,dsd\mu(P), \quad x \in \mathbb{R}^n,$$

in the form of the operator of Corollary 4.4 with $N = L+1$. Since the operator H of Example 5.2 is a finite sum of such an H', it is sufficient to prove the boundedness of H'. To prove this, we shall show that H' satisfies the assumptions of Corollary 4.4. Only condition (4.1) is to be checked. It is sufficient to show the following:

$$\text{(5.6)} \qquad \begin{cases} \forall (s^\circ, P^\circ, y^\circ, \xi^\circ) \in (0,\infty) \times V \times \Sigma^{n-1} \times \Sigma^{n-1}, \\ \exists j \in \{0, 1, \ldots, L\}, \ \left. \dfrac{\partial}{\partial u_j} \langle y^\circ, s\xi^\circ P \rangle \right|_{s=s^\circ,\, P=P^\circ} \neq 0, \end{cases}$$

where $\partial/\partial u_0 = \partial/\partial s$ and P is given by (5.5).

If $\langle y^\circ, \xi^\circ P^\circ \rangle \neq 0$, then we take $j = 0$, and we have

$$\left. \frac{\partial}{\partial s} \langle y^\circ, s\xi^\circ P \rangle \right|_{s=s^\circ,\, P=P^\circ} = \langle y^\circ, \xi^\circ P^\circ \rangle \neq 0.$$

If $\langle y^\circ, \xi^\circ P^\circ \rangle = 0$, then, by Lemma 5.1(1), there exists a $B \in \mathfrak{so}(n,\mathbb{R})$ such that $\langle y^\circ, \xi^\circ P^\circ B \rangle \neq 0$. Since $\mathfrak{so}(n,\mathbb{R})$ is the Lie algebra of $SO(n,\mathbb{R})$, there exists a coordinate neighborhood (W, ψ) such that $P^\circ \in W$ and $(\partial P/\partial v_1)|_{P=P^\circ} = P^\circ B$ for $P = \psi^{-1}(v_1, \ldots, v_L)$. Thus,

$$\left. \frac{\partial}{\partial v_1} \langle y^\circ, s\xi^\circ P \rangle \right|_{s=s^\circ,\, P=P^\circ} = \langle y^\circ, s^\circ \xi^\circ P^\circ B \rangle \neq 0$$

for $P = \psi^{-1}(v_1, \ldots, v_L)$. Since the change of variables $(u_1, \ldots, u_N) \leftrightarrow (v_1, \ldots, v_N)$ defined through $\varphi^{-1}(u_1, \ldots, u_N) = P = \psi^{-1}(v_1, \ldots, v_N)$ is a smooth diffeomorphism, we have $(\partial/\partial u_j)\langle y^\circ, s\xi^\circ P \rangle|_{s=s^\circ,\, P=P^\circ} \neq 0$ for some $j \in \{1, \ldots, L\}$ (see Remark 4.5). This completes the proof. □

The next example together with Lemma 5.1(3) will give Hausdorff operators whose number of averaging parameters are smaller than that of Example 5.2.

Example 5.3. Let $n \in \mathbb{N}$, let $n \geq 4$, let $0 < p < 1$, and let $M = [n/p - n/2] + 1$. Let $\mathcal{L}$ be a linear subspace of $\mathfrak{so}(n,\mathbb{R})$ that satisfies condition (5.1), and let $B_1, \ldots, B_m$ be a basis of $\mathcal{L}$. Take a $P^\circ \in SO(n,\mathbb{R})$, and define

$$P(u_1, \ldots, u_m) = P^\circ \exp \sum_{j=1}^{m} u_j B_j, \quad (u_1, \ldots, u_m) \in \mathbb{R}^m.$$

Assume that $\Phi : \mathbb{R}^{1+m} \to \mathbb{C}$ is a function of class C^M, and assume that

$$\text{(5.7)} \qquad \operatorname{supp} \Phi \subset \left\{ (u_0, u_1, \ldots, u_m) \in \mathbb{R}^{1+m} \ : \ u_0 \in [a,b], \ \sum_{j=1}^{m} |u_j| < \delta \right\},$$

with $0 < a < b < \infty$, and with $\delta > 0$ sufficiently small. Then the operator

$$(Hf)(x) = \int_{\mathbb{R}^{1+m}} \Phi(u_0, u_1, \ldots, u_m) f(u_0 x P(u_1, \ldots, u_m))\, du_0 du_1 \cdots du_m,$$

where $x \in \mathbb{R}^n$, is bounded in $H^p(\mathbb{R}^n)$.

Proof. We shall check that H satisfies the assumptions of Corollary 4.4. We write $u = (u_0, u_1, \ldots, u_m) \in \mathbb{R}^{1+m}$ and

$$A(u) = u_0 P(u_1, \ldots, u_m) = u_0 P^\circ \exp \sum_{j=1}^{m} u_j B_j.$$

It is sufficient to check condition (4.1). Let $y^\circ, \xi^\circ \in \Sigma^{n-1}$. If $\langle y^\circ, \xi^\circ P^\circ \rangle \neq 0$, then

$$\frac{\partial}{\partial u_0} \langle y^\circ, \xi^\circ A(u) \rangle \Big|_{u_1=\cdots=u_m=0} = \langle y^\circ, \xi^\circ P^\circ \rangle \neq 0.$$

If $\langle y^\circ, \xi^\circ P^\circ \rangle = 0$, then, by condition (5.1), there exists a $j \in \{1, \ldots, m\}$ such that $\langle y^\circ, \xi^\circ P^\circ B_j \rangle \neq 0$, and, with this j, we have

$$\frac{\partial}{\partial u_j} \langle y^\circ, \xi^\circ A(u) \rangle \Big|_{u_1=\cdots=u_m=0} = \langle y^\circ, u_0 \xi^\circ P^\circ B_j \rangle \neq 0$$

for $u_0 > 0$. Thus,

$$\max_{j \in \{0,1,\ldots,m\}} \left| \frac{\partial}{\partial u_j} \langle y^\circ, \xi^\circ A(u) \rangle \right|_{u_1=\cdots=u_m=0} > 0$$

for all $u_0 \in (0, \infty)$ and all $y^\circ, \xi^\circ \in \Sigma^{n-1}$. If $[a, b]$ is a compact subinterval of $(0, \infty)$, then by uniform continuity there exists a $\delta > 0$ such that

$$\max_{j \in \{0,1,\ldots,m\}} \left| \frac{\partial}{\partial u_j} \langle y^\circ, \xi^\circ A(u) \rangle \right| > 0$$

for all u with $u_0 \in [a, b]$ and $\sum_{j=1}^{m} |u_j| < \delta$, and for all $y^\circ, \xi^\circ \in \Sigma^{n-1}$. Hence, condition (4.1) holds provided that (5.7) holds for sufficiently small $\delta > 0$. This completes the proof. □

6. A quantitative version of a local theorem

In this section, we shall prove a quantitative version of Theorem 4.1. The theorem will be used to prove a global theorem for Hausdorff operators in the next section.

We shall write $\partial_j = (\partial / \partial u_j)$.

Theorem 6.1. *Let $n \in \mathbb{N}$, let $n \geq 2$, let $0 < p < 1$, and let $M = [n/p - n/2] + 1$. Let $N \in \mathbb{N}$, $\Phi : \mathbb{R}^N \to \mathbb{C}$ be a function of class C^M, let $A : \mathbb{R}^N \to M_n(\mathbb{R})$ be a mapping of class C^{M+1}, and let $0 < c_1, c_2, B < \infty$. Suppose there exist positive real numbers $T, S_1, \ldots, S_N$ for which the following hold:*

(i) *for all $u \in \operatorname{supp} \Phi$, $y \in \Sigma^{n-1}$, and $\xi \in \Sigma^{n-1}$, there exists a $j = j(u, y, \xi) \in \{1, \ldots, N\}$ such that $|\langle y, \xi S_j \partial_j A(u) \rangle| > c_1 T$;*

(ii) *$\| (S_j \partial_j)^\ell A(u) \| \leq c_2 T$ for all $j \in \{1, \ldots, N\}$, $\ell \in \{0, 1, \ldots, M+1\}$, and $u \in \operatorname{supp} \Phi$;*

(iii) *$\|(S_i \partial_i)(S_j \partial_j) A(u)\| \leq c_2 T$ for all $i, j \in \{1, \ldots, N\}$ and all $u \in \mathbb{R}^N$;*

(iv) *for all $j \in \{1, \ldots, N\}$ and $\ell \in \{0, 1, \ldots, M\}$,*

$$\int |(S_j \partial_j)^\ell \Phi(u)| \, |\det A(u)|^{-1/2} \, du \leq B T^{-n/p+n/2}.$$

Then there exists a constant c depending only on N, n, p, c_1, c_2 such that

$$\|\mathcal{H}_{\Phi,A} f\|_{H^p(\mathbb{R}^n)} \leq c B \|f\|_{H^p(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n). \tag{6.1}$$

Proof. The theorem will be proved in three steps.

Step 1. Estimate (6.1) holds if Φ has compact support, if conditions (ii) and (iv) hold with $T = S_1 = \cdots = S_N = 1$, and if the following stronger version of (i) holds:

(i′) For all $y, \xi \in \Sigma^{n-1}$, there exists a $j = j(y, \xi) \in \{1, \dots, N\}$ such that $|\langle y,\ \xi \partial_j A(u)\rangle| > c_1$ for all $u \in \operatorname{supp} \Phi$.

In fact, this claim can be proved by only carefully repeating the argument given in the proof of Theorem 4.1. We omit the details.

Step 2. Estimate (6.1) holds if Φ has compact support, and if conditions (i), (ii), (iii), and (iv) hold with $T = S_1 = \cdots = S_N = 1$.

To prove this, we incorporate the following notation:

$$\Omega = \mathbb{Z}^N, \quad Q_0 = (-1, 1)^N,$$
$$\eta\omega + \eta Q_0 = \{\eta\omega + \eta u : u \in Q_0\}, \quad \eta > 0,\ \ \omega \in \Omega.$$

We shall use the letter c to denote various positive constants which depend only on N, n, p, c_1, c_2.

First, observe that condition (iii) implies that the estimate

$$\|\partial_j A(u) - \partial_j A(\overline{u})\| \le c(N, n) c_2 |u - \overline{u}|$$

for all $u, \overline{u} \in \mathbb{R}^N$. By this estimate and by (ii), we can take a constant $\eta > 0$ depending only on N, n, c_1, and c_2 such that

(v) *if* $u, \overline{u} \in \mathbb{R}^N$, $y, \overline{y}, \xi, \overline{\xi} \in \Sigma$, $u - \overline{u} \in \eta Q_0$, $|y - \overline{y}| < \eta$, $|\xi - \overline{\xi}| < \eta$, *and if* u *or* $\overline{u}$ *is in* $\operatorname{supp}\Phi$, *then* $\left|\langle y, \xi\partial_j A(u)\rangle - \left\langle \overline{y}, \overline{\xi}\partial_j A(\overline{u})\right\rangle\right| < 3^{-1} c_1$ *for all* $j \in \{1, \dots, N\}$.

We take finite points $z_1, \dots, z_K \in \Sigma$ such that $\Sigma = \bigcup_{m=1}^{K} B_\Sigma(z_m, \eta)$. The number K can be taken depending only on N, n, c_1, and c_2.

We shall define a map

$$J : \Omega \times \{1, \dots, K\} \times \{1, \dots, K\} \to \{1, \dots, N\}.$$

Let $(\omega, \ell, m) \in \Omega \times \{1, \dots, K\} \times \{1, \dots, K\}$. If $\operatorname{supp}\Phi \cap (\eta\omega + \eta Q_0) \neq \emptyset$, then there exists a point $u \in \operatorname{supp}\Phi \cap (\eta\omega + \eta Q_0)$, and hence, by condition (i), there exists a $j \in \{1, \dots, N\}$ such that $|\langle z_\ell, z_m \partial_j A(u)\rangle| > c_1$. In this case, condition (v) implies that

$$|\langle z_\ell, z_m \partial_j A(\eta\omega)\rangle| > (2/3) c_1. \tag{6.2}$$

Thus, if $\operatorname{supp}\Phi \cap (\eta\omega + \eta Q_0) \neq \emptyset$, then we choose a $j \in \{1, \dots, N\}$ that satisfies (6.2) and define $J(\omega, \ell, m) = j$. If $\operatorname{supp}\Phi \cap (\eta\omega + \eta Q_0) = \emptyset$, then, just formally, we define $J(\omega, \ell, m) = 1$.

We write $\mathcal{E}$ to denote the set of all of the maps

$$I : \{1, \dots, K\} \times \{1, \dots, K\} \to \{1, \dots, N\}. \tag{6.3}$$

For each $\omega \in \Omega$, we define $J_\omega \in \mathcal{E}$ by

$$J_\omega(\ell, m) = J(\omega, \ell, m), \quad (\ell, m) \in \{1, \dots, K\} \times \{1, \dots, K\}.$$

For each $I \in \mathcal{E}$, we define

$$\Omega_I = \{\omega \in \Omega \ :\ J_\omega = I\}.$$

Thus, we have

$$\Omega = \bigcup_{I \in \mathcal{E}} \Omega_I, \tag{6.4}$$

where the right-hand side is a disjoint union.

We take a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\operatorname{supp} \varphi \subset Q_0 \text{ and } \sum_{\omega \in \Omega} \varphi(u - \omega) = 1 \quad \text{for all } u \in \mathbb{R}^N.$$

Then by (6.4) we have

$$\Phi(u) = \sum_{I \in \mathcal{E}} \sum_{\omega \in \Omega_I} \Phi(u)\varphi(\eta^{-1}u - \omega) = \sum_{I \in \mathcal{E}} \Phi_I(u),$$
$$\Phi_I(u) = \sum_{\omega \in \Omega_I} \Phi(u)\varphi(\eta^{-1}u - \omega).$$

Since $\mathcal{H}_{\Phi,A} = \sum_{I \in \mathcal{E}} \mathcal{H}_{\Phi_I,A}$ and since the cardinality of the set $\mathcal{E}$ is N^{K^2}, which depends only on N, n, c_1, and c_2, it is sufficient to show the estimate

$$\|\mathcal{H}_{\Phi_I,A} f\|_{H^p(\mathbb{R}^n)} \le cB\|f\|_{H^p(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n), \tag{6.5}$$

for each $I \in \mathcal{E}$.

We fix an $I \in \mathcal{E}$. We shall show that Φ_I and A satisfy condition (i′) of Step 1, and conditions (ii) and (iv) with $T = S_1 = \cdots = S_N = 1$. In fact, condition (ii) is obvious. As for condition (iv), we have

$$|\partial_j^\ell \Phi_I(u)| \le c \sum_{\ell'=0}^{\ell} |\partial_j^{\ell'} \Phi(u)|,$$

and hence condition (iv) for Φ implies condition (iv) for Φ_I. We shall show that Φ_I and A satisfy condition (i′). Take $y_0 \in \Sigma$ and $\xi_0 \in \Sigma$. There exist ℓ_0 and m_0 such that $y_0 \in B_\Sigma(z_{\ell_0}, \eta)$ and $\xi_0 \in B_\Sigma(z_{m_0}, \eta)$. We shall show that $i_0 = I(\ell_0, m_0)$ satisfies

$$|\langle y_0, \xi_0 \partial_{i_0} A(u)\rangle| > 3^{-1}c_1 \quad \text{for all } u \in \operatorname{supp} \Phi_I. \tag{6.6}$$

To show this, take an arbitrary $u_0 \in \operatorname{supp} \Phi_I$. Since

$$\operatorname{supp} \Phi_I \subset \operatorname{supp} \Phi \cap \bigcup_{\omega \in \Omega_I} \operatorname{supp} \varphi(\eta^{-1} \cdot -\omega) \subset \operatorname{supp} \Phi \cap \bigcup_{\omega \in \Omega_I} (\eta\omega + \eta Q_0),$$

there exists an $\omega_0 \in \Omega_I$ such that

$$u_0 \in \operatorname{supp} \Phi \cap (\eta\omega_0 + \eta Q_0).$$

Then, since $\omega_0 \in \Omega_I$, we have $J(\omega_0, \ell_0, m_0) = I(\ell_0, m_0) = i_0$. Thus, since $\operatorname{supp} \Phi \cap (\eta\omega_0 + \eta Q_0) \neq \emptyset$, the definition of the map J implies that

$$|\langle z_{\ell_0}, z_{m_0} \partial_{i_0} A(\eta\omega_0)\rangle| > (2/3)c_1. \tag{6.7}$$

On the other hand, since $u_0 \in \operatorname{supp} \Phi$, $u_0 - \eta\omega_0 \in \eta Q_0$, $|y_0 - z_{\ell_0}| < \eta$, and $|\xi_0 - z_{m_0}| < \eta$, claim (v) implies that

$$|\langle y_0, \xi_0 \partial_{i_0} A(u_0)\rangle - \langle z_{\ell_0}, z_{m_0} \partial_{i_0} A(\eta\omega_0)\rangle| < (1/3)c_1. \tag{6.8}$$

Inequalities (6.7) and (6.8) imply that $|\langle y_0, \xi_0 \partial_{i_0} A(u_0)\rangle| > (1/3)c_1$. Thus, (6.6) is proved.

We now can apply the result of Step 1 to $\mathcal{H}_{\Phi_I,A}$ to obtain the desired $H^p(\mathbb{R}^n)$ estimate. This proves the assertion of Step 2.

Step 3. Now, the claim of Theorem 6.1 will be proved in the general case.

First, assume that Φ has compact support. We define $\widetilde{\Phi}$ and $\widetilde{A}$ by

$$\widetilde{\Phi}(u) = S_1 \cdots S_N \Phi(S_1 u_1, \dots, S_N u_N),$$
$$\widetilde{A}(u) = T^{-1} A(S_1 u_1, \dots, S_N u_N).$$

Then a simple change of variables gives

$$(\mathcal{H}_{\Phi,A} f)(x) = T^{-n} (\mathcal{H}_{\widetilde{\Phi},\widetilde{A}} f)(T^{-1}x),$$

and thus

$$\|\mathcal{H}_{\Phi,A} f\|_{H^p(\mathbb{R}^n)} = T^{-n+n/p} \|\mathcal{H}_{\widetilde{\Phi},\widetilde{A}} f\|_{H^p(\mathbb{R}^n)}.$$

Thus, it is sufficient to show the estimate

$$\|\mathcal{H}_{\widetilde{\Phi},\widetilde{A}} f\|_{H^p(\mathbb{R}^n)} \le cBT^{-n/p+n} \|f\|_{H^p(\mathbb{R}^n)}. \tag{6.9}$$

But for $\mathcal{H}_{\widetilde{\Phi},\widetilde{A}}$, we can apply Step 2. In fact, as is easily seen, $\widetilde{\Phi}$ and $\widetilde{A}$ satisfy conditions (i), (ii), and (iii) with $T = S_1 = \cdots = S_N = 1$, and also satisfy condition (iv) with B replaced by $BT^{-n/p+n}$, i.e.,

$$\int \left| \left(\frac{\partial}{\partial u_j} \right)^{\ell} \widetilde{\Phi}(u) \right| |\det \widetilde{A}(u)|^{-1/2} \, du \le BT^{-n/p+n}.$$

Thus, (6.9) follows from the assertion of Step 2.

Finally, consider the case in which Φ does not have compact support. We take a function $\theta \in C_0^\infty(\mathbb{R}^N)$ such that $\theta(u) = 1$ for $|u| \le 1$, and we define

$$\Phi_R(u) = \Phi(u)\theta(R^{-1}u)$$

for $R > \max\{S_1, \dots, S_N\}$. Then Φ_R has a compact support and satisfies the conditions of the theorem uniformly in R as far as $R > \max\{S_1, \dots, S_N\}$. Hence, by what has already been proved, we have the H^p estimate of $\mathcal{H}_{\Phi_R,A}$ uniformly in $R > \max\{S_1, \dots, S_N\}$. Since condition (iv) for Φ implies that $L_A(\Phi) < \infty$, we have $L_A(\Phi - \Phi_R) \to 0$ as $R \to \infty$ and hence, by Proposition 2.1, $\mathcal{H}_{\Phi_R,A} f \to \mathcal{H}_{\Phi,A} f$ in $L^2(\mathbb{R}^n)$. Now the desired H^p estimate of $\mathcal{H}_{\Phi,A}$ follows by virtue of the Fatou property of $H^p(\mathbb{R}^n)$. This completes the proof of Theorem 6.1. □

7. A global theorem

In this section, we give a global theorem for multidimensional Hausdorff operators. Our method is based on Theorem 6.1. Although our method could be used to give more general theorems, here we shall give only a generalization of Theorem B.

As in Section 3, we consider the Lie group $SO(n, \mathbb{R})$ and the Haar measure μ on $SO(n, \mathbb{R})$. We write $L = \dim SO(n, \mathbb{R}) = n(n-1)/2$. We shall consider an operator of the following form:

$$(Tf)(x) = \int_{\substack{0<s<\infty \\ P \in SO(n,\mathbb{R})}} \Psi(s, P) s^{-n} f(s^{-1}xP) \, ds d\mu(P), \quad x \in \mathbb{R}^n,$$

where $\Psi : (0, \infty) \times SO(n, \mathbb{R}) \to \mathbb{C}$ is a given sufficiently smooth function. We take a function $\theta \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \theta \subset (2^{-1}, 2)$ and $\sum_{k \in \mathbb{Z}} \theta(2^{-k}s) = 1$ for all $s > 0$. For each $k \in \mathbb{Z}$, we define

$$(T_k f)(x) = \int_{\substack{0<s<\infty \\ P \in SO(n,\mathbb{R})}} \theta(2^{-k}s) \Psi(s, P) s^{-n} f(s^{-1}xP) \, ds d\mu(P), \quad x \in \mathbb{R}^n.$$

For the same reason as in Proposition 2.1, T_k is a well-defined bounded operator in $L^2(\mathbb{R}^n)$. At least formally, we have

$$Tf = \sum_{k\in\mathbb{Z}} T_k f.$$

With these notations, we shall prove the following theorem.

Theorem 7.1. *Let $n \in \mathbb{N}$, let $n \geq 2$, let $0 < p < 1$, and let $M = [n/p - n/2] + 1$. Suppose that $\Psi : (0,\infty) \times SO(n,\mathbb{R}) \to \mathbb{C}$ is a function of class C^M and satisfies the following estimates with some $\epsilon > 0$:*

(a) *for $\ell = 0, 1, \dots, M$,*

$$\left| \left(\frac{\partial}{\partial s}\right)^{\ell} \Psi(s,P) \right| \leq \min\{s^{\epsilon}, s^{-\epsilon}\} s^{-n/p+n-1-\ell};$$

(b) *for each $P_0 \in SO(n,\mathbb{R})$, there exists a coordinate neighborhood (V,φ) of $SO(n,\mathbb{R})$ such that $V \ni P_0$ and*

$$\left| \left(\frac{\partial}{\partial u_j}\right)^{\ell} \Psi\left(s, \varphi^{-1}(u_1,\dots,u_L)\right) \right| \leq \min\{s^{\epsilon}, s^{-\epsilon}\} s^{-n/p+n-1}$$

for $j = 1,\dots,L$ and $\ell = 0,1,\dots,M$.

Then, for each $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$, the series $\sum_{k\in\mathbb{Z}} T_k f$ converges unconditionally in $H^p(\mathbb{R}^n)$ and

$$\left\| \sum_{k\in\mathbb{Z}} T_k f \right\|_{H^p(\mathbb{R}^n)} \leq c_\epsilon \|f\|_{H^p(\mathbb{R}^n)}.$$

Proof. It suffices to show the estimate

$$\|T_k f\|_{H^p(\mathbb{R}^n)} \leq c \min\{2^{k\epsilon}, 2^{-k\epsilon}\} \|f\|_{H^p(\mathbb{R}^n)}, \quad f \in L^2 \cap H^p. \tag{7.1}$$

Since $SO(n,\mathbb{R})$ is compact, we can take a finite number of coordinate neighborhoods $\{(V_\nu, \varphi_\nu)\}$ that satisfy the condition of (c) such that $\{V_\nu\}$ covers $SO(n,\mathbb{R})$. We take smooth functions α_ν on $SO(n,\mathbb{R})$ such that $\operatorname{supp}\alpha_\nu \subset V_\nu$ and $\sum_\nu \alpha_\nu = 1$ on $SO(n,\mathbb{R})$. Using this partition of unity $\{\alpha_\nu\}$, we write $T_k f = \sum_\nu U_{k,\nu} f$, where $U_{k,\nu}$ is the operator defined in the same way as T_k but with $\Psi(s,P)\alpha_\nu(P)$ in place of $\Psi(s,P)$. To prove (7.1), it is sufficient to show the estimate

$$\|U_{k,\nu} f\|_{H^p(\mathbb{R}^n)} \leq c \min\{2^{k\epsilon}, 2^{-k\epsilon}\} \|f\|_{H^p(\mathbb{R}^n)}, \quad f \in L^2 \cap H^p,$$

for each ν.

We shall simplify the notation by omitting the letter ν. Thus, we assume that (V,φ) is a coordinate neighborhood satisfying the condition of (c), and that α is a smooth function on $SO(n,\mathbb{R})$ with $\operatorname{supp}\alpha \subset V$. The operator we consider is

$$(U_k f)(x) = \int_{\substack{0<s<\infty \\ P\in SO(n,\mathbb{R})}} \theta(2^{-k}s)\Psi(s,P)\alpha(P) s^{-n} f(s^{-1}xP)\, ds d\mu(P),$$

and we shall prove the estimate

$$\|U_k f\|_{H^p(\mathbb{R}^n)} \leq c \min\{2^{k\epsilon}, 2^{-k\epsilon}\} \|f\|_{H^p(\mathbb{R}^n)}, \quad f \in L^2 \cap H^p. \tag{7.2}$$

Writing $U_k f$ by using the local coordinate (V,φ) and writing, for notational convenience, $\tilde{\varphi} = \varphi^{-1}$ and

$$u = (u_0, u_1, \dots, u_L) \in \mathbb{R}^{1+L}, \quad u' = (u_1, \dots, u_L) \in \mathbb{R}^L,$$

we have

$$(7.3)\qquad \begin{aligned}&(U_k f)(x)\\ &= \int_{\mathbb{R}^{1+L}} \theta(2^{-k}u_0)\Psi(u_0, \tilde{\varphi}(u'))\alpha(\tilde{\varphi}(u'))u_0^{-n} f(u_0^{-1} x\tilde{\varphi}(u'))w(u')\, du,\end{aligned}$$

where $w(u')$ is the smooth nonnegative function such that $w(u')du' = d\mu(P)$ under the parametrization $P = \tilde{\varphi}(u')$. We define Φ_k and A_k by

$$(7.4)\qquad \begin{aligned}\Phi_k(u) &= \theta(2^{-k}u_0)\Psi(u_0, \tilde{\varphi}(u'))\alpha(\tilde{\varphi}(u'))w(u'),\\ A_k(u) &= \tilde{\theta}(2^{-k}u_0)u_0\tilde{\varphi}(u')\tilde{\alpha}(\tilde{\varphi}(u')),\end{aligned}$$

where $\tilde{\theta}$ is a function in $C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\tilde{\theta} \subset (2^{-1}, 2)$ and $\tilde{\theta} = 1$ in a neighborhood of $\operatorname{supp}\theta$, and $\tilde{\alpha}$ is a smooth function on $SO(n, \mathbb{R})$ such that $\operatorname{supp}\tilde{\alpha} \subset V$ and $\tilde{\alpha} = 1$ in a neighborhood of $\operatorname{supp}\alpha$. Notice that

$$(7.5)\qquad \operatorname{supp}\Phi_k \subset (2^{k-1}, 2^{k+1}) \times \operatorname{supp}(\alpha \circ \tilde{\varphi}).$$

Also notice that, in a neighborhood of $\operatorname{supp}\Phi_k$, we have

$$(7.6)\qquad A_k(u) = u_0\tilde{\varphi}(u'), \quad \det A_k(u) = u_0^n, \quad {}^t A_k(u)^{-1} = u_0^{-1}\tilde{\varphi}(u').$$

Thus, (7.3) can be written as

$$(7.7)\qquad \begin{aligned}(U_k f)(x) &= \int_{\mathbb{R}^{1+L}} \Phi_k(u)|\det A_k(u)|^{-1} f(x\, {}^t A_k(u)^{-1})\, du\\ &= (\mathcal{H}_{\Phi_k, A_k} f)(x).\end{aligned}$$

To apply Theorem 6.1 to operator (7.7), we check that $\Phi = \Phi_k$ and $A = A_k$ satisfy conditions (i)–(iv) of Theorem 6.1 with

$$N = 1 + L, \quad T = 2^k, \quad S_0 = 2^k, \quad S_1 = \cdots = S_L = 1, \quad B = c2^{-\epsilon|k|},$$

where c is a constant independent of $k \in \mathbb{Z}$. (Notice that the index set $\{1, \dots, N\}$ of Theorem 6.1 should be replaced by $\{0, 1, \dots, L\}$ in our present notation.) Condition (ii) is easy to check with the aid of (7.5) and (7.6). Estimate (iii) is easily seen from (7.4). We shall check (i). Since $A_k(u) = u_0\tilde{\varphi}(u')$ in a neighborhood of $\operatorname{supp}\Phi_k$ and since $u_0 \approx 2^k$ on $\operatorname{supp}\Phi_k$, condition (i) is equivalent to the estimate

$$|\langle y, \xi\tilde{\varphi}(u')\rangle| + \sum_{j=1}^{L} |\langle y, \xi\partial_j\tilde{\varphi}(u')\rangle| \ge c, \quad y, \xi \in \Sigma^{n-1}, \quad u' \in \operatorname{supp}(\alpha \circ \tilde{\varphi}).$$

But this estimate follows just from the argument of Section 5 (see, in particular, (5.6)) and from the compactness of Σ^{n-1} and $\operatorname{supp}(\alpha \circ \tilde{\varphi})$. Finally, we check condition (iv). Let $\ell \in \{0, 1, \dots, M\}$. From estimates (a) and (b) of Theorem 7.1, we have

$$\left|(2^k\partial_0)^\ell \Phi_k(u)\right| + \sum_{j=1}^{L} \left|\partial_j^\ell \Phi_k(u)\right| \lesssim \min\{2^{k\epsilon}, 2^{-k\epsilon}\}(2^k)^{-n/p+n-1}.$$

It follows from (7.6) that

$$|\det A_k(u)|^{-1/2} \approx (2^k)^{-n/2}, \quad u \in \operatorname{supp}\Phi_k.$$

These estimates and (7.5) give

$$\begin{aligned}
&\int \left(\left|(2^k\partial_0)^\ell \Phi_k(u)\right| + \sum_{j=1}^{L} \left|\partial_j^\ell \Phi_k(u)\right| \right) |\det A_k(u)|^{-1/2}\, du \\
&\lesssim \int_{\substack{2^{k-1}<u_0<2^{k+1} \\ u'\in \operatorname{supp}(\alpha\circ\tilde{\varphi})}} \min\{2^{k\epsilon}, 2^{-k\epsilon}\}(2^k)^{-n/p+n-1}(2^k)^{-n/2}\, du \\
&\lesssim \min\{2^{k\epsilon}, 2^{-k\epsilon}\}(2^k)^{-n/p+n/2}.
\end{aligned}$$

Thus, condition (iv) is satisfied.

Now, by using Theorem 6.1, estimate (7.2) follows. This completes the proof of Theorem 7.1. □

8. Some remarks on condition (4.1)

In this section, we give some remarks concerning the number N in condition (4.1). To simplify notation, we write $B_j = \partial A(u)/\partial u_j$. Thus, $B_1, \ldots, B_N$ are $n \times n$ real matrices. We consider the following condition:

$$\begin{cases} \text{for all } (y,\xi) \in \Sigma^{n-1} \times \Sigma^{n-1}, \quad \text{there exists a } j \in \{1,\ldots,N\} \\ \text{such that } \langle y, \xi B_j\rangle \neq 0. \end{cases} \tag{8.1}$$

We shall say that (8.1) is possible if there exist $B_1, \ldots, B_N \in M_n(\mathbb{R})$ which satisfy (8.1).

The following statement is valid.

Proposition 8.1.

(1) *Condition* (8.1) *is possible only if* $N \geq n$.
(2) *If* n *is odd and* $n \geq 3$, *then* (8.1) *is possible only if* $N \geq n+1$.
(3) *For all* $n \geq 2$, *condition* (8.1) *is possible with* $N = 1 + n(n-1)/2$. *If* $n \geq 4$, *then* (8.1) *is possible with an* $N < 1 + n(n-1)/2$.

Proof.

(1) If $N < n$, then for any given $\xi \in \Sigma^{n-1}$, there exists a $y \in \Sigma^{n-1}$ that is orthogonal to the N vectors $\xi B_1, \ldots, \xi B_N$, and hence (8.1) cannot hold.
(2) To prove this, assume on the contrary that $N = n \geq 3$ is odd, and (8.1) holds.

First, notice that each B_j is nonsingular. In fact, if, for example, B_1 is singular, then there exists a $\xi \in \Sigma^{n-1}$ such that $\xi B_1 = 0$. There exists a $y \in \Sigma^{n-1}$ that is orthogonal to $n-1$ vectors $\xi B_2, \ldots, \xi B_n$. Thus, $\langle y, \xi B_j\rangle = 0$ for all j, which contradicts (8.1).

Next, consider

$$P(\lambda) = \det(\lambda B_1 + B_2 + \cdots + B_n), \quad \lambda \in \mathbb{R}.$$

Then $P(\lambda)$ is a real polynomial in λ of degree n. Since n is odd, there exists a $\lambda_0 \in \mathbb{R}$ such that $P(\lambda_0) = 0$. Then the matrix $\lambda_0 B_1 + B_2 + \cdots + B_n$ is singular, and there exists a $\xi \in \Sigma^{n-1}$ such that $\xi(\lambda_0 B_1 + B_2 + \cdots + B_n) = 0$. This means that the n vectors $\xi B_1, \ldots, \xi B_n$ are linearly dependent, and hence there exists a $y \in \Sigma^{n-1}$ that is orthogonal to all ξB_j, $j = 1, \ldots, n$, which contradicts (8.1).

(3) These follow from Examples 5.2 and 5.3. □

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