

## VECTOR-VALUED MODULAR FORMS ON A THREE-DIMENSIONAL BALL

EBERHARD FREITAG AND RICCARDO SALVATI MANNI

ABSTRACT. In this paper we give a structure theorem for the module of vector valued modular forms in the case of a three dimensional ball with the action of the Picard modular group  $\Gamma_3[\sqrt{-3}]$ . The corresponding modular variety of dimension 3 is a copy of the Segre cubic.

### INTRODUCTION

Recently there has been growing interest in structure theorems for vector-valued modular forms. One reason may be Harder's conjecture ([5, Conjecture 3], in van der Geer's talk). Different methods have been developed to get structure theorems for modules of vector-valued modular forms [3, 6, 8, 13, 16–18]. At the beginning, the case of Siegel modular forms was studied, but since then other groups have also found interest. For example, in the paper [6], Cléry and van der Geer determined generators for some modules of vector-valued Picard modular forms on the two-dimensional ball. In this paper we consider the case of a three-dimensional ball with the action of the Picard modular group  $\Gamma_3[\sqrt{-3}]$  (see Section 3). The corresponding modular variety of dimension three is a copy of the Segre cubic. We obtain similar results as in [6] but with completely different methods.

Vector-valued Picard modular forms on the  $n$ -ball  $\mathcal{B}_n$  belong to the space of functions that have a transformation formula related to rational representations of the complexification of the maximal compact group of the unitary group  $U(1, n)$ , which is the group  $GL(1, \mathbb{C}) \times GL(n, \mathbb{C})$ . Here we consider the representation

$$\varrho_r(k_1, k_2) = k_1^r k_2 \quad (r \in \mathbb{Z}).$$

A similar representation in a Siegel case has been treated in [11]. We denote by  $\mathcal{M}(r)$  the space of modular forms  $f : \mathcal{B}_n \rightarrow \mathbb{C}^n$  which belong to this representation. The direct sum

$$\mathcal{M} = \bigoplus_{r \in \mathbb{Z}} \mathcal{M}(r)$$

is a module over the ring of scalar-valued modular forms.

In the case  $\Gamma_3[\sqrt{-3}]$  this ring is generated by five forms  $T_1, \dots, T_5$  of weight three which satisfy the relation of a Segre cubic [10, 15]. We will determine the structure of the module  $\mathcal{M}$ . For this we consider the submodule  $\mathcal{N}$  of  $\mathcal{M}$ , generated by ten Cohen-Rankin brackets  $\{T_i, T_j\}$ . They are elements of  $\mathcal{M}(5)$ . Our main result is that  $\mathcal{M}$  and  $\mathcal{N}$  agree.

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To get a proof, we first investigate  $\mathcal{N}$ . There are some obvious relations between the Cohen-Rankin brackets and also the Segre relation induces a relation between them. The fact that these simple relations are defining relations (see Proposition 5.1) rests on a pure algebraic statement about differential modules which is developed in Section 1. In Section 2 we develop the framework for vector-valued ball modular forms and in Section 3 we describe the group of our interest  $\Gamma_3[\sqrt{-3}]$ , the congruence group of level  $\sqrt{-3}$  in the full Picard modular group with respect to  $\mathbb{Q}(\sqrt{-3})$ . We describe its ring of modular forms, the relation to the Segre cubic, and the ramification locus.

In Section 4 we study some special modular forms which are related to the tangent bundle of the Segre cubic. They are needed for the proof of the equality of  $\mathcal{M}$  and its submodule  $\mathcal{N}$  which will be given in Section 5. Some computer calculations are necessary. In our main result, Theorem 5.5, we also give the Hilbert function of the module  $\mathcal{M}$ .

The method which we use works also in a Siegel case that has been treated in [7]. In Section 6 we describe this briefly.

We also correct some sign errors in [10]. They had no further influence to [10], but, in the present paper, the corrections are necessary.

## 1. DIFFERENTIAL MODULES OVER GRADED ALGEBRAS

Let  $A = \bigoplus_{d=0}^{\infty} A_d$  be a finitely generated graded algebra over a field  $K = A_0$  of characteristic zero. We assume that  $A$  is an integral domain and denote its field of fractions by  $Q(A)$ . We consider the Kähler differential module  $\Omega = \Omega(Q(A)/K)$ . Recall that this is a  $Q(A)$ -vector space together with a  $K$ -linear derivation  $d : Q(A) \rightarrow \Omega$ . The dimension of  $\Omega$  equals the transcendental degree of  $Q(A)$  and  $\Omega$  is generated by the image of  $d$ . In the following, we denote by  $\deg(f)$  the degree of a non-zero homogeneous element of  $A$ . For two non-zero homogeneous elements of positive degree  $f, g \in A$  we define  $\{f, g\} := \deg(g)gdf - \deg(f)fdg$ . Another way to write this is

$$\{f, g\} = \frac{g^{\deg(f)+1}}{f^{\deg(g)-1}} d\left(\frac{f^{\deg(g)}}{g^{\deg(f)}}\right).$$

This is a skew-symmetric  $K$ -bilinear pairing and it satisfies the following rule:

$$\deg(h)h\{f, g\} = \deg(g)g\{f, h\} + \deg(f)f\{h, g\}.$$

**Definition 1.1.** We denote by  $\mathcal{N}$  the  $A$ -module that is generated by all  $\{f, g\}$  where  $f, g$  are homogeneous elements of positive degree in  $A$ .

We are interested in a finite presentation of  $\mathcal{N}$ . There is no difficulty getting a finite system of generators. Let  $A = K[f_1, \dots, f_m]$ , ( $f_i$  homogeneous). Then  $\{f_i, f_j\}$  are generators of  $\mathcal{N}$ . It is more involved to get defining relations. We use the notation  $d_i = \deg(f_i)$ . A polynomial  $P \in K[X_1, \dots, X_m]$  is called isobaric of weight  $k$  (with respect to  $(d_1, \dots, d_m)$ ) if it is of the form

$$P = \sum_{d_1\nu_1 + \dots + d_m\nu_m = k} a_{\nu_1, \dots, \nu_m} X_1^{\nu_1} \dots X_m^{\nu_m}.$$

Then Euler's relation  $\sum_{\nu=1}^m d_\nu \frac{\partial P}{\partial X_\nu} X_\nu = kP$  holds.

The ideal of relations between  $f_1, \dots, f_m$  is generated by isobaric polynomials. Let  $R(f_1, \dots, f_m) = 0$  be an isobaric relation. Differentiation gives

$$\sum_{\nu=1}^m (\partial_\nu R) df_\nu = 0 \quad \text{where} \quad \partial_\nu R := \frac{\partial R}{\partial X_\nu}(f_1, \dots, f_m).$$

From this relation and Euler's relation we derive

$$\sum_{\nu=1}^m (\partial_\nu R) \{f_\nu, f_\mu\} = 0 \quad (\mu \text{ arbitrary}).$$

We want to formalize this and introduce a module  $\mathcal{N}'$  which is defined by the so far known relations.

**Definition 1.2.** We denote by  $\mathcal{N}'$  the  $A$ -module that is generated by symbols  $[f_i, f_j]$  with the following defining relations:

$$(1) \quad d_k f_k [f_i, f_j] = d_j f_j [f_i, f_k] + d_i f_i [f_k, f_j], \quad [f_i, f_j] + [f_j, f_i] = 0.$$

For each isobaric relation  $R$  between the  $f_1, \dots, f_m$  one has

$$(2) \quad \sum_{\nu=1}^m (\partial_\nu R) [f_\nu, f_\mu] = 0 \quad (\mu \text{ arbitrary}).$$

It is of course enough to take for  $R$  a system of generators of the ideal of all relations. There is a natural surjective homomorphism

$$\mathcal{N}' \longrightarrow \mathcal{N}, \quad [f_i, f_j] \longmapsto \{f_i, f_j\}.$$

We notice that  $\mathcal{N}$  is torsion free for trivial reasons, but it is not clear that  $\mathcal{N}'$  is torsion free too. Under certain circumstances,  $\mathcal{N}' \rightarrow \mathcal{N}$  is an isomorphism. To work this out, we consider an arbitrary relation in  $\mathcal{N}$

$$\sum_{i < j} P_{ij} \{f_i, f_j\} = 0, \quad P_{ij} \in A.$$

We multiply this relation by  $d_1 f_1$  and insert

$$d_1 f_1 \{f_i, f_j\} = d_i f_i \{f_1, f_j\} - d_j f_j \{f_1, f_i\}.$$

Then we obtain the relation

$$\sum_j P_j \{f_1, f_j\} = 0,$$

where the elements  $P_j \in A$  are defined as  $P_j = \sum_{i < j} d_i f_i P_{ij} - \sum_{i > j} d_i f_i P_{ji}$ .

Let  $n$  be the transcendental degree of  $Q(A)$ . We can assume that  $f_1, \dots, f_n$  are independent. Then each  $f_k, k > n$ , satisfies an algebraic relation  $R_k(f_1, \dots, f_n, f_k) = 0$ . Here  $R_k$  is an irreducible polynomial in the variables  $X_1, \dots, X_n, X_k$ . Now we make use of the relation

$$(\partial_k R_k) \{f_1, f_k\} + \sum_{\nu=1}^n (\partial_\nu R_k) \{f_1, f_\nu\} = 0.$$

We have to use the elements (from the ring  $A$ )

$$\Pi := \prod_{k=n+1}^m \partial_k R_k, \quad \Pi^{(k)} := \frac{\Pi}{\partial_k R_k}.$$

We multiply the original relation by  $\Pi$ :

$$\Pi \sum_j P_j \{f_1, f_j\} = 0.$$

For  $k > n$  we have the formula

$$\Pi \{f_1, f_k\} = \Pi^{(k)}(\partial_k R_k) \{f_1, f_k\} = -\Pi^{(k)} \sum_{j=1}^n (\partial_j R_k) \{f_1, f_j\}.$$

Now we can eliminate the  $\{f_1, f_k\}$  for  $k > n$  to produce a relation between the  $\{f_1, f_i\}$ ,  $2 \leq i \leq n$ . But these elements are independent. Hence the coefficients of the relation must vanish. A simple calculation now gives the following lemma.

**Lemma 1.3.** *Let*

$$\sum_{i < j} P_{ij} \{f_i, f_j\} = 0, \quad P_{ij} \in A.$$

*Then the elements*

$$P_j = \sum_{i < j} d_i f_i P_{ij} - \sum_{i > j} d_i f_i P_{ji}$$

*satisfy the following system of relations:*

$$P_j \Pi = \sum_{k=n+1}^m (\partial_j R_k) P_k \Pi^{(k)} \quad (1 \leq j \leq n).$$

**Supplement.** *Conversely, these relations imply in  $\mathcal{N}'$  the relation*

$$f_1 \Pi \sum_{i < j} P_{ij} [f_i, f_j] = 0.$$

For the proof of the Supplement we just have to notice that the calculations above only use the defining relations of  $\mathcal{N}'$ . □

Let us assume that multiplication by  $f_1 \Pi$  is injective on  $\mathcal{N}'$ . Then we see that  $\sum P_{ij} \{f_i, f_j\} = 0$  implies  $\sum P_{ij} [f_i, f_j] = 0$ . Hence  $\mathcal{N}' \rightarrow \mathcal{N}$  is an isomorphism and  $\mathcal{N}'$  must be torsion free. This gives the following result.

**Proposition 1.4.** *Assume that the  $f_1, \dots, f_n$  is a transcendental basis such that each  $f_k$ ,  $n < k \leq m$ , satisfies an irreducible algebraic relation*

$$R_k(f_1, \dots, f_n, f_k) = 0.$$

*The homomorphism  $\mathcal{N}' \rightarrow \mathcal{N}$  is an isomorphism if and only if  $\mathcal{N}'$  is torsion free. For this it suffices that multiplications by  $f_1$  and  $\partial_k R_k$  ( $n < k \leq m$ ) are injective on  $\mathcal{N}'$ .*

## 2. THE EXTENDED BALL

Let  $V$  be a complex vector space of dimension  $n + 1$  and let  $\langle \cdot, \cdot \rangle$  be a non-degenerate hermitian form of signature  $(1, n)$ . We consider the projective space  $\mathbb{P}(V) = (V - \{0\})/\mathbb{C}^*$  and the natural projection

$$V - \{0\} \longrightarrow \mathbb{P}(V), \quad v \longmapsto [v].$$

Let

$$\tilde{B} := \{v \in V; \quad \langle v, v \rangle > 0\}$$

be the set of all vectors of positive norm  $\langle v, v \rangle > 0$  and let  $\mathcal{B}$  be its image in the projective space. This is a model of the complex  $n$ -ball. The unitary group  $U(V)$  acts on  $\mathcal{B}$  and on  $\tilde{\mathcal{B}}$ .

We choose a vector  $e \in V$  with positive norm  $\langle e, e \rangle > 0$  and we consider the orthogonal complement  $\mathcal{Z} = e^\perp$  which is a negative definite space of dimension  $n$ . We have  $V = \mathbb{C}e \oplus \mathcal{Z}$ . Sometimes we write the elements  $v \in V$  in the form

$$v = Ce + z = \begin{pmatrix} C \\ z \end{pmatrix}.$$

Then we can write the elements of  $\text{End}(V)$  as matrices

$$p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \in \mathbb{C}, \quad b \in \mathcal{Z}^*, \quad c \in \mathcal{Z}, \quad d \in \text{End}(\mathcal{Z}),$$

such that the action on  $V = \mathbb{C}e + \mathcal{Z}$  is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C \\ z \end{pmatrix} = \begin{pmatrix} aC + b(z) \\ Cc + d(z) \end{pmatrix}.$$

For the multiplication of two of such matrices one has to make use of the canonical isomorphism  $\mathcal{Z} \otimes \mathcal{Z}^* \rightarrow \text{End}(\mathcal{Z})$ .

We denote by

$$\mathcal{B}_{\mathcal{Z}} := \{z \in \mathcal{Z}; \quad -\langle z, z \rangle < 1\}$$

the complex  $n$ -ball in the space  $\mathcal{Z}$  with respect to the positive definite form  $-\langle \cdot, \cdot \rangle$ . There is a natural bijection

$$\mathcal{B}_{\mathcal{Z}} \xrightarrow{\sim} \mathcal{B}, \quad z \mapsto [e + z].$$

We carry over the action of  $U(V)$  to  $\mathcal{B}_{\mathcal{Z}}$  and denote it by  $g\langle z \rangle$ ,

$$g\langle z \rangle := (a + b(z))^{-1}(c + d(z)).$$

Let  $g \in GL(V)$  be an element with the property  $g(e) = e$ . Then  $g$  acts on  $V/\mathbb{C}e$ . We denote by  $P \subset GL(V)$  the subgroup

$$P := \{p \in GL(V); \quad p(e) = e, \quad p \text{ acts as identity on } V/\mathbb{C}e\}.$$

The corresponding matrices then are of the form

$$p = \begin{pmatrix} 1 & b \\ 0 & \text{id}_{\mathcal{Z}} \end{pmatrix}, \quad b \in \mathcal{Z}^*.$$

The group  $P$  is a closed complex Lie subgroup. The quotient  $GL(V)/P$  carries a natural structure as a complex manifold. For  $g \in GL(V)$ , the element  $g(e)$  depends only on the coset  $gP$ . Hence, the subset

$$\mathcal{B}^* = \{gP \in GL(V)/P; \quad g(e) \in \tilde{\mathcal{B}}\}$$

is a well-defined subset of  $GL(V)/P$ . It is open and hence a complex manifold too. There are natural (holomorphic) maps

$$\mathcal{B}^* \longrightarrow \tilde{\mathcal{B}} \longrightarrow \mathcal{B}, \quad gP \mapsto g(e) \mapsto [g(e)].$$

We consider the group

$$K_{\mathbb{C}} = GL(\mathbb{C}e) \times GL(\mathcal{Z}) \cong \mathbb{C}^* \times GL(n, \mathbb{C})$$

as a subgroup of  $GL(V)$  in the obvious way. The corresponding matrices are of the form

$$k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Usually the element  $k_1$  will be identified with the corresponding complex number. The group  $K_{\mathbb{C}}$  is the complexification of the maximal compact subgroup

$$K := U(\mathbb{C}e) \times U(\mathcal{Z})$$

of  $U(V)$ .

The elements of  $K_{\mathbb{C}}$  fix the point  $[e] \in \mathbb{P}(V)$ . Hence we have the natural map  $K_{\mathbb{C}} \rightarrow \mathcal{B}^*$ .

**Lemma 2.1.** *The natural map  $K_{\mathbb{C}} \rightarrow \mathcal{B}^*$  gives a bijection between  $K_{\mathbb{C}}$  and the fibre of the natural projection  $\mathcal{B}^* \rightarrow \mathcal{B}$  over  $[e]$ .*

*Proof.* The elements which stabilize  $[e]$  are of the form

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

They can be written in a unique way in the form  $g = kp$ ,  $k \in K_{\mathbb{C}}$ ,  $p \in P$ . □

The group  $K_{\mathbb{C}}$  normalizes  $P$  and hence acts on  $G/P$  by multiplication from the right,

$$(gP, k) \mapsto gkP, \quad g \in GL(V), \quad k \in K_{\mathbb{C}}.$$

Hence  $\mathcal{B}^* \rightarrow \mathcal{B}$  is a principal fibre bundle with structural group  $K_{\mathbb{C}}$ .

As we mentioned already, the unitary group  $U(V)$  acts on  $\tilde{\mathcal{B}}$ . Hence it acts also on  $\mathcal{B}^*$  by multiplication from the left.

We can now define vector-valued automorphic forms. Since  $\mathcal{B}^*$  plays the role of an extension of the ball  $\mathcal{B}$ , from now on we use the letter  $z$  to denote the elements of  $\mathcal{B}^*$ . The action of  $U(V)$  is denoted by  $\gamma z$  and that of  $K_{\mathbb{C}}$  by  $zk$ .

**Definition 2.2.** Let  $\Gamma \subset U(V)$  be a subgroup, let  $\chi$  be a character of  $\Gamma$  and let  $\varrho : K_{\mathbb{C}} \rightarrow GL(\mathcal{U})$  be a rational representation of  $K_{\mathbb{C}}$  on some finite-dimensional complex vector space. An automorphic form for  $(\Gamma, \chi, \varrho)$  is a holomorphic function

$$g : \mathcal{B}^* \longrightarrow \mathcal{U}$$

with the transformation property

$$f(\gamma zk) = \chi(\gamma)\varrho(k)^{-1}f(z), \quad \text{with } \gamma \in \Gamma \text{ and } k \in K.$$

In the case  $n = 1$  the usual regularity condition at the cusps has to be added.

We denote the space of these forms by  $[\Gamma, \chi, \varrho]$ . For trivial  $\chi$  we simply write  $[\Gamma, \varrho]$ . It may happen that elements of the form  $\zeta \text{id}_V$ ,  $|\zeta| = 1$ , are contained in  $\Gamma$ . The corresponding transformations of  $\mathcal{B}^*$  come also from  $K_{\mathbb{C}}$ . Hence  $\chi$  and  $\varrho$  have to satisfy a compatibility condition if non-zero automorphic forms exist.

We explain briefly the relation to the notion of (scalar-valued) automorphic form as it has been used by Borchers; cf. [4, section 13]. An automorphic form in his sense is a holomorphic function  $f : \tilde{\mathcal{B}} \rightarrow \mathbb{C}$  with the transformation property  $f(\gamma z) = \chi(\gamma)f(z)$  and  $f(tz) = t^{-r}f(z)$ . The composition of  $f$  with the projection  $\mathcal{B}^* \rightarrow \tilde{\mathcal{B}}$  then gives an automorphic form in the sense of Definition 2.2 with respect to the representation  $\varrho(k_1, k_2) = k_1^r$ .

In older contexts, automorphic forms are functions on  $\mathcal{B}_{\mathcal{Z}}$  transforming with respect to an automorphy factor. We want to describe the link between the two approaches. For this we construct a section  $\mathcal{B}_{\mathcal{Z}} \rightarrow \mathcal{B}^*$ . First we construct a section  $\mathcal{B} \rightarrow \tilde{\mathcal{B}}$ . Each element of  $V$  can be written in the form  $v = Ce + z$  where  $C$  is a complex number and  $z \in \mathcal{Z}$ . From  $\langle v, v \rangle > 0$  it follows  $C \neq 0$ . Hence each element

of  $\mathcal{B}$  has a unique representative in  $\tilde{\mathcal{B}}$  with  $C = 1$ . This gives a section  $\mathcal{B} \rightarrow \tilde{\mathcal{B}}$ . Now let  $v = Ce + w \in \tilde{\mathcal{B}}$ . We associate to  $v$  a linear transformation  $g_v \in \text{GL}(V)$ , namely

$$g_v(xe + y) = Cxe + wx + y \quad (x \in \mathbb{C}, y \in \mathcal{Z}),$$

or, in matrix notation

$$g_v = \begin{pmatrix} C & 0 \\ w & \text{id}_{\mathcal{Z}} \end{pmatrix} \quad (v = Ce + w).$$

We have  $g_v(e) = v$ . Hence  $g_vP$  is contained in  $\mathcal{B}^*$ . This gives us the desired section  $\tilde{\mathcal{B}} \rightarrow \mathcal{B}^*$ . Combining it with  $\mathcal{B} \rightarrow \tilde{\mathcal{B}}$  we get a section  $\mathcal{B} \rightarrow \mathcal{B}^*$ . Moreover, using the isomorphism  $\mathcal{B}_{\mathcal{Z}} \cong \mathcal{B}$ , we get the map

$$\sigma : \mathcal{B}_{\mathcal{Z}} \rightarrow \mathcal{B}^*, \quad z \mapsto \begin{pmatrix} 1 & 0 \\ z & \text{id}_{\mathcal{Z}} \end{pmatrix} P.$$

**Lemma 2.3.** *There is a “canonical factor of automorphy”*

$$J_{can} : \text{U}(V) \times \mathcal{B}_{\mathcal{Z}} \rightarrow K_{\mathbb{C}}$$

with the property

$$\sigma(\gamma\langle z \rangle) J_{can}(\gamma, z) = \gamma\sigma(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It can be defined by the formula

$$J_{can} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) = \begin{pmatrix} a + b(z) & 0 \\ 0 & d - (a + b(z))^{-1}(c + d(z)) \otimes b \end{pmatrix}.$$

*Proof.* We have

$$\sigma(\gamma\langle z \rangle) = \begin{pmatrix} 1 & 0 \\ (a + b(z))^{-1}(c + d(z)) & \text{id} \end{pmatrix} P, \quad \gamma\sigma(z) = \begin{pmatrix} a + b(z) & b \\ c + d(z) & d \end{pmatrix} P.$$

The equation

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ (a + b(z))^{-1}(c + d(z)) & \text{id} \end{pmatrix} \begin{pmatrix} a + b(z) & b \\ 0 & d - (a + b(z))^{-1}(c + d(z)) \otimes b \end{pmatrix} \\ &= \begin{pmatrix} a + b(z) & b \\ c + d(z) & d \end{pmatrix} \end{aligned}$$

gives the second statement of Lemma 2.3. It also implies that  $J$  is an automorphy factor. □

We call  $J_{can}$  the *canonical automorphy factor*. For any representation  $\varrho$  of  $K_{\mathbb{C}}$  we then can define the automorphy factor

$$J_{\varrho}(\gamma, z) = \varrho(J_{can}(\gamma, z)).$$

If one takes for  $\varrho$  the tautological representation  $\text{id}_{K_{\mathbb{C}}}$ , one obtains back the canonical automorphy factor.

**Lemma 2.4.** *Let  $f : \mathcal{B}^* \rightarrow \mathcal{Z}$  be an automorphic form with respect to  $(\Gamma, \chi, \varrho)$ . Then  $F(z) = f(\sigma z)$  has the transformation property*

$$F(\gamma\langle z \rangle) = \chi(\gamma) J_{\varrho}(\gamma, z) F(z)$$

and every holomorphic  $F$  with this transformation property comes from an  $f$ .

*Proof.* For  $\gamma \in \Gamma$  we have

$$F(\gamma z) = f(\sigma\gamma\langle z \rangle) = f(\gamma\sigma(z)J(\gamma, z)^{-1}) = v(\gamma)\varrho(J(\gamma, z))f(z).$$

□

The *Jacobian transformation* (derivative)  $J_{Jac}(g, z)$  gives an automorphy factor of  $U(V)$  with values in  $GL(\mathcal{Z})$ . We want to relate it to the canonical automorphy factor.

**Proposition 2.5.** *Consider the representation*

$$\varrho : K_{\mathbb{C}} \longrightarrow GL(\mathcal{Z}), \quad (k_1, k_2) \longmapsto k_1^{-1}k_2.$$

(Here we consider  $k_1 \in GL(\mathbb{C}e) \cong \mathbb{C}^*$  as a complex number.) Then

$$J_{Jac}(g, z) = J_{\varrho}(g, z) \quad \text{for } g \in U(V).$$

*Proof.* We will prove this not only for  $g \in U(V)$  but for all  $g \in GL(V)$ . One has to observe that both sides can be considered for arbitrary  $g \in GL(V)$  as rational functions on  $\mathcal{B}_{\mathcal{Z}}$  with values in  $End(\mathcal{Z})$ . We verify the equality for generators of  $GL(V)$ .

1)  $g = k = (k_1, k_2) \in K_{\mathbb{C}}$ . We have  $J_{can}(k, z) = k$ . The formula  $k\langle z \rangle = k_1^{-1}k_2z$  shows

$$J_{Jac}(k, z) = k_1^{-1}k_2 = \varrho(k) = J_{\varrho}(k, z).$$

2)  $g = \begin{pmatrix} 1 & 0 \\ c & \text{id} \end{pmatrix}$ . This acts as a translation  $g\langle z \rangle = z + c$  and the Jacobian is the identity. By definition also  $J_{can}(g, z)$  is the identity.

3)  $g = \begin{pmatrix} 1 & b \\ 0 & \text{id} \end{pmatrix}$ . In this case we have

$$g\langle z \rangle = (1 + b(z))^{-1}z.$$

We have

$$J_{can}(g, z) = \begin{pmatrix} 1 + b(z) & 0 \\ 0 & \text{id} - (1 + b(z))^{-1}z \otimes b \end{pmatrix}$$

and hence

$$J_{\varrho}(g, z) = (1 + b(z))^{-1}(\text{id} - (1 + b(z))^{-2}z \otimes b).$$

It is easy to check by means of coordinates that this is the Jacobian of  $g$ . □

### 3. SOME EXAMPLES OF BALL QUOTIENTS

We consider  $V = \mathbb{C}^{n+1}$  and the hermitian form

$$\langle z, w \rangle = \bar{z}_0w_0 - \bar{z}_1w_1 - \cdots - \bar{z}_nw_n.$$

We denote by

$$\mathcal{E} := \mathbb{Z}[\zeta], \quad \zeta = e^{2\pi i/3},$$

the ring of Eisenstein integers and introduce the lattice

$$L_n = \mathcal{E}^{n+1} \subset V.$$

We denote the unitary group of  $L_n$  by  $\Gamma_n = U(L_n)$ . We also have to consider the congruence subgroup

$$\Gamma_n[a] = \text{kernel}(\Gamma_n \longrightarrow GL(n + 1, \mathcal{E}/a)) \quad (a \in \mathcal{E}).$$

The case  $a = \sqrt{-3}$  is of particular interest.



We are interested first in scalar-valued modular forms. They belong to the one-dimensional representation  $\varrho_r(k) = k_1^r$ . In this case we use the notation  $[\Gamma, \chi, r] = [\Gamma, \chi, \varrho_r]$  and we omit  $\chi$  when it is trivial. The ring of (scalar-valued modular forms) is

$$A(\Gamma) = \bigoplus_{r \in \mathbb{Z}} [\Gamma, r].$$

The structure of this ring has been determined in the four-dimensional case  $\Gamma_4[\sqrt{-3}]$  in [9] building on the paper [2]. The corresponding modular variety describes the variety of marked cubic surfaces. The ring  $A(\Gamma_4[\sqrt{-3}])$  is rather complicated and will not be considered here. But it is possible to derive from this four-dimensional case several interesting cases of lower dimension. The idea is to consider a subspace  $W \subset V$  of signature  $(1, n)$ ,  $n < 4$ , such that  $W \cap \mathcal{E}^5$  is a lattice (of rank  $n + 1$ ). The embedding

$$L_{n-1} \longrightarrow L_n, \quad a \longmapsto (a, 0),$$

gives an embedding  $\Gamma_{n-1}[\sqrt{-3}] \rightarrow \Gamma_n[\sqrt{-3}]$ . By restriction we obtain a ring homomorphism

$$A(\Gamma_n[\sqrt{-3}]) \longrightarrow A(\Gamma_{n-1}[\sqrt{-3}]).$$

A general result states that  $A(\Gamma_{n-1}[\sqrt{-3}])$  is the normalization of the image. In this way, when  $n = 3$ , one can prove the following result [10] (a different proof has been given in [15]).

**Theorem 3.1.** *The ring of modular forms  $A([\Gamma_3[\sqrt{-3}]])$  is generated by six modular forms  $T_1, \dots, T_6$  of weight three with the defining relations*

$$T_1 + \dots + T_6 = 0, \quad T_1^3 + \dots + T_6^3 = 0.$$

*The associated modular variety is a Segre cubic.*

Explicit expressions for the  $T_i$  have been given in [10, Proposition 8.5]. Unfortunately there is a sign error which we want to correct here. (This error does not influence the rest of the paper [10].) In [10, Definition 8.1], 15 Borcherds products  $B_1, \dots, B_{15}$  of weight one with respect to the congruence group  $\Gamma_3[3]$  have been introduced. The action of the group  $\Gamma_3$  on the  $B_i$  has been described there in Lemma 8.2. We give a corrected version.

The group  $\Gamma_3$  acts (from the right) on modular forms through  $(f, \gamma) \mapsto f^\gamma$ , where

$$f^\gamma(z) := f(\gamma z).$$

In [10] it has been described that, up to constant factors, the functions  $B_i$  are permuted under this action. Hence we can describe the action of an element  $g \in \Gamma_3$  by a list

$$\begin{pmatrix} \sigma_1 & \cdots & \sigma_{15} \\ \varepsilon_1 & \cdots & \varepsilon_{15} \end{pmatrix}.$$

This list has to be read as follows:

$$B_i^g = \varepsilon_{\sigma_i} B_{\sigma(i)}.$$

**Lemma 3.2.** *The transformation group corresponding to  $\Gamma_3$  on the forms  $B_i$  is generated by the following three transformations:*

$$\begin{pmatrix} 8 & 15 & 7 & 9 & 2 & 12 & 4 & 14 & 3 & 5 & 11 & 6 & 1 & 13 & 10 \\ \zeta & -\zeta & 1 & 1 & \bar{\zeta} & -\zeta & \bar{\zeta} & 1 & \zeta & -\bar{\zeta} & \bar{\zeta} & -\zeta & 1 & \bar{\zeta} & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 15 & 14 & 2 & 1 & 13 & 12 & 8 & 7 & 11 & 5 & 9 & 6 & 10 & 4 \\ \zeta & \bar{\zeta} & \bar{\zeta} & -1 & \bar{\zeta} & \zeta & 1 & -\zeta & -\bar{\zeta} & 1 & -\bar{\zeta} & 1 & 1 & -\bar{\zeta} & 1 \end{pmatrix},$$

$$\begin{pmatrix} 12 & 6 & 11 & 3 & 13 & 8 & 4 & 14 & 9 & 1 & 7 & 15 & 5 & 2 & 10 \\ -\zeta & -\zeta & -\zeta & -\zeta & \zeta & \bar{\zeta} & -1 & -1 & -1 & -\zeta & -1 & -\zeta & \bar{\zeta} & \bar{\zeta} & -\bar{\zeta} \end{pmatrix}.$$

In [10, Sect. 8], ten linear relations between the forms  $B_i^3$  have been described. By elimination, one can produce six modular forms  $T_i$  which are linked to the Segre cubic.

**Proposition 3.3.** *The assignments*

$$\begin{aligned} T_1 &\mapsto B_1^3 + B_{13}^3 - B_{15}^3, \\ T_2 &\mapsto B_1^3 - B_{13}^3 + B_{15}^3, \\ T_3 &\mapsto -B_2^3 - B_{13}^3 + B_{14}^3, \\ T_4 &\mapsto -B_2^3 + B_{13}^3 - B_{14}^3, \\ T_5 &\mapsto -B_4^3 - B_{11}^3 - B_{13}^3, \\ T_6 &\mapsto -B_6^3 + B_{10}^3 - B_{15}^3, \end{aligned}$$

define an isomorphism

$$\mathbb{C}[T_1, \dots, T_6] / \langle T_1 + \dots + T_6, T_1^3 + \dots + T_6^3 \rangle \xrightarrow{\sim} \mathbb{C}[B_1^3, \dots, B_{15}^3].$$

Moreover, the algebra  $A(\Gamma_3[\sqrt{-3}])$  is generated by the  $B_i^3$ .

Now one can reproduce Lemma 8.8 in [10]. We reformulate and extend it.

**Lemma 3.4.** *The isomorphism*

$$\mathbb{C}[T_1, \dots, T_6] / \langle T_1 + \dots + T_6, T_1^3 + \dots + T_6^3 \rangle \xrightarrow{\sim} \mathbb{C}[B_1^3, \dots, B_{15}^3]$$

is equivariant with respect to a surjective homomorphism  $\Gamma_3 \rightarrow S_6 \times \{\pm 1\}$ . Here  $S_6 \times \{\pm 1\}$  acts on the variables  $T_i$  by permutation in combination with the sign. This means that  $(\sigma, \varepsilon)$  acts by  $T_i \mapsto \varepsilon T_{\sigma(i)}$ . The three transformations in Lemma 3.2 map to the three pairs

$$(1, 6, 4, 2, 5, 3), \varepsilon = 1; (6, 5, 2, 4, 3, 1), \varepsilon = -1; (3, 2, 4, 6, 5, 1), \varepsilon = -1.$$

The kernel of  $\det^3$  is a subgroup of index two of  $\Gamma_3$  which does not contain the negative of the unit-matrix. For this subgroup  $\varepsilon$  is the sign of  $\sigma$ .

(Here  $(a_1, \dots, a_6)$  stands for the permutation  $i \mapsto a_i$ .)

We denote the Segre cubic defined in Proposition 3.3 by  $\mathcal{S}$  and by  $\mathcal{R} \subset \mathcal{S}$  the ramification locus. It can be described as follows. Let  $\gamma \in \Gamma_3[\sqrt{-3}]$  be an element of finite order which acts non-trivially on  $\mathcal{B}_3$ . By [1] it acts as a triflection on  $\mathcal{B}_3$  and its fixed point set is a so-called *short mirror*. From Definition 8.1 in [10] we can see that there is modular form of weight five on  $\Gamma_3[\sqrt{-3}]$  (but with a non-trivial multiplier system), namely  $\chi := B_1 B_8 B_{11} B_{13} B_{14}$ , whose set of zeros is the union of all short mirrors. The multiplicities are one.

**Proposition 3.5.** *The ramification locus  $\mathcal{S} \subset \mathcal{B}_3$  is the zero locus of a modular form  $\chi$  of weight five with respect to  $\Gamma_3[\sqrt{-3}]$  but with respect to a non-trivial multiplier system.*

We are interested in vector-valued modular forms with respect to the representation

$$\rho_r \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = k_1^r k_2.$$

We denote the space of modular forms by  $\mathcal{M}(r) = [\Gamma_3[\sqrt{-3}], \rho_r]$ . The direct sum

$$\mathcal{M} = \bigoplus_{r \in \mathbb{Z}} [\Gamma_3[\sqrt{-3}], \rho_r]$$

is a module over

$$A = A(\Gamma_3[\sqrt{-3}]).$$

We want to determine its structure.

#### 4. THE TANGENT BUNDLE OF THE SEGRE CUBIC

We study the following situation. Let  $P(X_0, \dots, X_n)$  be an irreducible homogeneous polynomial and let  $X \subset \mathbb{P}^n(\mathbb{C})$  be the associated hypersurface and  $X_{reg}$  its regular locus. Let  $D \subset \mathbb{C}^{n-1}$  be an open domain and let  $t_0, \dots, t_n$  be holomorphic functions on  $D$  without zeros and such that

$$D \longrightarrow X_{reg}, \quad z \longmapsto [t_0(z), \dots, t_n(z)],$$

is a holomorphic map onto  $X_{reg}$ . We want to describe the tangent space at a point  $[b] \in X_{reg}$ . The projective tangent space  $T_b X$  in  $\mathbb{P}^n(\mathbb{C})$  is defined by the equation

$$\sum_{i=0}^n (\partial_i P)(b) Y_i = 0.$$

Here  $\partial_i$  denotes the partial derivative by  $X_i$ . Since  $X$  is a hypersurface, any solution of

$$\sum_{i=0}^n C_i Y_i = 0 \quad (Y \in \text{inverse image of tangent space})$$

must be of the form

$$(C_0, \dots, C_n) = \alpha((\partial_0 P)(b), \dots, (\partial_n P)(b))$$

with a constant  $\alpha$

Now we write  $b = t(z)$ ,  $z \in D$ . The tangent space  $T_z D = \mathbb{C}^{n-1}$  maps into the space generated by the rows of

$$\begin{pmatrix} t_0(z) & \dots & t_n(z) \\ \partial_1 t_0(z) & \dots & \partial_1 t_n(z) \\ \vdots & \vdots & \vdots \\ \partial_{n-1} t_0(z) & \dots & \partial_{n-1} t_n(z) \end{pmatrix}.$$

We denote by  $G_i$ ,  $0 \leq i \leq n$ , the determinant of this matrix after cancellation of the  $i$ th column. Then we obtain

$$\det \begin{pmatrix} Y_0 & \cdots & Y_n \\ t_0(z) & \cdots & t_n(z) \\ \partial_1 t_0(z) & \cdots & \partial_1 t_n(z) \\ \vdots & \vdots & \vdots \\ \partial_{n-1} t_0(z) & \cdots & \partial_{n-1} t_n(z) \end{pmatrix} = 0$$

or

$$\sum_{i=1}^n G_i(z) Y_i = 0 \quad (Y \in \text{inverse image of tangent space}).$$

So we get

$$G_i(z) = f(z) \partial_i P(t(z)) \quad \text{where } f(z) \in \mathbb{C}.$$

It is clear that  $f(z)$  is a holomorphic function on  $D$  and that it is non-zero along the locus where the tangent map of  $D \rightarrow P^n \mathbb{C}$  is injective.

We want to apply this to the Segre cubic  $\mathcal{S}$ . Therefore we have to consider  $\mathcal{S}$  as a hypersurface in  $\mathbb{P}^4(\mathbb{C})$  (and not in  $\mathbb{P}^5(\mathbb{C})$  as in Theorem 3.1),

$$\mathcal{B}_3 \longrightarrow \mathcal{S} \subset P^4 \mathbb{C}, \quad z \longmapsto [T_1(z), \dots, T_5(z)].$$

The equation of  $\mathcal{S}$  with respect to this embedding is

$$S := T_1^3 + \cdots + T_5^3 - (T_1 + \cdots + T_5)^3.$$

We consider now the  $4 \times 5$ -matrix

$$\begin{pmatrix} T_1(z) & \cdots & T_5(z) \\ \partial_1 T_1(z) & \cdots & \partial_1 T_5(z) \\ \vdots & \vdots & \vdots \\ \partial_3 T_1(z) & \cdots & \partial_3 T_5(z) \end{pmatrix}.$$

Now  $G_i$ ,  $1 \leq i \leq 5$ , is the determinant of this matrix after cancellation of the  $i$ th column. The consideration above shows the following result.

**Lemma 4.1.** *We have*

$$G_i(z) = c \chi^2 \frac{\partial S}{\partial T_i} \quad (c \in \mathbb{C}).$$

*Proof.* We have shown above a formula  $G_i(z) = f(z)(\partial S/\partial T_i)$  with a holomorphic function  $f$  whose zero locus is inside the ramification. It is easy to check that  $f$  is a modular form. From Proposition 3.5 it follows that up to a constant factor it is a power of  $\chi$ . The exponent must be two as a weight consideration or the ramification index, studied in [10], shows. □

### 5. THE STRUCTURE THEOREM

We now can determine the structure of the  $A$ -module  $\mathcal{M} = \bigoplus_{r \in \mathbb{Z}} [\Gamma_3[\sqrt{-3}], \varrho_r]$ . Recall  $A = A(\Gamma_3[\sqrt{-3}])$ . The elements  $\{T_i, T_j\}$  can be considered as elements of  $\mathcal{M}(5)$ . We consider the submodule

$$\mathcal{N} = \sum_{ij} A\{T_i, T_j\}.$$

It is sufficient to restrict to  $1 \leq i < j \leq 5$ . Our goal is to understand the structures of  $\mathcal{M}$  and  $\mathcal{N}$ . First we determine defining relations of  $\mathcal{N}$ .

**Proposition 5.1.** *Defining relations for the module*

$$\mathcal{N} = \sum_{1 \leq i, j \leq 5} A\{T_i, T_j\}$$

are

- (1)  $T_k\{T_i, T_j\} = T_j\{T_i, T_k\} + T_i\{T_k, T_j\}, \quad \{T_i, T_j\} + \{T_j, T_i\} = 0,$
- (2)  $\sum_{\nu=1}^5 (\partial_\nu S)\{T_\nu, T_\mu\} = 0.$

We recall that

$$S := T_1^3 + \dots + T_5^3 - (T_1 + \dots + T_5)^3$$

is the equation of the Segre cubic (considered as a hypersurface in  $\mathbb{P}^4(\mathbb{C})$ ) and  $\partial_\nu S$  denotes its derivative by  $T_\nu$ .

*Proof of Proposition 5.1.* As in Section 1 we define a module

$$\mathcal{N}' = \sum_{1 \leq i, j \leq 5} A[T_i, T_j]$$

with symbols  $[T_i, T_j]$  that satisfy the relations described in the proposition. There is a natural homomorphism  $\mathcal{N}' \rightarrow \mathcal{N}$  and we have to show that this is an isomorphism. By Proposition 1.4 it is sufficient that multiplication by the variables  $T_i$  and the  $\partial_i S$  is injective. This can be done by means of a computer. □

In the following we will use the notation:

$$S_\nu := \partial_\nu S.$$

In Lemma 4.1 we proved

$$\det \begin{pmatrix} T_1 & \dots & T_4 \\ \partial_1 T_1 & \dots & \partial_1 T_4 \\ \vdots & & \vdots \\ \partial_3 T_1 & \dots & \partial_3 T_4 \end{pmatrix} = c\chi^2 S_5.$$

We obtain

$$\det \begin{pmatrix} T_1 & T_1 T_2 & \dots & T_1 T_4 \\ \partial_1 T_1 & T_1 \partial_1 T_2 & \dots & T_1 \partial_1 T_4 \\ \vdots & & & \vdots \\ \partial_3 T_1 & T_1 \partial_3 T_2 & \dots & T_1 \partial_3 T_4 \end{pmatrix} = c\chi^2 S_5 T_1^3.$$

If we multiply the first column by  $T_2$  and subtract it from the second one and so on, we obtain the following lemma.

**Lemma 5.2.** *We have*

$$\det(\{T_1, T_2\}, \{T_1, T_3\}, \{T_1, T_4\}) = c\chi^2 S_5 T_1^2.$$

Since the determinant is different from 0, every element of  $\mathcal{M}$  can be written in the form

$$g_1\{T_1, T_2\} + g_2\{T_1, T_3\} + g_3\{T_1, T_4\}$$

with meromorphic functions  $g_i$ . It is easy to check that these are meromorphic modular forms. In particular, they have trivial multipliers. From Lemma 5.2 we

get that the product  $h_i = g_i \chi^2 S_5 T_1^2$  is holomorphic. The multipliers of  $\chi$  are non-trivial on the triflections. They are third roots of unity. Hence  $h_i/\chi$  is holomorphic and, applying the same argument,  $h_i/\chi^2$  is holomorphic. We have shown that

$$\mathcal{M} \subset \frac{1}{T_1^2 S_5} \mathcal{N}.$$

During the proof we selected 1 and 5 from  $\{1, \dots, 5\}$ . Since we could have chosen other indices we obtain the following proposition.

**Proposition 5.3.** *We have*

$$\mathcal{M} = \bigcap_{1 \leq i \neq j \leq 5} \frac{1}{T_i^2 S_j} \mathcal{N}.$$

*Proof.* Since the elements on the right hand side are holomorphic, they must belong to  $\mathcal{M}$ . □

We know generators and defining relations of  $\mathcal{N}$ , thus the following lemma can be proved by means of SINGULAR.

**Lemma 5.4.** *For arbitrary  $1 \leq i < j \leq 5$  one has*

$$\mathcal{N} = \frac{1}{S_i} \mathcal{N} \cap \frac{1}{S_j} \mathcal{N} \quad \text{and} \quad \mathcal{N} = \frac{1}{T_i^2} \mathcal{N} \cap \frac{1}{T_j^2} \mathcal{N}.$$

Putting together Proposition 5.3 and Lemma 5.4 we obtain our main result.

**Theorem 5.5.** *The module  $\mathcal{M}$  is generated by the brackets  $\{T_i, T_j\}$ . Its Hilbert function is*

$$\frac{-t^{14} + 10t^5}{(t^3 - 1)^4} = 10t^5 + 40t^8 + 100t^{11} + 199t^{14} + 346t^{17} + 550t^{20} + \dots$$

*Recall that the  $T_i$  have degree three and the  $\{T_i, T_j\}$  are counted with degree five.*

### 6. THE IGUSA QUARTIC

The method which we used works in several other cases [11, 12]. We have been asked whether the method works also in the case of the Siegel modular group of genus two and level two. This case has been treated in the paper [7]. It turns out that the method works perfectly also in this case.

In the Siegel case the ring  $A$  has to be replaced by the ring of Siegel modular forms of even weight and trivial character. It is generated by five forms  $T_1, \dots, T_5$  of weight two with the defining relation

$$S = (T_1 T_2 + T_1 T_3 + T_2 T_3 - T_4 T_5)^2 - 4T_1 T_2 T_3 (T_1 + T_2 + T_3 + T_4 + T_5).$$

This is the defining equation of the Igusa quartic which is the dual variety of the Segre cubic. Hence one might expect similar results. In analogy to the ball case, the  $A$ -module  $\mathcal{M} = \bigoplus_{r \in \mathbb{Z}} \mathcal{M}(r)$  can be defined, where  $\mathcal{M}(r)$  now consists of the modular forms of transformation type

$$f(MZ) = \det(CZ + D)^{2+2r} (CZ + D) f(Z) (CZ + D)'.$$

The module  $\mathcal{M}(4)$  contains the brackets  $\{f, g\} = g^2(gdf - fdg)$  where  $f, g \in A(2)$  are modular forms of weight two. As in the ball case we can consider the submodule  $\mathcal{N} \subset \mathcal{M}$  that is generated by the  $\{T_i, T_j\}$ . The formula of Proposition 5.3

$$\mathcal{M} = \bigcap_{1 \leq i \neq j \leq 5} \frac{1}{T_i^2 S_j} \mathcal{N}$$

is correct also in the Siegel case. But now there is a big difference. Lemma 5.4 turns out to be false in the Siegel case. But it is not a problem to compute with the help of SINGULAR the intersection that gives  $\mathcal{M}$ . In this way we can reproduce results of [7].

**Theorem 6.1.** *The Hilbert function of  $\mathcal{M}$  is*

$$\frac{t^8 - 4t^6 + 15t^4}{t^8 - 4t^6 + 6t^4 - 4t^2 + 1} = 15t^4 + 56t^6 + 135t^8 + 264t^{10} + 455t^{12} + \dots$$

*The Hilbert function of  $\mathcal{N}$  is*

$$\frac{-t^{14} - t^{12} - t^{10} + 5t^8 + 10t^4}{t^8 - 4t^6 + 6t^4 - 4t^2 + 1} = 10t^4 + 40t^6 + 105t^8 + 219t^{10} + 395t^{12} + \dots$$

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MATHEMATISCHES INSTITUT, IM NEUENHEIMER FELD 288, UNIVERSITÄT HEIDELBERG, D69120  
HEIDELBERG, GERMANY  
*Email address:* `freitag@mathi.uni-heidelberg.de`

DIPARTIMENTO DI MATEMATICA, PIAZZALE ALDO MORO, 2, UNIVERSITÀ SAPIENZA, I-00185  
ROMA, ITALY  
*Email address:* `salvati@mat.uniroma1.it`