

ON LEWY EXTENSION FOR SMOOTH HYPERSURFACES IN $\mathbb{C}^n \times \mathbb{R}$

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ABSTRACT. We prove an analogue of the Lewy extension theorem for a real dimension $2n$ smooth submanifold $M \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$. A theorem of Hill and Taiani implies that if M is CR and the Levi-form has a positive eigenvalue restricted to the leaves of $\mathbb{C}^n \times \mathbb{R}$, then every smooth CR function f extends smoothly as a CR function to one side of M . If the Levi-form has eigenvalues of both signs, then f extends to a neighborhood of M . Our main result concerns CR singular manifolds with a nondegenerate quadratic part Q . A smooth CR f extends to one side if the Hermitian part of Q has at least two positive eigenvalues, and f extends to the other side if the form has at least two negative eigenvalues. We provide examples to show that at least two nonzero eigenvalues in the direction of the extension are needed.

1. INTRODUCTION

Let $M \subset \mathbb{C}^n \times \mathbb{R}$ be a smooth real hypersurface. A function $F: \mathbb{C}^n \times \mathbb{R} \rightarrow \mathbb{C}$ is a CR function if it is holomorphic in the first n variables. A natural question is: When does a smooth function $f: M \rightarrow \mathbb{C}$ extend to a smooth CR function on a neighborhood of M in $\mathbb{C}^n \times \mathbb{R}$, or at least to one side of M ? The question is the classical CR extension, but with a real parameter.

Let $(z, s) = (z_1, \dots, z_n, s)$ be the coordinates on $\mathbb{C}^n \times \mathbb{R}$. Define the standard CR structure on M , the Cauchy–Riemann equations restricted to M , as if it were embedded in \mathbb{C}^{n+1} . At each point $p \in M$, define

$$(1) \quad T_p^{(0,1)}M = \mathbb{C} \otimes T_pM \cap \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\}.$$

Generically, we expect $\dim T_p^{(0,1)}M = n - 1$, but $\dim T_p^{(0,1)}M = n$ is possible. If $\dim T_p^{(0,1)}M = n$ for all p , then M is a complex submanifold by the Newlander–Nirenberg theorem, and so it is locally equal to $\mathbb{C}^n \times \{s_0\}$ for some s_0 , and f extends to both sides of M if and only if f is holomorphic on M . Thus, assume that at least somewhere $\dim T_p^{(0,1)}M = n - 1$. The dimension $\dim T_p^{(0,1)}M$ is called the *CR dimension* of M at p .

The points where $\dim T_p^{(0,1)}M = n - 1$ are called CR points of M , and the points where $\dim T_p^{(0,1)}M = n$ are the so-called CR singularities. Write M_{CR} for the set of CR points of M . We need a definition of a CR function on a possibly CR singular submanifold, and we take the definition in the weakest possible sense: *A function $f: M \rightarrow \mathbb{C}$ is CR, if $vf = 0$ for all CR vector fields on M_{CR} .* Alternatively, we

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obtain the same definition if we allow v to be smooth vector fields on M such that $v_p \in T_p^{(0,1)}M$ for all p . Such vector fields in general vanish at the CR singularities. There are other possible definitions of a CR function on a CR singular manifold, but they imply the definition above. The condition that f be CR is clearly a necessary condition for f to extend to even one side of M in $\mathbb{C}^n \times \mathbb{R}$. We prove that under certain nondegeneracy conditions it is sufficient.

Notice $M \subset \mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$. When M and f are real-analytic and both CR, the function f extends holomorphically to a neighborhood of M in \mathbb{C}^{n+1} as a holomorphic function, via the classical theorem of Severi, and hence real-analytically as a CR function of a neighborhood of M in $\mathbb{C}^n \times \mathbb{R}$. If M and f are only smooth and CR, such an extension need not hold. For $M \subset \mathbb{C}^{n+1}$ not of infinite type, then f would extend into wedges, see for example Tumanov [29]. But the M we are considering is of infinite type, and f does not in general extend into any open subset of \mathbb{C}^{n+1} . The general Lewy extension in the CR case was solved by Hill and Taiani [13]: A CR function extends to a higher dimensional submanifold extending in a direction of nonzero eigenvalues of the Levi-form.

For every fixed s , and any set $X \subset \mathbb{C}^n \times \mathbb{R}$, define

$$(2) \quad (X)_{\{s\}} := \{z \in \mathbb{C}^n \mid (z, s) \in X\}.$$

For a fixed s , if $n \geq 2$, and the manifold $(M)_{\{s\}} \subset \mathbb{C}^n \times \{s\}$ is not Levi-flat (the Levi-form has at least one nonzero eigenvalue), then we obtain a local extension of f to at least one side of $(M)_{\{s\}}$ in $\mathbb{C}^n \times \{s\}$. The key is to tie these extensions together. In our setup, the result of Hill and Taiani [13] implies the following corollary. We state the theorem formally so that the reader can compare the CR result with our CR singular result below, Theorem 1.2, which is the main result of the paper.

Theorem 1.1 (Special case of Hill–Taiani). *Let $M \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a real smooth CR submanifold of real dimension $2n$ (a hypersurface) and of CR dimension $n-1$. Let $p = (z_0, s_0) \in M$ be a point. Then there exists a neighborhood $U \subset \mathbb{C}^n \times \mathbb{R}$ of p , such that given a smooth CR function $f: M \rightarrow \mathbb{C}$, we have:*

- (i) *If at z_0 , $(M)_{\{s_0\}}$ is a hypersurface whose Levi-form has at least one positive eigenvalue, then the side of M in U corresponding to the positive eigenvalue of the Levi-form of $(M)_{\{s_0\}}$ is a submanifold with boundary H_+ , where $\partial H_+ = M \cap U$, and there exists a smooth function $F: H_+ \rightarrow \mathbb{C}$ that is CR in $H_+ \setminus M$ and $f|_{M \cap U} = F|_{M \cap U}$.*
- (ii) *If at z_0 , $(M)_{\{s_0\}}$ is a hypersurface whose Levi-form has eigenvalues of both signs, then there exists a smooth CR function $F: U \rightarrow \mathbb{C}$, such that $f|_{M \cap U} = F|_{M \cap U}$.*

We provide a separate sketch of a proof in our notation as some of the ideas are reused in the CR singular case, and to make the paper more self contained. The proof follows by modification of the proof from [22], where a similar, but global, theorem was proved for $n \geq 1$; the case $n = 1$ requires an extra hypothesis.

Let us move to CR singularities. A real codimension two submanifold $M \subset \mathbb{C}^{n+1}$ has generically isolated CR singularities. Such manifolds were first studied in \mathbb{C}^2 by E. Bishop [1]. Especially in the elliptic case (when a family of discs attaches to M),

the work of Bishop was extended by Moser–Webster [26], Moser [25], Kenig–Webster [20], Gong [9], Huang–Krantz [15], Huang–Yin [16], Slapar [28], and many others.

For $n > 1$, the work mostly addressed normal form, see Huang–Yin [17–19], Gong–Lebl [10], Coffman [5], Burcea [2, 3]. In particular, it is not always possible to change variables to realize the manifold as a submanifold of $\mathbb{C}^n \times \mathbb{R}$, a so-called *flattening*, see Dolbeault–Tomassini–Zaitsev [6, 7], Huang–Yin [18, 19], and Fang–Huang [8]. If $M \subset \mathbb{C}^{n+1}$ is not *flattenable*, then even in the real-analytic case, an extension of CR functions does not in general exist near CR singularities. See Harris [12] and Lebl–Minor–Shroff–Son–Zhang [21].

The authors considered the extension of CR functions for hypersurfaces of $\mathbb{C}^n \times \mathbb{R}$ in the elliptic case [22], and in the general nondegenerate real-analytic case [23]. In this paper we address what happens in the nondegenerate smooth case. Via the results of [23], a formal extension always exists, but a smooth extension does not exist in all cases.

Locally, after a complex affine change of variables fixing $\mathbb{C}^n \times \mathbb{R}$, a smooth CR singular $M \subset \mathbb{C}^n \times \mathbb{R}$ is given by

$$(3) \quad M : \quad s = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} + E(z, \bar{z}) = Q(z, \bar{z}) + E(z, \bar{z}),$$

where A is a Hermitian (sesquilinear) form, B is a bilinear form, Q is the real quadratic form given by A and B , and E is a smooth real-valued function in $O(3)$. We require that Q is nondegenerate; that is, Q is represented by a nonsingular symmetric $2n \times 2n$ matrix.

Let

$$(4) \quad M^{quad} : \quad s = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} = Q(z, \bar{z})$$

be the quadric model of M . The manifold M^{quad} (and therefore M) has an isolated CR singularity because Q is nondegenerate (see Proposition 2.1). Suppose A has a positive and b negative eigenvalues. Diagonalize A , and write

$$(5) \quad M : \quad s = \sum_{j=1}^a |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z, z) + \overline{B(z, z)} + E(z, \bar{z}),$$

where $a = 0$ or $b = 0$ is interpreted appropriately. Unless A is positive definite we cannot always simultaneously diagonalize B . If the number a is normalized to $a \geq b$, then it is an invariant. To simplify the statement of the theorem, we do not make this requirement.

Define the manifold with boundary

$$(6) \quad H_+ : \quad s \geq \sum_{j=1}^a |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z, z) + \overline{B(z, z)} + E(z, \bar{z}).$$

The form A carries the Levi-form of the model manifold $(M^{quad})_{\{s\}}$; that is, A restricted to $T_p^{(0,1)}(M^{quad})_{\{s\}}$ is the Levi-form of $(M^{quad})_{\{s\}}$. Thus, a small perturbation of A gives the Levi-form of $(M)_{\{s\}}$. If $a \geq 2$, then the Levi-form of $(M)_{\{s\}}$ has at least one positive eigenvalue in the direction of H_+ . Therefore it is natural to expect an extension of f to H_+ , if A has at least two positive eigenvalues. We now state our main result.

Theorem 1.2. *Suppose M and H_+ are defined near the origin by (5) and (6), and suppose $n \geq 2$ and Q is nondegenerate. Then there exists a neighborhood U of the origin, such that given a smooth CR function $f: M \rightarrow \mathbb{C}$, we have:*

- (i) *If A has at least two positive eigenvalues ($a \geq 2$), then there exists a function $F \in C^\infty(H_+ \cap U)$ such that F is CR on $(H_+ \setminus M) \cap U$ and $F|_{M \cap U} = f|_{M \cap U}$.*
- (ii) *If A has at least two positive eigenvalues ($a \geq 2$) and at least two negative eigenvalues ($b \geq 2$), then there exists a CR function $F \in C^\infty(U)$ such that $F|_{M \cap U} = f|_{M \cap U}$.*

In either case, F has a formal power series in z and s at 0.

By changing s to $-s$, we change the roles of a and b . So if A has two negative eigenvalues, then f extends to the opposite side of M (below M). In particular, the first item immediately implies the second.

Some nondegeneracy is necessary. As in [22], $M \subset \mathbb{C}^n \times \mathbb{R}$ given by $s = \|z\|^4$, and $f: M \rightarrow \mathbb{C}$ given by $\|z\|^2 = \sqrt{s}$ is a counterexample. For this M , $(M)_{\{s\}}$ is strictly pseudoconvex for $s > 0$, and a continuous extension F exists, but the extension is not differentiable at the origin.

The conditions on the number of eigenvalues are necessary. In section 2, we show that if M is given by $s = |z_1|^2 - |z_2|^2$, then there exists a smooth CR function that extends to neither side. In this case, both $a = 1$ and $b = 1$. Furthermore, if M is given by $s = |z_1|^2 + |z_2|^2 - |z_3|^2$, then there exists a smooth CR function that extends only to H_+ and not to the opposite side. In section 8, we also provide an example where M is defined by $s = |z_1|^2 + |z_2|^2$, and there is a smooth CR function on M that extends to one side only near every point.

It is not clear if the isolated CR singularity of M and M^{quad} is required for the result, but our technique breaks down in several key steps without this requirement, notably in the application of Malgrange's theorem. The results of [23] show that a formal extension exists even when the CR singularity is not isolated (as long as A is nondegenerate).

We mostly discuss $n \geq 2$. When $n = 1$, the CR condition is vacuous, and every smooth function is a CR function. An extra condition is necessary for extension, and under an extra condition the authors proved an extension in the elliptic case in [22].

A natural question is whether a version of the well-known Baouendi-Trèves approximation theorem holds in the present setting. Example 2.4 shows that there is no straightforward analogue.

Let us outline the organization of the paper. We prove some preliminaries and give several examples in section 2 to justify the hypotheses of the theorems. In section 3 we sketch the proof of the extension at CR points. We construct certain affine analytic discs in section 4 that are useful to study the extension to $H_+ \setminus M$. Section 5 is devoted to showing the existence and regularity of the extension in $H_+ \setminus M$ near the CR singularity. The affine discs from section 4 are used to show a priori interior regularity of the extension and to show the existence of the extension for $s \leq 0$. Additionally we also use natural complex manifolds with boundary attached to M to show the existence of the extension for $s > 0$. In section 6 we use the results of [23] to show that a formal extension exists at CR singularities. We prove that the extension is regular up to the boundary in section 7. Finally, in section 8 we provide further relevant examples.

2. PRELIMINARIES AND EXAMPLES

The sesquilinear quadratic form $A(z, \bar{z})$ is represented by a Hermitian matrix, which we also call A , as $\bar{z}^t Az$, thinking of z as a column vector. Similarly, the bilinear form $B(z, z)$ is represented by a matrix B as $z^t Bz$. The matrix B is not unique, but it can be made unique by requiring that B is symmetric, or upper (or lower) triangular. Let us suppose that it is symmetric.

A linear transformation T in the z variables transforms the matrices A and B as T^*AT (where $*$ is the conjugate transpose) and T^tBT . Via Sylvester’s law of inertia, we change A into a diagonal matrix with only 1s, 0s, and -1 s on the diagonal. Both matrices cannot, in general, be diagonalized by the same T . However, it is a standard result in linear algebra that if A is positive definite, then the symmetric B can be diagonalized such that the diagonal elements are nonnegative numbers. See, e.g., [14, Theorem 7.6.5]. In other words, if A is positive definite, then M^{quad} can be put into the form

$$(7) \quad M^{quad} : \quad s = \|z\|^2 + \sum_{j=1}^n \lambda_j (z_j^2 + \bar{z}_j^2) = \sum_{j=1}^n (|z_j|^2 + \lambda_j (z_j^2 + \bar{z}_j^2)).$$

We start by showing that having an isolated CR singularity of M^{quad} is a generic condition. When A is positive definite, it is not difficult to see this by direct computation. In this case, M^{quad} has an isolated CR singularity if and only if $\lambda_j \neq \frac{1}{2}$ for all j , what is normally called “nonparabolic”. The general case is similar.

Proposition 2.1. *Let a quadric model M^{quad} be given by*

$$(8) \quad s = \sum_{j=1}^a |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z, z) + \overline{B(z, z)} = Q(z, \bar{z})$$

for a bilinear form B . Then, M^{quad} has an isolated CR singularity if and only if the real quadratic form $Q(z, \bar{z})$ is nondegenerate. In particular, the set of B for which M^{quad} has an isolated CR singularity is an open dense set. It is the complement of a proper real-algebraic subvariety of the set of all symmetric $n \times n$ matrices giving $B(z, z)$.

If M is a submanifold with quadric model M^{quad} such that Q is nondegenerate, then M also has an isolated CR singularity at the origin.

Proof. Suppose $s = Q(z, \bar{z})$ is the defining equation for M^{quad} . The real quadratic form Q is given by a real $2n \times 2n$ symmetric matrix. The set of CR singularities is the set where the z -plane is tangent to M^{quad} , and therefore, where the derivative of Q vanishes. The derivative of Q vanishes outside the origin if and only if the underlying real matrix is not of full rank. The conclusion about B follows at once. The final conclusion about M also follows at once; the z -plane M cannot be tangent to M unless all the z derivatives of $Q(z, \bar{z}) + E(z, \bar{z})$ vanish, and this is not possible in some punctured neighborhood of the origin if Q is nondegenerate. \square

Example 2.2. It is possible for a degenerate M to have an isolated CR singularity. The introduction provides the example $s = \|z\|^4$, which is degenerate in the sense that both A and Q are degenerate, but the CR singularity is the origin alone.

On the other hand, if A is nondegenerate, it is still possible for Q to be degenerate, and for the CR singularity to be large. For example for the “completely

parabolic” quadric manifold (all $\lambda_j = \frac{1}{2}$), the CR singularity is a totally real n -real-dimensional linear submanifold.

Let us show via examples that the conditions of Theorems 1.1 and 1.2 are necessary.

Example 2.3. *In the CR case, at least one nonzero eigenvalue is necessary.*

Let $(z, s) \in \mathbb{C} \times \mathbb{R}$, and M be given by $\text{Im } z = 0$. Define a smooth CR function $f: M \rightarrow \mathbb{C}$ by

$$(9) \quad f(z, s) = \begin{cases} \frac{e^{-1/s^2}}{z+is} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

Any extension must satisfy the same formula near M , simply plugging in complex z , being the unique extension along $(M)_{\{s\}}$. It is clear that f extends to neither side of M in $\mathbb{C} \times \mathbb{R}$. In this example $n = 1$, where clearly Hill–Taiani does not apply because there is no Levi-form. It is immediate that when $n > 1$ and the Levi-form is zero (Levi-flat), the extension does not occur.

Similarly, let $M \subset \mathbb{C}^2 \times \mathbb{R}$ and f be given by

$$(10) \quad \text{Im } z_1 = s |z_2|^2, \quad f(z, s) = \begin{cases} \frac{e^{-1/s^2}}{z_1+is} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

We obtain an example where M is not Levi-flat, the Levi-form is only zero when $s = 0$, and the extension to neither side is possible near the origin.

Example 2.4. *In the CR singular case, at least two eigenvalues of the same sign are necessary to guarantee an extension to some side.*

Let $M \subset \mathbb{C}^2 \times \mathbb{R}$ be given by

$$(11) \quad s = |z_1|^2 - |z_2|^2.$$

Let $f: M \rightarrow \mathbb{C}$ be given by

$$(12) \quad f(z, s) = \begin{cases} \frac{1}{z_1} e^{-1/s^2} & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ \frac{1}{z_2} e^{-1/s^2} & \text{if } s < 0. \end{cases}$$

Let us show that f is smooth. If $s > 0$, then clearly z_1 is not zero on M . Similarly if $s < 0$, then z_2 is not zero on M . Therefore, f is smooth when $s \neq 0$. By symmetry it is enough to show that f is smooth if $s \geq 0$ as we approach $s = 0$ from above. Suppose $s > 0$. Rewrite f in terms of z and \bar{z} ;

$$(13) \quad f(z, |z_1|^2 - |z_2|^2) = \frac{1}{z_1} e^{-1/(|z_1|^2 - |z_2|^2)^2}.$$

The derivatives of f are going to be of the form

$$(14) \quad \frac{P(z, \bar{z})}{z_1^d (|z_1|^2 - |z_2|^2)^k} e^{-1/(|z_1|^2 - |z_2|^2)^2}$$

for a polynomial P . As $s > 0$, and so $|z_1|^2 > |z_2|^2$, we bound

$$(15) \quad \left| \frac{1}{z_1} \right| \leq \frac{1}{|z_1|^2 - |z_2|^2}.$$

So all derivatives are bounded on the set where $s > 0$, as s approaches 0. Therefore f is smooth. Also f is CR as we wrote it as a function of z and s and not \bar{z} .

On the other hand, f cannot extend holomorphically into the set where $s > |z_1|^2 - |z_2|^2$, because there is a pole arbitrarily close to the origin (where $z_1 = 0$). Similarly f cannot extend holomorphically into the set $s < |z_1|^2 - |z_2|^2$. Hence there is no neighborhood of the origin and no CR function (of any regularity) on either side of M in $\mathbb{C}^2 \times \mathbb{R}$ inside this neighborhood that extends f .

This example shows that a straightforward analogue of the Baouendi–Trèves approximation theorem does not hold. In particular, the above CR function f cannot be a uniform limit of polynomials in z and s in some compact neighborhood $K \subset\subset M$ of the origin. Every point in $\mathbb{C}^2 \times \mathbb{R}$ lies on an analytic disc attached to M ; to see this, simply fix z_2 and s , and the intersection of the resulting line with M will give such a disc. Therefore, if f were such a limit, then it would extend to a neighborhood of the origin in $\mathbb{C}^2 \times \mathbb{R}$.

Example 2.5. *In the CR singular case, we need at least two eigenvalues of both signs to guarantee extension to both sides.*

Let $M \subset \mathbb{C}^3 \times \mathbb{R}$ be given by

$$(16) \quad s = |z_1|^2 + |z_2|^2 - |z_3|^2.$$

Let $f: M \rightarrow \mathbb{C}$ be given by

$$(17) \quad f(z, s) = \begin{cases} 0 & \text{if } s \geq 0, \\ \frac{1}{z_3} e^{-1/s^2} & \text{if } s < 0. \end{cases}$$

Let us show that f is smooth. If $s < 0$, then clearly z_3 is not zero on M . Therefore, f is smooth when $s \neq 0$. The computation is similar as in the previous example. Suppose $s < 0$. Rewrite f in terms of z and \bar{z} ;

$$(18) \quad f(z, |z_1|^2 + |z_2|^2 - |z_3|^2) = \frac{1}{z_3} e^{-1/(|z_1|^2 + |z_2|^2 - |z_3|^2)^2}.$$

The derivatives of f are going to be of the form

$$(19) \quad \frac{P(z, \bar{z})}{z_3^d (|z_1|^2 + |z_2|^2 - |z_3|^2)^k} e^{-1/(|z_1|^2 + |z_2|^2 - |z_3|^2)^2}$$

for a polynomial P . As $s < 0$, then $|z_1|^2 + |z_2|^2 > |z_3|^2$. We bound

$$(20) \quad \left| \frac{1}{z_3} \right| \leq \frac{1}{|z_3|^2 - (|z_1|^2 + |z_2|^2)}.$$

So all derivatives are bounded on the set where $s < 0$, as s approaches 0. Therefore f is smooth. Again f is CR as we wrote it as a function of z and s , and not \bar{z} .

As with the previous example, the function cannot extend to the set $s < |z_1|^2 + |z_2|^2 - |z_3|^2$, because there is a pole arbitrarily close to the origin (where $z_3 = 0$). By Theorem 1.2, it does extend to the other side smoothly.

Example 2.6. *Interior regularity may fail for a degenerate submanifold with a large CR singularity.*

Write $z = (z', z'')$, and define M by

$$(21) \quad s = (\|z'\|^2 - \|z''\|^2)^3.$$

Let $f: M \rightarrow \mathbb{C}$ be given by $f(z, s) = \sqrt[3]{s}$. This function is CR, is smooth on M , is in fact real-analytic, and extends as a continuous CR function to a neighborhood of the origin. But the extension is not differentiable when $s = 0$.

3. EXTENSION AT CR POINTS

In this section we provide a sketch of the proof of the first part of Theorem 1.1, from which the second part immediately follows. Suppose $p = (z_0, s_0) \in M$ is such that the Levi-form of the hypersurface $(M)_{\{s_0\}} \subset \mathbb{C}^n$ has at least one positive eigenvalue at p . There is a neighborhood U of p , where the Levi-form of $(M)_{\{s\}}$ has at least one positive eigenvalue at $(z, s) \in U$ corresponding to the same side H_+ .

Let $I = \{s \in \mathbb{R} \mid (z, s) \in H_+ \cap U \text{ for some } z \in \mathbb{C}^n\}$; U can be made small enough so that for each $s \in I$, every point of $(U \cap H_+ \setminus M)_{\{s\}}$ is contained in an analytic disc whose boundary lies on M . Using the Cauchy integral formula on these analytic discs, a smooth CR function $f: M \cap U \rightarrow \mathbb{C}$ extends to a function $F: H_+ \cap U \rightarrow \mathbb{C}$ that is holomorphic and smooth up to the boundary on each leaf $(H_+)_{\{s\}}$. This is Lewy's extension theorem. Furthermore, the analytic discs above can be chosen to vary smoothly with respect to s , showing that $F \in C^\infty((H_+ \setminus M) \cap U)$.

It only remains to be shown that F is smooth up to M in $H_+ \cap U$. We do this by an approach that is similar to but simpler than that employed in Lemma 4.2 from [22]. We present an outline below that incorporates the simplifications and the necessary modifications.

Claim 3.1. F is continuous $H_+ \cap U$.

Proof. It suffices to check for continuity up to M in $H_+ \cap U$. For $(z_0, s_0) \in M \cap U$ and $(z, s) \in (H_+ \setminus M) \cap U$,

$$(22) \quad |F(z_0, s_0) - F(z, s)| \leq |F(z_0, s_0) - F(z, s_0)| + |F(z, s_0) - F(z, s)|.$$

Since $F \in C^\infty((H_+ \setminus M) \cap U)$, it is enough to show that the first term can be made small. For z close to z_0 there is an analytic disc Δ passing through z and attached to $M \cap V$ for a neighborhood $V \subset U$ of (z_0, s_0) . Considering the smooth CR function $f - f(z_0, s_0)$ and using the maximum modulus principle, we have

$$(23) \quad |F(z, s_0) - F(z_0, s_0)| \leq \sup_{(\zeta, s_0) \in \partial\Delta} |f(\zeta, s_0) - f(z_0, s_0)|.$$

The conclusion follows since f is smooth on M . \square

Claim 3.2. The function F_s extends continuously to $H_+ \cap U$ and the restrictions $F_{z_k}|_{M \cap U}$ and $F_s|_{M \cap U}$ are smooth.

Proof. Let $(z_0, s_0) \in M \cap U$. Without loss of generality we may suppose that $\rho_{z_1} \neq 0$ in a neighborhood $V \subset U$ of (z_0, s_0) . Since $M \cap V$ is CR, there are smooth vector fields X_1, \dots, X_{n-1} that span $T^{(1,0)}(M \cap V)$. The only missing tangential direction, also called the bad tangent direction, is given by

$$(24) \quad X = \frac{\partial}{\partial z_1} - \left(\frac{\rho_{z_1}}{\rho_{\bar{z}_1}} \right) \frac{\partial}{\partial \bar{z}_1}.$$

Since F is holomorphic along each leaf, $F_{z_1}|_{M \cap V} = XF|_{M \cap V} = Xf$ is smooth. Also, $X_k F|_{M \cap V} = X_k f$, for $1 \leq k \leq n-1$, are smooth as well. So, F_{z_1}, \dots, F_{z_n} are smooth on $M \cap V$ since $\{\partial/\partial z_1, X, X_1, \dots, X_{n-1}, \bar{X}_1, \dots, \bar{X}_{n-1}\}$ is a smooth coordinate frame on each leaf in V .

To see that F_s is smooth on M , consider the vector field

$$(25) \quad Y = \frac{\partial}{\partial s} - \left(\frac{\rho_s}{\rho_{\bar{z}_1}} \right) \frac{\partial}{\partial \bar{z}_1}$$

on V . Let X be any smooth CR vector field on $M \cap V$. Then, $Xf \equiv 0$ because f is a CR function. Since X does not have a $\frac{\partial}{\partial s}$ component, neither does $[X, Y]$. That is, $[X, Y]$ differentiates along the leaves and contains only \bar{z} derivatives. Consequently, $X(Yf) = [X, Y]f = 0$ since F is holomorphic and smooth up to M in each leaf. So, Yf is a smooth CR function on $M \cap V$, and hence it extends to a continuous CR function G on $H_+ \cap U$ that is smooth inside, holomorphic on each leaf, and smooth up to M on each leaf. Since $F_s = G$ on $(H_+ \setminus M) \cap V$, F_s extends continuously to M and is smooth on M . \square

The earlier claim shows that F_{z_k} and F_s are continuous on $H_+ \cap U$ since they are smooth CR functions on $M \cap U$. That is, $F \in C^1(H_+ \cap U)$. Repeating this procedure, we find that F is smooth on $H_+ \cap U$.

4. AFFINE ANALYTIC DISCS

First we need a small proposition that is essentially a version of Thom’s transversality theorem for affine discs and real hypersurfaces.

Proposition 4.1. *Suppose $\Omega \subset \mathbb{C}^n$, $n \geq 2$, is a domain with smooth boundary and $L_\alpha(\xi) = \alpha\xi + \beta$ an affine map, where $\alpha \in \mathbb{C}^n$, $\beta \in \Omega$, and $\xi \in \mathbb{C}$. Denote by D_α , the connected component of $L_\alpha(\mathbb{C}) \cap \Omega$ containing β . Suppose D_α is bounded.*

Then given any $\epsilon > 0$ and any neighborhood U of the closure $\overline{D_\alpha}$, there exists an $\tilde{\alpha} \in \mathbb{C}^n$ with $\|\alpha - \tilde{\alpha}\| < \epsilon$, and $\alpha_k = \tilde{\alpha}_k$ for $k = 3, \dots, n$, such that $\overline{D_{\tilde{\alpha}}} \subset \subset U$, and $L_{\tilde{\alpha}}(\mathbb{C})$ intersects $\partial\Omega$ transversally.

Proof. Without loss of generality, by restricting to a two-dimensional subspace we assume $n = 2$ for simplicity, and we also assume that $\alpha \in S^3 \subset \mathbb{C}^2$. Write $L(\xi, \alpha) = L_\alpha(\xi) = \alpha\xi + \beta$, and treat $L: \mathbb{C} \times S^3 \rightarrow \mathbb{C}^2$ as a mapping of manifolds. Clearly, $L(\mathbb{C} \times S^3) = \mathbb{C}^2$, and hence L is transverse to $\partial\Omega \subset \mathbb{C}^2$. By Thom’s transversality theorem, then L_α is transverse to $\partial\Omega$ for almost all α . The compactness follows at once for $\tilde{\alpha}$ close enough to α . \square

In the next two propositions we study the topology of $(H_+)_{\{s\}}$, which is needed because we construct the extension on leaves $(H_+)_{\{s\}}$ separately.

Proposition 4.2. *Let C be an $m \times m$ symmetric nondegenerate real matrix, with k positive and ℓ negative eigenvalues ($m = k + \ell$). Given $s \in \mathbb{R}$, consider the manifold with boundary*

$$(26) \quad X_s = \{x \in \mathbb{R}^m \mid s \geq x^t C x\}.$$

Suppose that $k \geq 2$.

- (i) *If either $s > 0$, or $s < 0$ but $\ell \geq 3$, then X_s is connected and also simply connected, and ∂X_s is connected.*
- (ii) *If $s < 0$, but $\ell = 2$, then X_s and ∂X_s are connected, $\pi_1(X_s) = \mathbb{Z}$, and the generator $\gamma: S^1 \rightarrow X_s$ lies in ∂X_s . In fact, after the change of coordinates (27) it is the set given by $x_1 = \dots = x_k = 0$, $x_{m-1}^2 + x_m^2 = -s$.*
- (iii) *If $s < 0$ and $\ell = 1$, then X_s has exactly two simply connected components, each of which has connected boundary.*

Proof. Sylvester's law of inertia shows that without loss of generality we may suppose

$$(27) \quad x^t C x = x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_m^2.$$

Therefore, write

$$(28) \quad X_s = \{x \in \mathbb{R}^m \mid x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_m^2 \leq s\},$$

$$(29) \quad Y_s = \partial X_s = \{x \in \mathbb{R}^m \mid x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_m^2 = s\}.$$

If $\ell = 0$, then the set X_s is a ball, in which case the proposition is trivial, so for the rest of the proof suppose $\ell \geq 1$.

Via a rotation we continuously deform x so that $x_1 > 0$, $x_2 = \cdots = x_k = 0$, and $x_{k+1} \neq 0$, $x_{k+2} = \cdots = x_m = 0$, while still staying in Y_s . If $\ell \geq 2$, then we further continuously rotate until we also get $x_{k+1} > 0$.

If $s > 0$, we let x_{k+1} continuously go to 0, while x_1 goes to \sqrt{s} . That is, the set Y_s is connected.

Although if $s < 0$, we let x_1 go to zero and x_{k+1} go to either $\sqrt{-s}$ or $-\sqrt{-s}$. The second is a possibility only if $\ell = 1$, and it is clear that in this case Y_s has exactly two components, as there is no way to pass through a point where $x_{k+1} = 0$ if $s < 0$.

In either case, if Y_s is connected, then X_s must be connected as it is a manifold with boundary and $Y_s = \partial X_s$. Similarly if $\ell = 1$ and $s < 0$, we find that there must be exactly two components of X_s as there is no way to pass through the point x_{k+1} and still stay in the set X_s .

Let us consider the fundamental group of X_s . Suppose $\gamma: S^1 \rightarrow X_s$ is a loop in X_s . The set X_s is given by $x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_m^2 \leq s$. We continuously deform γ to make the first k components of the loop go 0, while staying in X_s . Now the loop is in the set Z defined by $x_1 = \cdots = x_k = 0$ and $-x_{k+1}^2 - \cdots - x_m^2 \leq s$. The set Z is \mathbb{R}^ℓ if $s > 0$, meaning the loop deforms to a point and X_s is simply connected.

If $s < 0$, then the set given by $-x_{k+1}^2 - \cdots - x_m^2 \leq s$ and $x_1 = \cdots = x_k = 0$ is $\mathbb{R}^\ell \setminus B(0, \sqrt{-s})$, which is simply connected if $\ell > 2$ (it has 2 components if $\ell = 1$). If $\ell = 2$, clearly then $\pi_1(X_s) = \mathbb{Z}$ as any loop in $\mathbb{R}^2 \setminus B(0, \sqrt{-s})$ is homotopic to a multiple of the circle. Consequently, the generator is given by $-x_{m-1}^2 - x_m^2 = s$ as claimed. \square

Proposition 4.3. *Suppose M and H_+ are defined near the origin by (5) and (6), $a \geq 2$, and Q is nondegenerate.*

Then there exists a neighborhood $U \subset \mathbb{C}^n \times \mathbb{R}$ of the origin such that for all s the manifold with boundary $(H_+ \cap U)_{\{s\}}$ either

- (i) *is empty,*
- (ii) *is connected with connected boundary, or*
- (iii) *has two components, each one with connected boundary, in which case $s < 0$.*

In cases (ii) and (iii), each connected component of $(H_+ \cap U)_{\{s\}}$ is simply connected, or it has a single generator of the first fundamental group, which is a loop in $M \cap U$.

Proof. Write M as

$$(30) \quad s = \sum_{j=1}^a |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + z^t B z + \overline{z^t B z} + E(z, \bar{z}) = Q(z, \bar{z}) + E(z, \bar{z})$$

for E in $O(3)$, where B is a complex symmetric matrix and $a \geq 2$. Via an n -by- n unitary matrix of the form $T \oplus I_{n-2}$, where T is a 2-by-2 unitary matrix and I_{n-2} is the identity matrix, which keeps A fixed, we put the symmetric matrix B into the form

$$(31) \quad \begin{bmatrix} \lambda_1 & 0 & * \\ 0 & \lambda_2 & * \\ * & * & * \end{bmatrix},$$

where $\lambda_j \geq 0$. Setting $z_3 = \dots = z_n = 0$, we find that Q becomes a real quadratic form

$$(32) \quad |z_1|^2 + |z_2|^2 + \lambda_1 z_1^2 + \lambda_2 z_2^2.$$

By direct computation, the symmetric real 4×4 matrix representing this quadratic form has at least two positive eigenvalues. Therefore the real $2n \times 2n$ matrix representing Q has at least two positive eigenvalues.

First we consider the case when $E = 0$. That is, we consider the set where $s \geq Q(z, \bar{z})$. This set satisfies the hypotheses of Proposition 4.2, and so the conclusion of the proposition holds.

If E is not zero, we note that since Q is nondegenerate, the Morse lemma implies that there exists a real smooth change of coordinates at the origin in \mathbb{C}^n such that $Q(z, \bar{z}) + E(z, \bar{z})$ becomes a quadratic form in the new coordinates. The result again follows from Proposition 4.2. □

We now show the existence of affine analytic discs through any point in a small neighborhood of zero in H_+ that are attached to the CR points of M . Such discs either continuously shrink towards a CR point of M , or continuously deform to discs through $(0, s)$ and living only in the first two coordinates of z .

Lemma 4.4. *Suppose M and H_+ are defined near the origin by (5) and (6), $a \geq 2$, and Q is nondegenerate.*

Given any neighborhood V of the origin, there exist neighborhoods $U_1 \subset U_2 \subset V$ of the origin such that for every point $(z, s) \in (H_+ \setminus M) \cap U_1$ there exists a complex affine function $L: \mathbb{C} \rightarrow \mathbb{C}^n \times \{s\}$, such that $L(0) = (z, s)$, $L^{-1}(M \cap U_2)$ is compact and connected, and such that $L(\mathbb{C})$ intersects $M \cap U_2$ transversally and intersects only CR points of M .

Furthermore, if $s \neq 0$ and W is a neighborhood of $M \setminus \{0\}$, then there exists a continuous family of such affine discs, which is a continuous family of affine functions $L_t: \mathbb{C} \rightarrow \mathbb{C}^n \times \{s\}$, $t \in [0, 1]$, such that $L_t^{-1}(M \cap U_2)$ is compact for all t , $L_1 = L$ and $L_0(\mathbb{C}) \cap U_2 \cap H_+ \subset W$, or otherwise $L_0(0) = (0, s)$ with $z_3 = \dots = z_n = 0$ on $L_0(\mathbb{C})$ (and also $s > 0$).

The fact that $L(\mathbb{C}) \cap H_+ \cap U_2$ are discs because $L(\mathbb{C}) \cap M \cap U_2$ is connected is not strictly necessary for our purposes, but it simplifies somewhat the terminology and proofs.

Proof. To simplify notation, we assume below that we can make V smaller so that $U_2 = V$. We also write U instead of U_1 . Using coordinates $(\zeta, \tau) \in \mathbb{C}^n \times \mathbb{R}$, suppose $M \subset \mathbb{C}^n \times \mathbb{R}$ is given by

$$(33) \quad \tau = \sum_{j=1}^a |\zeta_j|^2 - \sum_{j=a+1}^{a+b} |\zeta_j|^2 + B(\zeta, \zeta) + \overline{B(\zeta, \zeta)} + E(\zeta, \bar{\zeta}),$$

for E in $O(3)$, where $a \geq 2$.

As B is given by a complex symmetric matrix, then as above, we put the matrix representing B into the form

$$(34) \quad \begin{bmatrix} \lambda_1 & 0 & * \\ 0 & \lambda_2 & * \\ * & * & * \end{bmatrix},$$

where $\lambda_j \geq 0$.

Then $H_+ \setminus M$ is given by

$$(35) \quad \tau > \sum_{j=1}^a |\zeta_j|^2 - \sum_{j=a+1}^{a+b} |\zeta_j|^2 + B(\zeta, \zeta) + \overline{B(\zeta, \zeta)} + E(\zeta, \bar{\zeta}).$$

Take some $(z, s) \in H_+ \setminus M$. Let $\ell: \mathbb{C} \rightarrow \mathbb{C}^n$ be given by

$$(36) \quad \ell(\xi) = (z_1 + c_1\xi, z_2 + c_2\xi, z_3, \dots, z_n)$$

and $L: \mathbb{C} \rightarrow \mathbb{C}^n \times \{s\}$ be given by $L(\xi) = (\ell(\xi), s)$.

Let us plug L into (35) to get

$$(37) \quad s > |z_1 + c_1\xi|^2 + |z_2 + c_2\xi|^2 + \sum_{j=3}^a |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(\ell(\xi), \ell(\xi)) + \overline{B(\ell(\xi), \ell(\xi))} + E(\ell(\xi), \overline{\ell(\xi)}).$$

We expand the quadratic terms to get

$$(38) \quad s > Q(z, \bar{z}) + P(z, \bar{z})\xi + \overline{P(z, \bar{z})}\bar{\xi} + \alpha\xi^2 + \bar{\alpha}\bar{\xi}^2 + (|c_1|^2 + |c_2|^2)\xi\bar{\xi} + E(\ell(\xi), \overline{\ell(\xi)}),$$

where Q is a real quadratic form, P is real-linear, and

$$(39) \quad \alpha = \lambda_1 c_1^2 + \lambda_2 c_2^2.$$

We find one solution c_1, c_2 to $\alpha = 0, |c_1|^2 + |c_2|^2 = 1$. Because the coefficient in front of $\xi\bar{\xi}$ is 1, we rewrite the inequality as

$$(40) \quad s > |P(z, \bar{z}) + \xi|^2 - R(z, \bar{z}) + E(\ell(\xi), \overline{\ell(\xi)}),$$

where R is a real quadratic form. As E is of order 3, by making V smaller if necessary, we can without loss of generality restrict z (and therefore also ξ) to a small neighborhood, such that

$$(41) \quad \left| E(\ell(\xi), \overline{\ell(\xi)}) \right| \leq \frac{1}{2} \|z\|^2 + \frac{1}{2} |P(z, \bar{z}) + \xi|^2.$$

We obtain

$$(42) \quad s + R(z, \bar{z}) + \frac{1}{2} \|z\|^2 > \frac{1}{2} |P(z, \bar{z}) + \xi|^2.$$

The inequality (42) is satisfied at $\xi = 0$. The left-hand side of (42) does not depend on ξ and goes to zero as $(z, s) \rightarrow (0, 0)$ in H_+ . Therefore, picking a small enough

neighborhood U , if $(z, s) \in U \cap H_+$, then $L(\xi)$ is in H_+ only for points well inside V . In other words, equality in (37) is satisfied only in V , and so $L(\mathbb{C}) \cap M \cap V$ is compact, which is what we wanted.

If $s \neq 0$, then the origin is not in $L(\mathbb{C})$. As there are infinitely many solutions c_1, c_2 , and so infinitely many possible lines L , we pick one that does not go through the origin even if $s = 0$. Therefore, the line intersects only at CR points, and we make the intersection transversal by applying Proposition 4.1.

We still need to show that $L(\mathbb{C}) \cap M \cap V$ is connected so that we obtain analytic discs rather than more general one-dimensional manifolds with boundary. Write (33) as $\tau = \rho(\zeta, \bar{\zeta})$, and define

$$(43) \quad r(\xi, \bar{\xi}) = \rho(\ell(\xi), \overline{\ell(\xi)}).$$

The right-hand side of (37) is $r(\xi, \bar{\xi})$, so if V is small enough, r is subharmonic. We also assume V is convex.

Suppose for a contradiction that $L(\mathbb{C}) \cap M \cap V$ is disconnected. It is a one real dimensional smooth curve as the intersection is transversal. It is therefore composed of several curves, one of which, say C_0 , is the “outside” curve, and some curves inside, say one of these is C_1 . As V is convex, the interior of C_0 is contained within $L(\mathbb{C}) \cap V$. Write (33) as $\tau = \rho(\zeta, \bar{\zeta})$, and define

$$(44) \quad r(\xi, \bar{\xi}) = \rho(\ell(\xi), \overline{\ell(\xi)}).$$

If $r < s$ on both sides of C_1 , along which $r = s$, we would violate the maximum principle. Therefore $r > s$ on at least some points inside C_1 , and so r achieves a maximum inside C_1 , which is again a contradiction as r is subharmonic. Consequently, C_1 did not exist and $L(\mathbb{C}) \cap M \cap V$ is connected.

For the final result we replace z with tz in the above estimates and write $L_t(\xi) = (tz_1 + c_1\xi, tz_2 + c_2\xi, tz_3, \dots, tz_n, s)$. Then $L_1 = L$ as claimed, and L_t is a continuous family. Clearly $(tz, s) \in U$ for all $t \in [0, 1]$, so $L_t(\mathbb{C}) \cap M \cap V$ is compact. If $s < 0$, then in (42) we see that the left-hand side must become negative before t reaches zero. So for some t , the set $L_t(\mathbb{C}) \cap H_+ \cap V$ is empty, and hence for some slightly larger t , the set $L_t(\mathbb{C}) \cap H_+ \cap V \subset W$. If $s > 0$, and $L_t(\mathbb{C}) \cap H_+ \cap V$ is never empty for $t \in [0, 1]$, then $L_0(0) = (0, s)$. \square

5. THE EXTENSION NEAR A CR SINGULARITY

Using the affine discs we first show interior regularity of an extension, if it exists.

Lemma 5.1. *Suppose $\Omega \subset \mathbb{C}^n \times \mathbb{R}$ is a domain with smooth boundary and $f: \partial\Omega \rightarrow \mathbb{C}$ is a smooth function that is CR at CR points of $\partial\Omega$. Suppose for some $s_0 \in \mathbb{R}$, $L: \mathbb{C} \rightarrow \mathbb{C}^n \times \{s_0\} \subset \mathbb{C}^n \times \mathbb{R}$ is a complex affine mapping, and D is a bounded component of $L(\mathbb{C}) \cap \Omega$. Suppose $U \subset \mathbb{C}^n \times \mathbb{R}$ is a neighborhood of D and $F: U \cap \Omega \rightarrow \mathbb{C}$ is such that for any fixed s near s_0 , $z \mapsto F(z, s)$ is holomorphic and extends continuously up to the boundary $(\partial\Omega \cap U)_{\{s\}}$, where it agrees with f . Then F is smooth in a neighborhood of D in Ω .*

Proof. Making Ω smaller if necessary, we assume Ω is bounded and $D = L(\mathbb{C}) \cap \Omega$. Pick some point $(z_0, s_0) \in L(\mathbb{C}) \cap \Omega$, where $L(\xi) = (c\xi + z_0, s_0)$ and $c \in \mathbb{C}^n$. After a small perturbation of c , we assume $L(\mathbb{C})$ intersects $\partial\Omega$ transversally via Proposition 4.1. Without loss of generality assume $c = (1, 0, \dots, 0)$ and $z_0 = 0$, so $L(\xi) = (\xi, 0, \dots, 0, s_0)$. Write $L_{z',s}(\xi) = (\xi, z', s)$, where $z' \in \mathbb{C}^{n-1}$. As the intersection $L(\mathbb{C}) \cap \partial\Omega$ is transversal, for all s near s_0 and z' near 0, $L_{z',s}(\mathbb{C}) \cap \Omega$ is

bounded and connected, and $L_{z',s}(\mathbb{C}) \cap \partial\Omega$ is a smooth path. We write F using the Cauchy integral formula as an integral over this path. That is, for $(z, s) = (\xi, z', s)$ near (z_0, s_0) ,

$$(45) \quad F(z, s) = F(L_{z',s}(\xi)) = \frac{1}{2\pi i} \int_{L_{z',s}^{-1}(\partial\Omega)} \frac{f(L_{z',s}(\tau))}{\tau - \xi} d\tau.$$

The intersection $L_{z',s}(\mathbb{C}) \cap \partial\Omega$ is transversal if we change z' and s slightly. Therefore, $L_{z',s}^{-1}(\partial\Omega)$ varies smoothly with (z', s) and hence with (z, s) . Similarly, ξ varies smoothly as a function of z . Therefore, the function F is smooth in (z, s) near (z_0, s_0) and the lemma follows. \square

Lemma 5.2. *Suppose M and H_+ are defined near the origin by (5) and (6), $a \geq 2$, and Q is nondegenerate. Then there exists a neighborhood W of $M \setminus \{0\}$ such that smooth CR functions on M extend to CR functions on $W \cap H_+$.*

That is, given a C^∞ function $f: M \rightarrow \mathbb{C}$ which is CR outside the origin, there exists a function $F: W \cap H_+ \rightarrow \mathbb{C}$, such that the restrictions to leaves, $z \mapsto F(z, s)$, are holomorphic and continuous up to $(W \cap M)_{\{s\}}$ where it agrees with f .

Proof. Let M be defined by $0 = -s + A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} + E(z, \bar{z})$. We use the same function to define $(M)_{\{s\}}$. For points (z, s) near the origin, the Levi-form of $(M)_{\{s\}}$ is a small perturbation of the form A restricted to the $T_{(z,s)}^{(1,0)}(M)_{\{s\}}$. As A has two positive eigenvalues, then the Levi-form of $(M)_{\{s\}}$ must have at least one positive eigenvalue. This eigenvalue corresponds to the side $(H_+)_{\{s\}}$. Therefore for some neighborhood of the origin, we can apply Theorem 1.1 near all CR points of M . The extension is unique, so near CR points of M it can be patched together. In other words, shrinking M and H_+ to a smaller neighborhood of the origin if needed, we have an extension to some neighborhood of $M \setminus \{0\}$ in H_+ . That is, we know that there are a neighborhood W of $M \setminus \{0\}$ and an F defined on $W \cap H_+$, holomorphic along leaves, and continuous along leaves up to M . \square

Extension for $s < 0$ follows using the affine discs; however, for certain points where $s > 0$, it is necessary use different complex manifolds with boundary attached to M .

Lemma 5.3. *Suppose M and H_+ are defined near the origin by (5) and (6), $a \geq 2$, and Q is nondegenerate. Let W be as in Lemma 5.2.*

Then there exists a neighborhood U of the origin with the following property. Each point $(z, s) \in U$, where $s > 0$ and $z_3 = \dots = z_n = 0$, is connected via a path in $(U \cap H_+ \setminus M)_{\{s\}}$ to a point $(z', s) \in W$.

Additionally, if $f: M \rightarrow \mathbb{C}$ is a C^∞ function that is CR outside the origin, then F extends in $\mathbb{C}^n \times \{s\}$ by analytic continuation along the above paths.

Proof. Without loss of generality it is enough to consider $n = 2$. Let $M \subset \mathbb{C}^2 \times \mathbb{R}$ be given by

$$(46) \quad s = |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(\lambda_1 z_1^2 + \lambda_2 z_2^2) + E(z, \bar{z}).$$

We suppose that $s > 0$.

If $\lambda_1 = \lambda_2 = 0$, then the set $(H_+)_{\{s\}}$ is a ball of radius \sqrt{s} , and the extension follows by the standard Hartogs's extension theorem. So suppose that at least one λ_j is not zero.

Let $(z, s) \in H_+ \setminus M$ be some fixed point, where $\lambda_1 z_1^2 + \lambda_2 z_2^2 \neq 0$. We work in \mathbb{C}^2 , that is on one fixed leaf. Let

$$(47) \quad g(\zeta_1, \zeta_2) = \lambda_1 \zeta_1^2 + \lambda_2 \zeta_2^2.$$

We consider the one dimensional submanifolds $X_t \subset \mathbb{C}^2$ given by $g(\zeta) = t$ for nonzero t . Let $t_0 = g(z)$ (note that $t_0 \neq 0$).

Let us consider the set of points on X_t that correspond to $H_+ \setminus M$; that is, let us consider the set of points where $\zeta \in X_t$ and

$$(48) \quad s > \|\zeta\|^2 + 2 \operatorname{Re} g(\zeta) + E(\zeta, \bar{\zeta}).$$

Let us call this set of points $Y_t \subset X_t$. In other words

$$(49) \quad Y_t = \{\zeta \in X_t : s - 2 \operatorname{Re} t > \|\zeta\|^2 + E(\zeta, \bar{\zeta})\}.$$

As $E(\zeta, \bar{\zeta})$ is $o(\|\zeta\|^2)$, we pick a neighborhood U in which

$$(50) \quad |E(\zeta, \bar{\zeta})| < \frac{1}{2} \|\zeta\|^2.$$

Then (49) implies

$$(51) \quad 2(s - 2 \operatorname{Re} t) > \|\zeta\|^2.$$

We obtain that Y_t is a relatively compact subset of X_t . Denote by ∂Y_t the relative boundary of Y_t in X_t . If the neighborhood U is picked small enough; that is, if (z, s) is picked to be close enough to the origin, then the boundary ∂Y_{t_0} lies on $M_{\{s\}}$, and it also lies on $M_{\{s\}}$ (or is empty) for all t such that $\operatorname{Re} t \geq \operatorname{Re} t_0$.

Therefore we find a continuous family of Y_t such that $\partial Y_t \subset M_{\{s\}}$, and as we move t to make $\operatorname{Re} t$ larger, Y_t must be empty when $2(s - 2 \operatorname{Re} t) = 0$. Hence for some t , the set Y_t is nonempty and completely within $W_{\{s\}}$. Now move t towards t_0 along some path. Let t_1 be the first such t where Y_{t_1} contains at least one point not in $W_{\{s\}}$. Clearly such points must be in the interior and since $W_{\{s\}}$ is a neighborhood of $M_{\{s\}}$, $Y_{t_1} \setminus W_{\{s\}}$ is a compact set in Y_{t_1} . Let $\varphi_t: \mathbb{C} \setminus \{0\} \rightarrow X_t$ be the natural rational parametrization of X_t . For example, if $\lambda_j \neq 0$, it is

$$\varphi_t(\xi) = \left(\frac{\xi + \frac{t}{\xi}}{2\sqrt{\lambda_1}}, \frac{\xi - \frac{t}{\xi}}{2i\sqrt{\lambda_2}} \right).$$

In particular, φ_t varies analytically with t . There exists a smooth path $\Gamma \subset \varphi_{t_1}^{-1}(Y_{t_1}) \subset \mathbb{C} \setminus \{0\}$ that goes exactly once around $\varphi_{t_1}^{-1}(Y_{t_1} \setminus W_{\{s\}})$. We apply the Cauchy formula on X_t for t slightly before getting to t_1 (in the sense of moving towards t_0), so that $\Gamma \subset \varphi_t^{-1}(Y_t)$, and $z \in Y_t$:

$$(52) \quad F(z, s) = \frac{1}{2\pi i} \int_{\Gamma} \frac{F(\varphi_t(\tau))}{\tau - \varphi_t^{-1}(z)} d\tau.$$

The formula therefore holds for t in a neighborhood of t_1 , and $z \in Y_t \cap W_{\{s\}}$, and hence we define an extension of F in all of Y_t .

Hence we have the required extension F to all points (z, s) in some neighborhood U except perhaps points where $g(z) = 0$. We find an extension F into an open set minus the subvariety given by $g(z) = 0$. However, we notice that this subvariety, which is a union of two complex lines through the origin, must in fact intersect H_+ only in a bounded set, which is clear from (49). Thus the set to which we did not yet extend F is a compact subset of the leaf and we use the standard Hartogs's extension phenomenon. □

Lemma 5.4. *Suppose M and H_+ are defined near the origin by (5) and (6), $a \geq 2$, and Q is nondegenerate.*

Then there exists a neighborhood U of the origin with the following property. If $f: M \rightarrow \mathbb{C}$ is C^∞ and CR outside the origin, then there is a function $F \in C^\infty(H_+ \cap U \setminus \{0\})$ such that F is CR on $(H_+ \setminus M) \cap U$ and $F|_{M \cap U} = f|_{M \cap U}$.

Proof. We apply Lemma 5.2 to obtain W and define F in W , and we also apply Lemma 5.3 to show that we can analytically extend F to all points (z, s) where $s > 0$ and $z_3 = \cdots = z_n = 0$.

We must show that for some neighborhood U , for each s and for every point z in $(U \cap H_+ \setminus M)_{\{s\}}$, we can extend via analytic continuation from some point $w \in (U \cap W)_{\{s\}}$. We then show that this extension is unique, as long as U is small enough.

We find a small enough neighborhood V where Proposition 4.3 applies, and we assume that M and H_+ are closed submanifolds of V .

Let us suppose that $s \neq 0$. Via Lemma 4.4 there are neighborhoods U_1 and $U_2 \subset V$ such that for any given $(z, s) \in U_1$, we have a family of affine maps L_t , with $L_1(0) = (z, s)$ and such that the image $L_t(\mathbb{C}) \cap U_2 \cap H_+$ either ends up in W for $t = 0$, or for $t = 0$ it ends up in the set $z_3 = \cdots = z_n = 0$. If the images of L_t end up in W , then we may apply the Kontinuitätssatz (see, e.g., [27, p. 189]). As there exists a holomorphic function in a neighborhood of $(L_0(\mathbb{C}) \cap U_2 \cap H_+ \setminus M)_{\{s\}}$ in \mathbb{C}^n , there exists a holomorphic function in a neighborhood of $(L_1(\mathbb{C}) \cap U_2 \cap H_+ \setminus M)_{\{s\}}$. Hence the function extends via analytic continuation to (z, s) . We let $U = U_1$.

We need to now show that the extension is single valued. We pick U such that Proposition 4.3 applies there. Each component (possibly two) of $(U \cap H_+)_{\{s\}}$ is a submanifold with boundary whose boundary (the submanifold $M_{\{s\}}$) is connected. The proposition also says that $(U \cap H_+)_{\{s\}}$ and therefore $(U \cap H_+ \setminus M)_{\{s\}}$ is either simply connected or the fundamental group is \mathbb{Z} whose generator is a loop in the boundary $U \cap M_{\{s\}}$. If the neighborhood is simply connected the extension F is single valued. If it is not simply connected, we push the generator of the fundamental group from the boundary into $(U \cap H_+ \setminus M)_{\{s\}}$, but such that it still stays inside W . The function is therefore already single valued on the generator of the fundamental group. Therefore, the extension is single valued in $(U \cap H_+)_{\{s\}}$.

Finally suppose that $s = 0$. The first part of Lemma 4.4 applies in U even for $s = 0$. So if $(z, 0) \in (H_+ \setminus M) \cap U$, then there is an affine disc through $(z, 0)$ attached to $M_{\{0\}}$ and not attached to the CR singularity. The boundary of this disc falls into W ; that is, we have an extension near the boundary of this disc.

As an extension exists in a neighborhood of the CR points, even on the $s = 0$ leaf, we take a slightly smaller disc D_0 such that $\partial D_0 \subset (W \cap H_+ \setminus M)_{\{0\}}$; that is, F is holomorphic in a neighborhood of ∂D_0 . Let D_s be the identical disc, but on the s -leaf rather than the 0 leaf. For small $s \neq 0$, $\partial D_s \subset (U \cap H_+ \setminus M)_{\{s\}}$, and we know that F was extended above to the entire disc $\overline{D_s}$. Denote by L_0 and L_s the respective affine functions.

We define an extension \tilde{F} in D_0 via the Cauchy integral formula:

$$(53) \quad \tilde{F}(z, 0) = \frac{1}{2\pi i} \int_{L_0^{-1}(\partial D_0)} \frac{F(L_0(\tau))}{\tau - L_0^{-1}(z)} d\tau.$$

As F on D_s is given by

$$(54) \quad F(z, s) = \frac{1}{2\pi i} \int_{L_s^{-1}(\partial D_s)} \frac{F(L_s(\tau))}{\tau - L_s^{-1}(z)} d\tau$$

by continuity, $\tilde{F} = F$ where both are defined. Hence F extends through the $s = 0$ leaf in some neighborhood.

We have an extension F defined in $H_+ \cap U$ for some small enough neighborhood U of the origin. Regularity of F in $(H_+ \setminus M) \cap U$ follows via Lemma 5.1. Regularity near the CR points of M , that is regularity on $(H_+ \setminus \{0\}) \cap U$ follows from Theorem 1.1. □

6. FORMAL EXTENSION AT A CR SINGULARITY

The formal extension in [23] is stated for a nondegenerate A ; however, it also works as long as A has at least two positive eigenvalues and Q is nondegenerate. For completeness we give the statement in the notation that we need and a sketch of the proof.

Lemma 6.1. *Suppose $M \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, given by*

$$(55) \quad M : \quad s = \sum_{j=1}^a |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z, z) + \overline{B(z, z)},$$

$a \geq 2$, and Q is nondegenerate. Suppose $f(z, \bar{z})$ is a polynomial such that when considered as a function on M (parametrized by z), f is a CR function on M_{CR} .

Then there exists a polynomial $F(z, s)$ such that f and F agree on M ; that is,

$$(56) \quad f(z, \bar{z}) = F(z, A(z, \bar{z}) + B(z, z) + \overline{B(z, z)}).$$

Furthermore, if f is homogeneous of degree d , then F is weighted homogeneous of degree d ; that is,

$$(57) \quad F(z, s) = \sum_{j+2k=d} P_j(z) s^k$$

where P_j is a homogeneous polynomial of degree j .

The proof is a combination of the proofs of Lemma 3.2 and Lemma 3.3 from [23]. The main difficulties in [23] are 1) handling the case where A has one positive and one negative eigenvalue, and 2) handling the case where the CR singularity is large. In the case of the above lemma, neither issue arises.

Sketch of Proof. Restricting to $z_3 = \dots = z_n = 0$, we find a submanifold in $\mathbb{C}^2 \times \mathbb{R}$ with positive definite A . Let us therefore for the moment consider $M' \subset \mathbb{C}^2 \times \mathbb{R}$ given by

$$(58) \quad M' : \quad s = |z_1|^2 + |z_2|^2 + B'(z, z) + \overline{B'(z, z)}.$$

In [23] we noted that there exists a so-called elliptic direction $c \in \mathbb{C}^2$, in particular, for $s > 0$ the image of the map

$$(59) \quad \xi \mapsto (c\xi, s)$$

intersects M' in an ellipse and therefore induces an analytic disc. Let us call this disc $\Delta_{c,s}$. We will again apply the Kontinuitätssatz, but in this case we are allowed to move between leaves as all the data (M and f) is analytic (in fact polynomial).

For small $v \in \mathbb{C}^2$

$$(60) \quad \xi \mapsto (c\xi + v, s)$$

still meets M' in an ellipse, and thus induces a disc $\Delta_{c,s,v}$ attached to M' . Fixing such a v and letting s go to zero, we find that eventually $\Delta_{c,s,v}$ must become empty. Therefore we find a family of affine analytic discs attached to M' that shrink down to a point on M' that is not the origin.

We return to our original $M \subset \mathbb{C}^n \times \mathbb{R}$. We abuse notation somewhat by writing c instead of $(c, 0) \in \mathbb{C}^n$, and we still write $\Delta_{c,s} \subset \mathbb{C}^n \times \mathbb{R}$ for the analytic disc above.

We have a family of analytic discs attached to M_{CR} where one end of the family is a disc, $\Delta_{c,s}$, through a point $(0, s)$ and the other end of the family shrinks to point on M_{CR} .

We now complexify s to consider M as a subset of \mathbb{C}^{n+1} . As M and f are analytic, and f is CR it extends to a holomorphic function of a neighborhood of M_{CR} in \mathbb{C}^{n+1} . The *Kontinuitätssatz* therefore implies that f extends to a holomorphic function of a neighborhood of $\overline{\Delta_{c,s}}$. As the discs are always attached to M_{CR} , we find that the extension agrees with f on the intersection $\overline{\Delta_{c,s}} \cap M$.

Note that f therefore extends to a holomorphic function for a neighborhood of discs $\overline{\Delta_{c',s}}$ for all $c' \in C \subset \mathbb{C}^n$ in a neighborhood C of c .

Let $M_{c'} \subset \mathbb{C} \times \mathbb{C}$ be the manifold defined by the pullback

$$(61) \quad (\xi, s) \mapsto (\xi c', s).$$

In [22, Lemma 5.1], we proved that a polynomial $P(\xi, \bar{\xi})$, when considered as a function on the manifold $M_{c'}$ parametrized by ξ , extends to a polynomial in ξ and s .

Thus for each c' in C we find a polynomial $F_{c'}(\xi, s)$ which extends $f(c'\xi, \overline{c'\xi})$. Consider for a moment $\Delta_{c,1}$. There exists a holomorphic function $F(z, s)$ defined in a neighborhood $\Delta_{c,1}$ that extends f . For all $c' \in C$, $F(c'\xi, s) = F_{c'}(\xi, s)$ on an open set, and since C is an open set, then $F(z, s)$ agrees with a polynomial on an open set. See [22, Proposition 5.2]. □

The lemma implies the existence of a formal power series for an extension. The following proposition and its proof is essentially the same as Proposition 5.1 from [23] with the necessary modifications made for smooth functions rather than real-analytic functions.

Proposition 6.2. *Let $M \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a smooth submanifold given by (5), let Q be nondegenerate, let $a \geq 2$, and write (5) as $s = \rho(z, \bar{z})$.*

Suppose $f \in C^\infty(M)$ such that $f|_{M_{CR}}$ is a CR function. There exists a formal power series $F(z, s)$ for f at the origin; that is, $F(z, \rho(z, \bar{z})) = f(z, \bar{z})$ formally at the origin (parametrizing M by z).

Proof. Write M as

$$(62) \quad s = Q(z, \bar{z}) + E(z, \bar{z}),$$

where E is $O(3)$ and $Q(z, \bar{z}) = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)}$. Parametrizing M by z , decompose f as

$$(63) \quad f(z, \bar{z}) = f_k(z, \bar{z}) + \tilde{f}(z, \bar{z}),$$

where f_k is a real-homogeneous polynomial of degree k and \tilde{f} is $O(k + 1)$.

The rest of the proof is essentially identical to the proof of Proposition 5.1 from [23]. Let us go through it quickly. A basis of CR vector fields for CR points of M near the origin is given by vector fields of the form $X = (Q_{\bar{z}_j} + E_{\bar{z}_j}) \frac{\partial}{\partial \bar{z}_k} - (Q_{\bar{z}_k} + E_{\bar{z}_k}) \frac{\partial}{\partial \bar{z}_j}$, and similarly $X^{quad} = Q_{\bar{z}_j} \frac{\partial}{\partial \bar{z}_k} - Q_{\bar{z}_k} \frac{\partial}{\partial \bar{z}_j}$ for M^{quad} . Then

$$(64)$$

$$0 = Xf = X(f_k + \tilde{f}) = (Q_{\bar{z}_j}) \frac{\partial f_k}{\partial \bar{z}_k} - (Q_{\bar{z}_k}) \frac{\partial f_k}{\partial \bar{z}_j} + O(k + 1) = X^{quad} f_k + O(k + 1).$$

Therefore, $X^{quad} f_k = 0$ and $f_k(z, \bar{z})$ is a CR function on the model M^{quad} . By Lemma 6.1, we write $f_k(z, \bar{z}) = F_k(z, Q(z, \bar{z}))$ for some weighted homogeneous $F_k(z, s)$. The function $F_k(z, s)$ is CR on M . If we parametrize by z , $F_k(z, \rho(z, \bar{z}))$ has the same k th order terms as f , and

$$(65) \quad f(z, \bar{z}) - F_k(z, \rho(z, \bar{z}))$$

is a CR function on M vanishing to one higher order. We obtain a formal power series. □

7. REGULARITY OF THE EXTENSION AT A CR SINGULARITY

In this section we prove Theorem 1.2. Let M, H_+, f be as in the theorem. Lemma 5.4 gives us a neighborhood U and the extension F . From now on, we assume $H_+ = U \cap H_+$ and $M = U \cap M$. In the following, we parametrize M by z as usual when writing $f(z, \bar{z})$, and we compute the partial derivatives of f with respect to this z .

Claim 7.1. $F \in C(H_+)$.

Proof. The extension $F: H_+ \rightarrow \mathbb{C}$ from Lemma 5.4 is smooth (and hence continuous) on $H_+ \setminus \{0\}$. By Lemma 4.4, there exists a small neighborhood $U' \subset U$ of the origin such that for each point $(z, s) \in H_+ \cap U'$, there is an analytic disc Δ through (z, s) with $\partial\Delta \subset M$. Via the maximum principle

$$(66) \quad \begin{aligned} |F(z, s) - f(0)| &\leq \sup \left\{ |f(\zeta, \bar{\zeta}) - f(0)| : (\zeta, s) \in \partial\Delta \right\} \\ &\leq \sup \left\{ |f(\zeta, \bar{\zeta}) - f(0)| : (\zeta, s) \in M \right\}. \end{aligned}$$

As $s \rightarrow 0, M \ni (\zeta, s) \rightarrow 0$. □

The derivatives F_{z_j} and F_s extend smoothly to $H_+ \setminus \{0\}$. We need them to be smooth through the origin. First, let us show that their restrictions to M extend smoothly through the origin.

Claim 7.2. $F_{z_j}|_M, F_s|_M \in C^\infty(M)$, for $1 \leq j \leq n$.

Proof. Let $s = \rho(z, \bar{z})$ define M , then write

$$(67) \quad \xi_j = \rho_{z_j} = \epsilon_j z_j + v_j \cdot \bar{z} + h.o.t.,$$

for $\epsilon_j = -1, 0, 1$ and some constant vector v_j , where \cdot denotes dot product. Similarly, $\bar{\xi}_j = \rho_{\bar{z}_j} = \epsilon_j \bar{z}_j + \bar{v}_j \cdot z + h.o.t$. Because Q is nondegenerate we find that the ξ_j and $\bar{\xi}_j$ are linearly independent, and therefore $(\xi, \bar{\xi})$ gives a smooth change of variables at the origin.

We take derivatives outside the origin:

$$(68) \quad f_{\bar{z}_j} = F_s|_M \xi_j.$$

The function $f_{\bar{z}_j}$ is smooth through the origin, so the right-hand side is smooth as well. We need to show that $f_{\bar{z}_j}$ is divisible by ξ_j to show that $F_s|_M$ is smooth.

A formal solution to the extension problem exists, so the division is also true formally. Thus for any order m , we write f as

$$(69) \quad f(z, \bar{z}) = P(z, \rho(z, \bar{z})) + R(z, \bar{z}),$$

where $P(z, s)$ is a polynomial of degree m and R is $O(m + 1)$. So,

$$(70) \quad f_{\bar{z}_j} = P_s(z, \rho) \xi_j + R_{\bar{z}_j}(z, \bar{z}).$$

Hence, the lower order terms are not an obstruction to the division of $f_{\bar{z}_j}$ by ξ_j .

Because the variables $\xi, \bar{\xi}$ are a smooth change of variables, we now think of everything in terms of ξ and $\bar{\xi}$.

Consider the real part of ξ_j as a variable, and apply the Malgrange–Mather division theorem [24, Chapter V]. One obtains smooth functions q and r , where

$$(71) \quad f_{\bar{z}_j} = q \xi_j + r,$$

and r does not depend on the real part of ξ_j . If r had any finite order terms, the lowest order part of r would be divisible by ξ_j as we saw above and would thus depend on the real part of ξ_j . Hence r does not have any finite order terms.

Consider the ideal I generated by ξ_j . Outside the origin, $f_{\bar{z}_j}$ is divisible by ξ_j using (68). Thus, $f_{\bar{z}_j}$ is locally in the ideal at every point outside the origin. At the origin, r vanishes to infinite order in (71), and so $f_{\bar{z}_j}$ is in I formally, as its Taylor series is a Taylor series of a function that is in the ideal, namely $q \xi_j$. As the Taylor series of $f_{\bar{z}_j}$ at each point belongs formally to I , then a theorem of Malgrange [24, Theorem 1.1' in Chapter VI], implies $f_{\bar{z}_j} \in I$. So $f_{\bar{z}_j}$ is divisible by ξ_j , and hence $F_s|_M$ extends smoothly through the origin.

Write (on M)

$$(72) \quad f_{z_j} = F_{z_j}|_M + F_s|_M \bar{\xi}_j.$$

As $F_s|_M$ is smooth and f_{z_j} is smooth, then $F_{z_j}|_M$ is smooth through the origin. \square

The proof of Theorem 1.2 now follows from the next claim.

Claim 7.3. $F \in C^\infty(H_+)$.

Proof. By Theorem 1.1, $F \in C^\infty(H_+ \setminus \{0\})$. Let us first show $F \in C^1(H_+)$. Above we saw that $F \in C(H_+)$ and the derivatives of F_{z_j} and F_s restricted to M are smooth functions. They are also CR functions. Then F_{z_j} and F_s satisfy the hypothesis of Theorem 1.2, and so by Claim 7.1, F_{z_j} and F_s are continuous at the origin. Hence $F \in C^1(H_+)$. By iterating this procedure we find that $F \in C^\infty(H_+)$. \square

8. FURTHER EXAMPLES

Example 8.1. *Extension can be one sided at every point.*

Let $M \subset \mathbb{C}^2 \times \mathbb{R}$ be given by

$$(73) \quad s = |z_1|^2 + |z_2|^2 = \|z\|^2.$$

Let $g: S^3 \subset \mathbb{C}^2 \rightarrow \mathbb{C}$ be a smooth CR function defined on the sphere that does not extend holomorphically to the outside of the unit ball through any point of S^3 ; see [4] or [11]. Define

$$(74) \quad f(z, s) = \begin{cases} e^{-1/s^2} g\left(\frac{z}{\sqrt{s}}\right) & \text{if } s < 0, \\ 0 & \text{if } s = 0. \end{cases}$$

Again it is easy to see that f is CR. Furthermore, on M outside the origin we find

$$(75) \quad f(z, \|z\|^2) = e^{-1/\|z\|^4} g\left(\frac{z}{\|z\|}\right).$$

Clearly f is smooth outside the origin. Since S^3 is compact, all derivatives of g are bounded. Taking derivatives of f , we see that all derivatives of f are bounded near the origin, and so f is smooth at the origin as well. The function extends to be smooth and CR on H_+ given by $s \geq \|z\|^2$. However, for every $p \in M$, there is no neighborhood U of p such that f extends to be CR on U .

Example 8.2. *Extension fails in $n = 1$.*

We already saw that the extension fails in $n = 1$. However, let us give a further example in the smooth case. Suppose $M \subset \mathbb{C} \times \mathbb{R}$ is a nonparabolic Bishop surface given by

$$(76) \quad s = |z|^2 + \lambda z^2 + \lambda \bar{z}^2, \quad (\text{where } 0 \leq \lambda < \infty \text{ and } \lambda \neq \frac{1}{2}).$$

This M has a nondegenerate isolated CR singularity. Define a smooth $f: \mathbb{C} \rightarrow \mathbb{R}$ that is zero on the first quadrant of \mathbb{C} and positive elsewhere. Parametrizing M by z , we have that $f(z, \bar{z})$ is a smooth function on M . As the CR condition is vacuous, it can be considered a CR function. For every $s \neq 0$, the leaf

$$(77) \quad (H_+)_{\{s\}} = \{z \in \mathbb{C} \mid s \geq |z|^2 + \lambda z^2 + \lambda \bar{z}^2\}$$

is either empty, or has part of its boundary in the first quadrant. Clearly the function f cannot extend to this leaf holomorphically as it is not identically zero, but it is zero on a nontrivial arc of the boundary.

The next two examples show that the existence of the extension depends on the topology of the leaves $(H_+)_{\{s\}}$.

Example 8.3. *Without two positive eigenvalues, leaves may have disconnected boundary.*

The submanifold M given by

$$(78) \quad s = |z_1|^2 - |z_2|^2 + \lambda(z_1^2 + \bar{z}_1^2)$$

for $\lambda > 1/2$, has an isolated CR singularity. Both Q and A are nondegenerate, and A has only one positive and one negative eigenvalue. The leaves $(H_+)_{\{s\}}$ for $s > 0$ have a disconnected boundary with two components. Construct a smooth f that is a different constant along each boundary component of each leaf. Such an f is CR, but does not extend to H_+ .

Example 8.4. *Higher order terms may complicate the topology of the leaves.*

The submanifold M defined by

$$(79) \quad s = \sin\left(\|z\|^{-2}\right) e^{-\|z\|^{-2}}$$

is degenerate, and the CR singularity consists of the origin and concentric circles.

Let $f: M \rightarrow \mathbb{C}$ be $f(z, s) = \|z\|^2$. This f is smooth and CR on M . The set $(H_+)_{\{s\}}$ has generally several components and a disconnected boundary. The function f is equal to a different constant on each component of the boundary of any $(H_+)_{\{s\}}$, and therefore no extension exists to H_+ .

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