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RESIDUE FORMULAS FOR LOGARITHMIC FOLIATIONS AND APPLICATIONS

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Abstract. In this work we prove a Baum–Bott type formula for noncompact complex manifold of the form $\tilde{X} = X - \mathcal{D}$, where X is a complex compact manifold and $\mathcal{D}$ is a normal crossing divisor on X. As applications, we provide a Poincaré–Hopf type theorem and an optimal description for a smooth hypersurface $\mathcal{D}$ invariant by an one-dimensional foliation $\mathscr{F}$ on $\mathbb{P}^n$ satisfying $\mathrm{Sing}(\mathscr{F}) \subsetneq \mathcal{D}$.

1. Introduction

In [4] Baum and Bott developed a work about residues of singularities of holomorphic foliations on complex manifolds. In the case of one-dimensional holomorphic foliation $\mathscr{F}$, with isolate singularities, on an n-dimensional complex compact manifold X we have the following classical Baum–Bott formula:

$$\int_X c_n(T_X - T_{\mathscr{F}}) = \sum_{p\in \mathrm{Sing}(\mathscr{F})} \mu_p(\mathscr{F}) \qquad \text{(Baum–Bott formula)}, \tag{1}$$

where the $\mu_p(\mathscr{F})$ are the Milnor number of $\mathscr{F}$ in p. Baum–Bott formula is a generalization (for holomorphic vector fields) of the Poincaré–Hopf theorem

$$\int_X c_n(T_X) = \sum_{p\in \mathrm{Sing}(\mathscr{F})} \mu_p(\mathscr{F}),$$

where $\mathscr{F}$ is a foliation induced by a global holomorphic vector field, with isolated singularities, on X.

In this work we provide a Baum–Bott type formula for noncompact complex manifold of the form $\tilde{X} = X - \mathcal{D}$, where X is a complex compact manifold and $\mathcal{D}$ is an analytic divisor contained in X invariant by an one-dimensional holomorphic foliation $\mathscr{F}$, which is called by *logarithmic foliation* along $\mathcal{D}$. As an application, we obtained a Poincaré–Hopf type theorem for these noncompact manifolds. Furthermore, for logarithmic foliations on projective spaces, we prove a necessary and sufficient conditions for all singularities of the foliation occur in an analytic invariant hypersurface.

We prove the following result.

Theorem 1. *Let $\tilde{X}$ be an n-dimensional complex manifold such that $\tilde{X} = X - \mathcal{D}$, where X is an n-dimensional complex compact manifold and $\mathcal{D}$ is a smooth hypersurface on X. Let $\mathscr{F}$ be a foliation of dimension one on X, with isolated*

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singularities and logarithmic along $\mathcal{D}$. Suppose that $Ind_{\log \mathcal{D},\mathrm{p}}(\mathscr{F}) = 0$ for all $p \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}$. Then

$$\int_X c_n(T_X(-\log \mathcal{D}) - T_{\mathscr{F}}) = \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap (X \setminus \mathcal{D})} \mu_p(\mathscr{F}).$$

Here, $Ind_{\log \mathcal{D},\mathrm{p}}(\mathscr{F})$ denotes the logarithmic index of $\mathscr{F}$ on p; see section 2.4.

The classical Gauss–Bonnet theorem for a complex compact manifold X, proved by Chern in [9], says that

$$\int_X c_n(T_X) = \chi(X). \tag{2}$$

The following version of Gauss–Bonnet formula for noncompact manifolds was initially proposed by Iitaka [18] and proved by Norimatsu [22], Silvotti [25], and Aluffi [2].

Theorem (Norimatsu, Silvotti, and Aluffi). *Let $\tilde{X}$ be an n-dimensional complex manifold such that $\tilde{X} = X - \mathcal{D}$, where X is an n-dimensional complex compact manifold and $\mathcal{D}$ is a normally crossing hypersurface on X. Then*

$$\int_X c_n(T_X(-\log \mathcal{D})) \;\; = \;\; \chi(\tilde{X}),$$

where $\chi(\tilde{X})$ denotes the Euler characteristic given by

$$\chi(\tilde{X}) = \sum_{i=1}^{n} \dim\, H_c^i(\tilde{X}, \mathbb{C}).$$

Liao in [21] has provided more general formulas in terms of the Chern–Schwartz–MacPherson class of $\tilde{X}$.

In section 4 we consider the case in which $\mathcal{D}$ is a normal crossing hypersurface, and we prove the following Baum–Bott type formula.

Theorem 2. *Let $\tilde{X}$ be an n-dimensional complex manifold such that $\tilde{X} = X - \mathcal{D}$, where X is an n-dimensional complex compact manifold, and $\mathcal{D}$ is a normally crossing hypersurface on X. Let $\mathscr{F}$ be a foliation on X of dimension 1, with isolated singularities (nondegenerates) and logarithmic along $\mathcal{D}$. Then*

$$\int_X c_n(T_X(-\log \mathcal{D}) - T_{\mathscr{F}}) = \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap (\tilde{X})} \mu_p(\mathscr{F}). \tag{3}$$

As a consequence of Theorem 2 and the Norimatsu–Silvotti–Aluffi theorem, we obtain the following Poincaré–Hopf type theorem.

Corollary 1. *Let $\tilde{X}$ be an n-dimensional complex manifold such that $\tilde{X} = X - \mathcal{D}$, where X is an n-dimensional complex compact manifold, $\mathcal{D}$ is a reduced normal crossing hypersurface on X. Let $\mathscr{F}$ be a foliation on X of dimension 1 given by a global holomorphic vector field, with isolated singularities (nondegenerates) and logarithmic along $\mathcal{D}$. Then*

$$\chi(\tilde{X}) = \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap \tilde{X}} \mu_p(\mathscr{F}),$$

where $\mu_p(\mathscr{F})$ denotes the Milnor number of $\mathscr{F}$ on p.

Finally, in section 6 we prove a complete characterization in order that an invariant hypersurface contains all of the singularities of the projective foliation.

Theorem 3. *Let $\mathcal{D} \subset \mathbb{P}^n$ be a smooth and irreducible hypersurface, and let $\mathscr{F}$ be a foliation of dimension* 1 *on $\mathbb{P}^n$, with isolated singularities* (*nondegenerates*) *and logarithmic along $\mathcal{D}$. Then the following proprieties holds*

(1) *If n is odd, then*
  (a) $\#\,[\mathrm{Sing}(\mathscr{F}) \cap \mathbb{P}^n \backslash \mathcal{D}] > 0 \iff \deg(\mathcal{D}) < \deg(\mathscr{F}) + 1,$
  (b) $\#\,[\mathrm{Sing}(\mathscr{F}) \cap \mathbb{P}^n \backslash \mathcal{D}] = 0 \iff \deg(\mathcal{D}) = \deg(\mathscr{F}) + 1.$
(2) *If n is even, then*
  (a) $$\#\,[\mathrm{Sing}(\mathscr{F}) \cap \mathbb{P}^n \backslash \mathcal{D}] > 0 \iff \begin{cases} \deg(\mathcal{D}) \neq \deg(\mathscr{F}) + 1 \\ \textit{or} \\ \deg(\mathcal{D}) = \deg(\mathscr{F}) + 1, \\ \textit{with } \deg(\mathscr{F}) \neq 0; \end{cases}$$
  (b) $\#\,[\mathrm{Sing}(\mathscr{F}) \cap \mathbb{P}^n \backslash \mathcal{D}] = 0 \iff \deg(\mathcal{D}) = 1$ *and* $\deg(\mathscr{F}) = 0.$
(3) *In general, we have the formula*
$$\#\,[\mathrm{Sing}(\mathscr{F}) \cap \mathbb{P}^n \backslash \mathcal{D}] = \sum_{i=0}^{n} (-1)^i (\deg(\mathcal{D}) - 1)^i \deg(\mathscr{F})^{n-i}.$$

Observe that if n is odd, then $\mathrm{Sing}(\mathscr{F}) \subsetneq \mathcal{D}$ if and only if the Soares bound for the Poincaré problem is achieved [26]. For the Poincaré problem see, for instance, [7, 10–12, 15] and the references therein.

2. Preliminaries

2.1. **Logarithmic forms and logarithmic vector fields.** Let X be an n-dimensional complex manifold, and let $\mathcal{D}$ be a reduced hypersurface on X. Given a meromorphic q-form ω on X, we say that ω is a *logarithmic* q-form along $\mathcal{D}$ at $x \in X$ if the following conditions occur:

(i) ω is holomorphic on $X - \mathcal{D}$.
(ii) If $h = 0$ is a reduced equation of $\mathcal{D}$, locally at x, then $h\,\omega$ and $h\,d\omega$ are holomorphic.

Denoting by $\Omega^q_{X,x}(\log \mathcal{D})$ the set of germs of a logarithmic q-form along $\mathcal{D}$ at x, we define the following coherent sheaf of $\mathcal{O}_X$-modules
$$\Omega^q_X(\log \mathcal{D}) := \bigcup_{x \in X} \Omega^q_{X,x}(\log \mathcal{D}),$$
which is called by *sheaf of logarithmic q-forms along $\mathcal{D}$*. See [13, 19, 23] for details.

Now, given $x \in X$, let $v \in T_{X,x}$ be a germ at x of a holomorphic vector field on X. We say that v is a *logarithmic vector field along $\mathcal{D}$ at x* if v satisfies the following condition: if $h = 0$ is a equation of $\mathcal{D}$, locally at x, then the derivation $v(h)$ belongs to the ideal $\langle h_x \rangle \mathcal{O}_{X,x}$. Denoting by $T_{X,x}(-\log \mathcal{D})$ the set of germs of logarithmic vector field along $\mathcal{D}$ at x, we define the following coherent sheaf of $\mathcal{O}_X$-modules,
$$T_X(-\log \mathcal{D}) := \bigcup_{x \in X} T_{X,x}(-\log \mathcal{D}),$$
which is called by *sheaf of logarithmic vector fields along $\mathcal{D}$*.

It is known that $\Omega^1_X(\log \mathcal{D})$ and $T_X(-\log \mathcal{D})$ is always a reflexive sheaf; see [23] for more details. If $\mathcal{D}$ is an analytic hypersurface with normal crossing singularities, the sheaves $\Omega^1_X(\log \mathcal{D})$ and $T_X(-\log \mathcal{D})$ are locally free; furthermore, the Poincaré residue map

$$\mathrm{Res} : \Omega^1_{\mathrm{X}}(\log \mathcal{D}) \longrightarrow \mathcal{O}_{\mathcal{D}} \cong \bigoplus_{\mathrm{i}=1}^{\mathrm{N}} \mathcal{O}_{\mathcal{D}_{\mathrm{i}}}$$

gives the following exact sequence of sheaves on X:

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log \mathcal{D}) \xrightarrow{\mathrm{Res}} \bigoplus_{i=1}^{N} \mathcal{O}_{\mathcal{D}_i} \longrightarrow 0, \tag{4}$$

where Ω^1_X is the sheaf of holomorphics 1-forms on X and $\mathcal{D}_1, \ldots, \mathcal{D}_N$ are the irreducible components of $\mathcal{D}$.

Now, if $\mathcal{D}$ is such that $\mathrm{cod}(\mathrm{Sing}(\mathcal{D})) > 2$, then there exists the following exact sequence of sheaves on X (see Dolgachev [14]):

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log \mathcal{D}) \longrightarrow \mathcal{O}_{\mathcal{D}} \longrightarrow 0. \tag{5}$$

On the projective space $\mathbb{P}^n$, if $\mathcal{D}$ is a smooth hypersurface, then there exists the following exact sequence of sheaves (see Angelini [3]):

$$0 \longrightarrow T_{\mathbb{P}^n}(-\log \mathcal{D}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k) \longrightarrow 0, \tag{6}$$

where k is the degree of $\mathcal{D}$.

2.2. **Singular one-dimensional holomorphic foliations.**

Definition 2.1. Let X be a connected complex manifold. A one-dimensional holomorphic foliation is given by the following data,

- (i) an open covering $\mathcal{U} = \{U_\alpha\}$ of X,
- (ii) for each U_α a holomorphic vector field ζ_α,
- (iii) for every nonempty intersection, $U_\alpha \cap U_\beta \neq \emptyset$, a holomorphic function

$$f_{\alpha\beta} \in \mathcal{O}^*_X(U_\alpha \cap U_\beta),$$

such that $\zeta_\alpha = f_{\alpha\beta}\zeta_\beta$ in $U_\alpha \cap U_\beta$ and $f_{\alpha\beta}f_{\beta\gamma} = f_{\alpha\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$.

We denote by $K_{\mathscr{F}}$ the line bundle defined by the cocycle $\{f_{\alpha\beta}\} \in \mathrm{H}^1(X, \mathcal{O}^*)$. Thus, a one-dimensional holomorphic foliation $\mathscr{F}$ on X induces a global holomorphic section $\zeta_{\mathscr{F}} \in \mathrm{H}^0(X, T_X \otimes K_{\mathscr{F}})$.

The line bundle $T_{\mathscr{F}} := (K_{\mathscr{F}})^* \hookrightarrow T_X$ is called the *tangent bundle* of $\mathscr{F}$. The singular set of $\mathscr{F}$ is $\mathrm{Sing}(\mathscr{F}) = \{\zeta_{\mathscr{F}} = 0\}$. We will assume that $\mathrm{cod}(\mathrm{Sing}(\mathscr{F})) \geq 2$.

Definition 2.2. Let V be an analytic subspace of a complex manifold X. We say that V is invariant by a foliation $\mathscr{F}$ if $T_{\mathscr{F}}|_V \subset (\Omega^1_V)^*$. If V is a hypersurface, we say that $\mathscr{F}$ is *logarithmic along* V.

Definition 2.3. A foliation on a complex projective space $\mathbb{P}^n$ is called by *projective foliation*. Let $\mathscr{F}$ be a projective foliation with tangent bundle $T_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(r)$. The integer number $d := r + 1$ is called the *degree* of $\mathscr{F}$.

2.3. **The GSV index.** Gómez-Mont, Seade, and Verjovsky [17] introduced the *GSV index* for a holomorphic vector field over an analytic hypersurface, with isolated singularities, on a complex manifold, generalizing the (classical) Poincaré–Hopf index. The concept of the GSV index was extended to a holomorphic vector field on more general contexts. For example, Seade and Suwa in [24] defined the GSV index for a holomorphic vector field on an analytic subvariety type isolated complete intersection singularity. Brasselet, Seade, and Suwa in [5] extended the notion of the GSV index for vector fields defined in certain types of analytical subvariety with nonisolated singularities.

In [16] Gómez-Mont defined the *homological index* of a holomorphic vector field on an analytic hypersurface with isolated singularities, which coincides with the GSV index. There is also the *virtual index*, introduced by Lehmann, Soares, and Suwa [20], that via Chern-Weil theory can be interpreted as the GSV index. Brunella [8] also present the GSV index for foliations on complex surfaces by a different approach.

Let X be an n-dimensional complex manifold, let $\mathcal{D}$ be an isolated hypersurface singularity on X, and let $\mathscr{F}$ be a foliation on X of dimension 1, with isolated singularities. Suppose that $\mathscr{F}$ is logarithmic along $\mathcal{D}$, i.e., the analytic hypersurface $\mathcal{D}$ is invariant by each holomorphic vector field that is a local representative of $\mathscr{F}$. The GSV index of $\mathscr{F}$ in $x \in \mathcal{D}$ will be denoted by $\mathrm{GSV}(\mathscr{F}, \mathcal{D}, x)$. For definition and details on the GSV index we refer to [6, 27].

2.4. **The logarithmic index.** Recently, Aleksandrov introduced in [1] the notion of logarithmic index for a logarithmic vector field . Let $\mathscr{F}$ be a one-dimensional holomorphic foliation on X with isolated singularities that is logarithmic along $\mathcal{D}$. For a fixed point $x \in X$ let $v \in T_X(-\log \mathcal{D})|_U$ be a germ of vector field on (U, x) tangent to $\mathscr{F}$. The interior multiplication i_v induces the complex of logarithmic differential forms

$$0 \longrightarrow \Omega^n_{X,x}(\log \mathcal{D}) \xrightarrow{i_v} \Omega^{n-1}_{X,x}(\log \mathcal{D}) \xrightarrow{i_v} \cdots \xrightarrow{i_v} \Omega^1_{X,x}(\log \mathcal{D}) \xrightarrow{i_v} \mathcal{O}_{n,x}.$$

Since all singularities of v are isolated, the i_v-homology groups of the complex $\Omega^\bullet_X(\log \mathcal{D})$ are finite-dimensional vector spaces (see [1]). Thus, the Euler characteristic

$$\chi(\Omega^\bullet_X(\log \mathcal{D}), i_v) = \sum_{i=0}^{n} (-1)^i \dim H_i(\Omega^\bullet_{X,x}(\log D), i_v)$$

of the complex of logarithmic differential forms is well defined. Since this number does not depend on the local representative v of the foliation $\mathscr{F}$ at the point x, we define the logarithmic index of $\mathscr{F}$ at the point x by

$$Ind_{\log D,x}(\mathscr{F}) := \chi(\Omega^\bullet_X(\log \mathcal{D}), i_v).$$

It follows from the definition that $Ind_{\log D,x}(\mathscr{F}) = 0$ for all $x \in X - \mathrm{Sing}(\mathscr{F})$.

We have the following important property (see [1]).

Proposition 2.4 ([1]). *Let X, $\mathcal{D}$, and $\mathscr{F}$ be as described above. Then for each $x \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}$ we have*

$$Ind_{\log \mathcal{D},x}(\mathscr{F}) = \mu_x(\mathscr{F}) - \mathrm{GSV}(\mathscr{F}, \mathcal{D}, x) \tag{7}$$

where $\mu_x(v)$ and $\mathrm{GSV}(v, \mathcal{D}, x)$ denote, respectively, the Milnor number and the GSV index of v.

Remark 2.5. If $x \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}_{\mathrm{reg}}$, we obtain

$$Ind_{\log\, \mathcal{D},x}(\mathscr{F}) = \mu_x(\mathscr{F}) - \mu_x(\mathscr{F}|_{\mathcal{D}_{\mathrm{reg}}})$$

since, in this case, the GSV index of $\mathscr{F}$ in x coincides with the Milnor number of $\mathscr{F}|_{\mathcal{D}_{\mathrm{reg}}}$ in x. In particular,

$$Ind_{\log\, \mathcal{D},x}(\mathscr{F}) = 0$$

whenever x is a nondegenerate singularity of $\mathscr{F}$.

3. Proof of Theorem 1

To prove Theorem 1, we will first prove the following result.

Theorem 3.1. *Let X be an n-dimensional complex compact manifold, and let $\mathcal{D}$ be a smooth hypersurface on X. Then for all line bundles L on X we have*

$$\int_X c_n(T_X(-\log\, \mathcal{D}) - L) = \int_X c_n(T_X - L) - \int_{\mathcal{D}} c_{n-1}(T_X - [\mathcal{D}] - L). \tag{8}$$

Proof. By using properties of Chern class, we get

$$\begin{aligned}\int_X c_n(T_X(-\log\, \mathcal{D}) - L) &= \sum_{j=0}^{n} \int_X c_{n-j}(T_X(-\log\, \mathcal{D}))c_1(L^*)^j \\ &= \sum_{j=0}^{n}(-1)^{n-j} \int_X c_{n-j}(\Omega_X^1(\log\, \mathcal{D}))c_1(L^*)^j.\end{aligned}$$

On the one hand, since $\mathcal{D}$ is smooth, we can use the exact sequence (4) to obtain

$$c_i(\Omega_X^1(\log \mathcal{D})) = \sum_{k=0}^{i} c_{i-k}(\Omega_X^1)c_k(\mathcal{O}_{\mathcal{D}}) \qquad \forall i \in \{1, \dots, n\}.$$

On the other hand, the Chern classes of $\mathcal{O}_{\mathcal{D}}$ are

$$c_k(\mathcal{O}_{\mathcal{D}}) = c_k(\mathcal{O}_X - \mathcal{O}(-\mathcal{D})) = c_1([\mathcal{D}])^k, \qquad k = 1, \dots, n. \tag{9}$$

Thus,

$$\int_X c_n(T_X(-\log\, \mathcal{D}) - L) = \sum_{j=0}^{n}(-1)^{n-j} \int_X \left[\sum_{k=0}^{n-j} c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}])^k\right] c_1(L^*)^j.$$

Now we split this sum into two parts as follows:

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{n-j} \int_X \left[\sum_{k=0}^{n-j} c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}])^k\right] c_1(L^*)^j \\ &\quad= \sum_{j=0}^{n-1}(-1)^{n-j} \int_X \left[\sum_{k=1}^{n-j} c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}])^k\right] c_1(L^*)^j \\ &\qquad+ \sum_{j=0}^{n}(-1)^{n-j} \int_X c_{n-j}(\Omega_X^1)c_1(L^*)^j.\end{aligned}$$

In the first part appear all terms with $k \geq 1$, and in the second part are the terms with $k = 0$. By using the Poincaré duality, we compute the first part as follows:

$$\begin{aligned}
&\sum_{j=0}^{n-1}(-1)^{n-j}\int_X\left[\sum_{k=1}^{n-j} c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}])^k\right]c_1(L^*)^j \\
&\quad = \sum_{j=0}^{n-1}(-1)^{n-j}\int_{\mathcal{D}}\left[\sum_{k=1}^{n-j} c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}])^{k-1}\right]c_1(L^*)^j \\
&\quad = -\sum_{j=0}^{n-1}\int_{\mathcal{D}}\left[\sum_{k=1}^{n-j}(-1)^{n-j-k} c_{n-j-k}(\Omega_X^1)(-1)^{k-1}c_1([\mathcal{D}])^{k-1}\right]c_1(L^*)^j \\
&\quad = -\sum_{j=0}^{n-1}\int_{\mathcal{D}}\left[\sum_{k=1}^{n-j} c_{n-j-k}(T_X)c_1([\mathcal{D}]^*)^{k-1}\right]c_1(L^*)^j \\
&\quad = -\sum_{j=0}^{n-1}\int_{\mathcal{D}} c_{n-1-j}(T_X-[\mathcal{D}])c_1(L^*)^j \\
&\quad = -\int_{\mathcal{D}} c_{n-1}(T_X-[\mathcal{D}]-L).
\end{aligned}$$

Now, by basic proprieties of Chern classes we compute the second sum as follows:

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^{n-j}\int_X c_{n-j}(\Omega_X^1)c_1(L^*)^j &= \sum_{j=0}^{n}(-1)^{n-j}\int_X (-1)^{n-j}c_{n-j}(T_X)c_1(L^*)^j \\
&= \sum_{j=0}^{n}\int_X c_{n-j}(T_X)c_1(L^*)^j \\
&= \int_X c_n(T_X-L).
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^{n-j}\int_X\left[\sum_{k=0}^{n-j} c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}])^k\right]c_1(L^*)^j = &-\int_{\mathcal{D}} c_{n-1}(T_X-[\mathcal{D}]-L) \\
&+\int_X c_n(T_X-L),
\end{aligned}$$

and this proves the result. □

Now we will prove Theorem 1.

Proof. Since $\mathcal{D}$ is smooth, we can invoke formula (8) of Theorem 3.1 to obtain the following equality:

$$\int_X c_n(T_X(-\log\mathcal{D})-T_{\mathscr{F}}) = \int_X c_n(T_X-T_{\mathscr{F}}) - \int_{\mathcal{D}} c_{n-1}(T_X-[\mathcal{D}]-T_{\mathscr{F}}).$$

By hypothesis the one-dimensional foliation $\mathscr{F}$ is logarithmic along $\mathcal{D}$ and has only isolated singularities; then it follows from [27] that the top Chern number of restriction $(T_X-[\mathcal{D}]-T_{\mathscr{F}})|_{\mathcal{D}}$ coincides with the sum of the GSV index of $\mathscr{F}$

along $\mathcal{D}$. That is,

$$\int_{\mathcal{D}} c_{n-1}(T_X - [\mathcal{D}] - T_{\mathscr{F}}) = \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}} \mathrm{GSV}(\mathscr{F}, \mathcal{D}, p).$$

Hence,

$$\begin{aligned} \int_X c_n(T_X(-\log \mathcal{D}) - T_{\mathscr{F}}) &= \int_X c_n(T_X - T_{\mathscr{F}}) - \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}} \mathrm{GSV}(\mathscr{F}, \mathcal{D}, p) \\ &= \sum_{p \in \mathrm{Sing}(\mathscr{F})} \mu_p(\mathscr{F}) - \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}} \mathrm{GSV}(\mathscr{F}, \mathcal{D}, p), \end{aligned}$$

where in the last step we are using the Baum–Bott classical formula (1). Now, since $Ind_{\log \mathcal{D}, p}(\mathscr{F}) = 0$ for all $p \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}$, by Proposition 2.4 we get the following relation:

$$\mathrm{GSV}(\mathscr{F}, \mathcal{D}, p) = \mu_p(\mathscr{F}) \qquad \forall p \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}.$$

Therefore, we obtain

$$\sum_{p \in \mathrm{Sing}(\mathscr{F})} \mu_p(\mathscr{F}) - \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap \mathcal{D}} \mathrm{GSV}(\mathscr{F}, \mathcal{D}, p) = \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap (X \setminus \mathcal{D})} \mu_p(\mathscr{F}),$$

and the desired formula is proved. □

4. Proof of Theorem 2

In this section we will consider $\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_N$, an analytic hypersurface on X, with normal crossing singularities. Fixing an irreducible component, say, $\mathcal{D}_N$, we define

$$\hat{\mathcal{D}}_N := \bigcup_{j=1}^{N-1} \mathcal{D}_j$$

and

$$\hat{\mathcal{D}}_N | \mathcal{D}_N := \bigcup_{j=1}^{N-1} \mathcal{D}_j \cap \mathcal{D}_N.$$

We note that $\hat{\mathcal{D}}_N | \mathcal{D}_N$ is an analytic hypersurface on $\mathcal{D}_N$ with normal crossing singularities and $N-1$ irreducible components. We will use the following multiple index notation: for each multi-index $J = (j_1, \ldots, j_N)$ and $J' = (j'_1, \ldots, j'_{N-1})$, with $1 \le j_l, j'_t \le n$, we denote

$$\begin{aligned} c_1(\mathcal{D})^J &= c_1([\mathcal{D}_1])^{j_1} \cdots c_1([\mathcal{D}_N])^{j_N}, \\ c_1(\hat{\mathcal{D}}_N)^{J'} &= c_1([\mathcal{D}_1])^{j'_1} \cdots c_1([\mathcal{D}_{N-1}])^{j'_{N-1}}. \end{aligned}$$

Lemma 4.1. *In the above conditions, for each $i = 1, \ldots, n$, we have*

$$c_i(\Omega^1_X(\log \mathcal{D})) = \sum_{k=0}^{i} \sum_{|J|=k} c_{i-k}(\Omega^1_X) c_1(\mathcal{D})^J.$$

Proof. Since $\mathcal{D}$ is an analytic hypersurface with normal crossing singularities, the Poincaré residue map

$$\mathrm{Res} : \Omega^1_X(\log \mathcal{D}) \longrightarrow \mathcal{O}_{\mathcal{D}} \cong \bigoplus_{i=1}^{N} \mathcal{O}_{\mathcal{D}_i}$$

induces the following exact sequence:

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log \mathcal{D}) \xrightarrow{\text{Res}} \bigoplus_{i=1}^{N} \mathcal{O}_{\mathcal{D}_i} \longrightarrow 0.$$

By using this exact sequence, we get

$$\begin{aligned} c_i(\Omega^1_X(\log \mathcal{D})) &= \sum_{k=0}^{i} c_{i-k}(\Omega^1_X) c_k(\bigoplus_{i=1}^{N} \mathcal{O}_{\mathcal{D}_i}) \\ &= \sum_{k=0}^{i} c_{i-k}(\Omega^1_X) \left(\sum_{j_1+\cdots+j_N=k} c_{j_1}(\mathcal{O}_{\mathcal{D}_1}) \cdots c_{j_N}(\mathcal{O}_{\mathcal{D}_N}) \right) \\ &= \sum_{k=0}^{i} c_{i-k}(\Omega^1_X) \left(\sum_{j_1+\cdots+j_N=k} c_1([\mathcal{D}_1])^{j_1} \cdots c_1([\mathcal{D}_N])^{j_N} \right), \end{aligned}$$

where in last equality we use the following relations,

$$c_i(\mathcal{O}_{\mathcal{D}_j}) = c_1([\mathcal{D}_j])^i, \qquad i = 1, \dots, n,$$

which can be obtained from (9). ☐

Lemma 4.2. *In the above conditions, for each $i = 1, \dots, n-1$,*

$$c_i(\Omega^1_X)|_{\mathcal{D}_N} = c_i(\Omega^1_{\mathcal{D}_N}) - c_{i-1}(\Omega^1_{\mathcal{D}_N}) c_i([\mathcal{D}_N])|_{\mathcal{D}_N}.$$

Proof. The proof follows from taking the total Chern class in the exact sequence

$$0 \to T_{\mathcal{D}_N} \to T_X|_{\mathcal{D}_N} \to [\mathcal{D}_N]|_{\mathcal{D}_N} \to 0.$$ ☐

Lemma 4.3. *In the above conditions, if L is a holomorphic line bundle on X, then the following relations hold:*

(10)
$$\int_X c_n(T_X(-\log \mathcal{D}) - L) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \sum_{|J|=k} \int_X (-1)^{n-j} c_{n-j-k}(\Omega^1_X) c_1(\mathcal{D})^J c_1(L^*)^j.$$

In particular,

(11)
$$\int_X c_n(T_X(-\log \hat{\mathcal{D}}_N) - L) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \sum_{|J'|=k} \int_X (-1)^{n-j} c_{n-j-k}(\Omega^1_X) c_1(\hat{\mathcal{D}}_N)^{J'} c_1(L^*)^j$$

and

(12)
$$\begin{aligned} &\int_{\mathcal{D}_N} c_{n-1}(T_{\mathcal{D}_N}(-\log(\hat{\mathcal{D}}_N|\mathcal{D}_N)) - L|_{\mathcal{D}_N}) \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{n-1-j} \sum_{|J'|=k} \int_{\mathcal{D}_N} (-1)^{n-1-j} c_{n-1-j-k}(\Omega^1_{\mathcal{D}_N}) c_1(\hat{\mathcal{D}}_N)^{J'} c_1(L^*)^j. \end{aligned}$$

Proof. By basic proprieties of Chern classes we get

$$\begin{aligned}\int_X c_n(T_X(-\log\mathcal{D})-L) &= \int_X \sum_{j=0}^{n} c_{n-j}(T_X(-\log\mathcal{D}))c_1(L^*)^j \\ &= \int_X \sum_{j=0}^{n}(-1)^{n-j}c_{n-j}(\Omega_X^1(\log\mathcal{D}))c_1(L^*)^j.\end{aligned}$$

By Lemma 4.1 we get

$$c_{n-j}(\Omega_X^1(\log\mathcal{D})) = \sum_{k=0}^{n-j}\sum_{|J|=k} c_{n-j-k}(\Omega_X^1)c_1(\mathcal{D})^J.$$

Substituting this, we obtain (10). Relation (11) is obtained by taking $\mathcal{D}=\hat{\mathcal{D}}_N$ in relation (10). Analogously, applying relation (11), we can obtain (12) by taking $X=\mathcal{D}_N$ as a complex manifold of dimension $n-1$ and $\mathcal{D}=\hat{D}_N|D_N$ as an analytic subvariety of $\mathcal{D}_N$ with normal crossings. □

Proposition 4.4. *In the above conditions, if L is a holomorphic line bundle on X, then*

$$\begin{aligned}\int_X c_n(T_X(-\log\mathcal{D})-L) = &\int_X c_n(T_X(-\log(\hat{\mathcal{D}}_N)-L) \\ &-\int_{\mathcal{D}_N} c_{n-1}(T_{\mathcal{D}_N}(-\log(\hat{\mathcal{D}}_N|\mathcal{D}_N)-L|_{\mathcal{D}_N}).\end{aligned}$$

Proof. By Lemma 4.3 it is sufficient to show that the following equality occurs:

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{|J|=k}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\mathcal{D})^J c_1(L^*)^j \\ &\quad=\sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{|J'|=k}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\hat{\mathcal{D}}_N)^{J'}c_1(L^*)^j \\ &\quad\quad-\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\sum_{|J'|=k}\int_{\mathcal{D}_N}(-1)^{n-1-j}c_{n-1-j-k}(\Omega_{\mathcal{D}_N}^1)c_1(\hat{\mathcal{D}}_N)^{J'}c_1(L^*)^j.\end{aligned}$$

Indeed, we can decompose the sum on the left-hand side into the terms with $k=0$ and those with $k\geq 1$ as follows:

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{|J|=k}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\mathcal{D})^J c_1(L^*)^j \\ &\quad=\sum_{j=0}^{n}\int_X(-1)^{n-j}c_{n-j}(\Omega_X^1)c_1(L^*)^j \qquad (13)\\ &\quad\quad+\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{|J|=k}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\mathcal{D})^J c_1(L^*)^j.\end{aligned}$$

The second sum on the right-hand side can readily be computed. In fact,

$$\begin{aligned}\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{|J|=k}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\mathcal{D})^Jc_1(L^*)^j\\ =\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{|J'|=k}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\hat{\mathcal{D}}_N)^{J'}c_1(L^*)^j\\ +\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{\substack{|J|=k\\ j_N\geq 1}}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}_1])^{j_1}\cdots c_1([\mathcal{D}_N])^{j_N}c_1(L^*)^j.\end{aligned}$$

By using the fact that $c_1([\mathcal{D}_N])$ is Poincaré dual to the fundamental class of $\mathcal{D}_N$, we obtain

$$\begin{aligned}\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{|J|=k}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\mathcal{D})^Jc_1(L^*)^j\\ =\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{|J'|=k}\int_X(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\hat{\mathcal{D}}_N)^{J'}c_1(L^*)^j\\ +\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{\substack{|J|=k\\ j_N\geq 1}}\int_{\mathcal{D}_N}(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}_1])^{j_1}\cdots c_1([\mathcal{D}_N])^{j_N-1}c_1(L^*)^j.\end{aligned}$$

Now, using the relation of Lemma 4.2, we get

$$\begin{aligned}\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{\substack{|J|=k\\ j_N\geq 1}}\int_{\mathcal{D}_N}(-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1([\mathcal{D}_1])^{j_1}\cdots c_1([\mathcal{D}_N])^{j_N-1}c_1(L^*)^j\\ =\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{\substack{|J|=k\\ j_N\geq 1}}\int_{\mathcal{D}_N}(-1)^{n-j}c_{n-j-k}(\Omega_{\mathcal{D}_N}^1)c_1([\mathcal{D}_1])^{j_1}\cdots c_1([\mathcal{D}_N])^{j_N-1}c_1(L^*)^j\\ -\sum_{j=0}^{n-1}\sum_{k=1}^{n-j-1}\sum_{\substack{|J|=k\\ j_N\geq 1}}\int_{\mathcal{D}_N}(-1)^{n-j}c_{n-j-1-k}(\Omega_{\mathcal{D}_N}^1)c_1(\mathcal{D})^Jc_1(L^*)^j\\ =\sum_{j=0}^{n-1}\int_{\mathcal{D}_N}(-1)^{n-j}c_{n-j-1}(\Omega_{\mathcal{D}_N}^1)c_1(L^*)^j\\ +\sum_{j=0}^{n-1}\sum_{k=1}^{n-j-1}\sum_{|J'|=k}\int_{\mathcal{D}_N}(-1)^{n-j}c_{n-j-1-k}(\Omega_{\mathcal{D}_N}^1)c_1(\hat{\mathcal{D}}_N)^{J'}c_1(L^*)^j\\ =-\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\sum_{|J'|=k}\int_{\mathcal{D}_N}(-1)^{n-1-j}c_{n-1-j-k}(\Omega_{\mathcal{D}_N}^1)c_1(\hat{\mathcal{D}}_N)^{J'}c_1(L^*)^j.\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{|J|=k}\int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\mathcal{D})^J c_1(L^*)^j \\
&\quad= \sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{|J'|=k}\int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\hat{\mathcal{D}}_N)^{J'} c_1(L^*)^j \\
&\qquad -\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\sum_{|J'|=k}\int_{\mathcal{D}_N} (-1)^{n-1-j}c_{n-1-j-k}(\Omega_{\mathcal{D}_N}^1)c_1(\hat{\mathcal{D}}_N)^{J'} c_1(L^*)^j,
\end{aligned}$$

and we have completed the calculation of the second sum.

Replacing it in the initial equality (13), we obtain

$$\begin{aligned}
&\sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{|J|=k}\int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\mathcal{D})^J c_1(L^*)^j \\
&\quad= \sum_{j=0}^{n}\int_X (-1)^{n-j}c_{n-j}(\Omega_X^1)c_1(L^*)^j \\
&\qquad +\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\sum_{|J'|=k}\int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\hat{\mathcal{D}}_N)^{J'} c_1(L^*)^j \\
&\qquad -\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\sum_{|J'|=k}\int_{\mathcal{D}_N} (-1)^{n-1-j}c_{n-1-j-k}(\Omega_{\mathcal{D}_N}^1)c_1(\hat{\mathcal{D}}_N)^{J'} c_1(L^*)^j \\
&\quad= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{|J'|=k}\int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1)c_1(\hat{\mathcal{D}}_N)^{J'} c_1(L^*)^j \\
&\qquad -\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\sum_{|J'|=k}\int_{\mathcal{D}_N} (-1)^{n-1-j}c_{n-1-j-k}(\Omega_{\mathcal{D}_N}^1)c_1(\hat{\mathcal{D}}_N)^{J'} c_1(L^*)^j. \qquad \square
\end{aligned}$$

Now we will prove Theorem 2.

Proof of Theorem 2. We will prove by induction on the number of irreducible components of $\mathcal{D}$. Indeed, if the number of irreducible components of $\mathcal{D}$ is 1, then $\mathcal{D}$ is smooth. By hypothesis the singularities of $\mathscr{F}$ are nondegenerate, and thus the theorem follows from Theorem 1.

Let us suppose that for every analytic hypersurface on X satisfying the hypothesis of theorem and having $N-1$ irreducible components, formula (3) holds. Let $\mathcal{D}$ be an analytic hypersurface on X with N irreducible components satisfying the hypotheses of the theorem. We will prove that formula (3) is true for $\mathcal{D}$.

We know that $\hat{\mathcal{D}}_N$ is an analytic hypersurface on X and that $\hat{\mathcal{D}}_N|\mathcal{D}_N$ is an analytic hypersurface on $\mathcal{D}_N$, both with normal crossing singularities and having exactly $N-1$ irreducible components. Moreover, $\mathscr{F}$ and its restriction $\mathscr{F}|_{\mathcal{D}_N}$ on $\mathcal{D}_N$ are logarithmic along $\mathcal{D}_N$ and $\hat{\mathcal{D}}_N|\mathcal{D}_N$, respectively. Thus, we can use the induction hypothesis, and we obtain

$$\sum_{p\in \mathrm{Sing}(\mathscr{F})\cap\left(X\setminus\hat{\mathcal{D}}_N\right)} \mu_p(\mathscr{F}) = \int_X c_n(T_X(-\log\,\hat{\mathcal{D}}_N) - T_{\mathscr{F}}) \tag{14}$$

and

$$\sum_{p\in \mathrm{Sing}(\mathscr{F})\cap\left[\mathcal{D}_N\setminus\left(\hat{\mathcal{D}}_N|\mathcal{D}_N\right)\right]} \mu_p(\mathscr{F}) = \int_{\mathcal{D}_N} c_{n-1}(T_{\mathcal{D}_N}(-\log\,(\hat{\mathcal{D}}_N|\mathcal{D}_N)) - T_{\mathscr{F}}|_{\mathcal{D}_N}). \tag{15}$$

By using the following identity $X - \mathcal{D} = (X - \hat{\mathcal{D}}_N) - [\mathcal{D}_N - (\hat{\mathcal{D}}_N \cap \mathcal{D}_N)]$, we get

$$\sum_{p\in \mathrm{Sing}(\mathscr{F})\cap(X\setminus\mathcal{D})} \mu_p(\mathscr{F}) = \sum_{p\in \mathrm{Sing}(\mathscr{F})\cap\left(X\setminus\hat{\mathcal{D}}_N\right)} \mu_p(\mathscr{F}) - \sum_{p\in \mathrm{Sing}(\mathscr{F})\cap\left[\mathcal{D}_N\setminus\left(\hat{\mathcal{D}}_N|\mathcal{D}_N\right)\right]} \mu_p(\mathscr{F}).$$

Therefore, by (14) and (15) we get

$$\begin{aligned}\sum_{p\in \mathrm{Sing}(\mathscr{F})\cap(X\setminus\mathcal{D})} \mu_p(\mathscr{F}) &= \int_X c_n(T_X(-\log\,\hat{\mathcal{D}}_N) - T_{\mathscr{F}}) \\ &\quad - \int_{\mathcal{D}_N} c_{n-1}(T_{\mathcal{D}_N}(-\log\,(\hat{\mathcal{D}}_N|\mathcal{D}_N)) - T_{\mathscr{F}}|_{\mathcal{D}_N}),\end{aligned}$$

and we obtain the desired equality by applying Proposition 4.4. Thus, we prove that formula (3) is true for $\mathcal{D}$, and the proof of the theorem follows by induction. □

5. Application: A Poincaré–Hopf type theorem

In this section we will prove a Poincaré–Hopf type theorem for noncompact complex manifolds. More precisely, we prove the following.

Corollary 1. *Let $\tilde{X}$ be an n-dimensional complex manifold such that $\tilde{X} = X - \mathcal{D}$, where X is an n-dimensional complex compact manifold, and $\mathcal{D}$ is a reduced normal crossing hypersurface on X. Let $\mathscr{F}$ be a foliation on X of dimension 1 given by a global holomorphic vector field, with isolated singularities* (*nondegenerates*) *and logarithmic along $\mathcal{D}$. Then*

$$\chi(\tilde{X}) = \sum_{p\in \mathrm{Sing}(\mathscr{F})\cap\tilde{X}} \mu_p(\mathscr{F}),$$

where $\mu_p(\mathscr{F})$ denotes the Milnor number of $\mathscr{F}$ on p.

Proof. On the one hand, it follows from the Norimatsu–Silvotti–Aluffi theorem that

$$\int_X c_n(T_X(-\log\,\mathcal{D})) = \chi(\tilde{X}).$$

On the other hand, since $\mathcal{D}$ is a normal crossing hypersurface, it follows from Theorem 2 that

$$\int_X c_n(T_X(-\log\,\mathcal{D})) \quad = \sum_{p\in \mathrm{Sing}(\mathscr{F})\cap\tilde{X}} \mu_p(\mathscr{F}).$$

This shows the result. □

6. Application to one-dimensional projective foliations

In this section we give an optimal description for a smooth hypersurface $\mathcal{D}$ invariant by a one-dimensional foliation $\mathscr{F}$ on $\mathbb{P}^n$ satisfying $\mathrm{Sing}(\mathscr{F}) \subsetneq \mathcal{D}$. More precisely, we will prove Theorem 3.

Firstly, we need some preliminary results.

Lemma 6.1. *Let* $f(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n+1}{n-i-j} x^j y^i$, *with* $n \in \mathbb{N}$.

(i) *If* $x \neq y$, *then* $f(x,y) = \dfrac{(1+x)^{n+1} - (1+y)^{n+1}}{x-y}$.

(ii) *If* $x = y$, *then* $f(x,y) = (n+1)(1+x)^n$.

Proof. Developing the summation in the following triangular format, we get

$$\begin{aligned} f(x,y) = \binom{n+1}{n} + \binom{n+1}{n-1}x + \binom{n+1}{n-2}x^2 + \cdots + \binom{n+1}{0}x^n \\ + \binom{n+1}{n-1}y + \binom{n+1}{n-2}yx + \cdots + \binom{n+1}{0}yx^{n-1} \\ + \binom{n+1}{n-2}y^2 + \cdots + \binom{n+1}{0}y^2x^{n-2} \\ \vdots \\ + \binom{n+1}{0}y^n. \end{aligned}$$

We can put into evidence the common factor in each column and we get the following formula:

$$f(x,y) = \sum_{k=0}^{n} \left[\binom{n+1}{n-k} \left(\sum_{j=0}^{k} x^{k-j} y^j \right) \right]. \tag{16}$$

(i) Suppose that $x \neq y$. Since

$$x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k = \frac{x^{k+1} - y^{k+1}}{x-y}, \qquad 0 \leq k \leq n,$$

we obtain

$$\begin{aligned} f(x,y) &= \sum_{k=0}^{n} \left[\binom{n+1}{n-k} \left(\frac{x^{k+1} - y^{k+1}}{x-y} \right) \right] \\ &= \frac{1}{x-y} \left[\sum_{k=0}^{n} \binom{n+1}{n-k} x^{k+1} - \sum_{k=0}^{n} \binom{n+1}{n-k} y^{k+1} \right] \\ &= \frac{(1+x)^{n+1} - (1+y)^{n+1}}{x-y}, \end{aligned}$$

where in the last equality we have used the binomial theorem.

(ii) Consider the case in which $x = y$. By equality (16) we have

$$f(x,y) = \binom{n+1}{n} + \binom{n+1}{n-1}2x + \binom{n+1}{n-1}3x^2 + \cdots + \binom{n+1}{0}(n+1)x^n.$$

Hence, by using the binomial theorem we obtain

$$\begin{aligned} f(x,y) &= \frac{d}{dx}\left[(1+x)^{n+1}\right] \\ &= (n+1)(1+x)^n. \end{aligned}$$

□

Lemma 6.2. *Let k, d, and n be natural numbers, with $k \geq 1$, $d \geq 0$, and $n \geq 2$. Consider the natural number $\delta(k,d,n)$ defined by the relation*

$$\delta(k,d,n) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n+1}{n-i-j}(-k)^j(d-1)^i.$$

Then $\delta(k,d,n)$ satisfies the following conditions:

(i) *If n is odd, then*
  (a) $\delta(k,d,n) > 0 \iff k < d+1$;
  (b) $\delta(k,d,n) = 0 \iff k = d+1$;
  (c) $\delta(k,d,n) < 0 \iff k > d+1$.

(ii) *If n is even, then $\delta(k,d,n) \geq 0$ and moreover,*
  (a) $\delta(k,d,n) > 0 \iff \begin{cases} k \neq d+1 \\ \text{or} \\ k = d+1, \quad \text{with } d \neq 0 \end{cases}$;
  (b) $\delta(k,d,n) = 0 \iff k = 1$ *and* $d = 0$.

(iii) $\delta(k,d,n) = \sum_{i=0}^{n}(-1)^i(k-1)^i d^{n-i}$.

Proof. In order to prove the present lemma, we can consider $x = -k$ and $y = d-1$ in $f(x,y)$ of Lemma 6.1. Hence, we obtain

$$\delta(k,d,n) = \frac{(1-k)^{n+1} - d^{n+1}}{-k-d+1} \qquad \text{if } k \neq 1 \text{ or } d \neq 0,$$

and

$$\delta(k,d,n) = 0 \qquad \text{if } k = 1 \text{ and } d = 0. \tag{17}$$

The proof of items (i) and (ii) can readily be obtained by the study of the sign of the expression

$$\frac{(1-k)^{n+1} - d^{n+1}}{-k-d+1},$$

and also by using relation (17).

Now let us consider the summation $\sum_{i=0}^{n}(-1)^i(k-1)^i d^{n-i}$. We have

$$\begin{aligned} \sum_{i=0}^{n}(-1)^i(k-1)^i d^{n-i} &= d^n\left[\sum_{i=0}^{n}(-1)^i(k-1)^i d^{-i}\right] \\ &= d^n\left[\sum_{i=0}^{n}\left(\frac{(-1)(k-1)}{d}\right)^i\right]. \end{aligned}$$

By the property

$$\forall a \in \mathbb{Z}, \qquad 1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$$

we get

$$\sum_{i=0}^{n}(-1)^i(k-1)^i d^{n-i} \quad = \quad \frac{(k-1)^{n+1}-d^{n+1}}{-k-d+1}.$$

Hence, this proves the equality of item (iii). □

Lemma 6.3. *Let $\mathcal{D}\subset\mathbb{P}^n$ be a smooth and irreducible hypersurface of degree k. Then, for $l=1,\dots,n$, we obtain*

$$c_l(T_{\mathbb{P}^n}(-\log\mathcal{D})) = \left[\sum_{j=0}^{l}\binom{n+1}{l-j}(-1)^j k^j\right] c_1(\mathcal{O}_{\mathbb{P}^n}(1))^l. \tag{18}$$

Proof. Formula (18) can be obtained by considering the recursion

$$c_{j+1}(T_{\mathbb{P}^n}(-\log\mathcal{D})) = \binom{n+1}{j+1}c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{j+1} - c_j(T_{\mathbb{P}^n}(-\log\mathcal{D}))(k\,c_1(\mathcal{O}_{\mathbb{P}^n}(1))),$$

$j=0,\dots,n-1$, which can be obtained by considering the exact sequence (6):

$$0\longrightarrow T_{\mathbb{P}^n}(-\log\mathcal{D})\longrightarrow\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}\longrightarrow\mathcal{O}_{\mathbb{P}^n}(k)\longrightarrow 0.$$ □

Now, we will prove Theorem 3.

Proof of Theorem 3. Let $\deg(\mathcal{D})=k$, and let $\deg(\mathscr{F})=d$. On the one hand, we have

$$\int_{\mathbb{P}^n} c_n(T_{\mathbb{P}^n}(-\log\mathcal{D})-T_{\mathscr{F}}) = \sum_{i=0}^{n}\int_{\mathbb{P}^n} c_{n-i}(T_{\mathbb{P}^n}(-\log\mathcal{D}))c_1(T_{\mathscr{F}}^*)^i.$$

Now, by using formula (18) in each $c_{n-i}(T_{\mathbb{P}^n}(-\log\mathcal{D}))$ in the summation above, we obtain

$$\int_{\mathbb{P}^n} c_n(T_{\mathbb{P}^n}(-\log\mathcal{D})-T_{\mathscr{F}})=\sum_{i=0}^{n}\left[\sum_{j=0}^{n-i}\binom{n+1}{n-i-j}(-1)^j k^j\right]\int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{n-i}c_1(T_{\mathscr{F}}^*)^i.$$

On the other hand, the tangent bundle $T_{\mathscr{F}}$ of foliation on $\mathbb{P}^n$ is such that $T_{\mathscr{F}}=\mathcal{O}_{\mathbb{P}^n}(1-d)$. Therefore, we obtain $c_1(T_{\mathscr{F}}^*)=(d-1)c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Hence,

$$\begin{aligned}\int_{\mathbb{P}^n} c_n(T_{\mathbb{P}^n}(-\log\mathcal{D})-T_{\mathscr{F}}) &= \sum_{i=0}^{n}\left[\sum_{j=0}^{n-i}\binom{n+1}{n-i-j}(-1)^j k^j\right](d-1)^i\int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^n\\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n+1}{n-i-j}(-k)^j(d-1)^i,\end{aligned}$$

where in the last equality we have used the fact that $\int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^n=1$.

By hypothesis the singularities of $\mathscr{F}$ are nondegenerates. Then the number $\#\left[\mathrm{Sing}(\mathscr{F})\cap\mathbb{P}^n\backslash\mathcal{D}\right]$ corresponds to the sum of the numbers of Milnor of the singular points of $\mathscr{F}$ in $\mathbb{P}^n\backslash\mathcal{D}$. Moreover, for all $p\in\mathrm{Sing}(\mathscr{F})\cap\mathcal{D}_{\mathrm{reg}}$ we have $Ind_{\log\,\mathcal{D},p}(\mathscr{F})=0$ since the singularities are nondegenerates. Thus, it follows from Theorem 1 that

$$\#\left[\mathrm{Sing}(\mathscr{F})\cap\mathbb{P}^n\backslash\mathcal{D}\right] \quad = \quad \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n+1}{n-i-j}(-k)^j(d-1)^i.$$

Now the conclusion of the proof can readily be obtained by the signal study of $\delta(k,d,n)$ that was done in Lemma 6.2. □

Particularly, items (1)(b) and (2)(b) of Theorem 3 characterize the situations in which all of the singularities of $\mathscr{F}$ occur in the invariant hypersurface $\mathcal{D}$. We will present optimal examples.

Example 6.4. Let $\mathscr{F}$ be the foliation on $\mathbb{P}^3$ induced by the polynomial vector field

$$\begin{aligned} v &= (-z_1^{k-1} - z_2^{k-1} - z_3^{k-1})\frac{\partial}{\partial z_0} + (z_0^{k-1} - z_2^{k-1} - z_3^{k-1})\frac{\partial}{\partial z_1} \\ &\quad + (z_0^{k-1} + z_1^{k-1} - z_3^{k-1})\frac{\partial}{\partial z_2} + (z_0^{k-1} + z_1^{k-1} + z_2^{k-1})\frac{\partial}{\partial z_3}. \end{aligned}$$

The hypersurface $\mathcal{D} = \{z_0^k + z_1^k + z_2^k + z_3^k = 0\}$ is invariant by $\mathscr{F}$. It is not difficult to see that $\mathrm{Sing}(\mathscr{F}) \subset \mathcal{D}$. Note that $\deg(\mathcal{D}) = k$ and $\deg(\mathscr{F}) = k-1$, according to item (1)(b) of Theorem 3.

Example 6.5. Consider the foliation $\mathscr{F}$ induced by the vector field $v = \partial/\partial z_0$. For each $1 \le i \le n$ the hypersurface $\mathcal{D}_i = \{z_i = 0\}$ is invariant by $\mathscr{F}$. Moreover, for all $i = 1, \dots, n$ we have

$$\mathrm{Sing}(\mathscr{F}) = \{(1:0:\cdots:0)\} \subset \mathcal{D}_i.$$

Note that we have $\deg(\mathscr{F}) = 0$ and $\deg(\mathcal{D}_i) = 1$ for all i. Therefore, if we consider n even, we have the case of item (2)(b) of Theorem 3.

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