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ENTROPY DIMENSION OF MEASURE PRESERVING SYSTEMS

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ABSTRACT. The notion of metric entropy dimension is introduced to measure the complexity of entropy zero dynamical systems. For measure preserving systems, we define entropy dimension via the dimension of entropy generating sequences. This combinatorial approach provides us with a new insight for analyzing entropy zero systems. We also define the dimension set of a system to investigate the structure of the randomness of the factors of a system. The notion of a uniform dimension in the class of entropy zero systems is introduced as a generalization of a K-system in the case of positive entropy. We investigate joinings among entropy zero systems and prove the disjointness property among some classes of entropy zero systems using dimension sets. Given a topological system, we compare topological entropy dimension with metric entropy dimension.

1. INTRODUCTION

Since entropy was introduced by Kolmogorov from information theory, it has played an important role in the study of dynamical systems. Entropy measures the chaoticity or unpredictability of a system. It is well known as a complete invariant for the Bernoulli automorphism class. Properties of positive entropy systems have been studied in many different respects along with their applications. Compared with positive entropy systems, we have much less understanding and fewer tools for entropy zero systems. Entropy zero systems, which are called deterministic systems in the case of $\mathbb{Z}$-actions, cover a wide class of dynamical systems exhibiting different "random" behaviors or different level of complexities. They range from irrational rotations on a circle (and, more generally, isometry on a compact metric space) to horocycle flows. Also many of the physical systems studied recently show intermittent or weakly chaotic behavior [25, 32]. They have the property that a generic orbit has sequences of 0's with density 1, and hence we would say that they have very low complexity or randomness. They do not have finite invariant measures which are physically meaningful. Hence to analyze the complexity of these systems, the notion of algorithemic information content or Kolmogorov complexity has been employed instead of the entropy. It measures the information content of generic orbits of the system.

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Many general group actions, such as $\mathbb{Z}^n$-actions with entropy zero, have interesting subdynamics. They exhibit diverse complexities, and their noncocompact subgroup actions show very different behavior [2, 21–23]. We mention a few known examples of entropy zero with their properties in the case of $\mathbb{Z}^2$-actions:

(1) (a) $h(\sigma^{(p,q)}) = 0 \ \ \forall (p,q) \in \mathbb{Z}^2$,
(b) for any given $0 < \alpha < 2$,

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} H(\bigvee_{(i,j)\in R_n} \sigma^{-(i,j)} P) > 0,$$

(2) (a) $h(\sigma^{(p,q)}) = 0 \ \ \forall (p,q) \in \mathbb{Z}^2$,
(b)

$$\lim_{n\to\infty} \frac{1}{n} H(\bigvee_{(i,j)\in R_n} \sigma^{-(i,j)} P) > 0,$$

(3) (a) $h(\sigma^{(1,0)}) > 0$
(b) $h(\sigma^{(p,q)}) = 0 \quad \forall (p,q) \neq (n,0)$,
(c)

$$\lim_{n\to\infty} \frac{1}{n} H(\bigvee_{(i,j)\in R_n} \sigma^{-(i,j)} P) > 0,$$

where $\{\sigma^{(p,q)}\}_{(p,q)\in\mathbb{Z}^2}$ is the $\mathbb{Z}^2$-action, $h(\sigma^{(p,q)})$ is the entropy of the single transformation $\sigma^{(p,q)}$, R_n denotes the square of size $n \times n$ in $\mathbb{Z}^2$, and P is some finite measurable partition.

The first example is by Katok and Thouvenot [15] and the second one is constructed in [23]. Although the third example is not written anywhere explicitly, it is known that the example is by Ornstein and Weiss, also independently by Thouvenot.

In his study of cellular automaton maps [21], J. Milnor considered the cellular automaton maps together with horizontal shifts as $\mathbb{Z}^2$-actions of zero entropy. He introduced the notion of directional entropy and investigated the properties of the complexities of these systems via directional entropies and their entropy geometry. Boyle and Lind pursued the study of the entropy geometry further in [2]. Besides Milnor's examples we have many examples whose directional entropies are finite and continuous in all directions including irrational directions [21, 22]. And they have the property $\lim_{n\to\infty} \frac{1}{n} H(\bigvee_{(i,j)\in R_n} \sigma^{-(i,j)} P) > 0$. However, as was shown above in example (2), there are many $\mathbb{Z}^2$-actions whose directional entropy does not capture the complexity of a system. We may say that examples (2) and (3) have complexity in the order of n, while positive entropy systems have complexity in the order of n^2.

Cassaigne constructed a uniformly recurrent point and hence a minimal system of a given subexponential orbit growth rate [3]. In [6], inspired by the viewpoint of *topological independence* (see [13, 14]), the authors gave the definition of topological entropy dimension to analyze entropy zero systems. It measures subexponential but superpolynomial topological complexity via the growth rate of orbits. Together with the examples, some of the properties of the entropy zero topological systems have been investigated. As was shown in physical models, many examples of low complexity do not carry finite invariant measure. Their meaningful invariant measures are σ-finite. Examples of finite measure preserving systems of subexponential growth rate were first constructed in [8]. Katok and Thouvenot introduced the notion of slow entropy for $\mathbb{Z}^2$-actions to show that certain measure preserving $\mathbb{Z}^2$-actions are not realizable by two commuting Lipschitz continuous maps in [15]. Since the "natural" extension of the definition of entropy to slow entropy is not an isomorphism invariant, they use the number of ϵ-balls in the Hamming distance to define slow entropy. It is clear that their definition is easily applied to $\mathbb{Z}$-actions to differentiate the complexity.

In section 2 we will introduce the notion of an entropy generating sequence and a positive entropy sequence to understand the complexity of entropy zero systems. By definition, it is clear that the entropy generating sequence is a sequence along which the system has some independence. First we define the dimension of a subset of $\mathbb{N}$ of density zero and use the notion to define the entropy dimension of a system via the entropy generating sequence. It is clear that the properties should be further investigated to understand the structure of zero entropy systems. We hope that many of the tools developed for the study of the positive entropy class will be investigated in the class of a given entropy dimension. For example, we ask if we can have α-dimension Pinsker σ-algebra and α-dimension Bernoulli in the case that α-entropy exists. We ask also if we have some kind of regularity in the size of the atoms of the iterated partition of these systems. Moreover, since general group actions have many "natural" examples of entropy zero with diverse complexity, we need to extend our study to general group actions. We believe the study of entropy dimension together with the study of subgroup actions will lead us to the understanding of more challenging and interesting properties of entropy zero general group actions.

We briefly describe the content of the paper. In section 2, we introduce the notion of an entropy generating sequence and a positive entropy sequence, which are subsets of $\mathbb{N}$. For a given subset of $\mathbb{N}$, we introduce the notion of dimensions, upper and lower, of a subset to measure the size of the subset. This notion classifies the size of the subsets of density 0. We show (in Proposition 2.4) the relation between the dimensions of an entropy generating sequence and a positive entropy sequence. For a measure preserving system we will define the metric entropy dimension through the dimensions of an entropy generating sequence and a positive entropy sequence. We will study many of the basic properties of entropy dimension. In section 3, we define the dimension set of a system to understand the structure of the complexity of its factors. We also introduce the notion of a uniform dimension whose dimension set consists of a singleton. Using the dimension sets, we also study the property of disjointness among entropy zero systems. We prove a theorem which is more general than the disjointness between K-mixing systems and zero entropy systems. In section 4, for a compact metric space we consider the entropy dimension of a given open cover with respect to a measure and show that the topological entropy dimension is always bigger than or equal to the metric entropy dimension of a topological system. We provide a class of examples of uniform dimension in section 5. In a rough statement, we may say that the property without a factor of smaller entropy dimension corresponds to the K-mixing property without zero entropy factors. Our construction is based on the cutting and stacking method as in [8], but it demands technical arguments to guarantee that no partition has smaller entropy dimension. We need to make level sets of each step "spread out" through the columns of the later towers without increasing the subexponential growth rate of orbits.

We noticed recently that entropy dimension was first introduced in [4]. Another related concept, *scaled entropy*, was introduced to distinguish Bernoullian K-automorphisms with equal entropy by Vershik in [29]. For the study of completely integrable Hamiltonian systems, Marco [19] defined two entropy type invariants, polynomial entropy and weak polynomial entropy, that can be applied to measure polynomial scale of complexity. Since we started our work on the complexity of topological and metric entropy zero systems [1,6,7], there have been several papers published in different directions in the area [5,12,20]. Clearly, this is the beginning of the study of entropy zero systems with many more open questions.

2. Entropy dimension

Let $(X,\mathcal{B},\mu,T)$ be a measure-theoretical dynamical system (MDS, for short), and let $\alpha\in\mathcal{P}_X$, where $\mathcal{P}_X$ denotes the collection of finite measurable partitions of X.

In the case of zero entropy, we want to generalize the definition of entropy to measure the growth rate of the iterated partitions. However it was noticed in [8] that for $P\in\mathcal{P}_X$ the natural extension $C(T,P)=\inf\{\beta:\limsup_{n\to\infty}\frac{1}{n^\beta}H_\mu(\bigvee_{i=0}^{n-1}T^{-i}P)=0\}$ is not an isomorphic invariant. More precisely, the following was proved. If there exists a partition P such that $C(T,P)=\inf\{\beta:\limsup_{n\to\infty}\frac{1}{n^\beta}H_\mu(\bigvee_{i=0}^{n-1}T^{-i}P)=0\}=\alpha>0$, then for any $\alpha<\tau<1$ and $\epsilon>0$, there exists a partition $\tilde{P}$ such that

$$(1)\ |P-\tilde{P}|<\epsilon,\ \text{and}$$

$$(2)\ \inf\{\beta:\limsup_{n\to\infty}\frac{1}{n^\beta}H_\mu(\bigvee_{i=0}^{n-1}\tilde{P})=0\}=\tau.$$

To make $C(T,P)$ an isomorphic invariant, the authors count the number of ϵ-balls in the Hamming distance of n-names and take the limit of n's and ϵ's [8].

Before we introduce the notion of entropy dimension for a measure preserving system, we define the dimension of a subset S of positive integers $\mathbb{N}$. Let $S=\{s_1<s_2<\cdots\}$ be an increasing sequence of positive integers. For $\tau\geq 0$, we define

$$\overline{D}(S,\tau)=\limsup_{n\to\infty}\frac{n}{(s_n)^\tau}\qquad\text{and}\qquad\underline{D}(S,\tau)=\liminf_{n\to\infty}\frac{n}{(s_n)^\tau}.$$

It is clear that $\overline{D}(S,\tau)\leq\overline{D}(S,\tau')$ if $\tau\geq\tau'\geq 0$ and $\overline{D}(S,\tau)\notin\{0,+\infty\}$ for at most one $\tau\geq 0$. We define *the upper dimension of* S by

$$\overline{D}(S)=\inf\{\tau\geq 0:\overline{D}(S,\tau)=0\}=\sup\{\tau\geq 0:\overline{D}(S,\tau)=\infty\}.$$

Similarly, $\underline{D}(S,\tau)\leq\underline{D}(S,\tau')$ if $\tau\geq\tau'\geq 0$ and $\underline{D}(S,\tau)\notin\{0,+\infty\}$ for at most one $\tau\geq 0$. We define the *lower dimension of* S by

$$\underline{D}(S)=\inf\{\tau\geq 0:\underline{D}(S,\tau)=0\}=\sup\{\tau\geq 0:\underline{D}(S,\tau)=\infty\}.$$

Clearly, $0\leq\underline{D}(S)\leq\overline{D}(S)\leq 1$. When $\overline{D}(S)=\underline{D}(S)=\tau$, we say S has dimension τ. For example, if S has positive density, then $\overline{D}(S)=\underline{D}(S)=1$, and if $S=\{n^2|n=1,2,\ldots\}$, then clearly $\overline{D}(S)=\underline{D}(S)=\frac{1}{2}$.

In the following we will investigate the dimension of a special kind of sequence, which is called the *entropy generating sequence*.

Let $(X, \mathcal{B}, \mu, T)$ be an MDS, and let $\alpha \in \mathcal{P}_X$. We say an increasing sequence $S = \{s_1 < s_2 < \cdots\}$ of $\mathbb{N}$ is an *entropy generating sequence* of α if

$$\liminf_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-s_i}\alpha) > 0.$$

We say $S = \{s_1 < s_2 < \cdots\}$ of $\mathbb{N}$ is a *positive entropy sequence* of α if the *sequence entropy* of α along the sequence S, which is defined by

$$h_\mu^S(T, \alpha) := \limsup_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-s_i}\alpha),$$

is positive.

Denote by $\mathcal{E}_\mu(T, \alpha)$ the set of all entropy generating sequences of α, and denote $\mathcal{P}_\mu(T, \alpha)$ by the set of all positive entropy sequences of α. Clearly, $\mathcal{P}_\mu(T, \alpha) \supset \mathcal{E}_\mu(T, \alpha)$.

Definition 2.1. Let $(X, \mathcal{B}, \mu, T)$ be an MDS, and let $\alpha \in \mathcal{P}_X$. We define

$$\overline{D}_\mu^e(T, \alpha) = \begin{cases} \sup\limits_{S\in\mathcal{E}_\mu(T,\alpha)} \overline{D}(S) & \text{if } \mathcal{E}_\mu(T, \alpha) \neq \emptyset, \\ 0 & \text{if } \mathcal{E}_\mu(T, \alpha) = \emptyset, \end{cases}$$

$$\overline{D}_\mu^p(T, \alpha) = \begin{cases} \sup\limits_{S\in\mathcal{P}_\mu(T,\alpha)} \overline{D}(S) & \text{if } \mathcal{P}_\mu(T, \alpha) \neq \emptyset, \\ 0 & \text{if } \mathcal{P}_\mu(T, \alpha) = \emptyset. \end{cases}$$

Similarly, we define $\underline{D}_\mu^e(T, \alpha)$ and $\underline{D}_\mu^p(T, \alpha)$ by changing the upper dimension into lower dimension.

Definition 2.2. Let $(X, \mathcal{B}, \mu, T)$ be an MDS. We define

$$\overline{D}_\mu^e(X, T) = \sup_{\alpha\in\mathcal{P}_X} \overline{D}_\mu^e(T, \alpha), \qquad \underline{D}_\mu^e(X, T) = \sup_{\alpha\in\mathcal{P}_X} \underline{D}_\mu^e(T, \alpha),$$

$$\overline{D}_\mu^p(X, T) = \sup_{\alpha\in\mathcal{P}_X} \overline{D}_\mu^p(T, \alpha), \qquad \underline{D}_\mu^p(X, T) = \sup_{\alpha\in\mathcal{P}_X} \underline{D}_\mu^p(T, \alpha).$$

Since the sequence entropies along a given sequence are the same for mutually conjugated systems, we can deduce that these four quantities are also conjugacy invariants. But the following proposition shows that $\overline{D}_\mu^p(X, T)$ can only take trival values 0 and 1. An MDS $(X, \mathcal{B}, \mu, T)$ is said to be *null* if $h_\mu^S(T, \alpha) = 0$ for any sequence S of $\mathbb{N}$ and $\alpha \in \mathcal{P}_X$. A well known result by Kushnirenko [17] states that an MDS $(X, \mathcal{B}, \mu, T)$ has discrete spectrum if and only if it is null.

Proposition 2.3. *Let $(X, \mathcal{B}, \mu, T)$ be an MDS. Then*

$$\overline{D}_\mu^p(T, \alpha) = \begin{cases} 1 & \textit{if } \mathcal{P}_\mu(T, \alpha) \neq \emptyset, \\ 0 & \textit{if } \mathcal{P}_\mu(T, \alpha) = \emptyset, \end{cases} \quad \textit{for } \alpha \in \mathcal{P}_X.$$

Moreover, $\overline{D}_\mu^p(X, T) = 0$ or 1, and $\overline{D}_\mu^p(X, T) = 0$ if and only if $(X, \mathcal{B}, \mu, T)$ is null.

Proof. When $\mathcal{P}_\mu(T, \alpha) = \emptyset$, $\overline{D}_\mu^p(T, \alpha) = 0$. Now assume $\mathcal{P}_\mu(T, \alpha) \neq \emptyset$, thus there exists $S = \{s_1 < s_2 < \cdots\} \subset \mathbb{N}$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-s_i}\alpha) = a > 0.$$

Next we take $1 \le n_1 < n_2 < n_3 < \cdots$ such that $n_{j+1} \ge 2s_{n_j}$ for each $j \in \mathbb{N}$ and at the same time $\limsup_{j\to+\infty} \frac{1}{n_j} H_\mu(\bigvee_{i=1}^{n_j} T^{-s_i}\alpha) = a$. Then put

$$F = S \cup \{1, 2, \ldots, n_1\} \cup \bigcup_{i=1}^{\infty} \{s_{n_i} + 1, s_{n_i} + 2, \ldots, n_{i+1}\}.$$

For simplicity, we write $F = \{f_1 < f_2 < \cdots\}$. Notice that

$$F \cap [1, s_{n_j}] \subset [1, n_j] \cup (F \cap [n_j + 1, s_{n_j}]) \subset [1, n_j] \cup \{s_1, s_2, \ldots, s_{n_j}\},$$

hence $|F \cap [1, s_{n_j}]| \le 2n_j$. So we have

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-f_i}\alpha) &\ge \limsup_{j\to+\infty} \frac{H_\mu(\bigvee_{i=1}^{n_j} T^{-s_i}\alpha)}{|F \cap [1, s_{n_j}]|} \\ &\ge \limsup_{j\to+\infty} \frac{H_\mu(\bigvee_{i=1}^{n_j} T^{-s_i}\alpha)}{2n_j} = \frac{a}{2} > 0;\end{aligned}$$

therefore $F \in \mathcal{P}_\mu(T, \alpha)$. Since $n_{j+1} \ge 2s_{n_j}$ for each $j \in \mathbb{N}$, it is easy to see that $\overline{D}(F) = 1$. This implies $\overline{D}_\mu^p(T, \alpha) = 1$ as $F \in \mathcal{P}_\mu(T, \alpha)$. □

In the following, we investigate the relations among these dimensions.

Proposition 2.4. *Let $(X, \mathcal{B}, \mu, T)$ be an MDS, and let $\alpha \in \mathcal{P}_X$. Then*

$$\underline{D}_\mu^e(T, \alpha) \le \overline{D}_\mu^e(T, \alpha) = \underline{D}_\mu^p(T, \alpha) \le \overline{D}_\mu^p(T, \alpha).$$

Proof. (1) $\underline{D}_\mu^e(T, \alpha) \le \overline{D}_\mu^e(T, \alpha)$ and $\underline{D}_\mu^p(T, \alpha) \le \overline{D}_\mu^p(T, \alpha)$ are obvious by Definition 2.1.

(2) We will show that $\overline{D}_\mu^e(T, \alpha) \le \underline{D}_\mu^p(T, \alpha)$. If $\overline{D}_\mu^e(T, \alpha) = 0$, then it is obvious that $\overline{D}_\mu^e(T, \alpha) \le \underline{D}_\mu^p(T, \alpha)$. Now we assume that $\overline{D}_\mu^e(T, \alpha) > 0$, and $\tau \in (0, \overline{D}_\mu^e(T, \alpha))$ is given.

There exists $S = \{s_1 < s_2 < \cdots\} \in \mathcal{E}_\mu(T, \alpha)$ with $\overline{D}(S) > \tau$, i.e., $\limsup_{n\to+\infty} \frac{n}{s_n^\tau} = +\infty$. Hence

$$\limsup_{n\to+\infty} \frac{n}{n + s_n^\tau} = 1. \tag{2.1}$$

Next we put $F = S \cup \{\lfloor n^{\frac{1}{\tau}} \rfloor : n \in \mathbb{N}\}$, where $\lfloor r \rfloor$ denotes the largest integer less than or equal to r. Clearly, $\underline{D}(F) \ge \tau$.

Let $F = \{f_1 < f_2 < \cdots\}$. Then for each $n \in \mathbb{N}$ there exists a unique $m(n) \in \mathbb{N}$ such that $s_n = f_{m(n)}$. Since

$$\{s_1, s_2, \ldots, s_n\} \subseteq \{f_1, f_2, \ldots, f_{m(n)}\} \subseteq \{s_1, s_2, \ldots, s_n\} \cup \{\lfloor k^{\frac{1}{\tau}} \rfloor : k \le s_n^\tau\},$$

we have $n \le m(n) \le n + s_n^\tau$. Combining this with (2.1), we get

$$\limsup_{n\to+\infty} \frac{n}{m(n)} = 1. \tag{2.2}$$

Now we have

$$\begin{aligned}
\limsup_{m\to+\infty}\frac{H_\mu(\bigvee_{i=1}^{m}T^{-f_i}\alpha)}{m} &\geq \limsup_{n\to+\infty}\frac{H_\mu(\bigvee_{i=1}^{m(n)}T^{-f_i}\alpha)}{m(n)}\\
&\geq \limsup_{n\to+\infty}\frac{H_\mu(\bigvee_{i=1}^{n}T^{-s_i}\alpha)}{n}\frac{n}{m(n)}\\
&\geq (\liminf_{n\to+\infty}\frac{H_\mu(\bigvee_{i=1}^{n}T^{-s_i}\alpha)}{n})\cdot(\limsup_{n\to+\infty}\frac{n}{m(n)})\\
&= \liminf_{n\to+\infty}\frac{H_\mu(\bigvee_{i=1}^{n}T^{-s_i}\alpha)}{n} \quad \text{(by (2.2))}\\
&> 0 \qquad \text{(since } S\in\mathcal{E}_\mu(T,\alpha)).
\end{aligned}$$

This implies $F\in\mathcal{P}_\mu(T,\alpha)$. Hence $\underline{D}_\mu^p(T,\alpha)\geq\underline{D}(F)\geq\tau$. Since τ is arbitrary in $(0,\overline{D}_\mu^e(T,\alpha))$, we have $\overline{D}_\mu^e(T,\alpha)\leq\underline{D}_\mu^p(T,\alpha)$.

(3) We need to prove that $\underline{D}_\mu^p(T,\alpha)\leq\overline{D}_\mu^e(T,\alpha)$. If $\underline{D}_\mu^p(T,\alpha)=0$, then it is obvious that $\underline{D}_\mu^p(T,\alpha)\leq\overline{D}_\mu^e(T,\alpha)$. Now we assume that $\underline{D}_\mu^p(T,\alpha)>0$ and $\tau\in(0,\underline{D}_\mu^p(T,\alpha))$ is given.

In the following, we show that

Fact A. There exist a sequence $F=\{f_1<f_2<\cdots\}$ of natural numbers and a real number $d>0$ such that $\overline{D}(F)\geq\tau$, and for any $1\leq m_1\leq m_2$,

$$H_\mu(\bigvee_{i=m_1}^{m_2}T^{-f_i}\alpha)\geq(m_2+1-m_1)d. \tag{2.3}$$

Moreover by (2.3) we know $F\in\mathcal{E}_\mu(T,\alpha)$. Hence $\overline{D}_\mu^e(T,\alpha)\geq\overline{D}(F)\geq\tau$. Finally, since τ is arbitrary, we have $\overline{D}_\mu^e(T,\alpha)\geq\underline{D}_\mu^p(T,\alpha)$.

Now it remains to prove Fact A. First, there exists $S=\{s_1<s_2<\cdots\}\in\mathcal{P}_\mu(T,\alpha)$ with $\underline{D}(S)>\tau$, i.e., $\liminf_{n\to+\infty}\frac{n}{s_n^\tau}=+\infty$. Hence there exists $a>0$ such that

$$an\geq s_n^\tau \tag{2.4}$$

for all $n\in\mathbb{N}$.

Since $S\in\mathcal{P}_\mu(T,\alpha)$, there exist an increasing sequence $\{n_1<n_2<\cdots<n_k<\cdots\}$ of positive integers and $0<b<4$ such that $H_\mu(\bigvee_{i=1}^{n_k}T^{-s_i}\alpha)\geq n_kb$ for all $k\in\mathbb{N}$. Without loss of generality (if necessary, we choose a subsequence), we assume that $n_{k+1}\geq\frac{4(H_\mu(\alpha)+1)}{b}\sum_{j=1}^k n_j$ for all $k\in\mathbb{N}$. Let $c=\frac{b}{4(H_\mu(\alpha)+1)}$ and $n_0=0$. Then $0<c<1$, and we have the following.

Claim. For each $k\in\mathbb{N}$, there exist $l_k\in\mathbb{N}$ and

$$F_k:=\{i_1^k<i_2^k<\cdots<i_{l_k}^k\}\subseteq\{n_{k-1}+1,n_{k-1}+2,\ldots,n_k\}$$

such that $cn_k\leq l_k\leq n_k-n_{k-1}$ and $H_\mu(\bigvee_{i\in F_k'}T^{-s_i}\alpha)\geq|F_k'|\frac{b}{4}$ for each $\emptyset\neq F_k'\subseteq F_k$.

Proof of Claim. Assume that the Claim is not true. Then for some $k \in \mathbb{N}$ there exist $w \in \mathbb{N}$ and $E_1, E_2, \ldots, E_w \subseteq \{n_{k-1}+1, n_{k-1}+2, \ldots, n_k\}$ such that $1 \leq |E_1|, |E_2|, \ldots, |E_w| < cn_k$, $E_i \cap E_j = \emptyset$ for any $1 \leq i < j \leq w$ and $\bigcup_{i=1}^{w} E_i = \{n_{k-1}+1, n_{k-1}+2, \ldots, n_k\}$ and for $1 \leq j \leq w-1$, $H_\mu(\bigvee_{t \in E_j} T^{-s_t}\alpha) < |E_j|\frac{b}{4}$. This implies that

$$\begin{aligned}
H_\mu(\bigvee_{i=1}^{n_k} T^{-s_i}\alpha) &\leq H_\mu(\bigvee_{i=1}^{n_{k-1}} T^{-s_i}\alpha) + \sum_{j=1}^{w} H_\mu(\bigvee_{t \in E_j} T^{-s_t}\alpha) \\
&\leq n_{k-1}H_\mu(\alpha) + \sum_{j=1}^{w-1} |E_j|\frac{b}{4} + |E_w|H_\mu(\alpha) \leq \frac{b}{4}n_k + \frac{b}{4}(n_k - n_{k-1}) + cn_k H_\mu(\alpha) \\
&\leq \frac{b}{4}n_k + \frac{b}{4}n_k + \frac{b}{4}n_k < bn_k,
\end{aligned}$$

a contradiction. This completes the proof of the Claim. □

Let $F = \bigcup_{k=1}^{\infty}\{s_i : i \in F_k\}$. For simplicity, we write $F = \{f_1 < f_2 < \cdots\}$. Then

$$\begin{aligned}
\overline{D}(F,\tau) &= \limsup_{m \to +\infty} \frac{m}{f_m^\tau} \geq \limsup_{v \to +\infty} \frac{\sum_{k=1}^{v} l_k}{(f_{\sum_{k=1}^{v} l_k})^\tau} = \limsup_{v \to +\infty} \frac{\sum_{k=1}^{v} l_k}{(s_{i_{l_v}^v})^\tau} \\
&\geq \limsup_{v \to +\infty} \frac{l_v}{(s_{n_v})^\tau} \geq \limsup_{v \to +\infty} \frac{l_v}{an_v} \quad \text{(by (2.4))} \\
&\geq \limsup_{v \to +\infty} \frac{cn_v}{an_v} \geq \frac{c}{a} > 0.
\end{aligned}$$

Hence $\overline{D}(F) \geq \tau$.

For a given $m \in \mathbb{N}$, there exists a unique $k(m) \in \mathbb{N}$ such that $\sum_{k=0}^{k(m)-1} l_k < m \leq \sum_{k=1}^{k(m)} l_k$, where $l_0 = 0$. Set $r(m) = m - \sum_{k=0}^{k(m)-1} l_k$. Then $f_m = s_{i_{r(m)}^{k(m)}}$. Now for $1 \leq m_1 \leq m_2$, there are three cases.

Case 1. $k(m_1) = k(m_2)$. Then

$$\begin{aligned}
H_\mu(\bigvee_{i=m_1}^{m_2} T^{-f_i}\alpha) &= H_\mu(\bigvee_{j=r(m_1)}^{r(m_2)} T^{-s_{i_j^{k(m_1)}}}\alpha) \geq \frac{b}{4}(r(m_2) + 1 - r(m_1)) \quad \text{(by Claim)} \\
&= \frac{b}{4}(m_2 - m_1 + 1).
\end{aligned}$$

Case 2. $k(m_2) = k(m_1) + 1$. Then

$$\begin{aligned}
H_\mu(\bigvee_{i=m_1}^{m_2} T^{-f_i}\alpha) &= H_\mu(\bigvee_{j=r(m_1)}^{l_{k(m_1)}} T^{-s_{i_j^{k(m_1)}}}\alpha \vee \bigvee_{j=1}^{r(m_2)} T^{-s_{i_j^{k(m_2)}}}\alpha) \\
&\geq \frac{1}{2}\left(H_\mu(\bigvee_{j=r(m_1)}^{l_{k(m_1)}} T^{-s_{i_j^{k(m_1)}}}\alpha) + H_\mu(\bigvee_{j=1}^{r(m_2)} T^{-s_{i_j^{k(m_2)}}}\alpha)\right) \\
&\geq \frac{b}{8}\left((l_{k(m_1)} + 1 - r(m_1)) + r(m_2)\right) = \frac{b}{8}(m_2 - m_1 + 1).
\end{aligned}$$

Case 3. $k(m_2) \geq k(m_1)+2$. Then

$$\begin{aligned}
H_\mu(\bigvee_{i=m_1}^{m_2} T^{-f_i}\alpha) &\geq H_\mu(\bigvee_{j=1}^{l_{k(m_2)-1}} T^{-s_{i_j^{k(m_2)-1}}}\alpha \vee \bigvee_{j=1}^{r(m_2)} T^{-s_{i_j^{k(m_2)}}}\alpha) \\
&\geq \frac{1}{2}\left(H_\mu(\bigvee_{j=1}^{l_{k(m_2)-1}} T^{-s_{i_j^{k(m_2)-1}}}\alpha) + H_\mu(\bigvee_{j=1}^{r(m_2)} T^{-s_{i_j^{k(m_2)}}}\alpha)\right) \\
&\geq \frac{b}{8}(l_{k(m_2)-1}+r(m_2)) \geq \frac{b}{8}(cn_{k(m_2)-1}+r(m_2)) \text{ (by Claim)} \\
&\geq \frac{b}{8}(c\sum_{j=1}^{k(m_2)-1} l_j + cr(m_2)) \quad \text{(by Claim)} \\
&= \frac{bc}{8}m_2 \geq \frac{bc}{8}(m_2-m_1+1).
\end{aligned}$$

Let $d=\frac{bc}{8}$. Then (2.3) follows from the above three cases. □

The following theorem is a direct application of Proposition 2.4.

Theorem 2.5. *Let* $(X,\mathcal{B},\mu,T)$ *be an MDS. Then* $\overline{D}_e(X,T)=\underline{D}_p(X,T)$.

By Proposition 2.4 and Theorem 2.5, we have the following definitions.

Definition 2.6. Let $(X,\mathcal{B},\mu,T)$ be an MDS, and let $\alpha\in\mathcal{P}_X$. We define

$$\overline{D}_\mu(T,\alpha) := \overline{D}_\mu^e(T,\alpha) = \underline{D}_\mu^p(T,\alpha),$$

which is called the *upper entropy dimension of* α, and we define

$$\underline{D}_\mu(T,\alpha) := \underline{D}_\mu^e(T,\alpha)$$

to be the *lower entropy dimension of* α. When $\overline{D}_\mu(T,\alpha)=\underline{D}_\mu(T,\alpha)$, we note this quantity $D_\mu(T,\alpha)$, *the entropy dimension of* α.

Definition 2.7. Let $(X,\mathcal{B},\mu,T)$ be an MDS. We define

$$\overline{D}_\mu(X,T) = \sup_{\alpha\in\mathcal{P}_X} \overline{D}_\mu(T,\alpha),$$

which is called the *upper metric entropy dimension* of $(X,\mathcal{B},\mu,T)$, and we define

$$\underline{D}_\mu(X,T) = \sup_{\alpha\in\mathcal{P}_X} \underline{D}_\mu(T,\alpha),$$

which is called the *lower metric entropy dimension* of $(X,\mathcal{B},\mu,T)$. When $\overline{D}_\mu(X,T)=\underline{D}_\mu(X,T)$, we denote the quantity by $D_\mu(X,T)$ and call it the *metric entropy dimension* of $(X,\mathcal{B},\mu,T)$.

By Proposition 2.3 we have the following.

Theorem 2.8. *Let* $(X,\mathcal{B},\mu,T)$ *be a null MDS. Then* $D_\mu(X,T)=0$.

In the following, we study the basic properties of the entropy dimension of a measure preserving system. But since the upper dimension and the lower dimension do not agree in general, we discuss the properties of the upper dimension. We note that they hold for the entropy dimension.

Proposition 2.9. *Let $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{D},\nu,S)$ be two MDSs, and let $\alpha,\beta\in\mathcal{P}_X,\eta\in\mathcal{P}_Y$. Then*

(1) *If $\alpha\preceq\beta$, then $\overline{D}_\mu(T,\alpha)\le\overline{D}_\mu(T,\beta)$, where by $\alpha\preceq\beta$ we mean that every atom of β is contained in one of the atoms of α.*
(2) *For any $0\le m\le n$, $\overline{D}_\mu(T,\alpha)=\overline{D}_\mu(T,\bigvee_{i=m}^n T^{-i}\alpha)$.*
(3) *$\overline{D}_\mu(T,\alpha\vee\beta)=\max\{\overline{D}_\mu(T,\alpha),\overline{D}_\mu(T,\beta)\}$.*
(4) *$\overline{D}_\mu(X,T)=\sup\{\overline{D}_\mu(T,\alpha):\alpha\in\mathcal{P}_X^2\}$, where $\mathcal{P}_X^2$ denotes the set of all partitions by two measurable sets of X.*
(5) *$\overline{D}_{\mu\times\nu}(T\times S,\alpha\times\eta)=\max\{\overline{D}_\mu(T,\alpha),\overline{D}_\nu(S,\eta)\}$.*

Statements (1) *and* (2) *also hold for lower dimensions.*

Proof. By the definition, (1) and (2) are obvious. Also (5) follows from (3). For (3), we first have $\overline{D}_\mu(T,\alpha\vee\beta)\ge\max\{\overline{D}_\mu(T,\alpha),\overline{D}_\mu(T,\beta)\}$ by (1). Second, if $\overline{D}_\mu(T,\alpha\vee\beta)=0$, then it is clear that $\overline{D}_\mu(T,\alpha\vee\beta)=\max\{\overline{D}_\mu(T,\alpha),\overline{D}_\mu(T,\beta)\}$. Now we assume that $0<\overline{D}_\mu(T,\alpha\vee\beta)$. For any $\tau\in(0,\overline{D}_\mu(T,\alpha\vee\beta))$, there exists $S=\{s_1<s_2<\cdots\}\in\mathcal{P}_\mu(T,\alpha\vee\beta)$ with $\underline{D}(S)>\tau$.

Since $S\in\mathcal{P}_\mu(T,\alpha\vee\beta)$, $\limsup_{n\to+\infty}\frac{1}{n}H_\mu(\bigvee_{i=1}^n T^{-s_i}(\alpha\vee\beta))>0$. This implies

$$\limsup_{n\to+\infty}\frac{1}{n}H_\mu(\bigvee_{i=1}^n T^{-s_i}\alpha)>0\quad\text{or}\quad\limsup_{n\to+\infty}\frac{1}{n}H_\mu(\bigvee_{i=1}^n T^{-s_i}\beta)>0;$$

that is, $S\in\mathcal{P}_\mu(T,\alpha)$ or $S\in\mathcal{P}_\mu(T,\beta)$. Hence $\tau\le\underline{D}(S)\le\max\{\overline{D}_\mu(T,\alpha),\overline{D}_\mu(T,\beta)\}$. As τ is arbitary, we get $\overline{D}_\mu(T,\alpha\vee\beta)=\max\{\overline{D}_\mu(T,\alpha),\overline{D}_\mu(T,\beta)\}$.

Now we are to show (4). Clearly, $\overline{D}_\mu(X,T)\ge\sup\{\overline{D}_\mu(T,\alpha):\alpha\in\mathcal{P}_X^2\}$. Conversely, for any $\alpha=\{A_1,\dots,A_k\}\in\mathcal{P}_X$, let $\alpha_i=\{A_i,A_i^c\}$ for $i=1,2,\dots,k$. Then $\bigvee_{i=1}^k\alpha_i\succeq\alpha$. Hence by (1) and (3), we have

$$\overline{D}_\mu(T,\alpha)\le\max\{\overline{D}_\mu(T,\alpha_i):1\le i\le k\}\le\sup\{\overline{D}_\mu(T,\alpha):\alpha\in\mathcal{P}_X^2\}.$$

Finally, since α is arbitrary, we get (4). □

For two partitions $\alpha=\{A_1,A_2,\dots,A_k\}$ and $\beta=\{B_1,B_2,\dots,B_k\}\in\mathcal{P}_X$, denote by $\mu(\beta\Delta\alpha):=\sum_{i=1}^k\mu(B_i\Delta A_i)$.

Lemma 2.10. *Let $(X,\mathcal{B},\mu,T)$ be an MDS, and let $\alpha=\{A_1,A_2,\dots,A_k\}\in\mathcal{P}_X$. Then for any $\epsilon>0$, there exists $\delta>0$ such that for any $\beta=\{B_1,B_2,\dots,B_k\}\in\mathcal{P}_X$ with $\mu(\beta\Delta\alpha)<\delta$, it holds that*

(1) *$\overline{D}_\mu(T,\beta)>\overline{D}_\mu(T,\alpha)-\epsilon$,*
(2) *$\underline{D}_\mu(T,\beta)>\underline{D}_\mu(T,\alpha)-\epsilon$, and*
(3) *$D_\mu(T,\beta)>D_\mu(T,\alpha)-\epsilon$ when the dimensions exist.*

Proof. We only prove this for the upper dimension. If $\overline{D}_\mu(T,\alpha)=0$, it is obvious. Now assume that $\overline{D}_\mu(T,\alpha)>0$. For any $\epsilon>0$, there exists $S=\{s_1<s_2<s_3<\cdots\}\in\mathcal{E}_\mu(T,\alpha)$ with $\overline{D}(S)>\overline{D}_\mu(T,\alpha)-\epsilon$. There exists $\delta>0$ such that if $\beta=\{B_1,B_2,\dots,B_k\}\in\mathcal{P}_X$ and $\mu(\beta\Delta\alpha)<\delta$, then

$$H_\mu(\alpha|\beta)+H_\mu(\beta|\alpha)<\frac{1}{2}\liminf_{n\to\infty}\frac{1}{n}H_\mu(\bigvee_{i=1}^n T^{-s_i}\alpha)$$

(see Lemma 4.15 in [30]).

For any $\beta = \{B_1, B_2, \ldots, B_k\} \in \mathcal{P}_X$ and $\mu(\beta\Delta\alpha) < \delta$,

$$\begin{aligned}
&\liminf_{n\to+\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^n T^{-s_i}\beta) \\
&\geq \liminf_{n\to+\infty} \frac{1}{n}\left(H_\mu(\bigvee_{i=1}^n T^{-s_i}(\alpha\vee\beta)) - H_\mu(\bigvee_{i=1}^n T^{-s_i}\beta | \bigvee_{i=1}^n T^{-s_i}\alpha)\right) \\
&\geq \liminf_{n\to+\infty} \frac{1}{n}\left(H_\mu(\bigvee_{i=1}^n T^{-s_i}\alpha) - nH_\mu(\beta|\alpha)\right) \\
&\geq \frac{1}{2}\liminf_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^n T^{-s_i}\alpha) > 0,
\end{aligned}$$

that is, $S \in \mathcal{E}_\mu(T,\beta)$. Hence $\overline{D}_\mu(T,\beta) \geq \overline{D}(S) > \overline{D}_\mu(T,\alpha) - \epsilon$. □

Theorem 2.11. *Let $(X, \mathcal{B}, \mu, T)$ be an MDS.*

(1) *If $\{\alpha_i\}_{i\in\mathbb{N}} \subset \mathcal{P}_X$ and $\alpha \in \mathcal{P}_X$ satisfying $\alpha \preceq \bigvee_{i\in\mathbb{N}} \alpha_i$, then $\overline{D}_\mu(T,\alpha) \leq \sup_{i\geq 1} \overline{D}_\mu(T,\alpha_i)$.*

(2) *If $\{\alpha_i\}_{i\in\mathbb{N}} \subset \mathcal{P}_X$ and $\alpha_i \nearrow \mathcal{B} \pmod{\mu}$, then $\overline{D}_\mu(X,T) = \lim_{i\to+\infty} \overline{D}_\mu(T,\alpha_i)$. Moreover, if α is a generating partition, i.e., $\bigvee_{i=0}^\infty T^{-i}\alpha = \mathcal{B} \pmod{\mu}$, then $\overline{D}_\mu(X,T) = \overline{D}_\mu(T,\alpha)$.*

(3) *If $\{\alpha_i\}_{i\in\mathbb{N}} \subset \mathcal{P}_X$ and $\bigvee_{i\in\mathbb{N}} \alpha_i = \mathcal{B} \pmod{\mu}$, then $\overline{D}_\mu(X,T) = \sup_{i\geq 1} \overline{D}_\mu(T,\alpha_i)$.*

Proof. It is obvious that (1) implies (2) and (3). Now we show (1). Let $\alpha = \{A_1, A_2, \ldots, A_k\} \in \mathcal{P}_X$ and fix $\epsilon > 0$. By Lemma 2.10 there exists $\delta > 0$ such that if $\beta = \{B_1, B_2, \ldots, B_k\} \in \mathcal{P}_X$ and $\mu(\beta\Delta\alpha) := \sum_{i=1}^k \mu(B_i\Delta A_i) < \delta$, then $\overline{D}_\mu(T,\beta) > \overline{D}_\mu(T,\alpha) - \epsilon$. Since $\alpha \preceq \bigvee_{i\in\mathbb{N}} \alpha_i$, there exist $N \in \mathbb{N}$ and $\gamma = \{C_1, C_2, \ldots, C_k\} \preceq \bigvee_{i=1}^N \alpha_i$ such that $\mu(\gamma\Delta\alpha) < \delta$. Thus $\overline{D}_\mu(T,\gamma) \geq \overline{D}_\mu(T,\alpha) - \epsilon$, and so

$$\begin{aligned}
\sup_{i\geq 1} \overline{D}_\mu(T,\alpha_i) &\geq \max\{\overline{D}_\mu(T,\alpha_i) | i = 1, 2, \ldots, N\} = \overline{D}_\mu(T, \bigvee_{i=1}^N \alpha_i) \\
&\geq \overline{D}_\mu(T,\gamma) \geq \overline{D}_\mu(T,\alpha) - \epsilon.
\end{aligned}$$

Since the above inequality is true for any $\epsilon > 0$, we get $\sup_{i\geq 1} \overline{D}_\mu(T,\alpha_i) \geq \overline{D}_\mu(T,\alpha)$. □

Proposition 2.12. *Let $(X, \mathcal{B}, \mu, T)$ be an MDS, and let $\alpha \in \mathcal{P}_X$.*

(1) *For $k \in \mathbb{N}$, we have $\overline{D}_\mu(T^k,\alpha) = \overline{D}_\mu(T,\alpha)$. Moreover, $\overline{D}_\mu(X,T^k) = \overline{D}_\mu(X,T)$, and this is also true for lower dimensions and dimensions whenever the dimensions exist.*

(2) *When T is invertible, we have $\overline{D}_\mu(T,\alpha) = \overline{D}_\mu(T^{-1},\alpha)$ and hence $\overline{D}_\mu(X,T) = \overline{D}_\mu(X,T^{-1})$.*

Proof. (1) We only prove this for the upper dimension. Given $k \in \mathbb{N}$. Let $S \in \mathcal{E}_\mu(T^k,\alpha)$. Then $kS = \{ks : s \in S\} \in \mathcal{E}_\mu(T,\alpha)$. Since $\overline{D}(kS) = \overline{D}(S)$, $\overline{D}_\mu(T,\alpha) \geq \underline{D}(S)$. Finally, since S is arbitrary in $\mathcal{E}_\mu(T,\alpha)$, $\overline{D}_\mu(T,\alpha) \geq \overline{D}_\mu(T^k,\alpha)$.

Conversely, let $S = \{s_1 < s_2 < \cdots\} \in \mathcal{E}_\mu(T,\alpha)$. Without loss of generality, we assume $s_1 \geq k$. Set $S_1 = \{\lfloor \frac{s_i}{k} \rfloor : i \in \mathbb{N}\}$. For simplicity of notation, we write $S_1 = \{t_1 < t_2 < \cdots\}$. Then $\lfloor \frac{s_j}{k} \rfloor \leq t_j \leq \lfloor \frac{s_j + k}{k} \rfloor$ for all $j \in \mathbb{N}$.

Now

$$\liminf_{n\to+\infty}\frac{H_\mu(\bigvee_{j=1}^{n} T^{-kt_j}\alpha)}{n} \geq \liminf_{n\to+\infty}\frac{H_\mu(\bigvee_{i=0}^{k-1}\bigvee_{j=1}^{n} T^{-(kt_j+i)}\alpha)}{kn} \geq \liminf_{n\to+\infty}\frac{H_\mu(\bigvee_{j=1}^{n} T^{-s_j}\alpha)}{kn} > 0$$

as $S \in \mathcal{E}_\mu(T,\alpha)$. This implies that $S_1 \in \mathcal{E}_\mu(T^k,\alpha)$. Since $\overline{D}(S) = \overline{D}(S_1)$, $\overline{D}_\mu(T^k,\alpha) \geq \overline{D}(S_1) = \overline{D}(S)$. Finally, since S is arbitrary, $\overline{D}_\mu(T^k,\alpha) \geq \overline{D}_\mu(T,\alpha)$.

(2) Let T be invertible. By symmetry of T and T^{-1}, it is sufficient to show $\overline{D}_\mu(T^{-1},\alpha) \geq \overline{D}_\mu(T,\alpha)$. If $\overline{D}_\mu(T,\alpha) = 0$, this is obvious. Now we assume that $\overline{D}_\mu(T,\alpha) > 0$. Given $\tau \in (0, \overline{D}_\mu(T,\alpha))$.

By Fact A in the proof of Proposition 2.4, we know that there exists a sequence $S = \{s_1 < s_2 < \cdots\} \subset \mathbb{N}$ and $a > 0$ such that $\overline{D}(S) > \tau$ and for any $1 \leq m_1 \leq m_2$,

$$H_\mu(\bigvee_{i=m_1}^{m_2} T^{-s_i}\alpha) \geq (m_2 + 1 - m_1)a. \tag{2.5}$$

Since $\overline{D}(S) > \tau$, there exists a sequence $\{n_1 < n_2 < \cdots\}$ such that $n_1 \geq 2$, $n_{i+1} \geq 1 + 2\sum_{j=1}^{i} n_i$ and $n_i \geq s_{n_i}^{\tau}$ for all $i \in \mathbb{N}$. Let $n_0 = 0$, $s_0 = 0$, $f_0 = 0$, and $f_m = s_{n_j} - s_{n_j - m}$ if $n_{j-1} < m \leq n_j$ for some $j \in \mathbb{N}$. Put $F = \{f_m : m \in \mathbb{N}\}$.

Set $n_{-1} = 0$. Given $n \in \mathbb{N}$ with $n \geq n_1 + 1$, there exists $j \geq 2$ such that $n_{j-1} < n \leq n_j$. Now we have

$$\begin{aligned}
&H_\mu(\bigvee_{m=1}^{n} T^{f_m}\alpha)\\
&\geq \max\{H_\mu(\bigvee_{m=n_{j-2}+1}^{n_{j-1}} T^{f_m}\alpha), H_\mu(\bigvee_{m=n_{j-1}+1}^{n} T^{f_m}\alpha)\}\\
&= \max\{H_\mu(\bigvee_{m=n_{j-2}+1}^{n_{j-1}} T^{s_{n_{j-1}}-s_{n_{j-1}-m}}\alpha), H_\mu(\bigvee_{m=n_{j-1}+1}^{n} T^{s_{n_j}-s_{n_j-m}}\alpha)\}\\
&= \max\{H_\mu(\bigvee_{m=n_{j-2}+1}^{n_{j-1}} T^{-s_{n_{j-1}-m}}\alpha), H_\mu(\bigvee_{m=n_{j-1}+1}^{n} T^{-s_{n_j-m}}\alpha)\}\\
&\geq \max\{a(n_{j-1} - n_{j-2} - 1), a(n - n_{j-1})\} \qquad \text{(by (2.5))}\\
&\geq \max\{\frac{a}{2}n_{j-1}, \frac{a}{2}(n - n_{j-1})\}\\
&\geq \frac{a}{4}(n_{j-1} + n - n_{j-1})\\
&= \frac{a}{4}n,
\end{aligned}$$

that is,

$$H_\mu(\bigvee_{m=1}^{n} T^{f_m}\alpha) \geq \frac{a}{4}n. \tag{2.6}$$

Since (2.6) is true for any $n \geq n_1 + 1$, $F \in \mathcal{E}_\mu(T^{-1}, \alpha)$. Now note that $f_{n_j} = s_{n_j}$ for $j \in \mathbb{N}$, one has

$$\overline{D}(F, \tau) = \limsup_{m \to +\infty} \frac{m}{f_m^\tau} \geq \limsup_{j \to +\infty} \frac{n_j}{s_{n_j}^\tau} \geq 1.$$

Hence $\overline{D}(F) \geq \tau$. Moreover $\overline{D}_\mu(T^{-1}, \alpha) \geq \overline{D}(F) \geq \tau$ as $F \in \mathcal{E}_\mu(T^{-1}, \alpha)$. Finally, since τ is arbitrary in $(0, \overline{D}_\mu(T, \alpha))$, we have $\overline{D}_\mu(T^{-1}, \alpha) \geq \overline{D}_\mu(T, \alpha)$. □

3. Factors and joinings

In this section, we will introduce notions such as dimension sets, dimension σ-algebras, and uniform dimension systems to understand the structure of entropy zero systems.

When the metric entropy dimension of an MDS exists, the entropy dimensions of its factors still may not exist. One of the easy examples is a product system, one with entropy dimension but the other system with no entropy dimension. So in this section we are to consider the upper entropy dimension.

Definition 3.1. We define the *dimension set* of an MDS $(X, \mathcal{B}, \mu, T)$ by

$$\begin{aligned} \mathrm{Dims}_\mu(X, T) &= \{\overline{D}_\mu(T, \{A, X \setminus A\}) : A \in \mathcal{B} \text{ and } 0 < \mu(A) < 1\} \\ &= \{\overline{D}_\nu(Y, S) : (Y, \mathcal{D}, \nu, S) \text{ is a factor of } (X, \mathcal{B}, \mu, T)\}. \end{aligned}$$

Remark 3.2. It is clear that $\mathrm{Dims}_\mu(X, T) = \emptyset$ if and only if $(X, \mathcal{B}, \mu, T)$ is a trivial system, i.e., $\mathcal{B} = \{\emptyset, X\}$ (modμ). We use the convention $\sup\{\tau \in \mathrm{Dims}_\mu(X, T)\} = 0$ when $\mathrm{Dims}_\mu(X, T) = \emptyset$. Thus $\overline{D}_\mu(X, T) = \sup\{\tau \in \mathrm{Dims}_\mu(X, T)\}$.

Let $(X, \mathcal{B}, \mu, T)$ be an MDS. Let

$$P_\mu(T) = \{A \in \mathcal{B} : \text{ the measure-theoretic entropy of } \{A, X \setminus A\} \text{ is zero}\}.$$

It is a T-invariant sub-σ-algebra of $\mathcal{B}$ and is known as the *Pinsker σ-algebra* of $(X, \mathcal{B}, \mu, T)$. Moreover, for $k \geq 1$, $P_\mu(T) = P_\mu(T^k)$. If in addition T is invertible, then $P_\mu(T) = P_\mu(T^{-1})$. Recall that the MDS $(X, \mathcal{B}, \mu, T)$ is said to have *completely positive entropy* (c.p.e., for short) if $P_\mu(T) = \{X, \emptyset\}$. By the well-known Rohlin–Sinai Theorem ([26]), an MDS is a K-system if and only if it has c.p.e. For more details, one may see Parry or Glasner's books ([10, 24]) for references.

For $\tau \in [0, 1)$, we define

$$P_\mu^\tau(T) := \{A \in \mathcal{B} : \overline{D}_\mu(T, \{A, X \setminus A\}) \leq \tau\}.$$

It is clear that $P_\mu^{\tau_1}(T) \subseteq P_\mu^{\tau_2}(T) \subseteq P_\mu(T)$ for any $0 \leq \tau_1 \leq \tau_2 < 1$. The following theorem lists some basic properties of $P_\mu^\tau(T)$.

Theorem 3.3. *Let $(X, \mathcal{B}, \mu, T)$ be an MDS, and let $\tau \in [0, 1)$. Then*

(1) *$P_\mu^\tau(T)$ is a sub-σ-algebra of $\mathcal{B}$.*
(2) *$T^{-1} P_\mu^\tau(T) = P_\mu^\tau(T)$ (mod μ).*
(3) *For $k \geq 1$, $P_\mu^\tau(T) = P_\mu^\tau(T^k)$. If T is invertible, then $P_\mu^\tau(T) = P_\mu^\tau(T^{-1})$.*

Proof. (1) Clearly, $\emptyset, X \in P_\mu^\tau(T)$. Let $A, B \in P_\mu^\tau(T)$. Since $X \setminus (X \setminus A) = A$, $X \setminus A \in P_\mu^\tau(T)$.

Let $A_i \in P_\mu^\tau(T)$, $i \in \mathbb{N}$. Now we are to show that $\bigcup_{i=1}^\infty A_i \in P_\mu^\tau(T)$, i.e.,

$$\overline{D}_\mu(T, \{\bigcup_{i=1}^\infty A_i, X \setminus \bigcup_{i=1}^\infty A_i\}) \leq \tau.$$

Since $\{\bigcup_{i=1}^{\infty} A_i, X \setminus \bigcup_{i=1}^{\infty} A_i\} \subseteq \bigvee_{i=1}^{\infty}\{A_i, X \setminus A_i\}$, using Theorem 2.11(1), we get

$$\overline{D}_\mu(T, \{\bigcup_{i=1}^{\infty} A_i, X \setminus \bigcup_{i=1}^{\infty} A_i\}) \le \sup_{i\in\mathbb{N}} \overline{D}_\mu(T, \{A_i, X \setminus A_i\}) \le \tau.$$

Hence $\bigcup_{i=1}^{\infty} A_i \in P_\mu^\tau(T)$. This shows $P_\mu^\tau(T)$ is a sub-σ-algebra of $\mathcal{B}$.

(2) Since for $A \in \mathcal{B}$,

$$\overline{D}_\mu(T, \{A, X \setminus A\}) = \overline{D}_\mu(T, T^{-1}\{A, X \setminus A\}) = \overline{D}_\mu(T, \{T^{-1}A, X \setminus T^{-1}(A)\}),$$

we have $T^{-1}P_\mu^\tau(T) \subseteq P_\mu^\tau(T)$.

Conversely, let $A \in P_\mu^\tau(T)$. Then $A \in P_\mu^\tau(T) \subseteq P_\mu(T) = T^{-1}P_\mu(T)$. Hence there exists $B \in \mathcal{B}$ such that $A = T^{-1}B$. Now note that

$$\overline{D}_\mu(T, \{B, X \setminus B\}) = \overline{D}_\mu(T, \{T^{-1}B, T^{-1}(X \setminus B)\}) = \overline{D}_\mu(T, \{A, X \setminus A\}) \le \tau,$$

we have $B \in P_\mu^\tau(T)$, i.e., $A \in T^{-1}P_\mu^\tau(T)$. Therefore $P_\mu^\tau(T) = T^{-1}P_\mu^\tau(T)$.

(3) For $\alpha \in \mathcal{P}_X$ and $k \in \mathbb{N}$, we have $\overline{D}_\mu(T^k, \alpha) = \overline{D}_\mu(T, \alpha)$ (see Proposition 2.12). Hence $\alpha \subset P_\mu^\tau(T)$ if and only if $\alpha \subset P_\mu^\tau(T^k)$. This implies $P_\mu^\tau(T) = P_\mu^\tau(T^k)$. Finally, $P_\mu^\tau(T) = P_\mu^\tau(T^{-1})$ by Proposition 2.12. □

We call $P_\mu^\tau(T)$ the *τ^--dimensional sub-σ-algebra* of $(X, \mathcal{B}, \mu, T)$, and if $P_\mu^\tau(T) = \mathcal{B}$, we call $(X, \mathcal{B}, \mu, T)$ a *τ^--dimensional system.*

The following theorem states that the entropy dimension set must be right closed.

Theorem 3.4. *Let $(X, \mathcal{B}, \mu, T)$ be an invertible ergodic MDS. If $r_i \in Dims_\mu(X, T)$, $i \in \mathbb{N}$ and $r \in [0, 1]$ such that $r_i \nearrow r$, then $r \in Dims_\mu(X, T)$.*

Proof. For $i \in \mathbb{N}$, let $P_i = \{A_i, X \setminus A_i\}$ for some $A_i \in \mathcal{B}$ with $0 < \mu(A_i) < 1$ such that $r_i = \overline{D}_\mu(T, P_i) \in \mathrm{Dims}_\mu(X, T)$. We denote by $\mathcal{B}_i = \bigvee_{n=-\infty}^{\infty} T^{-n}P_i$, the σ-algebra generated by P_i. Let $\mathcal{D} = \bigvee_{i=1}^{\infty} \mathcal{B}_i$. For each $i \in \mathbb{N}$, $h_\mu(T, P_i) = 0$ since $r_i < 1$. Thus $h_\mu(T, \mathcal{D}) = 0$, moreover by Krieger's generator theorem [16], we have a partition $P = \{A, X \setminus A\}$ such that $\mathcal{D} = \bigvee_{n=-\infty}^{\infty} T^{-n}P$. Then by Theorem 2.11(1) and Proposition 2.9(2) we have

$$r_i \le \sup_{n\ge1} \overline{D}_\mu(T, \bigvee_{k=-n}^{n} T^{-k}P) = \overline{D}_\mu(T, P),$$

for all $i = 1, 2, \ldots$. Since $P \preceq \bigvee_{n\in\mathbb{N}} \bigvee_{j=-n}^{n} \bigvee_{i=1}^{n} T^{-j}P_i$, by Theorem 2.11(1) again,

$$\overline{D}_\mu(T, P) \le \sup_{n\ge1} \overline{D}_\mu(T, \bigvee_{j=-n}^{n} \bigvee_{i=1}^{n} T^{-j}P_i) = r.$$

Hence $\overline{D}_\mu(T, P) = r$ which shows that $r \in \mathrm{Dims}_\mu(X, T)$. □

In the following we will give a disjointness theorem via entropy dimension. Let us recall the related notions first.

Let $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$ be two MDSs. A probability measure λ on $(X \times Y, \mathcal{B} \times \mathcal{D})$ is a *joining* of $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$ if it is $T \times S$-invariant, and has μ and ν as marginals; i.e., $\mathrm{proj}_X(\lambda) = \mu$ and $\mathrm{proj}_Y(\lambda) = \nu$. We let $J(\mu, \nu)$ be the space of all joinings of $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$. We say $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$ are *disjoint* if $J(\mu, \nu) = \{\mu \times \nu\}$. More generally if $\{(X_i, \mathcal{B}_i, \mu_i, T_i)\}_{i\in I}$ is a collection of MDSs, a probability measure λ on $(\prod_{i\in I} X_i, \prod_{i\in I} \mathcal{B}_i)$ is a *joining* of $\{(X_i, \mathcal{B}_i, \mu_i, T_i)\}$ if it is $\prod_{i\in I} T_i$-invariant and has μ_i as marginals; i.e.,

$\text{proj}_{X_i}(\lambda) = \mu_i$ for every $i \in I$. We let $J(\{\mu_i\}_{i\in I})$ be the spaces of all these joinings. When $(X_i, \mathcal{B}_i, \mu_i, T_i) = (X, \mathcal{B}, \mu, T)$ for $i \in I$, we write $J(\{\mu_i\}_{i\in I})$ as $J(\mu; I)$ and call $\lambda \in J(\mu; I)$ I-fold self-joinings.

Lemma 3.5. *Let $(X, \mathcal{B}, \mu, T)$ be an MDS. If $\eta \in J(\mu; \mathbb{Z})$, then $\overline{D}_\eta(X^{\mathbb{Z}}, T^{\mathbb{Z}}) = \overline{D}_\mu(X, T)$.*

Proof. First there exists $\{\alpha_j\}_{j=1}^\infty \subseteq \mathcal{P}_X$ such that $\alpha_1 \preceq \alpha_2 \preceq \alpha_3 \cdots$ and $\bigvee_{j=1}^\infty \alpha_j = \mathcal{B}$ (modμ). Then for $i \in \mathbb{Z}$, let $\pi_i : X^{\mathbb{Z}} \to X$ be the ith coordinate projection. Let $\beta_i^j = \pi_i^{-1}(\alpha_j)$ for $i \in \mathbb{Z}$ and $j \in \mathbb{N}$. Then $\beta_i^j \in \mathcal{P}_{X^{\mathbb{Z}}}$. It is clear that $\bigvee_{i\in\mathbb{Z}, j\in\mathbb{N}} \beta_i^j = \mathcal{B}^{\mathbb{Z}}$ (mod η) and $\overline{D}_\mu(T, \alpha_j) = \overline{D}_\eta(T^{\mathbb{Z}}, \beta_i^j)$ for $i \in \mathbb{Z}, j \in \mathbb{N}$. Hence by Theorem 2.11(1),

$$\overline{D}_\eta(X^{\mathbb{Z}}, T^{\mathbb{Z}}) = \sup_{i\in\mathbb{Z}, j\in\mathbb{N}} \overline{D}_\eta(T^{\mathbb{Z}}, \beta_i^j) = \sup_{j\in\mathbb{N}} \overline{D}_\mu(T, \alpha_j) = \overline{D}_\mu(X, T).$$

This finishes the proof of the lemma. □

Theorem 3.6. *Let $(X, \mathcal{B}, \mu, T)$ be an invertible MDS, and let $(Y, \mathcal{D}, \nu, S)$ be an ergodic MDS. If* $\text{Dims}_\mu(X, T) > \overline{D}_\nu(Y, S)$ *(i.e., for any $\tau \in \text{Dims}_\nu(X, T)$, $\tau > \overline{D}_\nu(Y, S)$), then $(X, \mathcal{B}, \mu, T)$ is disjoint from $(Y, \mathcal{D}, \nu, S)$.*

Proof. We follow the arguments in [11, proof of Theorem 1]. Let λ be a joining of $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$. Let

$$\lambda = \int_X \delta_x \times \lambda_x d\mu(x)$$

be the disintegration of λ over μ, and define the probability measure λ_∞ on $X \times Y^{\mathbb{Z}}$ and ν_∞ on $Y^{\mathbb{Z}}$ by

$$\lambda_\infty = \int_X \delta_x \times (\cdots \times \lambda_x \times \lambda_x \times \cdots) d\mu(x)$$

and

$$\nu_\infty = \int_X (\cdots \times \lambda_x \times \lambda_x \times \cdots) d\mu(x).$$

Since λ is $T \times S$-invariant,

$$\lambda = (T \times S)\lambda = \int_X \delta_{Tx} \times S\lambda_x d\mu(x)$$
$$= \int_X \delta_x \times S\lambda_{T^{-1}x} d\mu(x).$$

By uniqueness of disintegration we have $\lambda_x = S\lambda_{T^{-1}x}$ for μ-a.e. $x \in X$, i.e., $S\lambda_x = \lambda_{Tx}$ for μ-a.e. $x \in X$. Moreover

$$S^{\mathbb{Z}}\nu_\infty = \int_X (\cdots \times S\lambda_x \times S\lambda_x \times \cdots) d\mu(x) = \int_X (\cdots \times \lambda_{Tx} \times \lambda_{Tx} \times \cdots) d\mu(x) = \nu_\infty.$$

This implies $\nu_\infty \in J(\nu, \mathbb{Z})$ since $\int_X \lambda_x d\mu(x) = \nu$. It is also clear that $\lambda_\infty \in J(\{\mu, \nu_\infty\})$, i.e., λ_∞ is a joining of $(X, \mathcal{B}, \mu, T)$ and $(Y^{\mathbb{Z}}, \mathcal{D}^{\mathbb{Z}}, \nu_\infty, S^{\mathbb{Z}})$.

Let

$$\mathcal{E} = \{E \in \mathcal{B} : \exists F \in \mathcal{D}^{\mathbb{Z}},\ \lambda_\infty((E \times Y^{\mathbb{Z}}) \Delta (X \times F)) = 0\}$$

and

$$\mathcal{F} = \{F \in \mathcal{D}^{\mathbb{Z}} : \exists E \in \mathcal{B},\ \lambda_\infty((E \times Y^{\mathbb{Z}}) \Delta (X \times F)) = 0\}.$$

Then $\mathcal{E}$ is a T-invariant sub-σ-algebra and $\mathcal{F}$ is an $S^{\mathbb{Z}}$-invariant sub-σ-algebra.

Now for any $E \in \mathcal{E}$, there exists $F \in \mathcal{D}^{\mathbb{Z}}$ such that $\lambda_\infty((E \times Y^{\mathbb{Z}})\Delta(X \times F)) = 0$. Now

$$\begin{aligned}\overline{D}_\mu(T, \{E, X \setminus E\}) &= \overline{D}_{\lambda_\infty}(T \times S^{\mathbb{Z}}, \{E \times Y^{\mathbb{Z}}, (X \setminus E) \times Y^{\mathbb{Z}}\}) \\ &= \overline{D}_{\lambda_\infty}(T \times S^{\mathbb{Z}}, \{X \times F, X \times (Y^{\mathbb{Z}} \setminus F)\}) \\ &= \overline{D}_{\nu_\infty}(S^{\mathbb{Z}}, \{F, Y^{\mathbb{Z}} \setminus F\}) \\ &\le \overline{D}_{\nu_\infty}(Y^{\mathbb{Z}}, S^{\mathbb{Z}}) \\ &= \overline{D}_\nu(Y, S) \qquad \text{(by Lemma 3.5)}.\end{aligned}$$

Since $\mathrm{Dims}_\mu(X,T) > \overline{D}_\nu(Y,S)$, we have $\mu(E)=0$ or 1. Hence $\mathcal{E} = \{\emptyset, X\} \pmod{\mu}$ and so $\mathcal{F} = \{\emptyset, Y^{\mathbb{Z}}\} \pmod{\nu_\infty}$.

Define a transformation $R : X \times Y^{\mathbb{Z}} \to X \times Y^{\mathbb{Z}}$ by $R(x, \mathbf{y}) = (x, \sigma\mathbf{y})$, where $\mathbf{y} = \{y_i\}_{i \in Z} \in Y^{\mathbb{Z}}$ and σ is the left shift on $Y^{\mathbb{Z}}$. Now if $f(x, y)$ is an R-invariant measurable function on $X \times Y^{\mathbb{Z}}$, then for every $x \in X$ the function $f_x(\mathbf{y}) = f(x, \mathbf{y})$ is a σ-invariant function on the Bernoulli $\mathbb{Z}$-system$(Y^{\mathbb{Z}}, \lambda_x^{\mathbb{Z}}, \sigma)$, hence a constant, $\lambda_x^{\mathbb{Z}}$-a.e.; i.e., $f(x, \mathbf{y}) = f(x)$, λ_∞-a.e.. Thus every R-invariant function on $X \times Y^{\mathbb{Z}}$ is $\mathcal{B} \times Y^{\mathbb{Z}}$-measurable.

For any $F \in \mathcal{D}^{\mathbb{Z}}$ with $\nu_\infty(\sigma^{-1}F\Delta F) = 0$, let $f(x, \mathbf{y}) = 1_F(\mathbf{y})$ for λ_∞-a.e. $(x, \mathbf{y}) \in X \times Y^{\mathbb{Z}}$. Then f is R-invariant and so f is $\mathcal{B} \times Y^{\mathbb{Z}}$-measurable. Thus there exists $E \in \mathcal{B}$ such that $f(x, y) = 1_E(x)$ for λ_∞-a.e. $(x, y) \in X \times Y^{\mathbb{Z}}$ since f is a characteristic function. This implies $F \in \mathcal{F} = \{\emptyset, Y^{\mathbb{Z}}\} \pmod{\nu_\infty}$, so $\nu_\infty(F) = 0$ or 1. Hence $(Y^{\mathbb{Z}}, \mathcal{D}^{\mathbb{Z}}, \nu_\infty, \sigma)$ is ergodic.

Moreover, since $(Y^{\mathbb{Z}}, \mathcal{D}^{\mathbb{Z}}, \lambda_x^{\mathbb{Z}}, \sigma)$ is ergodic for μ-a.e. $x \in X$ and $\nu_\infty = \int_X \lambda_x^{\mathbb{Z}} d\mu(x)$, we have $\lambda_x^{\mathbb{Z}} = \nu_\infty$ for μ-a.e. $x \in X$. Considering the projection of a zero coordinate, $\lambda_x = \nu$ for μ-a.e. $x \in X$. Hence $\lambda = \mu \times \nu$. Then it follows that $(X, \mathcal{B}, \mu, T)$ is disjoint from $(Y, \mathcal{D}, \nu, S)$. □

The following result is also obvious.

Theorem 3.7. *Let $\pi : (X, \mathcal{B}, \mu, T) \to (Y, \mathcal{D}, \nu, S)$ be a factor map between two MDSs. Then $\mathrm{Dims}_\mu(X,T) \supseteq \mathrm{Dims}_\nu(Y,S)$. In particular, the dimension set is invariant under a measurable isomorphism, and so is the entropy dimension.*

In the following, we consider a special case for the dimension set.

Definition 3.8. Let $\tau \in (0, 1]$. We call $(X, \mathcal{B}, \mu, T)$ a *τ-uniform entropy dimension system* (τ-u.d. system for short) if $\mathrm{Dims}_\mu(X,T) = \{\tau\}$, and we call $(X, \mathcal{B}, \mu, T)$ a *τ^+-dimension system* (τ^+-d. system for short) if $\mathrm{Dims}_\mu(X,T) \subset [\tau, 1]$. If $0 \notin \mathrm{Dims}_\mu(X,T)$, we will say $(X, \mathcal{B}, \mu, T)$ has *strictly positive entropy dimension.*

The motivation to consider the u.d. systems comes from the K-mixing systems. We can view the u.d. systems as the analogue of the K-mixing properties in zero entropy situation.

By Definition 3.8 and Theorem 3.7 we have

Proposition 3.9. *Let $\tau \in (0, 1]$. Then the following hold.*

1. *A nontrivial factor of a τ-u.d. system is also a τ-u.d. system.*
2. *A nontrivial factor of a τ^+-d. system is also a τ^+-d. system.*
3. *If a system has strictly positive entropy dimension, then any nontrivial factor of this system also has strictly positive entropy dimension.*

Lemma 3.10. *Let $(X,\mathcal{B},\mu,T)$ be an MDS. If $(X,\mathcal{B},\mu,T)$ has strictly positive entropy dimension, then $(X,\mathcal{B},\mu,T)$ is weakly mixing.*

Proof. It is well known that if $(X,\mathcal{B},\mu,T)$ is not weakly mixing, then there exists a nontrivial factor $(Y,\mathcal{D},\nu,S)$ of $(X,\mathcal{B},\mu,T)$ with discrete spectrum. By Kushnirenko [17], $(Y,\mathcal{D},\nu,S)$ is null. By Theorem 2.8, $\overline{D}_\nu(Y,S)=0$, a contradiction with Proposition 3.9(3). □

As a direct application of Theorem 3.6, we have the following.

Corollary 3.11.

(1) *α-u.d. invertible MDSs are disjoint from ergodic β-u.d. MDSs when $1\ge\alpha>\beta\ge 0$.*
(2) *An invertible MDS which has strictly positive entropy dimension is disjoint from all ergodic 0-entropy dimension MDSs.*

The following example shows that two systems with the same entropy dimension may also have disjointness property.

Example 3.12. Choose $0<r_i\nearrow 1$, and let $(X_i,\mathcal{B}_i,\mu_i,T_i)$ be an r_i-u.d. invertible MDS (in section 5 we will show existence of such MDSs). Let $(X,\mathcal{B},\mu,T)$ be the product system, i.e., $(X,\mathcal{B},\mu,T)=(\prod_{i=1}^\infty X_i,\prod_{i=1}^\infty \mathcal{B}_i,\prod_{i=1}^\infty \mu_i,\prod_{i=1}^\infty T_i)$. Then $\overline{D}_\mu(X,T)=1$ by Theorem 2.9. Also, since each $(X_i,\mathcal{B}_i,\mu_i,T_i)$ is weakly mixing, so is $(X,\mathcal{B},\mu,T)$, and hence it is ergodic. Since $h_\mu(X,T)=0$, any K-automorphism MDS (which is also a 1-u.d. system) is disjoint from the ergodic MDS $(X,\mathcal{B},\mu,T)$.

4. Metric entropy dimension of an open cover

By a *topological dynamical system* (TDS for short) (X,T) we mean a compact metrizable space X together with a surjective continuous map T from X to itself. Let (X,T) be a TDS, and let $\mu\in M(X,T)$, where $M(X,T)$ denotes the collection of invariant probability measures of (X,T). Denote by $\mathcal{C}_X$ the set of finite covers of X and by $\mathcal{C}_X^o$ the set of finite open covers of X. For a $\mathcal{U}\in\mathcal{C}_X$, we define

$$H_\mu(\mathcal{U})=\inf\{H_\mu(\alpha):\alpha\in\mathcal{P}_X \text{ and } \alpha\succeq\mathcal{U}\},$$

where by $\alpha\succeq\mathcal{U}$ we mean that every atom of α is contained in one of the elements of $\mathcal{U}$. We say an increasing sequence $S=\{s_1<s_2<\cdots\}$ of $\mathbb{N}$ is an *entropy generating sequence* of $\mathcal{U}$ w.r.t. μ if

$$\liminf_{n\to\infty}\frac{1}{n}H_\mu(\bigvee_{i=1}^n T^{-s_i}\mathcal{U})>0.$$

We say $S=\{s_1<s_2<\cdots\}$ of $\mathbb{N}$ is a *positive entropy sequence* of $\mathcal{U}$ w.r.t. μ if

$$h_\mu^S(T,\mathcal{U}):=\limsup_{n\to\infty}\frac{1}{n}H_\mu(\bigvee_{i=1}^n T^{-s_i}\mathcal{U})>0.$$

Denote by $\mathcal{E}_\mu(T,\mathcal{U})$ the set of all entropy generating sequences of $\mathcal{U}$, and by $\mathcal{P}_\mu(T,\mathcal{U})$ the set of all positive entropy sequences of $\mathcal{U}$. Clearly, $\mathcal{P}_\mu(T,\mathcal{U})\supset\mathcal{E}_\mu(T,\mathcal{U})$.

Definition 4.1. Let (X,T) be a TDS, let $\mu \in M(X,T)$, and let $\mathcal{U} \in \mathcal{C}_X$. We define

$$\overline{D}^e_\mu(T,\mathcal{U}) = \begin{cases} \sup\limits_{S\in\mathcal{E}_\mu(T,\mathcal{U})} \overline{D}(S) & \text{if } \mathcal{E}_\mu(T,\mathcal{U}) \neq \emptyset, \\ 0 & \text{if } \mathcal{E}_\mu(T,\mathcal{U}) = \emptyset, \end{cases}$$

$$\overline{D}^p_\mu(T,\mathcal{U}) = \begin{cases} \sup\limits_{S\in\mathcal{P}_\mu(T,\mathcal{U})} \overline{D}(S) & \text{if } \mathcal{P}_\mu(T,\mathcal{U}) \neq \emptyset, \\ 0 & \text{if } \mathcal{P}_\mu(T,\mathcal{U}) = \emptyset. \end{cases}$$

Similarly, we can define $\underline{D}^e_\mu(T,\mathcal{U})$ and $\underline{D}^p_\mu(T,\alpha)$ by changing the upper dimension into the lower dimension.

As in Proposition 2.4 we have the following.

Proposition 4.2. *Let (X,T) be a TDS, let $\mu \in M(X,T)$, and let $\mathcal{U} \in \mathcal{C}_X$. Then*

$$\underline{D}^e_\mu(T,\mathcal{U}) \leq \overline{D}^e_\mu(T,\mathcal{U}) = \underline{D}^p_\mu(T,\mathcal{U}) \leq \overline{D}^p_\mu(T,\mathcal{U}).$$

By Proposition 4.2, we have the following.

Definition 4.3. Let (X,T) be a TDS, let $\mu \in M(X,T)$, and let $\mathcal{U} \in \mathcal{C}_X$. We define

$$\overline{D}_\mu(T,\mathcal{U}) := \overline{D}^e_\mu(T,\mathcal{U}) = \underline{D}^p_\mu(T,\mathcal{U}),$$

which is called the upper entropy dimension of $\mathcal{U}$. Similarly, we have the definition of lower dimension and dimension.

Theorem 4.4. *Let (X,T) be a TDS, and let $\mu \in M(X,T)$. Then*

$$\overline{D}_\mu(X,T) = \sup_{\mathcal{U}\in\mathcal{C}^o_X} \overline{D}_\mu(T,\mathcal{U}),$$

where $\mathcal{C}^o_X$ is the set of finite open covers of X.

Proof. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\} \in \mathcal{C}^o_X$. For any $s = (s(1), \ldots, s(n)) \in \{0,1\}^n$, set $U_s = \bigcap_{i=1}^n U_i(s(i))$, where $U_i(0) = U_i$ and $U_i(1) = X \setminus U_i$. Let $\alpha = \{U_s : s \in \{0,1\}^n\}$. Then α is the Borel partition generated by $\mathcal{U}$ and $\overline{D}_\mu(X,T) \geq \overline{D}_\mu(T,\alpha) \geq \overline{D}_\mu(T,\mathcal{U})$. Since $\mathcal{U}$ is arbitrary, we get $\overline{D}_\mu(X,T) \geq \sup_{\mathcal{U}\in\mathcal{C}^o_X} \overline{D}_\mu(T,\mathcal{U})$.

For the other direction, let $\alpha = \{A_1, \ldots, A_k\} \in \mathcal{P}_X$. If $\overline{D}_\mu(T,\alpha) = 0$, it is obvious $\overline{D}_\mu(T,\alpha) \leq \sup_{\mathcal{U}\in\mathcal{C}^o_X} \overline{D}_\mu(T,\mathcal{U})$. Now assume that $\overline{D}_\mu(T,\alpha) > 0$. For any $\epsilon > 0$, there exists $S = \{s_1 < s_2 < s_3 < \cdots\} \in \mathcal{P}_\mu(T,\alpha)$ with $\underline{D}(S) > \overline{D}_\mu(T,\alpha) - \epsilon$.

Let $a := \frac{h^S_\mu(T,\alpha)}{2} > 0$. We have the following.

Claim. There exists $\mathcal{U} \in \mathcal{C}^o_X$ such that $H_\mu(T^{-i}\alpha|\beta) \leq a$ if $i \in \mathbb{Z}_+$ and $\beta \in \mathcal{P}_X$ satisfying $\beta \succeq T^{-i}\mathcal{U}$.

Proof of Claim. By [30, Lemma 4.15], there exists $\delta_1 = \delta_1(k,\epsilon) > 0$ such that if $\beta_i = \{B^i_1, \ldots, B^i_k\} \in \mathcal{P}_X, i = 1,2$ satisfies $\sum_{i=1}^k \mu(B^1_i \Delta B^2_i) < \delta_1$, then $H_\mu(\beta_1|\beta_2) \leq a$. Since μ is regular, we can take closed subsets $B_i \subseteq A_i$ with $\mu(A_i \setminus B_i) < \frac{\delta_1}{2k^2}$, $i = 1, \ldots, k$. Let $B_0 = X \setminus \bigcup_{i=1}^k B_i$, $U_i = B_0 \cup B_i, i = 1, \ldots, k$. Then $\mu(B_0) < \frac{\delta_1}{2k}$ and $\mathcal{U} = \{U_1, \ldots, U_k\} \in \mathcal{C}^o_X$.

Let $i \in \mathbb{Z}_+$. If $\beta \in \mathcal{P}_X$ is finer than $T^{-i}\mathcal{U}$, then we can find $\beta' = \{C_1, \ldots, C_k\} \in \mathcal{P}_X$ satisfying $C_j \subseteq T^{-i}U_j,\ j = 1, \ldots, k$ and $\beta \succeq \beta'$, and so $H_\mu(T^{-i}\alpha|\beta) \leq$

$H_\mu(T^{-j}\alpha|\beta')$. For each $j = 1, \ldots, k$, as $T^{-i}U_j \supseteq C_j \supseteq X \setminus \bigcup_{l\neq j} T^{-i}U_l = T^{-i}B_j$ and $T^{-i}A_j \supseteq T^{-i}B_j$, one has

$$\mu(C_j \Delta T^{-i}A_j) \le \mu(T^{-i}A_j \setminus T^{-i}B_j) + \mu(T^{-i}B_0) = \mu(A_j \setminus B_j) + \mu(B_0) < \frac{\delta_1}{2k} + \frac{\delta_1}{2k^2} \le \frac{\delta_1}{k}.$$

Thus, $\sum_{j=1}^k \mu(C_j \Delta T^{-i}A_j) < \delta_1$. It follows that $H_\mu(T^{-i}\alpha|\beta') \le a$, and so $H_\mu(T^{-i}\alpha|\beta) \le a$. □

For $n \in \mathbb{N}$, if $\beta_n \in \mathcal{P}_X$ with $\beta_n \succeq \bigvee_{i=1}^n T^{-s_i}\mathcal{U}$, then $\beta_n \succeq T^{-s_i}\mathcal{U}$ for each $i \in \{1, 2, \ldots, n\}$, and so using the Claim, one has

$$H_\mu(\bigvee_{i=1}^n T^{-s_i}\alpha) \le H_\mu(\beta_n) + H_\mu(\bigvee_{i=1}^n T^{-s_i}\alpha|\beta_n) \le H_\mu(\beta_n) + \sum_{i=1}^n H_\mu(T^{-s_i}\alpha|\beta_n) \le H_\mu(\beta_n) + na.$$

Moreover, since the above inequality is true for any $\beta_n \in \mathcal{P}_X$ with $\beta_n \succeq \bigvee_{i=1}^n T^{-s_i}\mathcal{U}$, one has $H_\mu(\bigvee_{i=1}^n T^{-s_i}\alpha) \le H_\mu(\bigvee_{i=1}^n T^{-s_i}\mathcal{U}) + na$. Recall that by $h_\mu^S(T,\alpha) = 2a$, one has $h_\mu^S(T,\mathcal{U}) \ge a > 0$. Also recall that $\overline{D}_\mu(T,\alpha) = \underline{D}_\mu^p(T,\alpha)$ (Definition 2.6), we then have

$$\overline{D}_\mu(T,\mathcal{U}) \ge \underline{D}(S) \ge \underline{D}_\mu^p(T,\alpha) - \epsilon = \overline{D}_\mu(T,\alpha) - \epsilon.$$

This implies $\sup_{\mathcal{U}\in\mathcal{C}_X^o} \overline{D}_\mu(T,\mathcal{U}) \ge \overline{D}_\mu(T,\alpha) - \epsilon$. Finally, since α and ϵ are arbitrary, we get $\sup_{\mathcal{U}\in\mathcal{C}_X^o} \overline{D}_\mu(T,\mathcal{U}) \ge \overline{D}_\mu(X,T)$. □

Now let us recall the corresponding notions in topological setttings, which appeared in [6].

Let (X,T) be a TDS, and let $\mathcal{U} \in \mathcal{C}_X^o$. We recall from [6] that an increasing sequence $S = \{s_1 < s_2 < \cdots\}$ of $\mathbb{N}$ is an *entropy generating sequence* of $\mathcal{U}$ if

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i}\mathcal{U}) > 0,$$

and $S = \{s_1 < s_2 < \cdots\}$ of $\mathbb{N}$ is a *positive entropy sequence* of $\mathcal{U}$ if

$$h_{\text{top}}^S(T,\mathcal{U}) := \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i}\mathcal{U}) > 0.$$

Here $\mathcal{N}(\mathcal{U})$ is the number of the sets in a subcover of $\mathcal{U}$ with smallest cardinality.

Denote by $\mathcal{E}(T,\mathcal{U})$ the set of all entropy generating sequences of $\mathcal{U}$, and by $\mathcal{P}(T,\mathcal{U})$ the set of all positive entropy sequences of $\mathcal{U}$. Clearly, $\mathcal{P}(T,\mathcal{U}) \supset \mathcal{E}(T,\mathcal{U})$.

Let (X,T) be a TDS, and let $\mathcal{U} \in \mathcal{C}_X^o$. We define

$$\overline{D}_e(T,\mathcal{U}) = \begin{cases} \sup_{S\in\mathcal{E}(T,\mathcal{U})} \overline{D}(S) & \text{if } \mathcal{E}(T,\mathcal{U}) \neq \emptyset, \\ 0 & \text{if } \mathcal{E}(T,\mathcal{U}) = \emptyset, \end{cases}$$

$$\underline{D}_p(T,\mathcal{U}) = \begin{cases} \sup_{S\in\mathcal{P}(T,\mathcal{U})} \underline{D}(S) & \text{if } \mathcal{P}(T,\mathcal{U}) \neq \emptyset, \\ 0 & \text{if } \mathcal{P}(T,\mathcal{U}) = \emptyset. \end{cases}$$

As in the proof of Proposition 2.4, we have $\overline{D}_e(T,\mathcal{U}) = \underline{D}_p(T,\mathcal{U})$ for any $\mathcal{U} \in \mathcal{C}_X^o$. Hence we define

$$\overline{D}(T,\mathcal{U}) := \overline{D}_e(T,\mathcal{U}) = \underline{D}_p(T,\mathcal{U}) \quad \text{and}$$
$$\overline{D}(X,T) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} \overline{D}(T,\mathcal{U}).$$

We call $\overline{D}(X,T)$ the upper entropy dimension of (X,T). Similarly, we have the definition of lower dimension and dimension.

Using Theorem 4.4, we have the following.

Theorem 4.5. *Let (X,T) be a TDS, and let $\mu \in M(X,T)$. Then*

$$\overline{D}_\mu(X,T) \leq \overline{D}(X,T).$$

Example 4.6. For any $\tau \in (0,1]$, there exists a minimal system (X,T) satisfying $D(X,T) = \tau$ and $D_\mu(X,T) = 0$ for any $\mu \in M(X,T)$.

Proof. Let (X,T) be the system generated by Cassaigne's model [3] (the uniformly recurrent one), then it is minimal. By taking $\phi(n) = \frac{n}{\log n}$ in this construction, we get $D(X,T) = 1$. By taking $\phi(n) = n^\tau$ in this construction, we get $D(X,T) = \tau$, for any $0 < \tau < 1$. In [1] it was shown that (X,T) is uniquely ergodic and with respect to the unique ergodic invariant measure μ, $D_\mu(X,T) = 0$. □

Definition 4.7. An invertible TDS (X,T) is *doubly minimal* if for all $x, y \in X$, $y \in \{T^n x\}_{n\in\mathbb{Z}}$, $\{(T^j x, T^j y)\}_{j\in\mathbb{Z}}$ is dense in $X \times X$.

The following result is [31, Theorem 5].

Lemma 4.8. *Any ergodic system $(Y,\mathcal{C},\nu,S)$ with $h_\nu(S) = 0$ has a uniquely ergodic topological model (X,T) that is doubly minimal.*

It is well known that any doubly minimal system has zero entropy (see [31]). However we have

Example 4.9. There exists a doubly minimal system with positive entropy dimension.

Proof. This comes directly from Lemma 4.8 and Theorem 4.5 since there exists an ergodic system with metric entropy dimension $0 < \tau <$ (see section 5). □

A TDS (X,T) with metric d is called *distal*, if $\inf_{n\geq 0} d(T^n x, T^n y) > 0$ for every $x \neq y \in X$. Let $(X,\mathcal{B},\mu,T)$ be an invertible ergodic MDS. A sequence $A_1 \supset A_2 \supset A_3 \cdots$ of sets in $\mathcal{B}$ with $\mu(A_n) > 0$ and $\mu(A_n) \to 0$ is called *a separating sieve* if there exists a subset $X_0 \subset X$ with $\mu(X_0) = 1$ such that for every $x, x' \in X_0$ the condition that for every $n \in \mathbb{N}$ there exists $k \in \mathbb{Z}$ with $T^k x, T^k x' \in A_n$ implies $x = x'$. We say that the invertible ergodic MDS $(X,\mathcal{B},\mu,T)$ is *measure distal* if either $(X,\mathcal{B},\mu,T)$ is finite or there exists a separating sieve. In [18] Lindenstrauss shows that every invertible ergodic measure distal MDS can be represented as a minimal topologically distal system.

It is well known that a distal TDS has zero topological entropy, and an invertible ergodic measure distal MDS has zero measure entropy. To end this section let us ask the following questions:

Question 4.10.

(1) Is the entropy dimension of a minimal distal TDS zero?
(2) Is the entropy dimension of an invertible ergodic measure distal MDS zero?

5. The existence of u.d. MDSs

In this section, our aim is to show that for every $\tau \in (0,1)$, there exists an MDS $(X, \mathcal{B}, \mu, T)$ having the property of τ-u.d. We mention that a K-mixing system is of u.d. for $\tau = 1$ and an irrational rotation is of u.d. for $\tau = 0$.

Our construction employs the so-called *cutting and stacking* method. Let X be the interval $[0,1)$, let $\mathcal{B}$ be the Borel σ-algebra on $[0,1)$, and let μ be the Lebesgue measure on $[0,1)$. In the cutting and stacking construction, $[0,1)$ will be cut into many subintervals, and all of them are left closed and right open. Let $B_i \subset [0,1), 1 \le i \le h$, be h disjoint subintervals of the same length. A *column* C is the ordered set of these intervals, i.e.,

$$C = \{B_1, B_2, \dots, B_h\} = \{B_i : 1 \le i \le h\}.$$

We can consider C obtained by "stacking" the B_i's one by one. We say the column C has *base* B_1, *top* B_h, *height* $h(C) = h$ and *width* $w(C) =$ the length of B_i. Denote $|C| = \bigcup_{i=1}^{h} B_i$. We call each B_i a *level set* of C. For the column C, the map T will map B_i linearly onto B_{i+1} for $1 \le i \le n-1$, and it is undefined on B_n. We call $C^0 = \{B_1^0, TB_1^0, \dots, T^{h-1}B_1^0\}$ a *subcolumn* of C if $B_1^0 \subset B_1$. A *tower* W is a finite collection of columns, which generally have different heights. In this paper, all the columns of a tower will have the same height. The width of tower W is

$$w(W) = \sum_{C \text{ is a column of } W} w(C).$$

The *cardinality* of a tower W, denoted by $\#W$, is the number of its columns. We denote by $|W|$ the union of all the level sets of its columns. The *base* of the tower W, which is denoted by base(W), is the union of all the bases of its columns. For the tower W, T is considered to be defined on each of its columns except on the tops of its columns. Hence for the tower W, T is undefined on a set of measure $w(W)$. The cutting and stacking method will construct a sequence of towers with widths tending to 0 so that T is invertible on the whole interval $[0,1)$ up to a set of measure zero. (One may see [9, 27, 28] for the basics of the cutting and stacking method.)

In our construction, we divide $[0,1)$ into three parts: $P_0 = [0, \frac{\xi}{2})$, $P_1 = [\frac{\xi}{2}, \xi)$, and $P_s = [\xi, 1)$, where we will decide ξ later (see (5.14) in section 5.4) and "s" stands for "spacer". This will be our initial tower and any level set of any other tower will be a subset of $P_0, P_1,$ or P_s. Due to the initial tower, we say that a level set B has a *name* "a" if $B \subset P_a, a = 0, 1, s$. The name of a column $C = \{B_1, B_2, \dots, B_h\}$ is a word $b = b_1 b_2 \cdots b_h \in \{0, 1, s\}^h$, where b_i is the name of B_i. The name of a tower is the collection of names of its columns. By $N(W)$ we denote the number of different names of columns of the tower W. We say two columns are *isomorphic* if they have the same name (they do not need to have the same widths). We say two towers $W = \{C_1, C_2, \dots, C_k\}$ and $W' = \{C_1', C_2', \dots, C_k'\}$ with the same cardinality are *isomorphic* if after some reordering, the columns C_i and C_i' are isomorphic and $w(C_i) = \lambda w(C_i')$ for some $\lambda > 0$ and all $1 \le i \le k$. If C^0 is a subcolumn of a column C of the tower W, then we may say that C^0 is a subcolumn of W. A *segment* S of height ℓ of a column $C = \{B_1, B_2, \dots, B_h\}$ is a collection of consecutive level sets $\{B_{\ell'}, B_{\ell'+1}, \dots, B_{\ell'+\ell-1}\}$ starting at some position ℓ' with $1 \le \ell' \le h - \ell + 1$. If $b = b_1 b_2 \cdots b_h \in \{0, 1, s\}^h$ is the name of the column C, then the word $b_{\ell'} b_{\ell'+1} \cdots b_{\ell'+\ell-1}$ is the name of the segment S. We also write S as

$S = B_{\ell'} B_{\ell'+1} \cdots B_{\ell'+\ell-1}$ or $S = b_{\ell'} b_{\ell'+1} \cdots b_{\ell'+\ell-1}$. Let W and W' be two towers such that the height of W' is bigger than that of W. We say a segment S of some column of W' is a W-segment if the name of S is the same as the name of some column C of W (in this case, we also say S is isomorphic to C).

Through cutting and stacking steps, we successively construct a sequence of towers to get an MDS with a given entropy dimension $\tau \in (0,1)$ by controlling the heights of independent and repetitive steps. We can see clearly from the construction what the entropy generating sequence is. We use three types of operations which will be described in the next section.

5.1. **Three kinds of operations.** Now we will describe the three kinds of operations we need.

5.1.1. *Independent cutting and stacking.* Let W^1 and W^2 be two towers with the same width w. Assume W^j has c_j-many columns $C_1^j, C_2^j, \dots, C_{c_j}^j$ for $j = 1, 2$. We divide each column of W^1 into subcolumns according to the distribution of the columns of W^2. That is, we divide each column C_i^1 into c_2-many subcolumns $C_{i,k}^1$ with width $w(C_{i,k}^1) = w(C_i^1)\frac{w(C_k^2)}{w}$, $i = 1, 2, \dots, c_1, k = 1, 2, \dots, c_2$. Likewise we divide each column C_k^2 into c_1-many subcolumns $C_{k,i}^2$ with width $w(C_{k,i}^2) = w(C_k^2)\frac{w(C_i^1)}{w}$, $i = 1, 2, \dots, c_1$, $k = 1, 2, \dots, c_2$. Since we have $w(C_{i,k}^1) = w(C_{k,i}^2)$, we stack each $C_{k,i}^2$ on top of $C_{i,k}^1$ to form a new column $C_{i,k}^1 * C_{k,i}^2$. Denote the new tower $\{C_{i,k}^1 * C_{k,i}^2\}$ by $W^1 *_{\text{ind}} W^2$.

For a tower W and an integer $e \geq 1$, we equally divide W into e-many subtowers $W^1, W^2, \dots, W^e$ (we divide each column of W into e-many subcolumns equally and take all the ith subcolumn to make the tower W^i). We call the tower $\text{Ind}(W, e) = W^1 *_{\text{ind}} W^2 *_{\text{ind}} \cdots *_{\text{ind}} W^e$ the e-many independent cuttings and stackings of W. We note that $\# \text{Ind}(W, e) = (\# W)^e, h(\text{Ind}(W, e)) = eh(W)$. In fact we can cut each column of W into $e(\# W)^{e-1}$-many subcolumns equally and then choose these subcolumns from different e-many combinations of columns of W to stack to form $\text{Ind}(W, e)$, i.e., the tower $\text{Ind}(W, e)$ is stacked by e-many W-segments independently.

5.1.2. *Repetitive cutting and stacking.* For a tower $W = \{C_1, C_2, \dots, C_c\}$ and an integer $r \geq 1$, we equally divide each column C_i of W into r-many subcolumns $C_{i,1}, C_{i,2}, \dots, C_{i,r}$ and stack them one by one to make a new column $C_{i,1} * C_{i,2} * \cdots * C_{i,r}$. Then we call the tower $\text{Rep}(W, r) = \{C_{i,1} * C_{i_2} * \cdots * C_{i,r} : i = 1, 2, \dots, c\}$ the r-many repetitive cuttings and stackings of W. We note that $\# \text{Rep}(W, r) = \# W$.

5.1.3. *Inserting spacers while independently cutting and stacking.* Let W be a tower with columns $\{C_1, C_2, \dots, C_c\}$, and let $e, h^* \geq 1$ be two integers. Due to the definition of $\text{Ind}(W, e)$, we can assume that the tower $\text{Ind}(W, e)$ is formed by columns $\overline{C}_{i_1} * \overline{C}_{i_2} * \cdots * \overline{C}_{i_e}$ for $i_1, i_2, \dots, i_e \in \{1, 2, \dots, c\}$, where $\overline{C}_i$ is a subcolumn of C_i. Cut each column $\overline{C}_{i_1} * \overline{C}_{i_2} * \cdots * \overline{C}_{i_e}$ of $\text{Ind}(W, e)$ into c-many subcolumns equally, which we denote by $(\overline{C}_{i_1} * \overline{C}_{i_2} * \cdots * \overline{C}_{i_e})_{i_{e+1}}$, $i_{e+1} = 1, 2, \dots, c$. Since $\text{Ind}(W, e)$ has c^e-many columns, the new tower has c^{e+1}-many columns. We write $(\overline{C}_{i_1} * \overline{C}_{i_2} * \cdots * \overline{C}_{i_e})_{i_{e+1}}$ as $\overline{C}_{i_1}^{i_{e+1}} * \overline{C}_{i_2}^{i_{e+1}} * \cdots * \overline{C}_{i_e}^{i_{e+1}}$ and call $\overline{C}_{i_k}^{i_{e+1}}$ the kth W-segment of the column $\overline{C}_{i_1}^{i_{e+1}} * \overline{C}_{i_2}^{i_{e+1}} * \cdots * \overline{C}_{i_e}^{i_{e+1}}$. Note that $\overline{C}_{i_k}^{i_{e+1}}$ is isomorphic to C_i. Now we will insert $e \cdot h^*$-many spacers altogether between these W-segments of $\overline{C}_{i_1}^{i_{e+1}} * \overline{C}_{i_2}^{i_{e+1}} * \cdots * \overline{C}_{i_e}^{i_{e+1}}$, where each spacer is an interval of length $w(\overline{C}_{i_1}^{i_{e+1}} * \overline{C}_{i_2}^{i_{e+1}} * \cdots * \overline{C}_{i_e}^{i_{e+1}})$ cut from P_s.

For $k=1,2,\ldots,e$, let $\ell = i_{k+1} \pmod{h^*}, 0 \le \ell \le h^*-1$, we insert ℓ-many spacers before the kth W-segment $\overline{C}_{i_k}^{i_{e+1}}$ and $(h^*-\ell)$-many spacers after. That is, we change each $\overline{C}_{i_k}^{i_{e+1}}$ into $s^\ell \overline{C}_{i_k}^{i_{e+1}} s^{h^*-\ell}$ (here we identify the inserted spacer with its name "s"). Denote the new tower by $\mathrm{Inds(W,e,h^*)}$. Each column of $\mathrm{Inds(W,e,h^*)}$ is formed by e-many such segments of the form $s^\ell \overline{C}_{i_k}^{i_{e+1}} s^{h^*-\ell}$. We should notice here that some columns of $\mathrm{Inds(W,e,h^*)}$ may have the same name. Furthermore,

$$(N(W))^e \le N(\mathrm{Inds(W,e,h^*)}) \le \#\mathrm{Inds(W,e,h^*)} = (\#\mathrm{W})^{\mathrm{e}+1}. \tag{5.1}$$

We write the new tower $\mathrm{Inds(W,e,h^*)}$ by $\overline{W}$. For convenience, we still call $s^\ell \overline{C}_{i_k} s^{h^*-\ell}$ the kth W-segment of the column of $\overline{W}$ or simply the kth W-segment of $\overline{W}$ ignoring the spacers.

If we denote by p_ℓ the probability of all the columns of $\overline{W}$ whose kth W-segment begins with ℓ-many spacers, then p_ℓ is either $\frac{\lfloor \frac{c}{h^*} \rfloor}{c}$ or $\frac{\lfloor \frac{c}{h^*} \rfloor + 1}{c}$, independent of k and ℓ. Since

$$\frac{|p_\ell - \frac{1}{h^*}|}{\frac{1}{h^*}} \le \frac{h^*}{c},$$

we may say that the number of beginning spacers of the kth W-segment of $\overline{W}$ is uniformly distributed on $\{0,1,\ldots,h^*-1\}$ within $\frac{h^*}{c}$-error.

5.2. **The choice of the parameters.** To construct an MDS $(X,\mathcal{B},\mu,T)$ with τ-u.d. for fixed $\tau \in (0,1)$, we need to define sequences of integer parameters $1 < r_1 < r_2 < \cdots$, $1 < e_0, e_1, e_2, \ldots$, $1 \le w_0 < w_1 < w_2 < \cdots$, $1 \le h_0 < h_1 < h_2 < \cdots$ and $1 < \tilde{h}_0 < \tilde{h}_1 < \tilde{h}_2 < \cdots$.

Given $\tau \in (0,1)$, we let $r_n = C_\tau n^2$, where C_τ is an integer such that $C_\tau^{\frac{\tau}{1-\tau}} > 4$. Let $1 < l_1 < n_1 < l_2 < n_2 \cdots$ be any sequence of integers satisfying that

$$\sum_{t=1}^{\infty} \frac{1}{(n_t)^{\frac{2\tau}{1-\tau}}} < \infty. \tag{5.2}$$

Put

$$e_0 = 2, h_0 = 1, w_0 = 1 \text{ and } h_1 = e_0.$$

Next we inductively construct $h_n, \tilde{h}_n, w_n, e_n$. For $n \ge 1$, put

$$\tilde{h}_n = h_n r_n,\ w_n = \begin{cases} \tilde{h}_n & \text{if } n \notin \{n_1, n_2, \ldots\}, \\ \tilde{h}_{n_t} + h_{l_t} & \text{if } n = n_t \text{ for some } t, \end{cases} \tag{5.3}$$

$$e_n = \lfloor \left(\frac{(w_n)^\tau}{e_0 e_1 \cdots e_{n-1}} \right)^{\frac{1}{1-\tau}} \rfloor, \tag{5.4}$$

and

$$h_{n+1} = w_n e_n. \tag{5.5}$$

Since $r_n \to +\infty$, it is clear from (5.3), (5.4), and (5.5) that

$$\lim_{n \to +\infty} \frac{w_n}{\tilde{h}_n} = 1. \tag{5.6}$$

By (5.4) we have

$$e_1 = \lfloor \left(\frac{(w_1)^\tau}{e_0} \right)^{\frac{1}{1-\tau}} \rfloor = \lfloor \left(\frac{(h_1 r_1)^\tau}{e_0} \right)^{\frac{1}{1-\tau}} \rfloor = \lfloor \left(\frac{(2C_\tau)^\tau}{2} \right)^{\frac{1}{1-\tau}} \rfloor = \lfloor \frac{C_\tau^{\frac{\tau}{1-\tau}}}{2} \rfloor \ge 2 \tag{5.7}$$

and for $n \geq 2$

$$\begin{aligned} e_n &\geq \lfloor (\frac{(w_{n-1}e_{n-1}r_n)^\tau}{e_0e_1\cdots e_{n-1}})^{\frac{1}{1-\tau}} \rfloor = \lfloor (\frac{(w_{n-1}e_{n-1})^\tau}{e_0e_1\cdots e_{n-1}})^{\frac{1}{1-\tau}} \cdot (r_n)^{\frac{\tau}{1-\tau}} \rfloor \\ &= \lfloor \frac{(\frac{(w_{n-1})^\tau}{e_0e_1\cdots e_{n-2}})^{\frac{1}{1-\tau}}}{e_{n-1}} \cdot (r_n)^{\frac{\tau}{1-\tau}} \rfloor \geq \lfloor (r_n)^{\frac{\tau}{1-\tau}} \rfloor. \end{aligned} \tag{5.8}$$

Hence we have $\lim_{n\to+\infty} e_n = +\infty$ and by the definition of e_n's,

$$\lim_{n\to+\infty} \frac{e_0e_1\cdots e_n}{(w_ne_n)^\tau} = 1. \tag{5.9}$$

Now by (5.6) and (5.9), we have

$$\begin{aligned} e_n &= \lfloor (\frac{(w_n)^\tau}{e_0e_1\cdots e_{n-1}})^{\frac{1}{1-\tau}} \rfloor = \lfloor (\frac{w_n}{\tilde{h}_n})^{\frac{\tau}{1-\tau}} \cdot (\frac{(w_{n-1}e_{n-1}r_n)^\tau}{e_0e_1\cdots e_{n-1}})^{\frac{1}{1-\tau}} \rfloor \\ &= \lfloor (\frac{w_n}{\tilde{h}_n})^{\frac{\tau}{1-\tau}} \cdot \frac{(\frac{(w_{n-1})^\tau}{e_0e_1\cdots e_{n-2}})^{\frac{1}{1-\tau}}}{e_{n-1}} \cdot (r_n)^{\frac{\tau}{1-\tau}} \rfloor \\ &\sim (r_n)^{\frac{\tau}{1-\tau}}. \end{aligned} \tag{5.10}$$

From (5.7), (5.8), and setting $r_n = C_\tau n^2$, we have $e_n \geq 2$ for every $n \geq 1$. Together with (5.4), we deduce that

$$w_n \geq (e_0e_1\cdots e_{n-1})^{\frac{1}{\tau}} \geq 2^{\frac{n}{\tau}}. \tag{5.11}$$

Note that from (5.10) and setting $r_n = C_\tau n^2$, both e_n and r_n have polynomial growth rate on n. Hence for any $\epsilon > 0$, we have that

$$\lim_{n\to+\infty} \frac{(w_n)^\epsilon}{e_n} = \lim_{n\to+\infty} \frac{(w_n)^\epsilon}{(r_n)^{\frac{\tau}{1-\tau}}} = +\infty. \tag{5.12}$$

5.3. **The construction.** Let $W_0 = \tilde{W}_0 = \{P_0, P_1\}$ be the 0th and $\tilde{0}$th step tower. We note here that W_0 and $\tilde{W}_0$ do not contain a subset of P_s. The construction consists of a sequence of steps, step n and step $\tilde{n}$, $n \in \mathbb{N}$. The steps occur in the following order: Step 1, Step $\tilde{1}$, Step 2, Step $\tilde{2}$, …, Step n, Step $\tilde{n}$, ….

At Step 1, we do e_0-many independent cuttings and stackings of $\tilde{W}_0$ to construct the first tower W_1 of height $h_1 = e_0$, i.e., $W_1 = \mathrm{Ind}(\tilde{W}_0, e_0)$. We have 2^{e_0}-many columns of all possible sequences of 0's and 1's as their names of equal width and height in W_1. Suppose after Step n we have obtained the tower W_n of height h_n. Then at Step $\tilde{n}$, we do r_n-many repetitive cuttings and stackings of W_n; i.e., if we denote the tower after this step by $\tilde{W}_n$, then $\tilde{W}_n = \mathrm{Rep}(W_n, r_n)$. This step could not increase the complexity too much. At Step $(n+1)$, if $n \notin \{n_1, n_2, \ldots\}$, we do e_n-many independent cuttings and stackings of $\tilde{W}_n$, i.e., $W_{n+1} = \mathrm{Ind}(\tilde{W}_n, e_n)$. If $n = n_t$ for some $t \geq 1$, we insert the spacers while doing e_n-many independent cuttings and stackings between the $\tilde{W}_{n_t}$-segments as mentioned in section 5.1 (with parameter $h^* = h_{l_t}$), i.e., we let $W_{n_t+1} = \mathrm{Inds}(\tilde{W}_{n_t}, e_{n_t}, h_{l_t})$. Since we need to show that any nontrivial partition $P = \{A, A^c\}$ has the same entropy dimension, we need to be careful not to generate some kind of *rotation* factor by level sets. This step makes a level set of a previous step spread out *almost uniformly* to the level sets of future steps.

Then we get an invertible MDS and denote it by $(X, \mathcal{B}, \mu, T)$, where μ is the Lebesgue measure on X, $\mathcal{B}$ is the σ-algebra of X generated by the level sets of the sequence of towers W_n, and T is the associated map.

5.4. **List of the parameters and notations.** We recall the following parameters and notations.

- e_n—We do e_n-many independent cuttings and stackings at Step $n+1$ if $n \notin \{n_1, n_2, \ldots\}$. We insert spacers while doing e_n-many independent cuttings and stackings at Step $n+1$ if $n \in \{n_1, n_2, \ldots\}$.
- r_n—We do r_n-many repetitive cuttings and stackings at Step $\tilde{n}$.
- W_n and $\tilde{W}_n$—These are towers after Step n and Step $\tilde{n}$, respectively.
- W_n-segments and $\tilde{W}_n$-segments—These are subcolumns of columns of the towers W_n and $\tilde{W}_n$, respectively, when seen from towers in further steps. If $n = n_t$ for some t, a $\tilde{W}_n$-segment S together with the adding spacers, which has the form $s^\ell S s^{h_{\ell_t} - \ell}$, is also called a $\tilde{W}_n$-segment.
- h_n and $\tilde{h}_n$—These are the heights of columns of the towers W_n and $\tilde{W}_n$, respectively.
- w_n—This is the height of $\tilde{W}_n$-segments when seen from towers in further steps (after Step $\tilde{n}$). If $n \neq n_t$ for any t, then $w_n = \tilde{h}_n$,; if $n = n_t$ for some t, then $w_n = \tilde{h}_{n_t} + h_{l_t}$.
- n_t, l_t—At step $n_t + 1$ for $t \geq 1$, we add spacers while we do independent cuttings and stackings; h_{l_t} is the parameter related to the number of the spacers.
- c_n $(= \#W_n = \#\tilde{W}_n)$—This is the total number of the columns of W_n or $\tilde{W}_n$.
- ξ_n $(= \mu(|W_n|) = \mu(|\tilde{W}_n|))$—This is the total Lebesgue measure of the level sets in the tower W_n or $\tilde{W}_n$. Recall that ξ is the total length of the intervals P_0 and P_1, i.e., $\xi = \mu(|W_0|)$. In the following we will determine ξ to make $\lim_{n\to+\infty} \xi_n = 1$. Since at each step $(n_t + 1)$ we add spacers of measure $\xi_{n_t} \cdot \frac{h_{l_t}}{\tilde{h}_{n_t}}$, the measures ξ_n's of the tower W_n's satisfy the following:

$$
\begin{aligned}
&\xi_1 = \xi_2 = \cdots = \xi_{n_1} = \xi, \\
&\xi_{n_t+1} = \xi_{n_t} \frac{w_{n_t}}{\tilde{h}_{n_t}} = \xi_{n_t}\left(1 + \frac{h_{l_t}}{\tilde{h}_{n_t}}\right), \qquad (5.13)\\
&\xi_{n_t+1} = \xi_{n_t+2} = \cdots = \xi_{n_{t+1}}, t \geq 1.
\end{aligned}
$$

Due to the choice of r_n, $\sum_{t=1}^{+\infty} \frac{1}{r_{n_t}}$ converges. So $\sum_{t=1}^{+\infty} \frac{h_{l_t}}{\tilde{h}_{n_t}} < \sum_{t=1}^{+\infty} \frac{1}{r_{n_t}}$ converges. Let

$$\xi = \prod_{t=1}^{+\infty} \left(1 + \frac{h_{l_t}}{\tilde{h}_{n_t}}\right)^{-1}. \qquad (5.14)$$

Then we have $0 < \xi < 1$ and $\lim_{n\to+\infty} \xi_n = 1$.

We note that $\mu(X) = \lim_{n\to+\infty} \xi_n = 1$, and it is not hard to see from the construction that μ is T-invariant. We will show that μ is in fact ergodic in Remark 5.5.

5.5. **The upper bound of the entropy dimension.** For convenience, for a finite collection $\mathcal{A}$ consisting of measurable sets in $\mathcal{B}$ (which needs not be a partition), we denote

$$H_\mu(\mathcal{A}) = \sum_{A \in \mathcal{A}} -\mu(A) \log \mu(A) \quad \text{and} \quad N_\mu(\mathcal{A}) = \#\{A \in \mathcal{A} : \mu(A) > 0\}.$$

The intersections of $\beta \in \mathcal{P}_X$ and $U \subseteq X$ are denoted by

$$\beta \cap U = \{B \cap U : B \in \beta\}.$$

Let $n, K \in \mathbb{N}$. We define

$$U_{K,n} = \begin{cases} |W_K| \setminus \Big(\bigcup_{i=1}^{n} T^{h_K - i}\big(\text{base}(W_K)\big)\Big), & \text{if } h_K > n, \\ \emptyset, & \text{if } h_K \le n. \end{cases}$$

Here we recall that h_K denotes the heights of the tower W_K. Then we have the following estimation.

Lemma 5.1. *Given $k \in \mathbb{N}$, let E be a level set of a column in W_k, and let $\alpha = \{E, X \setminus E\}$. Then for any $\epsilon > 0$, there exists a constant $M = M(\epsilon) > 0$ such that when n is sufficiently large,*

$$N_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha \bigcap U_{K,n}) \le (n^3 + 2n^2 + n)2^{Mn^{\tau+\epsilon}} \tag{5.15}$$

for any $K \in \mathbb{N}$.

Proof. First we define $\mathcal{C}(n, K)$ for given $n, K \in \mathbb{N}$. Let $n, K \in \mathbb{N}$. There are two cases. The first case is $h_K \le n$. In this case we put $\mathcal{C}(n, K) = 0$ and then

$$N_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha \bigcap U_{K,n}) \le \mathcal{C}(n, K)$$

since $U_{K,n} = \emptyset$.

The second case is $h_K > n$. In this case, for each column $C = \{B_1, B_2, \ldots, B_{h_K}\}$ of W_K, we can associate C an α-name $b = b_1 b_2 \cdots b_{h_K} \in \{u, v\}^{h_K}$ by

$$b_i = \begin{cases} u, & \text{if } B_i \subset E, \\ v, & \text{if } B_i \subset (X \setminus E). \end{cases}$$

Let $\tilde{E} \subset U_{K,n}$ be a level set of a column in the tower W_K and let $d = d_0 d_1 \cdots d_{n-1} \in \{u, v\}^n$ be the α-name of the segment $S = \{\tilde{E}, T\tilde{E}, T^2\tilde{E}, \ldots, T^{n-1}\tilde{E}\}$ (inherited from the α-name of the column of W_K that contains S). Note that for each $0 \le i \le n-1$,

$$d_i = \begin{cases} u, & \text{if } T^i\tilde{E} \subset E, \\ v, & \text{if } T^i\tilde{E} \subset (X \setminus E). \end{cases}$$

In fact, d is a subword of length n of α-names of W_K-segments.

We denote

$$\mathcal{C}(n, K) = \#\{d \in \{u, v\}^n : d \text{ is a subword of } \alpha\text{-names of } W_K\text{-segments}\}.$$

Since any element in the collection $\bigvee_{i=0}^{n-1} T^{-i}\alpha \cap U_{K,n}$ is a union of some level sets in W_K (mod μ), we have that

$$N_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha \cap U_{K,n}) \le \mathcal{C}(n, K). \tag{5.16}$$

In the following we will show that for any $\epsilon > 0$, there exists a constant $M = M(\epsilon) > 0$ such that when n is sufficiently large,

$$\mathcal{C}(n, K) \le (n^3 + 2n^2 + n)2^{Mn^{\tau+\epsilon}}$$

for any $K \in \mathbb{N}$. Thus combining this fact with (5.16), one has (5.15).

Recall that $c_n = \#W_n = \#\tilde{W}_n$ is the total number of the columns of the tower W_n or $\tilde{W}_n$. From our construction,

$$c_{n+1} = \begin{cases} (c_n)^{e_n}, & \text{if } n \notin \{n_1, n_2, \ldots\}, \\ (c_n)^{e_n+1}, & \text{if } n \in \{n_1, n_2, \ldots\}. \end{cases} \tag{5.17}$$

Hence c_n has the expression

$$\begin{aligned} c_n &= 2^{\left(\prod_{0\le i\le n-1, i\notin\{n_1,n_2,\ldots\}} e_i\right)\cdot\left(\prod_{0\le i\le n-1, i\in\{n_1,n_2,\ldots\}} (e_i+1)\right)} \\ &= 2^{\left(\prod_{i=0}^{n-1} e_i\right)\cdot\left(\prod_{0\le i\le n-1, i\in\{n_1,n_2,\ldots\}} (1+\frac{1}{e_i})\right)}. \end{aligned} \tag{5.18}$$

Given $n \gg h_k$ and $K \in \mathbb{N}$. We consider two cases separately.

Case I. *Suppose $\tilde{h}_\ell \le n < h_{\ell+1}$ for some $\ell \in \mathbb{N}$.*

If $h_K \le n$, then $\mathcal{C}(n, K) = 0$. Now we assume $h_K > n$. Then $K \ge \ell + 1$. Let

$$S = \{\tilde{E}, T\tilde{E}, T^2\tilde{E}, \ldots, T^{n-1}\tilde{E}\}$$

be any segment of height n of a column in W_K, and let d be the α-name of S. By our construction, any column of W_K is stacked by a sequence of $W_{\ell+1}$-segments. According to the positions of S, there are two subcases.

Subcase I.1. *S is completely contained in some $W_{\ell+1}$-segment.*

In this case S has the form $S = pS_1S_2\cdots S_mq$, where S_i is some $\tilde{W}_\ell$-segment for each $1 \le i \le m$, p is an ending part of some $\tilde{W}_\ell$-segment S_0, q is a beginning part of some $\tilde{W}_\ell$-segment S_{m+1}, and $m = \lfloor \frac{n}{w_\ell} \rfloor$ or $\lfloor \frac{n}{w_\ell} \rfloor - 1$. We should note that the $\tilde{W}_\ell$-segments here may contain the inserted spacers if $\ell = n_t$ for some t. The segment S in this case is determined by $S_0, S_1, \ldots, S_{m+1}$ and the length of p.

For each $0 \le i \le m+1$, if $\ell \ne n_t + 1$ for every t, then there are no more than $c_\ell = (c_{\ell-1})^{e_{\ell-1}}$-many choices for the $\tilde{W}_\ell$-segment S_i; if $\ell = n_t + 1$ for some t, there are no more than $c_\ell = (c_{\ell-1})^{e_{\ell-1}+1}$-many choices for S_i. $m \le \lfloor \frac{n}{w_\ell} \rfloor \le \lfloor \frac{h_{\ell+1}}{w_\ell} \rfloor = e_\ell$. There are w_ℓ-many choices for the length of p, which is no more than n. Then the total number of the α-names of such S's is bounded by

$$n\left((c_{\ell-1})^{e_{\ell-1}+1}\right)^{e_\ell+2}.$$

Subcase I.2. *S is not completely contained in any $W_{\ell+1}$-segment.*

Then there are two subcases.

Subcase I.2(a). S has overlaps with two W_t-segments for some $t \ge \ell + 1$. We can finally deduce that S has overlaps with two $W_{\ell+1}$-segments (there may exist spacers of a later step between them). Then S has the form $S = S_0s^rS_1$, where S_0 is an ending part of some $W_{\ell+1}$-segment, S_1 is a beginning part of some $W_{\ell+1}$-segment, and s^r is r-many spacers between the two $W_{\ell+1}$-segments. S is determined by S_0S_1, r and the height of S_0. By a similar discussion as in Subcase I.1, the number of the α-names of such S_0S_1 is bounded by $n\left((c_{\ell-1})^{e_{\ell-1}+1}\right)^{e_\ell+2}$. And both r and the height of S_0 have no more than n choices. Hence the total number of d's in this subcase is bounded by $n^3\left((c_{\ell-1})^{e_{\ell-1}+1}\right)^{e_\ell+2}$.

Subcase I.2(b). S begins with some spacers and is then followed by a beginning part of some $W_{\ell+1}$-segment or S begins with an ending part of some $W_{\ell+1}$-segment and is then followed by some spacers. Then S has the form $S = s^r S'$ or $S = S's^r$, where S' is a segment which is completely contained in some $W_{\ell+1}$-segment. By a similar discussion as in Subcase I.1, the α-name of S' has no more than $n\big((c_{\ell-1})^{e_{\ell-1}+1}\big)^{e_\ell+2}$-many choices and r has no more than n choices. The total number of d's in this subcase is bounded by $2n^2\big((c_{\ell-1})^{e_{\ell-1}+1}\big)^{e_\ell+2}$.

Summing up the above estimations, we have

$$\mathcal{C}(n,K) \le (n^3+2n^2+n)\big((c_{\ell-1})^{e_{\ell-1}+1}\big)^{e_\ell+2}.$$

Next by (5.18), we have

$$\big((c_{\ell-1})^{e_{\ell-1}+1}\big)^{e_\ell+2} = 2^{\left(\prod_{i=0}^{\ell} e_i\right)\cdot\left(\prod_{0\le i\le \ell-2,\, i\in\{n_1,n_2,\dots\}}(1+\frac{1}{e_i})\right)\cdot(1+\frac{1}{e_{\ell-1}})\cdot(1+\frac{2}{e_\ell})}.$$

By (5.2) and (5.10), the product $\prod_{i\in\{n_1,n_2,\dots\}}(1+\frac{1}{e_i})$ is bounded. By the definition of w_ℓ (see (5.3)), we have $w_\ell \le 2\tilde{h}_\ell \le 2n$. Hence

$$\prod_{i=0}^{\ell} e_i \le (w_\ell)^\tau \cdot e_\ell \text{ (by (5.11))} \tag{5.19}$$
$$< 2n^\tau \cdot M'(r_\ell)^{\frac{\tau}{1-\tau}} \text{ (by (5.10))},$$

where $M' > 0$ is a constant independent on n. By (5.12), for any given $\epsilon > 0$, when n is sufficiently large (hence so is ℓ),

$$(r_\ell)^{\frac{\tau}{1-\tau}} \le (w_\ell)^\epsilon \le 2n^\epsilon.$$

Hence for any $\epsilon > 0$, we can find a constant $M = M(\epsilon) > 0$, such that

$$\mathcal{C}(n,K) \le (n^3+2n^2+n)2^{Mn^{\tau+\epsilon}}$$

when n is sufficiently large.

Case II. *Suppose $h_\ell \le n < \tilde{h}_\ell$ for some $\ell \in \mathbb{N}$.*

As in Case I, we assume $h_K > n$ and then $K \ge \ell+1$. Let

$$S = \{\tilde{E}, T\tilde{E}, T^2\tilde{E}, \dots, T^{n-1}\tilde{E}\}$$

be any segment of height n of a column in W_K, and let d be the α-name of S. In this case, by our construction any column of W_K is stacked by a sequence of $\tilde{W}_\ell$-segments. As in Case I, according to the positions of S, there are two subcases.

Subcase II.1. *S is completely contained in some $\tilde{W}_\ell$-segment.*

Since $\tilde{W}_\ell$ is obtained by the repetitions of columns of W_ℓ, in this case S has the form $S = pS_0S_0\cdots S_0q$, where S_0 is some W_ℓ-segment, p is an ending part of S_0, and q is a beginning part of S_0. The segment S in this case is determined by S_0 and the height of p. There are c_ℓ-many choices for S_0 and at most n choices for the height of p. Then the total number of the α-names of such S's is no more than nc_ℓ.

Subcase II.2. *S is not completely contained in any $\tilde{W}_\ell$-segment.*

As in Subcase I.2, there are again two subcases.

Subcase II.2(a). S has overlaps with two $\tilde{W}_t$-segments for some $t \geq \ell$. We can finally deduce that S has overlaps with two $\tilde{W}_\ell$-segments (there may exist spacers of a later step between them). Then S has the form $S = S_0 s^r S_1$, where S_0 is an ending part of some $\tilde{W}_\ell$-segment, S_1 is a beginning part of some $\tilde{W}_\ell$-segment, and s^r is r-many spacers between the two $\tilde{W}_\ell$-segments. S is determined by S_0, S_1, and r. There are no more than nc_ℓ-many choices for S_0 and S_1 and no more than n-many choices for r. Hence the total number of d's in this subcase is bounded by $n^3(c_\ell)^2$.

Subcase II.2(b). S begins with some spacers and is then followed by a beginning part of some $\tilde{W}_\ell$-segment or S begins with an ending part of some $\tilde{W}_\ell$-segment and is then followed by some spacers. Then S has the form $S = s^r S'$ or $S = S' s^r$, where S' is a segment which is completely contained in some $\tilde{W}_\ell$-segment. By a similar discussion as in Subcase II.1, the α-name of S' has no more than nc_ℓ-many choices; r has no more than n choices. The total number of d's in this subcase is bounded by $2n^2 c_\ell$.

Summing up the above estimations, we have in this case

$$\mathcal{C}(n, K) \leq (n^3 + 2n^2 + n)(c_\ell)^2.$$

By (5.18) we have

$$c_\ell = 2^{\left(\prod_{i=0}^{\ell-1} e_i\right) \cdot \left(\prod_{0 \leq i \leq \ell-1, i \in \{n_1, n_2, \ldots\}} (1+\frac{1}{e_i})\right)}.$$

In this situation, noticing that $w_{\ell-1} \leq 2\tilde{h}_{\ell-1} < 2h_\ell \leq 2n$, we have

$$\begin{aligned} \prod_{i=0}^{\ell-1} e_i &\leq (w_{\ell-1})^\tau \cdot e_{\ell-1} \text{ (by (5.11))} \\ &< 2n^\tau \cdot M'(r_{\ell-1})^{\frac{\tau}{1-\tau}} \text{ (by (5.10))}, \end{aligned}$$

where $M' > 0$ is the same constant that appeared in (5.19). By (5.12) again, for any given $\epsilon > 0$, when n is sufficiently large (hence so is ℓ),

$$(r_{\ell-1})^{\frac{\tau}{1-\tau}} \leq (w_{\ell-1})^\epsilon \leq 2n^\epsilon.$$

As shown in Case I, for any $\epsilon > 0$ we can find a constant $M = M(\epsilon) > 0$, which depends on ϵ but is independent on n, such that

$$\mathcal{C}(n, K) \leq (n^3 + 2n^2 + n) 2^{Mn^{\tau+\epsilon}}$$

when n is sufficiently large. This finishes the proof of the lemma. □

With the help of Lemma 5.1, we are able to show that $\overline{D}_\mu(X, T) \leq \tau$ as the following Lemma 5.2. Hence for any partition $\beta = \{B, X \setminus B\} \in \mathcal{P}_X^2$ with $0 < \mu(B) < 1$, we have $\overline{D}_\mu(T, \beta) \leq \overline{D}_\mu(X, T) \leq \tau$.

Lemma 5.2. $\overline{D}_\mu(X,T) \le \tau$.

Proof. Since

$$\bigvee_{k=1}^{+\infty}\Big(\bigvee_{E \text{ is a level set of } W_k}\{E, X\setminus E\}\Big) = \mathcal{B} \pmod{\mu},$$

by (3) of Theorem 2.11,

$$\overline{D}_\mu(X,T) = \sup_{E \text{ is a level set of some } W_k} \overline{D}_\mu(T,\{E, X\setminus E\}).$$

Hence it is sufficient to show that $\overline{D}_\mu(T,\alpha) \le \tau$ for any $\alpha = \{E, X\setminus E\}$, where E is a level set of W_k for some $k \in \mathbb{N}$.

Given $k \in \mathbb{N}$, let E be a level set in W_k, and let $\alpha = \{E, X\setminus E\}$. In the following we show that $\underline{D}(S) \le \tau$ for any $S \in \mathcal{P}_\mu(T,\alpha)$, which implies $\overline{D}_\mu(T,\alpha) \le \tau$ by Definition 2.6.

If this is not true, then we can find $S = \{s_1 < s_2 < \cdots\} \in \mathcal{P}_\mu(T,\alpha)$ and $\epsilon > 0$ such that $\underline{D}(S) > \tau + \epsilon$. It is clear that $\liminf_{n\to+\infty} \frac{n}{(s_n)^{\tau+\epsilon}} = +\infty$.

By Lemma 5.1, there exists a constant $M = M(\epsilon) > 0$ and $N_\epsilon \in \mathbb{N}$ such that

$$N_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha \cap U_{K,n}) \le (n^3 + 2n^2 + n)2^{Mn^{\tau+\epsilon}} \tag{5.20}$$

for any $K \in \mathbb{N}$ and $n \ge N_\epsilon$.

Now for each $n \ge N_\epsilon$, since $\mu(U_{K,n}) = \xi_K \cdot (1 - \frac{n}{h_K})$ when $h_K > n$, we have

$$\mu(X \setminus U_{K,n}) = (1-\xi_K) + \frac{n}{h_K}\xi_K.$$

Hence we can choose $K = K(n)$ sufficiently large to satisfy that

$$\mu(X\setminus U_n) \le \frac{1}{2n\log 2}$$

and

$$-\mu(U_n)\log\mu(U_n) - \mu(X\setminus U_n)\log\mu(X\setminus U_n) \le \frac{1}{2},$$

where for simplicity we write U_n as $U_{K(n),n}$.

Together with the fact $\#(\bigvee_{i=0}^{n-1} T^{-i}\alpha) \le 2^n$, we have

$$\mu(X\setminus U_n)\log\#(\bigvee_{i=0}^{n-1} T^{-i}\alpha) - \mu(U_n)\log\mu(U_n) - \mu(X\setminus U_n)\log\mu(X\setminus U_n) \le 1.$$

Hence when $n \geq N_\epsilon$,

$$
\begin{aligned}
&H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha) \\
&\quad \leq H_\mu\Big(\bigvee_{i=0}^{n-1} T^{-i}\alpha \vee \{U_n, X\setminus U_n\}\Big) \\
&\quad = H_\mu\Big(\bigvee_{i=0}^{n-1} T^{-i}\alpha \cap U_n\Big) + H_\mu\Big(\bigvee_{i=0}^{n-1} T^{-i}\alpha \cap (X\setminus U_n)\Big) \\
&\quad \leq -\mu(U_n)\log\frac{\mu(U_n)}{N_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha\cap U_n)} - \mu(X\setminus U_n)\log\frac{\mu(X\setminus U_n)}{N_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha)} \\
&\quad = \mu(U_n)\log N_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha\cap U_n) + \mu(X\setminus U_n)\log \#(\bigvee_{i=0}^{n-1} T^{-i}\alpha) \\
&\qquad - \mu(U_n)\log\mu(U_n) - \mu(X\setminus U_n)\log\mu(X\setminus U_n) \\
&\quad \leq \mu(U_n)\log N_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha\cap U_n) + 1 \\
&\quad \leq \log\Big((n^3+2n^2+n)2^{Mn^{\tau+\epsilon}}\Big) + 1,
\end{aligned}
$$

where the last inequality comes from (5.20).

Now using the above estimation we have

$$
\begin{aligned}
h_\mu^S(T,\alpha) &= \limsup_{n\to+\infty}\frac{1}{n}H_\mu(\bigvee_{i=1}^{n} T^{-s_i}\alpha) \\
&\leq \limsup_{n\to+\infty}\frac{1}{n}H_\mu(\bigvee_{i=0}^{s_n} T^{-i}\alpha) \\
&\leq \limsup_{n\to+\infty}\frac{1}{n}\cdot\Big(\log\big(((s_n+1)^3+2(s_n+1)^2+s_n+1)2^{M(s_n+1)^{\tau+\epsilon}}\big)+1\Big) \\
&\leq \limsup_{n\to+\infty}\frac{1}{n}\cdot(s_n+1)^{\tau+\epsilon}\cdot M\log 2 = 0.
\end{aligned}
$$

That is, $h_\mu^S(T,\alpha)=0$, a contradiction with $S\in\mathcal{P}_\mu(T,\alpha)$. This implies $\overline{D}_\mu(T,\alpha)\leq \tau$. □

5.6. **The lower bound.** For $A,B\subset\mathbb{Z}$, let $A+B \triangleq \{a+b : a\in A, b\in B\}$, and let $|A|$ denote the number of integers in A. Recall that w_n is given in (5.3). For $t\geq 1$, let

$$
\begin{aligned}
&F_0^t = \{0, w_{n_t}, 2w_{n_t}, \dots, (e_{n_t}-1)w_{n_t}\}, \\
&F_k^t = F_{k-1}^t + \{0, w_{n_t+k}, 2w_{n_t+k}, \dots, (e_{n_t+k}-1)w_{n_t+k}\} \text{ for } k\geq 1, \text{ and} \\
&F^t = \bigcup_{k=0}^{+\infty} F_k^t.
\end{aligned}
\tag{5.21}
$$

We have $F_0^t\subset F_1^t\subset\cdots$ and $|F_k^t| = e_{n_t}e_{n_t+1}\cdots e_{n_t+k}$.

Lemma 5.3. *For any $t\geq 1$, $D(F^t)=\tau$.*

Proof. Given $t \geq 1$, let $F^t = \{t_1 < t_2 < \cdots\}$. For any $n \in \mathbb{N}$, there exists a unique $k = k(n)$ such that $t_n \in F^t_{k+1} \setminus F^t_k$. Then

$$e_{n_t} e_{n_t+1} \cdots e_{n_t+k} < n \leq e_{n_t} e_{n_t+1} \cdots e_{n_t+k+1}$$

and

$$h_{n_t+k+1} < t_n \leq h_{n_t+k+2}.$$

For any τ' with $0 \leq \tau' < \tau$,

$$\begin{aligned}
\underline{D}(F^t, \tau') &= \liminf_{n \to +\infty} \frac{n}{(t_n)^{\tau'}} \\
&\geq \liminf_{k \to +\infty} \frac{e_{n_t} e_{n_t+1} \cdots e_{n_t+k}}{(h_{n_t+k+2})^{\tau'}} = \liminf_{k \to +\infty} \frac{e_{n_t} e_{n_t+1} \cdots e_{n_t+k}}{(w_{n_t+k+1} e_{n_t+k+1})^{\tau'}} \\
&= \liminf_{k \to +\infty} \frac{e_0 e_1 \cdots e_{n_t+k+1}}{(w_{n_t+k+1} e_{n_t+k+1})^{\tau}} \cdot \frac{(w_{n_t+k+1} e_{n_t+k+1})^{\tau - \tau'}}{e_0 e_1 \cdots e_{n_t-1} e_{n_t+k+1}} \\
&= +\infty.
\end{aligned} \tag{5.22}$$

We note that the last equality comes from (5.9) and (5.12). Hence $\underline{D}(F^t) \geq \tau'$. Since this inequality is true for any $\tau' \in [0, \tau)$, we have $\underline{D}(F) \geq \tau$.

For any τ' with $\tau < \tau' < 1$,

$$\begin{aligned}
\overline{D}(F^t, \tau') &= \limsup_{n \to +\infty} \frac{n}{(t_n)^{\tau'}} \\
&\leq \limsup_{k \to +\infty} \frac{e_{n_t} e_{n_t+1} \cdots e_{n_t+k+1}}{(h_{n_t+k+1})^{\tau'}} = \limsup_{k \to +\infty} \frac{e_{n_t} e_{n_t+1} \cdots e_{n_t+k+1}}{(w_{n_t+k} e_{n_t+k})^{\tau'}} \\
&= \limsup_{k \to +\infty} \frac{e_0 e_1 \cdots e_{n_t+k}}{(w_{n_t+k} e_{n_t+k})^{\tau}} \cdot \frac{e_{n_t+k+1}}{e_0 e_1 \cdots e_{n_t-1} \cdot (w_{n_t+k} e_{n_t+k})^{\tau' - \tau}} \\
&= 0,
\end{aligned} \tag{5.23}$$

where the last equality comes again from (5.9) and (5.12). So $\overline{D}(F^t) \leq \tau'$. Since this inequality is true for any $\tau' \in (\tau, 1)$, we have $\overline{D}(F^t) \leq \tau$. Hence $D(F^t) = \tau$. □

Lemma 5.4. *Given $t > 0$ and $k \geq 0$, let $B \subset F^t_k$, and let E_b be a level set in W_{l_t} for $b \in B$ (E_b's need not have different names), then*

$$\mu\Big(\bigcap_{b \in B} T^{-b} E_b\Big) \leq \Big(1 + \frac{h_{l_t}}{c_{n_t}}\Big)^{|B|} \cdot \Big(\frac{1}{\xi_{l_t}}\Big)^{|B|} \prod_{b \in B} \mu(E_b). \tag{5.24}$$

Moreover, let U_b be a union of finite many level sets in W_{l_t} for $b \in B$, then

$$\mu\Big(\bigcap_{b \in B} T^{-b} U_b\Big) \leq \Big(1 + \frac{h_{l_t}}{c_{n_t}}\Big)^{|B|} \cdot \Big(\frac{1}{\xi_{l_t}}\Big)^{|B|} \prod_{b \in B} \mu(U_b). \tag{5.25}$$

Proof. Assume $B = \{b_1, b_2, \ldots, b_m\}$ and E_{b_i}'s ($i = 1, 2, \ldots, m$) are level sets in W_{l_t}. Then every E_{b_i} satisfies that

$$\mu(E_{b_i}) = \frac{\xi_{l_t}}{c_{l_t} \cdot h_{l_t}}.$$

Notice that the level sets E_{b_i}'s in W_{l_t} are all spread out into many much smaller level sets after sufficiently large steps. For the small level sets A_1's from E_{b_1}, to ensure the level sets $T^{b_i - b_1} A_1$'s ($2 \leq i \leq m$) are from E_{b_i} respectively, the W_{l_t}-segment which contains $T^{b_i - b_1} A_1$ must be isomorphic with the column that contains E_{b_i} in W_{l_t} for every $2 \leq i \leq m$. This situation happens with probability $\big(\frac{1}{c_{l_t}}\big)^{m-1}$. Also we need that, for each $2 \leq i \leq m$, the positions of E_{b_i}'s in W_{l_t} coincide with the positions of E_{b_i}'s in W_{l_t}-segments after inserting spacers. Since the numbers of beginning spacers of W_{l_t}-segments are uniformly distributed on $\{0,1,\dots,h_{l_t}-1\}$ within $\frac{h_{l_t}}{c_{n_t}}$-error, at most $(\frac{1}{h_{l_t}}+\frac{1}{c_{n_t}})^{m-1}$-portion of them coincide. So

$$\begin{aligned}\mu\big(\bigcap_{b\in B} T^{-b}E_b\big) &= \mu\big(T^{-b_1}E_{b_1}\cap T^{-b_2}E_{b_2}\cap\cdots\cap T^{-b_m}E_{b_m}\big)\\ &\le \mu(E_{b_1})\cdot\big(\frac{1}{c_{l_t}}\big)^{m-1}\cdot\big(\frac{1}{h_{l_t}}+\frac{1}{c_{n_t}}\big)^{m-1}\\ &= \big(1+\frac{h_{l_t}}{c_{n_t}}\big)^{m-1}\cdot\big(\frac{1}{\xi_{l_t}}\big)^{m-1}\mu(E_{b_1})\mu(E_{b_2})\cdots\mu(E_{b_m})\\ &\le \big(1+\frac{h_{l_t}}{c_{n_t}}\big)^{|B|}\cdot\big(\frac{1}{\xi_{l_t}}\big)^{|B|}\prod_{b\in B}\mu(E_b).\end{aligned}$$

Since U_b is a disjoint union of level sets in W_{l_t}, inequality (5.25) then follows from (5.24). □

Remark 5.5. From the above lemma, for any $p\in\mathbb{Z}^+$ and any two level sets E and $\tilde{E}$, there exists $n>0$ with $\mu(T^{-np}E\cap\tilde{E})>0$. We note that the σ-algebra $\mathcal{B}$ is generated by the level sets. Approximated by the union of these level sets, for any two sets A and $\tilde{A}$ with positive measures, there also exists $n>0$ with $\mu(T^{-np}A\cap\tilde{A})>0$. This implies that μ is an ergodic measure under T^p for any p.

In the following we will prove the u.d. property for the partition $\{A,A^c\}$, where A is a union of finite many level sets in W_ℓ for some ℓ.

Lemma 5.6. *Let $\alpha=\{A,A^c\}$, where A is a union of finite level sets in W_ℓ for some $\ell\in\mathbb{N}$ with $0<\mu(A)\le\frac{1}{2}\xi_\ell$. Then for sufficiently large t,*

$$\liminf_{n\to+\infty}\frac{1}{n}H_\mu\Big(\bigvee_{i=1}^{n}T^{-t_i}\alpha\Big)\ge -\frac{1}{2}\mu(A)\log\frac{\mu(A)}{1-\mu(A)}>0,$$

where $F^t=\{t_1<t_2<\cdots\}$ is given by (5.21). *Hence F^t is an entropy generating sequence of α. Moreover, $D_\mu(T,\alpha)=\tau$.*

Proof. Since $0<\mu(A)\le\frac{1}{2}\xi_\ell<\frac{1}{2}$, we have $-\mu(A)\log\frac{\mu(A)}{1-\mu(A)}>0$. Note that $l_k<n_k$ and $e_k\ge 2$ for any $k\in\mathbb{N}$. By (5.3), (5.9), and (5.18), one has $\lim_{t\to+\infty}\frac{h_{l_t}}{c_{n_t}}=0$. Thus combining this fact with $\lim_{n\to+\infty}\xi_n=1$, we can take t sufficiently large such that $l_t\ge\ell$ and

$$\log\Big((1+\frac{h_{l_t}}{c_{n_t}})\cdot\frac{1}{\xi_{l_t}}\Big)<-\frac{1}{2}\mu(A)\log\frac{\mu(A)}{1-\mu(A)}.$$

For convenience, let $A_0=A, A_1=A^c$. For any finite subset B of F^t and any finite sequence $s=(s_b)_{b\in B}\in\{0,1\}^B$, let

$$B_0(s)=\{b\in B: s_b=0\}\quad\text{and}\quad B_1(s)=\{b\in B: s_b=1\}.$$

Noticing that $A_0=A$ is a union of finitely many level sets in W_{l_t} (we note here that since $A_1=A^c$ is not a union of finitely many level sets in W_{l_t}, we cannot apply Lemma 5.4 to $\mu(\bigcap_{b\in B} T^{-b}A_{s_b})$), we have

$$\begin{aligned}\mu(\bigcap_{b\in B} T^{-b}A_{s_b}) &\le \mu(\bigcap_{b\in B_0(s)} T^{-b}A_0)\\ &\le (1+\frac{h_{l_t}}{c_{n_t}})^{|B_0(s)|}\cdot(\frac{1}{\xi_{l_t}})^{|B_0(s)|}\prod_{b\in B_0(s)}\mu(A_0) \text{ (by Lemma 5.4)}\\ &= \Big(\prod_{b\in B_0(s)}\mu(A_0)\Big)\cdot\Big(\prod_{b\in B_1(s)}\mu(A_1)\Big)\\ &\quad\cdot(1+\frac{h_{l_t}}{c_{n_t}})^{|B_0(s)|}\cdot(\frac{1}{\xi_{l_t}})^{|B_0(s)|}\cdot(\frac{1}{\mu(A_1)})^{|B_1(s)|}\\ &\le \Big(\prod_{b\in B}\mu(A_{s_b})\Big)\cdot(1+\frac{h_{l_t}}{c_{n_t}})^{|B|}\cdot(\frac{1}{\xi_{l_t}})^{|B|}\cdot(\frac{1}{\mu(A_1)})^{|B|}.\end{aligned}$$

So

$$\begin{aligned}H_\mu(\bigvee_{i=1}^m T^{-t_i}\alpha) &= \sum_{s\in\{0,1\}^m} -\mu(\bigcap_{i=1}^m T^{-t_i}A_{s_i})\log\Big(\mu(\bigcap_{i=1}^m T^{-t_i}A_{s_i})\Big)\\ &\ge \sum_{s\in\{0,1\}^m} -\mu(\bigcap_{i=1}^m T^{-t_i}A_{s_i})\\ &\qquad\cdot\log\Big(\big(\prod_{i=1}^m\mu(A_{s_i})\big)\cdot(1+\frac{h_{l_t}}{c_{n_t}})^m\cdot(\frac{1}{\xi_{l_t}})^m\cdot(\frac{1}{\mu(A_1)})^m\Big)\\ &= \Big(\sum_{s\in\{0,1\}^m} -\mu(\bigcap_{i=1}^m T^{-t_i}A_{s_i})\log\big(\prod_{i=1}^m\mu(A_{s_i})\big)\Big)\\ &\qquad - \log\Big((1+\frac{h_{l_t}}{c_{n_t}})^m\cdot(\frac{1}{\xi_{l_t}})^m\cdot(\frac{1}{\mu(A_1)})^m\Big).\end{aligned}$$

Since

$$\begin{aligned}&\sum_{s\in\{0,1\}^m} -\mu(\bigcap_{i=1}^m T^{-t_i}A_{s_i})\log\Big(\prod_{i=1}^m\mu(A_{s_i})\Big)\\ &=\sum_{j=1}^m\sum_{s\in\{0,1\}^m} -\mu(\bigcap_{i=1}^m T^{-t_i}A_{s_i})\log\Big(\mu(A_{s_j})\Big)\\ &=\sum_{j=1}^m\sum_{s_j\in\{0,1\}} -\mu(T^{-t_j}A_{s_j})\log\Big(\mu(A_{s_j})\Big)\\ &=mH_\mu(\alpha),\end{aligned}$$

we have

$$\begin{aligned}H_\mu(\bigvee_{i=1}^m T^{-t_i}\alpha) &\ge mH_\mu(\alpha) - \log\Big((1+\frac{h_{l_t}}{c_{n_t}})^m\cdot(\frac{1}{\xi_{l_t}})^m\cdot(\frac{1}{\mu(A_1)})^m\Big)\\ &> m\Big(H_\mu(\alpha)+\frac{1}{2}\mu(A)\log\frac{\mu(A)}{1-\mu(A)}+\log\big(1-\mu(A)\big)\Big)\\ &= m\Big(-\frac{1}{2}\mu(A)\log\frac{\mu(A)}{1-\mu(A)}\Big).\end{aligned}\tag{5.26}$$

Hence

$$\liminf_{n\to+\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-t_i}\alpha) \geq -\frac{1}{2}\mu(A)\log\frac{\mu(A)}{1-\mu(A)} > 0,$$

which implies that F^t is an entropy generating sequence of α. By Lemma 5.2 and 5.3, we have $D_\mu(T, \alpha) = \tau$. ☐

Theorem 5.7. *$(X, \mathcal{B}, \mu, T)$ is a τ-u.d. system.*

Proof. Let $\beta = \{B, X \setminus B\} \in \mathcal{P}_X^2$. We first consider the case for $0 < \mu(B) < \frac{1}{2}$. Then

$$c(\beta) := -\frac{1}{2}\mu(B)\log\frac{\mu(B)}{1-\mu(B)} > 0.$$

For any $0 < \epsilon < \frac{1}{2}c(\beta)$, by [30, Lemma 4.15], we can choose $\delta > 0$ small enough such that $H_\mu(\beta|\gamma) + H_\mu(\gamma|\beta) < \epsilon$, whenever $\gamma = \{E, X \setminus E\} \in \mathcal{P}_X^2$ satisfies that $\mu(B\Delta E) + \mu\big((X\setminus B)\Delta(X\setminus E)\big) < \delta$. When ℓ is sufficiently large, there is a subset A of X which is a union of level sets in W_ℓ such that $0 < \mu(A) \leq \frac{1}{2}\xi_\ell$ and

$$\mu(A\Delta B) < \frac{\delta}{2}, \mu\big((X\setminus A)\Delta(X\setminus B)\big) < \frac{\delta}{2}.$$

Let $\alpha = \{A, X \setminus A\}$, then $H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha) < \epsilon$. Moreover, we can make $c(\alpha) > \frac{1}{2}c(\beta)$ when δ is sufficiently small, where $c(\alpha) = -\frac{1}{2}\mu(A)\log\frac{\mu(A)}{1-\mu(A)} > 0$. By Lemmas 5.3 and 5.6, there exists $F = \{t_1 < t_2 < \cdots\} \subseteq \mathbb{N}$ such that $D(F) = \tau$ and

$$\liminf_{n\to+\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-t_i}\alpha) \geq c(\alpha).$$

Thus we have

$$\begin{aligned}
&\liminf_{n\to+\infty}\frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-t_i}\beta)\\
&\quad= \liminf_{n\to+\infty} \frac{1}{n}\left(H_\mu(\bigvee_{i=1}^{n} T^{-t_i}(\alpha\vee\beta)) - H_\mu(\bigvee_{i=1}^{n} T^{-t_i}\alpha|\bigvee_{i=1}^{n} T^{-t_i}\beta)\right)\\
&\quad\geq \liminf_{n\to+\infty} \frac{1}{n}\left(H_\mu(\bigvee_{i=1}^{n} T^{-t_i}\alpha) - nH_\mu(\alpha|\beta)\right)\\
&\quad\geq c(\alpha) - H_\mu(\alpha|\beta) \geq c(\alpha) - \epsilon\\
&\quad\geq c(\alpha) - \frac{1}{2}c(\beta) > 0,
\end{aligned}$$

which means that F is also an entropy generating sequence of β. Hence $\underline{D}_\mu(T, \beta) \geq \underline{D}(F) \geq \tau$. Combining with Lemma 5.2, $D_\mu(T, \beta) = \tau$.

Next we consider the case for $\mu(B) = \frac{1}{2}$. Assume $\underline{D}_\mu(T, \beta) < \tau$. Since μ is ergodic under both T and T^2 (Remark 5.5), we have $\mu(B \cap T^{-1}B) \neq 0, \frac{1}{2}$. Thus either $0 < \mu(B \cap T^{-1}B) < \frac{1}{2}$ or $0 < \mu(X \setminus (B \cap T^{-1}B)) < \frac{1}{2}$. From the previous case, we have $D_\mu\Big(T, \{B \cap T^{-1}B, X \setminus (B \cap T^{-1}B)\}\Big) = \tau$. Noticing that

$\{B \cap T^{-1}B, X \setminus (B \cap T^{-1}B)\} \preceq \beta \bigvee T^{-1}\beta$, by (1) and (2) of Proposition 2.9, we have

$$\begin{aligned} \underline{D}_\mu\Big(T,&\{B \cap T^{-1}B, X \setminus (B \cap T^{-1}B)\}\Big) \\ &\leq \underline{D}_\mu(T, \beta \vee T^{-1}\beta) \\ &= \underline{D}_\mu(T, \beta) < \tau, \end{aligned}$$

which leads to a contradiction. Hence we still have $D_\mu(T, \beta) = \tau$.

Since β is arbitrary, we conclude that $(X, \mathcal{B}, \mu, T)$ is a τ-u.d. system. □

Remark 5.8. By a similar method, we can also choose suitable parameters such that $(X, \mathcal{B}, T, \mu)$ is a 1-u.d. system with zero entropy.

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