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THE STRUCTURE OF RANDOM AUTOMORPHISMS OF COUNTABLE STRUCTURES

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Abstract. In order to understand the structure of the "typical" element of an automorphism group, one has to study how large the conjugacy classes of the group are. When typical is meant in the sense of Baire category, a complete description of the size of the conjugacy classes has been given by Kechris and Rosendal. Following Dougherty and Mycielski, we investigate the measure theoretic dual of this problem, using Christensen's notion of Haar null sets. When typical means random, that is, almost every with respect to this notion of Haar null sets, the behavior of the automorphisms is entirely different from the Baire category case. In this paper we generalize the theorems of Dougherty and Mycielski about S_∞ to arbitrary automorphism groups of countable structures isolating a new model theoretic property, the *cofinal strong amalgamation property*. As an application, we show that a large class of automorphism groups can be decomposed into the union of a meager and a Haar null set.

1. Introduction

The study of typical elements of Polish groups is a flourishing field with a large number of applications. The systematic investigation of typical elements of automorphism groups of countable structures was initiated by Truss [19]. He conjectured that the existence of a comeager conjugacy class can be characterized in model theoretic terms. This question was answered affirmatively by Kechris and Rosendal [15]. They, extending the work of Hodges et al. [13], also investigated the relation between the existence of comeager conjugacy classes in every dimension and other group theoretic properties, such as the small index property, uncountable cofinality, automatic continuity, and Bergman's property.

The existence and description of typical elements frequently have applications in the theory of dynamical systems as well. For example, it is easy to see that the automorphism group of the countably infinite atomless Boolean algebra is isomorphic to the homeomorphism group of the Cantor set, which is a central object in dynamics. Thus, from their general results, Kechris and Rosendal deduced the

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existence of a comeager conjugacy class in the homeomorphism group of the Cantor set. A description of an element with such a class was given by Glasner and Weiss [9], and from a different perspective by Bernardes and Darji [2].

Thus, it is natural to ask whether there exist measure theoretic analogues of these results. Unfortunately, on nonlocally compact groups, there is no natural invariant σ-finite measure. However, a generalization of the ideal of measure zero sets can be defined in every Polish group as follows.

Definition 1.1 (Christensen [3])**.** Let G be a Polish group, and let $B \subset G$ be Borel. We say that B is *Haar null* if there exists a Borel probability measure μ on G such that for every $g, h \in G$ we have $\mu(gBh) = 0$. An arbitrary set S is called Haar null if $S \subset B$ for some Borel Haar null set B.

It is known that the collection of Haar null sets forms a σ-ideal in every Polish group (see [4], [16]), and it coincides with the ideal of measure zero sets in locally compact groups with respect to every left (or equivalently right) Haar measure. Using this definition, it makes sense to talk about the properties of random elements of a Polish group. A property P of elements of a Polish group G is said to *hold almost surely* or *almost every element of G has property P* if the set $\{g \in G : g \text{ has property } P\}$ is co-Haar null.

Since we are primarily interested in homeomorphism and automorphism groups, and in such groups conjugate elements can be considered isomorphic, we are interested only in the conjugacy invariant properties of the elements of our Polish groups. Hence, in order to describe the random element, one must give a complete description of the size of the conjugacy classes with respect to the Haar null ideal. The investigation of this question has been started by Dougherty and Mycielski [8] in the permutation group of a countably infinite set, S_∞. If $f \in S_\infty$ and a is an element of the underlying set, then the set $\{f^k(a) : k \in \mathbb{Z}\}$ is called the *orbit of a (under f)*, while the cardinality of this set is called the *orbit length.* Thus, each $f \in S_\infty$ has a collection of orbits (associated with the elements of the underlying set). It is easy to show that two elements of S_∞ are conjugate if and only if they have the same (possibly infinite) number of orbits for each possible orbit length.

Theorem 1.2 (Dougherty and Mycielski [8])**.** *Almost every element of S_∞ has infinitely many infinite orbits, and only finitely many finite ones.*

Therefore, almost all permutations belong to the union of a countable set of conjugacy classes.

Theorem 1.3 (Dougherty and Mycielski [8])**.** *All of these countably many conjugacy classes are non-Haar null.*

Thus, the above theorems give a complete description of the non-Haar null conjugacy classes and the (conjugacy invariant) properties of a random element. The aim of our paper is to initiate a systematic study of the size of the conjugacy classes of automorphism groups of countable structures. Our work is centered around questions of the following type.

Question 1.4. Let $\mathcal{A}$ be a countable (first order) structure.

(1) What properties of $\mathcal{A}$ ensure that (an appropriate) generalization of the theorem of Dougherty and Mycielski holds for $\mathrm{Aut}(\mathcal{A})$?

(2) Describe the (conjugacy invariant) properties of almost every element of $\mathrm{Aut}(\mathcal{A})$: Which conjugacy classes of $\mathrm{Aut}(\mathcal{A})$ are non-Haar null? How many non-Haar null conjugacy classes are there? Is almost every element of $\mathrm{Aut}(\mathcal{A})$ contained in a non-Haar null class?

In this paper we answer the first question; see Section 3 and Theorem 4.14.

One can prove that in S_∞ the collection of elements that have no infinite orbits is a comeager set. This shows that the typical behavior in the sense of Baire category is quite different from the typical behavior in the measure theoretic sense. In particular, S_∞ can be decomposed into the union of a Haar null and a meager set. It is well known that this is possible in every locally compact group, but the situation is not clear in the nonlocally compact case. Thus, the below question of the first author arises naturally.

Question 1.5. Suppose that G is an uncountable Polish group. Can it be written as the union of a meager and a Haar null set?

We investigate this question for various automorphism groups and solve it for a large class; see Corollary 5.1.

The paper is organized as follows. First, in Section 2 we summarize facts and notations used later, then in Section 3 we give a detailed description of our results. For the sake of the transparency of the topic we also include in this section the results of two upcoming papers [6], [5]. Section 4 contains our main theorem, while in Section 5 we apply the general result to prove a theorem about Haar null-meager decompositions. After this, in Section 6 we investigate the possible cardinality of non-Haar null conjugacy classes of (locally compact and nonlocally compact) Polish groups. Finally, we conclude by listing a number of open questions in Section 7.

2. Preliminaries and notations

We will follow the notation of [14]. For a detailed introduction to the theory of Polish groups see [1, Chapter 1], while the model theoretic background can be found in [12, Chapter 7]. Nevertheless, we summarize the basic facts which we will use.

As usual, a countable structure $\mathcal{A}$ is a first order structure on a countable set with countably many constants, relations, and functions. The underlying set will be denoted by $\mathrm{dom}(\mathcal{A})$. The automorphism group of the structure $\mathcal{A}$ is denoted by $\mathrm{Aut}(\mathcal{A})$, which we consider a topological (Polish) group with the topology of pointwise convergence. Isomorphisms between topological groups are considered to be group automorphisms that are also homeomorphisms. The structure $\mathcal{A}$ is called *ultrahomogeneous* if every isomorphism between its finitely generated substructures extends to an automorphism of $\mathcal{A}$. The *age* of a structure $\mathcal{A}$ is the collection of the finitely generated substructures of $\mathcal{A}$. An injective homomorphism between structures will be called an *embedding*. A structure is said to be *locally finite* if every finite set of elements generates a finite substructure.

A countable set $\mathcal{K}$ of finitely generated structures of the same language is called a *Fraïssé class* if it satisfies the hereditary (HP), joint embedding (JEP), and amalgamation properties (AP) (see [12, Chapter 7]). We will need the notion of the strong amalgamation property: A Fraïssé class $\mathcal{K}$ satisfies the *strong amalgamation property* (*SAP*) if for every $\mathcal{B} \in \mathcal{K}$ and every pair of structures $\mathcal{C}, \mathcal{D} \in \mathcal{K}$ and embeddings $\psi : \mathcal{B} \to \mathcal{C}$ and $\chi : \mathcal{B} \to \mathcal{D}$, there exist $\mathcal{E} \in \mathcal{K}$ and embeddings $\psi' : \mathcal{C} \to \mathcal{E}$

and $\chi' : \mathcal{D} \to \mathcal{E}$ such that

$$\psi' \circ \psi = \chi' \circ \chi$$

and

$$\psi'(\mathcal{C}) \cap \chi'(\mathcal{D}) = (\psi' \circ \psi)(\mathcal{B}) = (\chi' \circ \chi)(\mathcal{B}).$$

For a Fraïssé class $\mathcal{K}$ the unique countable ultrahomogeneous structure $\mathcal{A}$ with $age(\mathcal{A}) = \mathcal{K}$ is called the *Fraïssé limit of* $\mathcal{K}$. If G is the automorphism group of a structure $\mathcal{A}$, we call a bijection p a *partial automorphism* or a *partial permutation* if it is an automorphism between two finitely generated substructures of $\mathcal{A}$ such that $p \subset g$ for some $g \in G$.

As mentioned before, S_∞ stands for the permutation group of the countably infinite set ω. It is well known that S_∞ is a Polish group with the pointwise convergence topology. This coincides with the topology generated by the sets of the form $[p] = \{f \in S_\infty : p \subset f\}$, where p is a finite partial permutation.

Let $\mathcal{A}$ be a countable structure. By the countability of $\mathcal{A}$, every automorphism $f \in \mathrm{Aut}(\mathcal{A})$ can be regarded as an element of S_∞, and it is not hard to see that in fact $\mathrm{Aut}(\mathcal{A})$ will be a closed subgroup of S_∞. Moreover, the converse is also true, namely every closed subgroup of S_∞ is isomorphic to the automorphism group of a countable structure.

Let G be a closed subgroup of S_∞. The *orbit* of an element $x \in \omega$ (under G) is the set $G(x) = \{y \in \omega : \exists g \in G\ (g(x) = y)\}$. For a set $S \subset \omega$ we denote the *pointwise stabilizer* of S by $G_{(S)}$, that is, $G_{(S)} = \{g \in G : \forall s \in S\ (g(s) = s)\}$. In the case in which $S = \{x\}$, we write $G_{(x)}$ instead of $G_{(\{x\})}$.

As in the case of S_∞, for a countable structure $\mathcal{A}$, an element $a \in \mathrm{dom}(\mathcal{A})$, and $f \in \mathrm{Aut}(\mathcal{A})$ the set $\{f^k(a) : k \in \mathbb{Z}\}$ is called the orbit of a and denoted by $\mathcal{O}^f(a)$, while the cardinality of this set is called the orbit length. The *collection of the orbits of* f, or the *orbits of* f is the set $\{\mathcal{O}^f(a) : a \in \mathrm{dom}(\mathcal{A})\}$. If $S \subset \mathrm{dom}(\mathcal{A})$, we will also use the notation $\mathcal{O}^f(S)$ for the set $\bigcup_{a \in S} \mathcal{O}^f(a)$.

We will constantly use the following fact.

Fact 2.1. Let $\mathcal{A}$ be a countable structure. A closed subset C of $\mathrm{Aut}(\mathcal{A})$ is compact if and only if for every $a \in \mathrm{dom}(\mathcal{A})$ the set $\{f(a), f^{-1}(a) : f \in C\}$ is finite.

We denote by $\mathcal{B}_\infty$ the countable atomless Boolean algebra, and by $(\mathbb{Q}, <)$ or $\mathbb{Q}$ the rational numbers as an ordered set. Let us use the notation $\mathcal{R}$ (or (V, R)) for the countably infinite random graph, that is, the unique countable graph with the following property: For every pair of finite disjoint sets $A, B \subset V$ there exists $v \in V$ such that $(\forall x \in A)(xRv)$ and $(\forall y \in B)(y \neg R v)$.

Let us consider the following notion of largeness.

Definition 2.2. Let G be a Polish topological group. A set $A \subset G$ is called *compact catcher* if for every compact $K \subset G$ there exist $g, h \in G$ such that $gKh \subset A$. A is *compact biter* if for every compact $K \subset G$ there exist an open set U and $g, h \in G$ such that $U \cap K \neq \emptyset$ and $g(U \cap K)h \subset A$.

The following easy observation is one of the most useful tools to prove that a certain set is not Haar null.

Fact 2.3. If A is compact biter, then it is not Haar null.

Proof. Suppose that this is not the case, let $B \supset A$ be a Borel Haar null set, and let μ be a witness measure for B. Then, by the regularity of μ, there exists a compact set $K \subset G$ such that $\mu(K) > 0$. Subtracting the relatively open μ measure zero subsets of K, we can suppose that for every open set U if $U \cap K \neq \emptyset$, then $\mu(U \cap K) > 0$. But, as A is compact biter, so is B; thus, for some open set U with $\mu(U \cap K) > 0$ there exist $g, h \in G$ such that $g(U \cap K)h \subset B$. This shows that μ cannot witness that B is Haar null, which is a contradiction. □

Note that the proof of Theorem 1.3 by Dougherty and Mycielski actually shows that every non-Haar null conjugacy class is compact biter and that the unique non-Haar null conjugacy class which contains elements without finite orbits is compact catcher.

It is sometimes useful to consider right and left Haar null sets: A Borel set B is *right* (*resp., left*) *Haar null* if there exists a Borel probability measure μ on G such that for every $g \in G$ we have $\mu(Bg) = 0$ (resp., $\mu(gB) = 0$). An arbitrary set S is called right (resp., left) Haar null if $S \subset B$ for some Borel right (resp., left) Haar null set B. The following observation will be used several times.

Lemma 2.4. *Suppose that B is a Borel set that is invariant under conjugacy. Then B is left Haar null if and only if it is right Haar null if and only if it is Haar null.*

Proof. Let μ be a measure witnessing that B is left Haar null. We check that it also witnesses the Haar nullness of B. Indeed, let $g, h \in G$ be arbitrary, and let $\mu(gBh) = \mu(ghh^{-1}Bh) = \mu(ghB) = 0$. The proof is analogous when B is right Haar null. □

3. Description of the results

We start by defining the crucial notion for the description of the orbits of a random element of an automorphism group. Informally, the following definition says that our structure is free enough: If we want to extend a partial automorphism defined on a finite set, there are only finitely many points, for which we have only finitely many options.

Definition 3.1. Let G be a closed subgroup of S_∞. We say that G *has the finite algebraic closure property* (*FACP*) if for every finite $S \subset \omega$ the set $\{b : |G_{(S)}(b)| < \infty\}$ is finite.

The following model theoretic property of Fraïssé classes turns out to be essentially a reformulation of the FACP for the automorphism groups of the limits.

Definition 3.2. Let $\mathcal{K}$ be a Fraïssé class. We say that $\mathcal{K}$ *has the cofinal strong amalgamation property* (*CSAP*) if there exists a subclass of $\mathcal{K}$ cofinal under embeddability, which satisfies the strong amalgamation property, or more formally: For every $\mathcal{B}_0 \in \mathcal{K}$ there exists a $\mathcal{B} \in \mathcal{K}$ and an embedding $\phi_0 : \mathcal{B}_0 \to \mathcal{B}$ such that the *strong amalgamation property holds over* $\mathcal{B}$; that is, for every pair of structures $\mathcal{C}, \mathcal{D} \in \mathcal{K}$ and embeddings $\psi : \mathcal{B} \to \mathcal{C}$ and $\chi : \mathcal{B} \to \mathcal{D}$ there exist $\mathcal{E} \in \mathcal{K}$ and embeddings $\psi' : \mathcal{C} \to \mathcal{E}$ and $\chi' : \mathcal{D} \to \mathcal{E}$ such that

$$\psi' \circ \psi = \chi' \circ \chi$$

and

$$\psi'(\mathcal{C}) \cap \chi'(\mathcal{D}) = (\psi' \circ \psi)(\mathcal{B}) = (\chi' \circ \chi)(\mathcal{B}).$$

A Fraïssé limit $\mathcal{A}$ is said to have the cofinal strong amalgamation property if $age(\mathcal{A})$ has the CSAP.

Generalizing the results of Dougherty and Mycielski, we show that the FACP is equivalent to some properties of the orbit structure of a random element of the group.

Theorem 3.3 (Theorem 4.14)**.** *Let $\mathcal{A}$ be a locally finite Fraïssé limit. Then the following are equivalent:*

(1) *Almost every element of* $\mathrm{Aut}(\mathcal{A})$ *has finitely many finite orbits.*
(2) $\mathrm{Aut}(\mathcal{A})$ *has the FACP.*
(3) $\mathcal{A}$ *has the CSAP.*

Moreover, any of the above conditions imply that almost every element of $\mathcal{A}$ has infinitely many infinite orbits.

Note that every relational structure, and also $\mathcal{B}_\infty$, is locally finite; moreover, it is well known that the ages of the structures $\mathcal{R}, (\mathbb{Q}, <)$ and $\mathcal{B}_\infty$ have the strong amalgamation property, which clearly implies the CSAP (it is also easy to directly check the FACP for these groups). Hence, we obtain the following corollary.

Corollary 3.4. *In* $\mathrm{Aut}(\mathcal{R}), \mathrm{Aut}(\mathbb{Q}, <)$*, and* $\mathrm{Aut}(\mathcal{B}_\infty)$ *almost every element has finitely many finite and infinitely many infinite orbits.*

As a corollary of our results, in Section 5 we show that a large number of groups can be partitioned in a Haar null and a meager set.

Corollary 3.5 (Corollary 5.1)**.** *Let G be a closed subgroup of S_∞ satisfying the FACP, and suppose that the set $F = \{g \in G : \mathrm{Fix}(g)$ is infinite$\}$ is dense in G. Then G can be decomposed into the union of an (even conjugacy invariant) Haar null and a meager set.*

Corollary 3.6 (Corollary 5.2)**.** $\mathrm{Aut}(\mathcal{R})$*,* $\mathrm{Aut}(\mathbb{Q}, <)$*, and* $\mathrm{Aut}(\mathcal{B}_\infty)$ *(and hence Homeo$(2^{\mathbb{N}})$) can be decomposed into the union of an (even conjugacy invariant) Haar null and a meager set.*

However, these results are typically far from the full description of the behavior of the random elements. We continue by summarizing our results from [5], [6] about two special cases, $\mathrm{Aut}(\mathbb{Q}, <)$ and $\mathrm{Aut}(\mathcal{R})$, where we gave a complete description of the Haar positive conjugacy classes.

3.1. Summary of the random behavior in $\mathrm{Aut}(\mathbb{Q}, <)$ and $\mathrm{Aut}(\mathcal{R})$.

In order to describe our results about $\mathrm{Aut}(\mathbb{Q}, <)$, we need the concept of orbitals (defined below; for more details on this topic see [10]). Let $p, q \in \mathbb{Q}$. The interval (p, q) will denote the set $\{r \in \mathbb{Q} : p < r < q\}$. For an automorphism $f \in \mathrm{Aut}(\mathbb{Q}, <)$ we denote the set of fixed points of f by $\mathrm{Fix}(f)$.

Definition 3.7. The set of *orbitals* of an automorphism $f \in \mathrm{Aut}(\mathbb{Q}, <)$, $\mathcal{O}^*_f$ consists of the convex hulls (relative to $\mathbb{Q}$) of the orbits of the rational numbers, that is,

$$\mathcal{O}^*_f = \{\mathrm{conv}(\{f^n(r) : n \in \mathbb{Z}\}) : r \in \mathbb{Q}\}.$$

It is easy to see that the orbitals of f form a partition of $\mathbb{Q}$, with the fixed points determining one element orbitals; hence, "being in the same orbital" is an equivalence relation. Using this fact, we define the relation $<$ on the set of orbitals by letting $O_1 < O_2$ for distinct $O_1, O_2 \in \mathcal{O}^*_f$ if $p_1 < p_2$ for some (and hence for all) $p_1 \in O_1$ and $p_2 \in O_2$. Note that $<$ is a linear order on the set of orbitals.

It is also easy to see that if $p, q \in \mathbb{Q}$ are in the same orbital of f, then $f(p) > p \Leftrightarrow f(q) > q$, $f(p) < p \Leftrightarrow f(q) < q$, and $f(p) = p \Leftrightarrow f(q) = q \Rightarrow p = q$. This observation makes it possible to define the *parity function*, $s_f : \mathcal{O}_f^* \to \{-1, 0, 1\}$. Let $s_f(O) = 0$ if O consists of a fixed point of f, let $s_f(O) = 1$ if $f(p) > p$ for some (and hence for all) $p \in O$, and let $s_f(O) = -1$ if $f(p) < p$ for some (and hence for all) $p \in O$.

Theorem 3.8 (See [6])**.** *For almost every element f of* $\mathrm{Aut}(\mathbb{Q}, <)$

(1) *for distinct orbitals $O_1, O_2 \in \mathcal{O}_f^*$ (see Definition* 3.7*) with $O_1 < O_2$ such that $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f^*$ with $O_1 < O_3 < O_2$ and $s_f(O_3) \neq s_f(O_1)$, and*

(2) (*following from Theorem* 4.14) *f has only finitely many fixed points.*

These properties characterize the non-Haar null conjugacy classes; i.e., a conjugacy class is non-Haar null if and only if one (*or equivalently each*) *of its elements has these properties.*

Moreover, every non-Haar null conjugacy class is compact biter and those non-Haar null classes in which the elements have no rational fixed points are compact catcher.

This yields the following surprising corollary (for details see [6]).

Corollary 3.9. *There are continuum many non-Haar null conjugacy classes in* $\mathrm{Aut}(\mathbb{Q}, <)$*, and their union is co-Haar null.*

Note that it was proved by Solecki [18] that in every nonlocally compact Polish group that admits a two-sided invariant metric there are continuum many pairwise disjoint non-Haar null Borel sets; thus, the above corollary is an extension of his results for $\mathrm{Aut}(\mathbb{Q}, <)$ (see also the case of $\mathrm{Aut}(\mathcal{R})$ below). We would like to point out that in a sharp contrast to this result, in $\mathrm{Homeo}^+([0, 1])$ (that is, in the group of order preserving homeomorphisms of the interval) the random behavior is quite different (see [7]), and more similar to the case of S_∞: There are only countably many non-Haar null conjugacy classes, and their union is co-Haar null.

The characterization of non-Haar null conjugacy classes of the automorphism group of the random graph appears to be similar to the characterization of the non-Haar null classes of $\mathrm{Aut}(\mathbb{Q}, <)$; however, their proofs are completely different.

Theorem 3.10 (See [5])**.** *For almost every element f of* $\mathrm{Aut}(\mathcal{R})$

(1) *for every pair of finite disjoint sets $A, B \subset V$ there exists a $v \in V$ such that $(\forall x \in A)(xRv)$ and $(\forall y \in B)(y \neg Rv)$ and $v \notin \mathcal{O}^f(A \cup B)$, i.e., the union of orbits of the elements of $A \cup B$, and*

(2) (*from Theorem* 4.14) *f has only finitely many finite orbits.*

These properties characterize the non-Haar null conjugacy classes, i.e., a conjugacy class is non-Haar null if and only if one (*or equivalently each*) *of its elements has these properties.*

Moreover, every non-Haar null conjugacy class is compact biter, and those non-Haar null classes in which the elements have no finite orbits are compact catcher.

It is not hard to see that this characterization again yields the following corollary (see [5]).

Corollary 3.11. *There are continuum many non-Haar null classes in* $\mathrm{Aut}(\mathcal{R})$*, and their union is co-Haar null.*

Table 1. Examples of various behaviors

	The union of the Haar null classes is Haar null		
	C	LC \ C	NLC
0	–	–	–
n	$\mathbb{Z}_n$	HNN	?
$\aleph_0$	?	$\mathbb{Z}$	S_∞
$\mathfrak{c}$	–	–	$\mathrm{Aut}(\mathbb{Q},<)$; $\mathrm{Aut}(\mathcal{R})$
	The union of the Haar null classes is not Haar null		
	C	LC \ C	NLC
0	2^ω	$\mathbb{Z}\times 2^\omega$	$\mathbb{Z}^\omega$
n	$\mathbb{Z}_n \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$	HNN $\times(\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$	$\mathbb{Z}_n \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Q}_d^\omega)$
$\aleph_0$	?	$\mathbb{Z} \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$	$S_\infty \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$
$\mathfrak{c}$	–	–	$\mathrm{Aut}(\mathbb{Q},<) \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$

3.2. **Various behaviors.** Examining any Polish group, we can ask the following questions.

Question 3.12.

(1) How many non-Haar null conjugacy classes are there?
(2) Is the union of the Haar null conjugacy classes Haar null?

Note that these are interesting even in compact groups. Table 1 summarizes our examples and the open questions as well (the left column indicates the number of non-Haar null conjugacy classes, while C, LC \ C, and NLC stand for compact, locally compact noncompact, and nonlocally compact groups, respectively). HNN denotes the well-known infinite group, constructed by Higmann, Neumann, and Neumann [11], with two conjugacy classes, while $\mathbb{Q}_d$ stands for the rationals with the discrete topology. The action ϕ of $\mathbb{Z}_2$ on $\mathbb{Z}_3^\omega$ and $\mathbb{Q}_d^\omega$ is the map defined by $a \mapsto -a$.

4. Main results

This section contains our generalization of the result of Dougherty and Mycielski to automorphism groups of countable structures. For the sake of simplicity we will use the following notation.

Definition 4.1. Let G be a closed subgroup of S_∞, and let $S \subset \omega$ be a finite subset. The *group theoretic algebraic closure* of S is

$$\mathrm{ACL}(S) = \{x \in \omega : \text{The orbit of } x \text{ under } G_{(S)} \text{ is finite}\}.$$

Obviously G has the finite algebraic closure property (see Definition 3.1) if and only if for every finite set S the set $\mathrm{ACL}(S)$ is finite. We start by proving a simple observation about the operator ACL.

Lemma 4.2. *If a group G has the FACP, then the corresponding operator* ACL *is idempotent.*

Proof. We have to show that for every finite set $S \subset \omega$ the identity $\mathrm{ACL}(\mathrm{ACL}(S)) = \mathrm{ACL}(S)$ holds. Let $S \subset \omega$ be an arbitrary finite set, and let $x \in \mathrm{ACL}(\mathrm{ACL}(S))$. We will show that x has a finite orbit under $G_{(S)}$, which implies that $x \in \mathrm{ACL}(S)$.

It is enough to show that $G_{(S)}(x)$ is finite. Enumerate the elements of $\mathrm{ACL}(S)$ as $\{x_1, x_2, \ldots, x_k\}$. The group $G_{(S)}$ acts on $\mathrm{ACL}(S)^k$ coordinatewise. Under this group action the stabilizer of the tuple $(x_1, x_2, \ldots, x_k)$ is $G_{(\mathrm{ACL}(S))}$. The orbit-stabilizer theorem states that for any group action the index of the stabilizer of an element in the whole group is the same as the cardinality of its orbit. This yields the result that the index $[G_{(S)} : G_{(\mathrm{ACL}(S))}]$ is the same as the cardinality of the orbit of $(x_1, x_2, \ldots, x_k)$. This orbit is finite because the whole space $\mathrm{ACL}(S)^k$ is finite. So $G_{(\mathrm{ACL}(S))}$ has finite index in $G_{(S)}$.

Let $g_1, g_2, \ldots, g_n \in G_{(S)}$ be a left transversal for $G_{(\mathrm{ACL}(S))}$ in $G_{(S)}$. Then $G_{(S)} = g_1 G_{\mathrm{ACL}(S)} \cup \cdots \cup g_n G_{\mathrm{ACL}(S)}$. Since $G_{(S)}(x) = g_1 G_{(\mathrm{ACL}(S))}(x) \cup g_2 G_{(\mathrm{ACL}(S))}(x) \cup \cdots \cup g_n G_{(\mathrm{ACL}(S))}(x)$ is a finite union of finite sets, it must be finite. ☐

Lemma 4.3. *The operator* ACL *is translation invariant in the following sense: If $S \subset \omega$ is a finite set and $g \in G$, then*

$$\mathrm{ACL}(gS) = g\,\mathrm{ACL}(S).$$

Proof. Let $x \in \omega$ be an arbitrary element. Then

$$\begin{aligned} &x \text{ and } y \text{ are in the same orbit under } G_{(S)} \Leftrightarrow \\ &\qquad \exists h \in G_{(S)} : h(y) = x \Leftrightarrow \exists h \in G_{(S)} : gh(y) = g(x) \Leftrightarrow \\ &\qquad \exists h \in G_{(S)} : ghg^{-1}(g(y)) = g(x) \Leftrightarrow \exists f \in G_{(gS)} : f(g(y)) = g(x) \Leftrightarrow \\ &\qquad\qquad g(x) \text{ and } g(y) \text{ are in the same orbit under } G_{(gS)}. \end{aligned}$$

So an element x has finite orbit under $G_{(S)}$ if and only if $g(x)$ has finite orbit under $G_{(gS)}$. ☐

Now we describe a process to generate a probability measure on G, a closed subgroup of S_∞ that has the FACP. This probability measure will witness that certain sets are Haar null (see Theorem 4.13).

Our random process will define a permutation $p \in G$ in stages. It depends on integer sequences $(M_i)_{i \in \omega}$ and $(N_i)_{i \in \omega}$ with $M_i, N_i \geq 1$.

We denote the partial permutation completed in stage i by p_i. We start with $p_0 = \mathrm{id}_{\mathrm{ACL}(\emptyset)}$ and maintain throughout the following property (i) for every $i \geq 1$, and properties (ii) and (iii) for $i \in \omega$:

(i) $p_{i-1} \subset p_i$,
(ii) $\mathrm{dom}(p_i)$ and $\mathrm{ran}(p_i)$ are finite sets such that $\mathrm{ACL}(\mathrm{dom}(p_i)) = \mathrm{dom}(p_i)$, $\mathrm{ACL}(\mathrm{ran}(p_i)) = \mathrm{ran}(p_i)$, and
(iii) there is a permutation $g \in G$ that extends p_i.

Let $O_0, O_1 \cdots \subset \omega$ be a sequence of infinite sets with the property that for every finite set $F \subset \omega$ and every infinite orbit O of $G_{(F)}$, the sequence $(O_i)_{i \in \omega}$ contains O infinitely many times. It is easy to see that such a sequence exists since there exist only countably many such finite sets F, and for each one there exist only countably many orbits of $G_{(F)}$.

At stage $i \geq 1$, we proceed the following way. First, suppose that i is even. We now choose a set $S_i \subset \omega$ with $|S_i| = M_i$ such that $S_i \cap \mathrm{ran}(p_{i-1}) = \emptyset$. If $i \equiv 0 \pmod 4$, we require that S_i contains the least M_i elements of $\omega \setminus \mathrm{ran}(p_{i-1})$, and if $i \equiv 2 \pmod 4$, we require S_i to contain the least M_i elements of $O_{(i-2)/4} \setminus \mathrm{ran}(p_{i-1})$. Now we will extend p_{i-1} to a partial permutation p_i such that

$$\mathrm{ran}(p_i) = \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup S_i). \tag{1}$$

Let us enumerate the elements of $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup S_i) \setminus \mathrm{ran}(p_{i-1})$ as $(x_1, \dots, x_j)$ such that if $x_1, \dots, x_{k-1}$ are already chosen, then we choose x_k such that

$$\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_k\}) \text{ is minimal with respect to inclusion.} \tag{2}$$

Claim 4.4. For every $1 \le k \le \ell \le m \le j$ if

$$x_m \in \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_k\}), \tag{3}$$

then

$${}_\ell \in \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{k-1}\} \cup \{x_m\}) \subset \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_k\}).$$

Thus, letting $\ell = k$ yields

$$\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{k-1}\} \cup \{x_m\}) = \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_k\}).$$

Proof. The last containment holds using Lemma 4.2 and (3). If $\ell = m$, then there is nothing to prove. Now suppose toward a contradiction that there exists an $\ell < m$ violating the statement of the claim, and suppose that ℓ is minimal with $k \le \ell < m$ and

$$x_\ell \notin \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{k-1}\} \cup \{x_m\}). \tag{4}$$

Using the minimality of ℓ, $\{x_1, \dots, x_{\ell-1}\} \subset \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{k-1}\} \cup \{x_m\})$, an application of Lemma 4.2 and the fact that $k \le \ell$ thus shows that $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{k-1}\} \cup \{x_m\}) = \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{\ell-1}\} \cup \{x_m\})$. By (4) it follows that $x_\ell \notin \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{\ell-1}\} \cup \{x_m\})$. Using this, the fact that $k \le \ell$, and (3), $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{\ell-1}\} \cup \{x_m\}) \subsetneq \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_\ell\})$, contradicting (2) since x_ℓ was chosen after $\{x_1, \dots, x_{\ell-1}\}$ to satisfy the condition that $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_\ell\})$ is minimal. ☐

We will determine the preimages of $(x_1, x_2, \dots, x_j)$ in this order. Denote the partial permutations defined in these substeps by $p_{i,k}$ so that $\mathrm{ran}(p_{i,k}) = \mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_k\}$ for $k = 0, \dots, j$. If the first k preimages are determined, then there are two possibilities for x_{k+1}:

(a) The set of possible preimages of x_{k+1} under $p_{i,k}$ is finite; that is, the set $\{g^{-1}(x_{k+1}) : g \in G, g \supset p_{i,k}\}$ is finite. Then choose one of them randomly with uniform distribution.

(b) The set of possible preimages of x_{k+1} under $p_{i,k}$ is infinite. Then choose one of the smallest N_i many possible values uniformly.

We note that the orbit of x_k under the stabilizer $G_{(\mathrm{ran}(p_{i-1}))}$ is infinite because $x_k \notin \mathrm{ran}(p_{i-1}) = \mathrm{ACL}(\mathrm{ran}(p_{i-1}))$ so

$$\text{possibility (b) must occur for at least } x_1. \tag{5}$$

Let $p_i = p_{i,j}$. Properties (i) and (iii) obviously hold for i. Let $g \in G$ be a permutation with $g \supset p_i$. Now $\mathrm{ran}(p_i) = \mathrm{ACL}(\mathrm{ran}(p_i))$ using (1) and Lemma 4.2. Then $\mathrm{dom}(p_i) = g^{-1}\,\mathrm{ran}(p_i)$; hence, using Lemma 4.3, $\mathrm{ACL}(\mathrm{dom}(p_i)) = \mathrm{ACL}(g^{-1}\,\mathrm{ran}(p_i)) = g^{-1}\,\mathrm{ACL}(\mathrm{ran}(p_i)) = g^{-1}\,\mathrm{ran}(p_i) = \mathrm{dom}(p_i)$, showing property (ii). This concludes the case in which i is even.

If i is odd, we let $S_i \subset \omega$ be the set of the least M_i elements of $\omega \setminus \mathrm{dom}(p_{i-1})$ if $i \equiv 1 \pmod 4$, and the least M_i elements of $O_{(i-3)/4} \setminus \mathrm{dom}(p_{i-1})$ if $i \equiv 3 \pmod 4$. We extend p_{i-1} to a partial permutation p_i such that

$$\mathrm{dom}(p_i) = \mathrm{ACL}(\mathrm{dom}(p_{i-1}) \cup S_i). \tag{6}$$

Again, we enumerate the elements of $\mathrm{ACL}(\mathrm{dom}(p_{i-1})\cup S_i)\setminus\mathrm{dom}(p_{i-1})$ as $(x_1,\dots,x_j)$ such that if $x_1,\dots,x_{k-1}$ are already chosen, then we choose x_k from the rest so that $\mathrm{ACL}(\mathrm{dom}(p_{i-1})\cup\{x_1,\dots,x_k\})$ is minimal with respect to inclusion. The proof of the following claim is analogous to the proof of Claim 4.4.

Claim 4.5. For every $1\le k\le \ell\le m\le j$, $x_m\in \mathrm{ACL}(\mathrm{dom}(p_{i-1})\cup\{x_1,\dots,x_k\})$ implies that $x_\ell\in\mathrm{ACL}(\mathrm{dom}(p_{i-1})\cup\{x_1,\dots,x_{k-1}\}\cup\{x_m\})\subset\mathrm{ACL}(\mathrm{dom}(p_{i-1})\cup\{x_1,\dots,x_k\})$. Thus, letting $\ell=k$ yields $\mathrm{ACL}(\mathrm{ran}(p_{i-1})\cup\{x_1,\dots,x_{k-1}\}\cup\{x_m\})=\mathrm{ACL}(\mathrm{ran}(p_{i-1})\cup\{x_1,\dots,x_k\})$.

We determine the images of $(x_1,x_2,\dots,x_j)$ in this order. Denote the partial permutations defined in these substeps by $p_{i,k}$ so that $\mathrm{dom}(p_{i,k})=\mathrm{dom}(p_{i-1})\cup\{x_1,\dots,x_k\}$ for $k=0,\dots,j$. If the first k images are determined, then there are two possibilities for x_{k+1}:

(a) The set of possible images of x_{k+1} under $p_{i,k}$ is finite; that is, the set $\{g(x_{k+1}) : g\in G, g\supset p_{i,k}\}$ is finite. Then choose one of them randomly with uniform distribution.
(b) The set of possible images of x_{k+1} under $p_{i,k}$ is infinite. Then choose one of the smallest N_i many possible values uniformly.

Again, the orbit of x_k under the stabilizer $G_{(\mathrm{dom}(p_{i-1}))}$ is infinite because $x_k\notin\mathrm{dom}(p_{i-1})=\mathrm{ACL}(\mathrm{dom}(p_{i-1}))$ for every k, so possibility (b) must occur for at least x_1.

Let $p_i=p_{i,j}$. Again, properties (i) and (iii) hold for i. Let $g\in G$ be a permutation with $g\supset p_i$. Now $\mathrm{dom}(p_i)=\mathrm{ACL}(\mathrm{dom}(p_i))$ using (6) and Lemma 4.2. Then using Lemma 4.3, $\mathrm{ACL}(\mathrm{ran}(p_i))=\mathrm{ACL}(g\,\mathrm{dom}(p_i))=g\,\mathrm{ACL}(\mathrm{dom}(p_i))=g\,\mathrm{dom}(p_i)=\mathrm{ran}(p_i)$, showing property (ii). This concludes the construction for odd i.

Now let $p=\bigcup_i p_i$. This makes sense using (i).

Claim 4.6. $p\in G$.

Proof. First, we show that $p\in S_\infty$. Using (iii), each p_i is a partial permutation, and hence injective. Using (i), p is the union of compatible injective functions; hence, p is an injective function. It is clear from the construction that $\{0,1,\dots,i-1\}\subset\mathrm{dom}(p_{4i})\cap\mathrm{ran}(p_{4i})$ for every i, and hence $p\in S_\infty$.

Using (iii), we can find an element $g_i\in G$ such that $g_i\supset p_i$. It is clear that $g_i\to p$, and since G is a closed subgroup of S_∞, we conclude that $p\in G$. ☐

The following lemma is crucial in proving that almost every element of G has finitely many finite and infinitely many infinite orbits.

Lemma 4.7. *Suppose that the parameters of the random process $M_1,\dots,M_i$ and $N_1,\dots,N_{i-1}$ are given along with the numbers $K\in\omega$ and $\varepsilon>0$. Then we can choose N_i so that for every set $S\subset\omega$ with $|S|=K$, the probability that $S\cap(\mathrm{dom}(p_i)\setminus\mathrm{dom}(p_{i-1}))\ne\emptyset$ if i is even, or that $S\cap(\mathrm{ran}(p_i)\setminus\mathrm{ran}(p_{i-1}))\ne\emptyset$ if i is odd, is at most ε.*

Proof. We suppose that i is even and prove the lemma only in this case. The proof for the case in which i is odd is analogous.

One can easily see using induction on i that if $M_1,\dots,M_{i-1}$ and $N_1,\dots,N_{i-1}$ are given, then the random process can yield only finitely many different p_{i-1} as a result.

Let p_{i-1} be one of the possible outcomes, and let $(x_1, x_2, \dots, x_j)$ denote the elements of $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup S_i) \setminus \mathrm{ran}(p_{i-1})$ enumerated in the same order as they appear during the construction. Note that this depends only on p_{i-1} and M_i. Let a_1 be the index for which $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1\}) = \mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{a_1}\}$, such an index exists using Claim 4.4. Hence, for every $m \leq a_1$, $x_m \in \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1\})$, and thus using Claim 4.4 again, it follows that

$$x_1 \in \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_m\}) \qquad \text{for every } 1 \leq m \leq a_1. \tag{7}$$

Claim 4.8. For every such m, there is a unique positive integer k_m such that if q is an extension of p_{i-1} with $\mathrm{ran}(q) = \mathrm{ran}(p_{i-1}) \cup \{x_m\}$ (such that $q \subset g$ for some $g \in G$), then $|\{g^{-1}(x_1) : g \in G, g \supset q\}| = k_m$.

Proof. Let $H = G_{(\mathrm{ran}(p_{i-1}) \cup x_m)}$, then

$$k = |\{g(x_1) : g \in H\}| = |\{g^{-1}(x_1) : g \in H\}| \tag{8}$$

is finite using (7) and the fact that H is a subgroup. It is enough to show that if q is an extension of p_{i-1} with $\mathrm{ran}(q) = \mathrm{ran}(p_{i-1}) \cup \{x_m\}$, then $|\{g^{-1}(x_1) : g \in G, g \supset q\}| = k$.

Let $g_1, \dots, g_k \in H$ with $g_\ell^{-1}(x_1) \neq g_n^{-1}(x_1)$ if $\ell \neq n$. If $h \in G$ is a permutation with $h \supset q$, then $g_n h \supset q$ for every $1 \leq n \leq k$. Then using the identity $(g_n h)^{-1}(x_1) = h^{-1}(g_n^{-1}(x_1))$, $(g_\ell h)^{-1}(x_1) \neq (g_n h)^{-1}(x_1)$ if $\ell \neq n$. This shows that $|\{g^{-1}(x_1) : g \in G, g \supset q\}| \geq k$.

To prove the other inequality, suppose toward a contradiction that there exist $g_1, \dots, g_{k+1}$ with $g_n \supset q$ for every $n \leq k+1$, and $g_\ell^{-1}(x_1) \neq g_n^{-1}(x_1)$ for every $\ell \neq n$. It is easy to see that $g_n g_1^{-1} \in H$ for every n, but the values $(g_n g_1^{-1})^{-1}(x_1) = g_1(g_n^{-1}(x_1))$ are pairwise distinct, contradicting (8). Thus, the proof of the claim is complete. ☐

Now let $k = \max\{k_2, k_3, \dots, k_{a_1}\}$ if $a_1 \geq 2$; otherwise let $k = 1$.

Claim 4.9. If $N_i > \frac{kKj}{\varepsilon}$, then for every fixed set $S \subset \omega$ with $|S| = K$ we have $\mathbb{P}(p_i^{-1}(x_m) \in S) < \frac{\varepsilon}{j}$ for every $1 \leq m \leq a_1$.

Proof. This is immediate for $m = 1$ since $k \geq 1$, and the preimage of x_1 is chosen from N_i many elements using (5). Now let $m > 1$, using Claim 4.8 and the fact that $k \geq k_m$, it follows that for every $y \in \omega$ $|\{g^{-1}(x_1) : g \in G, g \supset p_{i-1}, g(y) = x_m\}| \leq k$; hence, for the set $R = \{g^{-1}(x_1) : g \in G, g \supset p_{i-1}, g^{-1}(x_m) \in S\}$ $|R| \leq kK$. In order to be able to extend p_{i-1} to p_i with $p_i^{-1}(x_m) \in S$, we need to choose $p_i^{-1}(x_1)$ from R. Since during the construction of the random automorphism, $p_i^{-1}(x_1)$ is chosen uniformly from a set of size $N_i > \frac{kKj}{\varepsilon}$, we conclude that $\mathbb{P}(p_i^{-1}(x_m) \in S) \leq \mathbb{P}(p_i^{-1}(x_1) \in R) \leq \frac{|R|}{N_i} < \frac{\varepsilon}{j}$. ☐

For the rest of the proof we need to repeat the above argument until we reach j. If $a_1 < j$, let a_2 be the index satisfying $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{a_1+1}\}) = \mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{a_2}\}$; such an index exists using Claim 4.4 as before. Again, we can set a lower bound for N_i such that for every $a_1 < m \leq a_2$ $\mathbb{P}(p_i^{-1}(x_m) \in S) < \frac{\varepsilon}{j}$. Repeating the argument, we can choose N_i such that $\mathbb{P}(p_i^{-1}(x_m) \in S) < \frac{\varepsilon}{j}$ for every $1 \leq m \leq j$; thus, $\mathbb{P}(p_i^{-1}(\{x_1, \dots, x_j\}) \cap S \neq \emptyset) < \varepsilon$. This completes the proof of the lemma. ☐

Now we prove a proposition from which our main result will easily follow.

Proposition 4.10. *Let $G \leq S_\infty$ be a closed subgroup. If G has the FACP, then the sets*

$$\mathcal{F} = \{g \in G : g \text{ has finitely many finite orbits}\},$$
$$\mathcal{C} = \{g \in G : \forall F \subset \omega \text{ finite } \forall x \in \omega \ (\text{if } G_{(F)}(x) \text{ is infinite, then it is not covered by finitely many orbits of } g)\}$$

are co-Haar null.

The set $\mathcal{C}$ could seem unnatural for the first sight. However, from the above fact about the set $\mathcal{C}$ not only will our main theorem be deduced, but this fact also plays a crucial role in proving Theorem 3.10 (see [5]).

Proof. We first show the following lemma.

Lemma 4.11. *The sets $\mathcal{F}$ and $\mathcal{C}$ are conjugacy invariant Borel sets.*

Proof. The fact that $\mathcal{F}$ is conjugacy invariant follows from the fact that conjugation does not change the orbit structure of a permutation.

To show that $\mathcal{C}$ is conjugacy invariant, let $c \in \mathcal{C}$ and let $h \in G$, and we need to show that $h^{-1}ch \in \mathcal{C}$. Let $F \subset \omega$ be finite and let $x \in \omega$ so that $|G_{(F)}(x)| = \aleph_0$. Note that $G_{(h(F))}(h(x)) = hG_{(F)}h^{-1}(h(x)) = hG_{(F)}(x)$; hence, the first set is also infinite. By $c \in C$ there exists an infinite set $\{x_n : n \in \omega\} \subset G_{(h(F))}(h(x))$, so for $n \neq n'$ the points x_n and $x_{n'}$ are in different c orbits. But then the points $\{h^{-1}(x_n) : n \in \omega\} \subset G_{(F)}(x)$ are in pairwise distinct $h^{-1}ch$ orbits, as desired.

To show that $\mathcal{F}$ is Borel, notice that the set of permutations containing a given finite orbit is open for every finite orbit. Thus, for any finite set of finite orbits the set of permutations containing those finite orbits in their orbit decompositions is open: It can be obtained as the intersection of finitely many open sets. Thus, for every $n \in \omega$ the set of permutations containing at least n finite orbits is open: It can be obtained as the union of open sets (one open set for each possible set of n orbits). Thus, $S_\infty \setminus \mathcal{F}$ is G_δ: It is the intersection of the above open sets. Hence, $\mathcal{F}$ is Borel.

Now we show that $\mathcal{C}$ is also Borel. It is enough to show that if $H \subset \omega$ is arbitrary, then the set $H^* = \{g \in G : \text{Finitely many orbits of } g \text{ cannot cover } H\}$ is Borel since $\mathcal{C}$ can be written as the countable intersection of such sets. And H^* can be easily seen to be Borel for any H since its complement, $\{g \in G : \exists n \ \forall m \in H \ \exists k \ \exists i < n \ (g^k(i) = m)\}$, is $G_{\delta\sigma}$; hence, H^* is $F_{\sigma\delta}$. □

To prove the proposition, we use the above construction to generate a random permutation p. We set $M_i = 2^i$ for every $i \geq 1$, and we define $(N_i)_{i\geq 1}$ recursively. If $N_1, \dots, N_{i-1}$ are already defined, then, as before, the random process can yield only finitely many distinct p_{i-1}. Hence, there is a bound m_i depending only on $N_1, \dots, N_{i-1}$ such that $|\operatorname{dom}(p_i)| = |\operatorname{ran}(p_i)| \leq m_i$ since $|\operatorname{ran}(p_i)| = |\operatorname{ACL}(\operatorname{ran}(p_{i-1}) \cup S_i)|$ if i is even, and $|\operatorname{dom}(p_i)| = |\operatorname{ACL}(\operatorname{dom}(p_{i-1}) \cup S_i)|$ if i is odd, which is independent of N_i. Now we use Lemma 4.7 to choose N_i so that the conclusion of the lemma is true with $K = m_i$ and $\varepsilon = \frac{1}{2^i}$.

Using Lemma 2.4 and the fact that the sets $\mathcal{F}$ and $\mathcal{C}$ are conjugacy invariant, it is enough to show that

$$\mathbb{P}(ph \text{ has finitely many finite orbits}) = 1 \tag{9}$$

and

$$\mathbb{P}(\text{finitely many orbits of } ph \text{ do not cover } O) = 1 \tag{10}$$

for every $h \in G$, every finite $F \subset \omega$, and every infinite orbit O of $G_{(F)}$ since there exist only countably many such orbits. So let us fix $h \in G$ and an infinite orbit $O \subset \omega$ of $G_{(F)}$ for some finite $F \subset \omega$ for the rest of the proof.

For a partial permutation q, a *partial path* in q, is a sequence $(y, q(y), \dots, q^n(y))$ with $n \geq 1$, $q^n(y) \notin \operatorname{dom}(q)$, and $y \notin \operatorname{ran}(q)$. Note that $p_i h$ is considered a partial permutation with $\operatorname{dom}(p_i h) = h^{-1}(\operatorname{dom}(p_i))$ and $\operatorname{ran}(p_i h) = \operatorname{ran}(p_i)$.

During the construction of the random permutation an *event* occurs when the partial permutation is extended to a new element at some stage, regardless of whether it happens for possibility (a) or (b). Suppose that during an event, the partial permutation p' is extended to $p'' = p' \cup (x, y)$. We call this event *bad* if the number of partial paths decreases or $h^{-1}(x) = y$. Note that an event is bad if the extension connects two partial paths of $p'h$ or if it completes an orbit (possibly a fixed point).

Claim 4.12. Almost surely, only finitely many bad events happen.

Proof. Let i be fixed, and suppose first that it is even. It is easy to see that a bad event can happen at stage i only if a preimage is chosen from $h(\operatorname{ran}(p_{i-1}))$, which includes the case in which a fixed point is constructed. Note that $|\operatorname{ran}(p_i)| \leq m_i$; thus, the probability of choosing a preimage from this set is at most $\frac{1}{2^i}$ using Lemma 4.7.

We proceed similarly if i is odd. Then to connect partial paths or complete orbits, an image has to be chosen from the set $h^{-1}(\operatorname{dom}(p_{i-1}))$. Since $|\operatorname{dom}(p_i)| \leq m_i$, the probability of choosing from this set is at most $\frac{1}{2^i}$.

Using the Borel–Cantelli lemma, the number of i such that a bad event happens at stage i is almost surely finite. The fact that only a finite number of bad events can happen at a particular stage completes the proof of the claim. □

Since a finite orbit can be created only during a bad event, (9) follows immediately from the claim. Thus, $\mathcal{F}$ is co-Haar null.

Now we prove that $\mathcal{C}$ is also co-Haar null by showing (10). Let $n_0, n_1 \dots \in \omega$ be a sequence with $n_0 < n_1 < \cdots$ and $O_{n_i} = O$ for every $i \in \omega$. Let c_i be the number of partial paths of $p_{4n_i+2}h$ intersecting O. It is enough to show that the sequence $(c_i)_{i\in\omega}$ is unbounded almost surely since, using Claim 4.12, only finitely many of such partial paths can be connected in later stages; hence, infinitely many orbits of ph will intersect O, almost surely.

At stage $4n_i + 2$, p_{4n_i+1} is extended to p_{4n_i+2} with $\operatorname{ran}(p_{4n_i+2}) \setminus \operatorname{ran}(p_{4n_i+1}) \supset S_{4n_i+2}$, $|S_{4n_i+2}| = M_{4n_i+2} = 2^{4n_i+2}$ and $S_{4n_i+2} \subset O_{(4n_i+2-2)/4} = O_{n_i} = O$. Hence, it is enough to prove that apart from a finite number of exceptions, the elements of $\operatorname{ran}(p_{4n_i+2}) \setminus \operatorname{ran}(p_{4n_i+1})$ are in different partial paths in $p_{4n_i+2}h$, almost surely.

The proof of this fact is similar to the proof of Claim 4.12. An element $y \in O \cap (\operatorname{ran}(p_{4n_i+2}) \setminus \operatorname{ran}(p_{4n_i+1}))$ can be contained in a completed orbit (of $p_{4n_i+2}h$) only if $h^{-1}p^{-1}_{4n_i+2}(y) \in \operatorname{ran}(p_{4n_i+2})$, and hence $p^{-1}_{4n_i+2}(y) \in h(\operatorname{ran}(p_{4n_i+2}))$. Similarly, if $y, y' \in O \cap (\operatorname{ran}(p_{4n_i+2}) \setminus \operatorname{ran}(p_{4n_i+1}))$ are in the same partial path (in $p_{4n_i+2}h$) such that y is not the first element of this path, then $p^{-1}_{4n_i+2}(y) \in h(\operatorname{ran}(p_{4n_i+2}))$. Again using Lemma 4.7, the probability of this happening at stage $4n_i + 2$ is at most $\frac{1}{2^{4n_i+2}}$ since $|\operatorname{ran}(p_{4n_i+2})| \leq m_{4n_i+2}$. As before, the application of the Borel–Cantelli lemma completes the proof of (10). And thus the proof of the proposition is also complete. □

Theorem 4.13. *Let $G \leq S_\infty$ be a closed subgroup. If G has the FACP, then the sets*

$$\mathcal{F} = \{g \in G : g \text{ has finitely many finite orbits}\},$$
$$\mathcal{I} = \{g \in G : g \text{ has infinitely many infinite orbits}\}$$

are both co-Haar null. Moreover, if $\mathcal{F}$ is co-Haar null, then G has the FACP.

Proof. The fact that $\mathcal{F}$ is co-Haar null follows immediately from Proposition 4.10. Let $\mathcal{C}$ denote the set as in Proposition 4.10. If $g \in \mathcal{C}$, then g contains infinitely many orbits since otherwise finitely many orbits of g could cover ω, and hence every infinite orbit of $G_{(F)}$ for some finite $F \subset \omega$. It follows that the co-Haar null set $\mathcal{C} \cap \mathcal{F}$ is contained in $\mathcal{I}$, and hence $\mathcal{I}$ is also co-Haar null. And thus the proof of the first part of the theorem is complete.

Now we prove the second assertion. We have to show that if G does not have the FACP, then $\mathcal{F}$ is not co-Haar null. If G does not have the FACP, then there is a finite set $S \subset \omega$ such that $\mathrm{ACL}(S)$ is infinite. This means that all of the permutations in $G_{(S)}$ have infinitely many finite orbits, and hence $G_{(S)} \cap \mathcal{F} = \emptyset$. The stabilizer $G_{(S)}$ is a nonempty open set; thus, it cannot be Haar null. Therefore, the proof of the theorem is complete. □

Now we are ready to prove the main result of this section.

Theorem 4.14. *Let $\mathcal{A}$ be a locally finite Fraïssé limit. Then the following are equivalent:*

(1) *Almost every element of* $\mathrm{Aut}(\mathcal{A})$ *has finitely many finite orbits.*
(2) $\mathrm{Aut}(\mathcal{A})$ *has the FACP.*
(3) *$\mathcal{A}$ has the CSAP.*

Moreover, each of the above conditions implies that almost every element of $\mathcal{A}$ has infinitely many infinite orbits.

Proof. The equivalence (1) $\iff$ (2), and the last statement of the theorem is just the application of Theorem 4.13 to $G = \mathrm{Aut}(\mathcal{A})$. Thus, it is enough to show that (2) $\iff$ (3).

Let $\mathcal{K} = age(\mathcal{A})$. Since $\mathcal{A}$ is the limit of $\mathcal{K}$, using the condition that $\mathcal{A}$ is ultrahomogeneous, it follows that $\mathcal{K}$ has the extension property; that is, for every $\mathcal{B}, \mathcal{C} \in \mathcal{K}$ and for embeddings $\phi : \mathcal{B} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{A}$ there exists an embedding $\psi' : \mathcal{C} \to \mathcal{A}$ with $\psi' \circ \phi = \psi$. Thus, the embeddings between the structures in $\mathcal{K}$ can be considered partial automorphisms of $\mathcal{A}$.

(2) $\Rightarrow$ (3) Take an arbitrary $\mathcal{B}_0 \in \mathcal{K}$, and fix an isomorphic copy of it inside $\mathcal{A}$. Let $\mathcal{B} = \mathrm{ACL}(\mathrm{dom}(\mathcal{B}_0))$, and note that, by the fact that $\mathrm{Aut}(\mathcal{A})$ has the FACP, $\mathcal{B}$ is a finite substructure of $\mathcal{A}$. We will show that over $\mathcal{B}$ the strong amalgamation property holds (see Definition 3.2). In order to see this, let $\mathcal{C}, \mathcal{D} \in \mathcal{K}$, and let $\psi : \mathcal{B} \to \mathcal{C}$ and $\phi : \mathcal{B} \to \mathcal{D}$ be embeddings. By the extension property we can suppose that $\mathcal{B} < \mathcal{C} < \mathcal{A}$, $\mathcal{B} < \mathcal{D} < \mathcal{A}$, and $\psi = \phi = \mathrm{id}_{\mathcal{B}}$. By Lemma 4.2 $\mathrm{ACL}(\mathrm{dom}(\mathcal{B})) = \mathcal{B}$, and hence the $\mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}$ orbit of every point in $\mathrm{dom}(\mathcal{C}) \setminus \mathrm{dom}(\mathcal{B})$, is infinite. By Neumann's lemma [12, Corollary 4.2.2] $\mathrm{dom}(\mathcal{C}) \setminus \mathrm{dom}(\mathcal{B})$ has infinitely many pairwise disjoint copies under the action of $\mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}$. In particular, by the

pigeonhole principle there exists an $f \in \mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}$ such that $f(\mathcal{C}) \cap \mathcal{D} = \mathcal{B}$. Letting $\mathcal{E}$ be the substructure of $\mathcal{A}$ generated by $\mathrm{dom}(f(\mathcal{C})) \cup \mathrm{dom}(\mathcal{D})$, $\psi' = f|_{\mathcal{C}}$, and $\phi' = \mathrm{id}_{\mathcal{D}}$ shows that the SAP holds over $\mathcal{B}$, and hence the CSAP holds as well.

(2) $\Leftarrow$ (3) Let $S \subset \mathrm{dom}(\mathcal{A})$ be finite. Let $\mathcal{B}_0$ be the substructure generated by S. Clearly, $\mathcal{B}_0 \in \mathcal{K}$, and hence there exists a $\mathcal{B} \in \mathcal{K}$ over which the strong amalgamation property holds and which contains an isomorphic copy of $\mathcal{B}_0$. By the extension property of $\mathcal{A}$ we can suppose that $\mathcal{B}$ and all of the structures constructed later on in this part of the proof are substructures of $\mathcal{A}$ containing $\mathcal{B}_0$.

We claim that for every $b \in \mathrm{dom}(\mathcal{A}) \setminus \mathrm{dom}(\mathcal{B})$ the orbit $\mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}(b)$ is infinite. Indeed, let $\mathcal{C}$ be the substructure generated by $\mathrm{dom}(\mathcal{B}) \cup \{b\}$. Using the strong amalgamation property repeatedly, first for $\mathcal{B}, \mathcal{C}$, and $\mathcal{D} = \mathcal{C}$ obtaining an $\mathcal{E}_1$, then for $\mathcal{B}, \mathcal{C}$, and $\mathcal{D} = \mathcal{E}_1$ obtaining an $\mathcal{E}_2$, etc., for every n we can find a substructure $\mathcal{E}_n$ of $\mathcal{A}$ which contains $n+1$ isomorphic copies of $\mathcal{C}$ which intersect only in $\mathcal{B}$, and the isomorphisms between these copies fix $\mathcal{B}$. Extending the isomorphisms to automorphisms of $\mathrm{Aut}(\mathcal{A})$ shows that the orbit $\mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}(b)$ is infinite. □

Remark 4.15. It is not hard to construct countable Fraïssé classes to show that the CSAP is equivalent neither to the SAP nor to the AP. An example showing that the CSAP $\not\Rightarrow$ SAP is $age(\mathcal{B}_\infty)$. Indeed, using a result of Schmerl [17] that states that a Fraïssé class has the SAP if and only if its automorphism group has no algebraicity (that is, $\mathrm{ACL}(F) = F$ for every finite F), and $age(\mathcal{B}_\infty)$ cannot have the SAP.

To see that the AP $\not\Rightarrow$ CSAP, let $\mathcal{Z}$ be the structure on the set $\mathbb{Z}$ of integers with a relations R_n for each $n \geq 1$, $n \in \mathbb{N}$ satisfying the condition that $aR_nb \Leftrightarrow |a-b| = n$ for each $a, b \in \mathbb{Z}$ and $n \geq 1$. It can be easily checked that $age(\mathcal{Z})$ satisfies the AP, but $\mathrm{Aut}(\mathcal{Z})$ does not satisfy the FACP since the algebraic closure of any two points is $\mathbb{Z}$. Thus, Theorem 4.14 implies that $\mathcal{Z}$ cannot satisfy the CSAP.

5. An application to decompositions

In this section we present an application of our results: We use Theorem 4.13 to show that a large family of automorphism groups of countable structures can be decomposed into the union of a Haar null and a meager set.

Corollary 5.1. *Let G be a closed subgroup of S_∞ satisfying the FACP, and suppose that the set $F = \{g \in G : \mathrm{Fix}(g)$ is infinite$\}$ is dense in G. Then G can be decomposed into the union of an (even conjugacy invariant) Haar null and a meager set.*

Proof. Clearly, F is conjugacy invariant, and since it can be written as $F = \{g \in G : \forall n \in \omega\ \exists m > n\ \ (g(m) = m)\}$, F is G_δ. Using the assumptions of this corollary, it is dense G_δ, and hence comeager. Using Theorem 4.13, it is Haar null; hence, $F \cup (G \setminus F)$ is an appropriate decomposition of G. □

Corollary 5.2. $\mathrm{Aut}(\mathcal{R})$, $\mathrm{Aut}(\mathbb{Q}, <)$, *and* $\mathrm{Aut}(\mathcal{B}_\infty)$ *can be decomposed into the union of an (even conjugacy invariant) Haar null and a meager set.*

Proof. In order to show that the set of elements in these groups with infinitely many fixed points is dense, in each case it is enough to show that if p is a finite, partial automorphism, then there is another partial automorphism p' extending p such that $p' \supset p \cup (x, x)$ with $x \notin \mathrm{dom}(p)$.

For $\mathrm{Aut}(\mathbb{Q}, <)$, let x be greater than each element in $\mathrm{dom}(p) \cup \mathrm{ran}(p)$. Then it is easy to see that $p \cup (x, x)$ is also a partial automorphism.

For $\mathrm{Aut}(\mathcal{R})$ let x be an element different from each $\mathrm{dom}(p) \cup \mathrm{ran}(p)$ with the property that x is connected to every vertex in $\mathrm{dom}(p) \cup \mathrm{ran}(p)$. Then it is easy to see that $p \cup (x, x)$ is a partial automorphism.

For $\mathrm{Aut}(\mathcal{B}_\infty)$ let $a_0 \cup a_1 \cup \cdots \cup a_{n-1}$ be a partition of $\mathbf{1}$ with the property that $\mathrm{dom}(p) \cup \mathrm{ran}(p)$ is a subset of the algebra generated by $A = \{a_0, a_1, \ldots, a_{n-1}\}$. Then there is a permutation π of $\{0, 1, \ldots, n-1\}$ compatible with p; that is, $p(a_i) = a_{\pi(i)}$ for every i. Let us write each a_i as a disjoint union $a_i = a_i' \cup a_i''$ of nonzero elements. Again, a partial permutation can be described by a permutation of the elements $\{a_1', \ldots, a_n'\} \cup \{a_1'', \ldots, a_n''\}$. Hence, let p' be defined by $p'(a_i') = a'_{\pi(i)}$, $p'(a_i'') = a''_{\pi(i)}$. Then p' is a partial automorphism extending p with a new fixed point $\bigcup_{i<n} a_i'$. □

6. Various behaviors

It turns out that in natural Polish groups we may encounter very different behaviors of conjugacy classes with respect to the ideal of Haar null sets (see [7], [5], [6]). In this section we address the questions from 3.12—namely, given a Polish group, how many non-Haar null conjugacy classes are there?—and we decide whether the union of the Haar null classes is Haar null. Note that these questions make perfect sense in the locally compact case as well. In this section we construct a couple of examples.

If $(A, +)$ is an abelian group, we will denote by ϕ the automorphism of A defined by $a \mapsto -a$.

Proposition 6.1. *Let $(A, +)$ be an abelian Polish group such that for every $a \in A$ there exists an element b with $2b = a$. Observe that $\phi \in \mathrm{Aut}(A)$, $\phi^2 = id_A$, and $(\mathbb{Z}_2 \ltimes_\phi A, \cdot)$ can be partitioned into $\{0\} \times A$ and $\{1\} \times A$. Moreover, in the group $\mathbb{Z}_2 \ltimes_\phi A$ the conjugacy class of every element of $\{0\} \times A$ is of cardinality at most 2, whereas the set $\{1\} \times A$ is a single conjugacy class.*

Proof. Let $(0, a) \in \{0\} \times A$ and $(i, b) \in \mathbb{Z}_2 \ltimes_\phi A$ be arbitrary. We claim that the conjugacy class of $(0, a)$ is $\{(0, a), (0, -a)\}$. If $i = 0$, then $(0, a)$ and (i, b) commute, so let $i = 1$. By definition

$$(1, b)^{-1} \cdot (0, a) \cdot (1, b) = (1, b) \cdot (1, b + a) = (0, b - (b + a)) = (0, -a),$$

which shows our claim.

Now let $(1, a), (1, a') \in \mathbb{Z}_2 \ltimes_\phi A$ be arbitrary. Now for an arbitrary element $(1, b)$ we get

$$(1, b)^{-1} \cdot (1, a) \cdot (1, b) = (1, b) \cdot (0, -b + a) = (1, b - (-b + a)) = (1, 2b - a),$$

thus, choosing b so that $2b = a' + a$, we obtain

$$(1, b)^{-1} \cdot (1, a) \cdot (1, b) = (1, a'). \qquad \square$$

Corollary 6.2. *Let $A = \mathbb{Z}_3^\omega$ or $A = (\mathbb{Q}_d)^\omega$ (that is, the countable infinite power of the rational numbers with the discrete topology). Then $\mathbb{Z}_2 \ltimes_\phi A$ has a nonempty clopen conjugacy class, namely $\{(1, a) : a \in A\}$ and every other conjugacy class has cardinality at most 2. Hence, the union of the Haar null classes $\{(0, a) : a \in A\}$ is also nonempty clopen.*

Lemma 6.3. *Suppose that G_1 and G_2 are Polish groups, that $A_1 \subset G_1$ is Borel, and that $U \subset G_2$ is nonempty and open. Then $A_1 \times U$ is Haar null in $G_1 \times G_2$ if and only if A_1 is Haar null.*

Proof. Suppose first that A_1 is Haar null witnessed by a measure μ_1. Then, if μ' is the same measure copied to $G_1 \times \{1\}$, it is easy to see that μ' witnesses the Haar nullness of $A_1 \times G_2$, in particular, the Haar nullness of $A_1 \times U$.

Conversely, suppose that $A_1 \times U$ is Haar null witnessing the measure μ. Clearly, as countably many translates of U cover G_2, countably many translates of $A_1 \times U$ cover $A_1 \times G_2$; hence, $A_1 \times G_2$ is Haar null as well, and this is also witnessed with the measure μ. Let $\mu_1 = \mathrm{proj}_{G_1 *} \mu$. Then μ_1 witnesses the Haar nullness of A_1. □

Proposition 6.4. *If G is a Polish group with κ many non-Haar null conjugacy classes, then $G \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$ has κ many non-Haar null conjugacy classes and the union of the Haar null conjugacy classes is not Haar null.*

Proof. Clearly, the conjugacy classes of $G \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$ are of the form $C_1 \times C_2$, where C_1 is a conjugacy class in G and C_2 is a conjugacy class in $\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega$. By Corollary 6.2 we have every conjugacy class in the latter group being finite, with one exception—this exceptional conjugacy class is clopen; let us denote it by U. Now, since the finite sets are Haar null in $\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega$ by Lemma 6.3, the set of non-Haar null conjugacy classes in $G \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$ is equal to $\{C \times U : C \text{ is a non-Haar null conjugacy class in } G\}$; hence, the cardinality of the non-Haar null classes is κ. Moreover, the union of the Haar null conjugacy classes contains $G \times ((\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega) \setminus U)$, which is nonempty and open. Consequently, it is not Haar null. □

Finally, we would like to recall the following well-known theorem.

Theorem 6.5 (HNN extension [11])**.** *There exists a countably infinite group with two conjugacy classes.*

We denote such a group by HNN and consider it a discrete Polish group.

Combining Proposition 6.4, Corollaries 3.9, 3.11, and 6.2, Lemma 6.3, and Theorems 1.3 and 6.5, we obtain Table 1 (see the end of Section 3). (Recall that C, LC \ C, and NLC stand for compact, locally compact noncompact, and nonlocally compact, respectively.)

7. Open problems

We finish with a couple of open questions. In Section 6 we produced several groups with various numbers of non-Haar null conjugacy classes. However, our examples are somewhat artificial.

Question 7.1. Are there natural examples of automorphism groups with given cardinality of non-Haar null conjugacy classes?

The following question is maybe the most interesting one from the set theoretic viewpoint.

Question 7.2. Suppose that a Polish group has uncountably many non-Haar null conjugacy classes. Does it have continuum many non-Haar null conjugacy classes?

The answer is of course affirmative under, e.g., the continuum hypothesis. Since the definition of Haar null sets is complicated (the collection of non-Haar null closed sets can already be $\mathbf{\Sigma}_1^1$-hard and $\mathbf{\Pi}_1^1$-hard [18]), it is unlikely that this question can be answered with an absoluteness argument.

The characterization result of Section 4 and the similarity between Theorems 3.8 and 3.10 suggest that a general theory of the behavior of the random automorphism (similar to the one built by Truss, Kechris, and Rosendal) could exist.

Problem 7.3. Formulate necessary and sufficient model theoretic conditions which characterize the measure theoretic behavior of the conjugacy classes.

In particular, it would be very interesting to find a unified proof of the description of the non-Haar null classes of $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Aut}(\mathcal{R})$.

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