TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 372, Number 1, 1 July 2019, Pages 97–117
https://doi.org/10.1090/tran/7558

THE NORM PRINCIPLE FOR TYPE D_n GROUPS OVER COMPLETE DISCRETELY VALUED FIELDS

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Abstract. Let K be a complete discretely valued field with residue field k with $\mathrm{char}(k) \neq 2$. Assuming that the norm principle holds for extended Clifford groups $\Omega(q)$ for every even dimensional nondegenerate quadratic form q defined over any finite extension of k, we show that it holds for extended Clifford groups $\Omega(Q)$ for every even dimensional nondegenerate quadratic form Q defined over K.

1. Introduction

Let K be a field and T a commutative linear algebraic group defined over K. Given L/K, a finite separable field extension, one can define the *norm homomorphism* as $\mathrm{N}_{L/K} : T(L) \to T(K)$ which sends $t \rightsquigarrow \prod_\gamma \gamma(t)$, where γ runs over cosets of $\mathrm{Gal}\,(K^{sep}/L)$ in $\mathrm{Gal}\,(K^{sep}/K)$. The definition of the norm homomorphism can be extended to K-étale algebras in a similar manner. Note that if $T = \mathbb{G}_m$, then $\mathrm{N}_{L/K} : T(L) \to T(K)$ is precisely the usual norm $\mathrm{N}_{L/K} : L^* \to K^*$.

Now let G be a linear algebraic group defined over K, and let $f : G \to T$ be an algebraic group homomorphism defined over K. Consider the following diagram:

$$\begin{tikzcd}
G(L) \arrow[r, "f(L)"] & T(L) \arrow[d, "\mathrm{N}_{L/K}"] \\
G(K) \arrow[r, "f(K)"] & T(K)
\end{tikzcd}$$

We say that the *norm principle* holds for $f : G \to T$ over a finite separable field extension (or étale algebra) L/K if $\mathrm{N}_{L/K}(\mathrm{Im}\, f(L)) \subseteq \mathrm{Im}\, f(K)$. We say that the norm principle holds for $f : G \to T$ if for every finite separable field extension (equivalently for every étale algebra) L/K , $\mathrm{N}_{L/K}(\mathrm{Im}\, f(L)) \subseteq \mathrm{Im}\, f(K)$. Note that if G is also commutative, the norm principle clearly holds over all L/K because the norm homomorphism for G makes the above diagram commutative.

Suppose further that the commutator subgroup G' of G is defined over K, which for example is the case if G is connected ([Bo12], Chapter 1, Section 2.3). Then every homomorphism $f : G \to T$ factors through the natural homomorphism $\tilde{f} : G \to G/G'$, and it is an easy check that the norm principle for $\tilde{f}$ (over L/K) implies the *norm principle* for f (over L/K). We say that the norm principle holds for G (over L/K) if it holds for $\tilde{f}$ (over L/K).

Received by the editors October 11, 2017, and, in revised form, January 29, 2018.

2010 *Mathematics Subject Classification.* Primary 20G15; Secondary 11E72.

The second author was partially supported by the Canada Research Chairs Program and an NSERC research grant.

The work of the third author was supported by the NSF grant DMS #1160206.

Let Q be a quadratic form over K. The classical norm principle of Scharlau which asserts that norms of similarity factors of Q_L are themselves similarity factors of Q ([Lam05], Thm. 4.3) can be restated in this context to say that the norm principle holds for the multiplier map $\mathrm{M} : \mathrm{GO}(Q) \to \mathbb{G}_m$ where $\mathrm{GO}(Q)$ is the group of similitudes of Q ([KMRT98], Chapter 3, 12.A, p. 153; [Me96], section 0). Similarly Knebusch's norm principle, which states that norms of spinor norms of Q_L are spinor norms of Q ([Lam05], Remark 5.13), can be reformulated as the norm principle holding for the spinor norm map $\underline{\mu} : \Gamma^+(Q) \to \mathbb{G}_m$, where $\Gamma^+(Q)$ is the even Clifford group of Q ([Sch85], Ch. 9, Sec. 3, p. 385; [Me96], Section 0).

Norm principles have been previously studied in ([Gi93], [Me96]), especially in conjunction with the rationality or the R-triviality of the algebraic group in question. In [BM00], it was established that the norm principle holds in general for all reductive groups of classical type without D_n components. The D_n case was investigated in [Bh16], and a scalar obstruction defined up to spinor norms, whose vanishing would imply the norm principle, was given. However, the triviality of this scalar obstruction is far from clear and the question whether the norm principle holds for reductive groups with type D_n components still remains open.

If K is a number field, the norm principle was proved in full generality for all reductive groups by P. Gille ([Gi97]), so the first widely open and very interesting case is when K is the function field $k(C)$ of a curve C defined over a number field k and the group G in question is of classical type with the semisimple part $G' = \mathrm{Spin}(Q)$. As we show in the last section of the paper, the validity of the norm principle over K is closely related to the triviality of the kernel of the natural map $\mathrm{H}^1(K, G') \to \prod_v \mathrm{H}^1(K_v, G')$, where v runs through a set of discrete valuations of K. Therefore the (traditional) local-global approach leads us first to look in detail over completions K_v.

With this motivation in mind, in this paper, we investigate the D_n case over an arbitrary complete discretely valued field K with residue field k and $\mathrm{char}(k) \neq 2$, restricting ourselves to type D_n groups arising from quadratic forms. In the main result of the paper, we show that if the norm principle holds for such groups defined over all finite extensions of the residue field k, then it holds for such groups defined over K (cf. Theorem 5.1). This yields examples of complete discretely valued fields with residue fields of virtual cohomological dimension ≤ 2 over which the norm principle holds for the groups under consideration (cf. Corollary 6.3). As a further application, we also relate the possible failure of the norm principle to the nontriviality of certain Tate–Shafarevich sets (section 7).

Notations and definitions. Let K be a field of characteristic not 2 and (V, Q), a nondegenerate quadratic space of dimension $2n$ over K, where $n \in \mathbb{Z}_{>0}$. Let $a := (-1)^{\frac{(2n)(2n-1)}{2}} \det(Q)$ be the signed determinant of Q defined up to squares in K^*, and let Z denote the discriminant extension of Q which is given by $K[t]/(t^2 - a)$. Thus Z is a quadratic étale extension of K, and we let ψ denote the nontrivial K-automorphism of Z.

We would now like to recall the definition of $\Omega(Q)$, the extended Clifford group of Q, which is one of the main objects of study in this paper. To do so, we first recall some facts pertaining to certain related groups.

Let $\mathrm{PGO}^+(Q)$ denote the projective group of proper similitudes of Q and $(\mathrm{C}_0(Q), \underline{\sigma})$, the even Clifford algebra of Q along with its canonical involution.

By [KMRT98] (Thm. 8.10, Prop. 8.12, pp. 94-95), $(\mathrm{C}_0(Q), \underline{\sigma})$ is a central simple algebra over Z with orthogonal involution. Let $\mathrm{Sim}\,(\mathrm{C}_0(Q), \underline{\sigma}) = \{x \in \mathrm{C}_0(Q) | x\,\underline{\sigma}(x) \in Z^*\}$ denote its group of similitudes. It surjects onto the automorphism group of the even Clifford algebra, $\mathrm{Aut}_Z(\mathrm{C}_0(Q), \underline{\sigma})$, via the map induced by inner conjugation ([KMRT98], Thm. 12.15, p. 158) inducing the exact sequence:

$$1 \to R_{Z/K}\mathbb{G}_m \to \mathrm{Sim}\,(\mathrm{C}_0(Q), \underline{\sigma}) \xrightarrow{\mathrm{Int}} \mathrm{Aut}_Z(\mathrm{C}_0(Q), \underline{\sigma}) \to 0.$$

Recall the morphism $C : \mathrm{PGO}^+(Q) \to \mathrm{Aut}_Z\,(\mathrm{C}_0(Q), \underline{\sigma})$ ([KMRT98], Prop. 13.2 and 13.4, pp. 173–174). This is, in fact, injective if $\dim Q = 2n \geq 4$. Then, $\Omega(Q)$, the extended Clifford group[1] of Q, is defined to be the inverse image of the image of $\mathrm{PGO}^+(Q)$ in $\mathrm{Sim}\,(\mathrm{C}_0(Q), \underline{\sigma})$. More precisely,

$$\Omega(Q) := \{c \in \mathrm{Sim}\,(\mathrm{C}_0(Q), \underline{\sigma}) \ \mid \mathrm{Int}(c) \in C\left(\mathrm{PGO}^+(Q)\right)\}.$$

Thus $\Omega(Q)$ is a central extension of $\mathrm{PGO}^+(Q)$ by $\mathrm{R}_{Z/K}\,\mathbb{G}_m$ ([KMRT98], 13.19, p. 181)

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{R}_{Z/K}\,\mathbb{G}_m \arrow[r] \arrow[d, equal] & \Omega(Q) \arrow[r, "\chi'"] \arrow[d, hook] & \mathrm{PGO}^+(Q) \arrow[r] \arrow[d, hook, "C"] & 1 \\
1 \arrow[r] & \mathrm{R}_{Z/K}\,\mathbb{G}_m \arrow[r] & \mathrm{Sim}\,(\mathrm{C}_0(Q), \underline{\sigma}) \arrow[r, "\mathrm{Int}"] & \mathrm{Aut}_Z\,(\mathrm{C}_0(Q), \underline{\sigma}) \arrow[r] & 1
\end{tikzcd}$$

Finally, let μ denote the center of the spinor group $\mathrm{Spin}(Q)$. Recall that $\mu = \mathrm{R}_{Z/K}\,(\mu_2)$ when n is even, and $\mu = \mu_{4[Z]} := \mathrm{Ker}\left(\mathrm{R}_{Z/K}\,\mu_4 \xrightarrow{\mathrm{Norm}} \mu_4\right)$ when n is odd ([PR94], Chapter 6, Section 6.5, p. 332). If $\dim Q = 2n \geq 4$, the extended Clifford group $\Omega(Q)$ has center $\mathrm{R}_{Z/K}\,\mathbb{G}_m$ and is an *envelope* of $\mathrm{Spin}(Q)$ ([BM00], Ex. 4.4).

2. REDUCTIONS

2.1. **Another formulation of the norm principle.** Let $1 \to J \to G_1 \to G_2 \to 1$ be a central K-isogeny of reductive groups $G_1, G_2/K$. We recall another formulation of the norm principle for the connecting map $\delta : G_2(-) \to \mathrm{H}^1(-, J)$ and its relation to the norm principle of an associated map $f : G \to T$. This discussion is taken from ([Me96], 3.10).

Since J is commutative, one can define norm maps (corestriction) $\mathrm{N}_{L/K} : \mathrm{H}^1(L, J) \to \mathrm{H}^1(K, J)$ for finite separable extensions L/K. Consider the following diagram:

$$\begin{tikzcd}
G_2(L) \arrow[r, "\delta(L)"] & \mathrm{H}^1(L, J) \arrow[d, "\mathrm{N}_{L/K}"] \\
G_2(K) \arrow[r, "\delta(K)"] & \mathrm{H}^1(K, J)
\end{tikzcd}$$

We say that the norm principle holds for $\delta : G_2(-) \to \mathrm{H}^1(-, J)$ over a finite separable field extension (or étale algebra) L/K if $\mathrm{N}_{L/K}(\mathrm{Im}\,\delta(L)) \subseteq \mathrm{Im}\,\delta(K)$. We

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[1] If $\dim Q = 2$, we set $\Omega(Q)$ to be the commutative group $R_{Z/K}\mathbb{G}_m$, for which the norm principle holds.

say that the norm principle holds for δ if the norm principle holds for δ over every finite separable field extension (equivalently over every étale algebra) L/K. We again note that if G_2 is also commutative, then the norm principle clearly holds for δ over all L/K.

Let T' be a quasi-trivial[2] torus defined over K containing J. We can find such a T' for instance as follows: Recall that taking character groups yields an anti-equivalence between affine algebraic groups of multiplicative type and finitely generated abelian groups on which $\mathrm{Gal}(K^{sep}/K)$ acts continuously ([W79], Section 7.3). Under this correspondence, permutation lattices[3] correspond to quasi-trivial tori. Note that J is a group of multiplicative type by construction. Take any surjective map $P \to X(J)$ where P is a permutation lattice and $X(J)$ is the character group of J. This will give rise to a closed embedding $J \hookrightarrow T'$, where T' is the quasi-trivial torus corresponding to P.

The associated map $f : G \to T$ (where G/K is reductive and T is commutative) is determined by the following commutative diagram (Figure 1) with exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & J \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & T' \arrow[r] \arrow[d, "f'"] & G \arrow[r, "h"] \arrow[d, "f"] & G_2 \arrow[r] & 1 \\
 & T \arrow[r, "id"] \arrow[d] & T \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

FIGURE 1. Restating the norm principle

Lemma 2.1. *Let L/K be a finite separable extension, and let G_1, G_2, J, and $f : G \to T$ be as above. Then the norm principle holds for $f : G \to T$ over L/K if and only if it holds for $\delta : G_2(-) \to \mathrm{H}^1(-, J)$ over L/K.*

Proof. Let $1 \to T' \to G \xrightarrow{h} G_2 \to 1$ be the exact row as in the diagram above. Since T' is a quasi-trivial torus, $\mathrm{H}^1(L, T')$ is trivial for all L/K, and hence this exact sequence induces surjective maps $h(L) : G(L) \to G_2(L)$. Let $\delta_1 : T(-) \to \mathrm{H}^1(-, J)$ be the connecting map of the first exact column in the diagram above. By [Me96] (Lemma 3.11), the following square is anticommutative.

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[2] This is the product of Weil restrictions of split tori.

[3] That is, $\mathrm{Gal}(K^{sep}/K)$ modules which are free $\mathbb{Z}$ modules admitting a $\mathbb{Z}$ basis permuted by $\mathrm{Gal}(K^{sep}/K)$.

$$\begin{tikzcd}
g \in G(L) \arrow[r, "h(L)"] \arrow[d, "f(L)"] & g_2 \in G_2(L) \arrow[d, "\delta(L)"] \\
t \in T(L) \arrow[r, "\delta_1(L)"] & j \in \mathrm{H}^1(L, J)
\end{tikzcd}$$

Assume that the norm principle holds for f over L/K. Let $g_2 \in G_2(L)$ and $\delta(L)(g_2) = j$. Since $h(L)$ is surjective, pick $g \in G(L)$ such that $h(L)(g) = g_2$, and let $t = f(L)(g) \in T(L)$. Thus, $\delta_1(L)(t) = j^{-1}$, by the anticommutativity of the square. Since T is commutative, the norm principle holds for the map δ_1 over L/K. Thus $\delta_1(K)(\mathrm{N}_{L/K}(t)) = \mathrm{N}_{L/K}(j^{-1})$. Since the norm principle holds for f over L/K by assumption, $\mathrm{N}_{L/K}(t) = f(K)(\tilde{g})$ for some $\tilde{g} \in G(K)$. By ([Me96], Lemma 3.11) again, the following square over K is anticommutative.

$$\begin{tikzcd}
\tilde{g} \in G(K) \arrow[r, "h(K)"] \arrow[d, "f(K)"] & h(K)(\tilde{g}) \in G_2(K) \arrow[d, "\delta(K)"] \\
\mathrm{N}_{L/K}(t) \in T(K) \arrow[r, "\delta_1(K)"] & \mathrm{N}_{L/K}(j) \in \mathrm{H}^1(K, J)
\end{tikzcd}$$

Thus $\mathrm{N}_{L/K}(j) = \delta(K)[h(K)(\tilde{g})]$. Therefore, $\mathrm{N}_{L/K}(j) \in \operatorname{Im} \delta(K)$ and the norm principle holds for δ over L/K.

Conversely, let the norm principle hold for δ over L/K. Let $g \in G(L)$ and $f(L)(g) = t$. Set $h(L)(g) = g_2 \in G_2(L)$ and $\delta(L)(g_2) = j$. Thus $\delta_1(L)(t) = j^{-1}$ and $\delta_1(K)(\mathrm{N}_{L/K}(t)) = \mathrm{N}_{L/K}(j^{-1})$ as before. Since the norm principle holds for δ, $\mathrm{N}_{L/K}(j) = \delta(K)(\tilde{g_2})$ for some $\tilde{g_2} \in G_2(K)$. Since $h(K)$ is surjective, pick $\tilde{g} \in G(K)$ such that $h(K)(\tilde{g}) = \tilde{g_2}$. Then $\mathrm{N}_{L/K}(t) = f(K)(\tilde{g})f'(K)(t')$ for some $t' \in T'(K)$. Thus, $\mathrm{N}_{L/K}(t) \in \operatorname{Im} f(K)$, and hence the norm principle holds for f over L/K. $\square$

Let G be a reductive group defined over K whose simple components are of classical type, and let T be a commutative group defined K. Then [BM00] (Thm. 1.1) establishes that if the Dynkin diagram of G does not contain connected components D_n for $n \geq 4$, the norm principle holds for any group homomorphism $G \to T$. We would like to investigate the norm principle for $G \to T$ in the remaining case where G has simple components of type D_n under the further simplifying assumption that these simple components have simply connected covers $\mathrm{Spin}(Q_i)$ arising from quadratic forms Q_i over K.

By following the reductions in [BM00], it is easy to see that we need only to check whether the norm principle holds for the group $\Omega(Q)$; i.e., for the canonical map $\Omega(Q) \to \frac{\Omega(Q)}{[\Omega(Q),\Omega(Q)]}$ for any nondegenerate even dimensional quadratic form Q/K. If $\dim Q = 2$, as noted before, the norm principle holds for the commutative group $\Omega(Q)$.

2.2. **Maps S and α.** We now restate the norm principle for $\Omega(Q)$ when $\dim Q = 2n \geq 4$ in two other equivalent forms, which will be used in the rest of the paper.

Let T denote $\frac{\Omega(Q)}{[\Omega(Q),\Omega(Q)]}$, and let T' denote the quasi-trivial torus $R_{Z/K}\mathbb{G}_m$ containing μ. Using the fact that the semisimple part of $\Omega(Q)$ is $\mathrm{Spin}(Q)$, Figure 1 yields the following commutative diagram:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu \arrow[r] \arrow[d] & \mathrm{Spin}(Q) \arrow[r] \arrow[d] & \mathrm{PGO}^+(Q) \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & R_{Z/K}\mathbb{G}_m \arrow[r] \arrow[d, "f'"] & \Omega(Q) \arrow[r, "h"] \arrow[d, "f"] & \mathrm{PGO}^+(Q) \arrow[r] & 1 \\
 & T \arrow[r, "id"] \arrow[d] & T \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

Let $S : \mathrm{PGO}^+(Q)(-) \to \mathrm{H}^1(-, \mu)$ denote the connecting map of the first exact row. By Lemma 2.1, the norm principle for $\Omega(Q) \to T$ over L/K holds if and only if it holds for S over L/K.

For any L/K, the short exact sequence $1 \to \mu \to \mathrm{Spin}(Q) \to \mathrm{PGO}^+(Q) \to 1$ gives rise to the long exact sequence

$$\ldots \to \mathrm{PGO}^+(Q)(L) \xrightarrow{S(L)} \mathrm{H}^1(L, \mu) \xrightarrow{\alpha(L)} \mathrm{H}^1(L, \mathrm{Spin}(Q)) \to \ldots.$$

Hence we can deduce that the norm principle holds for S over L/K if and only if the following, which we call the norm principle for $\alpha : \mathrm{H}^1(-, \mu) \to \mathrm{H}^1(-, \mathrm{Spin}(Q))$ over L/K, holds:

For every $u \in \mathrm{Ker}\left(\mathrm{H}^1(L, \mu) \xrightarrow{\alpha(L)} \mathrm{H}^1(L, \mathrm{Spin}(Q))\right)$, *the element* $\mathrm{N}_{L/K}(u)$ *belongs to* $\mathrm{Ker}\left(\mathrm{H}^1(K, \mu) \xrightarrow{\alpha(K)} \mathrm{H}^1(K, \mathrm{Spin}(Q))\right)$.

Thus the above discussion gives the following:

Lemma 2.2. *Let L/K be a finite separable field extension. Then the following are equivalent:*

(1) *The norm principle holds for $\Omega(Q)$ over L/K.*
(2) *The norm principle holds for $S : \mathrm{PGO}^+(Q)(-) \to \mathrm{H}^1(-, \mu)$ over L/K.*
(3) *The norm principle holds for $\alpha : \mathrm{H}^1(-, \mu) \to \mathrm{H}^1(-, \mathrm{Spin}(Q))$ over L/K.*

2.3. **Auxiliary maps i and j.** We recall the (explicit) definitions of useful auxiliary maps $i : \mathrm{H}^1(-, \mu_2) \to \mathrm{H}^1(-, \mu)$ and $j : \mathrm{H}^1(-, \mu) \to \mathrm{H}^1(-, \mu_2)$.

Recall the following commutative diagram with the two complete rows and columns exact:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mu_2 \arrow[r, "id"] \arrow[d] & \mu_2 \arrow[d] & & \\
1 \arrow[r] & \mu \arrow[r] \arrow[d] & \mathrm{Spin}(Q) \arrow[r] \arrow[d] & \mathrm{PGO}^+(Q) \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathrm{O}^+(Q) \arrow[r] \arrow[d] & \mathrm{PGO}^+(Q) \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

For every L/K, the long exact sequence of cohomology gives rise to maps $i(L) : \mathrm{H}^1(L, \mu_2) \to \mathrm{H}^1(L, \mu)$ and $j(L) : \mathrm{H}^1(L, \mu) \to \mathrm{H}^1(L, \mu_2)$ which fit into the diagram

$$\begin{tikzcd}
 & \mathrm{H}^1(L, \mu_2) \arrow[r, "id"] \arrow[d, "i(L)"] & \mathrm{H}^1(L, \mu_2) \arrow[d, "i'(L)"] \\
\mathrm{PGO}^+(Q)(L) \arrow[r, "S(L)"] \arrow[d, "id"] & \mathrm{H}^1(L, \mu) \arrow[r, "\alpha(L)"] \arrow[d, "j(L)"] & \mathrm{H}^1(L, \mathrm{Spin}(Q)) \arrow[d] \\
\mathrm{PGO}^+(Q)(L) \arrow[r, "\mathrm{M}(L)"] & \mathrm{H}^1(L, \mu_2) \arrow[r] & \mathrm{H}^1(L, \mathrm{O}^+(Q))
\end{tikzcd}$$

FIGURE 2. Auxiliary maps i and j

Here, $\mathrm{M}(L) : \mathrm{PGO}^+(Q)(L) \to \mathrm{H}^1(L, \mu_2)$ is the multiplier map. The natural map $i'(L) : \mathrm{H}^1(L, \mu_2) \to \mathrm{H}^1(L, \mathrm{Spin}(Q))$ in the above diagram will also be used later.

Let $\mathrm{Sn}(L) : \mathrm{O}^+(Q)(L) \to \mathrm{H}^1(L, \mu_2)$ denote the spinor norm map. The maps i and j fit into the following commutative diagram with exact columns ([KMRT98], Props. 13.33 and 13.36).

$$\begin{tikzcd}
\mathrm{O}^+(Q)(L) \arrow[r, "\mathrm{Sn}(L)"] \arrow[d, "\pi(L)"] & \mathrm{H}^1(L, \mu_2) \arrow[d, "i(L)"] \\
\mathrm{PGO}^+(Q)(L) \arrow[r, "S(L)"] \arrow[d, "\mathrm{M}(L)"] & \mathrm{H}^1(L, \mu) \arrow[d, "j(L)"] \\
\mathrm{H}^1(L, \mu_2) \arrow[r, phantom, "="] & \mathrm{H}^1(L, \mu_2)
\end{tikzcd}$$

FIGURE 3. Relation to spinor norms

The explicit descriptions of i and j depend on the parity of $n = \dim(Q)/2$.

- If n is even, $\mathrm{H}^1(K,\mu) = \frac{Z^*}{Z^{*2}}$. Then, $i(K) : \frac{K^*}{K^{*2}} \to \frac{Z^*}{Z^{*2}}$ is the inclusion map and $j(K) : \frac{Z^*}{Z^{*2}} \to \frac{K^*}{K^{*2}}$ sends $[z] \rightsquigarrow [\mathrm{N}_{Z/K}(z)]$ for $z \in Z^*$.
- If n is odd, $\mathrm{H}^1(K,\mu) = \frac{U(K)}{U_0(K)}$, where $U \subset \mathbb{G}_m \times \mathrm{R}_{Z/K}\,\mathbb{G}_m$ is the subgroup defined by $U(K) = \{(f,z) \in K^* \times Z^* | f^4 = \mathrm{N}_{Z/K}(z)\}$ and $U_0 \subset U \subset \mathbb{G}_m \times \mathrm{R}_{Z/K}\,\mathbb{G}_m$ is the algebraic subgroup defined by $U_0(K) = \{(\mathrm{N}_{Z/K}(z), z^4) \in K^* \times Z^* | z \in Z^*\}$.

  Then $i(K) : \frac{K^*}{K^{*2}} \to \frac{U(K)}{U_0(K)}$ is the map sending $fK^{*2} \rightsquigarrow [f, f^2]$ for $f \in K^*$. Finally $j(K) : \frac{U(K)}{U_0(K)} \to \frac{K^*}{K^{*2}}$ sends $[f,z] \rightsquigarrow \mathrm{N}_{Z/K}(z_0)K^{*2}$, where $z_0 \in Z^*$ is such that $z_0\psi(z_0)^{-1} = f^{-2}z$.

We end this section with one more useful reduction, which is a direct consequence of Knebusch's norm principle.

Lemma 2.3. *Let L/K be a finite separable extension and $u \in \mathrm{Ker}(\alpha(L))$. If $j(L)(u) = 1 \in \mathrm{H}^1(L,\mu_2)$, then $\mathrm{N}_{L/K}(u) \in \mathrm{Ker}(\alpha(K))$.*

Proof. Since $u \in \mathrm{Ker}\,(\alpha(L))$, there exists $[g] \in \mathrm{PGO}^+(Q)(L)$ such that $S(L)([g]) = u$. Since $j(L)(u) = 1$, by Figure 3, we see that u is the image of a spinor norm of Q_L; i.e., $u = i(L)(\mathrm{Sn}(h))$ for some $h \in \mathrm{O}^+(Q)(L)$. By Knebusch's norm principle, $\mathrm{N}_{L/K}(\mathrm{Sn}(h)) = \mathrm{Sn}(\tilde{h})$ for some $\tilde{h} \in \mathrm{O}^+(Q)(K)$. Hence $\mathrm{N}_{L/K}(u) = i(K)\left(\mathrm{Sn}(\tilde{h})\right) = S(K)(\pi(\tilde{h}))$. Therefore $\alpha(K)(\mathrm{N}_{L/K}(u)) = 1$. ☐

2.4. **A square diagram.** Using the auxiliary maps i and j just defined, we construct a square diagram (SQ) by piecing together the two classical norm principles of Scharlau and Knebusch. It is shown that the norm principle holds for $\Omega(Q)$ precisely when the square is commutative for all finite separable extensions. This reformulation yields further reductions for the norm principle question.

Using Figure 2 in section 2.3, for each finite separable extension L/K, define the subgroup $H(L) \subseteq \mathrm{H}^1(L,\mu)$ as follows:

$$\begin{aligned} H(L) &:= \{x \in \mathrm{H}^1(L,\mu) \mid j(L)(x) \in \mathrm{Im}\,(\mathrm{M}(L))\} \\ &= \{x \in \mathrm{H}^1(L,\mu) \mid \alpha(L)(x) \in \mathrm{Im}\,(i'(L))\}. \end{aligned}$$

Recall that $i'(L) : \mathrm{H}^1(L,\mu_2) \to \mathrm{H}^1(L,\mathrm{Spin}(Q))$ induces the group homomorphism $i''(L) : L^*/L^{*2} \to L^*/\operatorname{Sn}(Q_L) \subseteq \mathrm{H}^1(L,\mathrm{Spin}(Q))$ sending $fL^{*2} \rightsquigarrow [f]$ for each $f \in L^*$. Thus $\alpha(L)$ induces the following map

$$\tilde{\alpha}(L) : H(L) \to L^*/\operatorname{Sn}(Q_L).$$

Lemma 2.4. *Here $\tilde{\alpha}(L) : H(L) \to L^*/\operatorname{Sn}(Q_L)$ is a group homomorphism.*

Proof. For $i = 1, 2$, let $z_i \in H(L)$ with $\tilde{\alpha}(L)(z_i) = x_i \in L^*/\operatorname{Sn}(Q_L)$. By the definition of $H(L)$ and a diagram chase of Figure 2 in section 2.3, there exist $[g_i] \in \mathrm{PGO}^+(L)$ such that $S([g_i^{-1}])z_i = i(L)(y_i)$ for $y_i \in \mathrm{H}^1(L,\mu_2)$. Thus $\tilde{\alpha}(L)(S([g_i^{-1}])z_i) = i'(L)(y_i) = i''(L)(y_i)$. Since $i''(L)$ is a group homomorphism and $H(L)$ is abelian, we see that

$$\begin{aligned} \tilde{\alpha}(L)(S([g_1^{-1}])z_1)\tilde{\alpha}(L)(S([g_2^{-1}])z_2) &= i''(L)(y_1)i''(L)(y_2) \\ &= i''(L)(y_1y_2) \\ &= \tilde{\alpha}(L)(S([g_1^{-1}])z_1S([g_2^{-1}])z_2) \\ &= \tilde{\alpha}(L)(S([g_2^{-1}g_1^{-1}])z_1z_2). \end{aligned}$$

Since $\mu = Z(\mathrm{Spin}(Q))$, we have $\tilde{\alpha}(L)(S([g_i^{-1}]z_i) = \tilde{\alpha}(L)(z_i)$ and $\tilde{\alpha}(L)(S([g_2^{-1}g_1^{-1}]z_1z_2) = \tilde{\alpha}(L)(z_1z_2)$ ([KMRT98], Corollary 28.4, p. 386), we conclude $\tilde{\alpha}(L)$ is a group homomorphism. □

From the definition of $H(L)$, it follows that $\mathrm{Im}\,(i(L)) \subseteq H(L)$. Further by a chase of the following square which is part of Figure 2, we see that $\tilde{\alpha}(L)(i(L)[fL^{*2}]) = [f]$ for each $f \in L^*$.

$$\begin{tikzcd}
fL^{*2} \in \mathrm{H}^1(L,\mu_2) \arrow[r, "id"] \arrow[d, "i(L)"] & fL^{*2} \in \mathrm{H}^1(L,\mu_2) \arrow[d, "i'(L)"] \\
\mathrm{H}^1(L,\mu) \arrow[r, "\alpha(L)"] & \in \mathrm{H}^1(L,\mathrm{Spin}(Q))
\end{tikzcd}$$

Similarly it is immediate to see that $\mathrm{Im}\,(S(L)) = \mathrm{Ker}\,(\alpha(L)) \subseteq H(L)$ and further is exactly $\mathrm{Ker}\,(\tilde{\alpha}(L))$.

Scharlau's norm principle implies that the norm map $\mathrm{N}_{L/K} : \mathrm{H}^1(L,\mu) \to \mathrm{H}^1(K,\mu)$ induces the map $\mathrm{N}_{L/K} : H(L) \to H(K)$. Similarly Knebusch's norm principle implies that the norm map $\mathrm{N}_{L/K} : L^* \to K^*$ induces the map $\mathrm{N}_{L/K} : L^*/\,\mathrm{Sn}(Q_L) \to K^*/\,\mathrm{Sn}(Q)$. Thus, the following square diagram (labelled SQ for L/K) is defined.

$$\begin{tikzcd}
H(L) \arrow[r, "\tilde{\alpha}(L)"] \arrow[d, "\mathrm{N}_{L/K}(\text{Scharlau})"] & L^*/\,\mathrm{Sn}(Q_L) \arrow[d, "\mathrm{N}_{L/K}(\text{Knebusch})"] \\
H(K) \arrow[r, "\tilde{\alpha}(K)"] & K^*/\,\mathrm{Sn}(Q)
\end{tikzcd}$$

Theorem 2.5. *Let L/K be a finite separable field extension. Then the following are equivalent:*

(1) *The norm principle holds for $\Omega(Q)$ over L/K.*
(2) *The square diagram (SQ for L/K) commutes.*

Proof. **(1) $\implies$ (2)**: By Lemma 2.2, the norm principle holds for α over L/K. Let $z \in H(L)$ and $\tilde{\alpha}(L)(z) = [x] \in L^*/\,\mathrm{Sn}(Q_L)$ for $x \in L^*$. As observed above, $x \in H(L)$ and $\tilde{\alpha}(L)(x) = [x] \in L^*/\,\mathrm{Sn}(Q_L)$. Since $\tilde{\alpha}(L)$ is a group homomorphism, $zx^{-1} \in \mathrm{Ker}(\tilde{\alpha}(L)) = \mathrm{Ker}(\alpha(L))$. Since we assume that the norm principle holds for α over L/K, $\alpha(K)(\mathrm{N}_{L/K}(zx^{-1})) = 1$. As $\mathrm{N}_{L/K}(z), \mathrm{N}_{L/K}(x), \mathrm{N}_{L/K}(zx^{-1}) \in H(K)$ and $\alpha(K)$ induces $\tilde{\alpha}(K)$, $\tilde{\alpha}(K)(\mathrm{N}_{L/K}(z)) = \tilde{\alpha}(K)(\mathrm{N}_{L/K}(x)) = [\mathrm{N}_{L/K}(x)] \in K^*/\,\mathrm{Sn}(Q)$. Hence (SQ for L/K) commutes.

$$\begin{tikzcd}
z \in H(L) \arrow[r, "\tilde{\alpha}(L)"] \arrow[d, "\mathrm{N}_{L/K}"] & {[x]} \in L^*/\,\mathrm{Sn}(Q_L) \arrow[d, "\mathrm{N}_{L/K}"] \\
\mathrm{N}_{L/K}(z) \in H(K) \arrow[r, "\tilde{\alpha}(K)"] & {[\mathrm{N}_{L/K}(x)]} \in K^*/\,\mathrm{Sn}(Q)
\end{tikzcd}$$

(2) $\implies$ (1): Let $u \in \mathrm{Ker}(\alpha(L)) = \mathrm{Ker}(\tilde{\alpha}(L))$. By the commutativity of the square (SQ for L/K), we have $\tilde{\alpha}(K)\big(\mathrm{N}_{L/K}(u)\big) = 1$. Hence the norm principle holds for α over L/K, and by Lemma 2.2 it holds for $\Omega(Q)$ over L/K. □

If Q/K is isotropic, then $\operatorname{Sn}(Q) = K^*$, and hence the square (SQ for L/K) commutes for all L/K yielding the following.

Corollary 2.6. *If Q/K is isotropic, then the norm principle holds for $\Omega(Q)$.*

2.4.1. *Reduction to quadratic extensions.* We now show that to prove the norm principle for $\Omega(Q)$, it suffices to consider separable quadratic extensions. More precisely, we prove the following.

Theorem 2.7. *Let Q/K be a quadratic form of even dimension. Suppose that for every finite separable field extension M/K, the norm principle holds for the M-group $\Omega(Q)_M$ over every separable quadratic field extension M'/M. Then the norm principle holds for $\Omega(Q)$.*

Proof. Let L/K be any finite separable field extension. By Theorem 2.5, it suffices to show that the square (SQ for L/K) commutes.

There exists a separable field extension M/K with $[M : K] = 2m + 1$ for[4] some $m \geq 0$ such that $LM := L \otimes_K M \simeq \prod_{i=1}^{r} M_i$, where each $[M_i : M] = 2^{m_i}$ with M_i/M separable field extensions filtered by quadratic extensions.

By assumption and Lemma 2.2, the norm principle holds for the map α over each M_i/M and hence over the étale algebra LM/M. Hence, by Theorem 2.5, the square (SQ for LM/M) commutes.

Note that the natural map $K^*/\operatorname{Sn}(Q) \to M^*/\operatorname{Sn}(Q_M)$ is injective. This is because if $f \in K^*$ becomes a spinor norm from Q_M, then by Knesbusch's norm principle, $\mathrm{N}_{M/K}(f) = f^{2m+1}$ is a spinor norm from Q and hence $[f] = 1 \in K^*/\operatorname{Sn}(Q)$ to begin with.

Look at the following cuboid which has commutative front vertical face (SQ for LM/M) as well as commutative side, top and bottom faces. A diagram chase and the injectivity of the map $K^*/\operatorname{Sn}(Q) \to M^*/\operatorname{Sn}(Q_M)$ shows that the back vertical face (SQ for L/K) commutes.

$$\begin{tikzcd}
H(L) \arrow[rr, "\tilde{\alpha}(L)"] \arrow[d, "\mathrm{N}"] \arrow[ddr, dashed] & & L^*/\operatorname{Sn}(Q_L) \arrow[d, "\mathrm{N}"] \arrow[ddr, dashed] & \\
H(K) \arrow[rr, "\tilde{\alpha}(K)"] \arrow[ddr, dashed] & & K^*/\operatorname{Sn}(Q) \arrow[ddr, dashed, hook] & \\
& H(LM) \arrow[rr, "\tilde{\alpha}(LM)"] \arrow[d, "\mathrm{N}"] & & (LM)^*/\operatorname{Sn}(Q_{LM}) \arrow[d, "\mathrm{N}"] \\
& H(M) \arrow[rr, "\tilde{\alpha}(M)"] & & M^*/\operatorname{Sn}(Q_M)
\end{tikzcd}$$

□

3. An inductive approach

In this section, we outline a possible inductive approach to the norm principle question.

---

[4] For instance, take M to be the fixed field of a 2-Sylow subgroup of $\operatorname{Gal}(N/L)$, where N is the Galois closure of L over K.

Let Q/K be a quadratic form of even dimension as before and let L/K be a finite separable field extension. Let $u \in \mathrm{Ker}(\alpha(L))$. We would like to show $\mathrm{N}_{L/K}(u) \in \mathrm{Ker}(\alpha(K))$. Set $j(L)(u) = [\lambda] \in \mathrm{H}^1(L, \mu_2)$ for some $\lambda \in L^*$. It follows from Figure 2 in section 2.3 that $Q_L \simeq \lambda Q_L$.

Suppose that there exist even dimensional quadratic forms f_u, g_u defined over K such that $Q \simeq f_u \perp g_u$ and $(f_u)_L \simeq \lambda\,(f_u)_L$. Note that this immediately implies $(g_u)_L \simeq \lambda\,(g_u)_L$. Let $R_u := \mathrm{O}^+(f_u) \times \mathrm{O}^+(g_u) \subset \mathrm{O}^+(Q)$. Define an intermediate new group $\tilde{R}_u$ to be the preimage of R_u under the canonical homomorphism $\mathrm{Spin}(Q) \to \mathrm{O}^+(Q)$.

We have the following diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, "id"] & \mu \arrow[r] \arrow[d, hook] & \mu_2 \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, "id"] & \tilde{R}_u \arrow[r] \arrow[d, hook] & R_u \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathrm{Spin}(Q) \arrow[r] & \mathrm{O}^+(Q) \arrow[r] & 1
\end{tikzcd}$$

Let $\gamma_u : \mathrm{H}^1(-, \mu) \to \mathrm{H}^1(-, \tilde{R}_u)$ be the induced map from the inclusion $\mu \hookrightarrow \tilde{R}_u$.

Lemma 3.1. *Let L/K be a finite separable field extension and $u \in \mathrm{Ker}(\alpha(L))$. Assume that there exist even dimensional quadratic forms $f_u, g_u/K$ such that $Q \simeq f_u \perp g_u$ and $j(L)(u) = [\lambda]$ for $\lambda \in L^*$ with $(f_u)_L \simeq \lambda\,(f_u)_L$ and $(g_u)_L \simeq \lambda\,(g_u)_L$. If $\mathrm{N}_{L/K}\left(\mathrm{Ker}\left(\gamma_u(L)\right)\right) \subset \mathrm{Ker}\left(\gamma_u(K)\right)$, then $\mathrm{N}_{L/K}(u) \in \mathrm{Ker}\left(\alpha(K)\right)$.*

Proof. Let $\gamma_u(L)(u) = v \in \mathrm{H}^1(L, \tilde{R}_u)$. Since $(f_u)_L \simeq \lambda\,(f_u)_L$ and $(g_u)_L \simeq \lambda\,(g_u)_L$, $[\lambda]$ goes to 1 in $\mathrm{H}^1(L, R_u)$. Hence v goes to 1 in $\mathrm{H}^1(L, R_u)$, and therefore there exists $a \in L^*$ such that $[a] \in \mathrm{H}^1(L, \mu_2)$ goes to v.

$$\begin{tikzcd}
 & \mathrm{H}^1(L,\mu_2) \arrow[rr, "i(L)"] \arrow[dl, "id"] \arrow[dd, dashed] & & u \in \mathrm{H}^1(L,\mu) \arrow[rr, "j(L)"] \arrow[dl, "\gamma_u(L)"] \arrow[dd, dashed, "\alpha(L)"] & & {[\lambda]} \in \mathrm{H}^1(L,\mu_2) \arrow[dl] \arrow[dd, dashed] \\
\mathrm{H}^1(L,\mu_2) \arrow[rr] \arrow[dr, "id"] & & v \in \mathrm{H}^1(L,\tilde{R}_u) \arrow[rr] \arrow[dr] & & 1 \in \mathrm{H}^1(L,R_u) \arrow[dr] & \\
 & \mathrm{H}^1(L,\mu_2) \arrow[rr] & & 1 \in \mathrm{H}^1(L,\mathrm{Spin}(Q)) \arrow[rr] & & \mathrm{H}^1(L,\mathrm{O}^+(Q))
\end{tikzcd}$$

As $\mu \subseteq Z(\tilde{R}_u)$, we have $i([a])^{-1}u$ is in the image of $\left(\tilde{R}_u/\mu\right)(L) \to \mathrm{H}^1(L, \mu)$ ([KMRT98], Corollary 28.4, p. 386). Hence $\gamma_u(L)\left(i([a])^{-1}u\right) = 1$, and therefore $\alpha(L)(i([a])^{-1}u) = 1 \in \mathrm{H}^1(L, \mathrm{Spin}(Q))$. Since $\alpha(L)(u) = 1$, we see that $\alpha(L)(i[a]) = 1$; i.e., a is a spinor norm of Q_L. By Knebusch's norm principle, $\mathrm{N}_{L/K}(a)$ is a spinor norm of Q and hence $\alpha(K)\left(i[N_{L/K}(a)]\right) = 1$.

By assumption on γ_u, we have $\gamma_u(K)\left(N_{L/K}\left(i([a])^{-1}u\right)\right) = 1 \in \mathrm{H}^1(K, \tilde{R}_u)$ and hence this element dies in $\mathrm{H}^1(L, \mathrm{Spin}(Q))$ also. That is, $\alpha(K)\left(\mathrm{N}_{L/K}\left(i([a])^{-1}u\right)\right) = 1$. This implies $\alpha(K)(\mathrm{N}_{L/K}(u)) = 1$. □

This leads to the following.

Theorem 3.2. *Let L/K be a finite separable field extension and $u \in \mathrm{Ker}(\alpha(L))$. Assume that there exist even dimensional quadratic forms $f_u, g_u/K$ such that $Q \simeq f_u \perp g_u$ and $j(L)(u) = [\lambda]$ for $\lambda \in L^*$ with $(f_u)_L \simeq \lambda\,(f_u)_L$ and $(g_u)_L \simeq \lambda\,(g_u)_L$.*

If the norm principle holds for $\Omega(f_u)$ and $\Omega(g_u)$ over L/K, then $\mathrm{N}_{L/K}(u) \in \mathrm{Ker}\,(\alpha(K))$.

Proof. Recall the exact sequence of algebraic K-groups $1 \to \mu \to \tilde{R}_u \to \tilde{R}_u/\mu \to 1$. By Lemma 3.1, it suffices to check that $\mathrm{N}_{L/K}\,(\mathrm{Ker}\,(\gamma_u(L))) \subset \mathrm{Ker}\,(\gamma_u(K))$, which is equivalent to verifying that the norm principle holds for $\tilde{R}_u/\mu(-) \to \mathrm{H}^1(-,\mu)$ over L/K (cf. Lemma 2.2). As in section 2.1, construct $f : G \to T$ given by the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu \arrow[r] \arrow[d] & \tilde{R}_u \arrow[r] \arrow[d] & \tilde{R}_u/\mu \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & R_{Z/K}\mathbb{G}_m \arrow[r] \arrow[d, "f'"] & G \arrow[r, "h"] \arrow[d, "f"] & \tilde{R}_u/\mu \arrow[r] & 1 \\
 & T \arrow[r, "id"] \arrow[d] & T \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

By Lemma 2.1, it suffices to check the norm principle for $f : G \to T$ over L/K or, more generally, the norm principle for G over L/K. Then [BM00], Proposition 5.2, along with the hypothesis that the norm principle holds for $\Omega(f_u)$ and $\Omega(g_u)$ concludes the proof. ☐

4. R-EQUIVALENCE CLASSES OF $\mathrm{PGO}^+(Q)(K)$

Let G be a linear algebraic group defined over K and L/K, a field extension. Then $x, y \in G(L)$ are said to be *R-equivalent* if there exists an L-rational map $f : \mathbb{A}^1_L \dashrightarrow G$ defined at 0 and 1 sending $0 \rightsquigarrow x$ and $1 \rightsquigarrow y$. This defines an equivalence relation on $G(L)$ ([Gi97], section II.1). Let $RG(L)$ denote the normal subgroup of elements in $G(L)$ which are R-equivalent to the identity e_G. We denote the quotient group $G(L)/RG(L)$ by $G(L)/R$. This is the group of R-equivalence classes introduced by Manin for the L-points of the variety underlying group G.

The norm principles in [Gi93] and [Me96] are stated in general for R-trivial elements (i.e, elements R-equivalent to e_G).

Theorem 4.1 (Gille, [Gi93]). *Let $1 \to \mu \to \tilde{G} \to G \to 1$ be an isogeny of semi-simple algebraic groups over K, and $\mathrm{N}_{L/K} : \mathrm{H}^1(L,\mu) \to \mathrm{H}^1(K,\mu)$ be the induced norm map for a field extension L/K. Let $RG(L)$ (resp., $RG(K)$) denote the elements of $G(L)$ (resp., $G(K)$) which are R-equivalent to the identity.*

$$\begin{tikzcd}
RG(L) \arrow[r, "\delta(L)"] & \mathrm{H}^1(L,\mu) \arrow[d, "\mathrm{N}_{L/K}"] \\
RG(K) \arrow[r, "\delta(K)"] & \mathrm{H}^1(K,\mu)
\end{tikzcd}$$

Then $\mathrm{N}_{L/K}\left(\operatorname{Im}\delta(L) : RG(L) \to \mathrm{H}^1(L,\mu)\right) \subseteq \left(\operatorname{Im}\delta(K) : RG(K) \to \mathrm{H}^1(K,\mu)\right)$.

A similar statement holds for the norm principles of morphisms $f : G \to T$ where G is a reductive linear algebraic group ([Me96], Thm. 3.9).

In [Me96(2)], the group of R-equivalence classes was computed for adjoint semisimple classical groups. These computations applied to the adjoint group $\mathrm{PGO}^+(Q)$ translates to a natural isomorphism $\mathrm{PGO}^+(Q)(K)/R \simeq \mathrm{G}(Q)/\left(K^{*2}\,\mathrm{Hyp}(Q)\right)$, where

- Q/K is a nondegenerate form of dim $2n$,
- $\mathrm{G}(Q)$ is the *group of similarities* $\{\lambda \in K^* \mid \lambda Q \simeq Q\}$, and
- $\mathrm{Hyp}(Q)$ is the subgroup generated by $\langle \mathrm{N}_{E/K}(E^*) \mid Q_E \simeq \mathbb{H}^n\rangle$, where E runs over finite extensions of K.

The following lemma identifies some quadratic forms which give rise to adjoint groups of type D_n with trivial group of R-equivalence classes over the base field.

Lemma 4.2. *Let K be a complete discretely valued field with ring of integers $\mathcal{O}_K$ and residue field k with* $\mathrm{char}(k) \neq 2$. *Let $\tau = \langle a_1, a_2, \ldots, a_r\rangle$ where each $a_i \in \mathcal{O}_K^*$. Let $\pi \in K^*$ be a parameter of K, and set $\xi = \tau \otimes \langle 1, \pi\rangle$. Then* $\mathrm{PGO}^+(\xi)(K)/R$ *is trivial.*

Proof. Let $\theta \in \mathrm{G}(\xi)$. Thus up to squares in K^*, $\theta = x\pi^\epsilon$ for some $x \in \mathcal{O}_K^*$ and $\epsilon \in \{0,1\}$. Since $\mathrm{PGO}^+(\xi)(K)/R \simeq \mathrm{G}(\xi)/\left(K^{*2}\,\mathrm{Hyp}(\xi)\right)$, we would like to show $x\pi^\epsilon \in \mathrm{Hyp}(\xi)K^{*2}$.

The totally ramified extension $L' = K(\sqrt{-\pi})$ splits ξ and $\pi = N_{L'/K}(\sqrt{-\pi})$. Thus $\pi \in \mathrm{Hyp}(\xi)$. Thus we can assume $\theta = x \in \mathrm{G}(\xi)$ for $x \in \mathcal{O}_K^*$.

Let $\overline{x}$ denote the image of x in k and $\overline{\tau}$, the image of τ in $W(k)$. Recall the second residue homomorphism $\delta_{2,\pi} : \mathrm{W}(K) \to \mathrm{W}(k)$ with respect to the parameter π. Then we have $[\overline{\tau}] = \delta_{2,\pi}(\xi) = \delta_{2,\pi}(x\pi^\epsilon\xi) = \delta_{2,\pi}(x\xi) = [\overline{x\tau}] \in W(k)$. This shows that $[\overline{\tau} \otimes \langle 1, -\overline{x}\rangle_k] = 0$ in $W(k)$.

Look at $L'' = K(\sqrt{-x\pi})$ which is a complete discretely valued field with residue field k. Recall the first residue homomorphism $\delta_{1,y} : \mathrm{W}(L'') \to \mathrm{W}(k)$ with respect to some parameter y of L''. Note that $\xi_{L''} \simeq \tau_{L''} \otimes \langle 1, -x\rangle_{L''}$ and $\delta_{1,y}(\xi_{L''}) = [\overline{\tau} \otimes \langle 1, -\overline{x}\rangle_k] = 0 \in W(k)$. Thus by Hensel's lemma, $[\xi_{L''}] = 0 \in W(L'')$.

Finally as $x\pi = N_{L''/K}(\sqrt{-x\pi})$, $x\pi \in \mathrm{Hyp}(\xi)$. Since $\mathrm{Hyp}(\xi)$ is the subgroup generated by norms from finite extensions which split S, $x \in \mathrm{Hyp}(\xi)$, which concludes the proof. □

5. Over complete discretely valued fields

In this section, we work over a complete discretely valued field K. We fix the convention of letting $\overline{\star}$ denote the image of $\star$ in the residue field for $\star$ defined over the ring of integers of a complete discretely valued field. We also let $\mathcal{O}_X$ denote the ring of integers of a complete discretely valued field X and $m_{\mathcal{O}_X}$, its maximal ideal. Further we call any nondegenerate quadratic form Q defined over X *unramified* if it can be obtained from a quadratic space over O_X by base change. This amounts to saying that Q admits a diagonalization $Q \simeq \langle x_1, x_2, \ldots, x_n\rangle$ where $x_i \in \mathcal{O}_X^*$ for each i. We now state the main result of this paper.

Theorem 5.1. *Let K be a complete discretely valued field with residue field k with* $\mathrm{char}(k) \neq 2$. *Assume that the norm principle holds for $\Omega(q)$ for every nondegenerate quadratic form q of even dimension defined over any finite extension of k. Then the* *norm principle holds for* $\Omega(Q)$ *for every nondegenerate quadratic form* Q *of even dimension over* K*.*

By Theorem 2.7, it suffices to show that the norm principle holds for $\Omega(Q)$ over separable quadratic extensions L/K. Fix a uniformizing parameter t of K. Write the nondegenerate quadratic form Q/K in the form $q \perp tp$ with $q \simeq \langle a_1, a_2, \ldots, a_{\dim q}\rangle$ and $p \simeq \langle b_1, b_2, \ldots, b_{\dim p}\rangle$ for $a_i, b_j \in \mathcal{O}_K^*$ since $\dim Q = 2n$, $\dim q$, and $\dim p$ have the same parity.

Note that L/K is a complete discretely valued field. Let ℓ/k denote its residue field. Then one of the following holds:

- L/K is an unramified extension and $\theta := t$ is a parameter of L.
- L/K is totally ramified and $\ell = k$. Further, $L \simeq K(\sqrt{ct})$ for some $c \in \mathcal{O}_K^*$, and $\theta := \sqrt{ct}$ is a parameter of L.

By Lemma 2.2, it suffices to verify the norm principle for α over L/K. Let $u \in \operatorname{Ker}(\alpha(L)) \subseteq H(L)$. Set $j(L)(u) = [\lambda] \in \mathrm{H}^1(L, \mu_2) \simeq L^*/L^{*2}$ for some representative $\lambda \in L^*$. We would like to show that $\alpha(K)\left(\mathrm{N}_{L/K}(u)\right) = 1 \in \mathrm{H}^1(K, \mathrm{Spin}(Q))$. By Corollary 2.6, we can assume q and p are anisotropic over K.

5.1. **Lemmata.** We begin by reducing to the case where λ is a unit in L.

Lemma 5.2. *Up to squares,* λ *can be assumed to be in* $\mathcal{O}_L^*$*.*

Proof. Without loss of generality, we can assume $\lambda \in \mathcal{O}_L^*$ or $\lambda = \tilde{\lambda}\theta$ for $\tilde{\lambda} \in \mathcal{O}_L^*$, where θ is a parameter of L.

If L/K is unramified and $\lambda = \tilde{\lambda}\theta$, where $\theta = t$, using $\lambda Q_L \simeq Q_L$ and the second residue map $\delta_{2,t} : W(L) \to W(\ell)$ for the parameter t of L, we see that

$$[\overline{p}_\ell] = \delta_{2,t}(Q_L) = \delta_{2,t}\left(\lambda Q_L\right) = [\overline{\tilde{\lambda}}\overline{q}_\ell] \in W(\ell).$$

Thus $Q_L \simeq q_L \otimes \langle 1, \tilde{\lambda}t\rangle_L$. By Lemma 4.2, $\mathrm{PGO}^+(Q)(L)/R = \{1\}$ and hence $\mathrm{RPGO}^+(Q)(L) = \mathrm{PGO}^+(Q)(L)$. Then Theorem 4.1 implies that the norm principle holds for our required map $S : \mathrm{PGO}^+(Q) \to \mathrm{H}^1(-, \mu)$ over the extension L/K and hence for $\Omega(Q)$ over L/K (Lemma 2.2).

If $L \simeq K(\sqrt{ct})$ for some $c \in \mathcal{O}_K^*$ and $\lambda = \tilde{\lambda}\theta$ where $\theta := \sqrt{ct}$ and $Q_L \simeq q_L \perp cp_L$, using $\lambda Q_L \simeq Q_L$ and the second residue map $\delta_{2,\theta} : W(L) \to W(k)$ for the parameter θ of L, we see that

$$0 = \delta_{2,\theta}(Q_L) = \delta_{2,\theta}\left(\lambda Q_L\right) = [\overline{\tilde{\lambda}}\overline{q} \perp \overline{\tilde{\lambda}}\overline{cp}] \in W(k).$$

Thus $[\overline{q} \perp \overline{cp}] = 0 \in W(k)$ and hence $[Q_L] = 0 \in W(L)$. This implies $\mathrm{PGO}^+(Q)(L)/R = \{1\}$ and hence $\mathrm{RPGO}^+(Q)(L) = \mathrm{PGO}^+(Q)(L)$. Then Theorem 4.1 and Lemma 2.2 imply that the norm principle holds for $\Omega(Q)$ over L/K. ☐

Next, we describe the shape of the element $u \in \operatorname{Ker}(\alpha(L))$ under consideration when Q_L is unramified. We construct a related element $u' \in \mathrm{H}^1(\mathcal{O}_L, \mu)$ which in fact also lives in $\operatorname{Ker}(\alpha(L))$. To do so, we make use of the explicit description of the maps $i(L)$ and $j(L)$ given in section 2.3.

Lemma 5.3. *Let* L *be a complete discretely valued field with ring of integers* $\mathcal{O}_L$*, parameter* θ*, and residue field* ℓ*. Let* $Q' = \langle x_1, x_2, \ldots, x_{2r}\rangle$ *be a quadratic form over* L *where each* $x_i \in \mathcal{O}_L^*$*. Let* u *be an element in the kernel of* $\alpha(L) : \mathrm{H}^1(L, \mu') \to \mathrm{H}^1(L, \mathrm{Spin}(Q'))$*, where* μ' *is the center of* $\mathrm{Spin}(Q')$*. Assume* $j(L)(u) = [\lambda] \in$

$\mathrm{H}^1(L,\mu_2)$ *for some* $\lambda \in \mathcal{O}_L^*$. *Then there exist* $u' \in \mathrm{H}^1(\mathcal{O}_L,\mu')$ *and* $\epsilon' \in \mathbb{Z}$ *such that* $u = u'\left(i(L)[\theta^{\epsilon'}]\right)$.

Proof. Let Z' denote the discriminant extension of Q'. Since Q' is unramified over L, Z' is an unramified (possibly split) quadratic extension of L. Thus θ is still a parameter of Z'.

Suppose first that $\dim Q' \cong 2 \mod 4$ and Z' is a field. Since $\mathrm{H}^1(L,\mu') = \frac{U(L)}{U_0(L)}$, there exist $\epsilon \in \mathbb{Z}$, $f \in \mathcal{O}_L^*$, and $z \in \mathcal{O}_{Z'}^*$ with $\mathrm{N}_{Z'/L}(z) = f^4$ such that $u = [f\theta^{\epsilon}, z\theta^{2\epsilon}]$. Define $u' := [f,z]$ in $\mathrm{H}^1(\mathcal{O}_L,\mu')$. Since $i(L)(\theta) = [\theta,\theta^2]$, it is clear that $u = u'\,(i(L)[\theta^{\epsilon}])$.

The case when $\dim Q' \cong 2 \mod 4$ but $Z' \simeq L \times L$ is a little more delicate. Again, clearly there exist $\epsilon, \epsilon_1 \in \mathbb{Z}$, $f, z_1, z_2 \in \mathcal{O}_L^*$ with $z_1 z_2 = f^4$ such that $u = [f\theta^{\epsilon}, z_1\theta^{\epsilon_1}, z_2\theta^{4\epsilon-\epsilon_1}]$. Define $u' := [f, z_1, z_2]$ in $\mathrm{H}^1(\mathcal{O}_L,\mu')$.

Since $j(L)(u) = [\lambda]$ and up to squares $\lambda \in \mathcal{O}_L^*$, we see that ϵ_1 introduced above is even.[5] Since $[ab, a^4, b^4] = [1] \in \mathrm{H}^1(L,\mu')$ and $i(L)(\theta) = [\theta,\theta^2,\theta^2]$, we see that

$$u = \begin{cases} u' & \text{if } \epsilon_1 \cong 0 \mod 4, \\ u'\,(i(L)[\theta]) & \text{if } \epsilon_1 \cong 2 \mod 4. \end{cases}$$

Suppose now that $\dim Q' \cong 0 \mod 4$. Since $\mathrm{H}^1(L,\mu') = Z'^*/Z'^{*2}$, there exist $\epsilon, \epsilon_1 \in \mathbb{Z}$, and $z \in \mathcal{O}_{Z'}^*$ (resp., $z_1, z_2 \in \mathcal{O}_L^*$) such that $u = [z\theta^{\epsilon}]$ (resp., $u = [z_1\theta^{\epsilon}, z_2\theta^{\epsilon_1}]$) if Z' is a field (resp., a split extension). Define $u' := [z]$ (resp., $[z_1, z_2]$) in $\mathrm{H}^1(\mathcal{O}_L,\mu')$. As $\lambda \in \mathcal{O}_L^*$ up to squares, we can assume[6] that $\epsilon = \epsilon_1$. Thus $u = u'\,(i(L)[\theta^{\epsilon}]) \in \mathrm{H}^1(L,\mu')$. □

The following lemma shows that u' defined above still lives in the kernel of $\alpha(L)$.

Lemma 5.4. *Let* u, u' *be as in Lemma* 5.3. *Then* $u' \in \mathrm{Ker}\,(\alpha(L))$.

Proof. We have $u = u'\left(i(L)[\theta^{\epsilon'}]\right) \in \mathrm{H}^1(L,\mu')$.

Since Q' is unramified over L, it can be obtained from a nondegenerate quadratic form $Q'_{\mathcal{O}_L}$ defined over $\mathcal{O}_L$ by base changing from $\mathcal{O}_L$ to L. Similarly $\mathrm{Spin}(Q'_L)$ can be obtained from a smooth[7] reductive group scheme $\mathcal{G} = \mathrm{Spin}(Q'_{\mathcal{O}_L})$ defined over $\mathcal{O}_L$ via the base change $\mathcal{O}_L \to L$.

Set $E := L(\sqrt{-\theta})$ if $\dim Q' \cong 2 \mod 4$ and $E := L(\sqrt{\theta})$ if $\dim Q' \cong 0 \mod 4$. In either case, E is a totally ramified extension of L with residue field ℓ. Since $i(E)(\theta) = [1]$, it is clear that u' is the image of u in $\mathrm{H}^1(E,\mu')$. Since $\alpha(L)(u) = 1$, we have $\alpha(E)(u') = 1$.

$$\begin{tikzcd}
u \in \mathrm{H}^1(L,\mu') \arrow[r, "\alpha(L)"] \arrow[d] & 1 \in \mathrm{H}^1(L,\mathrm{Spin}(Q')) \arrow[d] \\
u' \in \mathrm{H}^1(E,\mu') \arrow[r, "\alpha(E)"] & 1 \in \mathrm{H}^1(E,\mathrm{Spin}(Q'))
\end{tikzcd}$$

Since u' is defined over $\mathcal{O}_L$ and hence $\mathcal{O}_E$ and the kernel of the natural map $\mathrm{H}^1(\mathcal{O}_E,\mathcal{G}) \to \mathrm{H}^1(E,\mathrm{Spin}(Q'))$ is trivial ([Ni84]), we see that $u' \in \mathrm{Ker}\left(\alpha(\mathcal{O}_E) : \mathrm{H}^1(\mathcal{O}_E,\mu') \to \mathrm{H}^1(\mathcal{O}_E,\mathcal{G})\right)$.

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[5] $[f^{-2}z_1\theta^{\epsilon_1-2\epsilon}, f^{-2}z_2\theta^{2\epsilon-\epsilon_1}] = [f^{-2}z_1\theta^{\epsilon_1-2\epsilon}, 1]\psi[f^{-2}z_1\theta^{\epsilon_1-2\epsilon}, 1]^{-1}$ and $[\lambda] = [f^{-2}z_1\theta^{\epsilon_1-2\epsilon}]$.

[6] $[\lambda] = [z_1z_2\theta^{\epsilon+\epsilon_1}]$ when $Z' \simeq L \times L$.

[7] This is because the characteristic of the residue field of L is not 2.

$$\begin{tikzcd}
u' \in \mathrm{H}^1(\mathcal{O}_E, \mu') \arrow[r, "\alpha(\mathcal{O}_E)"] \arrow[d] & \mathrm{H}^1(\mathcal{O}_E, \mathcal{G}) \arrow[d] \\
u' \in \mathrm{H}^1(E, \mu') \arrow[r, "\alpha(E)"] & 1 \in \mathrm{H}^1(E, \mathrm{Spin}(Q'))
\end{tikzcd}$$

Specializing to the residue field of E, we see that $\overline{u'} \in \mathrm{Ker}\big(\alpha(\ell) : \mathrm{H}^1(\ell, \mu') \to \mathrm{H}^1(\ell, \overline{\mathcal{G}})\big)$. By Hensel's lemma, this implies $u' \in \mathrm{Ker}\big(\alpha(\mathcal{O}_L) : \mathrm{H}^1(\mathcal{O}_L, \mu') \to \mathrm{H}^1(\mathcal{O}_L, \mathcal{G})\big)$ which shows that $\alpha(L)(u') = 1$. $\square$

5.2. **Proof of the Theorem 5.1.** We proceed by induction on $\dim Q$. For the base case when $\dim Q = 2$, the norm principle holds for $\Omega(Q)$ because it is a commutative group. Assume that the norm principle holds for $\Omega(\tilde{Q})$ over L/K for all even dimensional quadratic forms $\tilde{Q}/K$ with $\dim \tilde{Q} < \dim Q$. We break up the proof into separate cases depending on the ramification of L/K.

Case I: L/K is unramified. Recall that we have reduced to the case where $\lambda \in \mathcal{O}_L^*$ up to squares. Using the second residue map again, we see that $[\overline{p}_\ell] = \delta_{2,t}(Q_L) = \delta_{2,t}(\lambda Q_L) = [\overline{\lambda p}_\ell] \in W(\ell)$. Similarly $\overline{q}_\ell \simeq \overline{\lambda q}_\ell$. Thus $\lambda \in \mathrm{G}(q_L) \cap \mathrm{G}(p_L)$, where recall $\mathrm{G}(f) = \{\lambda \in F^* \mid \lambda f \simeq f\}$ denotes the group of similarities of a quadratic form f defined over a field F.

Note that if $\dim q$ is odd, since $[\langle 1, -\lambda\rangle_L \otimes q_L] = 0 \in W(L)$, then by ([Sch85], Theorem 10.13), we have $\lambda \in L^{*2}$; i.e., $[\lambda] = 1 \in \mathrm{H}^1(L, \mu_2)$. Thus by Lemma 2.3, the norm principle holds.

So we assume that $\dim q$ is even. Therefore $\dim p$ is even too. Set $f_u = q$, $g_u = tp$, $R_u := \mathrm{O}^+(f_u) \times \mathrm{O}^+(g_u) \subset \mathrm{O}^+(Q)$ and $\tilde{R}_u$, the preimage of R_u under the canonical homomorphism $\mathrm{Spin}(Q) \to \mathrm{O}^+(Q)$. By Theorem 3.2, it suffices to show that the norm principle holds for $\Omega(f_u)$ and $\Omega(g_u)$ over L/K, which holds by induction if $\dim q, \dim p \neq 0$.

Without loss of generality,[8] suppose that $\dim p = 0$ and $Q_K \simeq q_K$. Since Q is unramified over K and hence Q_L is unramified over L, using Lemmata 5.3 and 5.4, we find $u' \in \mathrm{H}^1(\mathcal{O}_L, \mu)$ such that $u = u'\left(i(L)[t^{\epsilon'}]\right)$ for some $\epsilon' \in \mathbb{Z}$ and $\alpha(L)(u') = 1$. The proof of Lemma 5.4 in fact shows that we can specialize to the residue field and get $\overline{u'}$ in the kernel of $\alpha(\ell) : \mathrm{H}^1(\ell, \mu) \to \mathrm{H}^1(\ell, \mathrm{Spin}(\overline{Q_L}))$. Since the norm principle holds for α for the quadratic form $\overline{Q}$ over ℓ/k by assumption and Lemma 2.2, $\mathrm{N}_{\ell/k}(\overline{u'})$ is in the kernel of $\alpha(k) : \mathrm{H}^1(k, \mu) \to \mathrm{H}^1(k, \mathrm{Spin}(\overline{Q})$. Thus by Hensel's lemma, $\alpha(K)\left(\mathrm{N}_{L/K}(u')\right) = 1$. Lemma 2.3 implies that $\mathrm{N}_{L/K}(i(L)[t^{\epsilon'}]) \in \mathrm{Ker}\,(\alpha(K))$. Thus $\mathrm{N}_{L/K}(u) \in \mathrm{Ker}\,(\alpha(K))$.

Case II: L/K is ramified. Recall once again that we have reduced to the case where $\lambda \in \mathcal{O}_L^*$ up to squares. Since L/K is (totally) ramified, both fields L and K have the same residue field. Hence by Hensel's lemma, it follows that for every unit λ in $\mathcal{O}_L^*$, its class $\lambda \mathcal{O}_L^{*2}$ has a representative in $\mathcal{O}_K^*$. Thus we can and do assume in fact that $\lambda \in \mathcal{O}_K^*$. Further we can also assume $\lambda \notin L^{*2}$ as otherwise, we would be done by Lemma 2.3.

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[8] If $\dim q = 0$, Q is similar to the unramified form p over K and $\Omega(Q) \simeq \Omega(p)$. The same proof works in this case.

Subcase IIa. We first look at the situation when $\dim p \neq \dim q$. Then the following Lemma in conjunction with Theorem 3.2 and our induction hypothesis finishes the proof in this case.

Lemma 5.5. *Suppose that* $\dim p \neq \dim q$. *Then* $q \otimes \langle 1, -\lambda\rangle$ *or* $p \otimes \langle 1, -\lambda\rangle$ *is isotropic over* K. *Further,* $Q_K \simeq f_K \perp g_K$ *for* f, g *even dimensional quadratic forms over* K *with* $\lambda f_L \simeq f_L$ *and* $\lambda g_L \simeq g_L$.

Proof. Since λ is a multiplier for the form $q \perp cp$, we have $[\langle 1, -\lambda\rangle_L \otimes q \perp \langle 1, -\lambda\rangle_L \otimes cp_L] = 0 \in W(L)$. Without loss of generality, assume $\dim q > \dim p$. Hence $q \otimes \langle 1, -\lambda\rangle$ is isotropic over L.

Recall that $q \simeq \langle a_1, a_2, \ldots, a_{\dim q}\rangle$ for $a_i \in \mathcal{O}_K^*$ and $\lambda \in \mathcal{O}_K^*$. Thus $q \otimes \langle 1, -\lambda\rangle \simeq \langle a'_1, a'_2, \ldots, a'_{2\dim q}\rangle$ for some $a'_i \in \mathcal{O}_K*$. Hence there exist $w_i \in \mathcal{O}_L^* \cup \{0\}$ not all 0 and $t_i \in \mathbb{Z}$ such that $\sum a'_i w_i^2 \theta^{2t_i} = 0$, where θ is a parameter of L. By cancelling factors of θ^2 if necessary, we can assume that each $t_j \geq 0$ and at least one $t_i = 0$ (with corresponding $w_i \neq 0$). That is, $\overline{q} \otimes \langle 1, -\overline{\lambda}\rangle$ is isotropic over k. By Hensel's lemma, $q \otimes \langle 1, -\lambda\rangle$ is isotropic over K.

Since we have assumed q is anisotropic over K, this implies $q(u) = \lambda q(v)$ for $u, v \in K^{\dim q}$ and $q(u), q(v) \neq 0$. Note that if u, v are linearly dependent over K, then $\lambda \in K^{*2}$ contradicts our assumption that λ is not a square. Thus u, v span a two-dimensional K-vector space W. Let f be the K-quadratic form q restricted to W.

Let $q(v) = a, q(u) = \lambda a$ and $b_q(u, v) = x$, where b_q denotes the bilinear form corresponding to q. Then for the orthogonal basis $\{v, u - \frac{x}{a}v\}$, $f_K \simeq \langle a, \lambda a - \frac{x^2}{a}\rangle_K$ and for the orthogonal basis $\{u, v - \frac{x}{\lambda a}u\}$, $f_K \simeq \langle \lambda a, a - \frac{x^2}{\lambda a}\rangle_K$. Thus $\lambda f_L \simeq f_L$ and hence the lemma follows. ☐

Subcase IIb: Assume now that $\dim p = \dim q$. Since Q_L is unramified over L, use Lemmata 5.3 and 5.4 to find $u' \in \mathrm{H}^1(\mathcal{O}_L, \mu)$ such that $u = u'\left(i(L)[\theta^{\epsilon'}]\right)$ for some $\epsilon' \in \mathbb{Z}$ and $\alpha(L)(u') = 1$.

Lemma 5.6. $\mathrm{N}_{L/K}(u') = 1 \in \mathrm{H}^1(K, \mu)$.

Proof. We only give the proof in the case Z_L is a field. The proof when $Z_L \simeq L \times L$ is similar.

If $\dim q = \dim p$ is odd, then the discriminant extension Z of $Q = q \perp tp$ is a totally ramified quadratic extension of K, and $Z_L := Z \otimes_K L$ is an unramified (possibly split) quadratic extension of L. Thus θ is still a parameter of Z_L. Further, if the residue field of $Z_L = \ell'$, then the norm maps $\mathrm{N}_{Z_L/L} : Z_L^* \to L^*$ and $\mathrm{N}_{Z_L/Z} : Z_L^* \to Z^*$ induce the same map $\mathrm{N}_{\ell'/k} : \ell'^* \to k^*$ at the residue field level.

$$\begin{tikzcd}
L \arrow[r, "\text{unram}"] & Z_L \\
K \arrow[u, "\text{ram}"] \arrow[r, "\text{ram}"] & Z \arrow[u, "\text{unram}"]
\end{tikzcd}$$

Recall that $\mathrm{H}^1(L, \mu) = \frac{U(L)}{U_0(L)}$ and $u' = [f, z]$ for some $f \in \mathcal{O}_L^*$ and $z \in \mathcal{O}_{Z_L}^*$ with $\mathrm{N}_{Z_L/L}(z) = f^4$. Thus we have $\overline{\mathrm{N}_{Z_L/Z}(z)} = \mathrm{N}_{\ell'/k}(\overline{z}) = \overline{f}^4$. By Hensel's lemma, there exist $\tilde{f} \in \mathcal{O}_K^*$ and $x \cong 1 \mod (m_{\mathcal{O}_L})$ in $\mathcal{O}_L^*$ such that $f = x\tilde{f}$. Similarly, there exist $y, y' \cong 1 \mod (m_{\mathcal{O}_Z})$ in $\mathcal{O}_Z^*$ such that $\mathrm{N}_{Z_L/Z}(z) = \tilde{f}^4 y' = \tilde{f}^4 y^4$.

Now $\mathrm{N}_{L/K}(u') = [\mathrm{N}_{L/K}(f), \mathrm{N}_{Z_L/Z}(z)] = [\tilde{f}^2\, \mathrm{N}_{L/K}(x), \tilde{f}^4 y^4] = [\tilde{f}^2, \tilde{f}^4][\mathrm{N}_{L/K}(x), y^4] \in \mathrm{H}^1(K,\mu)$ where $\mathrm{N}_{Z/K}(y^4) = \mathrm{N}_{L/K}(x)^4$. Setting $a = \mathrm{N}_{Z/K}(y)\, \mathrm{N}_{L/K}(x)^{-1}$, we see that $a^4 = 1$ and $a \cong 1 \mod m_{\mathcal{O}_K}$. By Hensel's lemma yet again, $a = 1$ and hence $\mathrm{N}_{Z/K}(y) = \mathrm{N}_{L/K}(x)$. Since $[\mathrm{N}(b), b^4] = 1 \in \mathrm{H}^1(K,\mu)$ for every $b \in K^*$, we see that $\mathrm{N}_{L/K}(u') = 1$.

If $\dim q = \dim p$ is even, then the discriminant extension Z of $Q = q \perp tp$ is an unramified (possibly split) quadratic extension of K as also $Z_L/L := Z \otimes_K L/L$. Thus θ is still a parameter of Z_L. Note that $Z_L \simeq L \times L$ if and only if $Z \simeq K \times K$.

$$\begin{tikzcd} L \arrow[r, "\text{unram}"] & Z_L \\ K \arrow[u, "\text{ram}"] \arrow[r, "\text{unram}"] & Z \arrow[u, "\text{ram}"'] \end{tikzcd}$$

Recall that $\mathrm{H}^1(L,\mu) = Z_L^*/Z_L^{*2}$ and $u' = [z]$ in $\mathrm{H}^1(L,\mu)$ for some $z \in \mathcal{O}_{Z_L}^*$ if Z_L is a field (resp., a split extension). Since norms of units of totally ramified quadratic extensions are squares and Z_L/Z is ramified, $\mathrm{N}_{L/K}(u') = 1 \in \mathrm{H}^1(K,\mu)$. □

Lemma 2.3 implies that $\mathrm{N}_{L/K}(i(L)[\theta^{\epsilon'}]) \in \mathrm{Ker}\,(\alpha(K))$. Since $\mathrm{N}_{L/K}(u) = \mathrm{N}_{L/K}(u')\, \mathrm{N}_{L/K}\left(\left(i(L)[\theta^{\epsilon'}]\right)\right)$, we have $\mathrm{N}_{L/K}(u) \in \mathrm{Ker}\,(\alpha(K))$ which concludes the proof of Theorem 5.1.

6. Examples

Let G be a semisimple simply connected linear algebraic group defined over a field k. Then $\mathrm{H}^1(k, G)$ is trivial if k is a p-adic field ([K65]) or a global field of positive characteristic ([H75]). More generally, suppose that $\mathrm{char}(k) \neq 2$ and $\mathrm{cd}(k) \leq 2$. Then Bayer–Parimala's proof of Serre's conjecture II shows that $\mathrm{H}^1(k, G)$ is trivial if G is further assumed to be of classical type ([BP95]). Set $G := \mathrm{Spin}(q)$, where q is any even-dimensional nondegenerate quadratic form over k, and let μ be its center. It is immediate therefore that the norm principle holds for the map $\alpha : \mathrm{H}^1(-,\mu) \to \mathrm{H}^1(-, \mathrm{Spin}(q))$ and hence that it holds for the group $\Omega(q)$ defined over k.

Now let G be a semisimple adjoint linear algebraic group of classical type defined over a number field k. Then, $G(k)/R$, the group of R-equivalence classes of the k-points of G, is trivial ([Gi97] Corollaire III.4.2 and [KP08], p. 1). Set $G := \mathrm{PGO}^+(q)$, where q is any even dimensional nondegenerate quadratic form over k, and let μ be the center of $\mathrm{Spin}(q)$. It follows from Theorem 4.1 that the norm principle holds for the map $S : \mathrm{PGO}^+(q)(-) \to \mathrm{H}^1(-,\mu)$ and hence that it holds for the group $\Omega(q)$ defined over k.

Recall that a field k is said to have *virtual cohomological dimension* (vcd) $\leq n$ if the cohomological dimension of $k\left(\sqrt{-1}\right)$ is $\leq n$. Examples of vcd ≤ 2 fields include cd ≤ 2 fields and number fields. We begin by showing the following:

Lemma 6.1. *Let F be a real closed field and q' a nondegenerate even dimensional quadratic form over F. Then $\mathrm{PGO}^+(q')(F)/R$ is trivial.*

Proof. Without loss of generality we can assume that q' is anisotropic over F. Since for every $a \in F$, either a or $-a$ is a square in F, we can further assume that $q' \simeq \langle 1, \ldots, 1\rangle$. Then the variety of $\mathrm{PGO}^+(q')$ is stably rational over F ([Ch94]), whence the claim. □

Proposition 6.2. *Let k be a field with* $\mathrm{vcd}(k) \le 2$ *and q a nondegenerate even dimensional quadratic form over k. Then the norm principle holds for $\Omega(q)$.*

Proof. If $\mathrm{cd}(k) \le 2$, the discussion above already gives the proof. Hence we can assume that $\mathrm{cd}(k) \neq \mathrm{vcd}(k)$ and hence that $\mathrm{char}(k) = 0$ and k can be ordered (Theorem 1.1, [BP98]).

Let ℓ/k be a finite separable extension, μ, the center of $\mathrm{Spin}(q)$, and let $\alpha(-) : \mathrm{H}^1(-, \mu) \to \mathrm{H}^1(-, \mathrm{Spin}(q))$ be the natural map between the Galois cohomology sets. Let $\xi \in \mathrm{Ker}\,(\alpha(\ell))$ and set $\eta := N_{\ell/k}(\xi)$. We would like to show that $\alpha(k)\,(\eta)$ is trivial in $\mathrm{H}^1(k, \mathrm{Spin}(q))$.

Let Ω denote the set of all orderings v of k and k_v, the real closure of k at v. Then the Hasse principle result of Bayer–Parimala over perfect fields of vcd ≤ 2 for semisimple simply connected groups of classical type ([BP98]) gives in particular that the natural map $\mathrm{H}^1(k, \mathrm{Spin}(q)) \to \prod_{v\in\Omega} \mathrm{H}^1(k_v, \mathrm{Spin}(q))$ has trivial kernel.

Thus, it suffices to show that the image of η in $H^1(k_v, \mathrm{Spin}(q))$ is trivial for each $v \in \Omega$. Note that $\mathrm{Res}_{k_v}(\eta) = N_{\ell\otimes_k k_v/k_v}(\xi)$. By Lemma 6.1, $\mathrm{PGO}^+(q)(\ell \otimes_k k_v)/R$ is trivial and hence as the norm principle holds for R-trivial elements (Theorem 4.1), we can conclude that the image of $\mathrm{Res}_{k_v}(\eta)$ in $H^1(k_v, \mathrm{Spin}(q))$ is trivial. □

Since the virtual cohomological dimension behaves well with respect to finite extensions, Proposition 6.2, in conjunction with Theorem 5.1, immediately yields the following:

Corollary 6.3. *Let K be a complete discretely valued field with residue field k such that* $\mathrm{char}(k) \neq 2$ *and* $\mathrm{vcd}(k) \le 2$. *Then the norm principle holds for $\Omega(Q)$ for every even dimensional nondegenerate quadratic form Q/K.*

7. On the triviality of the Tate–Shafarevich set

Let G be a semisimple algebraic group over a number field K and V^K, the set of all places of K. One of the main finiteness results in the arithmetic theory of linear algebraic groups states that the natural global-to-local map

$$\rho_G : \mathrm{H}^1(K, G) \to \prod_{v\in V^K} \mathrm{H}^1(K_v, G)$$

is *proper*; i.e., the preimage of any finite set is finite; in particular, the corresponding Tate–Shafarevich set $\text{Ш}(G) := \mathrm{Ker}\ \rho_G$ is finite. Moreover, if in addition G is simply connected, then $\text{Ш}(G) = 1$; i.e., ρ_G is injective.

A natural question to ask is if, and to what extent, the above finiteness property can be extended to fields other than number fields. More precisely, let K be a finitely generated field. Can one equip K with a "natural" set V of discrete valuations such that for a given absolutely almost simple K-group G, the natural global-to-local map relative to V

$$\rho_{G,V} : \mathrm{H}^1(K, G) \to \prod_{v\in V} \mathrm{H}^1(K_v, G)$$

is proper? If the answer is affirmative, is it true that for a simply connected group G the kernel $\text{Ш}_V(G)$ of $\rho_{G,V}$ is trivial?

A natural candidate for such a V appears to be the set of discrete valuations associated with the prime divisors of a model of K; i.e., a smooth affine arithmetic scheme with function field K (we call such sets *divisorial*). It is known that divisorial sets V indeed work for adjoint inner forms of type A_ℓ; i.e. for $G = \mathrm{PGL}_{\ell+1}$, provided that char K does not divide $\ell+1$ (cf. [CRR13]); this relies on the finiteness of the unramified Brauer group ${}_{(\ell+1)}\operatorname{Br}(K)_V$ ([CRR16]). Until recently no other types have been considered.

Relating the norm principle and the Tate–Shafarevich set of spinor groups. The first interesting and widely open case is the one where K is the function field of a curve C defined over a number field and $G = \operatorname{Spin}(f)$ is the spinor group of a quadratic form f over K. In a recent paper ([CRR16(2)]) it was proved that for the special orthogonal group $\mathrm{O}^+(f)$ and for a divisorial set V, the global-to-local map $\rho_{\mathrm{O}^+(f),V}$ is proper. This result in conjunction with the exact sequence

$$\mathrm{O}^+(f)(K) \longrightarrow \mathrm{H}^1(K,\mu_2) \longrightarrow \mathrm{H}^1(K,\operatorname{Spin}(f)) \longrightarrow \mathrm{H}^1(K,\mathrm{O}^+(f)),$$

and twisting shows that the Tate–Shafarevich set $Ш_V(\operatorname{Spin}(f))$ is finite if and only if the group $\operatorname{LGC}(g) = \{[a] \in K^\times/\operatorname{Sn}(g) \mid a \text{ is a spinor norm of } g \text{ over } K_v \text{ for all } v \in V\}$ is finite for all quadratic forms g over K. Note that $\operatorname{LGC}(g)$ is a subset of $Ш_V(\operatorname{Spin}(g))$.

Note that the residue field $\kappa(v)$ of any valuation $v \in V$ is either a number field or the function field of a curve over a finite field; i.e., $\operatorname{vcd}(\kappa(v)) \leq 2$. Therefore, by Corollary 6.3, the norm principle for $\Omega(f)$ holds over the completion K_v. Thus by Theorem 2.5, the square diagram (SQ for X/K_v) is commutative for every finite separable extension X/K_v. Let L/K be a finite separable extension. It follows then that the obstruction to the commutativity of the square diagram (SQ for L/K) is a subgroup in $\operatorname{LGC}(f)$ having the property: *it is trivial if and only if the norm principle holds for the extension L/K.* Thus, the failure of the norm principle for $\operatorname{Spin}(f)$ would imply that $Ш_V(\operatorname{Spin}(f))$ is nontrivial.

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