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DECORATED MARKED SURFACES II: INTERSECTION NUMBERS AND DIMENSIONS OF HOMS

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ABSTRACT. We study derived categories arising from quivers with potential associated to a decorated marked surface $\mathbf{S}_\triangle$, in the sense taken in a paper by Qiu. We prove two conjectures from Qiu's paper in which, under a bijection between certain objects in these categories and certain arcs in $\mathbf{S}_\triangle$, the dimensions of morphisms between these objects equal the intersection numbers between the corresponding arcs.

1. INTRODUCTION

1.1. **The 3-Calabi–Yau categories from surfaces.** In this paper, we study a class of derived categories $\mathcal{D}_{fd}(\mathbf{S})$ associated to quivers with potential from triangulated marked surfaces $\mathbf{S}$. They are 3-Calabi–Yau and originally arose in the study of homological mirror symmetry. In type A_n (or equivalently, $\mathbf{S}$, an $(n+3)$-gon), such a category was first studied by Khovanov, Seidel, and Thomas [15, 20]. They showed that there is a faithful (classical) braid group action

$$\mathrm{ST}\,\mathcal{D}_{fd}(\mathbf{S}) \cong B_{n+1} = \mathrm{MCG}(\mathbf{S}, n)$$

on $\mathcal{D}_{fd}(\mathbf{S})$, where $\mathrm{ST}\,\mathcal{D}_{fd}(\mathbf{S})$ is the spherical twist group of $\mathcal{D}_{fd}(\mathbf{S})$ and $\mathrm{MCG}(\mathbf{S}, n)$ the mapping class group of the disk with n decorations. This plays a crucial role in understanding such categories and their spaces of stability conditions. More recently, Bridgeland and Smith [3] established a connection between dynamical systems of $\mathbf{S}$ and theory of stability conditions on $\mathcal{D}_{fd}(\mathbf{S})$. More precisely, they showed that

$$\mathrm{Stab}^\circ\,\mathcal{D}_{fd}(\mathbf{S}) / \mathrm{ST}\,\mathcal{D}_{fd}(\mathbf{S}) \cong \mathrm{Quad}(\mathbf{S}),$$

where $\mathrm{Stab}^\circ\,\mathcal{D}_{fd}(\mathbf{S})$ is the space of stability conditions and $\mathrm{Quad}(\mathbf{S})$ the moduli space of quadratic differentials. Moreover, Smith [21] showed that there is a fully faithful embedding

$$\mathcal{D}_{fd}(\mathbf{S}) \hookrightarrow \mathcal{D}\,\mathrm{Fuk}(\mathbf{X}),$$

where $\mathbf{X}$ is a symplectic 3-fold constructed from $\mathbf{S}$ and $\mathcal{D}\,\mathrm{Fuk}(\mathbf{X})$ its derived Fukaya category. This generalizes the symplectic construction of [15]. In the attempt of showing that $\mathrm{Stab}^\circ\,\mathcal{D}_{fd}(\mathbf{S})$ is the universal cover of $\mathrm{Quad}(\mathbf{S})$, [17] generalizes a result of [15] that $\mathrm{ST}\,\mathcal{D}_{fd}(\mathbf{S})$ can be identified with a subgroup of the mapping class group of $\mathbf{S}_\triangle$, the decorated version of $\mathbf{S}$:

$$\mathrm{ST}\,\mathcal{D}_{fd}(\mathbf{S}) \cong \mathrm{BT}(\mathbf{S}_\triangle) \subset \mathrm{MCG}(\mathbf{S}_\triangle).$$

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Under this embedding, the Homs between objects in $\mathcal{D}_{fd}(\mathbf{S})$ are in fact floer homology between the corresponding Lagrangian submanifolds. A (double graded) formula for calculating the dimension of Homs/floer homology (in type A or $\mathbf{S}$, a polygon) is given in [15]. Naturally, one would expect the corresponding formula for unpunctured $\mathbf{S}$, which is conjectured in [17] (an ungraded version). The main motivation of this paper is to prove such a conjecture and another closely related formula.

1.2. **Topological aspect of cluster theory.** Cluster algebras and quiver mutation were introduced by Fomin and Zelevinsky [9]. Derksen, Weyman, and Zelevinsky [6] further developed mutation of quivers with potential. During the last decade, the cluster phenomenon was spotted in various areas in mathematics, as well as in physics, including geometric topology and representation theory.

On the one hand, the geometric aspect of cluster theory was explored by Fomin, Shapiro, and Thurston [8]. They constructed a quiver $Q_{\mathbf{T}}$ (and later Labardini-Fragoso [16] gave a corresponding potential $W_{\mathbf{T}}$) from any (tagged) triangulation $\mathbf{T}$ of a marked surface $\mathbf{S}$. Moreover, they showed that mutation of quivers (with potential) is compatible with flip of triangulations. On the other hand, the categorification of cluster algebras leads to representations of quivers due to Buan et al. [5]. Later, Amiot [2] introduced generalized cluster categories via Ginzburg dg algebras associated to quivers with potential, where $\mathcal{D}_{fd}(\mathbf{S})$ fits into the following short exact sequence of triangulated categories:

$$0 \to \mathcal{D}_{fd}(\mathbf{S}) \to \operatorname{per}\mathbf{S} \to \mathcal{C}(\mathbf{S}) \to 0, \tag{1.1}$$

where $\mathcal{C}(\mathbf{S})$ is the generalized cluster category.

Several papers have been published concerning these categories associated to marked surfaces. Namely, the following correspondences were established (cf. Table 1):

- In the unpunctured case, Brüstle and Zhang [4] constructed a bijection between the set of open arcs on $\mathbf{S}$ and the set of indecomposables in the cluster category $\mathcal{C}(\mathbf{S})$.
- Qiu and Zhou [19] constructed such a bijection in the general case (i.e., with punctures).
- Qiu [17] constructed a bijection between the set of (simple) closed arcs on $\mathbf{S}_\triangle$ (the decorated version of $\mathbf{S}$) and the set of shift orbits of (reachable) spherical objects in $\mathcal{D}_{fd}(\mathbf{S})$.
- Qiu [18] constructed a bijection between the set of (simple) open arcs on $\mathbf{S}_\triangle$ and the set of (reachable) rigid indecomposable objects in $\mathcal{D}_{fd}(\mathbf{S})$.

Furthermore, there are several $\operatorname{Int} = \dim\operatorname{Hom}$ type formulae under these types of correspondences:

(1) Khovanov and Seidel [15] showed $\dim\operatorname{Hom}^{\mathbb{Z}}(\widetilde{X}_{\eta_1}, \widetilde{X}_{\eta_2}) = 2\operatorname{Int}(\eta_1, \eta_2)$ in the case of $\mathbf{S}$ of type A.
(2) Gadbled, Thiel, and Wagner [10] showed that the formula in (1) holds for $\mathbf{S}$ of extended affine type A.
(3) Zhang, Zhou, and Zhu [22] showed that $\dim\operatorname{Hom}^1(M_{\gamma_1}, M_{\gamma_2}) = \operatorname{Int}(\gamma_1, \gamma_2)$ for $\mathbf{S}$ unpunctured.
(4) Qiu and Zhou [19] showed that the formula in (3) holds for the punctured case.

TABLE 1. Topological model for categories associated to quivers with potential

Topological model	Correspondence	Categories
$\mathbf{S}_\triangle$ Closed arcs	$\widetilde{X}\colon \eta \xrightarrow{[17]} X_\eta[\mathbb{Z}]$	$\mathcal{D}_{fd}(\mathbf{S})$: 3-CY reachable spherical objects
$\mathbf{S}_\triangle$ Open arcs	$\widetilde{M}\colon \gamma \xrightarrow{[18]} \widetilde{M}_\gamma$	$\operatorname{per}\mathbf{S}$ reachable ind. objects
$\mathbf{S}$ Open arcs	$M\colon \mu \xrightarrow{[4,19]} M_\mu$	$\mathcal{C}(\mathbf{S})$: 3-CY ind. objects

Here $\operatorname{Hom}^{\mathbb{Z}}(-,-)$ denotes $\oplus_{m\in\mathbb{Z}}\operatorname{Hom}(-,-[m])$, which can be defined for shift orbits of objects. Note that the differences between $\operatorname{Hom}^{\mathbb{Z}}$ and Hom^1 reflect the different properties that $\mathcal{D}_{fd}(\mathbf{S})$ is 3-Calabi–Yau, while $\mathcal{C}(\mathbf{S})$ is 2-Calabi–Yau. The main techniques for proving these Int = dim Hom type formulae are the string models (or its variation/generalization) in representation theory. In this paper, we establish some framework of graded string models for certain 3-Calabi–Yau categories and prove the following two formulae (Theorems 4.5 and 4.9) of this type.

Theorem ([17, Conjectures 10.5 and 10.6])**.** *Under the correspondence $\widetilde{M},\widetilde{X}$ in Table* 1*, the following formulae hold:*

$$\dim\operatorname{Hom}^{\mathbb{Z}}(\widetilde{X}_{\eta_1},\widetilde{X}_{\eta_2}) = 2\operatorname{Int}(\eta_1,\eta_2),$$
$$\dim\operatorname{Hom}^{\mathbb{Z}}(\widetilde{M}_\gamma,\widetilde{X}_\eta) = \operatorname{Int}(\gamma,\eta).$$

1.3. **Context.** The paper is organized as follows. In Section 2, we review background materials. In Section 3, we prove the first formula for spherical objects under Assumption 3.2. In Section 4, we show that one can identify all sets of reachable spherical objects from different triangulations in a canonical way. This enables us to generalize the first formula to all cases, and we also prove the second formula as a byproduct. In the Appendix, we develop the graded string model, which is independent from the rest of the paper. The key result here are the calculations of a type of morphisms between (spherical) objects and the compositions of these morphisms. This appendix also serves as a technical section for the prequel [17].

2. PRELIMINARIES

2.1. **Triangulated 3-Calabi–Yau categories and spherical twists.** Fix an algebraically closed field $\mathbf{k}$ and all categories are $\mathbf{k}$-linear. A triangulated category $\mathcal{T}$ is called 3-Calabi–Yau if for any objects $X,Y\in\mathcal{T}$, there is a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(X,Y)\cong D\operatorname{Hom}_{\mathcal{T}}(Y,X[3])$$

where $D=\operatorname{Hom}_{\mathbf{k}}(-,\mathbf{k})$ is the $\mathbf{k}$-duality. An (indecomposable) object S in a triangulated 3-Calabi–Yau category $\mathcal{T}$ is called (3-)spherical if $\operatorname{Hom}_{\mathcal{T}}(S,S[n])$ equals $\mathbf{k}$ if $n=0$ or 3 and equals 0 otherwise. Recall from [20] that the twist functor of a spherical object S is defined by

$$\phi_S(X)=\operatorname{Cone}\left(S\otimes\operatorname{Hom}^{\mathbb{Z}}_{\mathcal{T}}(S,X)\to X\right),$$

with inverse

$$\phi_S^{-1}(X) = \mathrm{Cone}\left(X \to S \otimes \mathrm{Hom}^{\mathbb{Z}}_{\mathcal{T}}(X, S)\right)[-1].$$

2.2. **Quivers with potential, Ginzburg dg algebras, and associated categories.** A quiver with potential [6] is a pair (Q, W), where Q is a finite quiver and W is a linear combination of cycles in Q. We assume that Q does not have loops or 2-cycles. The Ginzburg dg algebra [7] $\Gamma = \Gamma(Q, W)$ associated to (Q, W) is defined as follows. Let $\overline{Q}$ be the graded quiver with the same set of vertices as Q and whose arrows are

- the arrows of Q with degree 0,
- an arrow $a^* : j \to i$ with degree -1 for each arrow $a : i \to j$ of Q,
- a loop $t_i : i \to i$ with degree -2 for each vertex i of Q.

The underlying graded algebra of Γ is the completion of the graded path algebra $\mathbf{k}\overline{Q}$ and the differential of Γ is determined uniquely by the following:

- $d(a) = 0$ and $d(a^*) = \partial_a W$ for a as an arrow of Q,
- $\sum_{i \in Q_0} d(t_i) = \sum_{a \in Q_1} [a, a^*]$.

Let $\mathcal{D}(\Gamma)$ be the derived category of Γ. We consider the following full subcategories of $\mathcal{D}(\Gamma)$:

- $\mathrm{per}\,\Gamma$: the perfect derived category of Γ,
- $\mathcal{D}_{fd}(\Gamma)$: the finite dimensional derived category of Γ.

It is known that $\mathrm{per}\,\Gamma$ is Krull–Schmidt [13] and $\mathcal{D}_{fd}(\Gamma)$ is 3-Calabi–Yau [11]. Let $\mathcal{H}_\Gamma$ be the canonical heart of $\mathcal{D}_{fd}(\Gamma)$ and $\mathrm{Sim}\,\mathcal{H}_\Gamma$ be the set of iso-classes of simple objects in $\mathcal{H}_\Gamma$. As in [17], we use the following notations:

- $\mathrm{ST}(\Gamma)$: the spherical twist group, which is the subgroup of $\mathrm{Aut}\,\mathcal{D}_{fd}(\Gamma)$ generated by ϕ_S for $S \in \mathrm{Sim}\,\mathcal{H}_\Gamma$;
- $\mathrm{Sph}(\Gamma)$: the set of reachable spherical objects in $\mathcal{D}_{fd}(\Gamma)$, i.e., $\mathrm{ST}(\Gamma) \cdot \mathrm{Sim}\,\mathcal{H}_\Gamma$.

Here all simple objects in $\mathcal{H}_\Gamma$ are spherical because the quiver Q has no loops; see [13, Lemma 2.15 or Theorem 6.2].

2.3. **Triangulations of marked surfaces.** An (unpunctured) marked surface $\mathbf{S}$ is an oriented compact surface with a finite set $\mathbf{M}$ of marked points lying on its nonempty boundary $\partial\mathbf{S}$ [8]. Up to homeomorphism, a marked surface $\mathbf{S}$ is determined by the following data:

- the genus g of $\mathbf{S}$,
- the number b of components of $\partial\mathbf{S}$,
- the partition of the number $m = |\mathbf{M}|$ describing the numbers of marked points on components of $\partial\mathbf{S}$.

An (open) arc γ in $\mathbf{S}$ is a curve in the surface satisfying the following:

- the endpoints of γ are in $\mathbf{M}$;
- except for its endpoints, γ is disjoint from $\partial\mathbf{S}$;
- γ has no self-intersections in $\mathbf{S} - \mathbf{M}$;
- γ is not isotopic to a point or a boundary segment.

The arcs are always considered up to isotopy. A triangulation $\mathbb{T}$ of $\mathbf{S}$ is a maximal collection of arcs in $\mathbf{S}$ which do not intersect each other in the interior of $\mathbf{S}$. We have

$$n := |\mathbb{T}| = 6g + 3b + m - 6,$$

and the number $\aleph$ of the triangles in $\mathbb{T}$ is $(2n + m)/3$.

There is a quiver with potential $(Q_{\mathbb{T}}, W_{\mathbb{T}})$ [8,16], associated to each triangulation $\mathbb{T}$ of $\mathbf{S}$ as follows:

- the vertices of $Q_{\mathbb{T}}$ are indexed by the arcs in $\mathbb{T}$;
- there is an arrow $i \to j$ whenever i and j are edges of the same triangle and j follows i clockwise; hence each triangle with three edges in $\mathbb{T}$ gives a 3-cycle (up to cyclic permutation);
- the potential $W_{\mathbb{T}}$ is the sum of such 3-cycles.

2.4. **Decorated marked surfaces.** A decorated marked surface $\mathbf{S}_\triangle$ is a marked surface with an extra set $\triangle$ of $\aleph$ decorating points in the interior of $\mathbf{S}$. A general closed arc η in $\mathbf{S}_\triangle$ is a curve in $\mathbf{S}$ such that

- its endpoints are in $\triangle$;
- except for its endpoints, γ is disjoint from $\triangle$ and from $\partial\mathbf{S}$;
- it is not isotopic to a point.

A closed arc in $\mathbf{S}_\triangle$ is a general closed arc whose endpoints do not coincide. An open arc γ in $\mathbf{S}_\triangle$ is a curve in $\mathbf{S}$ such that

- its endpoints are in $\mathbf{M}$;
- except for its endpoints, γ is disjoint from $\triangle$ and from $\partial\mathbf{S}$;
- it is not isotopic to a point or a boundary component.

We denote by $\mathrm{CA}(\mathbf{S}_\triangle)$, $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$, and $\mathrm{OA}(\mathbf{S}_\triangle)$ the set of simple closed, simple general closed, and simple open arcs in $\mathbf{S}_\triangle$, respectively. Recall from [17, Section 3.1] the notion of intersection numbers as follows.

- For an open arc γ and an (open or general closed) arc η, their intersection number is defined as the geometric intersection number in $\mathbf{S}_\triangle - \mathbf{M}$:
$$\mathrm{Int}(\gamma, \eta) = \min\{|\gamma' \cap \eta' \cap (\mathbf{S}_\triangle - \mathbf{M})| \mid \gamma' \sim \gamma, \eta' \sim \eta\}.$$
- For two general closed arcs α, β in $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$, their intersection number is a half integer in $\frac{1}{2}\mathbb{Z}$ and is defined as follows (following [15]):
$$\mathrm{Int}(\alpha, \beta) = \tfrac{1}{2}\,\mathrm{Int}_\triangle(\alpha, \beta) + \mathrm{Int}_{\mathbf{S}-\triangle}(\alpha, \beta),$$
where
$$\mathrm{Int}_{\mathbf{S}-\triangle}(\alpha, \beta) = \min\{|\alpha' \cap \beta' \cap (\mathbf{S} - \triangle)| \,\Big|\, \alpha' \sim \alpha, \beta' \sim \beta\}$$
and
$$\mathrm{Int}_\triangle(\alpha, \beta) = \sum_{Z \in \triangle} |\{t \mid \alpha(t) = Z\}| \cdot |\{r \mid \beta(r) = Z\}|.$$

A triangulation $\mathbf{T}$ is a maximal collection of open arcs in $\mathbf{S}_\triangle$ such that

- for any $\gamma_1, \gamma_2 \in \mathbf{T}$, $\mathrm{Int}(\gamma_1, \gamma_2) = 0$;
- each triangle of $\mathbf{T}$ contains exactly one (decorating) point in $\triangle$.

The forgetful map $F : \mathbf{S}_\triangle \to \mathbf{S}$ forgetting the decorating points induces a map from $\mathrm{OA}(\mathbf{S}_\triangle)$ to the set of arcs in $\mathbf{S}$, which sends a triangulation $\mathbf{T}$ of $\mathbf{S}_\triangle$ to a triangulation $\mathbb{T} = F(\mathbf{T})$ of $\mathbf{S}$. The quiver with potential $(Q_{\mathbf{T}}, W_{\mathbf{T}})$ associated to $\mathbf{T}$ is defined to be $(Q_{\mathbb{T}}, W_{\mathbb{T}})$.

Let γ be an (open) arc in a triangulation $\mathbf{T}$. The arc $\gamma^\sharp = \gamma^\sharp(\mathbf{T})$ (resp., $\gamma^\flat$) is the arc obtained from γ by moving its endpoints counterclockwise (resp., clockwise) along the quadrilateral in $\mathbf{T}$ whose diagonal is γ, to the next marked points. The forward (resp., backward) flip of a triangulation $\mathbf{T}$ at $\gamma \in \mathbf{T}$ is the triangulation $\mathbf{T}^\sharp_\gamma = \mathbf{T} \cup \{\gamma^\sharp\} - \{\gamma\}$ (resp., $\mathbf{T}^\flat_\gamma = \mathbf{T} \cup \{\gamma^\flat\} - \{\gamma\}$). See Figure 1, for example. The exchange graph $\mathrm{EG}(\mathbf{S}_\triangle)$ is the oriented graph whose vertices are triangulations of $\mathbf{S}_\triangle$ and whose arrows correspond to forward and backward flips. From now on, fix a connected component $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ of $\mathrm{EG}(\mathbf{S}_\triangle)$. When we say a triangulation $\mathbf{T}$ of $\mathbf{S}_\triangle$, we mean $\mathbf{T}$ is in $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$.

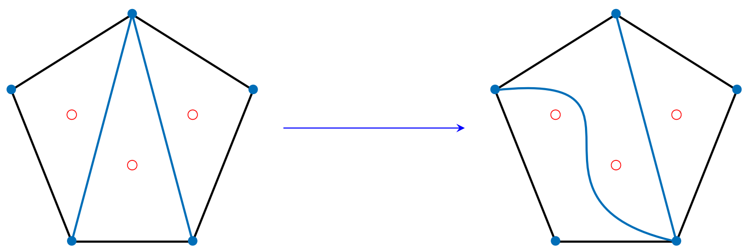

FIGURE 1. A forward flip of a triangulation

2.5. **The braid twists.** The mapping class group $\mathrm{MCG}(\mathbf{S}_\triangle)$ of $\mathbf{S}_\triangle$ consists of the isotopy classes of the homeomorphisms of $\mathbf{S}$ that fix $\partial\mathbf{S}$ pointwise and fix the set $\triangle$.

For any closed arc $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, the braid twist $B_\eta \in \mathrm{MCG}(\mathbf{S}_\triangle)$ along η is defined as in Figure 2. The braid twist group $\mathrm{BT}(\mathbf{S}_\triangle)$ is defined as the subgroup of $\mathrm{MCG}(\mathbf{S}_\triangle)$ generated by B_η for $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$. For a triangulation $\mathbf{T} = \{\gamma_1, \ldots, \gamma_n\}$ of $\mathbf{S}_\triangle$, its dual triangulation $\mathbf{T}^*$ consists of the closed arcs $s_1, \ldots, s_n$ in $\mathrm{CA}(\mathbf{S}_\triangle)$ satisfying that $\mathrm{Int}(\gamma_i, s_j)$ equals 1 for $i = j$ and equals 0 otherwise.

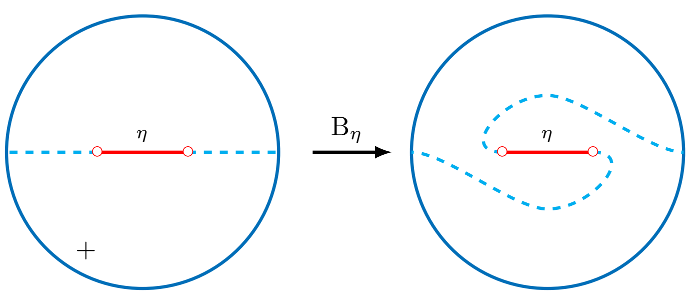

FIGURE 2. The braid twist

2.6. **Topological preparation.** We start with a lemma. Let

$$\mathbf{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

be a disk with three punctures, $P_0 = (0, \frac{1}{2})$, $P_1 = (0, -\frac{1}{2})$, $P_2 = (0, 0)$. Set

$$\mathbf{D}^{>0} = \{(x, y) \in \mathbf{D} \mid y > 0\}$$

and

$$\mathbf{D}^{<0} = \{(x, y) \in \mathbf{D} \mid y < 0\}.$$

In this subsection, when we mention a curve, we always mean a continuous map from $[0, 1]$ to $\mathbf{D}$ such that it is disjoint with the punctures except for its endpoints. Let $\eta : [0, 1] \to \mathbf{D}$ be a curve. Denote by $\overline{\eta}$ the curve defined as $\overline{\eta}(t) = \eta(1-t)$. The restriction $\eta|_{[t_1,t_2]}$ is the curve defined as $\eta|_{[t_1,t_2]}(t) = \eta((t_2 - t_1)t + t_1)$ if $t_1 < t_2$, and $\eta|_{[t_1,t_2]} = \overline{\eta|_{[t_2,t_1]}}$ if $t_1 > t_2$. For any two curves η_1, η_2 with $\eta_1(1) = \eta_2(0) \notin \{P_1, P_2, P_3\}$, their composition $\eta_2\eta_1$ is the curve defined by $\eta_2\eta_1(t) = \eta_1(2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$, and $\eta_2\eta_1(t) = \eta_2(2t - 1)$ for $\frac{1}{2} \leqslant t \leqslant 1$. For a simple curve η whose endpoints coincide, denote by D_η the disk (possibly with punctures) enclosed by η.

Lemma 2.1. *Let $\eta : [0, 1] \to \mathbf{D}$ be a simple curve with $\eta(0) = P_0$ and $\eta(1) = P_1$. Assume that η is in a minimal position w.r.t. $Y = \mathbf{D} \cap \{y = 0\}$, with $\mathrm{Int}(\eta, Y) > 2$. Then there exist two simple curves $\eta_0 \subset \mathbf{D}^{>0}$ and $\eta_1 \subset \mathbf{D}^{<0}$ satisfying the following:*

- *$\eta_0(0) = P_0$, $\eta_0(1) = \eta(s_0)$, $\eta_1(0) = \eta(s_1)$ and $\eta_1(1) = P_1$ for some $0 < s_i < 1$;*
- *$\eta_0 \not\sim \eta|_{[0,s_0]}$ and $\eta_1 \not\sim \eta|_{[s_1,1]}$;*
- *the curves η_0, η_1 intersect η at $\eta(s_0)$, $\eta(s_1)$, respectively, from different sides;*
- *the curve $\alpha_0 := \eta|_{[s_0,1]}\eta_0$ is isotopic to $\alpha_1 := \eta_1\eta|_{[0,s_1]}$ relative to $\{0, 1\}$.*

Proof. Let $\eta \cap Y = \{\eta(r_i) \mid 0 < r_1 < \cdots < r_m < 1\}$. As $m > 2$, we can connect P_0 to a point in a segment $\eta|_{[r_i,r_{i+1}]} \subset \mathbf{D}^{>0}$ for some $i > 0$ without intersecting η except for the endpoints to get an arc η_0. Similarly, we can get an arc η_1. Moreover, $\eta_0 \not\sim \eta|_{[0,s_0]}$ and $\eta_1 \not\sim \eta|_{[s_1,1]}$ since η is in a minimal position w.r.t. Y. Let $c_0 = \overline{\eta_0}\eta|_{[0,s_0]}$ and $c_1 = \eta|_{[s_1,1]}\overline{\eta_1}$. It follows that the corresponding disks D_{c_i} are not contractible and hence contain at least one puncture. Now we claim that $D_{c_0} \subset D_{c_1}$ or $D_{c_1} \subset D_{c_0}$. Otherwise, they are disjoint since c_0 and c_1 do not intersect transversely. So D_{c_i} does not contain P_0, P_1; hence both D_{c_0} and D_{c_1} have to contain P_2. This contradicts the fact that they are disjoint.

Without loss of generality, we assume that $D_{c_0} \subset D_{c_1}$ and η_0 intersects η at $\eta(s_0)$ from the left side. Then up to isotopy, there are three cases, as shown in Figure 3. In the first two cases, η_1 intersects η at $\eta(s_1)$ from the right side and the disk $D_{\overline{\alpha_1}\alpha_0}$ contains no punctures. Hence $\alpha_0 \sim \alpha_1$ relative to $\{0, 1\}$, and we are done. In case (c), we have $\eta \sim \eta_1\eta|_{[s_0,s_1]}\eta_0$. But $\eta_1\eta|_{[s_0,s_1]}\eta_0$ has fewer intersections with Y than η, which is a contradiction. This completes the proof. $\square$

Now we generalize [17, Lemma 3.14] to the case that $\mathbf{T}$ is an arbitrary triangulation of $\mathbf{S}_\triangle$. Recall that $\mathbf{T}^*$ is the dual triangulation of $\mathbf{T}$.

Lemma 2.2. *Let η be a closed arc in $\mathrm{CA}(\mathbf{S}_\triangle)$ which is not in $\mathbf{T}^*$. Then there are two arcs α, β in $\mathrm{CA}(\mathbf{S}_\triangle)$ such that*

(1) $\mathrm{Int}_{\mathbf{S}-\triangle}(\alpha, \beta) = 0$ *(so* $\mathrm{Int}(\alpha, \beta) \leqslant 1$*);*
(2) $\eta = B_\alpha(\beta)$ *or* $\eta = B_\alpha^{-1}(\beta)$*;*

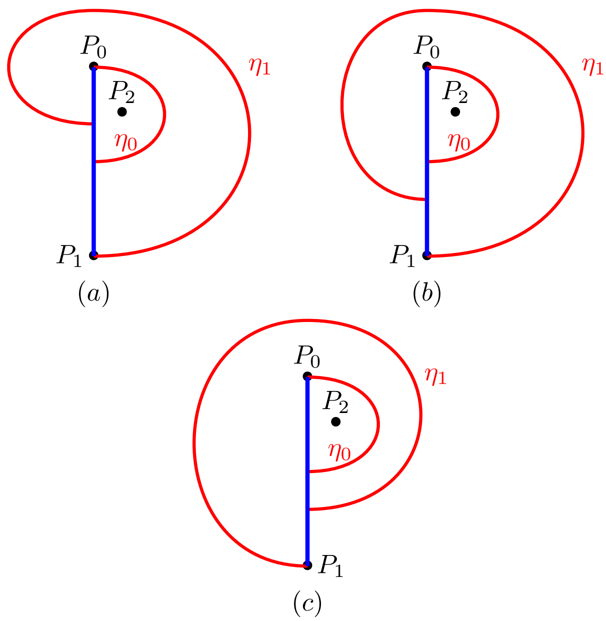

FIGURE 3. The three cases in the proof of Lemma 2.1 (topological view)

(3) $\mathrm{Int}(\gamma_i, \alpha) < \mathrm{Int}(\gamma_i, \eta)$ *and* $\mathrm{Int}(\gamma_i, \beta) < \mathrm{Int}(\gamma_i, \eta)$ *for any* $\gamma_i \in \mathbf{T}$.

Proof. Assume that η has minimal intersections with the arcs in $\mathbf{T}$ without loss of generality. If η intersects at least three triangles of $\mathbf{T}$, then the assertion holds by the proof of [17, Lemma 3.14]. Thus, we can suppose that η intersects exactly two triangles of $\mathbf{T}$.

Since the original marked surface $\mathbf{S}$ is not a once-punctured torus, these two triangles that intersect η cannot share three edges. On the other hand, if they share only one edge, say, i, then $\eta = s_i$ because η is contained in these two triangles. This contradicts our assumption. Therefore, these two triangles share exactly two edges, and they form an annulus A. As we care only about the interior of the union of these two triangles, we are in the same situation as with Lemma 2.1:

- the two boundaries of A correspond to $\partial\mathbf{D}$ and the puncture P_2, respectively;
- the endpoints of η correspond to punctures P_0 and P_1, respectively;
- the sharing edges topologically correspond to Y.

Now, since $\mathrm{Int}(\eta, \mathbf{T}) > 2$, there exist arcs η_0 and η_1 satisfying the conditions in Lemma 2.1. Let $\alpha = \alpha_1 \sim \alpha_2$, and let $\beta = \eta_1 \eta|_{[s_0,s_1]} \eta_0$. Then $\mathrm{Int}_{\mathbf{S}-\triangle}(\alpha, \beta) = 0$, $\mathrm{Int}_{\triangle}(\alpha, \beta) = 2$, and $\eta = B_\alpha(\beta)$ (for the case in which η_0 intersects η at $\eta(s_0)$ from the left side) or $\eta = B_\alpha^{-1}(\beta)$ (for the case in which η_0 intersects η at $\eta(s_0)$ from the right side). Note that η_0 and η_1 do not cross any arcs in $\mathbf{T}$. Then for any $\gamma_i \in \mathbf{T}$,

$$\mathrm{Int}(\gamma_i, \alpha) = \mathrm{Int}(\gamma_i, \eta|_{[0,s_1]}) = \mathrm{Int}(\gamma_i, \eta|_{[s_0,1]}) < \mathrm{Int}(\gamma_i, \eta)$$

and

$$\mathrm{Int}(\gamma_i, \beta) = \mathrm{Int}(\gamma_i, \eta|_{[s_0,s_1]}) < \mathrm{Int}(\gamma_i, \eta).$$

Thus, we complete the proof. □

As an immediate consequence, the proof in [17, Propositions 4.3 and 4.4] works for all cases (i.e., without Assumption 3.2).

Proposition 2.3 ([17, Propositions 4.3 and 4.4])**.** *For any triangulation* $\mathbf{T}$ *of* $\mathbf{S}_\triangle$*, we have*

$$\mathrm{BT}(\mathbf{S}_\triangle) = \mathrm{BT}(\mathbf{T})$$

and

$$\mathrm{CA}(\mathbf{S}_\triangle) = \mathrm{BT}(\mathbf{S}_\triangle) \cdot \mathbf{T}^*.$$

3. Intersection numbers and dimensions of Homs

Recall that in [17], the author gives a bijection from the set of closed arcs in $\mathbf{S}_\triangle$ to the set of reachable spherical objects in $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}_0})$ for any triangulation $\mathbf{T}_0$ such that any two of its triangles share at most one edge. In this section, we generalize this bijection to arbitrary triangulation $\mathbf{T}$ (of any decorated marked surface).

3.1. **The string model.** As in the Appendix, we have the following. For each $\eta \in \overline{\mathrm{CA}}(\mathbf{S}_\triangle)$, X_η is an object in $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$ which induces a map

$$\widetilde{X}_{\mathbf{T}}\colon\ \mathrm{CA}(\mathbf{S}_\triangle) \to \mathcal{D}_{fd}(\Gamma_{\mathbf{T}})/[1], \qquad \eta \mapsto X_\eta[\mathbb{Z}]. \tag{3.1}$$

The notation $X[\mathbb{Z}]$ means the shift orbit $\{X[i] \mid i \in \mathbb{Z}\}$. Moreover, let σ, τ be oriented general closed arcs in $\mathbf{S}_\triangle$ with $\mathrm{Int}_{\mathbf{S}-\triangle}(\sigma,\tau) = 0$ whose starting points coincide. Proposition A.8 gives a nonzero morphism

$$\varphi(\sigma,\tau) \in \mathrm{Hom}_{\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})}(X_\sigma, X_\tau[\upsilon]).$$

In the following, we keep the notations in the Appendix and upgrade Proposition A.11 first.

Proposition 3.1. *If* σ,τ *share their starting but not ending points, then there is a nonsplit triangle in* $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$*, whose image in* $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})/[1]$ *is*

$$\widetilde{X}(B_\sigma(\tau)) \longrightarrow \widetilde{X}(\sigma) \xrightarrow{\varphi(\sigma,\tau)} \widetilde{X}(\tau) \longrightarrow \widetilde{X}(B_\sigma(\tau)). \tag{3.2}$$

If σ,τ *share both of the endpoints, i.e.,* $\mathrm{Int}_\triangle(\sigma,\tau) = 2$*, then there is a nonsplit triangle in* $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$*, whose image in* $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})/[1]$ *is*

$$\widetilde{X}(B_\sigma(\tau)) \longrightarrow \widetilde{X}(\sigma) \oplus \widetilde{X}(\sigma) \xrightarrow{(\,\varphi(\sigma,\tau)\ \varphi(\overline{\sigma},\overline{\tau})\,)} \widetilde{X}(\tau) \longrightarrow \widetilde{X}(B_\sigma(\tau)), \tag{3.3}$$

where $\varphi(\sigma,\tau)$ *and* $\varphi(\overline{\sigma},\overline{\tau})$ *are linearly independent.*

Proof. For the case $\mathrm{Int}_\triangle(\sigma,\tau) = 1$, we have $B_\sigma(\tau) = \tau \wedge \sigma$. So the triangle in Proposition A.11 becomes (3.2).

Now consider the case $\mathrm{Int}_\triangle(\sigma,\tau) = 2$; see Figure 4. Let $\varsigma = \tau \wedge \sigma$. Then we have

$$\eta = B_\sigma(\tau) = \overline{\varsigma} \wedge \overline{\sigma}.$$

Similarly, let $\xi = \overline{\tau} \wedge \overline{\sigma}$, and then we have

$$\eta = B_\sigma(\tau) = \overline{\xi} \wedge \sigma.$$

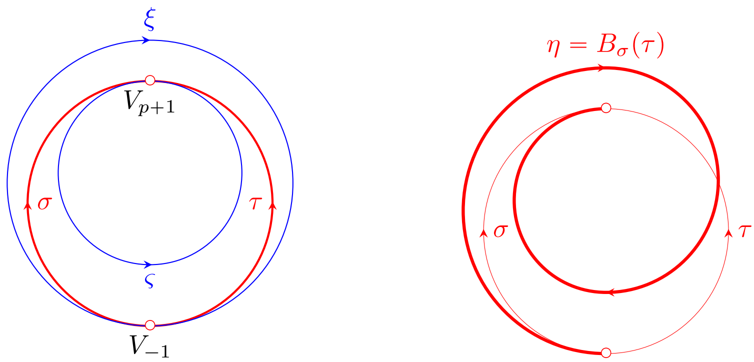

FIGURE 4. The braid twists as compositions of extensions

Note that the starting segments of $\overline{\sigma}$, $\overline{\tau}$, and $\overline{\varsigma}$ are in clockwise order at the common starting point; see Figure 4. By Corollary A.9, we have

$$\varphi(\overline{\tau},\overline{\varsigma}) \circ \varphi(\overline{\sigma},\overline{\tau}) = \varphi(\overline{\sigma},\overline{\varsigma}).$$

By Proposition A.11, the mapping cones of $\varphi(\overline{\tau},\overline{\varsigma})$, $\varphi(\overline{\sigma},\overline{\tau})$, and $\varphi(\overline{\sigma},\overline{\varsigma})$ are in $\widetilde{X}(\sigma)$, $\widetilde{X}(\xi)$, and $\widetilde{X}(\eta)$, respectively. Then applying octahedral axiom to this composition gives the following commutative diagram of (images in $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})/[1]$ of) triangles:

$$\begin{array}{ccccccc}
 & & \widetilde{X}(\varsigma) & = & \widetilde{X}(\varsigma) & & \\
 & & \downarrow & & \downarrow & & \\
\widetilde{X}(\xi) & \longrightarrow & \widetilde{X}(\eta) & \longrightarrow & \widetilde{X}(\sigma) & \longrightarrow & \widetilde{X}(\xi) \\
\| & & \downarrow & & \downarrow{\scriptstyle \varphi(\sigma,\tau)} & & \| \\
\widetilde{X}(\xi) & \longrightarrow & \widetilde{X}(\sigma) & \xrightarrow{\varphi(\overline{\sigma},\overline{\tau})} & \widetilde{X}(\tau) & \longrightarrow & \widetilde{X}(\xi) \\
 & & {\scriptstyle \varphi(\overline{\sigma},\overline{\varsigma})}\downarrow & & \downarrow{\scriptstyle \varphi(\overline{\tau},\overline{\varsigma})} & & \\
 & & \widetilde{X}(\varsigma) & = & \widetilde{X}(\varsigma) & &
\end{array}$$

Then we have the triangle (3.3). Since $\varphi(\overline{\tau},\overline{\varsigma}) \circ \varphi(\sigma,\tau) = 0$ by the triangle in the third column and $\varphi(\overline{\tau},\overline{\varsigma}) \circ \varphi(\overline{\sigma},\overline{\tau}) = \varphi(\overline{\sigma},\overline{\varsigma}) \neq 0$, we deduce that $\varphi(\sigma,\tau)$ and $\varphi(\overline{\sigma},\overline{\tau})$ are linearly independent. □

3.2. **The first formula.**

Assumption 3.2. Suppose that $\mathbf{S}$ admits a triangulation $\mathbf{T}_0$ such that any two triangles share at most one edge (i.e., there is no double arrow in the corresponding quiver $Q_{\mathbf{T}_0}$).

Let $\mathbf{T}_0 = \{\delta_1, \ldots, \delta_n\}$ and $\mathbf{T}_0^* = \{t_1, \ldots, t_n\}$ be the dual of $\mathbf{T}_0$. Write $\Gamma_0 = \Gamma_{\mathbf{T}_0}$, and denote by $\mathcal{H}_0$ the canonical heart of $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}_0})$ with simples T_i corresponding to t_i. By [17, Section 6], we have the following:

- The map $\widetilde{X}_0 = \widetilde{X}_{\mathbf{T}_0}$ in (3.1) induces a bijection $\widetilde{X}_0 \colon \mathrm{CA}(\mathbf{S}_\triangle) \xrightarrow{1\text{-}1} \mathrm{Sph}(\Gamma_0)/[1]$ and an isomorphism

$$\iota_0 \colon \mathrm{BT}(\mathbf{T}_0) \to \mathrm{ST}(\Gamma_0), \tag{3.4}$$

  sending the generator B_{t_i} to the generator ϕ_{T_i}.
- There is a commutative diagram

$$\begin{array}{ccc} \mathrm{BT}(\mathbf{S}_\triangle) & \xrightarrow{\quad\iota_0\quad} & \mathrm{ST}(\Gamma_0) \\ \circlearrowright & & \circlearrowright \\ \mathrm{CA}(\mathbf{S}_\triangle) & \xrightarrow{\quad\widetilde{X}_0\quad} & \mathrm{Sph}(\Gamma_0)/[1] \end{array}$$

  in the sense that, for any $b \in \mathrm{BT}(\mathbf{S}_\triangle)$ and $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, we have

$$\widetilde{X}_0\left(b(\eta)\right) = \iota_0(b)\left(\widetilde{X}_0(\eta)\right). \tag{3.5}$$

We will use X_η^0 to denote an object in the shift orbit $\widetilde{X_0}(\eta)$.

Lemma 3.3. *Let $\sigma, \tau \in \overline{\mathrm{CA}}(\mathbf{S}_\triangle)$ be two oriented arcs sharing the same starting point, and let $t_i \in \mathbf{T}_0^*$. Suppose that t_i, σ, τ are pairwise different and do not intersect each other in $\mathbf{S} - \triangle$ and that their start segments are in counterclockwise order at the common starting point* (*see Figure* 5)*. Then $\varphi(\sigma, \tau) \circ \varphi(t_i, \sigma) = 0$.*

Proof. Suppose that $f = \varphi(\sigma, \tau) \circ \varphi(t_i, \sigma) \neq 0$ in $\mathrm{Hom}(T_i, X_\tau^0[m])$ for some integer m. Since $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}_0})$ is 3-Calabi–Yau, there exists $f^* \in \mathrm{Hom}(X_\tau^0[m], T_i[3])$ such that $f^* \circ f \neq 0$. On the other hand, the start segments of σ, t_i, and τ are in clockwise order. Then by Corollary A.9, $\varphi(\sigma, \tau) = \varphi(t_i, \tau) \circ \varphi(\sigma, t_i)$, where the morphisms are properly shifted. Therefore, there is a nonzero composition

$$f^* \circ f \colon T_i \xrightarrow{\varphi(t_i,\sigma)} X_\sigma^0[l] \xrightarrow{\varphi(\sigma,t_i)} T_i[N] \xrightarrow{\varphi(t_i,\tau)} X_\tau^0[m] \xrightarrow{f^*} T_i[3].$$

Since T_i is a spherical object, we have $N = 0$ or $N = 3$. If $N = 0$, then the first two morphisms in the composition above must be identity (up to scale), which forces $X_\sigma^0[l] = T_i$. Since $\widetilde{X}_0$ is a bijection, this contradicts $\sigma \neq t_i$. If $N = 3$, similarly, we have $X_\tau^0[m] = T_i[3]$, which contradicts $\tau \neq t_i$. ☐

For any general closed curve $\sigma \in \overline{\mathrm{CA}}(\mathbf{S}_\triangle)$, we define $l_0(\sigma) = \mathrm{Int}(\sigma, \mathbf{T}_0) = \sum_{i=1}^n \mathrm{Int}(\sigma, t_i)$.

Lemma 3.4. *Let σ, τ be two general closed curves in $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$ sharing the same starting point, and let $\mathrm{Int}_{\mathbf{S}-\triangle}(\sigma, \tau) = 0$. Let $\eta = \tau \wedge \sigma$, and assume that $l_0(\eta) = l_0(\sigma) + l_0(\tau)$. If*

$$\mathrm{Int}(t_i, \eta) = \mathrm{Int}(t_i, \sigma) + \mathrm{Int}(t_i, \tau) \tag{3.6}$$

holds for some $t_i \in \mathbf{T}_0^*$, then we have

$$\dim \mathrm{Hom}^{\mathbb{Z}}(T_i, X_\eta^0) = \dim \mathrm{Hom}^{\mathbb{Z}}(T_i, X_\sigma^0) + \dim \mathrm{Hom}^{\mathbb{Z}}(T_i, X_\tau^0). \tag{3.7}$$

Proof. Use the notations for σ and τ in (A.2). By Proposition A.11, there is a nonsplit triangle

$$X^0_\sigma[-\upsilon] \xrightarrow{\varphi(\sigma,\tau)} X^0_\tau \xrightarrow{\varphi(\overline{\tau},\overline{\eta})} X^0_\eta[l] \xrightarrow{\varphi(\eta,\overline{\sigma})} X^0_\sigma[-\upsilon+1]$$

for some integers υ, l. Note that, in the case when $\sigma = \tau$, we will apply Proposition A.11 to (η, σ) instead of (σ, τ). Nevertheless, we will get the same triangle. So it is sufficient to prove that the map

$$\mathrm{Hom}(T_i[r], X^0_\sigma[-\upsilon]) \xrightarrow{\mathrm{Hom}(T_i[r],\varphi(\sigma,\tau))} \mathrm{Hom}(T_i[r], X^0_\tau) \tag{3.8}$$

is zero for any $r \in \mathbb{Z}$. Since $l_0(\eta) = l_0(\sigma) + l_0(\tau)$, by Construction A.5, we have that $\varphi(\sigma, \tau)$ is of the following form:

$$\begin{array}{ccccc} T_{k_0} & \overset{\pi_{a_1}}{\text{———————}} & T_{k_1}[\varrho_1] & \text{———————} & \cdots \\ \downarrow{\scriptstyle \varphi_0} & & & & \\ T_{j_0}[\upsilon] & \overset{(-1)^\upsilon \pi_{b_1}[\upsilon]}{\text{———————}} & T_{j_1}[\kappa_1+\upsilon] & \text{———————} & \cdots \end{array}$$

where φ_0 is induced from the triangle Λ_0 of $\mathbf{T}$ that contains the common starting point of σ and τ. Then for any $f : T_i[r] \to X^0_\sigma$, the composition $\varphi(\sigma, \tau) \circ f$ is given by $\varphi_0 \circ f_0$, where f_0 is the component of f from $T_i[r]$ to T_{k_0}. Thus, the map (3.8) being 0 is equivalent to $\varphi_0 \circ f_0 = 0$.

Case I. If the common starting point $V_{-1} = W_{-1}$ of σ and τ is not an endpoint of t_i, then γ_i is not an edge of the triangle Λ_0. So f_0 is not induced from Λ_0. Then by Lemma A.2, $\varphi_0 \circ f_0 = 0$, as required.

Case II. If $V_{-1} = W_{-1}$ is an endpoint of t_i, then f_0 is the unique nonzero component of $\varphi(t_i, \sigma)$ by Construction A.5. Furthermore, (3.6) implies that the segment V_0W_0 in η intersects t_i as shown in Figure 5. In particular, the start segments of t_i, σ, and τ are in counterclockwise order. Hence, by Lemma 3.3, $\varphi(\sigma, \tau) \circ \varphi(t_i, \sigma) = 0$, which implies that $\varphi_0 \circ f_0 = 0$, as required. ☐

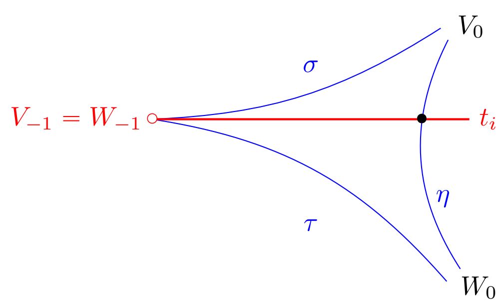

FIGURE 5. The composition of σ and τ

Proposition 3.5. *Let* $\eta \in \overline{\mathrm{CA}}(\mathbf{S}_\triangle)$. *Under Assumption* 3.2, *we have*

$$\dim \mathrm{Hom}^{\mathbb{Z}}(T_i, \widetilde{X}_0(\eta)) = 2\,\mathrm{Int}(t_i, \eta). \tag{3.9}$$

Proof. When $\mathrm{Int}(t_i, \eta) < 1$ (i.e., is 0 or 1/2), the formula is proved in [17, Proposition 5.9]. Now we assume that $\mathrm{Int}(t_i, \eta) \geqslant 1$. Use induction on $l_0(\eta)$ starting with the trivial cases $l_0(\eta) = 1$ (i.e., $\eta = t_j$ for some j). Now suppose that the formula holds for $l_0(\eta) \leqslant l$, with some $l \geqslant 1$. Consider the case $l_0(\eta) = l + 1$. Then there are three cases:

(I) η intersects a triangle (with decorating point Z) that does not intersect t_i. Apply [17, Lemma 3.14] to decompose η into σ and τ w.r.t. Z. Since Z is not an endpoint of t_i, (3.6) holds. So by Lemma 3.4, we have (3.7) for σ and τ (by properly choosing their orientations; the same holds for later use of this lemma). By the induction hypothesis, (3.9) holds for σ, τ and hence holds for η too by (3.7).

(II) $\mathrm{Int}_{\mathbf{S}-\triangle}(\eta, t_i) \neq 0$. Let Z be an endpoint of t_i such that the triangle containing Z contains intersections of η and t_i in $\mathbf{S} - \triangle$. Choose the closest intersection Y between η and t_i from Z. Apply [17, Lemma 3.14] w.r.t. Z and the line segment $YZ(\subset t_i)$ to decompose η into σ and τ. Again they satisfy condition (3.6) in Lemma 3.4, and thus (3.7) holds. By the induction hypothesis, (3.9) holds for σ, τ, and hence for η.

(III) Suppose the conditions in (I) and (II) both fail, i.e.,
- $\mathrm{Int}_{\mathbf{S}-\triangle}(\eta, t_i) = 0$ and
- η is contained in the two triangles of $\mathbf{T}_0$ which intersect t_i.

Note that these two triangles share exactly one edge by Assumption 3.2. Then since $\eta \neq t_i$, we deduce that η is a loop enclosing t_i (see Figure 6). In this case, $X_\eta^0 = \mathrm{Cone}(T_i \to T_i[3])[-1]$. A direct calculation shows that (3.9) holds. □

Corollary 3.6. *Under Assumption* 3.2*,*

$$\dim \mathrm{Hom}^{\mathbb{Z}}(\widetilde{X}_0(\eta_1), \widetilde{X}_0(\eta_2)) = 2\,\mathrm{Int}(\eta_1, \eta_2) \tag{3.10}$$

for any $\eta_i \in \mathrm{CA}(\mathbf{S}_\triangle)$.

Proof. By Proposition 2.3, there exist $t_i \in \mathbf{T}_0^*$ and $b \in \mathrm{BT}(\mathbf{T}_0)$ such that $\eta_1 = b(t_i)$. Then by (3.5), we have $\widetilde{X}_0(b(t_i)) = \iota_0(b)\left(\widetilde{X}_0(t_i)\right)$. Hence

$$\begin{aligned}\dim \mathrm{Hom}^{\mathbb{Z}}(\widetilde{X}_0(\eta_1), \widetilde{X}_0(\eta_2)) &= \dim \mathrm{Hom}^{\mathbb{Z}}(\widetilde{X}_0(t_i), \widetilde{X}_0(b^{-1}(\eta_2))) \\ &= 2\,\mathrm{Int}(t_i, b^{-1}(\eta_2)) \\ &= 2\,\mathrm{Int}(\eta_1, \eta_2).\end{aligned}$$

□

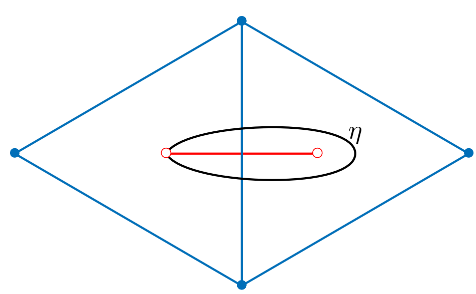

FIGURE 6. A loop encloses a closed arc

4. Main results

4.1. **Independence.** We still assume that Assumption 3.2 holds in this subsection; i.e., there is an initial triangulation $\mathbf{T}_0$ of $\mathbf{S}_\triangle$ such that any two triangles share at most one edge. Recall from [17] that two elements ψ and ψ' in $\operatorname{Aut}\mathcal{D}_{fd}(\Gamma_0)$ are *isotopic*, denoted by $\psi \sim \psi'$, if $\psi^{-1} \circ \psi'$ acts trivially on $\operatorname{Sph}(\Gamma_0)$. Let

$$\operatorname{Aut}^\circ \mathcal{D}_{fd}(\Gamma_0) = \operatorname{Aut}\mathcal{D}_{fd}(\Gamma_0)/\sim .$$

By [17, (2.6)], $\psi \sim \psi'$ is equivalent to the condition that $\psi^{-1} \circ \psi'$ acts trivially on $\operatorname{Sim}\mathcal{H}_0$. We will say an element φ in $\operatorname{Aut}\mathcal{D}_{fd}(\Gamma_0)$ is null-homotopic if $\varphi \sim \mathrm{id}$.

Now let $\mathbf{T}$ be an arbitrary triangulation. Keep the notations in Section 2.5. Denote by $\mathcal{H}_\mathbf{T}$ the canonical heart in $\mathcal{D}_{fd}(\Gamma_\mathbf{T})$ with simples $\{S_i\}$ corresponding to open arcs in $\mathbf{T}^* = \{s_i\}$. Denote by $\operatorname{Sph}(\Gamma_\mathbf{T})$ the set of reachable spherical objects.

Definition 4.1. We say two exact equivalences $\phi, \phi' \colon \mathcal{D}_{fd}(\Gamma_0) \to \mathcal{D}_{fd}(\Gamma_\mathbf{T})$ are isotopic if they differ only by null-homotopies; i.e., $\phi' = \varphi_1 \circ \phi \circ \varphi_0$ for some $\varphi_0 \in \operatorname{Aut}\mathcal{D}_{fd}(\Gamma_0)$ and $\varphi_1 \in \operatorname{Aut}\mathcal{D}_{fd}(\Gamma_\mathbf{T})$, which are null-homotopic.

Proposition 4.2. *There is a unique exact equivalence $\Phi_\mathbf{T} \colon \mathcal{D}_{fd}(\Gamma_0) \to \mathcal{D}_{fd}(\Gamma_\mathbf{T})$, up to isotopy and shifts, such that it induces a bijection*

$$\Phi_\mathbf{T} \colon \operatorname{Sph}(\Gamma_0)/[1] \to \operatorname{Sph}(\Gamma_\mathbf{T})/[1]$$

satisfying the following condition:

- *for any $s \in \mathbf{T}^*$, the corresponding simple in $\operatorname{Sim}\mathcal{H}_\mathbf{T}$ is in the shift orbit $\Phi_\mathbf{T}(\widetilde{X}_0(s))$.*

Proof. First, we show the uniqueness. Suppose that there are two such exact equivalences $\Phi_\mathbf{T}$ and $\Phi'_\mathbf{T}$. Then we have $\Phi_\mathbf{T}^{-1} \circ \Phi'_\mathbf{T}(T_i) = T_i[m_i]$ for any simple T_i in the canonical heart $\mathcal{H}_0$. By calculating the $\operatorname{Hom}^\mathbb{Z}$, we deduce that all m_i should coincide; i.e., $\Phi_\mathbf{T}^{-1} \circ \Phi'_\mathbf{T} \circ [-m]$ preserves $\operatorname{Sim}\mathcal{H}_0$ and hence $\operatorname{Sph}(\Gamma_0)$. In other words, $\Phi_\mathbf{T}^{-1} \circ \Phi'_\mathbf{T} \circ [-m]$ is null-homotopic in $\operatorname{Aut}\mathcal{D}_{fd}(\Gamma_0)$, as required.

Now we prove the existence by induction, on the minimal number of flips from $\mathbf{T}_0$ to $\mathbf{T}$, starting from the trivial case. Now suppose that $\mathbf{T}$ admits a required derived equivalence $\Phi_\mathbf{T}$, i.e.,

$$\Phi_\mathbf{T}(\widetilde{X}_0(s_i)) = S_i[\mathbb{Z}]. \tag{4.1}$$

Then we need to show only that there exists a required exact equivalence $\Phi_{\mathbf{T}'}$ for any flip $\mathbf{T}'$ of $\mathbf{T}$ in $\mathbf{S}_\triangle$.

Without loss of generality, suppose that $\mathbf{T}'$ is the forward flip of $\mathbf{T}$ w.r.t. an arc γ_j, and let s_j be the dual arc of γ_j in $\mathbf{T}^*$. By [13], there is an exact equivalence $\Phi \colon \mathcal{D}(\Gamma_\mathbf{T}) \to \mathcal{D}(\Gamma_{\mathbf{T}'})$ satisfying

$$\Phi\left((\mathcal{H}_\mathbf{T})^\sharp_{S_j}\right) = \mathcal{H}_{\mathbf{T}'},$$

where $(\mathcal{H}_\mathbf{T})^\sharp_{S_j}$ is the simple forward tilt of $\mathcal{H}_\mathbf{T}$ w.r.t. S_j (cf. [14, Section 5]).

Let $(\mathbf{T}')^*$ consist of closed arcs s'_i, and let $\operatorname{Sim}\mathcal{H}_{\mathbf{T}'}$ consist of the corresponding simples S'_i. By the tilting formula in [14, Proposition 5.2], we have

$$\Phi^{-1}(S'_i) = \begin{cases} \phi_{S_j}^{-1}(S_i) & \text{if there are arrows from } i \text{ to } j \text{ in } Q_\mathbf{T}, \\ S_j[1] & \text{if } i = j, \\ S_i & \text{otherwise.} \end{cases}$$

On the other hand, note that the indexing of s'_i is induced by the indexing of s_i via the Whitehead move (see [17, Figure 10]). It is straightforward to see that

$$s'_i = \begin{cases} B_{s_j}(s_i) & \text{if there are arrows from } i \text{ to } j \text{ in } Q_{\mathbf{T}}, \\ s_i & \text{otherwise.} \end{cases}$$

Taking $b = \mathrm{B}_{s_j}^{-1}$ with $\iota_0(b) = \phi^{-1}_{\widetilde{X}_0(s_j)}$ and $\eta = s_i$, (3.5) becomes

$$\phi^{-1}_{\widetilde{X}_0(s_j)}(\widetilde{X}_0(s_i)) = \widetilde{X}_0(\mathrm{B}_{s_j}^{-1}(s_i)). \tag{4.2}$$

Then by (4.1), we have

$$\begin{aligned} \Phi^{-1}(S'_i[\mathbb{Z}]) &= \phi^{-1}_{S_j}(S_i)[\mathbb{Z}] \\ &= \phi^{-1}_{\Phi_{\mathbf{T}}(\widetilde{X}_0(s_j))}\left(\Phi_{\mathbf{T}}(\widetilde{X}_0(s_i))\right) \\ &= \Phi_{\mathbf{T}}\left(\phi^{-1}_{\widetilde{X}_0(s_j)}(\widetilde{X}_0(s_i))\right) \\ &= \Phi_{\mathbf{T}}\left(\widetilde{X}_0(\mathrm{B}_{s_j}(s_i))\right) \\ &= \Phi_{\mathbf{T}}\left(\widetilde{X}_0(s'_i)\right) \end{aligned} \tag{4.3}$$

if there are arrows from i to j in $Q_{\mathbf{T}}$. Note that for other i, the equation above also holds automatically. Thus, $S'_i[\mathbb{Z}] = \Phi \circ \Phi_{\mathbf{T}}(\widetilde{X}_0(s'_i))$ and $\Phi_{\mathbf{T}'} = \Phi \circ \Phi_{\mathbf{T}}$ is the required equivalence. □

Recall that there is a bijection $\widetilde{X}_0\colon \mathrm{CA}(\mathbf{S}_\triangle) \to \mathrm{Sph}(\Gamma_0)/[1]$, and we proceed to discuss $\widetilde{X}_{\mathbf{T}}$.

Proposition 4.3. *$\widetilde{X}_{\mathbf{T}}$ induces a bijection $\widetilde{X}_{\mathbf{T}}\colon \mathrm{CA}(\mathbf{S}_\triangle) \to \mathrm{Sph}(\Gamma_{\mathbf{T}})/[1]$ that fits into the following commutative diagram:*

$$\begin{array}{ccccc} & & \mathrm{CA}(\mathbf{S}_\triangle) & & \\ & \swarrow^{\widetilde{X}_0} & & \searrow^{\widetilde{X}_{\mathbf{T}}} & \\ \mathrm{Sph}(\Gamma_0)/[1] & & \xrightarrow{\quad\Phi_{\mathbf{T}}\quad} & & \mathrm{Sph}(\Gamma_{\mathbf{T}})/[1] \end{array} \tag{4.4}$$

where $\Phi_{\mathbf{T}}$ is the bijection in Proposition 4.2.

Proof. Since $\widetilde{X}_0$ and $\Phi_{\mathbf{T}}$ are bijections, we need to prove only that

$$\widetilde{X}_{\mathbf{T}}(\eta) = \Phi_{\mathbf{T}} \circ \widetilde{X}_0(\eta). \tag{4.5}$$

Use induction on $l_0(\eta) = \mathrm{Int}(\eta, \mathbf{T})$. The starting step ($l_0(\eta) = 1$) is covered by Proposition 4.2. Now let us deal with the inductive step for some η with $l_0(\eta) > 1$ while assuming that (4.5) holds for any η' with $l_0(\eta') < l_0(\eta)$. By Lemma 2.2, there are α and β with the corresponding conditions there. Without loss of generality, assume that $\eta = B_\alpha(\beta)$. By inductive assumption, we have

$$\widetilde{X}_{\mathbf{T}}(\alpha) = \Phi_{\mathbf{T}} \circ \widetilde{X}_0(\alpha) \quad \text{and} \quad \widetilde{X}_{\mathbf{T}}(\beta) = \Phi_{\mathbf{T}} \circ \widetilde{X}_0(\beta). \tag{4.6}$$

Since by Corollary 3.6, we have

$$\dim \mathrm{Hom}^{\mathbb{Z}}(\widetilde{X}_0(\alpha), \widetilde{X}_0(\beta)) = 2\,\mathrm{Int}(\alpha, \beta), \tag{4.7}$$

the triangles in Proposition 3.1 imply that

$$\widetilde{X}_0(\eta) = \phi_{\widetilde{X}_0(\alpha)}\left(\widetilde{X}_0(\beta)\right). \tag{4.8}$$

Notice that by (4.6), equalities (4.7) and (4.8) also hold for $\widetilde{X}_{\mathbf{T}}$. Hence

$$\begin{aligned} \widetilde{X}_{\mathbf{T}}(\eta) &= \phi_{\widetilde{X}_{\mathbf{T}}(\alpha)}\left(\widetilde{X}_{\mathbf{T}}(\beta)\right) \\ &= \phi_{\Phi_{\mathbf{T}}\left(\widetilde{X}_0(\alpha)\right)}\left(\Phi_{\mathbf{T}} \circ \widetilde{X}_0(\beta)\right) \\ &= \Phi_{\mathbf{T}}\left(\phi_{\widetilde{X}_0(\alpha)}\left(\widetilde{X}_0(\beta)\right)\right) \\ &= \Phi_{\mathbf{T}} \circ \widetilde{X}_0(\eta), \end{aligned}$$

as required. □

Remark 4.4. By the proposition above, one can identify all sets $\mathrm{Sph}(\Gamma_{\mathbf{T}})$ of reachable spherical objects, for any $\mathbf{T}$ in $\mathrm{EG}^{\circ}(\mathbf{S}_{\triangle})$, using the canonical exact equivalences in Proposition 4.2 between $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$. Hence, such equivalences also allow us to identify all of the spherical twist groups $\mathrm{ST}(\Gamma_{\mathbf{T}})$. Note that here we will consider $\mathrm{ST}(\Gamma_{\mathbf{T}})$ as a subgroup of $\mathrm{Aut}^{\circ}\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$.

4.2. **The first formula revisit.**

Theorem 4.5 ([17, Conjecture 10.5])**.** *For any triangulation* $\mathbf{T}$ *and* $\eta_i \in \mathrm{CA}(\mathbf{S}_{\triangle})$*, we have*

$$\dim \mathrm{Hom}^{\mathbb{Z}}(\widetilde{X}_{\mathbf{T}}(\eta_1), \widetilde{X}_{\mathbf{T}}(\eta_2)) = 2\,\mathrm{Int}(\eta_1, \eta_2). \tag{4.9}$$

Proof. If Assumption 3.2 holds, the theorem is equivalent to Corollary 3.6 since one can identify all bijections $\widetilde{X}_{\mathbf{T}}$ as in Remark 4.4. For the special cases in which $\mathbf{S}$ does not satisfy Assumption 3.2, that is,

- either $\mathbf{S}$ is an annulus with one marked point in each boundary component
- or $\mathbf{S}$ is a torus with one boundary component and one marked point,

one can apply the same method in [17, Section 7]. More precisely, the formula holds for a higher rank surface (e.g., the surface obtained from $\mathbf{S}$ by adding a marked point) and hence also holds for $\mathbf{S}$. □

4.3. **Independence revisit.**

We use intersection formula (4.9) to prove (4.2) for $\mathbf{T}$.

Proposition 4.6. *For any* $\sigma, \tau \in \mathrm{CA}(\mathbf{S}_{\triangle})$ *with* $\mathrm{Int}_{\mathbf{S}-\triangle}(\sigma, \tau) = 0$*, we have*

$$\widetilde{X}_{\mathbf{T}}\left(B_{\tau}^{\varepsilon}(\sigma)\right) = \phi^{\varepsilon}_{\widetilde{X}_{\mathbf{T}}(\tau)}\left(\widetilde{X}_{\mathbf{T}}(\sigma)\right), \qquad \varepsilon \in \{\pm 1\}. \tag{4.10}$$

Proof. Without lose of generality, we prove the formula only for $\varepsilon = 1$. On the one hand, by (4.9), we have

$$\dim \mathrm{Hom}^{\mathbb{Z}}(\widetilde{X}_{\mathbf{T}}(\eta), \widetilde{X}_{\mathbf{T}}(\tau)) = \mathrm{Int}_{\triangle}(\sigma, \tau).$$

On the other hand, there is a triangle, i.e., (3.3) or (3.2), in Proposition 3.1 with $\eta = B_{\sigma}(\tau)$. Then $\widetilde{X}_{\mathbf{T}}(\eta) = \phi_{\widetilde{X}_{\mathbf{T}}(\sigma)}(\widetilde{X}_{\mathbf{T}}(\tau))$, as required. □

Proposition 4.7. *For any* $\mathbf{S}$ *and initial triangulation* $\mathbf{T}_0$ (*without Assumption* 3.2)*, Propositions* 4.2 *and* 4.3 *hold.*

Proof. Basically, we follow the same proof there. Note that (4.2) in the proof of Proposition 4.2 is now covered by the proposition above, which enables us do the generalization. □

4.4. **Intersection between open and closed arcs.** Let Γ be the Ginzburg dg algebra of some quiver with potential. A *silting set* $\mathcal{P}$ in a triangulated category $\mathcal{D}$ is an $\mathrm{Ext}^{>0}$-configuration, i.e., a maximal collection of nonisomorphic indecomposables such that $\mathrm{Ext}^i(P,T) = 0$ for any $P, T \in \mathcal{P}$ and integer $i > 0$. The silting object associated to $\mathcal{P}$ is $\bigoplus_{T\in\mathcal{P}} T$. By abuse of notation, we will not distinguish a silting set and its associated silting object. For example, Γ is the canonical silting object/set in $\operatorname{per}\Gamma$.

Moreover, one can forward/backward mutate a silting object to get new ones (see [1] for details). A silting set $\mathcal{P}$ in $\operatorname{per}\Gamma$ is *reachable* if it can be obtained by repeated mutations from Γ. Denote by $\mathrm{SEG}^\circ(\Gamma)$ the set of reachable silting sets in $\operatorname{per}\Gamma$, and denote by

$$\mathrm{RR}(\operatorname{per}\Gamma) = \bigcup_{\mathcal{P}\in\mathrm{SEG}^\circ(\Gamma)} \mathcal{P}$$

the set of *reachable rigid objects* in $\operatorname{per}\Gamma$. Recall a result from [18].

Lemma 4.8 ([18, Theorem 3.6])**.** *There is a canonical bijection*

$$\widetilde{M}_{\mathbf{T}}\colon \mathrm{OA}^\circ(\mathbf{S}_\triangle) \to \mathrm{RR}(\operatorname{per}\Gamma_{\mathbf{T}})$$

where $\mathrm{OA}^\circ(\mathbf{S}_\triangle)$ *is the subset of* $\mathrm{OA}(\mathbf{S}_\triangle)$ *consisting of the open arcs in some triangulation in* $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$.

We finish the paper by proving another conjecture in [17].

Theorem 4.9 ([17, Conjecture 10.6])**.** *For any triangulation* $\mathbf{T}$, $\gamma \in \mathrm{OA}^\circ(\mathbf{S}_\triangle)$, *and* $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, *we have*

$$\dim \mathrm{Hom}^{\mathbb{Z}}(\widetilde{M}_{\mathbf{T}}(\gamma), \widetilde{X}_{\mathbf{T}}(\eta)) = \mathrm{Int}(\gamma,\eta). \tag{4.11}$$

Proof. First, for any two triangulations $\mathbf{T}$ and $\mathbf{T}'$, we actually have a canonical identification $\Phi\colon \mathcal{D}_{fd}(\Gamma_{\mathbf{T}}) \to \mathcal{D}_{fd}(\Gamma_{\mathbf{T}'})$, as shown in Proposition 4.7. Note that there is a simple-projective duality between a silting set in $\operatorname{per}\Gamma$ and the set of simples of the corresponding heart in $\mathcal{D}_{fd}(\Gamma)$. Thus, as Φ preserves reachable spherical objects up to shift, Φ preserves reachable rigid objects up to shift. Second, by [17, Lemma 5.13], the theorem holds for $\gamma \in \mathbf{T}$ and any $\eta \in \mathrm{OA}(\mathbf{S})$. Now, choose any $\gamma \in \mathrm{OA}^\circ(\mathbf{S}_\triangle)$. Let $\mathbf{T}'$ be a triangulation in $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ that contains γ. Then we have

$$\dim \mathrm{Hom}^{\mathbb{Z}}_{\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})}(\widetilde{M}_{\mathbf{T}}(\gamma), \widetilde{X}_{\mathbf{T}}(\eta)) = \dim \mathrm{Hom}^{\mathbb{Z}}_{\mathcal{D}_{fd}(\Gamma_{\mathbf{T}'})}(\widetilde{M}_{\mathbf{T}'}(\gamma), \widetilde{X}_{\mathbf{T}'}(\eta)) = \mathrm{Int}(\gamma,\eta).$$

□

Appendix A. The string model

A.1. **Homological preparation.** Let $(Q,W) = (Q_{\mathbf{T}}, W_{\mathbf{T}})$ be the quiver with potential associated to a triangulation $\mathbf{T}$ of an unpunctured marked surfaces $\mathbf{S}$. Recall from Section 2.2 that there is an associated graded quiver $\overline{Q}$ and an associated Ginzburg dg algebra $\Gamma_{\mathbf{T}}$ whose underlying graded algebra is the completion of the graded path algebra $\mathbf{k}\overline{Q}$.

For each vertex i of $\overline{Q}$, denote by S_i the corresponding simple module of $\Gamma_{\mathbf{T}}$. There is a canonical heart $\mathcal{H}_{\mathbf{T}}$ in $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$ whose simples are $S_1, \ldots, S_n$. Let

$$S_{\mathbf{T}} = \bigoplus_{i=1}^{n} S_i$$

be the direct sum of the simples in $\mathcal{H}_{\mathbf{T}}$. Consider the differential graded endomorphism algebra $\mathfrak{E}_{\mathbf{T}} = \mathrm{RHom}(S_{\mathbf{T}}, S_{\mathbf{T}})$. By [12], we have the following exact equivalence:

$$\mathcal{D}_{fd}(\Gamma_{\mathbf{T}}) \underset{?\otimes^{\mathbf{L}}_{\mathfrak{E}_{\mathbf{T}}} S_{\mathbf{T}}}{\overset{\mathrm{RHom}_{\Gamma_{\mathbf{T}}}(S_{\mathbf{T}},?)}{\rightleftarrows}} \operatorname{per}\mathfrak{E}_{\mathbf{T}}. \tag{A.1}$$

In particular, the simples in $\mathcal{H}_{\mathbf{T}}$ become the indecomposable direct summands of $\mathfrak{E}_{\mathbf{T}}$. Then the Ext-algebra $\mathcal{E}_{\mathbf{T}} = \oplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_{\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})}(S_{\mathbf{T}}, S_{\mathbf{T}})$ is isomorphic to the homology algebra of $\mathfrak{E}_{\mathbf{T}}$. A basis of $\mathcal{E}_{\mathbf{T}}$ is indexed by the arrows and trivial paths in $\overline{Q}$ as follows.

Lemma A.1 ([13, Lemma 2.15])**.** *Let i, j be vertices of $\overline{Q}$, and let r be an integer. Then $\mathrm{Hom}_{\mathcal{D}_{fd}(\Gamma)}(S_i, S_j[r])$ has a basis*

$$\{\pi_b \mid b : i \to j \in \overline{Q}_1 \textit{ with } \deg b = 1 - r\} \cup \{\pi_{e_i} = \mathrm{id}_{S_i} \mid \textit{if } i = j \textit{ and } r = 0\}, \tag{A.2}$$

where e_i is the trivial path at i.

There is an A_∞ structure on $\mathcal{E}_{\mathbf{T}}$, induced by the differential of $\Gamma_{\mathbf{T}}$ (see [11, Appendix A.15]). In our case, this structures is as follows.

Lemma A.2. *The dg algebra $\mathfrak{E}_{\mathbf{T}}$ is formal and hence is quasi-isomorphic to $\mathcal{E}_{\mathbf{T}}$. Moreover, for any trivial paths e_i and e_j and any arrows x and y in $\overline{Q}$, we have the following.*

(1)

$$\pi_{e_j} \circ \pi_{e_i} = \begin{cases} \pi_{e_i} & \textit{if } i = j, \\ 0 & \textit{otherwise.} \end{cases} \tag{A.3}$$

(2)

$$\pi_y \circ \pi_{e_i} = \begin{cases} \pi_y & \textit{if } s(y) = i, \\ 0 & \textit{otherwise.} \end{cases} \tag{A.4}$$

$$\pi_{e_j} \circ \pi_x = \begin{cases} \pi_x & \textit{if } t(x) = j, \\ 0 & \textit{otherwise.} \end{cases} \tag{A.5}$$

(3)

$$\pi_y \circ \pi_x = \begin{cases} \pi_{\alpha^*} & \textit{if } xy\alpha \textit{ (up to cyclical equivalence) is a term in } W_{\mathbf{T}}, \\ \pi_{t_{s(x)}} & \textit{if } y = x^* \textit{ or } x = y^*, \\ 0 & \textit{otherwise.} \end{cases} \tag{A.6}$$

Here $s(\alpha)$ denotes the starting point of an arrow α, and $t(\alpha)$ denotes the ending point of α.

By Lemma A.2, there is an exact equivalence $\operatorname{per}\mathfrak{E}_{\mathbf{T}} \simeq \operatorname{per}\mathcal{E}_{\mathbf{T}}$ which, together with equivalence (A.1), gives an exact equivalence $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}}) \simeq \operatorname{per}\mathcal{E}_{\mathbf{T}}$. We will identify $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$ with $\operatorname{per}\mathcal{E}_{\mathbf{T}}$ when there is no confusion. In particular, $S_1, \ldots, S_n$ become the indecomposable direct summands of $\mathcal{E}_{\mathbf{T}}$ as dg $\mathcal{E}_{\mathbf{T}}$-modules. Since the differential of $\mathcal{E}_{\mathbf{T}}$ is 0, morphisms in (A.2) become homomorphisms of dg $\mathcal{E}_{\mathbf{T}}$-modules, and in particular maps.

Convention. Let $a\colon i \to j \in \overline{Q}_1$. By abuse of notation, $\pi_a[m]$ in

$$\mathrm{Hom}_{\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})}(S_i[m], S_j[m+1-\deg a])$$

will all be denoted by π_a for short. In particular, $\pi_b\pi_a$ makes sense whenever $ba \neq 0$. The approach is similar for other morphisms.

A.2. **The string model.** To each internal point A in an arc $\gamma_i \in \mathbf{T}$, we associate a vertex $\nu_A := i$. Let l be a segment in a triangle Δ of $\mathbf{T}$ whose endpoints are internal points in sides of Δ and which is not isotopic to a segment of any side of Δ. Let A, B be the endpoints of l such that from A to B the decorating point in Δ is to the right of l. We associate a graded arrow $\alpha(l)$ to be the unique arrow from ν_A to ν_B in $\overline{Q}$ induced from Δ. See Figure 7.

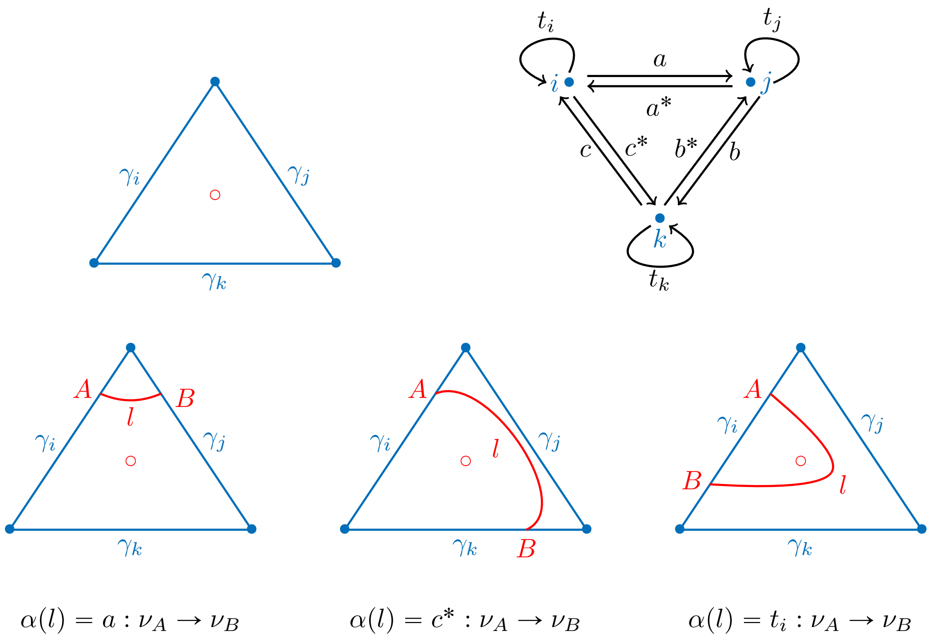

FIGURE 7. Segments inducing graded arrows

Construction A.3. Let σ be an oriented general closed arc such that it is in a minimal position w.r.t. $\mathbf{T}$ (i.e., there are no digons shown, as in Figure 8).

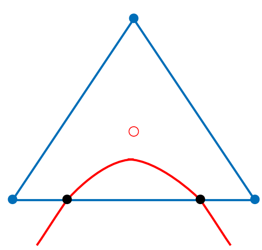

FIGURE 8. A digon intersected by σ and $\mathbf{T}$

- Suppose that σ intersects $\mathbf{T}$ at $V_0, \ldots, V_p$, accordingly, from its starting point to its ending point, where V_i is in the arc $\gamma_{k_i} \in \mathbf{T}$ for $0 \leqslant i \leqslant p$ and some $1 \leqslant k_i \leqslant n$ (see Figure 9). Moreover, denote by V_{-1} and V_{p+1} the starting and ending points of σ, respectively, by $\sigma(a, b)$ the segment of σ between V_a and V_b for $0 \leqslant a < b \leqslant p+1$, and by Λ_i the triangle containing the segment $\sigma(i-1, i)$ for $0 \leqslant i \leqslant p+1$.

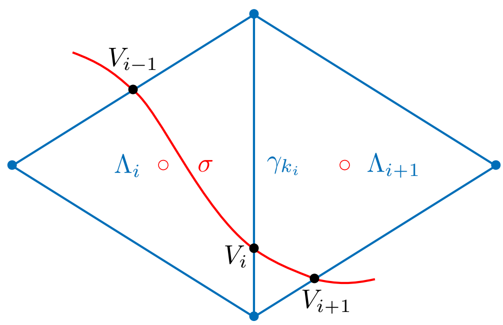

FIGURE 9. The intersections between σ and $\mathbf{T}$

- Each segment $\sigma(i-1, i)$ $(1 \leqslant i \leqslant p)$ of σ corresponds to a graded arrow $a_i := \alpha(\sigma(i-1, i))$ between k_{i-1} and k_i in $\overline{Q}$. Then we obtain a walk in $\overline{Q}$, called a string:

$$w(\sigma): \; k_0 \overset{a_1}{\text{———}} k_1 \overset{a_2}{\text{———}} \cdots \overset{a_p}{\text{———}} k_p \,. \tag{A.7}$$

  We define $\epsilon(a_i) = +$ if a_i points to the right, and $\epsilon(a_i) = -$ otherwise.
- The string $w(\sigma)$ induces a graded $\mathcal{E}_{\mathbf{T}}$-module $|X_\sigma|$ and a map d_σ on $|X_\sigma|$ of degree 1 as follows.
  - $|X_\sigma| = \oplus_{i=0}^{p} S_{k_i}[\varrho_i]$, where $\varrho_0 = 0$ and $\varrho_i = \varrho_{i-1} - \epsilon(a_i) \deg a_i$ for $1 \leqslant i \leqslant p$.
  - For each a_i, if $\epsilon(a_i) = +$, then the map $\pi_{a_i} : S_{k_{i-1}} \to S_{k_i}[1 - \deg a_i]$ induces a component $S_{k_{i-1}}[\varrho_{i-1}] \to S_{k_i}[\varrho_i]$ of d_σ; if $\epsilon(a_i) = -$, then the map $\pi_{a_i} : S_{k_i} \to S_{k_{i-1}}[1 - \deg a_i]$ induces a component $S_{k_i}[\varrho_i] \to S_{k_{i-1}}[\varrho_{i-1}]$ of d_σ. The other components of d_σ are 0.

Proposition A.4. *$X_\sigma := (|X_\sigma|, d_\sigma)$ is a perfect dg $\mathcal{E}_{\mathbf{T}}$-module in* $\operatorname{per}(\mathcal{E}_{\mathbf{T}})$.

Proof. We need to prove only that $d_\sigma^2 = 0$. By Lemma A.2, this follows from the fact that any two neighboring arc segments of σ are from different triangles. □

By construction, for any oriented general closed arc σ', if $\sigma' \sim \sigma$, then $X_{\sigma'} = X_\sigma$. Let $\overline{\sigma}$ be the oriented general closed arc obtained from σ by conversing the orientation. It is easy to see that $X_{\overline{\sigma}} \cong X_\sigma[l]$ for some l. Denote by $\widetilde{X}(\sigma)$ the shift orbit $X_\sigma[\mathbb{Z}]$ of X_σ. Then

$$\sigma \mapsto \widetilde{X}(\sigma)$$

is a well-defined map from the set $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$ to the set of objects in the orbit category $\operatorname{per} \mathcal{E}_{\mathbf{T}} /[1]$.

A.3. **Homomorphisms between strings.** Let σ be an oriented general closed arc as in Construction A.3, with the string (A.7) and the associated dg $\mathcal{E}_{\mathbf{T}}$-modules X_σ, whose underlying graded module is $\bigoplus_{i=0}^{p} S_{k_i}[\varrho_i]$.

Now, take another oriented general closed arc τ with $\mathrm{Int}_{\mathbf{S}-\triangle}(\sigma,\tau)=0$ which is in a minimal position w.r.t. $\mathbf{T}$ and σ. Note that τ may be isotopic to σ. Suppose that τ intersects $\mathbf{T}$ at $W_0,\dots,W_q$ in order, where W_i is in the arc $\gamma_{j_i}\in\mathbf{T}$, with starting point W_{-1} and ending point W_{q+1}. Then there is the associated string

$$w(\tau):\ j_0 \xrightarrow{\ b_1\ } j_1 \xrightarrow{\ b_2\ } \cdots \xrightarrow{\ b_q\ } j_q\ ,$$

where b_i is the arrow in $\overline{Q}_{\mathbf{T}}$ induced by the segment $\tau(i-1,i)$, and the associated dg $\mathcal{E}_{\mathbf{T}}$-module X_τ, whose underlying graded module is $\bigoplus_{i=0}^{q} S_{j_i}[\kappa_i]$.

Construction A.5. Suppose that $V_{-1}=W_{-1}$. Then there is an angle $\theta(\sigma,\tau)$ from σ to τ clockwise at this decorating point (see Figure 10). We will construct an element $\varphi(\sigma,\tau)$ in $\mathrm{Hom}^0(X_\sigma,X_\tau[\upsilon])$ induced by $\theta(\sigma,\tau)$. Here the value of $\upsilon=\upsilon(\sigma,\tau)$ is determined by the relative position of the segments $\sigma(-1,0)$ and $\tau(-1,0)$. There are four cases shown in Figure 10, where $\upsilon=0,1,2,3$, respectively.

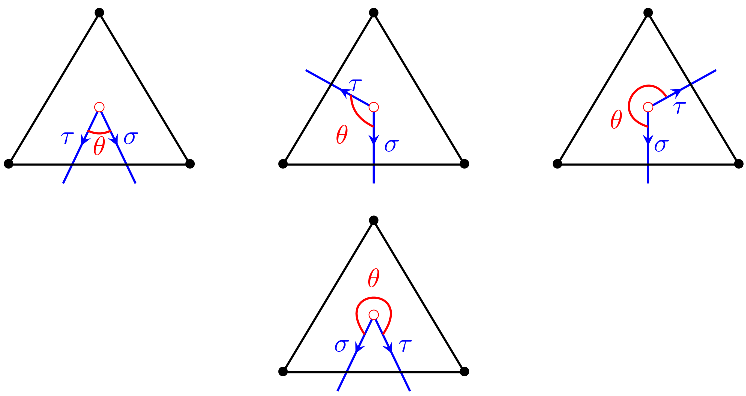

FIGURE 10. The four cases for the starting segments of σ and τ

Note that there is a unique $s\geqslant 0$ with a unique segment $l(\sigma,\tau)$ in the triangle Λ_s connecting V_s and W_s such that $l(\sigma,\tau)$, $\sigma(-1,s)$, and $\tau(-1,s)$ enclose a contractible triangle having $\theta(\sigma,\tau)$ as an internal angle. (A degenerate case is that when $\sigma\sim\tau$, $l(\sigma,\tau)$ is the decorating point in Λ_{p+1}.) It is clear that $s=0$ for the last three cases in Figure 10, and $s>0$ for the first case. We show in Figure 11 all of the possible subcases for Λ_s when $s>0$.

When the associated graded arrow $\alpha\left(l(\sigma,\tau)\right)$ exists and is from ν_{V_s} to ν_{W_s}, let $\varphi_s=\pi_{\alpha(l(\sigma,\tau))}: S_{k_s}\to S_{j_s}[\deg\varphi_s]$. By construction, for $i<s$, we have $S_{k_i}=S_{j_i}$ and let $\varphi_i=\mathrm{id}: S_{k_i}\to S_{j_i}$. We construct $\varphi(\sigma,\tau)$ in $\mathrm{Hom}^0(X_\sigma,X_\tau[\upsilon])$, whose nonzero components are $\varphi_i[\varrho_i]$. That is, when $s=0$ (i.e., the last three cases in

Figure 10), $\varphi(\sigma, \tau)$ has the following form,

$$\text{(A.8)}\qquad \begin{array}{ccccc} S_{k_0} & \xrightarrow{\quad \pi_{a_1} \quad} & S_{k_1}[\varrho_1] & \text{———} & \cdots \\ \Big\downarrow{\scriptstyle \varphi_0} & & & & \\ S_{j_0}[\deg\varphi_0] & \xrightarrow{(-1)^{\deg\varphi_0}\pi_{b_1}} & S_{j_1}[\kappa_1 + \deg\varphi_0] & \text{———} & \cdots \end{array}$$

and when $s > 0$ (i.e., the first case in Figure 10), $\varphi(\sigma, \tau)$ has the following form:

(A.9)

$$\begin{array}{ccccccccc} S_{k_0} & \text{———}\cdots\text{———} & S_{k_{s-1}}[\varrho_{s-1}] & \xrightarrow{\pi_{a_s}[\varrho_{s-1}]} & S_{k_s}[\varrho_s] & \xrightarrow{\pi_{a_{s+1}}[\varrho_s]} & S_{k_{s+1}}[\varrho_{s+1}] & \text{———} & \cdots \\ \Big\| & & \Big\| & & \Big\downarrow{\scriptstyle \varphi_s[\varrho_s]} & & & & \\ S_{k_0} & \text{———}\cdots\text{———} & S_{j_{s-1}}[\kappa_{s-1}] & \xrightarrow{\pi_{b_s}[\kappa_{s-1}]} & S_{j_s}[\kappa_s] & \xrightarrow{\pi_{b_{s+1}}[\kappa_s]} & S_{j_{s+1}}[\kappa_{s+1}] & \text{———} & \cdots \end{array}$$

where $\varphi_s[\varrho_s]$ exists if and only if one of the cases in (a) or (b) in Figure 11 occurs.

Lemma A.6. *$\varphi(\sigma, \tau)$ is in $Z^0 \operatorname{Hom}(X_\sigma, X_\tau[\upsilon])$.*

Proof. It suffices to prove that the components of $\varphi(\sigma, \tau)$ commute with the differentials of X_σ and X_τ. Since φ_s is not from the same triangle as $\pi_{a_{s+1}}$ or $\pi_{b_{s+1}}$, their compositions (if they exist) are 0. Hence we need to prove only that when $s > 0$, φ_s commutes with π_{a_s} and π_{b_s} in a suitable way. Consider the cases for Λ_s:

- Figure 11(a): $\epsilon(a_s) = \epsilon(b_s) = +$ and $\pi_{b_s} = \varphi_s \pi_{a_s}$, so $\varphi(\sigma, \tau) \in Z^0 \operatorname{Hom}(X_\sigma, X_\tau)$;
- Figure 11(b): $\epsilon(a_s) = \epsilon(b_s) = -$ and $\pi_{a_s} = \pi_{b_s} \varphi_s$, so $\varphi(\sigma, \tau) \in Z^0 \operatorname{Hom}(X_\sigma, X_\tau)$;
- Figure 11(c), (d), (e), or (f): $\epsilon(\pi_{a_s}) = +, \epsilon(\pi_{b_s}) = -$ (if it exists) and φ_s does not exist, so $\varphi(\sigma, \tau) \in Z^0 \operatorname{Hom}(X_\sigma, X_\tau)$. ☐

Lemma A.7. *$\varphi(\sigma, \tau)$ is not null-homotopic.*

Proof. For the first case in Figure 10, the identities in the form of (A.9) do not factor through π_α for any graded arrow α in $\overline{Q}$. Hence $\varphi(\sigma, \tau)$ is not null-homotopic.

For the second and third cases in Figure 10, since φ_0 in the form of (A.8) is of degree 1 or 2 and is not from the same triangle as a_1 or b_1, it does not factor through π_{a_1} or π_{b_1}. Hence $\varphi(\sigma, \tau)$ is not null-homotopic.

Assume that $\varphi(\sigma, \tau)$ is null-homotopic in the last case in Figure 10. Then there exist morphisms $\psi_{u,v} : S_{k_u} \to S_{j_v}[\kappa_v + 2]$ such that $\varphi_0 = \psi_{2,1} \circ \pi_{a_1} + \pi_{b_1} \circ \psi_{1,2}$ and $\pi_{a_i} \circ \psi_{i+1,i} + \psi_{i,i+1} \circ \pi_{b_i} + \psi_{i+2,i+1} \circ \pi_{a_{i+1}} + \pi_{b_{i+1}} \circ \psi_{i+1,i+2} = 0$ for $i \geqslant 1$. Let t be the maximal integer such that $a_i = b_i$ for $i < t$. Since $\deg\varphi_0 = 3$, by Lemma A.2 repeatedly, the morphisms $\pi_{a_i} \circ \psi_{i+1,i} + \psi_{i,i+1} \circ \pi_{b_i}$ are also nonzero and of degree 3 for $i < t$. Note that a_t and b_t are from the triangle Λ_t. All possible cases for Λ_t are shown in Figure 11, where s should be replaced by t and σ, τ should be switched for each other. It is checked case by case in the following that there is a contradiction. Hence $\varphi(\sigma, \tau)$ is not null-homotopic.

- Figure 11(a) or (b): We have $\epsilon(\pi_{a_t}) = \epsilon(\pi_{b_t})$, but $\deg\pi_{a_t} \neq \deg\pi_{b_t}$. By Lemma A.2, we have that $\pi_{a_t} \circ \psi_{t+1,t} + \psi_{t,t+1} \circ \pi_{b_t}$ is nonzero and of degree less than 3. Since a_{t+1} and b_{t+1} are not from Λ_t, by Lemma A.2, there are

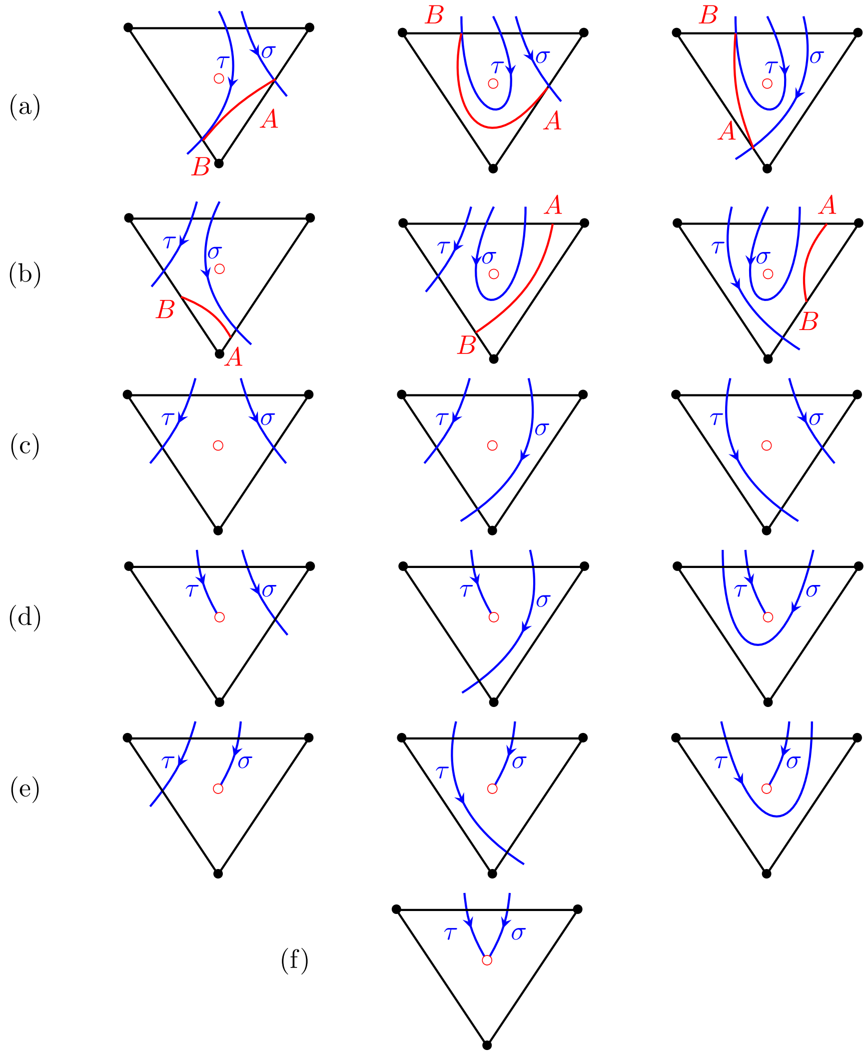

FIGURE 11. Relative positions of $\sigma(s-1,s)$ and $\tau(s-1,s)$ in the triangle Λ_s

no $\psi_{t+2,t+1}$ and $\psi_{t+1,t+2}$ satisfying $\pi_{a_t} \circ \psi_{t+1,t} + \psi_{t,t+1} \circ \pi_{b_t} + \psi_{t+2,t+1} \circ \pi_{a_{t+1}} + \pi_{b_{t+1}} \circ \psi_{t+1,t+2} = 0$. This is a contradiction.

- Figure 11(c), (d), (e), or (f): We have $\epsilon(\pi_{a_t}) = -$ and $\epsilon(\pi_{b_t}) = +$, if it exists. Then $\pi_{a_{t-1}} \circ \psi_{t,t-1} + \psi_{t-1,t} \circ \pi_{b_{t-1}}$ does not factor through π_{a_t} or π_{b_t} because of the directions of the maps. This is a contradiction. □

Combining the above two lemmas, we have the following result.

Proposition A.8. *Let σ, τ be oriented general closed arcs in $\mathbf{S}_\triangle$ with $\mathrm{Int}_{\mathbf{S}-\triangle}(\sigma,\tau) = 0$ whose starting points coincide. Then (the homotopy class of) $\varphi(\sigma,\tau)$ is a nonzero morphism in $\mathrm{Hom}_{\mathrm{per}\,\mathcal{E}_{\mathbf{T}}}(X_\sigma, X_\tau[\upsilon])$.*

In particular, $\varphi(\sigma,\tau)$ can be regarded as a morphism in $\mathrm{Hom}_{\mathrm{per}\,\mathcal{E}_{\mathbf{T}}/[1]}(\widetilde{X}(\sigma),\widetilde{X}(\tau))$.

Corollary A.9. *Let $\sigma_1,\sigma_2,\sigma_3$ be oriented general closed arcs in $\mathbf{S}_\triangle$ with $\mathrm{Int}_{\mathbf{S}-\triangle}(\sigma_i,\sigma_j)=0$ for any i,j which share the same starting point. If the start segments of σ_1,σ_2, and σ_3 are in clockwise order at the starting point, then*

$$\varphi(\sigma_2,\sigma_3)\circ\varphi(\sigma_1,\sigma_2)=\varphi(\sigma_1,\sigma_3). \tag{A.10}$$

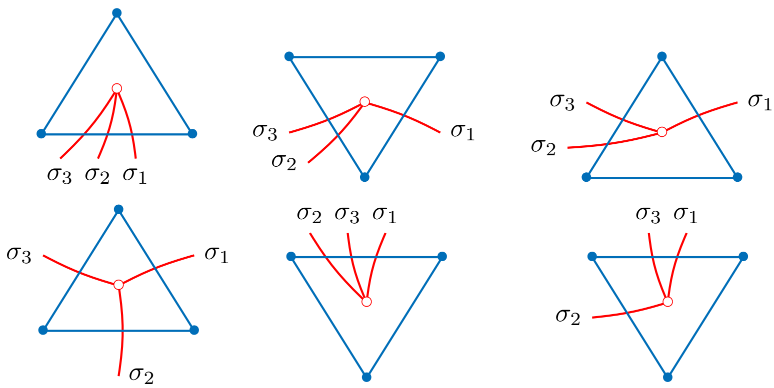

FIGURE 12. The relative position of σ_i

Proof. Consider the relative position of the first segments of σ_i. See Figure 12 for all essential cases (up to mirror). Then it is straightforward to check that $\varphi(\sigma_2,\sigma_3)\circ\varphi(\sigma_1,\sigma_2)$ is of the type in Construction A.5. Hence we are done. □

A.4. **The induced triangles.** Throughout this subsection, let σ,τ be oriented general closed arcs in $\mathbf{S}_\triangle$ with $\mathrm{Int}_{\mathbf{S}-\triangle}(\sigma,\tau)=0$. Suppose that σ and τ share the same starting point and do not coincide in $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$.

Definition A.10. The (*positive*) *extension* $\tau\wedge\sigma$ of τ by σ (w.r.t. the common starting point) is defined in Figure 13.

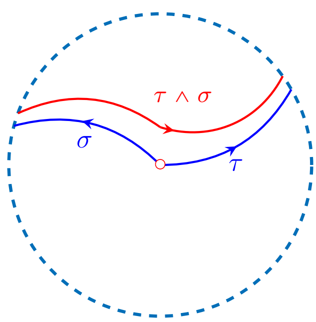

FIGURE 13. The extension

Proposition A.11. *There exists a nontrivial triangle in* $\operatorname{per}\mathcal{E}_{\mathbf{T}}$ *whose image in* $\operatorname{per}\mathcal{E}_{\mathbf{T}}/[1]$ *is*

$$\widetilde{X}(\tau\wedge\sigma)\xrightarrow{\varphi(\tau\wedge\sigma,\overline{\sigma})}\widetilde{X}(\sigma)\xrightarrow{\varphi(\sigma,\tau)}\widetilde{X}(\tau)\xrightarrow{\varphi(\overline{\tau},\overline{\tau\wedge\sigma})}\widetilde{X}(\tau\wedge\sigma).$$

Proof. Keep the notations for σ and τ in the previous subsection. Using homological algebra, the mapping cone of $\varphi(\sigma,\tau)$ is the dg $\mathcal{E}_{\mathbf{T}}$-module associated to the string arising from $\tau\wedge\sigma$. Hence we have the required triangle. □

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