

GENERALIZING THE MVW INVOLUTION, AND THE CONTRAGREDIENT

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ABSTRACT. For certain quasi-split reductive groups G over a general field F , we construct an automorphism ι_G of G over F , well-defined as an element of $\text{Aut}(G)(F)/jG(F)$, where $j : G(F) \rightarrow \text{Aut}(G)(F)$ is the inner-conjugation action of $G(F)$ on G . The automorphism ι_G generalizes (although only for quasi-split groups) an involution due to Mœglin, Vignéras, and Waldspurger for classical groups which takes any irreducible admissible representation π of $G(F)$ for G as a classical group and F a local field, to its contragredient π^\vee .

The paper also formulates a conjecture on the contragredient of an irreducible admissible representation of $G(F)$ for G as a reductive algebraic group over a local field F in terms of the (enhanced) Langlands parameter of the representation.

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1. INTRODUCTION

Let G be a reductive algebraic group over a local field F . The paper has two related goals. First, to construct an involutive automorphism ι_G of $G(F)$, generalizing an involution used by Mœglin, Vignéras, and Waldspurger in [MVW] for classical groups which takes an irreducible admissible representation π of $G(F)$ to $\iota_G(\pi) \cong \pi^\vee$, where π^\vee is the contragredient of π . We construct such an involution (well-defined as an element of $\text{Aut}(G)(F)/jG(F)$, where $j : G(F) \rightarrow \text{Aut}(G)(F)$ is the inner-conjugation action of $G(F)$ on G) for a certain class of quasi-split groups,

Received by the editors March 24, 2018.

2010 *Mathematics Subject Classification*. Primary 11F70; Secondary 22E55.

The final writing of this work was supported by a grant of the Government of the Russian Federation for the state support of scientific research carried out under the supervision of leading scientists, agreement 14.W03.31.0030, dated February 15, 2018.

in particular for those for which $2H^1(\text{Gal}(\bar{F}/F), Z(\bar{F})) = 0$, where Z is the center of G . This part of the paper—although requisitioned by representation theory of $G(F)$ for F as a local field—is purely an exercise in algebraic groups valid for a general field F . An example to keep in mind of what we do is best explained by the group $\text{SO}(2n)(F)$, for which $\text{Aut}(G)(F)/jG(F)$ is a large group but has a particularly useful element for representation theory, which is conjugation by any element of $\text{O}(2n)(F)$ which is not in $\text{SO}(2n)(F)$.

The involution ι_G , even when it exists, does not necessarily take an irreducible admissible representation of $G(F)$ to its contragredient (see Conjecture 1 for a precise statement), but perhaps it comes as close as it gets among automorphisms of $G(F)$ with this property; i.e., if there is an automorphism a_π of $G(F)$ with the property that $a_\pi(\pi) \cong \pi^\vee$, then $\iota_G(\pi) \cong \pi^\vee$.

If $F = \mathbb{R}$, there is a description by Adams in [Ad] of the contragredient in terms of what he calls a Chevalley involution—which fixes a fundamental torus H_f inside $G(\mathbb{R})$ and acts on it by $t \rightarrow t^{-1}$. We check that our involution coincides as an element of $\text{Aut}(G)(\mathbb{R})/G(\mathbb{R})$ with what Adams constructs if G is quasi-split over \mathbb{R} .

Second, we formulate a conjecture on the Langlands–Vogan parameter of the contragredient π^\vee of an irreducible admissible representation π of $G(F)$ in terms of the Langlands–Vogan parameter of π . The description of the Langlands–Vogan parameter of π^\vee has three ingredients: first and most obvious is that the parameter must go to the “dual parameter”, effected by the Chevalley involution on ${}^L G$; second, the character of the component group must go to the dual character; finally, we have to take into account the effect on “base point,” since if π has a Whittaker model for a character $\psi : N \rightarrow \mathbb{C}^\times$, π^\vee has a Whittaker model not for the character ψ , but for ψ^{-1} .

The two conjectural descriptions of the contragredient—one in terms of an automorphism of $G(F)$, and the other in terms of the Langlands–Vogan parameter of π , the first intrinsic to $G(F)$, the second quite extrinsic to the structure of $G(F)$ —are related in some sense, although neither seems to be a consequence of the other. Actually, the description of the contragredient in terms of the Langlands–Vogan parameter has larger applicability, as it sees characters of component groups which need not be self-dual, in which cases the involution ι_G fails to have the desired effect.

2. NOTATION AND OTHER PRELIMINARIES

In this paper, G will always stand for a connected reductive algebraic group over a local field F , which can be either archimedean or nonarchimedean. By a connected real reductive group, we will mean $G(\mathbb{R})$ for G as a connected reductive algebraic group defined over \mathbb{R} ; thus $G(\mathbb{R})$ might well be disconnected such as $\text{SO}(p, q)(\mathbb{R})$. We will use \mathbb{S} to denote the group of norm 1 elements in \mathbb{C}^\times and \mathbb{S}^n , its n th power.

Let W_F be the Weil group of F , and let W'_F be the Weil–Deligne group of F . Let ${}^L G(\mathbb{C}) = \widehat{G}(\mathbb{C}) \rtimes W_F$ be the L -group of G which comes equipped with a map onto W_F . Often we may simplify notation by denoting ${}^L G(\mathbb{C})$ by ${}^L G$, and $\widehat{G}(\mathbb{C})$ by \widehat{G} .

An admissible homomorphism $\varphi : W'_F \rightarrow {}^L G$ is called a Langlands parameter for $G(F)$. To an admissible homomorphism φ is associated the group of connected components $\pi_0(Z_\varphi) = Z_\varphi/Z_\varphi^0$, where Z_φ is the centralizer of φ in $\widehat{G}(\mathbb{C})$, and Z_φ^0 is its connected component containing the identity element. Two Langlands parameters

$\varphi_1, \varphi_2 : W'_F \rightarrow {}^L G(\mathbb{C})$ are said to be equivalent if $\varphi_1(w) = g_0 \varphi_2(w) g_0^{-1}$ for some $g_0 \in \widehat{G}(\mathbb{C})$.

An *enhanced Langlands parameter* is a pair (φ, μ) consisting of an admissible homomorphism $\varphi : W'_F \rightarrow {}^L G(\mathbb{C})$ and an irreducible representation μ of its group of connected components $\pi_0(Z_\varphi)$. According to the Langlands–Vogan parametrization, cf. [Vo], there should be a bijective correspondence (depending on fixing a base point, consisting of a pair (ψ, N) where ψ is a nondegenerate character of $N = N(F)$ which is a maximal unipotent subgroup in a fixed quasi-split group $G_0(F)$) between enhanced Langlands parameter (φ, μ) and pairs $(\pi, G(F))$ consisting of a pure innerform G of G_0 over F and an irreducible admissible representation π of $G(F)$. Beginning with the work of Harris, Taylor, and Henniart on the local Langlands correspondence for $\mathrm{GL}_n(F)$, the Langlands–Vogan parametrization is now supposed to be known via the work of Arthur, Mœglin, and others for all classical groups.

In this paper, we will have occasions to use the Jacobson–Morozov theorem, according to which, if G is a reductive algebraic group over a field F of characteristic 0 and $u \in G(F)$ is a unipotent element, then there is a homomorphism $\phi : \mathrm{SL}_2 \rightarrow G$ defined over F taking the upper triangular unipotent matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(F)$ to u . Given a unipotent element $u \in G(F)$, the homomorphism ϕ is unique up to conjugation by $Z_G(u)$, the centralizer of u .

If u is a regular unipotent element of $G(F)$ (which exists if and only if G is a quasi-split group over F), then the centralizer of u is up to the center of G , a unipotent group of dimension equal to the semisimple rank of $G(\bar{F})$ contained inside the unique Borel subgroup of G in which u lies. It follows that if $\phi : \mathrm{SL}_2 \rightarrow G$ corresponds to a regular unipotent element in $G(F)$ and takes the group of diagonal matrices in SL_2 inside a fixed maximal torus T of G , then such a homomorphism is unique, and hence is defined over F . Therefore, if G is a quasi-split group over F , $(G, B, T, \{X_\alpha\})$ is a pinning on G defined over F and there is a unique homomorphism $\phi : \mathrm{SL}_2 \rightarrow G$ defined over F corresponding to the regular nilpotent element $\sum X_\alpha$ in the Lie algebra of $G(F)$, taking the group of diagonal matrices in SL_2 to T .

Remark 1. It may be noted that for a quasi-split group G over any field F , if $B = TN$ is a Borel subgroup of G with unipotent radical N , and if we understand by an *algebraic Whittaker datum*, a morphism of algebraic groups $\psi : N(F) \rightarrow F$ which restricted to any simple root space in $N(\bar{F})$ is nontrivial, then there is a bijective correspondence between an algebraic Whittaker datum on G and $N(F)$ -conjugacy classes of pinnings $(G, B, T, \{X_\alpha\})$ on G , with B being the normalizer of N in G , by sending an algebraic Whittaker datum ψ to the collection of elements X_α in the α -root spaces in $N(\bar{F})$, with $\psi(X_\alpha) = 1$ for all simple roots α . For defining a homomorphism $\psi : N(F) \rightarrow F$ to be nondegenerate, there is no need to fix T . This is so because ψ is defined on $N(\bar{F})/[N(\bar{F}), N(\bar{F})]$, which is a module for $B/N \cong T$ (the “canonical” Cartan), hence decomposes as a sum of simple root spaces for $B/N \cong T$.

For F as a local field, fixing a nontrivial character $\psi_0 : F \rightarrow \mathbb{C}^\times$ gives a bijective correspondence between an algebraic Whittaker datum and a Whittaker datum on a reductive group G over F .

3. CONSTRUCTING A DUALITY INVOLUTION

The book [MVW] of Mœglin, Vignéras, and Waldspurger constructs an involution ι_G for all classical groups $G(F)$ (defined without using division algebras) which has the property that $\iota_G(\pi) \cong \pi^\vee$ for π any irreducible admissible representation of the group $G(F)$. This involution plays an important role in many results on classical groups and has often been called the MVW involution. The most well known of these involutions is for $\mathrm{GL}_n(F)$, in which case it can be taken to be $g \rightarrow {}^t g^{-1}$. It is a theorem of Gelfand and Kazhdan that this involution takes an irreducible admissible representation of $\mathrm{GL}_n(F)$ to its contragredient. For $\mathrm{Sp}_{2n}(F)$, the involution is the conjugation by an element of $\mathrm{GSp}_{2n}(F)$ with similitude -1 . Eventually, the MVW theorem is proved by a group theoretic assertion on classical groups that $\iota_G(g)$ is conjugate to g^{-1} for all $g \in G(F)$.

The aim of this section is to construct an involutive automorphism ι_G of $G(F)$ for any quasi-split group G over *any* field F which coincides with the MVW involution for classical groups, which we hope has the property that it takes an irreducible admissible representation π of $G(F)$, with F being a local field, to its contragredient (for certain groups G and certain representations π of $G(F)$ to be made precise later).

At the very least, for certain quasi-split reductive groups G , we will construct an involution ι_G on $G(F)$ which is well-defined as an element of $\mathrm{Aut}(G)(F)/G(F)$, where $\mathrm{Aut}(G)(F)$ is the group of automorphisms of G defined over F , and by abuse of notation, we denote here and elsewhere in the paper $\mathrm{Aut}(G)(F)/G(F)$ to be the quotient group of $\mathrm{Aut}(G)(F)$ by the group of inner automorphisms of G coming from elements of $G(F)$. (We call it abuse of notation because the latter group is not $G(F)$ but is $G(F)/Z(F)$.) Since the involution is being constructed to take a representation to its contragredient, its being well-defined as an element in $\mathrm{Aut}(G)(F)/G(F)$ is all that matters.

The involution ι_G will be defined using certain auxiliary data, and later we will check that it is well-defined as an element of $\mathrm{Aut}(G)(F)/G(F)$ in certain cases; it is only for such groups G for which ι_G is well-defined as an element of $\mathrm{Aut}(G)(F)/G(F)$ that ι_G plays a particularly useful role, and it is only in such cases that we will use it in this paper.

First, recall that for any reductive algebraic group G over an algebraically closed field $F = \bar{F}$, with T as a maximal torus, a Chevalley involution is an involutive automorphism of G preserving T and acting as $t \rightarrow t^{-1}$ on T . Existence of a Chevalley involution is a well-known theorem. A Chevalley involution takes any irreducible algebraic representation of G to its contragredient. Having fixed T , a Chevalley involution considered as an element in $\mathrm{Aut}(G)$ is unique up to multiplication by an inner automorphism of G by an element of T . We now fix a Borel subgroup B containing T , and will find it more convenient to multiply a Chevalley involution by an automorphism of G induced by conjugation by ω_G , where ω_G is a lift in G (in fact in $N(T)$, the normalizer of T in G) of the longest element in the Weyl group of T which takes B to the opposite Borel B^- . The resulting automorphism of G will act on T as $t \rightarrow w_G(t^{-1})$ and will preserve B , and by further multiplication by an automorphism of G induced by conjugation by an element of T , we can assume that the automorphism fixes a given pinning $\mathcal{P} = (G, B, T, \{X_\alpha\})$ of G , constructing a uniquely defined involutive automorphism $c_{G, \mathcal{P}}$ (perhaps trivial) of G , to be called the Chevalley involution associated to the pinning $\mathcal{P} = (G, B, T, \{X_\alpha\})$ of G . Since

any two pinning on G are conjugate under $G(\bar{F})$, this involutive automorphism is uniquely defined up to conjugation by an element of $G(\bar{F})$.

We will now define Chevalley involution for any quasi-split reductive group over a general field F which comes equipped with a pinning $(G, B, T, \{X_\alpha\})$.

We begin with a split group G_0 now over a general field F with a pinning $(G_0, B_0, T_0, \{X_\alpha\})$ over F . It is known that

$$\text{Aut}(G_0, B_0, T_0, \{X_\alpha\})(F) = \text{Aut}(G_0(\bar{F}), B_0(\bar{F}), T_0(\bar{F}), \{X_\alpha\}) = \text{Out}(G_0(\bar{F})),$$

where $\text{Out}(G_0(\bar{F}))$, is defined by the exact sequence,

$$1 \rightarrow \text{Int}(G_0(\bar{F})) \rightarrow \text{Aut}(G_0(\bar{F})) \rightarrow \text{Out}(G_0(\bar{F})) \rightarrow 1.$$

A quasi-split group over F is described by a (conjugacy class of) homomorphism

$$\lambda : \text{Gal}(\bar{F}/F) \rightarrow \text{Aut}(G_0(F), B_0(F), T_0(F), \{X_\alpha\}) = \text{Out}(G_0(\bar{F})),$$

allowing one to twist G_0 by the cocycle $\lambda \in H^1(\text{Gal}(\bar{F}/F), \text{Aut}(G_0))$ to construct a group G_λ over F . The twisted group G_λ continues to have $B_0(\bar{F})$ and $T_0(\bar{F})$ Galois invariant, and hence define subgroups B_λ and T_λ of G_λ over F ; further, the set of roots $\{X_\alpha\}$ is Galois invariant (although not pointwise).

The automorphism group of a quasi-split reductive group G_λ over F fixing a pinning $(G_\lambda, B_\lambda, T_\lambda, \{X_\alpha\})$ are the elements of $\text{Aut}(G_0(\bar{F}), B_0(\bar{F}), T_0(\bar{F}), \{X_\alpha\}) = \text{Out}(G_0(\bar{F}))$ which commute with the image of λ . The Chevalley involution $c_{G_0, \mathcal{P}}$ belongs to the center of $\text{Aut}(G_0(\bar{F}), B_0(\bar{F}), T_0(\bar{F}), \{X_\alpha\})$ and hence induces an involution of G_λ defined over F .

This allows one to introduce an automorphism $c_{G, \mathcal{P}}$ of $G = G_\lambda$ defined over F , fixing a pinning $\mathcal{P} = (G, B, T, \{X_\alpha\})$, which acts as $t \rightarrow w_G(t^{-1})$ on T , where w_G is the longest element in the Weyl group of T taking B to B^- , the opposite Borel subgroup. We call $c_{G, \mathcal{P}}$ the Chevalley involution associated to a pinning \mathcal{P} on G . Observe that $c_{G, \mathcal{P}} = 1$ if and only if $-1 \in W_G$ —thus in all simple groups except those of type $A_n, D_{2n+1}, E_6, c_{G, \mathcal{P}} = 1$.

Definition 1 (Duality involution). For a quasi-split reductive algebraic group G over a general field F , fix a pinning $\mathcal{P} = (G, B, T, \{X_\alpha\})$ on G , and let $c_{G, \mathcal{P}}$ be the Chevalley involution constructed above. Define an automorphism $\iota_{G, \mathcal{P}} \in \text{Aut}(G)(F)$ —to be called the duality involution—as a product of two commuting automorphisms $\iota_{G, \mathcal{P}} = \iota_- c_{G, \mathcal{P}}$, where $\iota_- \in T^{\text{ad}}(F)$ is the unique element which acts by -1 on all simple root spaces of T in B which is considered as an element of $\text{Aut}(G)(F)$ by the inner-conjugation action.

Example 1. We leave the pleasant task of identifying $\iota_{G, \mathcal{P}} = \iota_- c_{G, \mathcal{P}}$ with the MVW involution for any of the classical quasi-split groups $G = \text{Sp}_{2n}, \text{SO}_{2n}, \text{U}_n$ to the reader, except to point out that for $\text{Sp}_{2n}, c_{G, \mathcal{P}} = 1$; for $\text{SO}_{2n}, \iota_- \in T(F)$, and $c_{G, \mathcal{P}} = 1$ if n is even; for $\text{U}_n, \iota_- \in T(F)$ if n is odd.

Since any two pinning $\mathcal{P}, \mathcal{P}'$ on G are conjugate under $G^{\text{ad}}(F) = (G/Z)(F)$, the corresponding duality involutions, $\iota_{G, \mathcal{P}}$ and $\iota_{G, \mathcal{P}'}$ are related as $\iota_{G, \mathcal{P}'} = g \iota_{G, \mathcal{P}} g^{-1}$, where $g \in G^{\text{ad}}(F)$ is to be treated as an element of $\text{Aut}(G)(F)$. In more detail,

$$(*) \quad \iota_{G, \mathcal{P}'}(x) = g \iota_{G, \mathcal{P}}(g^{-1} x g) g^{-1} = g \iota_{G, \mathcal{P}}(g^{-1}) \iota_{G, \mathcal{P}}(x) \iota_{G, \mathcal{P}}(g) g^{-1} \quad \forall x \in G(\bar{F}).$$

Since $G(F)$ operates transitively on pairs (B, T) and for a given (B, T) , $T^{\text{ad}}(F)$ operates transitively on the choice of pinning $\{X_\alpha\}$, it follows that $G^{\text{ad}}(F) = G(F)T^{\text{ad}}(F)$; we record this in the following well-known lemma.

Lemma 1. *For a quasi-split reductive group G over a field F with center Z , if B is a Borel subgroup of G and T a maximal torus of B , then for $G^{\text{ad}} = G/Z$ and $T^{\text{ad}} = T/Z$, we have an isomorphism of groups,*

$$T^{\text{ad}}(F)/T(F) \cong G^{\text{ad}}(F)/G(F);$$

equivalently, $G^{\text{ad}}(F) = T^{\text{ad}}(F)G(F)$.

Write $g \in G^{\text{ad}}(F)$ as $g = g_0t$, with $g_0 \in G(F)$ and $t \in T^{\text{ad}}(F)$. Then $g\iota_{G,\mathcal{P}}(g^{-1}) = g_0t\iota_{G,\mathcal{P}}(t^{-1})\iota_{G,\mathcal{P}}(g_0^{-1})$. Since the action of $\iota_{G,\mathcal{P}}$ on T is given by $t \rightarrow w_G(t^{-1})$, $t\iota_{G,\mathcal{P}}(t^{-1}) = tw_G(t)$. Therefore, if we can prove that $tw_G(t) \in T(F)$ for all $t \in T^{\text{ad}}(F)$, it would follow from (*) that for different choices of \mathcal{P} , the automorphisms $\iota_{G,\mathcal{P}}$ differ by $G(F)$. In fact, the automorphisms $\iota_{G,\mathcal{P}}$, as we vary the pinning \mathcal{P} , differ by $G(F)$ if and only if $tw_G(t) \in T(F)$ for all $t \in T^{\text{ad}}(F)$.

From the short exact sequence of commutative algebraic groups, $1 \rightarrow Z \rightarrow T \rightarrow T^{\text{ad}} \rightarrow 1$, we get the long exact sequence of Galois cohomology groups:

$$1 \rightarrow T^{\text{ad}}(F)/T(F) \rightarrow H^1(F, Z) \rightarrow H^1(F, T).$$

It follows that $tw_G(t) \in T(F)$ for all $t \in T^{\text{ad}}(F)$ if and only if the automorphism of $H^1(F, Z)$ arising from the automorphism $z \rightarrow w_G(z^{-1})$ of Z is trivial on $\ker\{H^1(F, Z) \rightarrow H^1(F, T)\}$. Since the action of w_G on Z is trivial, the automorphism $z \rightarrow w_G(z^{-1})$ of Z is nothing but $z \rightarrow z^{-1}$. Hence the action of the automorphism $z \rightarrow w_G(z^{-1})$ on $H^1(F, Z)$ is nothing but $a \rightarrow -a$ (using additive notation now).

We record this conclusion as a proposition.

Proposition 1. *For G as a quasi-split group over a field F , the automorphism $\iota_{G,\mathcal{P}}$ constructed using a pinning \mathcal{P} on G is independent of the pinning chosen (as an element of $\text{Aut}(G)(F)/G(F)$) if and only if $2\ker\{H^1(F, Z) \rightarrow H^1(F, T)\} = 0$, in particular if $2H^1(F, Z(G)) = 0$.*

Corollary 1. *If $F = \mathbb{R}$ or $Z = Z(G)$ is an elementary abelian 2-group or $Z = Z(G)$ is a split torus or more generally an induced torus, then the duality involution $\iota_{G,\mathcal{P}}$ (as an element of $\text{Aut}(G)(F)/G(F)$) is independent of the pinning \mathcal{P} .*

Example 2. The condition $2H^1(F, Z(G)) = 0$ in Proposition 1 is satisfied for all classical groups $G = \text{GL}_n, \text{U}_n, \text{SO}_n, \text{Sp}_{2n}$. On the other hand, it is not satisfied for SL_n , and as we discuss in some detail in Example 3, there is no good duality involution on representations of $\text{SL}_n(F)$.

Using Remark 1, relating pinnings to algebraic Whittaker datum, the involution $\iota_{G,\mathcal{P}}$ can also be constructed using algebraic Whittaker datum. We quickly recall this.

Let $\psi : N(F) \rightarrow F$ be an algebraic Whittaker datum. The group G has an automorphism $\iota_{B,T,N,\psi}$ defined over F which takes the pair (T, B) to itself, and ψ to ψ^{-1} , and which operates on T as $t \rightarrow w_G(t^{-1})$; its effect on simple roots of T on $N(\bar{F})$ is that of the automorphism $-w_G$, where w_G is the longest element in the Weyl group of G over \bar{F} ; i.e., $\iota_{B,T,N,\psi}(\alpha) = -w_G(\alpha)$ for all simple roots of T inside $N(\bar{F})$. The automorphism $\iota_{B,T,N,\psi}$ can be taken to be $\iota_{G,\mathcal{P}}$ for the pinning \mathcal{P} associated to the algebraic Whittaker datum by Remark 1.

Conjecture 1. *For G as a quasi-split group over a local field F , let ι_G be the duality automorphism of G constructed above, which we assume is independent of*

the pinning used to define it as an element of $\text{Aut}(G)(F)/G(F)$ (thus imposing conditions on G such as in Proposition 1). Let π be an irreducible admissible generic representation of $G(F)$ (for any Whittaker datum); then $\pi^\vee \cong \iota_G(\pi)$. For π as any irreducible admissible representation of $G(F)$, if the Langlands parameter φ associated to π has the property that $\pi_0(Z_\varphi)$ is an elementary abelian 2-group (or its irreducible representations are self-dual), $\pi^\vee \cong \iota_G(\pi)$.

We end this section with the following question.

Question 1. If G_0, G are reductive groups over a nonarchimedean local field F which are innerforms of each other, it follows, for example, from Kaletha’s paper [Ka, Lemma 3.1] that $G_0^{\text{ad}}(F)/G_0(F)$ is naturally isomorphic to $G^{\text{ad}}(F)/G(F)$. It is possible that if G_0 is a quasi-split group, then there is a natural injection of the “outer automorphism groups” $\text{Aut}(G)(F)/G(F) \hookrightarrow \text{Aut}(G_0)(F)/G_0(F)$, with explicit knowledge of when there is an equality (perhaps when G is obtained as a pure innerform of G_0 , i.e., when G is obtained by twisting with respect to a cohomology class in $H^1(F, G_0)$?). If the group G has no outer automorphism over \bar{F} , then $\text{Aut}(G)(F) = G^{\text{ad}}(F)$, and [Ka, Lemma 3.1], mentioned at the beginning of this paragraph, already answers the question. If $\text{Aut}(G)(F)/G(F) \cong \text{Aut}(G_0)(F)/G_0(F)$, then the duality automorphism constructed in this paper as an element of the group $\text{Aut}(G_0)(F)/G_0(F)$ can be transported to an element of $\text{Aut}(G)(F)/G(F)$ and may play a role similar to what it does here, namely for any irreducible admissible representation π of $G(F)$, if the Langlands parameter φ associated to π has the property that $\pi_0(Z_\varphi)$ is an elementary abelian 2-group (or its irreducible representations are self-dual), $\pi^\vee \cong \iota_G(\pi)$; [MVW] as well as the work of Adams [Ad] in the real case supports this expectation. (Even for quasi-split groups, we are not suggesting that $\iota_G(\pi) \cong \pi^\vee$, so we will not do so for general groups.)

4. CONJECTURE ON THE CONTRAGREDIENT

We begin with the following basic lemma regarding the contragredient.

Lemma 2. *Let $G(F)$ be a quasi-split reductive algebraic group over a local field F of characteristic 0, with $B = TN$ as a Borel subgroup of G , and $\psi : N(F) \rightarrow \mathbb{C}^\times$ as a nondegenerate character. Then if an irreducible admissible representation π of $G(F)$ has a Whittaker model for the character ψ , then the contragredient π^\vee has a Whittaker model for the character $\bar{\psi} = \psi^{-1} : N(F) \rightarrow \mathbb{C}^\times$.*

Proof. Although this lemma must be a standard one, the author has not found a proof in the literature, so here is one. Observe that if π is unitary, then π^\vee is nothing but the complex conjugate $\bar{\pi}$ of π ; i.e., $\bar{\pi} = \pi \otimes_{\mathbb{C}} \mathbb{C}$, where \mathbb{C} is considered as a \mathbb{C} module using the complex conjugation action. By applying complex conjugation to a linear form $\ell : \pi \rightarrow \mathbb{C}$ on which $N(F)$ operates by ψ , we see that $\bar{\ell}$ defines a $(\bar{\psi} = \psi^{-1})$ -linear form on $\bar{\pi} = \pi^\vee$, proving the lemma for unitary representations, in particular for representations of any reductive group which are tempered up to a twist. The general case of the lemma follows by realizing a generic representation of $G(F)$ as an irreducible principal series representation $\pi = \text{Ind}_{P(F)}^{G(F)} \mu$ for an irreducible representation μ of $M(F)$ (M is a Levi subgroup of P) which is tempered up to a twist by a character of $M(F)$; that this can be done is what is called the *standard module conjecture*, for a proof of which, see [HO, Corollary 1.2] for

p -adic fields, and [Vo2] for $F = \mathbb{R}$. (It is in this appeal to [HO] that we use the characteristic zero hypothesis in the statement of the lemma.)

Now note that if N_M is the unipotent radical of a Borel subgroup in M , then there is a general recipe due to Rodier (recalled in the fourth paragraph of section 5), constructing a character $\psi_M : N_M(F) \rightarrow \mathbb{C}^\times$ from a character $\psi : N(F) \rightarrow \mathbb{C}^\times$ with the property that the induced principal series $\text{Ind}_{P(F)}^{G(F)} \mu$ is ψ -generic if and only if μ is ψ_M -generic. The recipe of Rodier has the property that if $\psi_M : N_M(F) \rightarrow \mathbb{C}^\times$ is associated to the character $\psi : N(F) \rightarrow \mathbb{C}^\times$, then $\bar{\psi}_M : N_M(F) \rightarrow \mathbb{C}^\times$ is associated to $\bar{\psi} : N(F) \rightarrow \mathbb{C}^\times$. This proves the lemma. \square

We next recall that in [GGP, section 9], denoting as G^{ad} the adjoint group of G (assumed without loss of generality at this point to be quasi-split since we want to construct something for the L -group), there is constructed a homomorphism $G^{\text{ad}}(F)/G(F) \rightarrow \pi_0(Z_\varphi)^\vee$, denoted as $g \mapsto \eta_g$ which has the following (conjectural) property on the character of component groups associated to a representation π : $\mu(\pi^g) = \mu(\pi) \otimes \eta_g$. We recall the map $g \rightarrow \eta_g$ here. For doing this, let \widehat{G}^{sc} be the universal cover of \widehat{G} :

$$1 \rightarrow \pi_1(\widehat{G}) \rightarrow \widehat{G}^{\text{sc}} \rightarrow \widehat{G} \rightarrow 1.$$

By the definition of universal cover, any automorphism of \widehat{G} lifts to an automorphism of \widehat{G}^{sc} uniquely, and thus if we are given an action of W_F on \widehat{G} (through a parameter $\varphi : W_F \rightarrow {}^L G$), it lifts uniquely to an action of W_F on \widehat{G}^{sc} , preserving $\pi_1(\widehat{G})$. Treating the above as an exact sequence of W_F -modules and taking W_F -cohomology, we get the boundary map

$$\widehat{G}^{W_F} = Z_\varphi \rightarrow H^1(W_F, \pi_1(\widehat{G})),$$

which gives rise to the map

$$\pi_0(Z_\varphi) \rightarrow H^1(W_F, \pi_1(\widehat{G}))$$

and corresponds to the map $Z_\varphi \times W_F \rightarrow \pi_1(\widehat{G})$, which is $(z, w) \rightarrow w(\tilde{z}) \cdot \tilde{z}^{-1}$, where \tilde{z} is an arbitrary lift of an element $z \in Z_\varphi$ to \widehat{G}^{sc} .

On the other hand, by the Tate duality (using the identification $Z(G)^\vee = \pi_1(\widehat{G})$), there is a perfect pairing:

$$H^1(W_F, Z(G)) \times H^1(W_F, \pi_1(\widehat{G})) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

This pairing allows one to think of elements in $H^1(W_F, Z(G))$ as characters on $H^1(W_F, \pi_1(\widehat{G}))$, and therefore using the map $\pi_0(Z_\varphi) \rightarrow H^1(W_F, \pi_1(\widehat{G}))$, as characters on $\pi_0(Z_\varphi)$. Finally, given the natural map $G^{\text{ad}}(F)/G(F) \rightarrow H^1(F, Z(G))$, we have constructed a group homomorphism from $G^{\text{ad}}(F)/G(F)$ to the group of characters on $\pi_0(Z_\varphi)$.

Remark 2. For later use, we compare the homomorphism $G^{\text{ad}}(F)/G(F) \rightarrow \pi_0(Z_\varphi)^\vee$ with a similar homomorphism constructed for a Levi subgroup in case the parameter $\varphi : W'_F \rightarrow {}^L G$ factors through ${}^L M$ as $W'_F \xrightarrow{\varphi'} {}^L M \hookrightarrow {}^L G$. (We will continue to assume G to be a quasi-split group.)

For this, we begin by noting that although there is no obvious map between $G^{\text{ad}}(F)/G(F)$ and $M^{\text{ad}}(F)/M(F)$, if Z is the center of G and T a maximal torus in G containing the maximally split torus, the natural map from $(T/Z)(F)/T(F)$ to $G^{\text{ad}}(F)/G(F)$ is an isomorphism (cf. Lemma 1). Since there is a natural map from

$(T/Z)(F)/T(F)$ to $M^{\text{ad}}(F)/M(F)$, we have a natural map from $G^{\text{ad}}(F)/G(F)$ to $M^{\text{ad}}(F)/M(F)$. This gives rise to commutative diagrams of natural homomorphisms,

$$\begin{array}{ccccc}
 G^{\text{ad}}(F)/G(F) & \longrightarrow & H^1(F, Z(G)) & \longrightarrow & \pi_0(Z_\varphi)^\vee \\
 \downarrow & & \downarrow & & \downarrow \\
 M^{\text{ad}}(F)/M(F) & \longrightarrow & H^1(F, Z(M)) & \longrightarrow & \pi_0(Z_{\varphi'})^\vee.
 \end{array}$$

Let g_0 be the unique conjugacy class in $G^{\text{ad}}(F)$ representing an element in $T^{\text{ad}}(F)$ (with T^{ad} as a maximally split, maximal torus in $G^{\text{ad}}(F)$) which acts by -1 on all simple root spaces of T on B . Denote the corresponding η_{g_0} as η_{-1} , a character on $\pi_0(Z_\varphi)$, which will be the trivial character, for example, if g_0 can be lifted to $G(F)$.

Let $c_{\widehat{G}}$ be the automorphism of \widehat{G} preserving a fixed pinning $(\widehat{G}, \widehat{B}, \widehat{T}, \{X_\alpha\})$ of \widehat{G} and acting as $t \rightarrow w_{\widehat{G}}(t^{-1})$ on \widehat{T} , where $w_{\widehat{G}}$ is the longest element in the Weyl group of \widehat{T} taking \widehat{B} to \widehat{B}^- , the opposite Borel subgroup. Observe that $c_{\widehat{G}} = 1$ if $-1 \in W_G = W_{\widehat{G}}$ —thus, in all simple groups except those of type A_n, D_{2n+1}, E_6 , $c_{\widehat{G}} = 1$.

The automorphism $c_{\widehat{G}}$ defined using a fixed pinning $(\widehat{G}, \widehat{B}, \widehat{T}, \{X_\alpha\})$ of \widehat{G} belongs to the center of the group of diagram automorphisms of \widehat{G} and hence extends to an automorphism of ${}^L G(\mathbb{C}) = \widehat{G}(\mathbb{C}) \rtimes W_F$, which will again be denoted by $c_{\widehat{G}}$ and will be called the Chevalley involution of ${}^L G$.

Conjecture 2. *Let G be a reductive algebraic group over a local field F which is a pure innerform of a quasi-split group G_0 over F which comes equipped with a fixed triple (B_0, N_0, ψ_0) used to parametrize representations on all pure innerforms of G_0 . For an irreducible admissible representation π of $G(F)$ with Langlands–Vogan parameter (φ, μ) , the Langlands–Vogan parameter of π^\vee is $(c_{\widehat{G}} \circ \varphi, (c_{\widehat{G}} \circ \mu)^\vee \otimes \eta_{-1})$, where $c_{\widehat{G}}$ is the Chevalley involution of ${}^L G(\mathbb{C})$. (Since the Chevalley involution $c_{\widehat{G}}$ acts as $z \rightarrow z^{-1}$ on $Z(\widehat{G})$, it follows that the character of the component group $(c_{\widehat{G}} \circ \mu)^\vee \otimes \eta_{-1}$ defines the same pure innerform of G_0 as that defined by μ .)*

Remark 3. A consequence of the conjecture is that if the component groups are known to be elementary abelian 2-groups for a quasi-split reductive group $G_0(F)$, such as for any reductive group if $F = \mathbb{R}$, or for classical groups defined using fields alone for any local field F , then all irreducible admissible representations of $G(F)$, a pure innerform of $G_0(F)$, are self-dual if and only if -1 belongs to the Weyl group of \widehat{G} (equivalently, $c_{\widehat{G}} = 1$), and there is an $\iota_- \in T_0(F)$, where T_0 is a maximal torus in B_0 , which acts by -1 on simple roots of T_0 in B_0 .

Example 3. We explicate Conjecture 2 for $\text{SL}_n(F)$, which in this case is an easy exercise. Let ℓ_a denote the natural (outer automorphism) action of $F^\times/F^{\times n}$ on $\text{SL}_n(F)$ (corresponding to the conjugation action of $\text{GL}_n(F)/F^\times \text{SL}_n(F) \cong F^\times/F^{\times n}$ on $\text{SL}_n(F)$, up to inner-conjugation action of $F^\times \text{SL}_n(F)$ on $\text{SL}_n(F)$). Clearly, $(\ell_a \pi)^\vee \cong \ell_a \pi^\vee$ for any irreducible representation π of $\text{SL}_n(F)$. Now fix an

irreducible representation π of $\mathrm{SL}_n(F)$ which has a Whittaker model for the character $\psi : N \rightarrow \mathbb{C}^\times$, where N is the group of upper triangular unipotent matrices in $\mathrm{SL}_n(F)$ and $\psi(n) = \psi_0(n_{1,2} + \cdots + n_{n-1,n})$ for a fixed nontrivial character ψ_0 of F . Then by Lemma 2, π^\vee has a Whittaker model for ψ^{-1} . Thus, if τ is the outer automorphism of $\mathrm{SL}_n(F)$ given by $\tau(g) = J^t g^{-1} J^{-1}$, where J is the antidiagonal matrix with each entry 1, so that τ takes (B, N, ψ) to (B, N, ψ^{-1}) , then $\tau(\pi) = \pi^\vee$, by a combination of the Gelfand–Kazhdan theorem regarding the contragredient of an irreducible admissible representation of $\mathrm{GL}_n(F)$ and the uniqueness of Whittaker model in an L -packet (i.e., given $\psi : N \rightarrow \mathbb{C}^\times$, there is a unique member in an L -packet of $\mathrm{SL}_n(F)$ with this Whittaker model) since π^\vee is the unique irreducible representation of $\mathrm{SL}_n(F)$ in its L -packet with the Whittaker model by ψ^{-1} . Having understood the action of τ on π , how about on other members $\ell_a \pi$ of the L -packet of π ? Observe that $\tau \circ \ell_a = \ell_{a^{-1}} \circ \tau$ (up to an element of $\mathrm{SL}_n(F)$). Therefore, $\tau(\ell_a \pi) = \ell_{a^{-1}}(\tau \pi) = \ell_{a^{-1}}(\pi^\vee)$. The upshot is that although τ takes π to its contragredient π^\vee , it takes $\ell_a(\pi)$ not to its contragredient but to the contragredient of $\ell_{a^{-1}}(\pi)$. Our conjectures say exactly this for $\mathrm{SL}_n(F)$, and in general too it says something similar, except that we cannot prove it for other groups since unlike for $\mathrm{SL}_n(F)$, where Whittaker models can be used to describe all members in an L -packet of representations, this is not the case in general. (The change that we see for $\mathrm{SL}_n(F)$ from ℓ_a to $\ell_{a^{-1}}$ is in general the dual representation on the component group.)

Remark 4. Conjecture 2 allows for the possibility, for groups such as G_2, F_4 , and E_8 to have non-self-dual representations arising out of component groups (which can be $\mathbb{Z}/3, \mathbb{Z}/4$, and $\mathbb{Z}/5$ in these respective cases). Indeed G_2, F_4 , and E_8 are known to have non-self-dual representations over p -adic fields arising from compact induction of certain cuspidal unipotent representations of corresponding finite groups of Lie type. An important special case of both Conjectures 1 and 2 asserts that generic representations of these groups are self-dual; if the representation happens to be generic and supercuspidal, we are totally at a loss on this question. (Global methods seem helpful here: globalizing supercuspidal generic representations π and π^\vee to globally generic cuspidal representations unramified at all finite primes except this one, a strong multiplicity one which one expects for globally generic representations, would prove $\pi \cong \pi^\vee$.)

Remark 5. For a nonarchimedean local field F , the projective (or adjoint) symplectic group $\mathrm{PGSp}_{2n}(F)$ has a unique nontrivial pure innerform defined using the unique quaternion division algebra D over F , call it $\mathrm{PGSp}_n(D)$. Every irreducible representation of $\mathrm{PGSp}_{2n}(F)$ is self-dual (by a mild variation of the theorem in [MVW, page 91] for $\mathrm{Sp}_{2n}(F)$ to $\mathrm{GSp}_{2n}(F)$, according to which for any irreducible admissible representation π of $\mathrm{GSp}_{2n}(F)$, $\pi^\vee \cong \pi \otimes \omega_\pi^{-1}$, where ω_π is the central character of π treated as a character of $\mathrm{GSp}_{2n}(F)$ using the similitude character $\mathrm{GSp}_{2n}(F) \rightarrow F^\times$.) This is in conformity with our conjecture, as $\mathrm{PGSp}_{2n}(F)$ is an adjoint group, with no diagram automorphisms; the L -group in this case is $\mathrm{Spin}_{2n+1}(\mathbb{C})$, and the possible component groups are easily seen to be extensions of subgroups of the component groups for $\mathrm{SO}_{2n+1}(\mathbb{C})$ (which are elementary abelian 2-groups) by $\mathbb{Z}/2$, but since we are looking at $\mathrm{PGSp}_{2n}(F)$, the character of the component group is supposed to be trivial on the $\mathbb{Z}/2$ coming from the center of $\mathrm{Spin}_{2n+1}(\mathbb{C})$; therefore, such representations of the component group are actually representations of an elementary abelian 2-group, in particular self-dual.

In the work [LST], the authors prove that there is no analogue of the MVW theorem for $\mathrm{Sp}_n(D)$, $n \geq 3$, or for $\mathrm{SO}_n(D)$, $n \geq 5$. Looking at their proof, it is clear that their argument also proves that not every irreducible representation of $\mathrm{PGSp}_n(D)$, or of $\mathrm{PGSO}_n(D)$, is self-dual. How does this compare with Conjecture 2? The only way out is to have more complicated component groups for $\mathrm{Spin}_{2n+1}(\mathbb{C})$, in particular having non-self-dual representations for the component groups. The author has not seen any literature on component groups for spin groups. Since $\mathrm{Spin}_5(\mathbb{C}) \cong \mathrm{Sp}_4(\mathbb{C})$, the component groups are abelian in this case; therefore, the component group in $\mathrm{Spin}_{2n+1}(\mathbb{C})$ could have non-self-dual representations only for $n \geq 3$, a condition consistent with [LST].

5. REDUCTION TO THE TEMPERED CASE

The aim of this section is to prove by using standard techniques that it suffices to prove Conjecture 2 about the contragredient for tempered representations. In doing this, we begin with a general discussion of (enhanced) Langlands parameters of Langlands quotients.

Note that if an irreducible admissible representation π of a reductive group G is obtained as the Langlands quotient of a principal series $\mathrm{Ind}_P^G \sigma$, then the Langlands parameter of π is the same as the Langlands parameter of σ considered as a parameter in ${}^L G$ via the natural embedding of the L -group ${}^L M$ in ${}^L G$ of a standard Levi subgroup M of G . Here we remind ourselves that since π is the Langlands quotient of the principal series representation $\mathrm{Ind}_P^G \sigma$, the exponent of the representation σ of M (i.e., the absolute value of the character by which the maximal split central torus of M acts on σ) is strictly positive. Under this condition, the centralizer of such a parameter φ inside \widehat{M} is the same as its centralizer in \widehat{G} . Calling these two centralizers $Z_\varphi(\widehat{M})$ and $Z_\varphi(\widehat{G})$, we have $Z_\varphi(\widehat{M}) = Z_\varphi(\widehat{G})$, and hence an equality of the group of their connected components $\pi_0(Z_\varphi(\widehat{M})) = \pi_0(Z_\varphi(\widehat{G}))$, as well as of the set of irreducible representations of their component groups.

To have the enhanced local Langlands correspondence, we also need to fix Whittaker datum for the quasi-split groups G and M . For this, we fix a Borel subgroup $B = TN$ of G with a nondegenerate character $\psi : N \rightarrow \mathbb{C}^\times$. Let w_G be a fixed element of $G(F)$ normalizing T and taking B to the opposite Borel $B^- = TN^-$; note that for the action of w_G on T as an element in the Weyl group, we have $w_G t w_G^{-1} = w_G(t)$. Similarly, for a standard Levi subgroup M of G , let w_M be a fixed element of $M(F)$ normalizing T and taking the positive roots of T in M to negative roots of T in M .

By a theorem of Rodier, it is known that a principal series representation $\pi = \mathrm{Ind}_P^G \sigma$ has a Whittaker model for a character $\psi : N \rightarrow \mathbb{C}^\times$ if and only if σ has a Whittaker model for the character $\psi_M : M \cap N \rightarrow \mathbb{C}^\times$, where ψ_M is the character of $M \cap N$ defined by $\psi_M(n) = \psi(w_G w_M n w_M^{-1} w_G^{-1})$. It is to be noted that the element w_G (as well as w_M) is well-defined only up to an element of $T(F)$, and hence the character ψ_M too is well-defined only up to conjugation by an element of $T(F)$, which is all that matters for fixing the enhanced Langlands correspondence for $M(F)$. The enhanced local Langlands correspondence will be considered for the groups G and M using a fixed nondegenerate character $\psi : N \rightarrow \mathbb{C}^\times$, and the corresponding nondegenerate character $\psi_M : M \cap N \rightarrow \mathbb{C}^\times$ for M .

The enhanced local Langlands correspondence is set up so that the representations π and σ have under the natural inclusion ${}^L M \hookrightarrow {}^L G$ the same parameters as well as the representations of their naturally isomorphic component groups.

By [BT, Theorem 4.13], the natural map from $G(F)$ to $(G/P)(F)$ is surjective, from which one deduces that the natural map from $H^1(\text{Gal}(\bar{F}/F), P)$ to $H^1(\text{Gal}(\bar{F}/F), G)$ is injective. Therefore, if M is a Levi subgroup in G , then the natural maps $H^1(\text{Gal}(\bar{F}/F), M) \rightarrow H^1(\text{Gal}(\bar{F}/F), P) \rightarrow H^1(\text{Gal}(\bar{F}/F), G)$ are all injective. As a consequence, pure innerforms of M give rise to distinct pure innerforms of G in which they lie as Levi subgroups. This allows one to associate to a Vogan L -packet on pure innerforms M_α of a quasi-split Levi group M_0 , a Vogan L -packet on pure innerforms G_α of a quasi-split group G_0 by taking the Langlands quotient of the corresponding principal series representations on G_α with M_α as the Levi subgroup.

We now come to the following crucial lemma about the contragredient in terms of Langlands quotients.

Lemma 3. *Assume that an irreducible admissible representation π of a reductive group $G = G(F)$ is obtained as a Langlands quotient $\text{Ind}_P^G \sigma \rightarrow \pi$, with σ being an irreducible representation of a Levi subgroup M of P which is tempered up to twisting by a character of M , and that the exponent of the irreducible representation σ of M restricted to the maximal split torus in the center of M is strictly positive. Let P^- be the parabolic “opposite” to P , i.e., with $P^- \cap P = M$. Let $P' = w_G w_M(P^-) = w_G(P^-)$. Then P' is a standard parabolic in G , i.e., contains the Borel subgroup B fixed earlier, and is an associate of P with $M' = w_G w_M(M) = w_G(M)$. The contragredient representation π^\vee is obtained as a Langlands quotient $\text{Ind}_{P^-}^G \sigma^\vee \rightarrow \pi^\vee$, as well as the Langlands quotient of $\text{Ind}_{P'}^G(\sigma'^\vee) \rightarrow \pi^\vee$, where σ' is the representation of $M' = w_G w_M(M)$ obtained from the representation σ of M through conjugation by $w_G w_M$.*

Proof. From [BW, chapter XI, Proposition 2.6 and Corollary 2.7], it follows that the representation π appears in $\text{Ind}_P^G \sigma$ as a quotient as well as a submodule in $\text{Ind}_{P^-}^G \sigma$; dualizing, we realize π^\vee as a quotient of $\text{Ind}_{P^-}^G \sigma^\vee$.

We note that the principal series representation $\text{Ind}_{P^-}^G \sigma^\vee$ is a standard module with strictly positive exponents, so the map $\text{Ind}_{P^-}^G \sigma^\vee \rightarrow \pi^\vee$ realizes π^\vee as a Langlands quotient.

We leave the simple checking that $P' = w_G w_M(P^-)$ is a standard parabolic in G to the reader. The natural isomorphism induced by a conjugation action by $w_G w_M$ in G taking P^- to P' , M to M' , and σ to σ' proves the assertion on π^\vee being the Langlands quotient of $\text{Ind}_{P'}^G(\sigma'^\vee)$. □

Lemma 4. *Let \widehat{G} be a connected reductive algebraic group over \mathbb{C} with a fixed Borel subgroup $\widehat{B} = \widehat{T}\widehat{N}$. Let $\widehat{P} \supset \widehat{B}$ be a standard parabolic with $\widehat{P} = \widehat{M}\widehat{N}$. Fix a pinning $(\widehat{G}, \widehat{B}, \widehat{T}, \{X_\alpha\})$ of \widehat{G} , which restricts us to giving a pinning $(\widehat{M}, \widehat{B}_{\widehat{M}}, \widehat{T}, \{X_\beta\})$ of \widehat{M} . Let $c_{\widehat{G}}$ be the automorphism of \widehat{G} preserving the pinning $(\widehat{G}, \widehat{B}, \widehat{T}, \{X_\alpha\})$ and acting as $t \rightarrow w_{\widehat{G}}(t^{-1})$ on \widehat{T} . Similarly, let $c_{\widehat{M}}$ be the automorphism of \widehat{M} preserving the pinning $(\widehat{M}, \widehat{B}_{\widehat{M}}, \widehat{T}, \{X_\beta\})$ and acting as $t \rightarrow w_{\widehat{M}}(t^{-1})$ on \widehat{T} . Fix representatives of Weyl group elements $w_{\widehat{G}}$ and $w_{\widehat{M}}$ in \widehat{G} and \widehat{M} , respectively, and by an abuse of language, denote them also as $w_{\widehat{G}}$ and $w_{\widehat{M}}$. Then there is an element*

$t_0 \in \widehat{T}$ such that

$$t_0 c_{\widehat{M}}(m) t_0^{-1} = w_{\widehat{G}} w_{\widehat{M}}(c_{\widehat{G}}(m)) w_{\widehat{M}}^{-1} w_{\widehat{G}}^{-1} \quad \forall m \in \widehat{M}. \quad (\star)$$

Proof. Observe that because $w_{\widehat{G}}(t) = w_{\widehat{G}} w_{\widehat{M}} w_{\widehat{M}}(t)$ for $t \in \widehat{T}$, the two homomorphisms of \widehat{M} into \widehat{G} ,

$$\begin{aligned} m &\longrightarrow c_{\widehat{M}}(m), \\ m &\longrightarrow w_{\widehat{G}} w_{\widehat{M}}(c_{\widehat{G}}(m)) w_{\widehat{M}}^{-1} w_{\widehat{G}}^{-1}, \end{aligned}$$

preserve \widehat{T} and are the same on \widehat{T} ; further, both of them take the root spaces α of \widehat{T} in \widehat{M} to root spaces $-w_{\widehat{G}}(\alpha)$. Therefore, the two homomorphisms of \widehat{M} into \widehat{G} differ by conjugation by an element of \widehat{T} , proving the lemma. \square

In the next section (Corollary 2), we will improve this lemma to an assertion on L -groups by being more precise about t_0, w_G, w_M .

6. EMBEDDING OF L -GROUPS

This section fills in a small detail on embedding of L -groups from embedding of their identity component required to extend Lemma 4 to an assertion on L -groups, and then to prove that Conjecture 2 is compatible with parabolic induction.

We begin by proving that for a standard (relevant) Levi subgroup \widehat{M} inside \widehat{G} , conjugation by $w_{\widehat{G}} w_{\widehat{M}}$ in \widehat{G} (for suitably chosen representatives of these elements in the Weyl group) extends to an automorphism of ${}^L G(\mathbb{C}) = \widehat{G} \rtimes W_F$ as an L -group which is the identity map on W_F ; only such morphisms of L -groups which are identities on W_F will be considered in this section. Presumably these considerations have occurred in the literature when one identifies an associate class of parabolics in a quasi-split group with an associate class of *relevant* parabolics in the L -group.

Let ${}^L G(\mathbb{C}) = \widehat{G} \rtimes W_F$ be an L -group. Denote the action of an element $\sigma \in W_F$ on $g \in \widehat{G}$ by g^σ . An automorphism of \widehat{G} given by the inner-conjugation action of $g \in \widehat{G}$ need not extend to an automorphism of ${}^L G$. Here is the lemma regarding it.

Lemma 5. *An automorphism of \widehat{G} given by the inner-conjugation action of $g_0 \in \widehat{G}$ extends to an automorphism of ${}^L G$ (which is the identity on W_F) if and only if for all $\sigma \in W_F$, $g_0^\sigma = g_0 \cdot z(\sigma)$ for some $z(\sigma) \in Z(\widehat{G})$.*

Proof. The proof is clear. \square

Next, we see some situations in which we do have elements in \widehat{G} satisfying the condition in Lemma 5.

Observe that ${}^L G(\mathbb{C}) = \widehat{G} \rtimes W_F$ comes equipped with a W_F -invariant pinning $(\widehat{G}, \widehat{B}, \widehat{T}, \{X_\alpha\})$ on \widehat{G} , which restricts to give a W_F -invariant pinning $(\widehat{M}, \widehat{B}_{\widehat{M}}, \widehat{T}, \{X_\beta\})$ on \widehat{M} for all relevant Levi subgroups in ${}^L G$.

Since \widehat{T} is invariant under W_F , its normalizer $N_{\widehat{G}}(\widehat{T})$ in \widehat{G} is also invariant under W_F , giving rise to an action of W_F on the Weyl group $W_{\widehat{G}}(\widehat{T}) = N_{\widehat{G}}(\widehat{T})/\widehat{T}$; more generally, since a relevant Levi subgroup \widehat{M} is invariant under W_F , the normalizer $N_{\widehat{M}}(\widehat{T})$ of \widehat{T} in \widehat{M} is also invariant under W_F , giving rise to an action of W_F on the Weyl group $W_{\widehat{M}}(\widehat{T}) = N_{\widehat{M}}(\widehat{T})/\widehat{T}$.

The Weyl group $W_{\widehat{M}}(\widehat{T})$ has the element $w_{\widehat{M}}$, which takes the Borel subgroup $\widehat{B}_{\widehat{M}}$ to its opposite Borel subgroup. We claim that these elements $w_{\widehat{M}}$ for any relevant Levi subgroup in ${}^L G$ are invariant (as elements of the Weyl group $W_{\widehat{M}}(\widehat{T})$) under W_F . For this, it suffices to note that for any $\sigma \in W_F$, $w_{\widehat{M}}^\sigma$ still takes the Borel subgroup $\widehat{B}_{\widehat{M}}$ to its opposite Borel, which is clear by applying σ to the equality $w_{\widehat{M}}(\widehat{B}_{\widehat{M}}) = \widehat{B}_{\widehat{M}}^-$, and noting that both $\widehat{B}_{\widehat{M}} = \widehat{B} \cap \widehat{M}$ and $\widehat{B}_{\widehat{M}}^- = \widehat{B}^- \cap \widehat{M}$ are invariant under W_F .

Lemma 6. *Let ${}^L G(\mathbb{C}) = \widehat{G} \rtimes W_F$ be an L -group. Any element in the Weyl group of \widehat{G} which is invariant under W_F is represented by an element in \widehat{G}^{W_F} up to $Z(\widehat{G})$. In particular, the longest element $w_{\widehat{M}}$ in a relevant Levi subgroup \widehat{M} in ${}^L G$ has a lift to $N_{\widehat{M}}(\widehat{T})$ which is W_F -invariant up to $Z(\widehat{G})$; i.e., there is a lift $n_{\widehat{M}}$ of $w_{\widehat{M}}$ to $N_{\widehat{M}}(\widehat{T})$ such that for all $\sigma \in W_F$, $n_{\widehat{M}}^\sigma = n_{\widehat{M}} z(\sigma)$ for some $z(\sigma) \in Z(\widehat{G})$.*

Proof. It suffices to prove the lemma assuming that \widehat{G} is an adjoint group, and further that it is an adjoint simple group, allowing us to give a proof case by case since we now have to deal only with groups of types A_n, D_n, E_6 . It is well known in these cases that W^σ is the Weyl group of \widehat{G}^σ , completing the proof of the lemma. \square

Corollary 2. *For a standard (relevant) Levi subgroup \widehat{M} inside \widehat{G} , the element $t_0 \in \widehat{T}$ in Lemma 4 has the property that $t_0^\sigma/t_0 \in Z(\widehat{M})$ for all $\sigma \in W_F$; hence we have a commutative diagram of homomorphisms of L -groups where the vertical arrows are the natural embedding of ${}^L M(\mathbb{C})$ in ${}^L G(\mathbb{C})$, the top right arrow is conjugation by t_0 , and the bottom right arrow is conjugation by $w_{\widehat{G}} w_{\widehat{M}}$, with both of these making sense on L -groups by Lemma 5.*

$$\begin{array}{ccccc}
 {}^L M(\mathbb{C}) & \xrightarrow{c_{\widehat{M}}} & {}^L M(\mathbb{C}) & \xrightarrow{t_0} & {}^L M(\mathbb{C}) \\
 \downarrow & & & & \downarrow \\
 {}^L G(\mathbb{C}) & \xrightarrow{c_{\widehat{G}}} & {}^L G(\mathbb{C}) & \xrightarrow{w_{\widehat{G}} w_{\widehat{M}}} & {}^L G(\mathbb{C}).
 \end{array}$$

Proof. Applying $\sigma \in W_F$ to the identity in Lemma 4,

$$t_0 c_{\widehat{M}}(m) t_0^{-1} = w_{\widehat{G}} w_{\widehat{M}} (c_{\widehat{G}}(m)) w_{\widehat{M}}^{-1} w_{\widehat{G}}^{-1} \quad \forall m \in \widehat{M},$$

and noting that the Weyl group elements $w_{\widehat{G}}$ and $w_{\widehat{M}}$ have lifts (by Lemma 6) to \widehat{G} which are σ -invariant for all $\sigma \in W_F$, it is clear that $t_0^\sigma/t_0 \in Z(\widehat{M})$ for all $\sigma \in W_F$.

The commutativity of the diagram now is just this identity from Lemma 4 interpreted for L -groups. \square

Proposition 2. *Let M be a quasi-split reductive group over a local field F , with \widehat{M} being the dual group, \widehat{T} the standard maximal torus in \widehat{M} , and \widehat{Z} the center of \widehat{M} . Then the natural map on cohomologies, $H^1(W_F, \widehat{Z}) \rightarrow H^1(W_F, \widehat{T})$, is injective.*

Proof. By the long exact sequence associated to the short exact sequence of W_F -modules: $0 \rightarrow \widehat{Z} \rightarrow \widehat{T} \rightarrow \widehat{T}/\widehat{Z} \rightarrow 0$, it suffices to prove that the map $H^0(W_F, \widehat{T}) \rightarrow H^0(W_F, \widehat{T}/\widehat{Z})$ is surjective and, equivalently, that the map $H^0(W, \widehat{T}) \rightarrow H^0(W, \widehat{T}/\widehat{Z})$ is surjective where W is the finite quotient of W_F through which it operates on \widehat{T} . The surjectivity of the map $H^0(W, \widehat{T}) \rightarrow H^0(W, \widehat{T}/\widehat{Z})$ would follow if we can prove that $(\widehat{T}/\widehat{Z})^W$ is a connected torus over \mathbb{C} , which is what we do now.

Note that there is a natural isomorphism

$$\widehat{T}/\widehat{Z} \cong \prod_{\alpha \text{ simple}} \mathbb{G}_m,$$

which is W -equivariant, where W operates on $\prod_{\alpha \text{ simple}} \mathbb{G}_m$, by permuting coordinates. Therefore,

$$(\widehat{T}/\widehat{Z})^W \cong \prod_{\langle \alpha \rangle} \mathbb{G}_m,$$

where the product is taken over the orbits of W on the set of simple roots. This completes the proof of the proposition. □

Corollary 3. *There is a choice of the element $t_0 \in \widehat{T}$ appearing in Lemma 4, with*

$$t_0 c_{\widehat{M}}(m) t_0^{-1} = w_{\widehat{G}} w_{\widehat{M}}(c_{\widehat{G}}(m)) w_{\widehat{M}}^{-1} w_{\widehat{G}}^{-1} \quad \forall m \in \widehat{M},$$

for which $t_0^\sigma = t_0$ for all $\sigma \in W_F$, and hence the top right arrow in the commutative diagram in Corollary 2 can be considered an identity at the level of Langlands parameters.

Proof. By Corollary 2, we have $t_0^\sigma/t_0 \in \widehat{Z}$ for all $\sigma \in W_F$, defining a 1-cocycle of W_F in \widehat{Z} which is a coboundary in \widehat{T} , and hence is a coboundary in \widehat{Z} by Proposition 2; i.e., there is a $z_0 \in \widehat{Z}$ with $t_0^\sigma/t_0 = z_0/z_0^\sigma$ for all $\sigma \in W_F$. Therefore, $(z_0 t_0)^\sigma = (z_0 t_0)$ for all $\sigma \in W_F$. One can replace t_0 by $z_0 t_0$ in the identity

$$t_0 c_{\widehat{M}}(m) t_0^{-1} = w_{\widehat{G}} w_{\widehat{M}}(c_{\widehat{G}}(m)) w_{\widehat{M}}^{-1} w_{\widehat{G}}^{-1}, \quad \forall m \in \widehat{M},$$

and for such a t_0 now with $t_0^\sigma = t_0$ for all $\sigma \in W_F$, the top right arrow in the commutative diagram above is clearly identity at the level of Langlands parameters (by the definition of equivalence of Langlands parameters). □

We finally have the following proposition we have been after in this section.

Proposition 3. *Assume that an irreducible admissible representation π of a reductive group $G = G(F)$ is obtained as a Langlands quotient $\text{Ind}_P^G \sigma \rightarrow \pi$, with σ being an irreducible representation of a Levi subgroup M of P which is tempered up to twisting by a character of M , and that the exponents of the representation σ of M restricted to the maximal split torus in the center of M are strictly positive. Then if Conjecture 2 holds true for the representation σ of M , it does also for the representation π of G .*

Proof. By Lemma 3 and the discussion in the beginning of the section 5, the enhanced Langlands parameter of π^\vee is the same as that of σ^\vee conjugated by $w_G w_M$ in ${}^L G$. By Lemma 6, the Weyl group element $w_G w_M$ has a lift in \widehat{G} which is W_F -invariant, and therefore conjugation by $w_G w_M$ is the identity map on parameters for G .

Since we are assuming that Conjecture 2 is true for tempered representations, the enhanced Langlands parameter of σ^\vee with values in ${}^L M$ arises by using the Chevalley involution on ${}^L M$. We are now done regarding the assertion on the Langlands parameter of π^\vee by the commutative diagram of L -groups contained in Corollary 2 combined with Corollary 3. Remark 2 fixes the twisting by the character η_- which appears in the conjecture. \square

Remark 6. Most of the difficulties of this section would not be there if our groups were innerforms of split groups instead of quasi-split groups.

Remark 7. If $\text{Ind}_P^G \sigma = \pi$, with σ being a unitary tempered representation of $M(F)$, is irreducible, then again the Langlands parameter of σ with values in ${}^L M$ when considered inside ${}^L G$ through the natural embedding of ${}^L M$ inside ${}^L G$ is the Langlands parameter of π ; further, for π to be irreducible, presumably their component groups are isomorphic too. Therefore, once again, Conjecture 2 for the representation σ of $M(F)$ implies Conjecture 2 for the representation π of $G(F)$.

7. COMPARING WITH THE WORK OF ADAMS

J. Adams in [Ad] has constructed the contragredient of any representation of $G(\mathbb{R})$ in terms of what he calls the Chevalley involution C : an automorphism of $G(\mathbb{R})$ of order 2 which leaves a fundamental Cartan subgroup H_f invariant and acts by $h \rightarrow h^{-1}$ on H_f . (A fundamental Cartan subgroup of $G(\mathbb{R})$ is a Cartan subgroup of G defined over \mathbb{R} which is of minimal split rank; such Cartan subgroups are conjugate under $G(\mathbb{R})$.) One of the main theorems of Adams is that $\pi^C \cong \pi^\vee$ for any irreducible representation π of $G(\mathbb{R})$. He uses this to prove that every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if -1 belongs to the Weyl group of H_f in $G(\mathbb{R})$ (in particular -1 belongs to the Weyl group of $G(\mathbb{C})$).

Assuming G to be a quasi-split reductive group over \mathbb{R} , with (B, T) being a pair of a Borel subgroup and a maximal torus in it, our recipe for constructing the contragredient is in terms of the duality involution ι_G of $G(\mathbb{R})$, which has the form $\iota_G = \iota_- \cdot c_G$, where ι_- is in an element in $T^{\text{ad}}(\mathbb{R})$ which acts by -1 on all simple root spaces of T in B , and where $c_G = c_{G, \mathcal{P}}$ is the Chevalley automorphism of $\mathcal{P} = (G, B, T, \{X_\alpha\})$. In this section, we check that the duality involution constructed out of consideration of the Whittaker model is the same as that of Adams constructed using fundamental tori. (This gives credence to our recipe on the duality involution ι_G .) Much of the section uses the “principal SL_2 ” arising from a regular unipotent element by the Jacobson–Morozov theorem, and transporting an analysis inside this principal SL_2 to the whole of G . Along the way, we also give a proof of the theorem of Adams on the existence of Chevalley involution for any reductive group over \mathbb{R} .

The following well-known lemma has the effect of saying that the contragredient which appears in the character of component group in Conjecture 2 plays no role for real groups.

Lemma 7. *Let G be a connected real reductive group with a Langlands parameter $\varphi : W_{\mathbb{R}} \rightarrow {}^L G$. Then the group $\pi_0(Z_\varphi)$ is an elementary abelian 2-group.*

Proof. From the structure of $W_{\mathbb{R}} = \mathbb{C}^\times \cdot \langle j \rangle$, with $j^2 = -1$, the group Z_φ can be considered the fixed points of the involution, which is $\varphi(j)$, on the centralizer of $\varphi(\mathbb{C}^\times)$ in \widehat{G} . The group $\varphi(\mathbb{C}^\times)$ being connected, abelian, and consisting of

semisimple elements, its centralizer in \widehat{G} is a connected reductive group. It suffices then to prove the following lemma. \square

Lemma 8. *Let H be a connected reductive group over \mathbb{C} , and let j be an involution on H with H^j as its fixed points. Then the group of connected components $\pi_0(H^j)$ is an elementary abelian 2-group.*

Proof. According to Cartan, the involution j gives rise to a real structure on H , i.e., there is a connected real reductive group H_j (with H being its complexification) on which j operates—and operates as a Cartan involution on $H_j(\mathbb{R})$; thus, $H_j(\mathbb{R})^j$ is a maximal compact subgroup of $H_j(\mathbb{R})$, and H^j is the complexification of the compact group $H_j(\mathbb{R})^j$. Therefore, $\pi_0(H_j(\mathbb{R})) = \pi_0(H_j(\mathbb{R})^j) = \pi_0(H^j)$. Finally, it suffices to note that the group of connected components of an algebraically connected real reductive group is an elementary abelian 2-group; cf. [BT, corollaire 14.5], whose authors attribute the result to Matsumoto. \square

Here is the main result of this section proving the existence of Chevalley involution (in the sense of Adams) for all real reductive groups, and at the same time identifying it as our duality involution ι_G when G is quasi-split over \mathbb{R} .

Proposition 4. *Let $H_f(\mathbb{R})$ be a fundamental maximal torus in a real reductive group $G(\mathbb{R})$. Then there exists an involutive automorphism C of $G(\mathbb{R})$ which acts as $t \rightarrow t^{-1}$ on H_f . Further, if $G(\mathbb{R})$ is quasi-split with a Borel subgroup B of $G(\mathbb{R})$, and T as a maximal torus in B , then we can choose H_f as well as a pinning $(G, B, T, \{X_\alpha\})$ on $G(\mathbb{R})$ so that $C = \iota_G = \iota_- \cdot c_G$.*

Proof. First we prove the result assuming $G(\mathbb{R})$ to be quasi-split with a Borel subgroup B defined over \mathbb{R} containing a maximal torus T defined over \mathbb{R} .

Using the Jacobson–Morozov theorem, we construct $\phi : \mathrm{SL}_2(\mathbb{R}) \rightarrow G(\mathbb{R})$, associated to a regular unipotent element inside the Borel subgroup B of $G(\mathbb{R})$. We can assume that ϕ commutes with diagram automorphisms of $G(\mathbb{R})$, i.e., with automorphism group of $\mathcal{P} = (G, B, T, \{X_\alpha\})$, in particular with the diagram automorphism corresponding to $c_G = c_{G, \mathcal{P}}$, the Chevalley involution. We will consider the semidirect product $G(\mathbb{R}) \rtimes \langle c_G \rangle$.

Observe that a regular unipotent element in $B(\mathbb{R})$ has nonzero components in each of the (relative) simple root spaces. An element $t_0 \in T(\mathbb{C})$ which operates on all simple roots of B by -1 is unique up to $Z(\mathbb{C})$ and therefore can be taken to be the image of $j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ up to central elements.

Let $\mathbb{S}^1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \subset \mathrm{SL}_2(\mathbb{R})$ (with $a^2 + b^2 = 1$). We claim that the centralizer of $\phi(\mathbb{S}^1)$ inside $G(\mathbb{C})$ is a fundamental torus; i.e., the centralizer is a maximal torus defined over \mathbb{R} such that its split rank over \mathbb{R} is the minimal possible among maximal tori over \mathbb{R} . For this, we first prove that the centralizer of $\phi(\mathbb{S}^1)$ in $G(\mathbb{C})$ is a maximal torus. To prove this, it suffices to prove a similar statement for the centralizer in $G(\mathbb{C})$ of the image of the diagonal torus in $\mathrm{SL}_2(\mathbb{R})$, which is a standard and simple result. Next, we prove that the centralizer of $\phi(\mathbb{S}^1)$ inside $G(\mathbb{C})$ is a fundamental torus. For this, we can assume that $\phi(\mathbb{S}^1)$ lands inside K^0 , the connected component of identity of a maximal compact subgroup K of $G(\mathbb{R})$. Since the centralizer of $\phi(\mathbb{S}^1)$ inside $G(\mathbb{R})$ is a torus, this is also the case for the centralizer of $\phi(\mathbb{S}^1)$ inside K^0 . Thus, the centralizer of $\phi(\mathbb{S}^1)$ inside K^0 is a

maximal torus of K^0 , but any maximal torus of $G(\mathbb{R})$ containing a maximal torus of K^0 is a fundamental torus of $G(\mathbb{R})$, proving that the centralizer of $\phi(\mathbb{S}^1)$ in G is a fundamental torus. Denote the centralizer of $\phi(\mathbb{S}^1)$ inside $G(\mathbb{R})$ as $H_f(\mathbb{R})$.

The fundamental torus H_f in $G(\mathbb{R})$ is normalized by $\phi(j)$. Since $\phi(\mathbb{S}^1)$ commutes with c_G , c_G leaves H_f invariant. Since c_G operates on T as $-w_G$ and there is an element in $G(\mathbb{R})$ which operates on T as w_G , there is an element in $G(\mathbb{R}) \rtimes \langle c_G \rangle$ which acts by -1 on T . Since all maximal tori are conjugate in $G(\mathbb{C})$, there is an element in $G(\mathbb{C}) \rtimes \langle c_G \rangle$ which acts by -1 on H_f . Call this element $t_0 \cdot c_G$. Since c_G commutes with $\phi(\mathbb{S}^1)$ and $t_0 \cdot c_G$ acts by -1 on H_f , it follows that t_0 acts by -1 on $\phi(\mathbb{S}^1)$, and the same for $\phi(j)$. Therefore, $\phi(j)t_0^{-1}$ commutes with $\phi(\mathbb{S}^1)$ and so belongs to $H_f(\mathbb{C})$. Write $\phi(j) = s^{-1}t_0$, with $s \in H_f(\mathbb{C})$. Then $t_0 \cdot c_G = s\phi(j) \cdot c_G$ acts by -1 on $H_f(\mathbb{C})$; therefore, since $s \in H_f(\mathbb{C})$, so does $\phi(j) \cdot c_G$, which is an automorphism of $G(\mathbb{R})$ since the inner-conjugation action of $\phi(j)$ is real, proving the existence of the Chevalley involution (in the sense of Adams, i.e., one which operates on H_f as $t \rightarrow t^{-1}$) for quasi-split groups, and at the same time realizing this Chevalley involution as $\phi(j) \cdot c_G$, where c_G is the Chevalley involution associated to a pinning on G . Since $\phi(j)$ acts by -1 on all simple root spaces of T in B , $\phi(j)$ can be taken to be ι_- .

For a general $G(\mathbb{R})$ with G_0 as a quasi-split innerform, we appeal to a result of Borovoi (cf. [Bor]), according to which $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), H_f) \rightarrow H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G_0)$ is surjective. We will use this result of Borovoi for the adjoint group $G_0^{\text{ad}} = G_0/Z$, and as a result, any innerform of $G_0(\mathbb{R})$ is obtained by twisting by a cocycle with values in $(H_f/Z)(\mathbb{C})$. It can easily be seen that if G_c is obtained by twisting $G_0(\mathbb{R})$ by an element $c \in H_f(\mathbb{C})$ with $c\bar{c} \in Z(\mathbb{C})$, then if C is a Chevalley involution (in the sense of Adams) on $G_0(\mathbb{R})$, $C' : x \rightarrow C(c^{-1}xc)$ is a Chevalley involution on $G_c(\mathbb{R})$ acting as -1 on H_f , which continues to be a fundamental torus in $G_c(\mathbb{R})$. We check that C' is defined over \mathbb{R} for the real structure defining G_c . If $x \rightarrow \bar{x}$ is the original complex conjugation and $x \rightarrow \underline{x}$ is associated to the real structure G_c , then by definition of twisting, $\underline{x} = c\bar{x}c^{-1}$. To check that C' is defined over \mathbb{R} , we need to check that $C'(\underline{x}) = \overline{C'(x)}$, i.e.,

$$C'(c\bar{x}c^{-1}) = \overline{C'(x)}c^{-1}.$$

Using the definition of $C'(x) = C(c^{-1}xc) = cC(x)c^{-1}$, we can rewrite the above as

$$C(\bar{x}) = c\bar{c}\overline{C(x)}c^{-1},$$

but since $c\bar{c} \in Z(C)$ and C is defined over \mathbb{R} , this equation holds. □

ACKNOWLEDGMENTS

This work is of older vintage and was part of my work [DP], which is still incomplete. The work of Kaletha [Ka] already contains Conjecture 2, and Kaletha arrived at it independently. Since this work and that of Kaletha do not have much else in common, it is hoped that this paper may still have some utility. There is also the work of Adams and Vogan [AV] formulating and proving (for $F = \mathbb{R}$) Conjecture 2 just for Langlands parameters. A recent preprint [Li] of Li deals with the recipe for the contragredient in the positive characteristic. The author thanks one of the referees of this paper for a very meticulous job.

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