

## COMPARE TRIANGULAR BASES OF ACYCLIC QUANTUM CLUSTER ALGEBRAS

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ABSTRACT. Given a quantum cluster algebra, we show that its triangular bases defined by Berenstein and Zelevinsky and those defined by the author are the same for the seeds associated with acyclic quivers. This result implies that the Berenstein–Zelevinsky basis contains all of the quantum cluster monomials.

We also give an easy proof that the two bases are the same for the seeds associated with bipartite skew-symmetrizable matrices.

### 1. INTRODUCTION

**1.1. Cluster algebras.** In [FZ02], Fomin and Zelevinsky invented cluster algebras as a combinatorial approach to dual canonical bases of quantum groups [Lus90, Kas90]. The quantum cluster algebras were later introduced in [BZ05]. These algebras possess many seeds, which are constructed recursively by an algorithm called mutation. Every seed consists of some skew-symmetrizable matrix and a collection of generators called (quantum) cluster variables. We might view these seeds as analogues of local charts of algebraic varieties.<sup>1</sup>

There are many attempts to “good” bases of cluster algebras; see [GLS11, GLS12, GLS13, MSW13, Thu14, Nak11, KQ14, Qin14, LLZ14, LLRZ14, GHKK18, Qin17, KKKO18]. In view of the original motivation of Fomin and Zelevinsky, a good basis should contain all of the quantum cluster monomials (monomials of quantum cluster variables belonging to the same seed).

**1.2. Berenstein–Zelevinsky’s triangular basis approach.** In [BZ14], Berenstein and Zelevinsky proposed the following new approach to good bases of quantum cluster algebras:

- Inspired by the Kazhdan–Lusztig theory, construct a triangular basis  $C^t$  in each seed  $t$  such that it contains all of the quantum cluster monomials in that seed. More precisely, first construct a basis consisting of some ordered products of quantum cluster variables. Then Lusztig’s lemma [BZ14, Theorem 1.1], [Lus90, 7.10] guarantees a unique new basis whose transition matrix from the old one is unitriangular, whence the name triangular basis.

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<sup>1</sup>In fact, we have a family of varieties called cluster varieties, whose local charts are tori, cluster variables are local coordinate functions, and transition maps are determined by the matrices in the seeds; see [FG09].

- Prove that these triangular bases give rise to a common basis for all seeds.

If this approach works, then we have a common triangular basis containing the quantum cluster monomials in all seeds. However, Berenstein–Zelevinsky’s construction works only for those special seeds of acyclic type; see Section 2.3 for the definition. They arrived at a common basis for the acyclic seeds, which we call the BZ-basis and denote by  $C$ .

On the other hand, it is known that the quantum cluster algebras associated with an acyclic quiver and  $z$ -coefficient pattern are isomorphic to some quantum unipotent subgroups and, consequently, inherit the dual canonical bases; see [GLS13, KQ14]. In [KQ14], Kimura and Qin showed that, for such quantum cluster algebras, the dual canonical bases contain all of the quantum cluster monomials. It is natural to propose the following conjecture.

**Conjecture 1.1.** *For quantum cluster algebras associated with an acyclic quiver and  $z$ -coefficient pattern, its dual canonical basis agrees with Berenstein–Zelevinsky’s triangular basis  $C$ .*

The verification of this conjecture would imply the desired property, that Berenstein–Zelevinsky’s triangular basis contains all quantum cluster monomials.

**1.3. Different triangular bases in monoidal categorification.** Inspired by this new approach of Berenstein and Zelevinsky, in [Qin17], in order to prove monoidal categorification conjectures of quantum cluster algebras, Qin introduced very different triangular bases for injective-reachable quantum cluster algebras. For every seed  $t$ , we can define such a triangular base  $\mathbf{L}^t$ ; see Section 2.2.

There are two crucial differences between the common triangular basis  $\mathbf{L}$  in [Qin17] and the basis  $C$  of Berenstein and Zelevinsky:

- (1) The basis is unique, but its existence cannot be guaranteed because Lusztig’s lemma does not apply.
- (2) The expectation from the Fock–Goncharov basis conjecture is included in the definition and plays an important role.

**1.4. Results.** We have two very different constructions of triangular bases. It is desirable to compare these bases, which are both defined for acyclic seeds. The main result of this paper claims that they are the same for quantum cluster algebras arising from acyclic skew-symmetric matrices (or, equivalently, from acyclic quivers).

**Theorem 1.2** (Main result). *Let  $\mathcal{A}$  be a quantum cluster algebra who has a seed  $t$  with an acyclic skew-symmetric matrix  $B(t)$ . Then in this seed, its triangular basis  $\mathbf{L}^t$  in [Qin17] agrees with Berenstein–Zelevinsky’s triangular basis  $C$ .*

Notice that, for the quantum cluster algebra arising from an acyclic quiver and  $z$ -coefficient pattern, its common triangular bases in [Qin17] is the dual canonical basis. Therefore, our main result Theorem 1.2 implies Conjecture 1.1.

Our proof is based on ideas and techniques developed by the author in [Qin17], in particular, the maximal degree tracking and the composition of unitriangular transitions. The triangular bases treated in this paper are much easier than those in [Qin17], and our paper does not depend on the long proof there. In particular, we give a self-contained proof that the triangular bases  $\mathbf{L}^t$  in different acyclic seeds  $t$  are the same; see Theorem 3.4.

We could further propose the following natural conjecture.

**Conjecture 1.3.** *The triangular basis  $\mathbf{L}^t$  agrees with Berenstein–Zelevinsky’s triangular basis  $C$  in seeds associated with acyclic skew-symmetrizable seeds.*

In a previous private communication with Zelevinsky, Qin pointed out that for bipartite orientation, this conjecture is true. The details will be given in the last part of the paper; see Theorem 3.10.

2. PRELIMINARIES

**2.1. Quantum cluster algebras.** We recall the definition of quantum cluster algebras by [BZ05] and follow the convention in [Qin17]. Let  $[x]_+$  denote  $\max(x, 0)$ . Let  $\tilde{B}$  be an  $m \times n$  integer matrix with  $n \leq m$ . Its  $n \times n$  upper submatrix  $B$  is called the principal part. Assume that  $\tilde{B}$  is of rank  $n$  and that  $B$  is skew-symmetrizable (namely, there exists a diagonal matrix with strictly positive integer diagonal entries such that its product with  $B$  is skew-symmetric). We can choose  $\Lambda$  as an  $m \times m$  skew-symmetric integer matrix such that  $\tilde{B}^T \Lambda = (D \ 0)$  for some diagonal matrix  $D$  with strictly positive integer diagonal entries. Such a pair  $(\tilde{B}, \Lambda)$  is called a *compatible pair*.

**Definition 2.1.** A quantum seed  $t$  is a triple  $(\tilde{B}(t), \Lambda(t), (X_i(t))_{1 \leq i \leq m})$ , where  $(\tilde{B}, \Lambda)$  is a compatible pair and  $X_i(t)$ ,  $1 \leq i \leq m$  are indeterminates (called  $X$ -variables).

We often omit the symbol  $(t)$  when the context is clear.

Let  $\{e_i\}$  denote the natural basis of  $\mathbb{Z}^m$  and  $X(t)^{e_i} = X_i(t)$ . Given any quantum seed  $t$ , we define the corresponding quantum torus  $\mathcal{T}(t)$  to be the Laurent polynomial ring  $\mathbb{Z}[q^{\pm \frac{1}{2}}][X(t)^g]_{g \in \mathbb{Z}^m}$  with the usual addition “+”, the usual multiplication “.”, and the twisted product

$$X(t)^g * X(t)^h = q^{\frac{1}{2}\Lambda(t)(g,h)} X(t)^{g+h},$$

where  $\Lambda(t)(\ , \ )$  denote the bilinear form on  $\mathbb{Z}^m$  such that

$$\Lambda(t)(e_i, e_j) = \Lambda(t)_{ij}.$$

$\mathcal{T}(t)$  admits a bar-involution  $\overline{(\ )}$  which is  $\mathbb{Z}$ -linear such that

$$\overline{q^s X(t)^g} = q^{-s} X(t)^g.$$

Notice that all Laurent monomials in  $\mathcal{T}(t)$  commute with each other up to a  $q$ -power. We say that they  $q$ -commute.

Let  $b_{ij}$  denote the  $(i, j)$ -entry of  $\tilde{B}(t)$ . We define the  $Y$ -variables to be the following Laurent monomials:

$$Y_k(t) = X(t)^{\sum_{1 \leq i \leq m} [b_{ik}]_+ e_i - \sum_{1 \leq j \leq m} [-b_{jk}]_+ e_j}.$$

By [BZ05, Proposition 4.7], for any integer  $1 \leq k \leq n$ , the following operation (called the mutation  $\mu_k$ ) gives us a new seed  $t' = \mu_k t = ((X_i(t'))_{1 \leq i \leq m}, \tilde{B}(t'), \Lambda(t'))$ :

- $X_i(t') = X_i(t)$  if  $i \neq k$ ;
- $X_k(t') = X(t)^{-e_k + \sum_i [b_{ik}]_+ e_i} + X(t)^{-e_k + \sum_j [-b_{jk}]_+ e_j}$ ;
- $\tilde{B}(t') = (b'_{ij})$  is determined by  $\tilde{B}(t) = (b_{ij})$ :
 
$$\begin{cases} b'_{ik} &= -b_{ki} \\ b'_{ij} &= b_{ij} + [b_{ik}]_+ [b_{kj}]_+ - [-b_{ik}]_+ [-b_{kj}]_+ \quad \text{if } i, j \neq k; \end{cases}$$

- $\Lambda(t')$  is skew-symmetric and satisfies

$$\begin{cases} \Lambda(t')_{ij} &= \Lambda(t)_{ij}, & i, j \neq k, \\ \Lambda(t')_{ik} &= \Lambda(t)(e_i, -e_k + \sum_j [-b_{jk}]_+ e_j), & i \neq k. \end{cases}$$

Notice that the mutation is an involution:  $\mu_k(\mu_k(t)) = t$ .

Similarly, the quantum torus  $\mathcal{T}(t')$  for the new seed  $t'$  is defined to be the Laurent polynomial ring  $\mathbb{Z}[q^{\pm\frac{1}{2}}][X(t')^g]_{g \in \mathbb{Z}^m}$  with the usual addition “+”, the usual multiplication “·”, and the twisted product

$$X(t')^g * X(t')^h = q^{\frac{1}{2}\Lambda(t')(g,h)} X(t')^{g+h}.$$

Notice that, by [BZ05, Proposition 6.2], any  $Z \in \mathcal{T}(t) \cap \mathcal{T}(t')$  is bar-invariant in  $\mathcal{T}(t)$  if and only if it is bar-invariant in  $\mathcal{T}(t')$ .

We define a quantum cluster algebra  $\mathcal{A}$  as the following:

- Choose an initial quantum seed  $t_0 = ((X_1, \dots, X_m), \tilde{B}, \Lambda)$ .
- All of the quantum seeds  $t$  are obtained from  $t_0$  by iterated mutations at directions  $1 \leq k \leq n$ .
- $\mathcal{A} = \mathbb{Z}[q^{\pm\frac{1}{2}}][X_{n+1}^{-1}, \dots, X_m^{-1}][X_i(t)]_{t, 1 \leq i \leq m}$ .

The  $X$ -variables  $X_i(t)$  in the seeds are called the quantum cluster variables. We call  $X_{n+1}, \dots, X_m$  the frozen variables, or the coefficients.

*Remark 2.2.* By the definition of mutation, all quantum cluster variables are obtained from the initial ones  $(X_i)_{1 \leq i \leq m}$  by iteratively applying rational maps. Therefore, they belong to the skew-field of fractions of the initial quantum torus  $\mathcal{T}(t_0) = \mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm}, \dots, X_m^{\pm}]$ . Consequently, the quantum cluster algebra is a  $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ -subalgebra of this skew-field.

The correction technique developed in [Qin14, Section 9] provides a convenient tool for studying the bases of  $\mathcal{A}$ ; see [Qin17, Section 5] for a summary. It tells us that most phenomena and properties of bases keep unchanged when we change the coefficient part of the seed  $t$ , namely, the lower  $(m - n) \times n$  submatrix  $B^c(t)$  of  $\tilde{B}(t)$ , or when we change  $\Lambda(t)$ .

Finally, we recall the correspondence between quivers and matrices. By a quiver, we mean a finite oriented graph, which we always assume to contain no loops or 2-cycles throughout this paper. Its rank is defined as the number of its vertices. Then, to each rank  $n$  quiver  $Q$ , we can associate an  $n \times n$  skew-symmetric matrix  $B$  such that its entry  $b_{ij}$  is given by the difference of the number of arrows from  $i$  to  $j$  with that of  $j$  to  $i$ . All skew-symmetric matrices arise in this way. So, if the matrix  $B(t)$  of a seed  $t$  is skew-symmetric, we say that  $t$  is skew-symmetric or  $t$  arises from a quiver; if  $B(t)$  is skew-symmetrizable, we say  $t$  is skew-symmetrizable.

**2.2. Triangular basis.** Choose any seed  $t$ . We recall the following notions introduced in [Qin17, Section 3.1].

**Definition 2.3** (Pointed elements and normalization). A Laurent polynomial  $Z$  in the quantum torus  $\mathcal{T}(t)$  is said to be pointed if it takes the form

$$(2.1) \quad Z = X(t)^g \cdot \left(1 + \sum_{0 \neq v \in \mathbb{N}^n} c_v Y(t)^v\right)$$

for some coefficients  $c_v \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ .

In this case,  $Z$  is said to be pointed at degree  $g$ , and we denote  $\text{deg}^t Z = g$ .

If  $Z = q^s X(t)^g (1 + \sum_{0 \neq v \in \mathbb{N}^n} c_v Y(t)^v)$  for some  $s \in \frac{\mathbb{Z}}{2}$ , we use  $[Z]^t$  to denote the pointed element  $q^{-s}Z$  and call it the normalization of  $Z$  in  $\mathcal{T}(t)$ .

Recall that the symbol “ $\cdot$ ” in (2.1) denotes the commutative multiplication in the Laurent polynomial ring  $\mathcal{T}(t)$ . If one uses the twisted product “ $\ast$ ” instead, then the coefficients  $c_v$  need to be shifted by appropriate  $q$ -factors.

By the existence of  $F$ -polynomials with constant term 1 [DWZ10, GHKK18] and, consequently, the existence of quantum  $F$ -polynomials [Tra11], all of the quantum cluster variables are pointed. Furthermore, the vectors  $g$  in (2.1) for different cluster variables take different values, and we call them the  $g$ -vectors.

In order to say that a pointed element has a unique maximal degree, we need to introduce the following partial order.

**Definition 2.4** (Degree lattice and dominance order). We call  $\mathbb{Z}^m$  the degree lattice and denote it as  $D(t)$ . Its dominance order  $\prec_t$  is defined to be the partial order such that  $g' \prec_t g$  if and only if  $g' = g + \deg^t Y(t)^v$  for some  $0 \neq v \in \mathbb{N}^n$ .

Notice that this order is well defined because the matrix  $\tilde{B}$  is of full rank.

We might omit the symbol  $t$  in  $X_i(t)$ ,  $I_k(t)$ ,  $\prec_t$ ,  $\deg^t$ , or  $[\ ]^t$  for simplicity.

**Lemma 2.5** ([Qin17, Lemma 3.1.2]). *For any  $g' \preceq_t g$  in  $\mathbb{Z}^m$ , there exist finitely many  $g'' \in \mathbb{Z}^m$  such that  $g' \preceq_t g'' \preceq_t g$ .*

Assume that, in  $\mathcal{T}(t)$ , we have (possibly infinitely many) elements  $\mathbb{L}_j$  pointed in different degrees. Any such element  $\mathbb{L}_j$  can be written as  $\sum_{g \in \mathbb{Z}^m} c_{g;j} X^g$ , where  $c_{g;j} \in \mathbb{Z}[q^{\pm \frac{1}{2}}]$ . A linear combination  $\sum_j a_j \mathbb{L}_j$ , with  $a_j \in \mathbb{Z}[q^{\pm \frac{1}{2}}]$ , is well defined and contained in  $\mathcal{T}(t)$  if  $\sum_j a_j c_{g;j}$  is a finite sum for all  $g \in \mathbb{Z}^m$  and vanishes except for finitely many  $g$ .

**Definition 2.6** (Triangularity, unitriangularity). Assume that  $Z$  is a Laurent polynomial in  $\mathcal{T}(t)$  such that it is a well defined linear combination of  $\mathbb{L}_j$ :

$$(2.2) \quad Z = \sum_j a_j \mathbb{L}_j, \quad a_j \in \mathbb{Z}[q^{\pm \frac{1}{2}}].$$

We say that this decomposition is  $\prec_t$ -triangular if there exists a unique  $\prec_t$ -maximal element  $\deg^t \mathbb{L}_0$  in  $\{\deg^t \mathbb{L}_j\}$ . It is further called  $\prec_t$ -unitriangular if  $a_0 = 1$ , or  $(\prec_t, \mathbf{m})$ -triangular if  $a_j \in \mathbf{m} = q^{-\frac{1}{2}} \mathbb{Z}[q^{-\frac{1}{2}}]$  for  $j \neq 0$ . A set  $\{Z\}$  is said to be  $(\prec_t, \mathbf{m})$ -unitriangular to  $\{\mathbb{L}_j\}$  if all of its elements  $Z$  have such a property.

**Lemma 2.7** ([Qin17, Remark 3.1.8]). *If the decomposition (2.2) is  $\prec_t$ -triangular, then it is the unique  $\prec_t$ -triangular decomposition of  $Z$  in  $\{\mathbb{L}_j\}$ .*

*Proof.* Thanks to Lemma 2.5, we can recursively determine all of the coefficients  $a_j$  of  $\mathbb{L}_j$  in (2.2), starting from the higher  $\prec_t$ -order Laurent degrees; see [Qin17, Remark 3.1.8]. □

The following lemma will be useful. It allows us to switch to the desired dominance order.

**Lemma 2.8** ([Qin17, Lemma 3.1.9]).

- (i) *If (2.2) is a finite decomposition of a pointed element  $Z$ , then it is  $\prec_t$ -unitriangular.*

- (ii) If, further, all but one coefficient in (2.2) belong to  $\mathbf{m}$ , then (2.2) is  $(\prec_t, \mathbf{m})$ -unitriangular.

*Proof.*

- (i) We recall the proof in [Qin17, Lemma 3.1.9]. Comparing maximal degrees of both sides of a finite decomposition, we obtain that the finite set  $\{\deg \mathbb{L}_j\}$  contains a unique maximal element  $\deg \mathbb{L}_0$  for some  $\mathbb{L}_0$  such that  $\deg \mathbb{L}_0 = \deg Z$ . So this decomposition is  $\prec_t$ -triangular. Finally,  $a_0 = 1$  because  $Z$  has coefficient 1 in its leading degree.
- (ii) By (i),  $Z$  admits a  $\prec_t$ -unitriangular decomposition. The hypothesis in (ii) simply tells us that the coefficients other than the leading coefficient (equal to 1) belong to  $\mathbf{m}$ . □

Recall that different cluster variables have different  $g$ -vectors, that is, are pointed at different degrees. For any  $1 \leq k \leq n$ , let  $I_k(t)$  denote<sup>2</sup> the unique quantum cluster variable (if it exists) such that  $\text{pr}_n \deg^t I_k(t) = -e_k$ , where  $\text{pr}_n$  is the projection of  $\mathbb{Z}^m$  onto the first  $n$ -components. The quantum cluster algebra  $\mathcal{A}$  is said to be *injective reachable* if  $I_k(t)$  exists for any  $1 \leq k \leq n$ . This property is independent of the choice of the seed  $t$ , and the proofs can be found in [Qin17, Proposition 5.1.4] or [Mull16, Theorem 3.2.1, Corollary 3.2.2], based on cluster categories for quiver cases [Pla11] and scattering diagrams in general [GHKK18]. In this case, the quantum cluster variables  $I_k(t)$ ,  $1 \leq k \leq n$ ,  $q$ -commute with each other because they belong to the same seed (denoted by  $t[1]$  in [Qin17]).

*Remark 2.9.* In the convention of Section 2.3, if  $B(t)$  is acyclic, we can obtain the quantum cluster variables  $I_k \forall 1 \leq k \leq n$  by applying the sequence of mutations on each vertex  $1, \dots, n$  such that their order increases with respect to  $\triangleleft$ . In particular, the corresponding cluster algebra is injective reachable. See Example 3.6 for an explicit calculation.

**Definition 2.10** (Triangular basis [Qin17, Definition 6.1.1]). A triangular basis  $\mathbf{L}^t$  for the seed  $t$  is defined to a  $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -basis of the quantum cluster algebra  $\mathcal{A}$  such that the following apply:

- *Cluster structure.* The quantum cluster monomials  $[\prod_{1 \leq i \leq m} X_i(t)^{u_i}]^t$ ,  $[\prod_{1 \leq k \leq n} I_k(t)^{v_k}]^t$  belong to  $\mathbf{L}^t \forall u_i, v_k \in \mathbb{N}$ .
- *Bar-invariance.* The basis elements are invariant under the bar involution in  $\mathcal{T}(t)$ .
- *Parametrization.* The basis elements are pointed, and we have the bijection

$$\deg^t : \mathbf{L}^t \simeq D(t) = \mathbb{Z}^m.$$

- *Triangularity.* For any  $X_i(t)$  and  $S \in \mathbf{L}^t$ , we have

$$[X_i(t) * S]^t = b + \sum c_{b'} \cdot b',$$

where  $\deg^t b' \prec_t \deg^t b = \deg^t X_i(t) + \deg^t S$  and the coefficients  $c_{b'} \in \mathbf{m} = q^{-\frac{1}{2}} \mathbb{Z}[q^{-\frac{1}{2}}]$ .

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<sup>2</sup>We use the notation  $I_k$  because this cluster variable corresponds to the  $k$ th indecomposable injective module of a quiver with potential [DWZ08, DWZ10].

It is easy to show that if  $\mathbf{L}^t$  exists, then it is unique by the triangularity and bar-invariance; see [Qin17, Lemma 6.2.6(i)]. In order to study  $\mathbf{L}^t$ , [Qin17] introduced the injective pointed set  $\mathbf{I}^t$  in the seed  $t$ :

$$\mathbf{I}^t = \{\mathbf{I}^t(f, u, v) \mid f \in \mathbb{Z}^{[n+1, m]}, u, v \in \mathbb{N}^{[1, n]}, u_k v_k = 0 \quad \forall k \in [1, n]\}$$

$$\mathbf{I}^t(f, u, v) = \left[ \prod_{n+1 \leq i \leq m} X_i(t)^{f_i} * \prod_{1 \leq k \leq m} X_k(t)^{u_k} * \prod_{1 \leq k \leq m} I_k(t)^{v_k} \right]^t.$$

This is a linearly independent family of pointed elements contained in  $\mathcal{A}$ . By the triangularity of  $\mathbf{L}^t$ , the set of pointed elements  $\mathbf{I}^t$  is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{L}^t$ . It follows that  $\mathbf{L}^t$  is also  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{I}^t$ ; see [Qin17, Lemma 3.1.11].

**Example 2.11** (Type  $A_3$ ). Consider the matrix

$$\tilde{B} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

which is the matrix of the ice quiver in Figure 1.

In the convention of [KQ14], its principal part is an acyclic type  $A_3$  quiver, and its coefficient part the  $z$ -pattern. By [GLS13], [KQ14, Proposition 6.2.1], there is a natural matrix  $\Lambda$  such that  $(\tilde{B}, \Lambda)$  is compatible:

$$\Lambda = \begin{pmatrix} 0 & -1 & -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix can be calculated via bilinear forms in the derived category of the representations of the quiver in Figure 2; see [KQ14, Section 6] for details. Furthermore, by [KQ14, Proposition 7.0.2], the corresponding quantum cluster algebra  $\mathcal{A}$  is isomorphic to the quantum unipotent subgroup  $A_q(\mathfrak{n}(c^2))$  localized at the coefficients  $X_4, X_5, X_6$ , where the Coxeter word  $c = s_3 s_2 s_1$  (read from right to left).

The quantum cluster variables  $I_1, I_2, I_3$  are obtained from consecutive mutations at 1, 2, 3. Our pointed element  $\mathbf{I}(f, u, v)$ ,

$$\mathbf{I}(f, u, v) = [X_4^{f_4} * X_5^{f_5} * X_6^{f_6} * X_1^{u_1} * X_2^{u_2} * X_3^{u_3} * I_1^{v_1} * I_2^{v_2} * I_3^{v_3}],$$

is a localized dual Poincaré–Birkhoff–Witt (PBW) basis element (rescaled by a  $q$ -power), and the triangular basis is the localized (rescaled) dual canonical basis; see [KQ14].

**Lemma 2.12** (Substitution [Qin17, Lemma 6.4.4]). *If a pointed element  $Z$  is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{L}^t$ , then so is  $[\prod_{n+1 \leq i \leq m} X_i^{f_i} * X^u * Z * I^v]$  for any  $f \in \mathbb{Z}^{[n+1, m]}, u, v \in \mathbb{N}^n$ .*

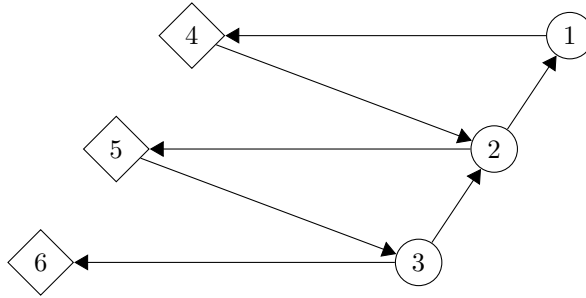


FIGURE 1. Acyclic  $A_3$  quiver with  $z$ -pattern

*Proof.*  $Z$  is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{I}^t$  and admits a  $(\prec_t, \mathbf{m})$ -unitriangular decomposition

$$Z = \sum_s a_s \mathbf{I}^t(f^{(s)}, u^{(s)}, v^{(s)}).$$

Replacing  $Z$  with this decomposition in  $[\prod_{n+1 \leq i \leq m} X^{f_i} * X^u * Z * I^v]$ , the result is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{L}^t$ , by the triangularity of  $\mathbf{L}^t$  and comparison of  $q$ -powers (see [Qin17, Lemma 6.2.4]).  $\square$

**2.3. Berenstein–Zelevinsky’s triangular basis.** Work is in some chosen seed  $t$ , whose symbol we often omit. Assume that its principal part  $B = B(t)$  is acyclic; namely, there exists an order  $\triangleleft$  on the vertex  $\{1, \dots, n\}$  such that  $b_{ij} \leq 0$  whenever  $i \triangleleft j$ . In this case,  $t$  is called an acyclic seed. If  $i \triangleleft j$ , we say that  $i$  is  $\triangleleft$ -inferior to  $j$  and also denote  $j \triangleright i$ .

A vertex  $j \in [1, n]$  is said to be a source point in  $t$  if  $j$  is  $\triangleleft$ -maximal, namely,  $j \triangleright k$  for all  $1 \leq k \leq n$ . Similarly, it is called a sink point in  $t$  if  $j$  is  $\triangleleft$ -minimal, namely,  $j \triangleleft k$  for all  $1 \leq k \leq n$ .

**Example 2.13.** In Example 2.11, we have order  $1 \triangleleft 2 \triangleleft 3$ . The vertex 1 is  $\triangleleft$ -minimal, and a sink point, 3, is  $\triangleleft$ -maximal and a source point. Notice that, in the quiver  $Q(t)$  associated with the principal part matrix  $B(t)$ , a sink point has no outgoing arrows and a source point has no incoming arrow; see Figure 2.

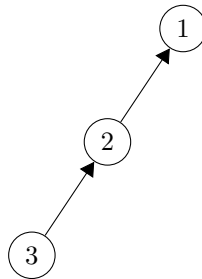


FIGURE 2. Acyclic quiver with a sink point and a source point



For any  $1 \leq k \leq n$ , let  $b_k = \widetilde{B}e_k$  denote the  $k$ th column of  $\widetilde{B}$ . Let  $S_k = S_k(t)$  denote<sup>3</sup> the quantum cluster variable  $X_k(\mu_k t)$ . Notice that  $S_k = X^{-e_k + [-b_k]_+} \cdot (1 + Y_k)$ , and we have  $\deg S_k = -e_k + [-b_k]_+$ , where  $[-b_k]_+$  denotes  $([-b_{j_k}]_+)_{1 \leq j \leq m}$ .

For any  $a \in \mathbb{Z}^m$ , Berenstein and Zelevinsky defined the standard monomials  $E_a = [\prod_{n < j \leq m} X^{a_j} * \prod_{1 \leq k \leq n} X_k^{[a_k]_+} * \prod_{1 \leq k \leq n} S_k^{[-a_k]_+}]$ , where the last factor is the product with increasing  $\triangleleft$  order; see [BZ14, (1.17), (1.22), Remark 1.3].

Define  $r(a) = \sum_{1 \leq k \leq n} [-a_k]_+$ . Define partial order  $a \prec_{\text{BZ}} a'$  if and only if  $r(a) < r(a')$ .

**Definition 2.14.** The Berenstein–Zelevinsky’s acyclic triangular basis for the seed  $t$  is defined to be the basis  $C^t = \{C_a\}$  of  $\mathcal{A}$  such that each  $C_a$  is bar-invariant and  $(\prec_{\text{BZ}}, q^{\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}])$ -triangular to the basis  $\{E_a\}$ .

We call  $C^t$  the BZ-basis for simplicity. Applying the bar involution, we obtain that  $C^t$  is  $(\prec_{\text{BZ}}, \mathbf{m})$ -triangular to  $\{\overline{E}_a\}$ , where

$$\overline{E}_a = [ \prod_{1 \leq k \leq n}^{\triangleright} S_k^{[-a_k]_+} * \prod_{1 \leq k \leq n} X_k^{[a_k]_+} * \prod_{n < j \leq m} X^{a_j} ],$$

where the first factor is the product with decreasing  $\triangleleft$  order.

**Example 2.15.** Let us continue to work on Example 2.11. The standard monomials, after the bar involution, give us

$$\overline{E}_a = [S_3^{[-a_3]_+} * S_2^{[-a_2]_+} * S_1^{[-a_1]_+} * X_1^{[a_1]_+} * X_2^{[a_2]_+} * X_3^{[a_3]_+} * X_4^{a_4} * X_5^{a_5} * X_6^{a_6}].$$

Notice that  $X_4, X_5, X_6$   $q$ -commute with all of the factors.

**Theorem 2.16** ([BZ14, Theorem 1.4]). *The Berenstein–Zelevinsky’s triangular basis  $C^t$  is independent of the chosen acyclic seed  $t$ .*

By this result, we can write  $C$  instead of  $C^t$ .

### 3. COMPARING TRIANGULAR BASES

**3.1. Basic results.** Let we choose and work with any seed  $t$  whose matrix  $B(t)$  is acyclic.

**Lemma 3.1.** *For any acyclic seed  $t$ , each  $C_a$  is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\{\overline{E}_a\}$ .*

*Proof.* Each  $C_a$  is a finite linear combination of  $\{\overline{E}_a\}$  with one term of coefficient 1 and others of coefficients in  $\mathbf{m}$ . This decomposition is  $\prec_t$ -triangular by Lemma 2.8. □

**Lemma 3.2.** *If  $n$  is a source point, then  $\overline{E}_a$  remains pointed in  $t' = \mu_n t$ .*

*Proof.* It might be possible to deduce this result from the existence of common Berenstein–Zelevinsky triangular bases in  $t$  and  $t'$ . Let us give an alternative elementary verification.

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<sup>3</sup>We use the symbol  $S_k$  because this cluster variable corresponds to the  $k$ th simple  $S_k$  in an associated quiver with potential.

In order to show that the  $q$ -normalization factor produced by the factors of  $\overline{E}_a$  remains unchanged in  $\mathcal{T}(t')$ , it suffices to show that, for any  $1 \leq i, j \leq m$ ,  $1 \leq l < k \leq n$ ,  $i \neq k$ , we have

$$(3.1) \quad \Lambda(t)(\deg^t X_i, \deg^t X_j) = \Lambda(t')(\deg^{t'} X_i, \deg^{t'} X_j),$$

$$(3.2) \quad \Lambda(t)(\deg^t X_i, \deg^t S_k) = \Lambda(t')(\deg^{t'} X_i, \deg^{t'} S_k),$$

$$(3.3) \quad \Lambda(t)(\deg^t S_l, \deg^t S_k) = \Lambda(t')(\deg^{t'} S_l, \deg^{t'} S_k).$$

Notice that we have  $\deg^t S_l = -e_l + \sum_s [-b_{sl}]_+ e_s$ , where all  $e_s$  appearing have  $s \neq n$ . Therefore, we deduce that  $\deg^{t'} S_l = \deg^t S_l \ \forall l < n$  from the tropical transformation of  $g$ -vectors; see [Qin17, Section 3.2], [FG09], [FZ07, (7.18)]. The first two equations simply follow from the mutation rule from  $\Lambda(t)$  to  $\Lambda(t')$ . It remains to check (3.3). By using (3.2), we obtain

$$\begin{aligned} & \Lambda(t)(\deg^t S_l, \deg^t S_k) \\ &= \Lambda(t)(-\deg^t X_l + \sum_s [-b_{sl}]_+ \deg^t X_s, \deg^t S_k) \\ &= -\Lambda(t)(\deg^t X_l, \deg^t S_k) + \sum_s [b_{-sl}]_+ \Lambda(t)(\deg^t X_s, \deg^t S_k) \\ &= -\Lambda(t')(\deg^{t'} X_l, \deg^{t'} S_k) + \sum_s [b_{-sl}]_+ \Lambda(t')(\deg^{t'} X_s, \deg^{t'} S_k) \\ &= \Lambda(t')(\deg^{t'} S_l, \deg^{t'} S_k). \end{aligned} \quad \square$$

The following statement is the main result of [KQ14] accompanied by the coefficient correction technique in [Qin14].

**Theorem 3.3** ([KQ14, Qin14]). *If the principal part  $B(t)$  of a seed  $t$  is acyclic and skew-symmetric, then the triangular basis  $\mathbf{L}^t$  for  $t$  exists. Moreover, it contains all the quantum cluster monomials.*

*Proof.* When we choose the special coefficient pattern  $B^c(t)$  to be a  $z$ -pattern as in [KQ14], the quantum cluster algebra is isomorphic to a subalgebra of a quantized enveloping algebra [GLS13]. Under this identification,  $X_i(t)$ ,  $I_k(t)$  are the factors of the dual PBW basis element, and the triangular basis  $\mathbf{L}^t$  is just the restriction of the dual canonical basis on this subalgebra (and localized at the coefficients  $(X_{n+1}, \dots, X_m)$ ). By [KQ14], this basis contains all the quantum cluster monomials.

By the correction technique in [Qin14], we can change the coefficient pattern  $B^c(t)$  and  $\Lambda(t)$  while keeping the claim true. □

The following statement is implied by the general result in [Qin17, Theorem 9.4.1]. We sketch a much easier proof for this special case.

**Theorem 3.4.** *Let  $t$  and  $t'$  be two seeds such that  $t' = \mu_k t$  for some  $1 \leq k \leq n$  and  $B(t)$ ,  $B(t')$  are acyclic and skew-symmetric. Then the quantum cluster algebra has a basis  $\mathbf{L}$  which is the triangular basis for both  $t$  and  $t'$ .*

*Proof.* Because  $t$  and  $t'$  are acyclic, by Theorem 3.3, we know that the triangular bases  $\mathbf{L}^t$  and  $\mathbf{L}^{t'}$  for  $t$  and  $t'$  exist. Moreover, the quantum cluster monomials  $X_k^{t'd} = X_k(t')^d$ ,  $I_k^{t'd} = I_k(t')^d$  belong to  $\mathbf{L}^t$ , where  $d \in \mathbb{N}$ . Therefore,  $X_k^{t'd}$  and  $I_k^{t'd}$

have  $(\prec_t, \mathbf{m})$ -unitriangular decomposition in the injective pointed set  $\mathbf{I}^t$ . These are the only new factors of elements in  $\mathbf{I}^{t'}$  which are not factors of elements in  $\mathbf{I}^t$ .

Easy calculation shows that elements in  $\mathbf{I}^{t'}$  remain pointed in  $\mathcal{T}(t)$ ; see [Qin17, Lemma 5.3.2]. Substituting their new factors  $X_k^{t'd}$  and  $I_k^{t'd}$  by the decomposition in  $\mathbf{I}^t$ , we deduce that  $\mathbf{I}^{t'}$  is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{I}^t$  by Lemma 2.12.

Also, notice that  $\mathbf{L}^{t'}$  is  $(\prec_{t'}, \mathbf{m})$ -unitriangular to  $\mathbf{I}^{t'}$  and that  $\mathbf{I}^t$  is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{L}^t$ . Composing these three transitions, we obtain that any  $S' \in \mathbf{L}^{t'}$  is a finite combination of elements  $S, S_i$  in  $\mathbf{L}^t$ ,

$$S' = S + \sum_i a_i S_i,$$

with coefficient  $a_i \in \mathbf{m}$ .

Now by the bar-invariance of  $\mathbf{L}^t$  and  $\mathbf{L}^{t'}$ , we must have  $a_i = 0$  and  $S' = S$ . It follows that the two triangular bases  $\mathbf{L}^t$  and  $\mathbf{L}^{t'}$  are the same.  $\square$

**3.2. Proof of the main result (Theorem 1.2).** Recall that the matrix  $\tilde{B}(t)$  of the given seed  $t$  is assumed to be of full rank, acyclic, and skew-symmetric. For any chosen  $1 \leq j \leq n$ , let  $t[j^{-1}]$  denote the seed obtained from  $t$  by deleting the  $j$ th column in the matrix  $\tilde{B}(t)$ . This operation is called freezing the vertex  $j$ . We have the corresponding quantum cluster algebra  $\mathcal{A}(t[j^{-1}])$ , in which the frozen variable  $X_j$  is invertible. Observe that the normalization  $[ \ ]^{t[j^{-1}]} = [ \ ]^t$  because  $\Lambda(t[j^{-1}]) = \Lambda(t)$  by construction. Moreover, the partial order  $\prec_{t[j^{-1}]}$  implies  $\prec_t$  by definition. We can define similarly, for  $f \in \mathbb{Z}^{\{j\} \cup [n+1, m]}$ ,  $u, v \in \mathbb{N}^{[1, n] - \{j\}}$ , where  $u_k v_k = 0$  for any  $k$ :

$$\begin{aligned} & \mathbf{I}^{t[j^{-1}]}(f, u, v) \\ &= [ \prod_{n+1 \leq i \leq m} X_i^{f_i} * X_j^{f_j} * \prod_{1 \leq k \leq n, k \neq j} X_k^{u_k} * \prod_{1 \leq k \leq n, k \neq j} I_k(t[j^{-1}])^{v_k} ]^{t[j^{-1}]} . \end{aligned}$$

We want to compare this new injective pointed set  $\mathbf{I}^{t[j^{-1}]}$  with the old one  $\mathbf{I}^t$ . One has to pay attention to the possible localization at  $X_j$  in the seed  $t[j^{-1}]$ .

Assume the vertex  $n$  to be  $\leftarrow$ -maximal—namely, a source point—then  $I_k(t[n^{-1}]) = I_k(t)$  for all  $1 \leq k < n$  (see Remark 2.9), and, moreover,  $(\deg Y_i)_n = b_{ni} \geq 0 \ \forall 1 \leq i \leq n$ . It follows that the Laurent monomials of  $I_k(t) \ \forall k \neq n$  have nonnegative degrees in  $X_n$ .

Notice that, for a source point  $n$ , if  $f_n \geq 0$ , then  $\mathbf{I}^{t[n^{-1}]}(f, u, v) \in \mathbf{I}^t$ .

**Lemma 3.5.** *Assume that  $n$  is a source point and that a pointed element  $Z \in \mathcal{A}(t[n^{-1}])$  has a finite combination of*

$$Z = \sum_s a_s \mathbf{I}^{t[n^{-1}]}(f^{(s)}, u^{(s)}, v^{(s)}).$$

*If  $(\deg Z)_n \geq 0$ , then we have  $f_n^{(s)} \geq 0$  whenever  $a_s \neq 0$ . Consequently, all  $\mathbf{I}^{t[n^{-1}]}(f^{(s)}, u^{(s)}, v^{(s)})$  appearing in the combination are contained in  $\mathbf{I}^t$ .*

*Proof.* Recall that  $\mathbf{I}^{t[n^{-1}]}$  is a linearly independent family of pointed elements with distinguished leading degrees. By Lemma 2.8(i), the given decomposition of  $Z$  is

$\prec_t$ -unitriangular with a unique leading term  $\mathbf{I}^{t[n^{-1}]}(f^{(0)}, u^{(0)}, v^{(0)})$  whose leading degree equals  $\deg Z$ . So the leading degrees of all  $\mathbf{I}^{t[n^{-1}]}(f^{(s)}, u^{(s)}, v^{(s)})$  appearing are  $\prec_t$ -inferior or equal to  $\deg Z$ . Since  $(\deg Z)_n \geq 0$  and  $(\deg Y_i)_n \geq 0 \ \forall 1 \leq i \leq n$ , they are all nonnegative in the  $n$ th components.

Notice that  $\text{pr}_n \deg I_k(t) = -e_k$  by definition and, in particular, the leading degree  $\deg I_k(t) \ \forall k < n$  vanishes in the  $n$ th components. It follows that  $\deg \mathbf{I}^{t[n^{-1}]}(f^{(s)}, u^{(s)}, v^{(s)})$  has nonnegative  $n$ th component if and only if  $f_n^{(s)} \geq 0$ . The claim follows.  $\square$

*Proof of Theorem 1.2.* We prove the claim by induction on the rank  $n$  of  $\tilde{B}(t)$ . The case  $n = 0$  is trivial.

Up to relabeling vertices, let us assume that  $n$  is a source point in  $t$ . Denote  $t' = \mu_n t$ .

It suffices to show that every  $\bar{E}_a, a \in \mathbb{Z}^m$ , is  $(\prec_t, \mathbf{m})$ -triangular to  $\mathbf{L}^t$ . If so, combined with Lemma 3.1, we obtain that every bar-invariant element  $C_a$  is  $(\prec_t, \mathbf{m})$ -triangular to  $\mathbf{L}^t$ . Consequently, each  $C_a$  admits a  $(\prec_t, \mathbf{m})$ -triangular decomposition  $C_a = S + \sum_i a_i S_i$ , where  $S, S_i \in \mathbf{L}^t$  are bar-invariant for all indices  $i$ , the coefficients  $a_i \in \mathbf{m}$ . Because  $C_a$  is bar-invariant, we must have that  $a_i$  vanishes for all  $i$ . Therefore, all elements  $C_a$  of the basis  $C$  are also elements of the basis  $\mathbf{L}^t$ . It follows that the two bases  $\mathbf{L}^t$  and  $C$  are the same.

- (i) Assume that  $a_n \geq 0$ . Consider the seed  $t[n^{-1}]$  obtained by freezing the vertex  $n$  in  $t$ . It is an acyclic seed whose matrix  $\tilde{B}(t[n^{-1}])$  has rank  $n - 1$ . By induction hypothesis, its triangular basis  $\mathbf{L}^{t[n^{-1}]}$  agrees with its BZ-basis  $C^{t[n^{-1}]}$ . Notice that the corresponding standard monomial  $E_a$  is also a standard monomial for seed  $t[n^{-1}]$ . Therefore,  $\bar{E}_a$  admits a finite decomposition in  $C^{t[n^{-1}]} = \mathbf{L}^{t[n^{-1}]}$  with one term of coefficient 1 and other terms of coefficient in  $\mathbf{m}$ . Recall that  $\mathbf{L}^{t[n^{-1}]}$  is  $\prec_{t[n^{-1}]}$ -unitriangular to  $\mathbf{I}^{t[n^{-1}]}$ . Composing these two transitions, we see that  $\bar{E}_a$  has a finite decomposition in  $\mathbf{I}^{t[n^{-1}]}$  with one term of coefficient 1 and others of coefficient in  $\mathbf{m}$ . Further notice that  $(\deg \bar{E}_a)_n \geq 0$ , by Lemma 3.5, with the decomposition terms appearing to belong to  $\mathbf{I}^t$ . By Lemma 2.8,  $\bar{E}_a$  is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{I}^t$ , and consequently  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{L}^t$ .
- (ii) When  $a_n < 0$ , let us rewrite  $\bar{E}_a$  as  $[S_n^{[-a_n]^+} * \bar{E}_{a_{\widehat{n}}}]^t$ , where  $a_{\widehat{n}}$  denotes the vector obtained from  $a$  by setting the  $n$ th component to 0. Notice that  $\bar{E}_a$  is also pointed in  $t'$  by Lemma 3.2, namely,  $\bar{E}_a = [S_n^{[-a_n]^+} * \bar{E}_{a_{\widehat{n}}}]^{t'}$ . For the seed  $t'$ , we freeze the vertex  $n$  and repeat the argument in (i). It follows that  $\bar{E}_{a_{\widehat{n}}}$  is  $(\prec_{t'}, \mathbf{m})$ -unitriangular to the triangular basis  $\mathbf{L}^{t'}$  of the seed  $t'$ . Notice that  $S_n$  is the  $n$ th cluster variable in the seed  $t'$ . By Lemma 2.12, we obtain that  $\bar{E}_a$  is  $(\prec_{t'}, \mathbf{m})$ -unitriangular to the triangular basis  $\mathbf{L}^{t'}$  of the seed  $t'$ . Because  $\mathbf{L}^t = \mathbf{L}^{t'}$  by Theorem 3.4,  $\bar{E}_a$  is  $(\prec_t, \mathbf{m})$ -unitriangular to  $\mathbf{L}^t$  by Lemma 2.8.  $\square$

**Example 3.6** (Kronecker quiver type). Let us look at the quantum cluster algebra with the seed  $t$  given by  $\tilde{B} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$  and  $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . We have the set of source points  $V_0 = \{1\}$  and the set of sink points  $V_1 = \{2\}$ .

Its seed  $t' = \mu_{V_1} t$  has the matrices  $\tilde{B} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$  and  $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . The vertex 2 is the source point in  $t'$ . It is easy to compute that

$$\begin{aligned} S_1(t') &= X(t')^{-e_1} + X(t')^{-e_1+2e_2}, \\ S_2(t') &= X(t')^{-e_2+2e_1} + X(t')^{-e_2}, \\ Y_1(t') &= X(t')^{2e_2}, \\ Y_2(t') &= X(t')^{-2e_1}. \end{aligned}$$

By [BZ14, (6.4)], [DX12], we have the following bar-invariant pointed element  $X_\delta$  in the BZ-basis  $C$ , given by

$$\begin{aligned} X_\delta &= q^{\frac{1}{2}} S_1(t') * S_2(t') - q^{\frac{3}{2}} X_2(t') * X_1(t') \\ &= X(t')^{e_1-e_2} \cdot (1 + Y_2(t') + Y_1(t')Y_2(t')) \\ &= X(t')^{e_1-e_2} + X(t')^{-e_1-e_2} + X(t')^{e_2-e_1}. \end{aligned}$$

Taking the bar-involution, we obtain

$$\begin{aligned} X_\delta &= q^{-\frac{1}{2}} S_2(t') * S_1(t') - q^{-\frac{3}{2}} X_1(t') * X_2(t') \\ &= [S_2(t') * S_1(t')]^{t'} - q^{-2} [X_1(t') * X_2(t')]^{t'}. \end{aligned}$$

We have

$$\begin{aligned} S_2(t') &= X_2(t), \\ S_1(t') &= I_1(t) \\ &= X(t)^{-e_1}(1 + Y_1(t) + (q + q^{-1})Y_1(t)Y_2(t) + Y_1(t)Y_2(t)^2), \\ X_2(t') &= I_2(t) \\ &= X(t)^{-e_2}(1 + Y_2(t)), \\ X_1(t') &= X_1(t). \end{aligned}$$

Then  $X_\delta$  can be rewritten as

$$\begin{aligned} X_\delta &= [X_2(t) * I_1(t)]^t - q^{-2} [X_1(t) * I_2(t)]^t \\ &= X(t)^{-e_1+e_2}(1 + Y_1(t) + (1 + q^{-2})Y_1(t)Y_2(t) + q^{-2}Y_1(t)Y_2(t)^2) \\ &\quad - q^{-2} X^{e_1-e_2}(1 + Y_2(t)) \\ &= X(t)^{-e_1+e_2}(1 + Y_1(t) + Y_1(t)Y_2(t)) \\ &= X(t)^{-e_1+e_2} + X(t)^{-e_1-e_2} + X(t)^{e_1-e_2}. \end{aligned}$$

Notice that the normalization factors do not change:

$$\begin{aligned} \Lambda(t)(\deg^t X_2(t), \deg^t I_1(t)) &= \Lambda(t)(e_2, -e_1) \\ &= 1 \\ &= \Lambda(t')(\deg^{t'} S_2(t'), \deg^{t'} S_1(t')), \\ \Lambda(t)(\deg^t X_1(t), \deg^t I_2(t)) &= \Lambda(t)(e_1, -e_2) \\ &= -1 \\ &= \Lambda(t')(\deg^{t'} X_1(t'), \deg^{t'} X_2(t')). \end{aligned}$$

Therefore, the pointed element  $X_\delta$  is  $(\prec_t, \mathbf{m})$ -unitriangular to the injective pointed set  $\mathbf{I}^t$  and, consequently,  $(\prec_t, \mathbf{m})$ -unitriangular to the triangular basis  $\mathbf{L}^t$ . It follows from its bar-invariance that  $X_\delta$  belongs to the triangular basis  $\mathbf{L}^t$ .

For the Kronecker quiver in Example 3.6, we have identified the set of cluster variables  $\{X_1(t'), X_2(t'), S_1(t'), S_2(t')\}$  with  $\{X_1(t), I_2(t), I_1(t), X_2(t)\}$ . One can use such identification to compare the two triangular bases in bipartite cases. We shall give details in Lemma 3.7.

**3.3. Bipartite skew-symmetrizable case.** We say that the seed  $t$  has a bipartite orientation (we say  $t$  is bipartite for short) if we have  $\{1, \dots, n\} = V_0 \sqcup V_1$  such that all of the vertices in  $V_0$  are source points and those in  $V_1$  are sink points.

Assume that  $t$  is bipartite. Let us denote by  $t'$  the seed obtained from  $t$  by mutating at all the vertices in  $V_1$ , namely,

$$\begin{aligned} \mu_{V_1} &= \prod_{k \in V_1} \mu_k, \\ t' &= \mu_{V_1} t. \end{aligned}$$

Notice that the mutations  $\mu_k, k \in V_1$  commute with each other.

The following lemma follows from the definitions of the corresponding cluster variables; see Figure 3 for identification of cluster variables, where  $i \in V_0, j \in V_1$ . Figure 3 is inspired by the knitting algorithm for acyclic quivers; see [Kel08].

**Lemma 3.7.** *We have, for any  $1 \leq i, j \leq n$ ,*

(3.4)  $X_i(t') = X_i(t), \quad i \in V_0,$

(3.5)  $X_j(t') = I_j(t), \quad j \in V_1,$

(3.6)  $S_i(t') = I_i(t), \quad i \in V_0,$

(3.7)  $S_j(t') = X_j(t), \quad j \in V_1.$

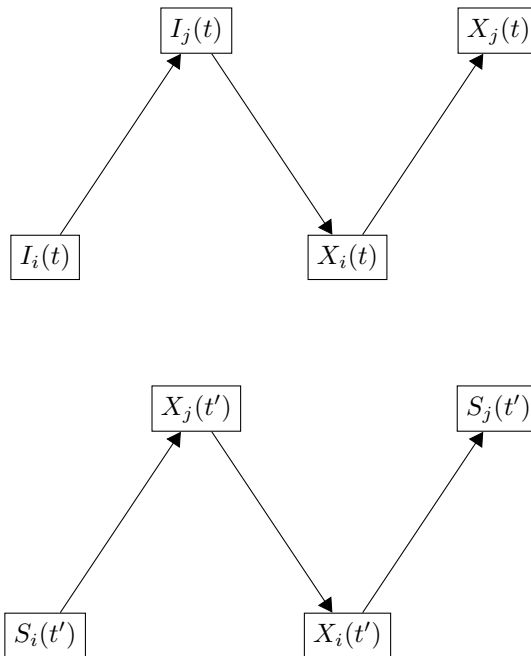


FIGURE 3. Part of knitting graphs for the seeds  $t$  and  $t'$

It follows from Lemma 3.7 that those  $S(t'), i \in V_0$   $q$ -commute with each other, and  $S_j(t'), j \in V_1$   $q$ -commute with each other.

Notice that  $t'$  is still bipartite, with the vertices in  $V_0$  being sink points and the vertices in  $V_1$  being source points.

**Lemma 3.8.**

- (1) For any  $1 \leq k \neq j \leq n$  such that  $j \in V_1$ ,  $X_k(t)$  and  $I_j(t)$   $q$ -commute.
- (2) For any  $1 \leq i \neq k \leq n$  such that  $i \in V_0$ ,  $X_i(t)$  and  $I_k(t)$   $q$ -commute

*Proof.*

- (1)  $X_k(t)$  and  $I_j(t)$  are quantum cluster variables in the same seed  $\mu_j t$ .
- (2) By (1), it remains to check the case  $i, k \in V_0$ . Notice that  $V_0$  consists of sink points in  $t' = \mu_{V_1} t$ .  $X_i(t)$  and  $I_k(t)$  are quantum cluster variables in the same seed  $\mu_k t'$ . □

**Lemma 3.9.** *The pointed element  $\bar{E}_a$  defined in  $t'$  remains pointed in  $t = \mu_{V_1} t'$ .*

*Proof.* The vertices in  $V_1$  are source points in  $t'$  which are not connected by arrows. We simply repeat the proof of Lemma 3.2. □

**Theorem 3.10.** *For bipartite  $t$ , the Berenstein–Zelevinsky’s triangular basis  $C$  is also the triangular basis  $\mathbf{L}^t$ .*

*Proof.* Notice that, in the seed  $t'$ , the vertices in  $V_1$  are source points and  $\leftarrow$ -superior than those in  $V_0$ . Using Lemma 3.8(ii), we have, for any  $a \in \mathbb{Z}^m$ ,

$$(3.8) \quad \bar{E}_a = \left[ \prod_{j \in V_1} S_j(t')^{[-a_j]_+} * \prod_{i \in V_0} S_i(t')^{[-a_i]_+} * \prod_{j \in V_1} X_j(t')^{[a_j]_+} * \prod_{i \in V_0} X_i(t')^{[a_i]_+} * \prod_{n+1 \leq j \leq m} X_j(t')^{a_j} \right]^{t'}$$

$$(3.9) \quad = \left[ \prod_{j \in V_1} X_j(t)^{[-a_j]_+} * \prod_{i \in V_0} I_i(t)^{[-a_i]_+} * \prod_{j \in V_1} I_j(t)^{[a_j]_+} * \prod_{i \in V_0} X_i(t)^{[a_i]_+} * \prod_{n+1 \leq j \leq m} X_j(t')^{a_j} \right]^{t'}$$

$$(3.10) \quad = \left[ \prod_{j \in V_1} X_j(t)^{[-a_j]_+} * \prod_{i \in V_0} X_i(t)^{[a_i]_+} * \prod_{i \in V_0} I_i(t)^{[-a_i]_+} * \prod_{j \in V_1} I_j(t)^{[a_j]_+} * \prod_{n+1 \leq j \leq m} X_j(t')^{a_j} \right]^{t'}$$

$$(3.11) \quad * \prod_{n+1 \leq j \leq m} X_j(t')^{a_j} \Big]^{t'}$$

By Lemma 3.9,  $\bar{E}_a$  remains to be pointed in  $t$ . Then the last equation above tells us that it belongs to the injective pointed set  $\mathbf{I}^t$ . All elements of  $\mathbf{I}^t$  take this form. So we see that the BZ-basis  $C$  verifies conditions (i), (ii), and (iv) in Definition 2.10. A closer examination tells us that condition (iii) in Definition 2.10 is also verified by the basis  $C$ ; see [BZ14]. So  $C$  is the triangular basis  $\mathbf{L}^t$  for the seed  $t$ . □

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