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TOPOLOGICAL FORMULA OF THE LOOP EXPANSION OF THE COLORED JONES POLYNOMIALS

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ABSTRACT. We give a formula of the loop expansion of the colored Jones polynomials based on homological representation of braid groups. This gives a direct proof of the Melvin-Morton-Rozansky conjecture, and a connection between entropy of braids and quantum representations.

1. Introduction

For $\alpha \in \{2,3,4,\ldots\}$ and an oriented knot K in S^3, let $J_{K,\alpha}(q) \in \mathbb{Z}[q,q^{-1}]$ be the α-colored Jones polynomial of K, normalized so that $J_{\mathsf{Unknot},\alpha}(q) = 1$. As Melvin-Morton proved [MeMo], by putting $q = e^{\hbar}$, the colored Jones polynomials can be expanded as a power series of two independent variables $\hbar\alpha$ and $\hbar$, as

$$J_{K,\alpha}(e^{\hbar}) = \sum_{i=0}^{\infty} D^{(i)}(\hbar\alpha)\hbar^i = \sum_{i=0}^{\infty}\left(\sum_{k=0}^{\infty} d_k^{(i)}(\hbar\alpha)^k\right)\hbar^i.$$

Further, by putting $z = e^{\hbar\alpha}$ we write the colored Jones polynomials as a function on $\hbar$ and z,

$$CJ_K(z,\hbar) = J_{K,\alpha}(e^{\hbar}) = \sum_{i=0}^{\infty} V_K^{(i)}(z)\hbar^i.$$

We call CJ_K the *colored Jones function*, or, *perturbative expansion of the colored Jones polynomial*, or, the *loop expansion of quantum $\mathfrak{sl}_2$-invariant*. It coincides with the $\mathfrak{sl}_2$ weight system reduction of the loop expansion of the Kontsevich invariant. In particular, the ith coefficient $V^{(i)}(z)$ corresponds to the $(i+1)$st loop part of the loop expansion [Oh]. In [Ov] Overbay studied the expansion by investigating an expansion of the R-matrix from $U_q(\mathfrak{sl}_2)$, and computed $V_K^{(1)}(z)$ and $V_K^{(2)}(z)$ for knots up to 10 crossings.

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Let $\Delta_K(z)$ be the Alexander-Conway polynomial of K, characterized by the skein relation

$$\Delta_{\nearrow\!\!\!\!\nwarrow}(z) - \Delta_{\nwarrow\!\!\!\!\nearrow}(z) = (z^{\frac{1}{2}} - z^{-\frac{1}{2}})\nabla_{)\,(}(z), \quad \Delta_{\mathsf{Unknot}}(z) = 1.$$

The Alexander-Conway polynomial appears as one of the basic building blocks of CJ_K. The Melvin-Morton-Rozansky conjecture [MeMo] (MMR conjecture, for short), proven in [BG], states that the $V^{(0)}(z)$ is equal to $\Delta_K(z)^{-1}$. More generally, $V^{(i)}(z)$ is a rational function whose denominator is $\Delta_K(z)^{2i+1}$ [Ro1].

In a theory of quantum invariants, this appearance of the Alexander-Conway polynomial is well-understood. The aforementioned rationality of $V^{(i)}(z)$ follows from Rozansky's rationality conjecture [Ro2] of the loop expansion of the Kontsevich invariant, proven in [Kri]: The Århus integral computation of the Kontsevich (LMO) invariant [BGRT] based on a surgery presentation of knots, provides the desired rationality (see [GK, Section 1.2] for a brief summary of Kricker's argument).

The clasper surgery [Ha] explains a geometric connection between the loop expansion and infinite cyclic covering [GR]. A null-clasper, a clasper with null-homologous leaves in the knot complement, lifts to a clasper in the infinite cyclic covering, and the loop expansion nicely behaves under the clasper surgery along null-claspers. Thus schematically speaking, the loop expansion is a $\mathbb{Z}$-equivariant LMO invariant [LMO], so it is not surprising that the Alexander-Conway polynomial appears in the loop expansion.

Nevertheless, in a purely topological point of view it is still somewhat mysterious why the Alexander polynomial appears in such a particular and direct form. In a known proof of the MMR conjecture, one uses quantum-invariant-like treatments of the Alexander-Conway polynomial, such as, state-sum, R-matrix, or weight systems so its topological content is often indirect.

In this paper, we give a formula of CJ_K by using homological braid group representations (Theorem 3.1). Our starting point is a recent result in [I, Koh] that identifies certain homological representations introduced by Lawrence [La] with generic $U_q(\mathfrak{sl}_2)$ representations.

Our approach is similar to Lawrence-Bigelow's approach of the Jones polynomial [Big2, La2], but there are several important differences. We give a formula of the loop expansion but do not provide a formula of the individual colored Jones polynomials. We use closed braid representatives and trace, whereas Lawrence and Bigelow used plat representatives and intersection products. It also should be emphasized that quantum representations from finite dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules are *not* identified with homological representations so translating a construction of quantum invariants in terms of homological representation is by no means routine.

Our formula leads to several insights. First, the MMR conjecture is a direct consequence of our formula. At $\hbar = 0$, topological considerations show that the homological representations is equal to the symmetric powers of the reduced Burau representation, so they naturally lead to the Alexander-Conway polynomial. Second, our formula gives a direct way to calculate $CJ_K(z, \hbar)$ without knowing or computing the individual colored Jones polynomial $J_{K,\alpha}(q)$, although a general calculation is difficult and impractical. Finally, as we discuss in Section 4, an identification with homological and quantum representations leads to an inequality between the entropy of braids and quantum $\mathfrak{sl}_2$ invariants of its closure (Theorem 4.3). This justifies a naive intuition that a knot with "complicated" quantum $\mathfrak{sl}_2$ invariants is represented as a closure of a "complicated" braid, namely, a braid with large entropy.

2. A topological description of generic quantum $\mathfrak{sl}_2$ representation

In this section we review the result in [I] that identifies a generic quantum $\mathfrak{sl}_2$ representation given in [JK] with Lawrence's homological representation and some additional arguments to treat the non-generic case.

Throughout the paper, we use the following notation and conventions. The q-numbers, q-factorials, and q-binomial coefficients are defined by

$$[n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q![j]_q!},$$

respectively. The quantum parameter q corresponds to q^2 in [I, JK]. We always assume that the braid group B_n is acting from the left.

Let R be a commutative ring. For R-modules (resp., RB_n-modules) V and W, we denote $V \cong_{\mathcal{Q}} W$ if they are isomorphic over the quotient field $\mathcal{Q}$ of R, that is, $V \otimes_R \mathcal{Q}$ and $W \otimes_R \mathcal{Q}$ are isomorphic as $\mathcal{Q}$-modules (resp., $\mathcal{Q}B_n$-modules).

For a subring $R \subset \mathbb{C}$, let $R[x^{\pm 1}]$ be the Laurent polynomial ring, and for an $R[x^{\pm 1}]$-module V and $c \in \mathbb{C}$, we denote the specialization of the variable x to complex parameter c by $V|_{x=c}$.

2.1. **Generic quantum representation.** Let $\mathbb{C}[[\hbar]]$ be the algebra of the complex formal power series in one variable $\hbar$, and we put $q = e^{\hbar}$, as usual. A quantum enveloping algebra $U_\hbar(\mathfrak{sl}_2)$ is a topological Hopf algebra over $\mathbb{C}[[\hbar]]$ generated by H, E, F subjected to the relations

$$\begin{cases} [H,E] = 2E, \quad [H,F] = -2F, \\ [E,F] = \dfrac{\sinh(\frac{\hbar H}{2})}{\sinh(\frac{\hbar}{2})} = \dfrac{e^{\frac{\hbar H}{2}} - e^{-\frac{\hbar H}{2}}}{e^{\frac{\hbar}{2}} - e^{-\frac{\hbar}{2}}}. \end{cases}$$

Let $\mathcal{R} \in U_\hbar(\mathfrak{sl}_2) \otimes U_\hbar(\mathfrak{sl}_2)$ be a *universal R-matrix* given by

$$\mathcal{R} = e^{\frac{\hbar}{4}(H \otimes H)} \left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{4}} \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n}{[n]_q!} E^n \otimes F^n \right). \tag{2.1}$$

(Strictly speaking, here we need to use the topological tensor product $\widetilde{\otimes}$, the $\hbar$-adic completion of $U_\hbar(\mathfrak{sl}_2) \otimes U_\hbar(\mathfrak{sl}_2)$. To make notation simple, in the rest of the paper $\otimes$ should be regarded as the topological tensor product, if we should do so.)

For $\lambda \in \mathbb{C}^*$, let V_λ be the Verma module of highest weight λ, a topologically free $U_\hbar(\mathfrak{sl}_2)$-module generated by a highest weight vector v_0 with $Hv_0 = \lambda v_0$, and $Ev_0 = 0$. We regard λ as an abstract variable. Let $\widehat{V_{\hbar,\lambda}}$ be a $\mathbb{C}[\lambda][[\hbar]]$-module freely generated by $\{\widehat{v}_0, \widehat{v}_1, \ldots,\}$, equipped with a $U_\hbar(\mathfrak{sl}_2)$-module structure

$$\begin{cases} H\widehat{v}_i = (\lambda - 2i)\widehat{v}_i, \\ E\widehat{v}_i = \widehat{v}_{i-1}, \\ F\widehat{v}_i = [i+1]_q[\lambda - i]_q \widehat{v}_{i+1}. \end{cases} \tag{2.2}$$

Here we put

$$[\lambda - i]_q = \frac{\sinh(\frac{1}{2}\hbar(\lambda - i))}{\sinh(\frac{1}{2}\hbar)} = \frac{e^{\frac{1}{2}\hbar(\lambda-i)} - e^{-\frac{1}{2}\hbar(\lambda-i)}}{e^{\frac{1}{2}\hbar} - e^{-\frac{1}{2}\hbar}}.$$

We call $\widehat{V_{\hbar,\lambda}}$ a *generic Verma module.*

For $j = 0, 1, \ldots,$ define

$$v_j = [\lambda]_q[\lambda - 1]_q \cdots [\lambda - j + 1]_q \widehat{v_j}$$

and let $V_{\hbar,\lambda}$ be the sub-$U_\hbar(\mathfrak{sl}_2)$-module of $\widehat{V_{\hbar,\lambda}}$ spanned by $\{v_0, v_1, \ldots\}$, with the action of $U_\hbar(\mathfrak{sl}_2)$

$$\begin{cases} Hv_i = (\lambda - 2i)v_i, \\ Ev_i = [\lambda + 1 - i]_q v_{i-1}, \\ Fv_i = [i + 1]_q v_{i+1}. \end{cases} \tag{2.3}$$

For $c \notin \mathbb{C}^* - \{1, 2, \ldots\}$, $\widehat{V_{\hbar,\lambda}}|_{\lambda=c}$ is isomorphic to $V_{\hbar,\lambda}|_{\lambda=c}$ because $[\lambda]_q[\lambda-1]_q \cdots [\lambda-j+1]_q$ is invertible for all j. On the other hand, for $c \in \{1, 2, \ldots, \}$, $v_j = 0$ if $j > c$ and (2.3) shows that $V_{\hbar,\lambda}|_{\lambda=c}$ is nothing but the standard irreducible $U_\hbar(\mathfrak{sl}_2)$-module of dimension $(c + 1)$ whereas $\widehat{V_{\hbar,\lambda}}|_{\lambda=c}$ is infinite dimensional.

Let us define $\mathsf{R} : \widehat{V_{\hbar,\lambda}} \otimes \widehat{V_{\hbar,\lambda}} \to \widehat{V_{\hbar,\lambda}} \otimes \widehat{V_{\hbar,\lambda}}$ by $\mathsf{R} = e^{-\frac{1}{4}\hbar\lambda^2} T\mathcal{R}$, where $T : \widehat{V_{\hbar,\lambda}} \otimes \widehat{V_{\hbar,\lambda}} \to \widehat{V_{\hbar,\lambda}} \otimes \widehat{V_{\hbar,\lambda}}$ is the transposition map $T(v \otimes w) = w \otimes v$, and $\mathcal{R}$ is the universal R-matrix (2.1).

By putting $z = q^{\lambda-1} = e^{\hbar(\lambda-1)}$, the action of R is given by

(2.4)

$$\begin{cases} \mathsf{R}(\widehat{v}_i \otimes \widehat{v}_j) = z^{-\frac{i+j}{2}} q^{-\frac{i+j}{2}} \displaystyle\sum_{n=0}^{i} F(i,j,n) \prod_{k=0}^{n-1} (z^{\frac{1}{2}} q^{-\frac{1+k+j}{2}} - z^{-\frac{1}{2}} q^{\frac{1+k+j}{2}}) \widehat{v}_{j+n} \otimes \widehat{v}_{i-n}, \\ \mathsf{R}(v_i \otimes v_j) = z^{-\frac{i+j}{2}} q^{-\frac{i+j}{2}} \displaystyle\sum_{n=0}^{i} F(i,j,n) \prod_{k=0}^{n-1} (z^{\frac{1}{2}} q^{\frac{1-i+k}{2}} - z^{-\frac{1}{2}} q^{-\frac{1-i+k}{2}}) v_{j+n} \otimes v_{i-n}, \end{cases}$$

where we put $F(i,j,n) = q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \begin{bmatrix} n+j \\ n \end{bmatrix}_q$.

Let $\mathbb{L} = \mathbb{Z}[q^{\pm 1}, z^{\pm 1}] = \mathbb{Z}[e^{\pm\hbar}, e^{\pm\hbar(\lambda-1)}] \subset \mathbb{C}[\lambda][[\hbar]]$ and let $V_{\mathbb{L}}$ and $\widehat{V_{\mathbb{L}}}$ be the sub-free $\mathbb{L}$-module of $\widehat{V_{\hbar,\lambda}}$ and $V_{\hbar,\lambda}$, spanned by $\{\widehat{v_0}, \ldots\}$ and $\{v_0, \ldots\}$, respectively.

Since all the coefficients of the action of R (2.4) lie in $\mathbb{L}$, $\widehat{V_{\mathbb{L}}}$ and $V_{\mathbb{L}}$ are equipped with an $\mathbb{L}B_n$-module structure. We denote the corresponding braid group representations by

$$\widehat{\varphi_{\mathbb{L}}} : B_n \to \mathrm{GL}(\widehat{V_{\mathbb{L}}}^{\otimes n}), \quad \varphi_{\mathbb{L}} : B_n \to \mathrm{GL}(V_{\mathbb{L}}^{\otimes n}).$$

These are decomposed as finite dimensional representations as follows. For $m \geq 0$, define $\widehat{V_{n,m}} \subset \widehat{V_{\mathbb{L}}}^{\otimes n}$ and $V_{n,m} \subset V_{\mathbb{L}}^{\otimes n}$ by

$$\begin{cases} \widehat{V_{n,m}} = \ker\,(q^{\frac{H}{2}} - q^{\frac{n\lambda-2m}{2}}) = \mathrm{span}\{\widehat{v}_{i_1} \otimes \cdots \otimes \widehat{v}_{i_n} \mid i_1 + \cdots + i_n = m\}, \\ V_{n,m} = \ker\,(q^{\frac{H}{2}} - q^{\frac{n\lambda-2m}{2}}) = \mathrm{span}\{v_{i_1} \otimes \cdots \otimes v_{i_n} \mid i_1 + \cdots + i_n = m\}. \end{cases}$$

By (2.4), the B_n-action preserves both $\widehat{V_{n,m}}$ and $V_{n,m}$ so we have linear representations

$$\widehat{\varphi^V_{n,m}} : B_n \to \mathrm{GL}(\widehat{V_{n,m}}) \quad \text{and} \quad \varphi^V_{n,m} : B_n \to \mathrm{GL}(V_{n,m}).$$

We call the $\mathbb{L}B_n$-module $\widehat{V_{n,m}}$ the (*generic*) *weight space* of weight $q^{n\lambda-2m}$.

By definition, as $\mathbb{L}B_n$-modules, $\widehat{V_{\mathbb{L}}}^{\otimes n}$ and $V_{\mathbb{L}}^{\otimes n}$ split as

$$\widehat{V_{\mathbb{L}}}^{\otimes n} \cong \bigoplus_{m=0}^{\infty} \widehat{V_{n,m}}, \quad V_{\mathbb{L}}^{\otimes n} \cong \bigoplus_{m=0}^{\infty} V_{n,m}. \tag{2.5}$$

Finally, we define the *space of* (*generic*) *null-vectors* $\widehat{W_{n,m}}$ by

$$\widehat{W_{n,m}} = \operatorname{Ker}(E) \cap \widehat{V_{n,m}}.$$

Since the action of B_n commutes with the action of $U_q(\mathfrak{sl}_2)$, we have linear representation

$$\widehat{\varphi^W_{n,m}} : B_n \to \mathrm{GL}(\widehat{W_{n,m}}).$$

In [JK, Lemma 13], it is shown that for $k = 1, \ldots, m$, the map $F^{m-k} : \widehat{W_{n,k}} \to \widehat{V_{n,m}}$ is injective and that over the quotient field, $\widehat{V_{n,m}}$ splits as

$$\widehat{V_{n,m}} \cong_{\mathcal{Q}} \bigoplus_{k=0}^{m} F^{m-k}\widehat{W_{n,k}} \cong_{\mathcal{Q}} \bigoplus_{k=0}^{m} \widehat{W_{n,k}}, \tag{2.6}$$

hence combining with (2.5), we conclude that the $\mathbb{L}B_n$-module $\widehat{V_{\mathbb{L}}}^{\otimes n}$ splits, over the quotient field,

$$\widehat{V_{\mathbb{L}}}^{\otimes n} \cong_{\mathcal{Q}} \bigoplus_{m=0}^{\infty}\bigoplus_{k=0}^{m} \widehat{W_{n,k}}. \tag{2.7}$$

2.2. **Lawrence's homological representations.** Here we briefly review the definition of (geometric) Lawrence's representation $L_{n,m}$. An explicit matrix of $L_{n,m}(\sigma_i)$ and some details will be given in the appendix.

For $i = 1, 2, \ldots, n$, let $p_i = i \in \mathbb{C}$ and let $D_n = \{z \in \mathbb{C} \mid |z| \leq n+1\} - \{p_1, \ldots, p_n\}$ be the n-punctured disc. We identify the braid group B_n with the mapping class group of D_n so that the standard generator σ_i corresponds to the *right-handed* half Dehn twist that interchanges the ith and $(i+1)$st punctures.

For $m > 0$, let $C_{n,m}$ be the unordered configuration space of m-points in D_n,

$$C_{n,m} = \{(z_1, \ldots, z_m) \in D_n \mid z_i \neq z_j \ (i \neq j)\}/S_m.$$

Here S_m is the symmetric group acting as permutations of the indices. For $i = 1, \ldots, n$, let $d_i = (n+1)e^{(\frac{3}{2}+i\varepsilon)\pi\sqrt{-1}} \in \partial D_n$, where $\varepsilon > 0$ is a sufficiently small number, and we take $\mathbf{d} = \{d_1, \ldots, d_m\}$ as a base point of $C_{n,m}$.

The first homology group $H_1(C_{n,m}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{\oplus n} \oplus \mathbb{Z}$, where the first n components correspond to the meridians of the hyperplanes $\{z_1 = p_i\}$ $(i = 1, \ldots, n)$ and the last component corresponds to the meridian of the discriminant $\bigcup_{1 \leq i < j \leq n}\{z_i = z_j\}$.

Let $\alpha : \pi_1(C_{n,m}) \to \mathbb{Z}^2 = \langle x, d\rangle$ be the homomorphism obtained by composing the Hurewicz homomorphism $\pi_1(C_{n,m}) \to H_1(C_{n,m}; \mathbb{Z})$ and the projection

$$H_1(C_{n,m}; \mathbb{Z}) = \mathbb{Z}^{\oplus n} \oplus \mathbb{Z} = \langle x_1, \ldots, x_n\rangle \oplus \langle d\rangle \to \langle x_1 + \cdots + x_n\rangle \oplus \langle d\rangle = \langle x\rangle \oplus \langle d\rangle.$$

Let $\pi : \widetilde{C}_{n,m} \to C_{n,m}$ be the covering corresponding to $\operatorname{Ker}\alpha$. We fix a lift $\widetilde{\mathbf{d}} \in \pi^{-1}(\mathbf{d}) \subset \widetilde{C}_{n,m}$ and use $\widetilde{\mathbf{d}}$ as a base point of $\widetilde{C}_{n,m}$. By identifying x and d as deck translations, $H_m(\widetilde{C}_{n,m}; \mathbb{Z})$ is a free $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$-module of rank $\binom{m+n-2}{m}$.

We will actually use $H^{lf}_m(\widetilde{C}_{n,m};\mathbb{Z})$, the homology of locally finite chains, and consider a free sub-$\mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$-module $\mathcal{H}_{n,m} \subset H^{lf}_m(\widetilde{C}_{n,m};\mathbb{Z})$ of rank $\binom{m+n-2}{m}$, spanned by homology classes represented by certain geometric objects called *multiforks*. The subspace $\mathcal{H}_{n,m}$ is preserved by B_n actions, hence by using a natural basis of $\mathcal{H}_{n,m}$ called *standard multiforks*, we get a linear representation

$$L_{n,m}\colon\ B_n \to \mathrm{GL}(\mathcal{H}_{n,m}) = \mathrm{GL}(\tbinom{m+n-2}{m}; \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]),$$

which we call (*geometric*) *Lawrence's representation.*

In the case $m = 1$ the discriminant $\bigcup_{1\leq i<j\leq n}\{z_i = z_j\}$ is empty so the variable d does not appear, and

$$L_{n,1}\colon\ B_n \to \mathrm{GL}(n-1; \mathbb{Z}[x^{\pm 1}])$$

coincides with the reduced Burau representation. The representation $L_{n,2}$ is often called the *Lawrence-Krammer-Bigelow representation*, which is extensively studied in [Big1, Kra, Kra2] and known to be faithful.

Remark 2.1. In general, the braid group representations $\mathcal{H}_{n,m}$, $H_m(\widetilde{C}_{n,m};\mathbb{Z})$ and $H^{lf}_m(\widetilde{C}_{n,m};\mathbb{Z})$ are not isomorphic to each other. However, all representations are *generically* identical, namely, there is an open dense subset $U \subset \mathbb{C}^2$ such that if we specialize x and d to complex parameters in U, then these three representations are isomorphic [Koh]. In particular, $\mathcal{H}_{n,m} \cong_{\mathcal{Q}} H_m(\widetilde{C}_{n,m};\mathbb{Z}) \cong_{\mathcal{Q}} H^{lf}_m(\widetilde{C}_{n,m};\mathbb{Z})$.

The following result gives a topological background why the MMR conjecture is true.

Proposition 2.2. *At $d = -1$, Lawrence's representation $L_{n,m}$ is equal to $\mathrm{Sym}^m L_{n,1}$, the mth symmetric power of the reduced Burau representation $L_{n,1}$.*

Proposition 2.2 seems to be known to experts although we do not know suitable references. This is directly seen by the formula of $L_{n,m}(\sigma_i)$ in the appendix. Roughly speaking, when we specialized $d = -1$, we ignored the discriminant $\bigcup_{1\leq i<j\leq n}\{z_i = z_j\}$ and a natural inclusion $C_{n,m} \to C^m_{n,1}/S_m$ induces an isomorphism

$$H_m(\widetilde{C_{n,m}};\mathbb{Z})|_{d=-1} \to H_m(\widetilde{C_{n,1}}^m/S_m;\mathbb{Z}) \cong H_1(\widetilde{C_{n,1}};\mathbb{Z})^{\otimes m}/S_m = \mathrm{Sym}^m H_1(\widetilde{C_{n,1}};\mathbb{Z})$$

of the braid group representations.

Here, we remark that a somewhat confusing minus sign of d comes from the convention of the orientation of submanifolds representing an element of $H_m(\widetilde{C_{n,m}};\mathbb{Z})$, as we will explain in the appendix.

2.3. **Identification and specializations of quantum and homological representations.** Here we summarize relations of braid group representations introduced in previous sections. First, generically a quantum representation is identified with Lawrence's representation. This was conjectured by Jackson and Kerler [JK] and proved by Kohno [Koh].

Theorem 2.3 ([Koh], [I, Corollary 4.6])**.** *As a braid group representation, there is an isomorphism*

$$\widehat{W_{n,m}} \cong \mathcal{H}_{n,m}|_{x=z^{-1}q, d=-q}.$$

For $\alpha \in \{2,3,\ldots\}$, let V_α be the α-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module and let $\varphi_\alpha : B_n \to \mathrm{GL}(V_\alpha^{\otimes n})$ be the quantum representation. Let $e : B_n \to \mathbb{Z}$ be the exponent sum map given by $e(\sigma_i^{\pm}) = \pm 1$. The usual quantum representation φ_α is recovered from a version of generic quantum representation as follows.

Proposition 2.4. *Let $\beta \in B_n$ and $\alpha \in \{2,3,\ldots\}$. Then*

$$e^{\frac{1}{4}\hbar(\alpha-1)^2 e(\beta)}\varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1} = \varphi_\alpha(\beta).$$

Proof. As we have seen, as $U_q(\mathfrak{sl}_2)$-module we have an isomorphism $V_\alpha \cong V_{\mathbb{L}}|_{\lambda=\alpha-1}$. The formula follows from this observation and the definition $\mathsf{R} = e^{-\frac{1}{4}\hbar\lambda^2}T\mathcal{R}$. □

Next we observe when we specialize λ as an integer, a certain symmetry appears.

Lemma 2.5. *For $\alpha \in \{2,3,\ldots\}$, $V_{n,m}|_{\lambda=\alpha-1} \cong V_{n,n\alpha-n-m}|_{\lambda=\alpha-1}$.*

Proof. Let us put

$$\mathsf{R}(v_i \otimes v_j) = \sum_{n=0}^{\infty} a_n v_{j+n} \otimes v_{i-n}, \quad \mathsf{R}(v_{\lambda-j} \otimes v_{\lambda-i}) = \sum_{n=0}^{\infty} b_n v_{\lambda-i+n} \otimes v_{\lambda-j-n},$$

where $a_n, b_n \in \mathbb{L}|_{z=q^{\alpha-1}} \cong \mathbb{Z}[q^{\pm 1}]$. We show $a_n = b_n$ for all i,j. This shows an equivalence of R-operators hence proves the desired isomorphism.

Note that when the weight variable λ is specialized as a positive integer $\alpha - 1$, $v_k = 0$ whenever $k \geq \lambda$, so $a_n = b_n = 0$ if $n > \min\{i, \lambda - j\}$. Hence we consider the case $n \leq \min\{i, \lambda - j\}$.

By (2.4), with putting $z = q^{\lambda-1}$, we have

$$\begin{aligned} b_n &= q^{\frac{\lambda}{2}(2\lambda-i-j)} q^{(\lambda-j-n)(\lambda-i+n)} q^{\frac{n(n-1)}{4}} \begin{bmatrix} n+\lambda-i \\ n \end{bmatrix}_q \prod_{k=0}^{n-1} (q^{\frac{1}{2}(j+k+1)} - q^{-\frac{1}{2}(j+k+1)}) \\ &= q^{-\frac{\lambda}{2}(i+j)} q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[n+\lambda-i]_q!}{[n]_q![\lambda-i]_q!} \prod_{k=0}^{n-1} (q^{\frac{1}{2}(j+k+1)} - q^{-\frac{1}{2}(j+k+1)}). \end{aligned}$$

Since

$$\prod_{k=0}^{n-1} (q^{\frac{1}{2}(j+k+1)} - q^{-\frac{1}{2}(j+k+1)}) = \frac{[j+1]_q \cdots [j+n]_q}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n} = \frac{[j+n]_q!}{[j]_q!(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n}$$

we conclude

$$\begin{aligned} b_n &= q^{-\frac{\lambda}{2}(i+j)} q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[n+\lambda-i]_q!}{[n]_q![\lambda-i]_q!} \cdot \frac{[j+n]_q!}{[j]_q!(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n} \\ &= q^{-\frac{\lambda}{2}(i+j)} q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[j+n]_q!}{[n]_q![j]_q!} \cdot \frac{[n+\lambda-i]_q!}{[\lambda-i]_q!(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n} \\ &= q^{-\frac{\lambda}{2}(i+j)} q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \begin{bmatrix} n+j \\ n \end{bmatrix}_q \prod_{k=0}^{n-1} (q^{\frac{1}{2}(\lambda-i+k+1)} - q^{-\frac{1}{2}(\lambda-i+k+1)}) \\ &= a_n. \end{aligned}$$

□

3. A topological formula for the loop expansion of the colored Jones polynomials

Now we are ready to prove our main result, a formula of the loop expansion of the colored Jones polynomials based on homological representation.

Theorem 3.1. *Let K be an oriented knot in S^3 represented as a closure of an n-braid β. Then the loop expansion of the colored Jones polynomial is given by*

$$CJ_K(\hbar, z) = \frac{z^{-\frac{1}{2}e(\beta)}q^{\frac{1}{2}e(\beta)}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{m=0}^{\infty} (z^{\frac{n}{2}} q^{\frac{1-n-2m}{2}} - z^{-\frac{n}{2}} q^{\frac{-1+n+2m}{2}}) \operatorname{trace} L_{n,m}(\beta)|_{x=qz^{-1}, d=-q}.$$

Here we put $q = e^{\hbar}$.

Proof. The colored Jones polynomial $J_{K,\alpha}(q)$ is defined by

$$J_{K,\alpha}(q) = \frac{1}{[\alpha]_q} q^{-\frac{1}{4}(\alpha^2-1)e(\beta)} \operatorname{trace}(q^{\frac{H}{2}} \varphi_\alpha(\beta)).$$

By Proposition 2.4,

$$J_{K,\alpha}(q) = \frac{1}{[\alpha]_q} q^{-\frac{1}{4}(\alpha^2-1)e(\beta)} q^{\frac{1}{4}(\alpha-1)^2 e(\beta)} \operatorname{trace}(q^{\frac{H}{2}} \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1}).$$

The colored Jones function $CJ_K(z, \hbar)$ is obtained by taking the limit $\alpha \to \infty$ keeping $z = e^{\hbar\alpha}$ constant, namely treating $\hbar\alpha$ as an independent variable:

$$CJ_K(\hbar, z) = \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} z^{-\frac{1}{2}e(\beta)} q^{\frac{1}{2}e(\beta)} \lim_{\substack{\alpha \to \infty \\ \hbar\alpha:\text{constant}}} \operatorname{trace}(q^{\frac{H}{2}} \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1}).$$

We compute the limit as follows (see Figure 1 for a diagrammatic summary of the computation).

Since $V_{n,i}|_{\lambda=\alpha-1} = 0$ if $i > n\lambda = n(\alpha-1)$, by (2.7), we have an $\mathbb{L}B_n$-module isomorphism $V_{\mathbb{L}}^{\otimes n} \cong \bigoplus_{i=0}^{n\lambda} V_{n,i}$. Moreover, $q^{\frac{H}{2}}$ acts on $V_{n,i}$ as a scalar multiple by $q^{\frac{n\lambda-2i}{2}}$. So

$$\operatorname{trace}(q^{\frac{H}{2}} \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1}) = \sum_{i=0}^{n\lambda} q^{\frac{n\lambda-2i}{2}} \operatorname{trace}(\varphi^V_{n,i}(\beta)|_{\lambda=\alpha-1}).$$

By Lemma 2.5, we identify the braid group representation $V_{n,n\lambda-k}$ as $V_{n,k}$ for $k \leq \frac{n\lambda}{2}$, and then regard each $V_{n,i}$ as a sub-$\mathbb{L}B_n$-module of $\widehat{V_{n,i}}$. Recall that $\operatorname{trace}(\varphi^V_{n,i}(\beta)|_{\lambda=\alpha-1})$ is equal to $\operatorname{trace}(\widehat{\varphi^V_{n,i}}(\beta)|_{\lambda=\alpha-1})$ when α is treated as an independent variable. By (2.6), over the quotient field, $\widehat{V_{n,i}}$ splits as $\bigoplus_{m=0}^{i} \widehat{W_{n,m}}$, hence

$$\operatorname{trace}(\widehat{\varphi^V_{n,i}}(\beta)|_{\lambda=\alpha-1}) = \sum_{m=0}^{\min\{m, n\lambda-m\}} \operatorname{trace}(\widehat{\varphi^W_{n,m}}(\beta)|_{\lambda=\alpha-1}).$$

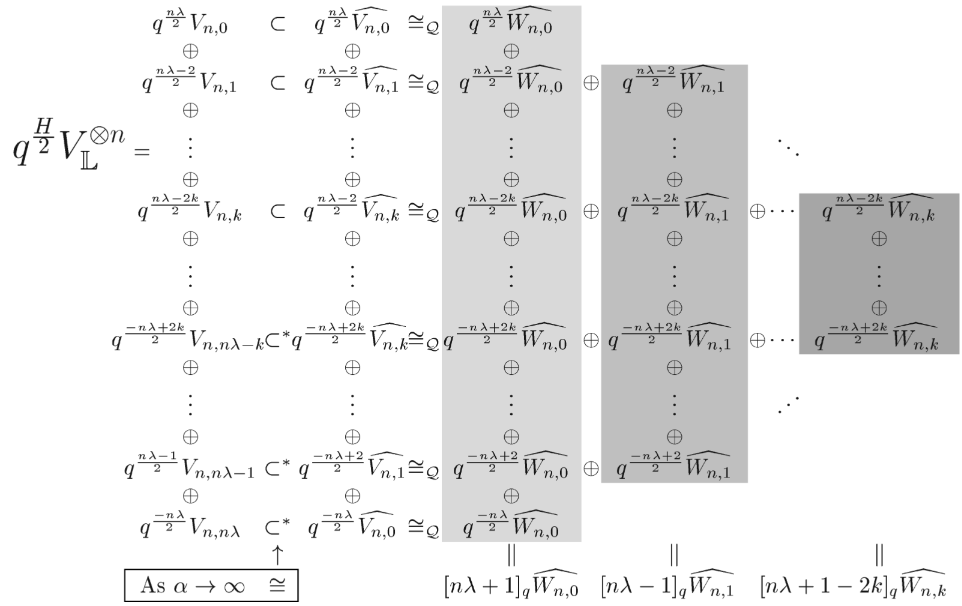

FIGURE 1. This diagram explains how to compute the desired limit. In the diagram, all representations are understood as taking specialization $\lambda = \alpha - 1$. The notation $\subset^*$ means that we regard $V_{n,n\lambda-i}$ as a submodule of $\widehat{V_{n,i}}$, by using the isomorphism $V_{n,n\lambda-i} \cong V_{n,i}$ in Lemma 2.5.

This shows

$$\begin{aligned}
\sum_{i=0}^{\alpha-1} q^{\frac{n\lambda-2i}{2}} \operatorname{trace}(\widehat{\varphi_{n,i}^{V}}(\beta)|_{\lambda=\alpha-1}) &= \sum_{i=0}^{\alpha-1} q^{\frac{n\lambda-2i}{2}} \sum_{m=0}^{\min\{m,n\lambda-m\}} \operatorname{trace}(\widehat{\varphi_{n,i}^{W}}(\beta)|_{\lambda=\alpha-1}) \\
&= \sum_{m=0}^{\frac{n\lambda}{2}} \sum_{i=m}^{n\lambda-m} q^{\frac{n\lambda-2i}{2}} \operatorname{trace}(\widehat{\varphi_{n,m}^{W}}(\beta)|_{\lambda=\alpha-1}) \\
&= \sum_{m=0}^{\frac{n\lambda}{2}} [n\lambda+1-2m]_q \operatorname{trace}(\widehat{\varphi_{n,m}^{W}}(\beta)|_{\lambda=\alpha-1}).
\end{aligned}$$

By Theorem 2.3, when we treat α as an independent variable, $\operatorname{trace}(\widehat{\varphi_{n,i}^{W}}(\beta)) = \operatorname{trace}(L_{n,m}(\beta)|_{x=z^{-1}q,d=-q})$ hence

$$\begin{aligned}
&\lim_{\substack{\alpha\to\infty \\ \hbar\alpha:\text{constant}}} \operatorname{trace}(q^{\frac{H}{2}} \circ \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1}) \\
&\qquad = \lim_{\substack{\alpha\to\infty \\ \hbar\alpha:\text{constant}}} \sum_{m=0}^{\min\{m,n\lambda-m\}} [n\lambda+1-2m]_q \operatorname{trace}(L_{n,m}(\beta)|_{x=z^{-1}q,d=-q}) \\
&\qquad = \sum_{m=0}^{\infty} \frac{z^{\frac{n}{2}} q^{\frac{1}{2}(1-n-2m)} - z^{-\frac{n}{2}} q^{-\frac{1}{2}(1-n-2m)}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \operatorname{trace}(L_{n,m}(\beta)|_{x=z^{-1}q,d=-q}).
\end{aligned}$$

Therefore we conclude that $CJ_K(\hbar, z)$ is written as

$$\frac{z^{-\frac{1}{2}e(\beta)}q^{\frac{1}{2}e(\beta)}}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}}\sum_{m=0}^{\infty}(z^{\frac{n}{2}}q^{\frac{1}{2}(1-n-2m)}-z^{-\frac{n}{2}}q^{-\frac{1}{2}(1-n-2m)})\operatorname{trace}(L_{n,m}(\beta)|_{x=z^{-1}q,d=-q}).$$

□

As we have mentioned, Theorem 3.1 provides an alternative, direct method to compute the loop expansion of the colored Jones polynomial. Actual computation may be quite hard and impractical, since one needs to compute $L_{n,m}(\beta)$ for all m. Here we give a sample calculation.

Example 3.2 ($(2,p)$-torus knot)**.** Let us consider $(2,p)$-torus knot $T(2,p)$ represented as a closure of 2-braid σ_1^p. The trace of Lawrence's representation is given by $\operatorname{trace} L_{2,m}(\sigma_1^p) = (-x^p)^m(-d^p)^{\binom{m}{2}}$ so

$$CJ_{T(2,p)}(z,\hbar) = \frac{z^{-\frac{1}{2}p}q^{\frac{1}{2}p}}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}}\sum_{m=0}^{\infty}(zq^{\frac{-1-2m}{2}}-z^{-1}q^{\frac{1+2m}{2}})(-z^{-p})^m q^{\binom{m+1}{2}p}. \tag{3.1}$$

To compute the 1-loop part, let us put $\hbar = 0$. Then

$$\begin{aligned} V^{(0)}_{T(2,p)}(z) &= \frac{z^{-\frac{1}{2}p}}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}}\sum_{m=0}^{\infty}(z-z^{-1})(-z^{-p})^m = (z^{\frac{1}{2}}+z^{-\frac{1}{2}})z^{-\frac{p}{2}}\sum_{m=0}^{\infty}(-z^{-p})^m \\ &= (z^{\frac{1}{2}}+z^{-\frac{1}{2}})z^{-\frac{p}{2}}\frac{1}{1+z^{-p}} = \frac{z^{\frac{1}{2}}+z^{-\frac{1}{2}}}{z^{\frac{1}{2}p}+z^{-\frac{1}{2}p}} \end{aligned}$$

which is equal to the inverse of the Alexander-Conway polynomial of $T(2,p)$.

To compute the 2-loop part, we put $q = e^{\hbar}$ and look at the coefficient of $\hbar$ in (3.1). Then

$$\begin{aligned} V^{(1)}_{T(2,p)}(z) &= \frac{z^{-\frac{1}{2}p}}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}}\Bigg(\frac{z}{2}\sum_{m=0}^{\infty}[pm^2+(p-2)m+(p-1)](-z^{-pm}) \\ &\qquad -\frac{z^{-1}}{2}\sum_{m=0}^{\infty}[pm^2+(p+2)m+(p+1)](-z^{-pm})\Bigg). \end{aligned}$$

In the ring of formal power series $\mathbb{C}[[z,z^{-1}]]$,

$$\sum_{m=0}^{\infty}z^m = \frac{1}{1-z},\ \sum_{m=0}^{\infty}mz^m = \frac{z}{(1-z)^2},\ \sum_{m=0}^{\infty}m^2z^m = \frac{z+z^2}{(1-z)^3},$$

hence

$$V^{(1)}_{T(2,p)}(z) = \frac{(p-1)(z^{p+1}-z^{-p-1})-(p+1)(z^{p-1}-z^{-p+1})}{2(z^{\frac{1}{2}}-z^{-\frac{1}{2}})(z^{\frac{1}{2}p}+z^{-\frac{1}{2}p})^3}.$$

For example,

$$V^{(1)}_{T(2,3)}(z) = \frac{(z^4-z^{-4})-2(z^2-z^{-2})}{(z^{\frac{1}{2}}-z^{-\frac{1}{2}})(z^{\frac{3}{2}}+z^{-\frac{3}{2}})^3} = \frac{(z^2-2z+2-2z^{-1}+z^{-2})}{(z-1+z^{-1})^3}.$$

This coincides with the computation of Overbay [Ov].

It is a direct consequence that the 1-loop part is the inverse of the Alexander-Conway polynomial.

Corollary 3.3 (Melvin-Morton-Rozansky conjecture).

$$V^{(0)}(z) = \frac{1}{\Delta_K(z)}.$$

Proof. The 1-loop part $V^{(0)}(z)$ is obtained by putting $\hbar = 0$ in the formula of Theorem 3.1. Since $d = -q = -e^{\hbar}$, in a homological representation, putting $\hbar = 0$ corresponds to putting $d = -1$. As we have pointed out in Proposition 2.2,

$$L_{n,m}(\beta)|_{d=-1} = \mathrm{Sym}^m L_{n,1}(\beta).$$

Therefore, by Theorem 3.1, the 1-loop part $V^{(0)}(z)$ is written as

$$\begin{aligned} V^{(0)}(z) &= z^{-\frac{1}{2}e(\beta)} \frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{i=0}^{\infty} \mathrm{trace}\, L_{n,i}(\beta)|_{x=z^{-1},d=-1} \\ &= z^{-\frac{1}{2}e(\beta)} \frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{i=0}^{\infty} \mathrm{trace}(\mathrm{Sym}^i L_{n,1})(\beta)|_{x=z^{-1}}. \end{aligned}$$

The MacMahon Master theorem says that

$$\sum_{i=0}^{\infty} \mathrm{trace}(\mathrm{Sym}^i L_{n,1})(\beta) = \det(I - L_{n,1}(\beta))^{-1}$$

hence

$$V^{(0)}(z) = z^{-\frac{1}{2}e(\beta)} \frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \frac{1}{\det(I - L_{n,1}(\beta)|_{x=z^{-1}})} = \frac{1}{\Delta_K(z^{-1})} = \frac{1}{\Delta_K(z)}.$$

□

4. Entropy and colored Jones polynomials

In this section we give an application of topological interpretation of quantum representations.

4.1. **Entropy estimates from configuration space.** For a homeomorphism of a compact topological space or a metric space $f : X \to X$, there is a fundamental numerical invariant $h(f) \in \mathbb{R}$ of topological dynamics called the (*topological*) *entropy.*

Let $\overline{C}_m(X)$ and $C_m(X)$ be the *ordered* and *unordered configuration space* of m-points of X,

$$\overline{C}_m(X) = \{(x_1 \dots, x_m) \in X^m \mid x_i \neq x_j\}, \quad C_m(X) = \overline{C}_m(X)/S_m,$$

where S_m is the symmetric group that acts as permutations of the coordinates. Then f induces the continuous maps $\overline{C}_m(f) : \overline{C}_m(X) \to \overline{C}_m(X)$ and $C_m(f) : C_m(X) \to C_m(X)$, respectively.

Note that $\overline{C}_m(X) \subset X^m$ is invariant under $f^{\times m} : X^m \to X^m$ so

$$h(\overline{C}_m(f)) \leq h(f^{\times m}) = mh(f).$$

The unordered configuration space $\overline{C}_m(X)$ is a finite cover of $C_m(X)$ so $h(C_m(f)) = h(\overline{C}_m(f))$.

Now for $A \in \mathrm{GL}(n;\mathbb{C})$ let $\rho(A)$ be its spectral radius of A, the maximum of the absolute value of the eigenvalues of A. It is known that if $C_m(f)$ is nice enough (see [Fr] for sufficient conditions for inequality (4.1) to hold), then

$$\log \rho(C_m(f)_* : H_*(C_m(f), \mathbb{Z}) \to H_*(C_m(f), \mathbb{Z})) \leq h(C_m(f)). \tag{4.1}$$

Hence by using configuration spaces we have an estimate of the entropy

$$\log \rho(C_m(f)_*) \leq mh(f). \tag{4.2}$$

The above considerations fit for the braid groups. Let us regard the braid group B_n as the mapping class group of n-punctured disc D_n. The entropy of braid $\beta \in B_n$ is defined by the infimum of entropy of homeomorphisms representing β,

$$h(\beta) = \inf\{h(f) \,|\, f : D_n \to D_n, [f] = \beta \in MCG(D_n) = B_n\}.$$

By the Nielsen-Thurston classification [FLP, Th], there is a representative homeomorphism f_β that attains the infimum so $h(\beta) = h(f_\beta)$. In particular, if β is pseudo-Anosov, then a pseudo-Anosov representative attains the infimum. By abuse of notation, we will use the same symbol β to mean its representative homeomorphism f_β that attains the infimum of the entropy.

As Koberda shows in [Kob], the inequality (4.1) holds in the case X is a surface. This implies that Lawrence's representation gives an estimate of entropy.

Theorem 4.1. *For an n-braid β,*

$$\sup_{|x|=1,|d|=1} \log \rho(L_{n,m}(\beta)) \leq mh(\beta)$$

Proof. Let $\widetilde{C}$ be a finite covering of the unordered configuration space $C_{n,m} = C_m(D_n)$. If the action of β on $C_{n,m}$ lifts, then by (4.2), $\log \rho(\widetilde{\beta_{\widetilde{C}*}}) \leq mh(\beta)$ holds, where $\widetilde{\beta_{\widetilde{C}}} : \widetilde{C} \to \widetilde{C}$ denotes the lift of a homeomorphism β.

For non-negative integers A, B, let $\widetilde{C} = \widetilde{C}_{A,B}$ be a finite abelian covering of $C_{n,m}$ that corresponds to the kernel of $\alpha_{A,B} : \pi_1(C_{n,m}) \to \mathbb{Z}/A\mathbb{Z} \oplus \mathbb{Z}/B\mathbb{Z}$, where $\alpha_{A,B}$ is given by the compositions

$$\pi_1(C_{n,m}) \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \cong \langle x \rangle \oplus \langle d \rangle \longrightarrow (\langle x \rangle / x^A) \oplus (\langle d \rangle / d^B) = \mathbb{Z}/A\mathbb{Z} \oplus \mathbb{Z}/B\mathbb{Z}.$$

The standard topological argument, using the eigenspace decompositions for the deck translations (see [BB] for the case $\mathbb{Z}$-covering, the case of the reduced Burau representation $L_{n,1}$. The same argument applies to the case $\mathbb{Z}^2$-covering) shows that for $a = 1, \dots, A-1$ and $b = 1, \dots, B-1$,

$$\rho(L_{n,m}(\beta)|_{x=e^{\frac{2\pi a\sqrt{-1}}{A}}, d=e^{\frac{2\pi b\sqrt{-1}}{B}}}) \leq \rho(\widetilde{\beta}_{\widetilde{C}_{A,B}}).$$

The sets $\{(e^{\frac{2\pi a\sqrt{-1}}{A}}, e^{\frac{2\pi b\sqrt{-1}}{B}}) \in \ | \ A, a, B, b \in \mathbb{Z}\}$ are dense in $S^1 \times S^1 = \{(x, d) \in \mathbb{C}^2 \mid |x| = |d| = 1\}$, hence we get the desired inequality. □

Generically one can identify the quantum representation with Lawrence's representation, so quantum representations also provide estimates of entropy.

Theorem 4.2. *Let β be an n-braid.*

1. $\sup_{|q|=1,|z|=1} \log \rho(\widehat{\varphi^W_{n,m}}(\beta)) \leq mh(\beta)$.
2. $\sup_{|q|=1,|z|=1} \log \rho(\widehat{\varphi^V_{n,m}}(\beta)) \leq mh(\beta)$.
3. $\sup_{|q|=1} \log \rho(\varphi_\alpha(\beta)) \leq \frac{n\alpha - n}{2} h(\beta)$.

Proof. The assertions (1) and (2) follow from Theorem 4.1 and Theorem 2.3. To see (3), recall that as an $\mathbb{L}B_n|_{z=q^{\alpha-1}} = \mathbb{C}[q^{\pm 1}]B_n$-module, we have

$$V_\alpha^{\otimes n} \cong V_{\mathbb{L}}|_{z=q^{\alpha-1}} \subset \bigoplus_{m=0}^{n(\alpha-1)} V_{n,m}|_{\lambda=\alpha-1}.$$

Moreover, by Lemma 2.5 $V_{n,m}|_{\lambda=\alpha-1} \cong V_{n,n(\alpha-1)-m}|_{\lambda=\alpha-1}$. Therefore,

$$\begin{aligned}\sup_{|q|=1} \rho(\varphi_\alpha(\beta)) &\leq \max_{1\leq m\leq \frac{n(\alpha-1)}{2}} \sup_{|q|=1} (\varphi^V_{n,m}(\beta)|_{z=q^{\alpha-1}}) \\ &\leq \max_{1\leq m\leq \frac{n(\alpha-1)}{2}} \sup_{|q|=1} \rho(\widehat{\varphi^V_{n,m}}(\beta)|_{z=q^{\alpha-1}}).\end{aligned}$$

By (1), we conclude $\sup_{|q|=1} \log \rho(\varphi_\alpha(\beta)) \leq \frac{n(\alpha-1)}{2} h(\beta)$. □

The estimates in Theorem 4.1 and Theorem 4.2 are not really practical since the left-hand side is quite hard to compute. We do not know whether Theorem 4.1 and Theorem 4.2 give strictly better estimates than the previously known Burau estimate, the case $m = 1$.

4.2. **Quantum $\mathfrak{sl}_2$ invariants and entropy.** An estimate in Theorem 4.2 suggests a relationship between quantum invariants and entropy of braids.

For $\alpha \in \{2, 3 \ldots, \}$, let $Q^{\mathfrak{sl}_2;V_\alpha}_K(q) = \mathrm{trace}(q^{\frac{H}{2}}\varphi_\alpha(\beta)) = [\alpha]_q J_{\alpha,K}(q)$ be the quantum $(\mathfrak{sl}_2, V_\alpha)$-invariant of the knot K, another common normalization of the colored Jones polynomials used to define quantum invariants of 3-manifolds.

Theorem 4.3. *Let K be a knot represented as the closure of an n-braid β, and $\alpha \in \{2, 3, \ldots\}$. Then*

$$\sup_{|q|=1} \log |Q^{\mathfrak{sl}_2;V_\alpha}_K(q)| \leq n \log \alpha + \log \rho(\varphi_\alpha(\beta)) \leq n \log \alpha + \frac{n(\alpha-1)}{2} h(\beta).$$

Proof. By definition of the spectral radius,

$$|Q^{\mathfrak{sl}_2;V_\alpha}_K(q)| = |\,\mathrm{trace}(q^{\frac{H}{2}}\varphi_\alpha(\beta))| \leq \alpha^n \rho(q^{\frac{H}{2}}\varphi_\alpha(\beta)) \leq \alpha^n \rho(q^{\frac{H}{2}})\rho(\varphi_\alpha(\beta)).$$

Here the last inequality follows from the fact that $q^{\frac{H}{2}}$ and $\varphi_\alpha(\beta)$ commute. For $|q| = 1$, $\rho(q^{\frac{H}{2}}) = 1$ hence by Theorem 4.2 (3), we conclude

$$\sup_{|q|=1} |\,\mathrm{trace}(q^{\frac{H}{2}}\varphi_\alpha(\beta))| \leq \sup_{|q|=1} \alpha^n \rho(\varphi_\alpha(\beta)) \leq \alpha^n e^{\frac{n(\alpha-1)}{2}h(\beta)}.$$

□

As we have mentioned, Theorem 4.3 justifies an intuitive statement

> "a knot with complicated quantum $\mathfrak{sl}_2$ invariants (colored Jones polynomial) is a closure of a complicated (large entropy) braid".

By an analogy of the famous volume conjecture [Ka, MuMu], it is interesting to look at the asymptotic behavior of $|Q^{\mathfrak{sl}_2;V_\alpha}_K(q)|$. By Theorem 4.3, we have

$$\frac{\sup_{|q|=1} \log |Q^{\mathfrak{sl}_2;V_\alpha}_K(q)|}{\alpha} \leq n\frac{\log \alpha}{\alpha} + \frac{\sup_{|q|=1} \log \rho(\varphi_\alpha(\beta))}{\alpha} \leq n\frac{\log \alpha}{\alpha} + \frac{n(\alpha-1)}{2\alpha} h(\beta).$$

This shows

$$\limsup_{\alpha\to\infty} \frac{\sup_{|q|=1} \log |Q^{\mathfrak{sl}_2;V_\alpha}_K(q)|}{\alpha} \leq \limsup_{\alpha\to\infty} \frac{\sup_{|q|=1} \log \rho(\varphi_\alpha(\beta))}{\alpha} \leq \frac{n}{2} h(\beta). \tag{4.3}$$

It is interesting to ask when the limit converges and when the inequalities (4.3) yield the equalities, Is there a closed braid representative $\widehat{\beta}$ of K with (4.3) equalities? The second inequality is related to the question when the quantum representation estimate of the entropy is (asymptotically) sharp.

A. Appendix: Multiforks for Lawrence's representation $L_{n,m}$

In this appendix, we present multiforks in Lawrence's representation $L_{n,m}$ and explicit matrices of $L_{n,m}(\sigma_i)$. For the basics of geometric treatments of Lawrence's representation, see [I, Section 2].

Let Y be the Y-shaped graph with four vertices c, r, v_1, v_2 and oriented edges as shown in Figure 2(1). A *fork* F based on $d \in \partial D_n$ is an embedded image of Y into $D^2 = \{z \in \mathbb{C} \mid |z| \leqslant n+1\}$ such that:

- All points of $Y \setminus \{r, v_1, v_2\}$ are mapped to the interior of D_n.
- The vertex r is mapped to d_i.
- The other two external vertices v_1 and v_2 are mapped to the puncture points.

The image of the edge $[r, c]$ and the image of $[v_1, v_2] = [v_1, c] \cup [c, v_2]$ regarded as a single oriented arc, are denoted by $H(F)$ and $T(F)$. We call $H(F)$ and $T(F)$ the *handle* and the *tine* of the fork F, respectively.

A *multifork* of dimension m is an ordered tuple of m forks $\mathbb{F} = (F_1, \dots, F_m)$ such that

- F_i is a fork based on d_i.
- $T(F_i) \cap T(F_j) \cap D_n = \emptyset$ $(i \neq j)$.
- $H(F_i) \cap H(F_j) = \emptyset$ $(i \neq j)$.

Figure 2 (2) illustrates an example of a multifork of dimension 3. We represent multiforks consisting of k parallel forks by drawing a single fork labeled by k, as shown in Figure 2 (3).

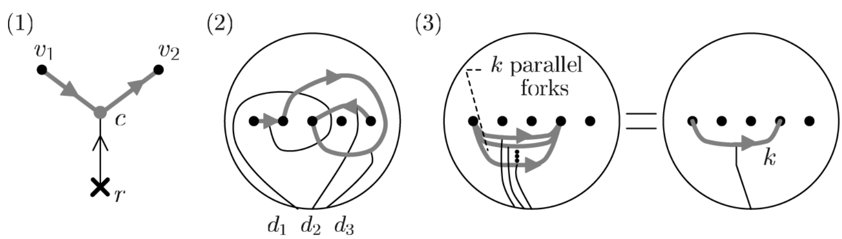

Figure 2. Multiforks: to distinguish tines from handles, we often denote the tines of forks by a bold gray line.

We regard the handle $H(F_i)$ of the fork F_i as a path $\gamma_i \colon [0,1] \to D_n$. Then the handles of $\mathbb{F}$ define a path $H(\mathbb{F}) = \{\gamma_1, \dots, \gamma_m\} \colon [0,1] \to C_{n,m}$. Take a lift of $H(\mathbb{F})$, $\widetilde{H(\mathbb{F})} \colon [0,1] \to \widetilde{C}_{n,m}$ so that $\widetilde{H(\mathbb{F})}(0) = \widetilde{\mathbf{d}}$.

Let $\Sigma(\mathbb{F}) = \{\{z_1, \dots, z_m\} \in C_{n,m} \mid z_i \in T(F_i)\}$, and let $\widetilde{\Sigma}(\mathbb{F})$ be the m-dimensional submanifold of $\widetilde{C}_{n,m}$ which is the connected component of $\pi^{-1}(\Sigma(\mathbb{F}))$ containing $\widetilde{H(\mathbb{F})}(1)$. The submanifold $\widetilde{\Sigma}(\mathbb{F})$ represents an element of $H^{lf}_m(\widetilde{C}_{n,m};\mathbb{Z})$. By abuse of notation, we use $\mathbb{F}$ to represent both a multifork and its representing homology class $[\widetilde{\Sigma}(\mathbb{F})] \in H^{lf}_m(\widetilde{C}_{n,m};\mathbb{Z})$.

Here the orientation of $\widetilde{\Sigma}(\mathbb{F})$ is defined so that a canonical homeomorphism $T(F_1) \times \cdots \times T(F_m) \to \Sigma(\mathbb{F})$ is orientation preserving. Thus, for a fork $\mathbb{F}_\tau = (F_{\tau(1)}, \dots, F_{\tau(m)})$ obtained by permuting its coordinate by a permutation $\tau \in S_m$, we have $\mathbb{F}_\tau = \mathsf{sgn}(\tau)\mathbb{F} \in H^{lf}_m(\widetilde{C}_{n,m};\mathbb{Z})$.

For $\mathbf{e} = (e_1, \ldots, e_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ with $e_1 + \cdots + e_{n-1} = m$, we assign a multifork $\mathbb{F}_{\mathbf{e}} = \{F_1, \ldots, F_m\}$ in Figure 3 and call $\mathbb{F}_{\mathbf{e}}$ a *standard multifork*. The set of standard multiforks spans a $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$-submodule $\mathcal{H}_{n,m}$ of $H_m^{lf}(\widetilde{C_{n,m}}; \mathbb{Z})$, which is free of dimension $\binom{n+m-2}{2}$ and is invariant under the B_n-action. This defines a (geometric) Lawrence's representation $L_{n,m} : B_n \to \mathrm{GL}\left(\binom{n+m-2}{2}; \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]\right)$.

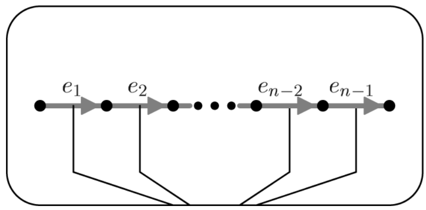

FIGURE 3. Standard multifork $\mathbb{F}_{\mathbf{e}}$ for $\mathbf{e} = (e_1, \ldots, e_{n-1})$

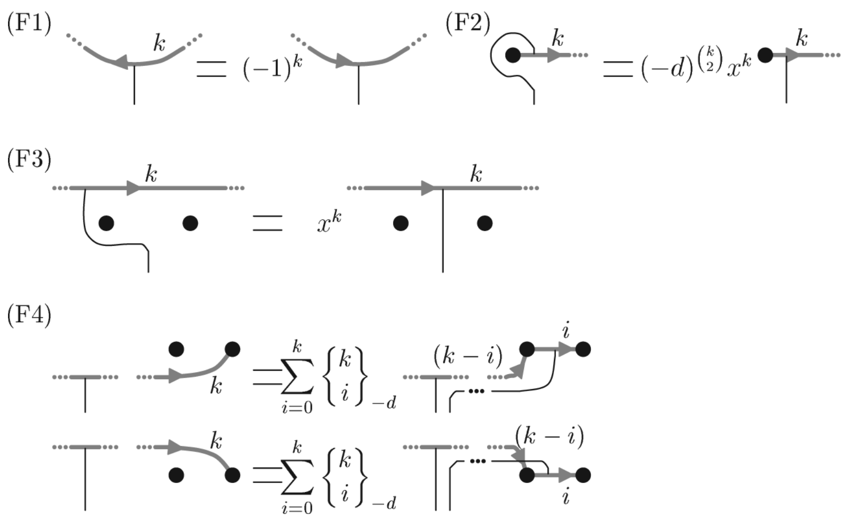

FIGURE 4. Geometric rewriting formula for multiforks. Here $\{a\}_q = \frac{q^a - 1}{q-1}$ is a different version of a q-integer, and $\left\{ {a \atop b} \right\}_q = \frac{\{a\}_q!}{\{a-b\}_q!\{b\}_q!}$ is the version of a q-binomial coefficient.

From the definition of the submanifold $\widetilde{\Sigma}(\mathbb{F})$ and by using noodle-fork pairing (the homology intersection $H_m(\widetilde{C_{n,m}}, \partial\widetilde{C_{n,m}}; \mathbb{Z}) \times H_m^{lf}(\widetilde{C_{n,m}}; \mathbb{Z}) \to \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$; see [Big1, I]) computation, we graphically express relations among homology classes represented by multiforks as in Figure 4. This allows us to express a given multifork as a sum of standard multiforks (see [Kra, Big2] for the case $m = 2$). In particular, these formulae lead to a formula of an explicit matrix representative of $L_{n,m}(\sigma_i)$:

$$\begin{aligned}
&L_{n,m}(\sigma_1)(\mathbb{F}_{e_1,\dots,e_{n-1}})\\
&\quad = \sum_{l=0}^{e_2} (-1)^{e_1}(-d)^{\binom{e_1}{2}} x^{e_1} \begin{Bmatrix} e_2 \\ l \end{Bmatrix}_{-d} \mathbb{F}_{e_1+e_2-l,l,\dots}\\
&L_{n,m}(\sigma_i)(\mathbb{F}_{e_1,\dots,e_{n-1}}) \qquad\qquad (i=2,\dots,n-2)\\
&\quad = \sum_{k=0}^{e_{i-1}} \sum_{l=0}^{e_{i+1}} (-1)^{e_i}(-d)^{\binom{e_i+k}{2}} x^{e_i+k} \begin{Bmatrix} e_{i-1} \\ k \end{Bmatrix}_{-d} \begin{Bmatrix} e_{i+1} \\ l \end{Bmatrix}_{-d} \mathbb{F}_{\dots,e_{i-1}-k,e_i+k+e_{i+1}-l,l,\dots}\\
&L_{n,m}(\sigma_{n-1})(\mathbb{F}_{e_1,\dots,e_{n-1}})\\
&\quad = \sum_{k=0}^{e_{n-2}} (-1)^{e_{n-1}}(-d)^{\binom{e_{n-1}+k}{2}} x^{e_{n-1}+k} \begin{Bmatrix} e_{n-2} \\ k \end{Bmatrix}_{-d} \mathbb{F}_{\dots,e_{n-2}-k,e_{n-1}+k}.
\end{aligned}$$

Then Proposition 2.2 follows from the formula. However, a multifork expression gives a direct way to see Proposition 2.2. Note that by orientation convention of $\widetilde{\Sigma}(\mathbb{F})$, when $d=-1$ the homology class represented by a multifork $\mathbb{F}=(F_1,\dots,F_m)$ is independent of a choice of indices of forks. Namely, a fork $\mathbb{F}_\tau=(F_{\tau(1)},\dots,F_{\tau(m)})$ obtained by permuting its coordinate by a permutation $\tau\in S_m$ is equal to the original fork $\mathbb{F}$. Therefore, the correspondence between multifork $\mathbb{F}=(F_1,\dots,F_m)$ that represents an element of $\mathcal{H}_{n,m}$ and a family of m forks $\{F_1,\dots,F_k\}$ that represents an element of $\operatorname{Sym}^m \mathcal{H}_{n,1}$ gives rise to the desired isomorphism $\mathcal{H}_{n,m}\to \operatorname{Sym}^m \mathcal{H}_{n,1}$.

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