TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 372, Number 1, 1 July 2019, Pages 503–543
https://doi.org/10.1090/tran/7586
Article electronically published on March 19, 2019

INTEGRATION OF 2-TERM REPRESENTATIONS UP TO HOMOTOPY VIA 2-FUNCTORS

OLIVIER BRAHIC AND CRISTIAN ORTIZ

Abstract. Given a representation up to homotopy of a Lie algebroid on a 2-term complex of vector bundles, we define the corresponding holonomy as a strict 2-functor from a Weinstein path 2-groupoid to the gauge 2-groupoid of the underlying 2-term complex. We construct a corresponding transformation 2-groupoid, and we prove that the 1-truncation of this 2-groupoid is isomorphic to the Weinstein groupoid of the $\mathcal{VB}$-algebroid associated with a representation up to homotopy.

Contents

1. Introduction

In recent years, the theory of Lie groupoids received a lot of attention, providing various applications in differential geometry, in particular, in the context of Poisson geometry. The differential geometric nature of Lie groupoids makes the notion of *vector bundle* fundamental to the theory; as it turns out, it also relates nicely to the notion of representation up to homotopy. Our main aim in this work is to understand the integrability problem in this setting.

Below, we expound on how the theory of representations, and of representations up to homotopy, of a Lie groupoid relate to $\mathcal{VB}$-groupoids and $\mathcal{VB}$-algebroids, which are the correct notion of vector bundles in the category of Lie groupoids and Lie algebroids. Special attention is paid to recent developments pertaining to the integration problem.

Received by the editors March 16, 2017, and, in revised form, March 20, 2018.
2010 *Mathematics Subject Classification.* Primary 58H05, 22A22; Secondary 53D17.
Key words and phrases. Differential geometry, Poisson geometry.
The first author was supported in part by CNPq grant 401253/2012-0.
The second author was supported by CAPES-COFECUB (grant 763/13) and CNPq (grant 4827967/2013-8).

Lie algebroids and groupoids representations. Given a Lie algebroid $A \to M$, a representation of A is a vector bundle $E \to M$ together with a Lie algebroid morphism $A \to D(E)$, where $D(E)$ is the Lie algebroid of derivations of E. This is equivalent to an A-connection $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E); (a, e) \mapsto \nabla_a e$ on the vector bundle E, with vanishing curvature (i.e., $\nabla_{[a,b]} - [\nabla_a, \nabla_b] = 0$). There are two important objects associated with a Lie algebroid representation; namely,

- the cohomology $H^\bullet(A, E)$ of A with coefficients in (E, ∇),
- the *transformation Lie algebroid* $A \ltimes E$, which is a Lie algebroid over E.

There is a global counterpart to the notion of Lie algebroid representation, which is that of a Lie groupoid representation. A representation of a Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of a vector bundle $E \to M$ together with a Lie groupoid morphism $\mathcal{G} \to \mathrm{Gau}(E)$. Here $\mathrm{Gau}(E)$ is the gauge groupoid of E, where objects are points of M, and arrows from x to y are linear isomorphisms $E_x \to E_y$ between the fibers. As with Lie algebroid representations, with any Lie groupoid representation, one can associate

- the groupoid cohomology $H^\bullet(\mathcal{G}, E)$ with coefficients in E;
- the *transformation groupoid* $\mathcal{G} \ltimes E$, which is a Lie groupoid over E.

Notice that, although both constructions contain the same information, one may think of the cohomology as the *algebraic* counterpart to the notion of representation, while the transformation algebroid/groupoid has a more *geometric* flavor.

In this context, one can discuss the integration problem in the following way. Regarding the transformation object, one can pass from Lie groupoids to Lie algebroids by differentiation and, provided that A is integrable, one can pass from Lie algebroids to Lie groupoids as well, by an integration process. Moreover, both operations of integration and those of taking the corresponding transformation object commute, which can be summarized in a simple diagram, as follows:

$$\begin{array}{ccc}
\begin{array}{c}\text{Lie algebroid}\\ \text{representation:}\\ A \circlearrowleft E\end{array} & \xrightarrow{\text{integration}} & \begin{array}{c}\text{Lie groupoid}\\ \text{representation:}\\ \mathcal{G}(A) \circlearrowleft E\end{array} \\
\text{transformation object}\ \Big\downarrow & & \Big\downarrow\ \text{transformation object} \\
\begin{array}{c}\text{transformation}\\ \text{Lie algebroid:}\\ A \ltimes E\end{array} & \xrightarrow{\text{integration}} & \begin{array}{c}\text{transformation}\\ \text{Lie groupoid:}\\ \mathcal{G}(A) \ltimes E = \mathcal{G}(A \ltimes E)\end{array}
\end{array}$$

Note that given a representation of a Lie algebroid/groupoid over M, one can also build the corresponding *semidirect product* Lie algebroid (resp., Lie groupoid) which is a Lie algebroid (resp., Lie groupoid) over M. In this case as well, a similar diagram as above makes sense, and the semidirect product operation commutes with both the Lie functor and the integration. In fact, we shall see that the notion of representation up to homotopy allows us to unify the two notions (see Examples 5.3 and 5.4).

Representations up to homotopy. Unlike with Lie algebras, the notion of adjoint representation of a Lie algebroid is not well defined *as a representation.* However, as shown in [1], it is well defined *as a representation up to homotopy.* This basic fact justifies the importance of representations up to homotopy in the representation theory of Lie algebroids and Lie groupoids.

A representation up to homotopy is a generalization of the notion of a representation, where a Lie algebroid is represented on a graded vector bundle $\mathcal{E} = \bigoplus E_i$, rather than on a single vector bundle. It is defined as a degree 1 operator on the complex $\Omega^\bullet(A) \otimes \Gamma(\mathcal{E})$, which squares to zero and satisfies a natural derivation condition. See [1] for a precise definition. As can be seen from this definition, the notion of representation up to homotopy of a Lie algebroid stems naturally from the *cohomological* aspect of a usual representation.

Degree 2 representations up to homotopy and $\mathcal{VB}$-algebroids. In the case of a 2-term representation up to homotopy, meaning that the graded vector bundle $\mathcal{E}$ is concentrated in degrees -1 and 0 (i.e., $\mathcal{E} = E_{-1} \oplus E_0$), Gracia-Saz and Mehta [17] obtained a nice geometric interpretation in terms of $\mathcal{VB}$-algebroids (see below). The concept of a $\mathcal{VB}$-algebroid was introduced by Pradines in [24], and it roughly corresponds to a vector bundle object in the category of Lie algebroids. They play a fundamental role in the theory of Lie algebroids, similar to that of vector bundles for manifolds.

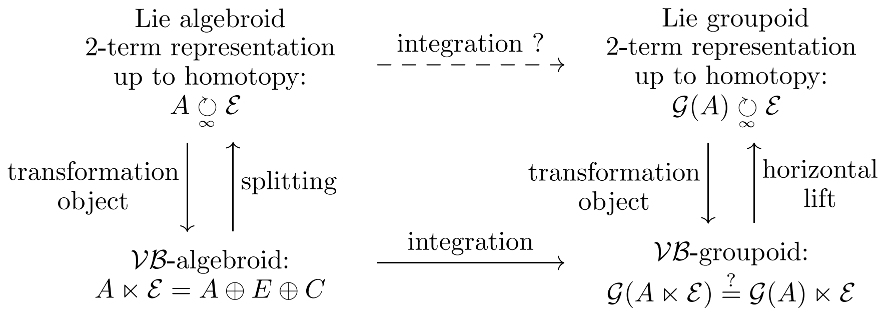
Since we want to break free from the cohomological approach, and in order to relate to the work of Gracia-Saz and Mehta [17], we shall forget about degrees, and we denote by $E := E_{-1}$ and by $C := E_0$ the degree -1 and 0 components, respectively.

What was shown in [17] is that a representation up to homotopy of A on $\mathcal{E} = E \oplus C$ induces a $\mathcal{VB}$-algebroid structure on the double vector bundle $D = A \oplus E \oplus C$. Conversely, given a $\mathcal{VB}$-algebroid D over A, with core C and side E, the choice of a splitting $D \cong A \oplus E \oplus C$ induces a 2-term representation up to homotopy of A on $E \oplus C$. In fact, there is an equivalence of categories between 2-term representations up to homotopy of a Lie algebroid A and $\mathcal{VB}$-algebroids over A (see [16, 17]). For this reason, one may denote by $A \ltimes \mathcal{E} = A \oplus E \oplus C$ the $\mathcal{VB}$-algebroid structure induced by a 2-term representation up to homotopy, and one may think of the operation of passing from a 2-term representation up to homotopy of A to a (split) $\mathcal{VB}$-algebroid as that of building a transformation Lie algebroid.

The discussion above above makes sense for Lie groupoids as well. Namely, a representation up to homotopy of a Lie groupoid $\mathcal{G}$ can be defined similarly to a degree 1 differential operator on a complex of the form $C^\bullet(\mathcal{G}, \mathcal{E})$. Then in the category of Lie groupoids, the obvious object to play the role of a transformation object is the corresponding $\mathcal{VB}$-groupoid structure, defined on $\mathbf{t}^*C \oplus_{\mathcal{G}} \mathbf{s}^*E$, which we denote by $\mathcal{G} \ltimes \mathcal{E}$. What plays the role of a splitting for a 2-term representation up to homotopy of a Lie groupoid is called *right-horizontal lift* in [18, sec. 3].

Besides their role as transformation objects, $\mathcal{VB}$-algebroids and $\mathcal{VB}$-groupoids bring a deep understanding on the representation theory of Lie algebroids and groupoids. In particular, while the adjoint and coadjoint representations up to homotopy of a Lie algebroid (resp., groupoid) depend on the choice of a connection, the $\mathcal{VB}$-algebroids (resp., $\mathcal{VB}$-groupoids) corresponding to these representations up to homotopy are the tangent and cotangent $\mathcal{VB}$-algebroids (resp., $\mathcal{VB}$-groupoids), which are entirely canonical. This relation between $\mathcal{VB}$-algebroids and representations up to homotopy has been used to study the infinitesimal picture of Lie groupoids equipped with multiplicative structures, e.g., multiplicative foliations [16].

The integration problem. Given the above discussion, one may expect to be able to reproduce the integration procedure of usual representations, so as to pass from a 2-term representation up to homotopy of a Lie algebroid to a 2-term representation up to homotopy of a Lie groupoid, in such a way that this operation would commute with that of taking the corresponding transformation objects, as follows:



Unfortunately, unlike for usual representations, there are some problems occurring:

(1) Even if A is integrable, the $\mathcal{VB}$-algebroid $A \ltimes \mathcal{E}$ may not be integrable.
(2) Even if $A \ltimes \mathcal{E}$ is integrable, a 2-term representation up to homotopy of A on $\mathcal{E}$ does not determine a 2-term representation up to homotopy of $\mathcal{G}(A)$ on $\mathcal{E}$.

The first item was completely addressed in [10], showing explicit obstructions to the integrability of $\mathcal{VB}$-algebroids, and hence providing integrability criteria for the integration of 2-term representations up to homotopy of an integrable Lie algebroid.

The second item deserves more attention. First, note that any 2-term representation up to homotopy of $\mathcal{G}(A)$ differentiates to a 2-term representation up to homotopy of A (see [3]). In this case, one can see that $\mathcal{G}(A) \ltimes \mathcal{E}$ integrates $A \ltimes \mathcal{E}$, both as a groupoid and as a $\mathcal{VB}$-groupoid [10]. So there is no serious issue when performing the differentiation process.

Second, recall from [11] that whenever $D = A \ltimes \mathcal{E}$ is integrable (as a mere Lie algebroid), the integrating groupoid $\mathcal{G}(A \ltimes \mathcal{E})$ comes naturally with a $\mathcal{VB}$-groupoid structure over $\mathcal{G}(A)$. In other words, under the integrability assumption, the $\mathcal{VB}$ nature carries through the integration process.

However, given a $\mathcal{VB}$-groupoid $\mathcal{G}(A \ltimes \mathcal{E})$ over $\mathcal{G}(A)$, the corresponding 2-term representation up to homotopy of $\mathcal{G}(A)$ is obtained by choosing a right-horizontal lift. Hence, given a 2-term representation up to homotopy of A, after we have integrated $A \ltimes \mathcal{E}$ to $\mathcal{G}(A \ltimes \mathcal{E})$, we still have to choose a right-horizontal lift of $\mathcal{G}(A \ltimes \mathcal{E})$ (see [18, sec. 3]) in order to obtain a 2-term representation up to homotopy of $\mathcal{G}(A)$.

We claim that such a right-horizontal lift is not determined by the infinitesimal data. Namely, the splitting of the $\mathcal{VB}$-algebroid $A \ltimes \mathcal{E} = A \oplus E \oplus C$ does not determine a right-horizontal lift of the corresponding $\mathcal{VB}$-groupoid $\mathcal{G}(A \ltimes \mathcal{E})$. It follows that one cannot obtain a 2-term representation up to homotopy of $\mathcal{G}(A)$ out of a 2-term representation up to homotopy of A without involving further choices. From this point of view, 2-term representations up to homotopy of a Lie algebroid A do not integrate to 2-term representations up to homotopy of $\mathcal{G}(A)$, at least not in a canonical way.

In [4], Arias Abad and Schätz proposed an integration scheme for Lie algebroid representations up to homotopy of any degree, the output of which, when applied to the 2-term case, is indeed not a 2-term representation up to homotopy of a Lie groupoid, but instead a representation up to homotopy of an ∞-groupoid. By contrast, our procedure provides a 2-representation of a strict 2-groupoid. In Arias Abad and Schätz's work [4], the notion of transformation object is not discussed, and it is not clear how one can recover a representation up to homotopy of a Lie groupoid, or a $\mathcal{VB}$-groupoid, with this procedure.

The aim of this paper is to provide an alternative integration scheme for 2-term representations up to homotopy of a Lie algebroid, which, being more intuitive, makes straightforward what the corresponding transformation object should be. We also derive from this construction the $\mathcal{VB}$-groupoid integrating a $\mathcal{VB}$-algebroid in a natural way.

Outline of the paper. The integration of usual representations can be described using functors as follows. Let $\mathcal{G}(A)$ denote the Weinstein groupoid of a Lie algebroid $A \to M$. A representation of A is a Lie algebroid morphism $\rho : A \longrightarrow \mathrm{D}(E)$, so it integrates to a morphism of groupoids

$$\mathrm{hol} : \mathcal{G}(A) \longrightarrow \mathrm{Gau}(E),$$

which is given by the holonomy of the flat A-connection ∇ induced ρ. In this way, we obtain a representation of $\mathcal{G}(A)$ on E, and the above holonomy functor can be easily used in order to describe an integration of the transformation Lie algebroid $A \ltimes E$.

In this paper, we explain how a 2-term representation up to homotopy of a Lie algebroid can be integrated with a strict 2-functor between 2-groupoids (see Theorem 1). We use such a 2-functor to construct the corresponding transformation 2-groupoid, and out of this object, we derive a natural description of the Weinstein groupoid of the $\mathcal{VB}$-algebroid associated with a 2-term representation up to homotopy of a Lie algebroid (see Theorem 2). More applications are further discussed as well.

The paper is organized as follows. In section 2, we briefly recall the notion of representation up to homotopy and its relation to both $\mathcal{VB}$-algebroids and $\mathcal{VB}$-groupoids. Section 3 is devoted to the study of $\mathcal{VB}$-algebroids in connection with Lie algebroid extensions and fibrations, notions on which some crucial points for our main construction rely. We show in Proposition 3.14 that any $\mathcal{VB}$-algebroid D gives rise to a Lie algebroid fibration. As a consequence, we prove Theorem 3.15, which describes the homotopy long exact sequence of a $\mathcal{VB}$-algebroid and states that the monodromy groups of D fit into an exact sequence:

$$\mathrm{Im}(\delta_2, e) \hookrightarrow \widetilde{\mathcal{N}}(D, e) \overset{p}{\twoheadrightarrow} \widetilde{\mathcal{N}}(A, m),$$

where δ_2 is a transgression map associated with the underlying Lie algebroid fibration $D \to A$.

In section 4, we are concerned with 2-groupoids and their representations. We introduce two natural examples of 2-groupoids, the so-called Weinstein 2-groupoid 2-$\mathcal{P}(A)$ of a Lie algebroid A and the *gauge 2-groupoid* 2-$\mathrm{Gau}(\mathcal{E})$ of a 2-term complex of vector bundles $\mathcal{E} = (C \to E)$ over a fixed manifold. More precisely, we have as follows:

- In 2-$\mathcal{P}(A)$, objects are points in M, 1-morphisms are thin homotopy classes of A-paths, and 2-morphisms are homotopy classes of A-homotopies between A-paths.
- Objects in 2-Gau$(\mathcal{E})$ are points in M, 1-morphisms $x \to y$ are given by *invertible* chain maps $(C_x, E_x) \to (C_y, E_y)$, and 2-morphisms are chain homotopies.

The main property of 2-$\mathcal{P}(A)$ is presented in Proposition 4.14, which says that the 1-truncation of the Weinstein 2-groupoid, i.e., the quotient space of 1-morphisms by 2-morphisms, coincides with the Weinstein groupoid of A. Then we introduce the notion of 2-representation, or representation of a 2-groupoid 2-$\mathcal{G}$, as a strict 2-functor 2-$\mathcal{G} \to$ 2-Gau$(\mathcal{E})$. We state the first main result of this paper, which says that a 2-term representation up to homotopy of a Lie algebroid A can be integrated to a representation of the Weinstein 2-groupoid of A. More precisely, we prove the following result.

Theorem 1. *Any 2-term representation up to homotopy of A on $\mathcal{E}$ integrates to a strict 2-functor*

$$2\text{-}\mathcal{P}(A) \xrightarrow{\text{hol}} 2\text{-Gau}(\mathcal{E}).$$

The strict 2-functor of Theorem 1 is referred to as the *holonomy 2-representation.* Essentially, this result allows one to bypass the dotted top arrow in the previous diagram, leading to a 2-representation of the 2-groupoid 2-$\mathcal{P}(A)$ rather than a representation up to homotopy of $\mathcal{G}(A)$. From there on, when following the right side of the diagram, it is natural to consider a transformation object 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ which is a 2-groupoid as well; this is the purpose of subsection 4.5.

In the subsequent section 5, we explain how the transformation 2-groupoid 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ can be used in order to recover the $\mathcal{VB}$-groupoid integrating the $\mathcal{VB}$-algebroid $D = A \ltimes \mathcal{E}$ underlying a 2-term representation up to homotopy of a Lie algebroid A. This is the statement of the second main result of this work.

Theorem 2. *Given a representation up to homotopy of a Lie algebroid A on a 2-term complex $\mathcal{E}$, the Weinstein groupoid of the associated $\mathcal{VB}$-algebroid $D = A \ltimes \mathcal{E}$ identifies with the 1-truncation groupoid of the transformation 2-groupoid 2-$\mathcal{P}(A) \ltimes \mathcal{E}$.*

This point of view has both abstract and practical advantages. On the one hand, it allows us to interpret the integrability of a $\mathcal{VB}$-algebroid as the vanishing of the second transgression map in the homotopy long exact sequence. On the other hand, as we shall illustrate with various examples, the procedure can be implemented in order to obtain the $\mathcal{VB}$-groupoid of a $\mathcal{VB}$-algebroid in a quite explicit way.

As a final note, the construction of the holonomy as a strict 2-functor was inspired by the construction of Schreiber and Waldorf [28], although their results would be hard to apply when the boundary map $\partial : C \to E$ does not have constant rank. Also, the 2-functoriality of the holonomy could probably be deduced from either [28] or [4] by working leafwise, which we avoid by using a direct proof. This approach allows us to compute explicitly the holonomy, which could be hard to deduce from [28] or [4].

Our approach is original, as it is based on Lie algebroid extensions and fibrations, which we believe to be more natural for differential geometers. Furthermore, the construction of a strict transformation 2-groupoid is original in this context. In fact, similar constructions are studied in [25–27]; however, as we explain, they lead to different types of integrations that seem difficult to relate to $\mathcal{VB}$-groupoids at first sight.

2. Generalities

2.1. 2-term representations up to homotopy. The notion of representation up to homotopy of a Lie algebroid was introduced in [1]. In our case of interest—namely, when the underlying graded vector bundle is concentrated in degrees -1 and 0—the definition boils down to the following.

We consider a Lie algebroid A over M and $\mathcal{E} = (\partial : C \to E)$, a 2-term complex of vector bundles over M, concentrated in degrees -1 and 0.

Definition 2.1. A 2-*term representation up to homotopy* of A on $\mathcal{E}$ is a triple $(\nabla^E, \nabla^C, \omega)$ where ∇^E, ∇^C are A-connections on E and C, respectively, and $\omega \in \Omega^2(A, \mathrm{Hom}(E, C))$ such that the following compatibility conditions are satisfied:

$$\partial \circ \nabla^C = \nabla^E \circ \partial, \tag{2.1}$$

$$\partial \circ \omega = \omega_E, \tag{2.2}$$

$$\omega \circ \partial = \omega_C, \tag{2.3}$$

$$\nabla\omega = 0. \tag{2.4}$$

Here ω_E and ω_C denote the respective curvatures of ∇^E and ∇^C, while $\nabla\omega \in \Omega^3(A, \mathrm{Hom}(E, C))$ denotes the covariant derivative naturally induced on $\Omega^\bullet(A, \mathrm{Hom}(E, C))$ by ∇^E and ∇^C.

Remark 2.2. In the original definition [1], the boundary operator $\partial : C \to E$ is part of the data that defines a representation up to homotopy. In this work, it is relevant to set the boundary operator apart and think of A acting on a 2-term complex via $(\nabla^E, \nabla^C, \omega)$.

There also exists a notion of representation up to homotopy of Lie groupoids [2, 18]. In the case in which the underlying graded vector bundle is concentrated in degrees -1 and 0, it can be described as follows.

Recall that any vector bundle $E \to M$ determines a smooth category $L(E)$ whose objects are elements of M, and morphisms between $x, y \in M$ are linear maps $E_x \to E_y$ (not necessarily invertible). Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, a *unital quasi action* of $\mathcal{G}$ on E is a smooth map $\Delta : \mathcal{G} \to L(E)$ that commutes with the unit, source, and target maps. Note that Δ does not necessarily commute with the multiplication. When this is the case, the unital quasi action is said to be *flat.* Also, $\Delta_g : E_{\mathbf{s}(g)} \to E_{\mathbf{t}}(g)$ is not required to be invertible. A *representation* of $\mathcal{G}$ on E is just a flat unital quasi action $\Delta : \mathcal{G} \to L(E)$ such that $\Delta_g : E_{\mathbf{s}(g)} \to E_{\mathbf{t}(g)}$ is invertible for each $g \in \mathcal{G}$.

Let us now fix a Lie groupoid $\mathcal{G} \rightrightarrows M$ and $\mathcal{E} = (\partial : C \to E)$, a 2-term complex of vector bundles over M, concentrated in degrees -1 and 0. We denote by $\mathcal{G}^{(2)} = \{(g, h) : \mathbf{s}(g) = \mathbf{t}(h)\}$ the space of composable arrows of $\mathcal{G}$.

Definition 2.3. A 2-term representation up to homotopy of $\mathcal{G}$ on $\mathcal{E} = (\partial : C \to E)$ is given by a triple $(\Delta^C, \Delta^E, \Omega)$, where Δ^C and Δ^E are unital quasi actions on C

and E, respectively, and $\Omega \in C^{\infty}(\mathcal{G}^{(2)}, \mathrm{Hom}(E,C))$ is a normalized cochain (i.e., $\Omega_{g_1,g_2} = 0$ if either g_1 or g_2 is a unit), satisfying the following conditions,

$$\begin{aligned} \partial \circ \Delta^C_{g_1} &= \Delta^E_{g_1} \circ \partial, \\ \partial \circ \Omega_{g_1,g_2} &= \Delta^E_{g_1 g_2} - \Delta^E_{g_2} \circ \Delta^E_{g_1}, \\ \Omega_{g_1,g_2} \circ \partial &= \Delta^C_{g_1 g_2} - \Delta^C_{g_2} \circ \Delta^C_{g_1}, \\ \Omega_{g_1 g_2, g_3} - \Omega_{g_1, g_2 g_3} &= \Delta^C_{g_1} \circ \Omega_{g_2,g_3} - \Omega_{g_1,g_2} \circ \Delta^E_{g_3}, \end{aligned}$$

for every triple $g_1, g_2, g_3 \in \mathcal{G}$ where the above multiplications make sense.

2.2. $\mathcal{VB}$**-groupoids.** Closely related to 2-term representations up to homotopy of Lie algebroids and groupoids are the notions of a $\mathcal{VB}$-algebroid and a $\mathcal{VB}$-groupoid, which we briefly recall. Detailed expositions can be found in [17, 18, 21].

Definition 2.4. A *$\mathcal{VB}$-groupoid* is a square

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{q_H} & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ E & \xrightarrow{q_E} & M, \end{array} \tag{2.5}$$

where double arrows denote Lie groupoid structures and single arrows denote vector bundles. It is required that the structure mappings (source, target, multiplication, unit section, and inversion) that define the Lie groupoid $\mathcal{H} \rightrightarrows E$ be morphisms of vector bundles over the corresponding structure mappings defining the Lie groupoid $\mathcal{G} \rightrightarrows M$.

A $\mathcal{VB}$-groupoid as in (2.5) will be denoted by $(\mathcal{H}; \mathcal{G}, E; M)$.

Example 2.5 (Tangent groupoid)**.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. The tangent bundle $T\mathcal{G}$ has a canonical Lie groupoid structure over TM. The structural maps of $T\mathcal{G} \rightrightarrows TM$ are defined by applying the tangent functor to each of the structural maps of $\mathcal{G} \rightrightarrows M$. It can be easily checked that with respect to these maps, the quadruple $(T\mathcal{G}; \mathcal{G}, TM; M)$ is a $\mathcal{VB}$-groupoid, referred to as the *tangent groupoid* of $\mathcal{G}$.

Example 2.6 (Cotangent groupoid)**.** Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ with Lie algebroid A, the cotangent bundle $T^*\mathcal{G}$ is equipped with a groupoid structure over A^*. An explicit description of this groupoid can be found also in [12]. The source and target maps are defined by

$$\tilde{s}(\alpha_g)u = \alpha_g(Tl_g(u - Tt(u)))$$

and

$$\tilde{t}(\alpha_g)v = \alpha_g(Tr_g(v)),$$

where $\alpha_g \in T^*_g\mathcal{G},\ u \in A_{s(g)}\mathcal{G}$, and $v \in A_{t(g)}\mathcal{G}$. The multiplication on $T^*\mathcal{G}$ is defined by

$$(\alpha_g \circ \beta_h)(X_g \bullet Y_h) = \alpha_g(X_g) + \beta_h(Y_h)$$

for $(X_g, Y_h) \in T_{(g,h)}\mathcal{G}_{(2)}$. We refer to $T^*\mathcal{G}$ with the groupoid structure over A^* as the *cotangent groupoid* of $\mathcal{G}$.

2.2.1. *$\mathcal{VB}$-groupoids and 2-term representations up to homotopy of Lie groupoids.* Given a 2-term representation up to homotopy $(\Delta^E, \Delta^C, \Omega)$ of a Lie groupoid $\mathbf{s}, \mathbf{t} : \mathcal{G} \rightrightarrows E$ on $\mathcal{E} = (\partial : C \to E)$, one can construct a $\mathcal{VB}$-groupoid $(\mathcal{G} \ltimes E; \mathcal{G}, E; M)$ in the following way [18]. Consider the direct sum of the pull-back bundles $\mathcal{G} \ltimes E := \mathbf{t}^*C \oplus \mathbf{s}^*E$, which is a vector bundle over $\mathcal{G}$. Then $\mathcal{G} \ltimes E$ comes equipped with a groupoid structure over E whose source, target, and identity maps are given by

$$\tilde{s}(c, g, e) = e, \quad \tilde{t}(c, g, e) = \partial(c) + \Delta^E_g(e), \quad \tilde{1}(e) = 0 \oplus e,$$

and with the following groupoid multiplication:

$$(c_1, g_1, e_1) \cdot (c_2, g_2, e_2) = \left(c_1 + \Delta^C_{g_1}(c_2) - \Omega_{g_1,g_2}(e_2), g_1 \cdot g_2, e_2\right).$$

One may think of $\mathcal{G} \ltimes E \rightrightarrows E$ as a transformation groupoid associated with the 2-term representation up to homotopy $(\Delta^E, \Delta^C, \Omega)$.

Conversely, given a $\mathcal{VB}$-groupoid $(\mathcal{H}, \mathcal{G}, E, M)$, the *core* bundle is the vector bundle $C := \mathrm{Ker}(s_{\mathcal{H}})|_M$. The *core sequence* of $\mathcal{H}$ is the exact sequence of vector bundles over $\mathcal{G}$

$$0 \to \mathbf{t}^*C \to \mathcal{H} \to \mathbf{s}^*E \to 0,$$

where the first map is $(g, c_{\mathbf{t}(g)}) \mapsto c_{\mathbf{t}(g)} 0^{\mathcal{H}}_g$, and the second map is the bundle map induced by the source $\mathbf{s}_{\mathcal{H}} : \mathcal{H} \to E$. A right-horizontal lift of $\mathcal{H}$ is defined in [18, Def. 3.8] as a right splitting of the core sequence, which coincides with the canonical splitting $\mathcal{H}_x = C_x \oplus E_x$ along any unit $x \in \mathcal{G}$. This allows one to identify $\mathcal{H}$ with a $\mathcal{VB}$-groupoid of the form $\mathbf{t}^*C \oplus \mathbf{s}^*E$ as above for some 2-term representation up to homotopy $(\Delta^E, \Delta^C, \Omega)$. Right-horizontal lifts always exist, and we refer to [18] for more details.

2.3. **$\mathcal{VB}$-algebroids.** Let us recall some definitions regarding double vector bundles and their special sections. We refer to [20, 21, 24] for a more detailed treatment.

Definition 2.7. A *double vector bundle* (DVB) is a commutative square

$$\begin{array}{ccc} D & \xrightarrow{p} & A \\ {\scriptstyle p_D}\downarrow & & \downarrow{\scriptstyle p_A} \\ E & \xrightarrow{p_E} & M, \end{array} \tag{2.6}$$

where all four sides are vector bundles, where p_D is a vector bundle morphism over p_A, and where $+_E : D \times_E D \longrightarrow D$ is a vector bundle morphism over $+ : A \times_M A \longrightarrow A$. Here $+_E$ is the addition map for the vector bundle $D \longrightarrow E$.

In other words, the quadruple $(D; A, E; M)$ is a $\mathcal{VB}$-groupoid where $D \longrightarrow E$ and $A \longrightarrow M$ are equipped with the Lie groupoid structure induced by the corresponding vector bundle structures.

Given a DVB $(D; A, E; M)$, the vector bundles A and E are called the *side bundles*. The zero sections are denoted by $0^A : M \longrightarrow A$, $0^E : M \longrightarrow E$, $0^D_A : A \longrightarrow D$ and $0^D_E : E \longrightarrow D$. Elements of D are written $(d; a, e; m)$, where $d \in D$, $m \in M$, $a = p(d) \in A_m$, and $e = p_D(d) \in E_m$.

The core C of a DVB is the intersection of the kernels of p and p_D. It has a natural vector bundle structure over M, the projection of which we call $p_C : C \longrightarrow M$. The inclusion $C \hookrightarrow D$ is usually denoted by

$$C_m \ni c \longmapsto \overline{c} \in p^{-1}(0^A_m) \cap p_D^{-1}(0^E_m).$$

Given a DVB $(D; A, E; M)$, the space of sections $\Gamma(E, D)$ is generated as a $C^\infty(E)$-module by two distinguished classes of sections; namely, linear sections and core sections, which we now describe.

Definition 2.8. For a section $c : M \to C$, the corresponding *core section* $\hat{c} : E \to D$ is defined as

$$\hat{c}(e_m) = 0^D_E(e_m) +_A \overline{c(m)}, \quad m \in M, \quad e_m \in E_m. \tag{2.7}$$

We denote the space of core sections by $\Gamma_c(E, D)$.

Definition 2.9. A section $\chi \in \Gamma(E, D)$ is called *linear* if $\chi : E \to D$ is a vector bundle morphism covering a section $a : M \longrightarrow A$. The space of linear sections is denoted by $\Gamma_\ell(E, D)$.

Definition 2.10. A $\mathcal{VB}$*-algebroid* is a DVB $(D; A, E; M)$ as follows:

$$\begin{array}{ccc} D & \xrightarrow{p} & A \\ {\scriptstyle p_D}\downarrow & & \downarrow{\scriptstyle p_A} \\ E & \xrightarrow{p_E} & M, \end{array} \tag{2.8}$$

where $D \longrightarrow E$ and $A \longrightarrow M$ are equipped with Lie algebroid structures such that the anchor map $\rho_D : D \longrightarrow TE$ is a bundle morphism over $\rho_A : A \longrightarrow TM$ and the following bracket conditions are satisfied:

(i) $[\Gamma_\ell(E, D), \Gamma_\ell(E, D)]_D \subset \Gamma_\ell(E, D)$,
(ii) $[\Gamma_\ell(E, D), \Gamma_c(E, D)]_D \subset \Gamma_c(E, D)$,
(iii) $[\Gamma_c(E, D), \Gamma_c(E, D)]_D = 0$.

Example 2.11. Given a Lie algebroid $A \longrightarrow M$, the application of the tangent functor to each of the structural maps defining the vector bundle $A \longrightarrow M$ determines a DVB $(TA; A, TM; M)$. Moreover, the Lie algebroid structure of $p_A : A \longrightarrow M$ can be lifted to $Tp_A : TA \longrightarrow TM$, making the quadruple $(TA; A, TM; M)$ into a $\mathcal{VB}$-algebroid called the *tangent algebroid* of A. For more details, see [21].

Example 2.12. Given a Lie algebroid $A \to M$, the cotangent bundle T^*A inherits a Lie algebroid structure over A^*, yielding a $\mathcal{VB}$-algebroid $(T^*A; A; A^*, M)$ referred to as the *cotangent algebroid* of A. For more details, see [21] and the references therein.

2.3.1. *$\mathcal{VB}$-algebroids and 2-term representations up to homotopy of Lie algebroids.* This section follows closely [17]. Let $(D; A, E; M)$ be a $\mathcal{VB}$-algebroid. The space of linear sections $\Gamma_\ell(E, D)$ is locally free as a $C^\infty(M)$-module. Hence, there exists a vector bundle $\hat{A} \to M$ whose space of sections $\Gamma(\hat{A})$ identifies with $\Gamma_\ell(E, D)$ as a $C^\infty(M)$-module. Also, every linear section $\chi : E \to D$ covers a section $a : M \to A$, inducing an exact sequence of vector bundles over M,

$$\mathrm{Hom}(E, C) \hookrightarrow \hat{A} \twoheadrightarrow A. \tag{2.9}$$

One observes that $\hat{A} \to M$ is a Lie algebroid with bracket $[\chi_1, \chi_2]_{\hat{A}} = [\chi_1, \chi_2]_D$ and anchor map $\rho_{\hat{A}}(\chi) = \rho_A(a)$ for every $\chi \in \Gamma(\hat{A})$ linear section covering $a \in \Gamma(A)$. Also, the vector bundle $\mathrm{Hom}(E, C) \to M$ is equipped with a Lie algebroid structure with zero anchor map and bracket $[\phi, \psi] = \phi \circ \partial \circ \psi - \psi \circ \partial \circ \phi$, where

$\partial : C \to E; c \mapsto \rho_D(c)$ is the *core anchor* of D. Thus, the sequence (2.9) is an extension of Lie algebroids over M.

Example 2.13. Given a Lie algebroid $A \to M$, one can consider the tangent $\mathcal{VB}$-algebroid $(TA; A, TM; M)$. In this case, the exact sequence (2.9) reads

$$\mathrm{Hom}(TM, A) \hookrightarrow \mathfrak{J}(A) \twoheadrightarrow A,$$

where $\mathfrak{J}(A) \to M$ is the first jet algebroid associated with A.

There exists a natural representation of $\hat{A}$ on C:

$$\Phi_\chi(c) := [\chi, \hat{c}]_D. \tag{2.10}$$

Similarly, there is a representation of $\hat{A}$ on E, defined as follows. Since $\rho_D : D \to TE$ is a DVB-morphism, for each linear section $\chi \in \Gamma_l(E, D)$, the vector field $\rho_D(\chi) \in \mathfrak{X}(E)$ is linear, and hence $\mathcal{L}_{\rho_D(\chi)}(\xi) \in \Gamma(E^*)$ for every $\xi \in \Gamma(E^*) \simeq C^\infty_{lin}(E)$, viewed as a fiberwise linear function on E. Thus, we obtain a representation of $\hat{A}$ on E by the formula

$$\langle \xi, \Psi_\chi(e) \rangle = \rho_{\hat{A}}(\chi)\langle \xi, e \rangle - \langle \mathcal{L}_{\rho_D(\chi)}\xi, e \rangle \tag{2.11}$$

for every $\chi \in \Gamma_l(E, D)$, $\xi \in \Gamma(E^*)$, and $\Gamma(e)$.

Now, following [17], we proceed to briefly explain the relation between $\mathcal{VB}$-algebroids over A and 2-term representations up to homotopy of A. Let $(D; A, E; M)$ be a $\mathcal{VB}$-algebroid. This determines a 2-term complex $\partial : C \to E$ defined by the core anchor map. A splitting $h : A \to \hat{A}$ of the sequence (2.9) is referred to as a *horizontal lift*. Notice that horizontal lifts are not required to be morphisms of Lie algebroids. The failure for h being a Lie algebroid morphism is controlled by the element $\omega \in \Gamma(\bigwedge^2 A^* \otimes \mathrm{Hom}(E, C))$ defined by

$$\omega(a, b) := h([a, b]_A) - [h(a), h(b)]_D. \tag{2.12}$$

The choice of a horizontal lift $h : A \to \hat{A}$ defines A-connections on E and C by pulling back via $h : A \to \hat{A}$ the representations (2.10) and (2.11), respectively. That is, the A-connection ∇^C on C is given by

$$\nabla^C_a c := [h(a), \hat{c}]_D, \tag{2.13}$$

and, similarly, the A-connection on E is defined by

$$\nabla^E_a e := \Psi_{h(a)} e, \tag{2.14}$$

where Ψ is as in (2.11). These A-connections are not flat unless $\omega = 0$.

As explained in [17], the A-connections ∇^E, ∇^C, together with the element $\omega \in \Gamma(\bigwedge^2 A^* \otimes \mathrm{Hom}(E, C))$ given by (2.12), define a representation up to homotopy of the Lie algebroid A on the 2-term complex $\partial : C \to E$. See Definition 2.1. Conversely, a representation up to homotopy $(\nabla^E, \nabla^C, \omega)$ of a Lie algebroid A on a 2-term complex $\partial : C \to E$ induces a Lie algebroid structure on the split DVB $D = A \times_M \times C \times_M E$. This is summarized in the following result.

Theorem 2.14 ([17]). *Let $A \to M$ be a Lie algebroid, and let E, C be vector bundles over M. The formulas* (2.12), (2.13), *and* (2.14) *define a one-to-one correspondence between split $\mathcal{VB}$-algebroid structures $D = A \times_M C \times_M E$ with core anchor $\partial : C \to E$ and representations up to homotopy $(\nabla^E, \nabla^C, \omega)$ of A on the complex $\partial : C \to E$ as above.*

The notion of morphism between representations up to homotopy can be found in [17]. As we have already observed, given a $\mathcal{VB}$-algebroid $(D; A, E; M)$ with core bundle C, the choice of a horizontal lift $h : A \to \hat{A}$ determines a 2-term representation up to homotopy on $C \oplus E$. Moreover, isomorphism classes of $\mathcal{VB}$-algebroid structures on the DVB $(D; A, E; M)$ with core C, are in one-to-one correspondence with isomorphism classes of representation up to homotopy of A on $C \oplus E$ (see, e.g., [17]). Indeed, this correspondence is given by an equivalence of categories [16].

2.4. **$\mathcal{VB}$-groupoids vs. $\mathcal{VB}$-algebroids.** As explained in [21], given a $\mathcal{VB}$-groupoid $(\mathcal{H}; \mathcal{G}, E; M)$, the application of the Lie functor to each of the groupoids $\mathcal{H} \rightrightarrows E$ and $\mathcal{G} \rightrightarrows M$ yields a $\mathcal{VB}$-algebroid $(A\mathcal{H}; A\mathcal{G}, E; M)$. In this case, we say that the $\mathcal{VB}$-groupoid $(\mathcal{H}; \mathcal{G}, E; M)$ integrates the $\mathcal{VB}$-algebroid $(A\mathcal{H}; A\mathcal{G}, E; M)$.

Definition 2.15. A $\mathcal{VB}$-algebroid $(D; A, E; M)$ is called *integrable* if there exists a $\mathcal{VB}$-groupoid $(\mathcal{H}; \mathcal{G}, E; M)$ such that $(A\mathcal{H}; A\mathcal{G}, E; M)$ is isomorphic to $(D; A, E; M)$.

The following result can be found in [11].

Theorem 2.16 ([11])**.** *Let $(D; A, E; M)$ be a $\mathcal{VB}$-algebroid. Assume that $D \to E$ integrates to a source simply connected Lie groupoid $\mathcal{G}(D) \rightrightarrows E$. Then the Lie algebroid $A \to M$ is integrable, and $(\mathcal{G}(D); \mathcal{G}(A), E; M)$ is a $\mathcal{VB}$-groupoid integrating $(D; A, E; M)$, where $\mathcal{G}(A)$ is the source simply connected integration of A.*

Theorem 2.16 says essentially that whenever D is integrable, the $\mathcal{VB}$-groupoid structure on $\mathcal{G}(D)$ comes for free. Note, however, that it says nothing about how to decide whether a given $\mathcal{VB}$-algebroid algebroid is integrable or not. On that matter, the following was proved in [10].

Theorem 2.17 ([10])**.** *Let (D, A, E, M) be a $\mathcal{VB}$-algebroid. Then D is integrable if and only if A is integrable and $\mathcal{N}(D) \cap \ker p = \{0^D_E\}$. Here $\mathcal{N}(D) \subset D$ denotes the monodromy groups of D, $p : D \to A$ the $\mathcal{VB}$-algebroid projection, and 0^D_E the zero section of $D \to E$.*

Note that both results are concerned only about $\mathcal{VB}$-algebroids and their integration to $\mathcal{VB}$-groupoids. If one is rather interested in 2-term representations up to homotopy, one may wonder if it is possible to integrate a 2-term representation up to homotopy of a Lie algebroid to a 2-term representation up to homotopy of a Lie groupoid in a canonical way (meaning, freely of choices). Surprisingly, the answer is "no" (see Remark 5.10), which was one of the motivations for this work. We will come back to this point below.

3. $\mathcal{VB}$-algebroids as Lie algebroid fibrations

Similarly, as vector bundles are special types of fiber bundles, $\mathcal{VB}$-algebroids form a special class of Lie algebroid fibrations. As such, they admit a homotopy long exact sequence (Theorem 3.6). They enjoy notable features, though, essentially due to the particular structure of their kernel (Proposition 3.12). This allows not only for an explicit integration of their kernel (3.13) but also for a nice description of their monodromy groups, which is quite remarkable amongst fibrations. The aim of this section is to study $\mathcal{VB}$-algebroids from the point of view of fibrations. Besides setting some useful notations, this will lay the path to some technical results from [8–10] crucial for our main construction.

3.1. **Generalities on Lie algebroid fibrations.** We first recall the notions of extension and fibration for Lie algebroids [8, 9].

Definition 3.1. A Lie algebroid *extension* is a surjective Lie algebroid morphism $p : A_E \twoheadrightarrow A$ covering a surjective submersion $p_E : E \to M$.

The *kernel* of an extension $p : A_E \twoheadrightarrow A$, is defined by $\mathcal{K} := \ker p$. One can show [8] that $\mathcal{K}$ is a Lie subalgebroid of A_E whose orbits are included in the fibers of the base submersion $p_E : E \to M$. Let us stress the fact that $\mathcal{K}$ coincides with the kernel of p as a vector bundle map; in particular, $\mathcal{K}$ is a Lie algebroid over E. Therefore, taking into account base manifolds, we obtain a short exact sequence of Lie algebroids

$$
\begin{array}{ccccc}
\mathcal{K} & \xhookrightarrow{\;i\;} & A_E & \xrightarrow{\;p\;} & A \\
\downarrow & & \downarrow & & \downarrow \\
E & \xrightarrow{\mathrm{id}_E} & E & \xrightarrow{p_E} & M.
\end{array}
\tag{3.1}
$$

Definition 3.2. Given a Lie algebroid extension, an *Ehresmann connection* is given by a vector subbundle $H \to E$ complementary to $\mathcal{K}$ in A_E, that is, $\mathcal{K} \oplus H = A_E$.

As is easily seen, a connection is characterized by a $C^\infty(M)$-linear map $\mathrm{hor} : \Gamma(A) \to \Gamma(H)$ referred to as the *horizontal lifting*. This determines a $C^\infty(M)$-linear map $\mathcal{D} : \Gamma(A) \to \mathrm{Der}(\mathcal{K})$, called the *covariant derivative* and defined by $\mathcal{D}_a := [\mathrm{hor}(a), \cdot\,]_{\mathcal{K}}$. Then the *curvature* of the connection is the element $\omega_{\mathcal{K}} \in \Omega^2(A, \Gamma(\mathcal{K}))$, defined by $\omega_{\mathcal{K}}(a, b) = \mathrm{hor}([a, b]_A) - [\mathrm{hor}(a), \mathrm{hor}(b)]_{A_E}$.

The couple $(\mathcal{D}, \omega_{\mathcal{K}})$ entirely determines the extension and, as a consequence of the Jacobi identity on A_E, satisfies the following conditions:

$$
\mathcal{D}_{[a,b]_A} - [\mathcal{D}_a, \mathcal{D}_b] = [\omega_{\mathcal{K}}(a, b),\ ]_{\mathcal{K}}, \tag{3.2}
$$

$$
\oint_{a,b,c} \mathcal{D}_a\, \omega_{\mathcal{K}}(b, c) + \omega_{\mathcal{K}}([a, b]_A, c) = 0. \tag{3.3}
$$

Definition 3.3. An Ehresmann connection is *complete* if, for any section $a \in \Gamma(A)$ such that $\rho_A(a)$ is a complete vector field, then $\rho_{A_E}(\mathrm{hor}(a))$ is a complete vector field as well.

For a complete connection, the notion of parallel transport along A-paths is well defined [8].

Definition 3.4. Let $A \to M$ be a Lie algebroid. An *A-path* is a Lie algebroid morphism $adt : TI \to A$, where $a : I \to A$ and $I = [0, 1]$.

The holonomy along an A-path $a : I \to A$ between $m_0, m_1 \in M$ can be constructed as follows. One can always choose a compactly supported time-dependent section α_t of A extending a, i.e., such that $\alpha_t(\gamma(t)) = a(t)$, where $\gamma : I \to M$ denotes the base path covered by a. Provided that the connection is complete, the corresponding time-dependent covariant derivative $\mathcal{D}_{\alpha_t} \in \mathrm{Der}(\mathcal{K})$ has a well-defined flow $\Phi^{\mathcal{D}_\alpha}_{1,0} : \mathcal{K} \to \mathcal{K}$. The holonomy $\mathrm{hol}^{\mathcal{K}}_a : \mathcal{K}|_{E_{m_0}} \to \mathcal{K}|_{E_{m_1}}$ along a is defined as the restriction of $\Phi^{\mathcal{D}_\alpha}_{1,0}$ to $\mathcal{K}|_{E_{m_0}}$, where $E_{m_i} := p_E^{-1}(m_i)$ denotes the fiber over m_i. It is independent of the choice of α_t extending a as a consequence of the horizontal lifting map $\mathrm{hor} : \Gamma(A) \to \mathrm{Der}(\mathcal{K})$ being $C^\infty(M)$-linear. Furthermore, $\mathrm{hol}^{\mathcal{K}}_a$ always defines a Lie algebroid morphism because $\mathcal{D}_{\alpha_t}$ are derivations of $\mathcal{K}$.

The presence of a complete connection also allows one to lift A-paths to *horizontal* A_E-paths. In [9], this lifting path property motivated the following definition.

Definition 3.5. A Lie algebroid *fibration* is an extension $p : A_E \longrightarrow\!\!\!\!\rightarrow A$ that admits a complete Ehresmann connection.

The fundamental groups $\pi_\bullet(A)$ of a Lie algebroid are obtained by taking A-spheres up to homotopy, where both A-spheres and homotopies are defined as Lie algebroid maps $TI^n \to A$ with certain boundary conditions (see subsection 4.2). Recall that in general, $\pi_n(A)$ is a bundle of abelian groups over M. The main result of [9] establishes a homotopy long exact sequence for Lie algebroid fibrations as follows.

Theorem 3.6 ([9]). *Given a Lie algebroid fibration $p : A_E \longrightarrow\!\!\!\!\rightarrow A$ with kernel $\mathcal{K}$, there are transgression maps $\delta_n : p_E^*\pi_n(A) \to \pi_{n-1}(\mathcal{K})$ that fit into a homotopy long exact sequence of groupoids:*

$$\cdots \to \pi_n(\mathcal{K}) \xrightarrow{i} \pi_n(A_E) \xrightarrow{p} p_E^*\pi_n(A) \xrightarrow{\delta_n} \pi_{n-1}(\mathcal{K}) \to \cdots$$
$$\cdots \to p_E^*\pi_2(A) \xrightarrow{\delta_2} \pi_1(\mathcal{K}) \xrightarrow{i} \pi_1(A_E) \xrightarrow{p} \pi_1(A).$$

Here $p_E^*\pi_n(A)$ denotes the pull-back bundle of groups along the projection $p_E : E \to M$. This pull-back comes into play when lifting A-spheres to A_E-spheres since a base point in E needs to be chosen.

3.2. **$\mathcal{VB}$-algebroids and Lie algebroid extensions.** We now consider a $\mathcal{VB}$-algebroid $(D; A, E; M)$ as in (2.8). Since the underlying map $p : D \to A$ is a surjective Lie algebroid morphism, covering a surjective submersion $p_E : E \to M$, we see that a $\mathcal{VB}$-algebroid is a special case of a Lie algebroid extension.

Proposition 3.7. *Any $\mathcal{VB}$-algebroid $(D; A, E; M)$ with core C defines a Lie algebroid extension $p : D \longrightarrow\!\!\!\!\rightarrow A$ whose kernel $\mathcal{K} \to E$ is canonically identified with $p_E^*C \to E$ as a vector bundle.*

Proof. By the definition of the core (subsection 2.3), we see that C injects into $\mathcal{K}$ as the restriction to the zero section of $p_E : E \to M$; namely, $C = \mathcal{K}|_{0^E}$. Then one can use the addition $+_A : D \times_A D \to D$ for the vector bundle structure on $D \to A$ to obtain an isomorphism $\mathcal{K} = p_E^*C$. Note that this identification is canonical in the sense that it does not depend on the choice of splitting of D. □

Remark 3.8. Note that the extension of Proposition 3.7 is quite different from (2.9). For instance, D is an extension of A with different base spaces (3.1) which will play an important role in this work.

The above result applies, in particular, to the differential $dp_E : TE \to TM$ of the projection of an arbitrary vector bundle $p_E : E \to M$. In that case, we recover the following well-known identification.

Proposition 3.9. *For any vector bundle $p_E : E \to M$, the vertical bundle* $\ker dp_E \to E$*, where $dp_E : TE \longrightarrow\!\!\!\!\rightarrow TM$ denotes the differential of the projection, identifies with $p_E^*E \to E$ as a vector bundle.*

3.3. **The kernel of a $\mathcal{VB}$-algebroid.** The kernel of an extension induced by a $\mathcal{VB}$-algebroid enjoys interesting properties. In order to explain this, we denote by $\mathbf{Vect_M}$ the category of real smooth vector bundles over a smooth manifold M, where morphisms are smooth vector bundle maps covering the identity of M. The following discussion was adapted from [6], to which we refer for the details.

Definition 3.10. A 2*-vector bundle* over M is a category internal to $\mathbf{Vect_M}$.

Namely, a 2-vector bundle is a category $E_1 \rightrightarrows E_0$ where the spaces both of objects E_0 and of arrows E_1 are vector bundles over M, and where all structure maps (source, target, unit, and composition) are vector bundle morphisms covering the identity. Morphisms between 2-vector bundles over M are *linear functors*, that is, functors which are vector bundle maps at the level of both objects and arrows. In this way, we obtain a category $\mathbf{2\text{-}Vect_M}$ of 2-vector bundles over M. We also denote by $\mathbf{2\text{-}Term_M}$ the category of 2-term complexes of vector bundles over M.

Proposition 3.11. *There is an equivalence of categories* DK : $\mathbf{2\text{-}Vect_M} \to \mathbf{2\text{-}Term_M}$.

Proof. As explained in [6], this equivalence is an instance of the Dold–Kan correspondence. We only sketch the proof and refer to [6] for more details. With any 2-vector bundle $E_1 \rightrightarrows E_0$, one can associate a 2-term complex of vector bundles over M given by $\mathcal{E} := (\partial : C \to E_0)$, where $C := \mathrm{Ker}(\mathbf{s})$ and $\partial := \mathbf{t}|_C$. Conversely, given a 2-term complex $\mathcal{E} = (\partial : C \to E)$ of vector bundles over M, we associate a 2-vector bundle $C \oplus E \rightrightarrows E$, where an arrow (c, e) has source $\mathbf{s}(c, e) = e$ and target $\mathbf{t}(c, e) = e + \partial c$. The multiplication of composable arrows is given by $(c_2, e + \partial c_1) \cdot (e, c_1) = (c_1 + c_2, e)$, while units are given by $\mathbf{1}_e = (0, e)$. Both correspondences are functorial and induce an equivalence of categories (see [6]). □

As a consequence of Proposition 3.11, we conclude that a 2-vector bundle is necessarily a Lie groupoid. The inversion map $C \oplus E \to C \oplus E$ is defined by $(c, e)^{-\mathbf{1}} = (-c, e + \partial c)$.

The next result gives a first hint of the relevance of treating a $\mathcal{VB}$-algebroid as a Lie algebroid extension with different bases as in (3.1) rather than looking at the corresponding jet Lie algebroid (2.9).

Proposition 3.12. *Let $(D; A, E; M)$ be a $\mathcal{VB}$-algebroid. The kernel $\mathcal{K}$ of the induced extension integrates to a* 2*-vector bundle whose associated* 2*-term complex is $\mathcal{E} = (\partial : C \to E)$.*

Proof. First, we use Proposition 3.7 to identify $\mathcal{K}$ with $p_E^*C = C \oplus E$. In this way, core sections are exactly the sections of $\mathcal{K}$ of the form $\hat{c} : e \mapsto (c \circ p_E(e), e)$ for some $c \in \Gamma(C)$. Note that they span $\Gamma(\mathcal{K})$ as a $C^\infty(E)$-module, so they entirely determine the Lie algebroid structure on $\mathcal{K}$. Furthermore, we easily deduce from condition (iii) in Definition 2.10 and the discussion in subsection 2.3 that

$$\rho_{\mathcal{K}}(c_1, e) = (\partial c_1, e),$$
$$[\hat{c}_1, \hat{c}_2]_{\mathcal{K}} = 0$$

for any $c_i \in \Gamma(C)$. In the above equation, $(\partial c_1, e) \in p_E^*E$, where $p_E^*E = E \oplus E$ is identified with $\ker dp_E \subset TE$ by Proposition 3.9. As can be easily checked, we recover the bracket and anchor of the Lie algebroid associated with $p_E^*C \rightrightarrows E$ as in the proof of the Proposition 3.11. Since the latter is source-simply connected, we conclude that $\mathcal{G}(\mathcal{K}) = p_E^*C \rightrightarrows E$. □

Remark 3.13. The following gives an explicit integration procedure for $\mathcal{K}$-paths that will be useful later on. Given a $\mathcal{K}$-path $ads : I \to \mathcal{K}$, where $a(s) = (c(s), e(s))$, there is an obvious way to extend a into an s-dependent section of $\mathcal{K}$, at least fiberwise, simply by taking the corresponding time-dependent core section. Since core sections commute, as with the integration of an abelian Lie algebra, it is easily checked that a is always $\mathcal{K}$-homotopic to a "straight" path—namely, $s \mapsto (\overline{c}, e(0) + s\partial(\overline{c}))$, where $\overline{c}$ is given by $\overline{c} := \int_0^1 c(s)ds$. It follows that the quotient map by $\mathcal{K}$-homotopies can be explicitly written as an integral:

$$\begin{aligned} P_1(\mathcal{K}) = P_1(p_E^*C) &\longrightarrow \mathcal{G}(\mathcal{K}) = p_E^*C \\ \big(c(s), e(s)\big)ds &\longmapsto \Big(\int_0^1 c(s)ds, e(0)\Big). \end{aligned} \tag{3.4}$$

3.4. **Relating connections and curvatures.** We now explain how a connection on a $\mathcal{VB}$-algebroid induces an Ehresmann connection on the underlying extension and how their respective horizontal lifts, curvatures, and holonomy are related.

Given a split $\mathcal{VB}$-algebroid $(D = A \oplus E \oplus C; A, E; M)$, there is a natural Ehresmann connection whose horizontal subbundle is given by $H = A \oplus E \to E$. Then necessarily, the horizontal lifting map $\Gamma(A) \to \Gamma(H)$ has values in linear sections of D, coinciding with the map $A \to \hat{A} \subset \Gamma_\ell(E, D),\ a \mapsto \mathrm{hor}(a)$. Also, it follows that the curvatures of a such a connection, seen both as a linear connection on a $\mathcal{VB}$-algebroid and as an Ehresmann connection, are related by

$$\omega_{\mathcal{K}}(a, b)(e) = \big(\omega(a, b)(e), e\big), \qquad e \in E.$$

Proposition 3.14. *For any $\mathcal{VB}$-algebroid $(D; A, E; M)$, the underlying extension*

$$\begin{array}{ccccc} p_E^*C & \xhookrightarrow{\ i\ } & D & \xrightarrow{\ p\ } & A \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ E & \xrightarrow{\ \mathrm{id}_E\ } & E & \xrightarrow{\ p_E\ } & M \end{array} \tag{3.5}$$

is a fibration.

Proof. Given an A-path $adt : TI \to A$, extended to a time-dependent section $a_t \in \Gamma(A)$, the parallel transport $\mathrm{hol}_a^{\mathcal{K}} : \mathcal{K}|_{E_0} \to \mathcal{K}|_{E_1}$ (if well defined) is given by the trajectories of the time-dependent ordinary differential operator defined on sections of $\mathcal{K}$ by $\mathcal{D}_a : \Gamma(\mathcal{K}) \to \Gamma(\mathcal{K}),\ k \to [\mathrm{hor}(a), k]_D$ (see [8]). Recall that $\mathcal{D}_a$ is a derivation of $\mathcal{K}$, and that such an operator is equivalent to a linear vector field on $\mathcal{K} \to E$ whose flow is a Lie algebroid morphism. In the case of a $\mathcal{VB}$-algebroid with a linear connection, the symbol of $\mathcal{D}_a$ is itself a linear vector field on E since it is given by $\rho_D(\mathrm{hor}(a))$. In particular, $\rho_D(\mathrm{hor}(a))$ is complete whenever its projection $\rho_A(a)$ to M is complete. Thus, any linear connection on a $\mathcal{VB}$-algebroid induces a complete Ehresmann connection. □

For a split $\mathcal{VB}$-algebroid, parallel transport is always well defined. Clearly, the holonomy in $\mathcal{K}$ induced by the Lie algebroid fibration is related to the linear holonomy in E and C in the following way:

$$\mathrm{hol}_a^{\mathcal{K}}(c, e) = \left(\mathrm{hol}_a^{C}(c), \mathrm{hol}_a^{E}(e)\right), \qquad (c, e) \in \mathcal{K} = p_E^*C. \tag{3.6}$$

3.5. **The homotopy sequence of a $\mathcal{VB}$-algebroid.** Since $\mathcal{VB}$-algebroids are special cases of Lie algebroid fibrations, one may specialize the long exact sequence of the homotopy groups given by Theorem 3.6.

Theorem 3.15. *Given a $\mathcal{VB}$-algebroid $(D; A, E; M)$, the following assertions hold:*

(i) *The homotopy long exact sequence associated with the fibration $p : D \longrightarrow\!\!\!\!\rightarrow A$ reads*

$$\pi_k(D) \simeq p_E^* \pi_k(A) \qquad \forall k \geq 3,$$

$$\pi_2(D) \hookrightarrow p_E^* \pi_2(A) \xrightarrow{\delta_2} \mathcal{G}(\mathcal{K}) \xrightarrow{\tilde{\iota}} \mathcal{G}(A_E) \xrightarrow{\pi} \mathcal{G}(A). \tag{3.7}$$

(ii) *The monodromy group $\widetilde{\mathcal{N}}(D, e)$ of D at $e \in E$ fits into an exact sequence of groups*

$$\mathrm{Im}(\delta_2, e) \hookrightarrow \widetilde{\mathcal{N}}(D, e) \xrightarrow{p} \widetilde{\mathcal{N}}(A, m), \tag{3.8}$$

where $\widetilde{\mathcal{N}}(A, m)$ denotes the monodromy group of A at $m := p_E(e)$.

Proof. It follows from Proposition 3.12 that the fundamental groups $\pi_k(\mathcal{K})$ are trivial for any $k \geq 2$. The first assertion then comes from the exactness of the homotopy sequence of Theorem 3.6.

In order to prove the second assertion, we denote by L_D the orbit of D through e, and by L_A the orbit of A through m. As a general fact [9, Cors. 3 and 12], the monodromy groups fit into exact sequences, as given by the horizontal lines of the following diagram:

$$\begin{array}{ccccc} \pi_2(D, e) & \hookrightarrow & \pi_2(L_D, e) & \xrightarrow{\tilde{\delta}_2^{CF}} & \widetilde{\mathcal{N}}(D)_e \\ \downarrow & & \downarrow & & \downarrow \\ \pi_2(A, m) & \hookrightarrow & \pi_2(L_A, m) & \xrightarrow{\delta_2^{CF}} & \widetilde{\mathcal{N}}(A)_m, \end{array} \tag{3.9}$$

where $\tilde{\delta}_2^{CF}$, δ_2^{CF} denote the transgression maps of [14] and where the vertical lines are naturally induced by the projection $p : D \to A$.

In this diagram, one shall first observe that the map $\pi_2(L_D, e) \to \pi_2(L_A, m)$ is an isomorphism, which can be argued as follows: by working leafwise, one may assume that A is transitive with unique leaf L_A and then choose a splitting $\sigma : TL_A \to A$ of the corresponding Atiyah sequence. By composing σ with ∇^E and then restricting to L_D, we obtain an Ehresmann connection on the fibration $L_D \to L_A$ whose curvature has values in the distribution $i_{L_D}^* \mathrm{Im}\, \partial \subset TL_D$. The leaves of this distribution being affine spaces (this is a consequence of Proposition 3.12), they have trivial topology, from which we deduce that $\pi_2(L_D, e) \simeq \pi_2(L_A, m)$.

Observe now that the map $\pi_2(D, e) \to \pi_2(A, m)$ is an injection and is such that $\pi_2(A, m)/\pi_2(D, e) \simeq \mathrm{Im}(\delta_2, e)$ (this follows from (3.7)). The exact sequence (3.8) then easily follows by diagram chasing in (3.9). □

4. Strict 2-groupoids and 2-representations

In this section, we introduce the notion of a representation of a 2-groupoid, which can be thought of as a categorified version of the usual notion of a representation of a Lie groupoid. The idea is the following: just as Lie groupoids can be represented on a vector bundle, Lie 2-groupoids are represented in 2-vector bundles. That is, a 2-representation is an action of a 2-groupoid via symmetries of a 2-vector bundle.

4.1. **Strict 2-groupoids.** Recall that a strict 2-category is defined as a category enriched over the category of small categories **Cat**.

Definition 4.1. A *strict* 2-*groupoid* is a strict 2-category where all 1-morphisms and 2-morphisms have inverses.

More explicitly, a strict 2-groupoid is given by a space of objects M, a space of 1-morphisms $\mathcal{G}_1$ between objects, and a space of 2-morphisms $\mathcal{G}_2$ between 1-morphisms. The space of 1-morphisms comes equipped with a strictly associative composition law and strict inverses, so 1-morphisms form a groupoid over objects, while the space of 2-morphisms comes equipped with a horizontal and a vertical composition, so $\mathcal{G}_2$ is a groupoid both over $\mathcal{G}_1$ and over M.

Furthermore, it is required that horizontal and vertical compositions commute; namely,

$$(\sigma_4 \underset{\mathbf{V}}{\bullet} \sigma_3) \underset{\mathbf{H}}{\bullet} (\sigma_2 \underset{\mathbf{V}}{\bullet} \sigma_1) = (\sigma_4 \underset{\mathbf{H}}{\bullet} \sigma_2) \underset{\mathbf{V}}{\bullet} (\sigma_3 \underset{\mathbf{H}}{\bullet} \sigma_1),$$

for any $\sigma_i, i = 1, \dots, 4 \in \mathcal{G}^{(2)}$, for which the above makes sense. We shall denote a strict 2-groupoid by 2-$\mathcal{G}$, and we represent 2-$\mathcal{G}$ diagrammatically as follows:

$$\text{2-}\mathcal{G}: \quad \begin{array}{ccc} \mathcal{G}_2 & \overset{\mathbf{s_V},\mathbf{t_V}}{\rightrightarrows} & \mathcal{G}_1 \\ & {\scriptstyle \mathbf{s_H},\mathbf{t_H}} \searrow\!\!\searrow \quad \swarrow\!\!\swarrow {\scriptstyle \mathbf{s},\mathbf{t}} & \\ & \mathcal{G}_0 & \end{array} . \tag{4.1}$$

Consider now a 2-groupoid 2-$\mathcal{G}$ as in (4.1). We call the 1-*truncation* of 2-$\mathcal{G}$ the orbit space $\mathcal{G}_1/\mathcal{G}_2$, where two 1-morphisms are identified if there exists a 2-morphism between them. The following proposition is straightforward.

Proposition 4.2. *Let* 2-$\mathcal{G}$ *be a* 2-*groupoid as in* (4.1). *The* 1-*truncation* $\mathcal{G}_1/\mathcal{G}_2$ *inherits a natural groupoid structure over* M.

4.2. **The Weinstein 2-groupoid of a Lie algebroid.** In this section, we are concerned with 2-groupoids associated with Lie algebroids. More precisely, to any Lie algebroid $p_A : A \to M$, we define its Weinstein 2-groupoid. The construction is very similar to that of [19] for Hausdorff topological spaces. See also [22, 28] for similar constructions in the case $A = TM$.

Given an A-path $adt : TI \to A$ with $a : I \to A$, we denote by $\mathbf{s}(adt) = p_A \circ a(0)$ its source, and by $\mathbf{t}(adt) = p_A \circ a(1)$ its target. An A-path with source x and target y will be denoted by $a : x \to y$. The *inverse* of a, denoted by $a^{-1}dt$, is defined to be the A-path

$$a^{-\mathbf{1}}(t)dt := -a(1-t)dt.$$

Every point $x \in M$ determines a trivial A-path $\mathbf{1}_x := 0_x dt$, where $0_x \in A_x$. The A-path $\mathbf{1}_x$ is called the *unit path* at x. Note that, at the moment, both notions of units and inverses stand as formal ones. However, they become genuine ones only after modding out by thin homotopies; see below.

Definition 4.3. An A-path $adt : TI \to A$ is *flat at its boundaries* if, for any smooth section $\theta \in \Gamma(A^*)$, the map $\langle \theta, a \rangle : I \to \mathbb{R}$ vanishes flatly at $t = 0$ and $t = 1$. That is, $\langle \theta, a \rangle$ vanishes at $t = 0$ and $t = 1$ as well as at all of its higher derivatives.

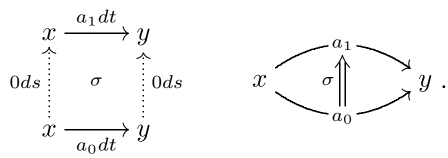
Definition 4.4. Given two A-paths $adt, bdt : TI \to A$ with $\mathbf{t}(adt) = \mathbf{s}(bdt)$, their *concatenation* $(a \cdot b)dt$ is defined by

$$(b \cdot a)(t)dt = \begin{cases} 2a(2t)dt & \text{if } t \in [0, 1/2], \\ 2b(2t-1)dt & \text{if } t \in [1/2, 1]. \end{cases}$$

Obviously, $(b\cdot a)dt$ is smooth whenever a and b are flat at their boundaries. In order to concatenate arbitrary A-paths, one can replace them by their *reparametrizations.* For that, we consider a cutoff function $\tau : I \to I$—namely, τ can be taken to be the restriction of any smooth function $f : \mathbb{R} \to [0,1]$ such that $f|_{(-\infty,0)} = 0$ and $f|_{(1,\infty)} = 0$—so that all the derivatives of τ vanish at 0 and 1. Then the reparametrization a^τ is the A-path given by $a^\tau(t)dt := a \circ d\tau : TI \to A$, where $a^\tau(t) := \tau'(t)(a \circ \tau)(t)$.

Definition 4.5. An *A-homotopy* between two A-paths a_0dt and a_1dt is a Lie algebroid morphism $\sigma = adt + bds : TI^2 \to A$ such that $a_0dt = adt|_{\{s=0\}}$ and $a_1dt = adt|_{\{s=1\}}$, and satisfying the boundary conditions $b|_{\{t=0,1\}} = 0$.

We shall denote by $\mathbf{s}_\mathbf{V}(\sigma) = a_0dt$ and $\mathbf{t}_\mathbf{V}(\sigma) = a_1dt$ the *source* and *target* of σ, and we shall write $\sigma : a_0 \Rightarrow a_1$. A simple way to picture an A-homotopy is as follows:



Definition 4.6. An A-homotopy $\sigma : TI^2 \to A$ is called *thin* if the induced map $\bigwedge^2 \sigma : \bigwedge^2 TI^2 \to \bigwedge^2 A$ is trivial. Two A-paths are said to be *thin homotopic* if there exists a thin homotopy $\sigma : a_0 \Rightarrow a_1$.

Definition 4.7. An A-homotopy $\sigma = adt + bds : TI^2 \to A$ is said to be flat at its boundary if, for any smooth section $\eta \in \Gamma(A^*)$, the following conditions hold:

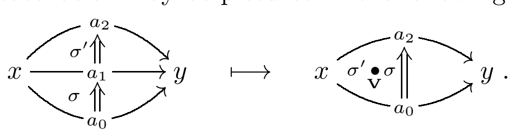
(i) the application $\langle \eta, b\rangle : I \to \mathbb{R}$ vanishes flatly at $\{s = 0, 1\}$,
(ii) the application $\langle \eta, a\rangle : I \to \mathbb{R}$ vanishes flatly at $\{t = 0, 1\}$.

In particular, both A-paths $\mathbf{s}_\mathbf{V}(\sigma)$ and $\mathbf{t}_\mathbf{V}(\sigma)$ are flat at their boundaries provided that σ is.

Definition 4.8. Given two A-homotopies $\sigma = adt + bds$ and $\sigma' = a'dt + b'ds$ with $\mathbf{t}_\mathbf{V}(\sigma) = \mathbf{s}_\mathbf{V}(\sigma')$, we define their *vertical concatenation* by

$$(\sigma' \underset{\mathbf{V}}{\bullet} \sigma)(t,s) = \begin{cases} a(t,2s)dt + 2b(t,2s)ds & \text{if } s \in [0, 1/2], \\ a'(t,2s-1)dt + 2b'(t,2s-1) & \text{if } s \in [1/2, 1]. \end{cases}$$

The vertical concatenation may be pictured in the following way:



The vertical concatenation is smooth whenever σ and σ' are flat at their boundaries (see [9, Lems. 3 and 7]). In order to concatenate smoothly two arbitrary A-homotopies, one has to replace them with their respective reparametrizations, where the reparametrization of an A-homotopy σ is given by $\sigma^\tau := \sigma \circ d(\tau \times \tau)$; namely, $\sigma^\tau = \tau'(t)a(\tau(t), \tau(s))dt + \tau'(s)b(\tau(t), \tau(s))ds$.

Definition 4.9. The *vertical inverse* of an A-homotopy $\sigma = adt + bds$ is given by

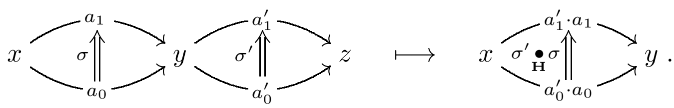
$$\big(\sigma^{-\mathbf{1_V}}\big)(t,s) = a(t,s)dt - b(t, 1-s)ds.$$

The *vertical unit* at an A-path adt is the A-homotopy $\mathbf{1}^V_{adt} := a(t)dt + 0_{\gamma(t)}ds$ (so $\mathbf{1}^V_{adt} : adt \Rightarrow adt$).

Definition 4.10. The horizontal source and target maps $\mathbf{s_H}, \mathbf{t_H}$ of an A-homotopy are given by $\mathbf{s_H} = \mathbf{s} \circ \mathbf{s_V}$ and $\mathbf{t_H} = \mathbf{t} \circ \mathbf{s_V}$. Given two A-homotopies $\sigma = adt + bds$ and $\sigma' = a'dt + b'ds$ such that $\mathbf{t_H}(\sigma) = \mathbf{s_H}(\sigma')$, one defines their *horizontal concatenation* in the following way:

$$\big(\sigma' \underset{\mathbf{H}}{\bullet} \sigma\big)(t,s) = \begin{cases} 2\,a(2t,s)dt + b(2t,s)ds & \text{if } t \in [0,1/2], \\ 2\,a'(2t-1,s)dt + b'(2t-1,s)ds & \text{if } t \in [1/2,1]. \end{cases}$$

As can be easily checked, the horizontal concatenation is smooth whenever σ, σ' are flat at their boundaries, and it may be illustrated as follows:



Definition 4.11. The *horizontal inverse* of an A-homotopy is defined by

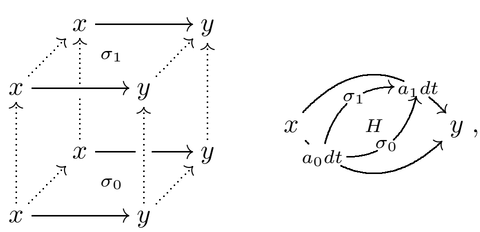
$$\big(\sigma^{-\mathbf{1_H}}\big)(t,s) = -a(1-t,s)dt + b(1-t,s)ds.$$

Given $x \in M$, the *horizontal unit* is the A-homotopy $\mathbf{1}^H_x := 0_x dt + 0_x ds$, between the unit path $\mathbf{1}_x$ and itself.

Definition 4.12. We use the term "3-homotopy" for any Lie algebroid morphism $H = H_1dt + H_2ds + H_3du : TI^3 \to A$ satisfying the boundary conditions $H_3du|_{\{t=0,1\}} = 0\,du$, and $H_3du|_{\{s=0,1\}} = 0\,du$.

In that case, H defines a 3-homotopy between $\sigma_0 := (H_1dt + H_2ds)|_{\{u=0\}}$ and $\sigma_1 := (H_1dt + H_2ds)|_{\{u=1\}}$. Note that, as a consequence of H being a Lie algebroid morphism, σ_1 is an A-homotopy if and only if σ_0 is (see [9]). In this work, we shall consider only 3-homotopies between A-homotopies. Note also that any A-homotopy σ is 3-homotopic to its reparametrization σ^τ.

A 3-homotopy between two A-homotopies σ_0 and σ_1 can be pictured in the following way:



where the dotted squares represent trivial morphisms.

Definition 4.13. Given a Lie algebroid $A \to M$, we use the term "Weinstein 2-groupoid of A", denoted by 2-$\mathcal{P}(A)$, for the strict 2-groupoid with M as space of objects, where 1-morphisms are thin homotopy classes of A-paths, and where

2-morphisms are given by 3-homotopy classes of A-homotopies, with horizontal and vertical compositions given by the concatenations along the t and s variables, respectively.

Routine computations show that 2-$\mathcal{P}(A)$ is indeed a strict 2-groupoid; details will be left to the reader. Let us rather emphasize the notations: in the sequel, we shall denote by $P_1(A)$ the space of thin homotopy classes of A-paths, and by $P_2(A)$ the space of 3-homotopy classes of A-homotopies, so that 2-$\mathcal{P}(A)$ denotes the strict 2-groupoid structure with $P_1(A)$ as 1-morphisms, and $P_2(A)$ as 2-morphisms:

$$2\text{-}\mathcal{P}(A): \quad \begin{array}{ccc} P_2(A) & \overset{\mathbf{s_V},\mathbf{t_V}}{\rightrightarrows} & P_1(A) \\ & {\scriptstyle \mathbf{s_H},\mathbf{t_H}} \searrow\!\!\searrow \quad \swarrow\!\!\swarrow {\scriptstyle \mathbf{s},\mathbf{t}} & \\ & M & \end{array} .$$

As explained in Proposition 4.2, the 1-truncation of a 2-groupoid with objects M inherits the structure of a groupoid over M. Applying this construction to the Weinstein 2-groupoid of a Lie algebroid yields the following result.

Proposition 4.14. *Let $A \to M$ be a Lie algebroid. The 1-truncation $P_1(A)/P_2(A)$ of the Weinstein 2-groupoid of A coincides with the Weinstein groupoid $\mathcal{G}(A)$ of A.*

Proof. The proof is straightforward, and it is a consequence of combining the following facts: any A-path is thin homotopic to its reparametrization, and thin homotopies are genuine homotopies. □

The Weinstein 2-groupoid can be used also to recover the fundamental groups $\pi_2(A, m)$ defined in [9], as the isotropies in $P_2(A)$ at identities. Namely, for every $m \in M$, we have

$$\pi_2(A, m) = \{\sigma \in P_2(A) \mid \sigma : 0_m dt \Rightarrow 0_m dt\} .$$

Remark 4.15. In this work, we will not define a differential structure on 2-$\mathcal{P}(A)$; hence, neither shall we differentiate a representation of 2-$\mathcal{P}(A)$ on a 2-term complex $\mathcal{E}$ to a 2-term representation of A. This was done in the case $A = TM$ in [28] in a slightly different context (see the precise relation in [5]), where a similar path 2-groupoid of a smooth manifold is constructed, as well as a smooth structure on it.

In [28], the smooth structure is defined in terms of diffeological spaces, and it seems clear that their construction extends to the case of an integrable Lie algebroid. It would be interesting to see if the diffeological structure is also well defined in the nonintegrable case.

Additionally, note that the space of 2-morphisms defined in this work is slightly smaller than that of [5, 28] since we take quotients by 3-homotopies. The basic reason is to obtain a 2-groupoid as small as possible for 2-term representations. In the terminology of [28], we essentially deal here with flat 2-connections (meaning that the curvature 3-form vanishes (2.4)), and hence it is natural to mod out the useless information (see Lemma A.8).

4.3. **The gauge 2-groupoid of a 2-term complex.** Let M be a smooth manifold. According to Proposition 3.11, the category $\mathbf{2\text{-}Vect_M}$ of 2-vector bundles over M is equivalent to the category $\mathbf{2\text{-}Term_M}$ of 2-term complexes of vector bundles over M. In this section, we study symmetries of 2-vector bundles via their description as a 2-term complex. Just as a vector bundle has a groupoid of symmetries—namely, its gauge groupoid—a 2-vector bundle has a 2-*groupoid* of symmetries, its gauge 2-groupoid.

Let $\mathcal{E} = (C \xrightarrow{\partial} E)$ be a 2-term complex of vector bundles over M. The gauge 2-groupoid of $\mathcal{E}$ is defined as the strict 2-groupoid 2-Gau($\mathcal{E}$) whose space of objects is M, and we have as follows.

- 1-morphisms: A 1-morphism $F : x \to y$ between objects $x, y \in M$ is an invertible chain map. That is, a couple of invertible linear maps $F = (F^C, F^E)$, where $F^C \in \mathrm{Iso}(C_x, C_y)$, $F^E \in \mathrm{Iso}(E_x, E_y)$, and such that $F^E \circ \partial_x = \partial_y \circ F^C$. The composition and inverses of chain maps is
$$\big(G^C, G^E\big) \cdot \big(F^C, F^E\big) = \big(G^C \circ F^C, G^E \circ F^E\big),$$
$$\big(F^C, F^E\big)^{-1} = \big((F^C)^{-1}, (F^E)^{-1}\big).$$
- 2-morphisms: For any objects $x, y \in M$ and 1-morphisms $F_0, F_1 : x \to y$, a 2-morphism between F_0 and F_1 is given by a chain homotopy $\phi : F_0 \Rightarrow F_1$, i.e., a linear application $\phi \in \mathrm{Hom}(E_x, C_y)$ satisfying
$$F_1^C - F_0^C = \phi \circ \partial,$$
$$F_1^E - F_0^E = \partial \circ \phi.$$
- Vertical composition: Given 2-morphisms $F_0 \overset{\phi}{\Rightarrow} F_1 \overset{\psi}{\Rightarrow} F_2$, where F_0, F_1, F_2 are 1-morphisms $x \to y$, the vertical composition between ϕ and ψ is the 2-morphism $\psi \underset{\mathbf{V}}{\bullet} \phi : F_0 \Rightarrow F_2$ defined by
$$\psi \underset{\mathbf{V}}{\bullet} \phi = \psi + \phi.$$
- Vertical unit: For any 1-morphism $A : x \to y$, the vertical unit at F is the 2-morphism $1^{\mathbf{V}}_A : F \Rightarrow F$ given by $1^{\mathbf{V}}_F = 0 \in \mathrm{Hom}(E_x, C_y)$.
- Vertical inverse: Given a 2-morphism $\phi : F_0 \Rightarrow F_1 \in \mathrm{Hom}(E_x, C_y)$, the vertical inverse of ϕ is the 2-morphism $\phi^{-1^{\mathbf{V}}} : F_1 \Rightarrow F_0$ given by $\phi^{-1^{\mathbf{V}}} := -\phi$.
- Horizontal composition: Given 2-morphisms $F_0 \overset{\phi}{\Rightarrow} F_1$, $F_0' \overset{\phi'}{\Rightarrow} F_1'$, where $F_0, F_1 : x \to y$ and F_0', $F_1' : y \to z$ are 1-morphisms, the horizontal composition $\phi' \underset{\mathbf{H}}{\bullet} \phi : F_0' \circ F_0 \Rightarrow F_1' \circ F_1$ is given by
$$\phi' \underset{\mathbf{H}}{\bullet} \phi = \phi' \circ F_0 + F_1' \circ \phi.$$
- Horizontal unit: For any $x \in M$, the horizontal unit $1^{\mathbf{H}}_x$ is the 2-morphism $1^{\mathbf{H}} : \mathrm{id} \Rightarrow \mathrm{id}$ given by the trivial morphism $1^{\mathbf{H}} := 0 \in \mathrm{Hom}(E_x, C_x)$.
- Horizontal inverse: Given a 2-morphism $\phi : F_0 \Rightarrow F_1 \in \mathrm{Hom}(E_x, C_y)$, the horizontal inverse of ϕ is the 2-morphism $\phi^{-1^{\mathbf{H}}} : F_0^{-1} \Rightarrow F_1^{-1}$ given by
$$\phi^{-1^{\mathbf{H}}} := -F_1^{-1} \circ \phi \circ F_0^{-1} \in \mathrm{Hom}(E_y, C_x).$$

It is easy to check that 2-Gau($\mathcal{E}$) indeed defines a strict 2-groupoid. This is left to the reader.

Remark 4.16. It is easily seen by adapting the results of Baez and Crans [6] that 2-Gau($\mathcal{E}$) is isomorphic to the strict 2-groupoid 2-Gau($\mathcal{K}$) of linear automorphisms of $\mathcal{K} = p_E^* C$, where $\mathcal{K}$ is seen as a 2-vector bundle by Proposition 3.12. Here 2-Gau($\mathcal{K}$) is obtained by taking objects to be points in M, 1-morphisms to be invertible linear functors $\mathcal{K}_x \to \mathcal{K}_y$, and 2-morphisms to be linear natural transformations between invertible linear functors.

Therefore, one can of think of 2-Gau($\mathcal{E}$) as a *frame* 2*-groupoid* for the 2-vector bundle $\mathcal{K}$. Since our main concern is with 2-term representations up to homotopy, we will work with 2-Gau($\mathcal{E}$) rather than 2-Gau($\mathcal{K}$), although the latter would be more natural from the point of view of fibrations.

Remark 4.17. In the special case in which $\partial : C \to E$ has constant rank, both spaces of 1-morphisms and of 2-morphisms of 2-Gau($\mathcal{E}$) come with an obvious structure of a (finite dimensional) smooth manifold. Furthermore, all of the structure maps are easily seen to be smooth. In that case, 2-Gau($\mathcal{E}$) is a Lie 2-groupoid. This assumption, however, is quite restrictive in view of the applications. For instance, one of the main motivations for introducing representations up to homotopy was to get a model for the adjoint and coadjoint representations of a Lie algebroid $A \to M$. In that case, $\partial = \rho_A : A \to TM$ coincides with the anchor map, and therefore it does not have constant rank unless A is regular.

Since this condition is irrelevant for the algebraic constructions of the next sections to hold, we shall not assume that $\partial : C \to E$ has constant rank.

4.4. **The holonomy 2-representation.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A representation of $\mathcal{G}$ consists on a vector bundle $E \to M$ equipped with a linear action of $\mathcal{G}$, i.e., a groupoid morphism $\mathcal{G} \to \mathrm{Gau}(E)$. If one moves to 2-groupoids, the corresponding notion of linear action is that of a 2-representation.

Definition 4.18. Let 2-$\mathcal{G}$ be a 2-groupoid as in (4.1). A 2*-representation* of 2-$\mathcal{G}$ consists of a 2-term complex $\mathcal{E} = (\partial : C \to E)$, equipped with a strict 2-functor $\Phi : 2\text{-}\mathcal{G} \to 2\text{-Gau}(\mathcal{E})$.

More precisely, if $\mathcal{G}^{(2)}$, $\mathcal{G}^{(1)}$, and M denote the spaces of 2-morphisms, 1-morphisms and objects of 2-$\mathcal{G}$, respectively, then a 2-representation is given by the following assignment:

- With any 1-morphism $g : x \to y$, one associates an invertible chain map $\Phi_g = (F_g^C, F_g^E)$,
$$\begin{array}{ccc} C_x & \xrightarrow{\partial_x} & E_x \\ {\scriptstyle F_g^C}\downarrow & & \downarrow{\scriptstyle F_g^E} \\ C_y & \xrightarrow{\partial_y} & E_y, \end{array}$$
in such a way that $\Phi_{gh} = \Phi_g \circ \Phi_h$ for every composable pair of 1-morphisms $g, h \in \mathcal{G}^{(1)}$.
- With any 2-morphism $\sigma \in \mathcal{G}^{(2)}$ with $\mathbf{s}_V(\sigma) = g$, $\mathbf{t}_V(\sigma) = h$, $\mathbf{s}_H(\sigma) = x$ and $\mathbf{t}_H(\sigma) = y$, one associates a chain homotopy $\Phi_\sigma \in \mathrm{Hom}(E_y, C_x)$ between the chain maps Φ_g and Φ_h. The assignment $\sigma \mapsto \Phi_\sigma$ is compatible with both the vertical and the horizontal composition, as well as with inverses.

Note that a usual representation $E \to M$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ can be thought of as a 2-representation of the 2-groupoid 2-$\mathcal{G}$ (objects are points of M, the space of 1-morphisms is $\mathcal{G}$, and the space of 2-morphisms is also $\mathcal{G}$) on the 2-term complex $(C := \{0\} \to E)$.

A 2-term representation up to homotopy $(\nabla^E, \nabla^C, \omega)$ of a Lie algebroid A on a 2-term complex $\mathcal{E} = (\partial : C \to E)$ gives rise to a 2-representation of the Weinstein

2-groupoid 2-$\mathcal{P}(A)$ on $\mathcal{E}$. Indeed, in that situation one defines the corresponding *holonomy*, denoted by hol, first as an assignment, in the following way:

- With any A path $adt : TI \to A$ from x to y, the holonomy associates the couple
$$\mathrm{hol}(a) := (\mathrm{hol}^C_a, \mathrm{hol}^E_a)\,,$$
where $\mathrm{hol}^C_a : C_x \to C_y$ (resp., $\mathrm{hol}^E_a : E_x \to E_y$) denotes the holonomy of ∇^C (resp., ∇^E) along a.
- With any A-homotopy $\sigma = adt + bds : TI^2 \to A$, the holonomy associates the application $\mathrm{hol}(\sigma) \in \mathrm{Hom}(E_x, C_y)$ defined by

$$\mathrm{hol}(\sigma) := \int_0^1\!\!\int_0^1 \mathrm{hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}\, dt\, ds. \tag{4.2}$$

Here, the notations in (4.2) are detailed in Appendix A. We can now state one of the central results of this work.

Theorem 4.19. *Let $(\nabla^E, \nabla^C, \omega)$ be a representation up to homotopy of a Lie algebroid A on a 2-term complex of vector bundles $\mathcal{E} = (\partial : C \to E)$. The holonomy defined above descends to a strict 2-functor* $\mathrm{hol} : 2\text{-}\mathcal{P}(A) \to 2\text{-}\mathrm{Gau}(\mathcal{E})$ *covering the identity on M.*

For clarity, the proof of Theorem 4.19 is postponed to Appendix A.

Definition 4.20. The 2-functor $\mathrm{hol} : 2\text{-}\mathcal{P}(A) \to 2\text{-}\mathrm{Gau}(\mathcal{E})$ is referred to as the holonomy 2-representation associated with the 2-term representation up to homotopy $(\nabla^E, \nabla^C, \omega)$ of A on $\mathcal{E} = (\partial : C \to E)$.

Remark 4.21. If $A = TM$, both $E = M \times \mathbb{R}^n$ and $C = M \times \mathbb{R}^k$ are trivial vector bundles and $\partial : C \to E; (x, v) \to (x, f(v))$, where $f : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map, Theorem 4.19 recovers the work of Schreiber and Waldorf [28] (by which Theorem 4.19 was directly inspired). An alternative proof should also follow from the integration procedure of Arias Abad and Schätz [4] (see also [5], which relates the two approaches).

Note also that working directly with the Weinstein 2-groupoid avoids, after pulling back the structure along a morphism $TI^2 \to A$, the choice of a trivialization in order to express the local connection forms as in [29] (although the gluing of 2-functors is a quite remarkable feature).

4.5. **Transformation 2-groupoid associated with a 2-representation.** Given that a 2-representation $\Phi : 2\text{-}\mathcal{G} \to 2\text{-}\mathrm{Gau}(\mathcal{E})$ of a 2-groupoid, there exists a transformation object $2\text{-}\mathcal{G} \ltimes \mathcal{E}$ whose outcome is a 2-groupoid. Since we are mainly interested in holonomy 2-representations as in Theorem 4.19, we shall present this construction only in that case, although it is easily seen to make sense in general.

For that, consider a Lie algebroid $A \to M$, and let 2-$\mathcal{P}(A)$ be the Weinstein 2-groupoid of A defined in subsection 4.2. Fix a 2-term representation up to homotopy $(\nabla^E, \nabla^C, \omega)$ of a A on a 2-term complex $\mathcal{E} = (\partial : C \to E)$. Let $\mathrm{hol} : 2\text{-}\mathcal{P}(A) \to 2\text{-}\mathrm{Gau}(\mathcal{E})$ be the corresponding holonomy 2-representation given by Theorem 4.19.

Definition 4.22. The *transformation 2-groupoid* associated with $\mathrm{hol} : 2\text{-}\mathcal{P}(A) \to 2\text{-}\mathrm{Gau}(\mathcal{E})$ is the strict 2-groupoid $2\text{-}\mathcal{P}(A) \ltimes \mathcal{E}$ defined as follows.

- Objects: The space of objects of $2\text{-}\mathcal{P}(A) \ltimes \mathcal{E}$ is given by E.

- 1-morphisms: The space of 1-morphisms of 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ is

$$P_1(A) \ltimes \mathcal{E} := \mathbf{t}^* C \oplus \mathbf{s}^* E,$$

where $\mathbf{s}, \mathbf{t} : P_1(A) \to M$ are the source and target maps of 1-morphisms of the Weinstein 2-groupoid. The source, target, and inverse maps of 1-morphisms are given as follows,

$$\begin{aligned} \tilde{\mathbf{s}}(c, a, e) &= e, \\ \tilde{\mathbf{t}}(c, a, e) &= \mathrm{hol}_a^E(e) + \partial c \\ (c, a, e)^{-\mathbf{1}} &= \left(-\mathrm{hol}_{a^{-1}}^C(c), a^{-1}, \mathrm{hol}_a^E(e) + \partial c\right), \end{aligned}$$

two 1-morphisms (c_1, a_1, e_1) and (c_0, a_0, e_0) are composable provided that $e_1 = \mathrm{hol}_{a_0}^E(e_0) + \partial c_0$, and their multiplication is given by

$$(c_1, a_1, e_1) \cdot (c_0, a_0, e_0) = \left(c_1 + \mathrm{hol}_{a_1}^C(c_0), a_1 \cdot a_0, e_0\right).$$

- 2-morphisms: The space of 2-morphisms in 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ is given by

$$P_2(A) \ltimes \mathcal{E} := \mathbf{t}_{\mathbf{H}}^* C \oplus \mathbf{s}_{\mathbf{H}}^* E,$$

where $\mathbf{s}_{\mathbf{H}}, \mathbf{t}_{\mathbf{H}} : P_2(A) \to M$ are the horizontal source and target maps of the Weinstein 2-groupoid.
- Vertical structure maps: The vertical source and target maps $\tilde{\mathbf{s}}_{\mathbf{V}}, \tilde{\mathbf{t}}_{\mathbf{V}} : P_2(A) \ltimes \mathcal{E} \to P_1(A) \ltimes \mathcal{E}$ are given by

$$\begin{aligned} \tilde{\mathbf{s}}_{\mathbf{V}}(c, \sigma, e) &:= (c, \mathbf{s}_{\mathbf{V}}(\sigma), e), \\ \tilde{\mathbf{t}}_{\mathbf{V}}(c, \sigma, e) &:= (c - \mathrm{hol}(\sigma)_e, \mathbf{t}_{\mathbf{V}}(\sigma), e). \end{aligned}$$

Two 2-morphisms (c_2, σ_2, e_2) and (c_1, σ_1, e_1) are composable vertically provided that $\mathbf{s}_{\mathbf{V}}(\sigma_2) = \mathbf{t}_{\mathbf{V}}(\sigma_1) \in P_1(A)$ and $c_2 = c_1 + \mathrm{hol}(\sigma_1)(e_1)$, and their vertical multiplication is given by

$$\left(c_2, \sigma_2, e_2\right) \underset{\mathbf{V}}{\bullet} \left(c_1, \sigma_1, e_1\right) = \left(c_1, \sigma_2 \underset{\mathbf{V}}{\bullet} \sigma_1, e_1\right).$$

The vertical inverse of a 2-morphism (c, σ, e) is defined by the following formula:

$$\left(c, \sigma, e\right)^{-\mathbf{1}_{\mathbf{V}}} = \left(c - \mathrm{hol}(\sigma)_e, \sigma^{-\mathbf{1}_{\mathbf{V}}}, e\right).$$

- Horizontal structure maps: The horizontal source and target maps are given by

$$\begin{aligned} \tilde{\mathbf{s}}_{\mathbf{H}}(c, \sigma, e) &= e, \\ \tilde{\mathbf{t}}_{\mathbf{H}}(c, \sigma, e) &= \mathrm{hol}_{\mathbf{t}_{\mathbf{V}}(\sigma)}(e) + \partial c. \end{aligned}$$

The horizontal composition and horizontal inverse are given by

$$\left(c, \sigma, e\right) \underset{\mathbf{H}}{\bullet} (b, \tau, f) = \left(c + \mathrm{hol}_{\mathbf{s}_{\mathbf{V}}(\sigma)}^C(b), \sigma \underset{\mathbf{H}}{\bullet} \tau, f\right),$$

$$\left(c, \sigma, e\right)^{-\mathbf{1}_{\mathbf{H}}} = \left(\mathrm{hol}_{\mathbf{s}_{\mathbf{V}}(\sigma)}^C(c), \sigma^{-\mathbf{1}_{\mathbf{H}}}, e\right).$$

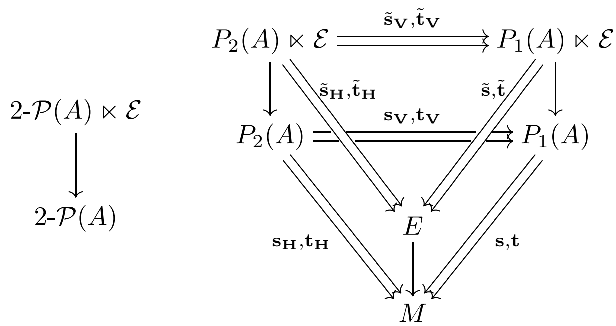
The next result is a direct consequence of the definition of 2-$\mathcal{P}(A)$.

Theorem 4.23. *Let A be a Lie algebroid over M. If* $\mathrm{hol} : 2\text{-}\mathcal{P}(A) \to 2\text{-}\mathrm{Gau}(\mathcal{E})$ *is the holonomy 2-representation associated with a 2-term representation up to homotopy of A, then 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ described above is a strict 2-groupoid.*

Remark 4.24. Quite interestingly, one can think of the semidirect product in Definition 4.22 as a *strict* analogue of the construction of a Grothendieck fibration (with cleavage) associated with a pseudo functor. In fact, one may see a $\mathcal{VB}$-groupoid as a special kind of Grothendieck fibration, for which the choice of a horizontal lift gives a cleavage (so that the corresponding groupoid 2-representation is indeed a special kind of pseudo functor).

Part of this work was motivated by the fact that this construction may not be the most natural from the point of view of $\mathcal{VB}$-algebroids and $\mathcal{VB}$-groupoids, essentially because the splittings of a $\mathcal{VB}$-algebroid do not integrate canonically to horizontal lifts of the corresponding $\mathcal{VB}$-groupoid (as we shall see in Remark 5.11).

Although it is not finite dimensional, there is a remarkable feature enjoyed by the semidirect product 2-$\mathcal{P}(A) \ltimes \mathcal{E}$—namely, the fact that 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ is a 2-$\mathcal{VB}$-groupoid over 2-$\mathcal{P}(A)$ in the following sense. Note first that the spaces of 1-morphisms and of 2-morphisms in 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ are $\mathcal{VB}$-groupoids over $P_2(A)$ and $P_1(A)$, respectively. Both even come with a left-horizontal splitting by construction, for which the curvature Ω vanishes. Then, taking into account the 2-groupoid structures on 2-$\mathcal{P}(A)$ and 2-$\mathcal{P}(A) \ltimes \mathcal{E}$, we obtain a diagram as follows:



,

where the upper and lower triangles are strict 2-groupoids, all vertical maps are vector bundles, the three faces are (canonically split, flat) $\mathcal{VB}$-groupoids, and all structures are compatible with each other in the obvious sense.

5. Application to the integration of $\mathcal{VB}$-algebroids

5.1. **$\mathcal{VB}$-groupoids as 1-truncations.** Consider a Lie algebroid $A \to M$, together with a 2-term representation up to homotopy $(\nabla^E, \nabla^C, \omega)$ on a complex $\mathcal{E} = (\partial : C \to E)$, with holonomy 2-representation hol : 2-$\mathcal{P}(A) \to$ 2-$\operatorname{Gau}(\mathcal{E})$. Let $D = A \oplus E \oplus C$ be the $\mathcal{VB}$-algebroid over A associated with such a 2-term representation up to homotopy.

The following result shows how to pass from the transformation 2-groupoid 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ to the Weinstein groupoid of D.

Theorem 5.1. *The Weinstein groupoid $\mathcal{G}(D)$ of D identifies with the 1-truncation of the transformation 2-groupoid 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ associated with the holonomy 2-representation.*

Remark 5.2. More precisely, the groupoid structure on $P_1(A) \ltimes \mathcal{E} \rightrightarrows E$ descends to an isomorphism

$$\mathcal{G}(D) = (P_1(A) \ltimes \mathcal{E})/_\sim,$$

where the groupoid structure on the right-hand side is obtained by 1-truncation, as explained in Proposition 4.2. Theorem 5.1 says that this quotient can be identified as a groupoid, with the *topological* Weinstein groupoid $\mathcal{G}(D)$ of the total space D of the corresponding $\mathcal{VB}$-algebroid. Note that this identification is free of choice once the 2-representation is fixed.

Proof of Theorem 5.1. The proof relies on a construction from [8, sec. 4] of which we will outline only the main ideas. Note, however, that we use here slightly different conventions, this in order to obtain the holonomy as a covariant 2-functor.

As detailed in [8], given an arbitrary Lie algebroid fibration $D \to A$, the choice of an Ehresmann connection $D = \mathcal{K} \oplus \mathrm{Hor}$ induces an identification of the form

$$P_1(D) \simeq P_1(\mathcal{K}) \rtimes P_1(A), \tag{5.1}$$

where $P_1(\mathcal{K}) \rtimes P_1(A)$ is a short notation for the fibered product $\{(a_\mathcal{K}, a) \in P_1(\mathcal{K}) \times P_1(A) : p_E(\mathbf{t}(a_\mathcal{K})) = \mathbf{s}(a)\}$. By using the identification (5.1), it is possible to describe both the concatenation and the homotopies of D-paths directly in $P_1(\mathcal{K}) \rtimes P_1(A)$ (see [8, Props. 4,1 and 4.3]). More explicitly, the following statements hold for any Lie algebroid fibration:

(a) The concatenation of D-paths reads through (5.1) in the following way:

$$(a_\mathcal{K}, a) \cdot (b_\mathcal{K}, b) = \big(a_\mathcal{K} \cdot (b_\mathcal{K} \circ \mathrm{hol}^{\mathcal{K}}_a)\,, a \cdot b\big). \tag{5.2}$$

(b) The equivalence relation by D-homotopies factors through $\mathcal{G}(\mathcal{K}) \rtimes P_1(A)$ as follows:

$$\begin{array}{ccc} P_1(\mathcal{K}) \rtimes P_1(A) & \twoheadrightarrow & \mathcal{G}(\mathcal{K}) \rtimes P_1(A) \\ & \searrow \quad \swarrow & \\ & \mathcal{G}(D), & \end{array}$$

where $\mathcal{G}(\mathcal{K}) \rtimes P_1(A)$ denotes the fibered product $\{(v, a) \in \mathcal{G}(\mathcal{K}) \times P_1(A) : p_E(\mathbf{t}(v)) = \mathbf{s}(a)\}$, and where the map $P_1(\mathcal{K}) \rtimes P_1(A) \to \mathcal{G}(\mathcal{K}) \rtimes P_1(A)$ is the identity on the $P_1(A)$ factor and takes a $\mathcal{K}$-path $a_\mathcal{K}$ to its $\mathcal{K}$-homotopy class $v := [a_\mathcal{K}]$ of the $\mathcal{K}$-path.

(c) The factored map $\mathcal{G}(\mathcal{K}) \rtimes P_1(A) \to \mathcal{G}(D)$ is the quotient by the equivalence relation $(v_0, a_0) \sim (v_1, a_1)$ if and only if there exists an A-homotopy $\sigma : a_0 \Rightarrow a_1$ such that $g_1^{-1} \cdot v_0 = \partial_{ext}(\sigma, \mathbf{s}(v_0))$. Here $\partial_{ext}(\sigma, \mathbf{s}(v_0)) \in \mathcal{G}(\mathcal{K})$ is represented by a $\mathcal{K}$-path obtained by solving a differential equation (see the definition below in our case of interest).

In the case of a fibration induced by a $\mathcal{VB}$-algebroid, we know from Proposition 3.12 that $\mathcal{G}(\mathcal{K}) = E \oplus C$, so two elements $v_0 = ((e'_0, c_0), a_0)$ and $v_1 = ((e'_1, c_1), a_1)$ in $\mathcal{G}(\mathcal{K}) \rtimes P_1(A)$ are equivalent if and only if $(c_1 - c_0, e'_0) = \partial_{ext}(\sigma, e'_0)$ for some A-homotopy $\sigma : a_0 \Rightarrow a_1$, and $e'_1 = \mathbf{t}(\partial_{ext}(\sigma, e'_0))$. Furthermore, by combining the definition of ∂_{ext} in [8, Prop. 4.3] with the Remark 3.13, one may define directly ∂_{ext} as follows:

$$\partial_{ext}(\sigma, e') := (\mathrm{hol}(\sigma)(e), e') \in \mathcal{G}(\mathcal{K}) = p_E^* C, \qquad \text{where } e := \mathrm{hol}^{E}_{a_0^{-1}}(e'). \tag{5.3}$$

Finally, one shall work with $P_1(A) \ltimes \mathcal{E}$, rather than $\mathcal{G}(\mathcal{K}) \rtimes P_1(A)$. Note that both spaces are in bijection by the map $g = ((e', c), a) \mapsto (c, a, e = \mathrm{hol}_a^{-1}(e'))$, so we have to transport the equivalence relation of item (c) above only from $\mathcal{G}(\mathcal{K}) \rtimes P_1(A)$ to $P_1(A) \ltimes \mathcal{E}$. We obtain

$$(c_0, a_0, e_0) \sim (c_1, a_1, e_1) \iff \begin{cases} \exists\, (\sigma : a_0 \Rightarrow a_1) \in P_2(A) : c_1 - c_0 = \mathrm{hol}(\sigma)(e_0), \\ \text{and } e_0 = e_1. \end{cases}$$

As one can see, the result then follows directly from the construction of the semidirect 2-groupoid 2-$\mathcal{P}(A) \ltimes \mathcal{E}$, since there exists a 2-morphism in 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ between (c_0, a_0, e_0) and (c_1, a_1, e_1) precisely if the above condition holds.

The fact that the groupoid structure also descends is a consequence of the construction of 2-$\mathcal{P}(A) \ltimes \mathcal{E}$ as well. Essentially, we transport the concatenation from $P_1(D)$ to $P_1(\mathcal{K}) \rtimes P_1(A)$ obtaining (5.2) then mod out in the first factor to $\mathcal{G}(\mathcal{K}) \rtimes P_1(A)$ and, finally, transport it to $P_1(A) \ltimes \mathcal{E}$, where we recover the composition law of 1-morphisms. □

As we will see in subsections 5.3 and 5.4, Theorem 5.1 can be used in order to integrate $\mathcal{VB}$-algebroids in a rather explicit manner. For now, let us point out the following degenerate cases.

Example 5.3. Assume that C is the zero vector bundle over M. Then necessarily $\omega = 0$, and ∇^E is a flat A-connection. In that case, we recover the Weinstein groupoid of the transformation algebroid $A \ltimes E \to E$.

Example 5.4. Assume that E is the zero vector bundle over M. Then necessarily $\omega = 0$, and ∇^C is a flat A-connection. In that case, we recover the Weinstein groupoid of the semidirect product Lie algebroid $A \ltimes C \to M$.

5.2. **Integral criterium for integrability as the image of a transgression map.** As already mentioned, not every 2-term representation up to homotopy of an integrable Lie algebroid A comes from a 2-term representation up to homotopy of the Weinstein groupoid $\mathcal{G}(A)$. This problem was addressed as an integrability problem for the corresponding $\mathcal{VB}$-algebroid in [10], where the following result was proved (see [10, Thm. 3.22 and equation (3.6)]).

Theorem 5.5 ([10])**.** *Let $(\nabla^E, \nabla^C, \omega)$ be a representation up to homotopy of A on a complex $\mathcal{E}$, and let D denote the corresponding $\mathcal{VB}$-algebroid. Then D is integrable if and only if A is integrable and, for any $[\sigma] \in \pi_2(A)$, $\sigma = adt + bds$, the periods of ω along σ vanish:*

$$\int_0^1\!\!\int_0^1 \mathrm{hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}(x)dtds = 0.$$

Here we recover formula (4.2) for the holonomy applied to A-spheres. In fact, the holonomy along A-spheres has a nice interpretation in terms of Lie algebroid fibrations: it coincides with the transgression map of Theorem 3.6.

Proposition 5.6. *Let $(D; A, E; M)$ be a $\mathcal{VB}$-algebroid. The transgression map*

$$\delta_2 : p_E^*\pi_2(A) \longrightarrow \mathcal{G}(\mathcal{K}) \simeq C \oplus E$$

associated with the underlying fibration $p : D \twoheadrightarrow A$ coincides with the holonomy along A-spheres (4.2) induced by any splitting of D. Namely, for any A-sphere $\sigma \in \pi_2(A, m)$ based at a point $m \in M$ seen as an element of $P_2(A)$, we have

$$\delta_2[\sigma]_e = (\mathrm{hol}(\sigma)(e), e) \qquad \forall e \in p_E^{-1}(m) \subset E.$$

Proof. We shall use consistently the same notations as in the proof of Theorem 5.1. By construction [8, 9], the transgression map δ_2 coincides with the restriction to $p_E^*\pi_2(A)$ of the operator ∂_{ext} defined in (5.3). For an A-sphere σ, we have $\partial_{ext}(\sigma, e) = (\mathrm{hol}(\sigma)(e), e)$ since σ defines a homotopy $\sigma : a_0 \Rightarrow a_1$, where a_0 and a_1 are trivial A-paths, so e' and $e = \mathrm{hol}^E_{a_0^{-1}}(e')$ do coincide. □

Proof of Theorem 5.5. According to Theorem 3.15, the monodromy groups fit into an exact sequence

$$\mathrm{Im}(\delta_2, e) \hookrightarrow \widetilde{\mathcal{N}}(D, e) \twoheadrightarrow \widetilde{\mathcal{N}}(A, m)\ , \tag{5.4}$$

where $\delta_2(\sigma, e) = (\mathrm{hol}(\sigma)(e), e)$ by Proposition 5.6. Assume that there exists an A-sphere σ_0 with $\mathrm{hol}(\sigma_0) \neq 0$, and let $e_0 \in E$ be such that $\mathrm{hol}(\sigma_0)(e_0) \neq 0$. Then we can define a sequence of nontrivial elements in the monodromy groups as follows:

$$\xi_n := \Big(\frac{e_0}{n}, \mathrm{hol}(\sigma_0)\big(\frac{e_0}{n}\big)\Big) \in \mathrm{Im}(\delta_2, \frac{e_0}{n}) \subset \widetilde{\mathcal{N}}(D, \frac{e_0}{n}) \qquad (n \in \mathbb{N}).$$

By the linearity of the application $\mathrm{hol}(\sigma_0)$, we see that (ξ_n) defines a sequence of nontrivial elements of the monodromy groups of D that converges to a trivial element—namely, $0^A_M(m_0)$, where $m_0 := p_E(e_0)$. As a consequence, D cannot be integrable. Furthermore, if D is integrable, then A is also integrable because the zero section $0^D_A : A \to D$ defines a Lie algebroid morphism by the axioms of a $\mathcal{VB}$-algebroid.

Reciprocally, if $\mathrm{hol}(\sigma) = 0$ for any A-sphere σ, then $\mathrm{Im}(\delta_2, e))$ is trivial for any $e \in E$, and it is clear from (5.4) that D is integrable provided that A is. □

5.3. **Integration of type 1 $\mathcal{VB}$-algebroids.** In the next two subsections, we explain how Theorem 5.1 can be effectively applied in order to obtain explicit integrations of $\mathcal{VB}$-algebroids of types 1 and 0. First, we recall the definition of a $\mathcal{VB}$-algebroid of type 1 from [17].

Definition 5.7. A $\mathcal{VB}$-algebroid $(D; A, E; M)$ is said to be of *type* 1 if the core anchor $\partial : C \to E$ is a vector bundle isomorphism.

As explained in [17, sec. 6], given a $\mathcal{VB}$-algebroid $(D; A, E; M)$ of type 1, one may use $\partial : E \to C$ in order to identify E with C. By doing so, it follows from (2.1) that, given a splitting of D, the induced connections ∇^E and ∇^C identify with each other, while ω identifies with the curvature ω_E of ∇^E. Consequently, any $\mathcal{VB}$-algebroid of type 1 identifies with a pull-back Lie algebroid, $D \simeq p_E^! A$. As a consequence, D is integrable if and only if A is integrable, and hence $\mathcal{G}(D)$ can be obtained as a pull-back groupoid. We explain now how this integration can be recovered from Theorem 5.1. For that, consider the 2-term representation up to homotopy $(\nabla^E, \nabla^E, \omega_E)$ associated with a $\mathcal{VB}$-algebroid of type 1, and let $2\text{-}\mathcal{P}(A) \ltimes \mathcal{E}$ be the transformation 2-groupoid defined by the corresponding holonomy 2-representation.

Proposition 5.8. *The* 1*-truncation of* $2\text{-}\mathcal{P}(A) \ltimes \mathcal{E}$ *identifies with the pull-back groupoid* $p_E^!\mathcal{G}(A)$.

Proof. First, one may observe using successively the identification $E \simeq C$ described above together with Lemma A.2, that the holonomy 2-functor is given by

$$\mathrm{hol}(a) = (\mathrm{hol}^E_a, \mathrm{hol}^E_a), \qquad a \in P_1(A). \tag{5.5}$$

$$\mathrm{hol}(\sigma) = \mathrm{hol}^E_{\mathbf{t}_V(\sigma)} - \mathrm{hol}^E_{\mathbf{s}_V(\sigma)}, \qquad \sigma \in P_2(A). \tag{5.6}$$

In particular, since an A-sphere is an A-homotopy between trivial paths, the holonomy along any $\sigma \in \pi_2(A)$ is trivial. It follows that the transgression map of Proposition 5.6 vanishes, and we recover the fact that D is integrable as a consequence of Theorem 5.5.

Finally, the fact that D integrates to a pull-back groupoid can be obtained by Theorem 5.1 in the following way. By the definitions, two 1-morphisms (c_0, a_0, x_0) and (c_1, a_1, x_1) in $P_1(A) \ltimes \mathcal{E}$ are joined by a 2-morphism if and only if the following conditions hold: $x_0 = x_1$, and there exists an A-homotopy $\sigma : a_0 \Rightarrow a_1$ such that $c_0 = c_1 + \mathrm{hol}(\sigma)(x_0)$.

Taking the identification $E \simeq C$ and formula (5.6) into account, it follows that the quotient space of 1-morphisms by 2-morphisms identifies with $p_E^!\mathcal{G}(A)$ by the map

$$\begin{aligned} P_1(A) \ltimes \mathcal{E}/_{\sim} &\longrightarrow \quad p_E^!\mathcal{G}(A), \\ [c, a, e] \quad &\longmapsto (c + \mathrm{hol}_a^E(e), [a], e). \end{aligned}$$

Then it is easily checked that the groupoid structure on $P_1(A) \ltimes \mathcal{E} \rightrightarrows E$ descends to the pull-back groupoid structure on $p_E^!\mathcal{G}(A)$. □

In order to illustrate how the integration of a Lie algebroid 2-term representation up to homotopy by a 2-functor differs from a 2-term representation up to homotopy of the corresponding Weinstein groupoid, which may have been expected as the natural integrating structure, let us take a look at an example proposed in [18, ex. 2.6].

Example 5.9. Consider the following $\mathcal{VB}$-algebroid:

$$\begin{array}{ccc} TTS^2 & \xrightarrow{dp_{TS^2}} & TS^2 \\ {\scriptstyle p_{TTS^2}}\downarrow & & \downarrow{\scriptstyle p_{TS^2}} \\ TS^2 & \xrightarrow{p_{TS^2}} & S^2, \end{array}$$

where $p_{TS^2} : TS^2 \to S^2$ and $p_{TTS^2} : TTS^2 \to TS^2$ denote the natural projection, and dp_{TS^2} the differential of p_{TS^2}. In that case, we have $\mathcal{E} = (\mathrm{id} : TS^2 \to TS^2)$; therefore, the $\mathcal{VB}$-algebroid D is of type 1 and integrates to a pull-back groupoid as follows:

$$\begin{array}{ccc} TS^2 \times TS^2 & \longrightarrow & S^2 \times S^2 \\ \downarrow\downarrow & & \downarrow\downarrow \\ S^2 \times S^2 & \longrightarrow & S^2, \end{array}$$

where both $TS^2 \times TS^2 \rightrightarrows TS^2$ and $S^2 \times S^2 \to S^2$ are pair groupoids.

A splitting of D gives rise to a linear connection ∇^E on E with curvature ω_E. The corresponding 2-term representation up to homotopy is given by $(\nabla^E, \nabla^E, \omega_E)$. As explained above, the integrating 2-functor is simply given by $\mathrm{hol}(\dot\gamma) = (\mathrm{hol}^E_\gamma, \mathrm{hol}^E_\gamma)$ and $\mathrm{hol}(\sigma) = \mathrm{hol}^E_{\gamma_0} - \mathrm{hol}^E_{\gamma_1}$.

Remark 5.10. Comparing Example 5.9 with [18, Ex. 2.6], one may notice that the holonomy 2-functor that we obtain is entirely determined by the infinitesimal data $(\nabla^E, \nabla^E, \omega_E)$, while the construction of an explicit 2-term representation up to homotopy of $S^2 \times S^2 \rightrightarrows S^2$ requires further choices (a Riemannian metric, for instance, as in [18]).

In fact, although the choice of a right-horizontal lift can be differentiated to a 2-term representation up to homotopy of the underlying Lie algebroid, the converse is *not* true. Namely, the choice of a splitting of a $\mathcal{VB}$-algebroid does not integrate to a right-horizontal lift, at least not without involving further choices. It follows that, given a 2-term representation up to homotopy of a Lie algebroid A whose corresponding $\mathcal{VB}$-algebroid is assumed to be integrable, we still *do not* obtain a representation up to homotopy of $\mathcal{G}(A)$ in a canonical way.

Remark 5.11. There is another point that we believe is worth clarifying, which is the following. A 2-term representation up to homotopy of a Lie groupoid, say, $\mathcal{G}(A)$, involves maps $\Delta^E : \mathcal{G}(A) \to \mathrm{Hom}(E, E)$ and $\Delta^C : \mathcal{G}(A) \to \mathrm{Hom}(C, C)$, usually thought of as the holonomy in $\mathcal{E}$ along elements of $\mathcal{G}(A)$.

As emphasized in [18, Ex. 2.6], in order to cover a general enough notion of 2-term representation up to homotopy for Lie groupoids, one needs to allow Δ^E and Δ^C to take values in *noninvertible* homomorphisms.

This may be quite confusing in view of Theorem 4.19, since not only do $\mathrm{hol}^E : P_1(A) \to \mathrm{Hom}(E, E)$ and $\mathrm{hol}^C : P_1(A) \to \mathrm{Hom}(C, C)$ *not* descend to maps $\mathcal{G}(A) \to \mathrm{Hom}(E, E)$ and $\mathcal{G}(A) \to \mathrm{Hom}(C, C)$, but hol^E_a and hol^C_a *always* define invertible maps.

Again, the apparent contradiction comes from the fact that the choice of a splitting of D is not enough to induce a right-horizontal lift of $\mathcal{G}(D)$. As a consequence, the corresponding quasi action (Δ^E, Δ^C) is not entirely determined by the infinitesimal data (∇^E, ∇^C). It is not hard to see, however, that in general, $(\mathrm{hol}^E_a, \mathrm{hol}^C_a)$ and $(\Delta^E_{[a]}, \Delta^C_{[a]})$ coincide up to a chain homotopy.

5.4. **Integration of type 0 $\mathcal{VB}$-algebroids.** We now explain how to integrate a $\mathcal{VB}$-algebroid $(D; A, E; M)$ of type 0.

Definition 5.12 ([17])**.** A $\mathcal{VB}$-algebroid $(D; A, E; M)$ is said to be of *type* 0 if the core anchor $\partial : C \to E$ vanishes.

Given a splitting of a $\mathcal{VB}$-algebroid of type 0, it follows from the axioms that the associated A-connections ∇^E, ∇^C have vanishing curvatures. Hence, both ∇^E and ∇^C are representations of A. Furthermore, ω defines a 2-cocycle with values in the induced representation of A on $\mathrm{Hom}(E, C)$. Note that ∇^E, ∇^C are canonical in the sense that they are independent of the choice of a splitting of D, while only the induced class $[\omega] \in H^2(A, \mathrm{Hom}(E, C))$ depends on this choice. See [17] for more details.

Since both E and C are honest representations, it makes sense to integrate ω along any A-sphere σ by usual integrals. Note that the resulting integral, called *period along* σ, depends only on the cohomology class $[\omega] \in H^2(A, \mathrm{Hom}(E, C))$ and the homotopy class of σ in $\pi_2(A)$. For the sake of simplicity, we shall assume that E and C are trivial vector bundles $E = M \times E_0$ and $C = M \times C_0$ on which A acts trivially. Then for any A-path, the holonomy in E and C is the identity. That is,

$$\mathrm{hol}(a) \simeq (\mathrm{id}_{E_0}, \mathrm{id}_{C_0}).$$

This simplifies formula (4.2) for the holonomy along 2-morphisms, justifying the notation

$$\mathrm{hol}(\sigma) = \iint_\sigma \omega.$$

The integration of a type 0 $\mathcal{VB}$-algebroid can then be summarized as follows.

Proposition 5.13. *Let $(D; A, E; M)$ be a $\mathcal{VB}$-algebroid of type 0, where both E and C are trivial representations, $E = M \times E_0$ and $C = M \times C_0$, of A. Then the following assertions hold:*

(i) *D is an integrable Lie algebroid if and only if A is integrable and, for any $[\sigma] \in \pi_2(A)$, the periods of ω along $[\sigma]$ vanish.*
(ii) *the (possibly topological) Weinstein groupoid $\mathcal{G}(D)$ identifies with the quotient of $C_0 \times P_1(A) \times E_0$ by the following equivalence relation:*

$$(c_0, a_0, e) \sim (c_1, a_1, e) \iff \begin{cases} \text{there exists an } A\text{-homotopy } \sigma : a_0 \Rightarrow a_1 \text{ such that} \\ \qquad c_1 - c_0 = \left(\iint_\sigma \omega\right)(e) \end{cases}.$$

Proof. Part (i) follows from Theorem 5.5, while the explicit description of the Weinstein groupoid $\mathcal{G}(D)$ is a direct application of Theorem 5.1. □

Remark 5.14. Notice the analogy with the construction of the groupoid integrating a prequantization Lie algebroid given by Crainic (see [13, Remark 3.3], and see also [15]). In fact, for any $\mathcal{VB}$-algebroid $(D; A, E; M)$ of type 0, D can be obtained as a *central* extension in the following way.

First, we denote by $A \ltimes E \to E$ the transformation Lie algebroid. As a general fact [8, Ex. 2.15], $A \ltimes E$ fits into a fibration $A \ltimes E \twoheadrightarrow A$ whose kernel is the trivial Lie algebroid over E. Next, observe that there is an obvious representation of $A \ltimes E \to E$ on $p_E^* C \to E$ induced by that of A on C by pull-back. Furthermore, $\omega \in \Omega^2(A \ltimes E, C)$ can be seen as a 2-cocycle on $A \ltimes E$ with values in $p_E^* C$ by using the obvious inclusion $\Omega^2(A, \mathrm{Hom}(E, C)) \subset \Omega^2(A \ltimes E, p_E^* C)$.

Then it is easily seen from the brackets that D identifies with a central extension $D \simeq (A \ltimes E) \ltimes_\omega p_E^* C$ with twisting cocycle ω, so D fits into the following extension:

$$\begin{array}{ccccc} p_E^* C & \xhookrightarrow{\ i\ } & D & \xrightarrow{\ p\ } & A \ltimes E \\ \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{\mathrm{id}_E} & E & \xrightarrow{\mathrm{id}_E} & E. \end{array} \tag{5.7}$$

Note that, in this exact sequence, all Lie algebroids are over the same base.

Example 5.15. There is a 2-representation up to homotopy that one can associate with any finite dimensional Lie algebra $\mathfrak{g}$, as was proposed by Sheng and Zhu [25] in relation to string 2-algebras. The construction goes as follows.

Consider the 2-term complex with trivial boundary $\mathcal{E} := (C := \mathbb{R} \xrightarrow{0} E := \mathfrak{g}^*)$ and the 2-term representation up to homotopy of $\mathfrak{g}$ on $\mathcal{E}$ where

- $\nabla^E := \mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ is the coadjoint representation;
- $\nabla^C := 0 : \mathfrak{g} \to \mathrm{End}(\mathbb{R})$ is the trivial representation;
- $\omega := [\ ,\ ]_\mathfrak{g}$ is given as the Lie algebra bracket on $\mathfrak{g}$, seen as an element in $\bigwedge^2 \mathfrak{g}^* \otimes \mathrm{Hom}(\mathfrak{g}^*, \mathbb{R})$.

The associated $\mathcal{VB}$-algebroid is given by $D = \mathfrak{g} \times \mathfrak{g}^* \times \mathbb{R}$, and it fits into a Lie algebroid fibration:

$$\begin{array}{ccccc} \mathbb{R} \times \mathfrak{g}^* & \hookrightarrow & \mathfrak{g} \times \mathfrak{g}^* \times \mathbb{R} & \twoheadrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* & \longrightarrow & \{*\}. \end{array}$$

Here D is indeed of type 0, so the kernel $\mathcal{K} = \mathfrak{g}^* \times \mathbb{R}$ of the fibration is simply a bundle of abelian Lie algebras over $\mathfrak{g}^*$.

The central extension (5.7) of Remark 5.14 takes a particularly interesting form in this example, as we now explain. In order to put things into context, recall that, given a Poisson manifold (M, π), one usually considers the Lie algebroid structure on T^*M. However, one might see (M, π) as a Jacobi manifold [15] as well. This amounts to seeing π as a 2-cocycle on T^*M with values in the trivial representation $M \times \mathbb{R}$. Then the Lie algebroid structure of (M, π), seen as a Jacobi manifold, coincides with the corresponding extension class. More explicitly, it is defined on $T^*M \times \mathbb{R}$ with anchor $\rho(\alpha, f) = \pi^\sharp(\alpha)$ and bracket

$$\big[(\alpha, f), (\beta, g)\big]_{T^*M \oplus \mathbb{R}} := \big([\alpha, \beta]_{T^*M}, \mathcal{L}_{\pi^\sharp\alpha} g - \mathcal{L}_{\pi^\sharp\beta} f + \pi(\alpha, \beta)\big),$$

where $[\ ,\ ]_{T^*M}$ denotes the standard Lie algebroid bracket on T^*M induced by π. We obtain this way an extension of the form $M \times \mathbb{R} \hookrightarrow T^*M \times \mathbb{R} \twoheadrightarrow T^*M$, where all of the Lie algebroids are over the same base M. In the case of a linear Poisson structure $M = \mathfrak{g}^*$, one easily checks from the definitions that the Lie algebroid structure on $T^*M \times \mathbb{R}$ coincides with $D = \mathfrak{g} \times \mathfrak{g}^* \times \mathbb{R}$, and we recover the diagram (5.7) as follows:

$$\begin{array}{ccccc} \mathbb{R} \times \mathfrak{g}^* & \hookrightarrow & \mathfrak{g} \times \mathfrak{g}^* \times \mathbb{R} & \twoheadrightarrow & \mathfrak{g} \times \mathfrak{g}^* \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^*. \end{array}$$

Finally, note that by applying Proposition 5.13 to D, we recover a result by Cranic and Zhu [15, Thm. 4, Ex. 4] describing the integration of $\mathfrak{g}^*$ as a Jacobi manifold.

Appendix A. Proof of Theorem 4.19

The proof will be divided in several lemmas. In particular, the main functoriality properties are established in Lemmas A.6 and A.7. The fact that the holonomy is well defined, independently of the thin homotopy class of an A-path and of the higher homotopy class of an A-homotopy, follows from Lemmas A.4 and A.8.

Lemma A.1. *For any A-path $a : TI \to A$, $\mathrm{hol}(a) = (\mathrm{hol}^E_a, \mathrm{hol}^C_a)$ is a chain homotopy.*

Proof. We have to show that $\mathrm{hol}^E_a \circ \partial = \partial \circ \mathrm{hol}^C_a$, which obviously follows by integrating the relation $\nabla^E \circ \partial = \partial \circ \nabla^C$. □

We need to introduce some notation. Given an A-homotopy $adt + bds : TI^2 \to A$, we shall denote by γ_t^s, rather than $\gamma(t,s)$, the base map. Similarly, for any $t, t' \in [0,1]$, we will denote by $a^s_{t',t} : T[t,t'] \to A$ the A-path $a|_{[t',t]\times\{s\}}dt$, while

$$\begin{aligned} \mathrm{hol}^E_{a^s_{t',t}} &: E_{\gamma^s_t} \to E_{\gamma^s_{t'}}, \\ \mathrm{hol}^C_{a^s_{t',t}} &: C_{\gamma^s_t} \to C_{\gamma^s_{t'}} \end{aligned}$$

will denote their corresponding holonomies.

Lemma A.2. *For an arbitrary A-connection ∇^E on a vector bundle $E \to M$ with curvature $\omega_E \in \Omega^2(A, \mathrm{Hom}(E,E))$, and any A-homotopy $adt + bds : TI^2 \to A$, the parallel transport and the curvature are related as follows:*

$$\frac{d}{ds}\mathrm{hol}^E_{a^s_{1,0}}(x) = \int_0^1 \mathrm{hol}^E_{a^s_{1,t}} \circ \omega_E(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}(x)dt. \tag{A.1}$$

A detailed proof of Lemma A.2 can be found in [23] in the case of usual linear connections (i.e., TM-connection). The case of an arbitrary A-connection follows by pulling back the A-connection along morphisms $TI^2 \to A$.

Remark A.3. Notice the global nature of equation (A.1), as opposed to the local definition of the curvature $\omega_E = [\nabla^E, \nabla^E] - \nabla^E_{[\ ,\ ]}$. It seems not to be a very popular equation,[1] although it goes back to the work of Nijenhuis [23] and is a fundamental one for our purposes.

Lemma A.4. *The holonomy along an A-path depends only on its thin homotopy class.*

Proof. Given a thin homotopy $adt + bds : TI^2 \to A$, the term $\omega_E(\alpha,\beta)$ on the right-hand side of equation (A.1) vanishes. It follows that $\mathrm{hol}^E_{a^s_{1,0}}$ is independent of s, and the same for $\mathrm{hol}^C_{a^s_{1,0}}$. We conclude that $\mathrm{hol}(a) = (\mathrm{hol}^E_a, \mathrm{hol}^C_a)$ depends only on the thin homotopy class of a. ☐

In order to keep simple notation, in the sequel, we denote by a_0 and a_1 the paths $a^{s=0}_{1,0}$ and $a^{s=1}_{1,0}$ (notice that the lower index in a_0 and a_1 then refers to the s variable). This way, $\sigma = adt + bds$ is an A-homotopy $\sigma : a_0 \Rightarrow a_1$.

Lemma A.5. *Given an A-homotopy $\sigma : a_0 \Rightarrow a_1$,* $\mathrm{hol}(\sigma)$ *defines a chain homotopy* $\mathrm{hol}(a_0) \Rightarrow \mathrm{hol}(a_1)$ *in* $2\text{-}\mathrm{Gau}(C \xrightarrow{\partial} E)$.

Proof. By using successively Lemma A.1, equations (2.2) and (2.3), and then Lemma A.2, we easily compute

$$\begin{aligned} \int_0^1 \mathrm{hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}\, dt \circ \partial &= \frac{d}{ds}\mathrm{hol}^C_{a^s_{1,0}}, \\ \partial \circ \int_0^1 \mathrm{hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}\, dt &= \frac{d}{ds}\mathrm{hol}^E_{a^s_{1,0}}. \end{aligned}$$

[1] See the comment in [7, sec. 15.4.1].

The result then follows by integrating along the s variable:

$$\begin{aligned}\mathrm{hol}(\sigma)\circ\partial &= \mathrm{hol}^C_{a_1} - \mathrm{hol}^C_{a_0},\\ \partial\circ\mathrm{hol}(\sigma) &= \mathrm{hol}^E_{a_1} - \mathrm{hol}^E_{a_0}.\end{aligned}$$

Note that this lemma was proved for $A = TM$ in [1, Prop. 3.13], where it is stated in a slightly different way. ☐

Lemma A.6. *The holonomy commutes with the vertical inverse and composition of* 2*-morphisms.*

Proof. Consider two A-homotopies $a_0 \overset{\sigma}{\Rightarrow} a_1 \overset{\sigma'}{\Rightarrow} a_2$. Recall that the vertical composition is given by concatenation along the s variable. It follows from the additivity of the integral that

$$\begin{aligned}\mathrm{hol}\big(\sigma \underset{\mathbf{V}}{\bullet} \sigma'\big) &= \int_0^1\!\!\int_0^1 \mathrm{hol}^C_{a^s_{1,t}}\circ\omega(a,b)_{\gamma^s_t}\circ\mathrm{hol}^E_{a^s_{t,0}}\,dtds\\ &\quad+\int_0^1\!\!\int_0^1 \mathrm{hol}^C_{a'^s_{1,t}}\circ\omega(a',b')_{\gamma'^s_t}\circ\mathrm{hol}^E_{a'^s_{t,0}}\,dtds = \mathrm{hol}(\sigma)\underset{\mathbf{V}}{\bullet}\mathrm{hol}(\sigma').\end{aligned}$$

The fact that the holonomy commutes with the vertical inverse is obvious. ☐

Lemma A.7. *The holonomy commutes with the horizontal inverse and composition of* 2*-morphisms.*

Proof. Given $a_0 \overset{\sigma}{\Rightarrow} a_1$ and $a'_0 \overset{\sigma'}{\Rightarrow} a'_1$ (where a_0 and a'_0 (resp., a_1 and a'_1) are composable paths), the horizontal composition is obtained by concatenation along the t coordinate.

Using the additivity of the integral, we see that

$$\begin{aligned}\mathrm{hol}\big(h \underset{\mathbf{H}}{\bullet} h'\big) &= \underbrace{\int_0^1\!\!\int_0^1 \mathrm{hol}^C_{a'^s_{1,0}}\circ\mathrm{hol}^C_{a^s_{1,t}}\circ\omega(a,b)_{\gamma^s_t}\circ\mathrm{hol}^E_{a^s_{t,0}}\,dtds}_{=:\,A}\\ &\quad+\underbrace{\int_0^1\!\!\int_0^1 \mathrm{hol}^C_{a'^{s'}_{1,t'}}\circ\omega(a',b')_{\gamma^{s'}_{t'}}\circ\mathrm{hol}^E_{a'^{s'}_{t',0}}\circ\mathrm{hol}^E_{a^{s'}_{1,0}}\,dt'ds'.}_{=:\,B}\end{aligned}$$

In the term A, we substitute the following relation:

$$\begin{aligned}\mathrm{hol}^C_{a'^s_{1,0}} &= \mathrm{hol}^C_{a'^{s=0}_{1,0}} + \int_0^s\!\!\int_0^1 \mathrm{hol}^C_{a'^{s'}_{1,t'}}\circ\omega_C(a',b')\circ\mathrm{hol}^C_{a'^{s'}_{t',0}}\,dt'ds'\\ &= \mathrm{hol}^C_{a'^{s=0}_{1,0}} + \int_0^s\!\!\int_0^1 \mathrm{hol}^C_{a'^{s'}_{1,t'}}\circ\omega(a',b')\circ\mathrm{hol}^E_{a'^{s'}_{t',0}}\,dt'ds'\circ\partial.\end{aligned}$$

We obtain

$$\begin{aligned}
A =& \int_0^1 \int_0^1 \mathrm{hol}^C_{a'^{s=0}_{1,0}} \circ \mathrm{hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}\, dtds \\
&+ \int_0^1 \int_0^1 \Big(\int_0^s \int_0^1 \mathrm{hol}^C_{a'^{s'}_{1,t'}} \circ \omega(a',b') \circ \mathrm{hol}^E_{a'^{s'}_{t',0}}\, dt'ds' \Big) \\
&\quad \circ \partial \circ \mathrm{hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}\, dtds \\
=& \mathrm{hol}^C_{a'^{s=0}_{1,0}} \circ \int_0^1 \int_0^1 \mathrm{hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}\, dtds \\
&+ \int_0^1 \int_0^1 \mathrm{hol}^C_{a'^{s'}_{1,t'}} \circ \omega(a',b')_{\gamma^{s'}_{t'}} \circ \mathrm{hol}^E_{a'^{s'}_{t',0}} \circ \\
&\quad \Big(\int_{s'}^1 \int_0^1 \mathrm{hol}^E_{a^s_{1,t}} \circ \omega_E(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}\, dtds \Big) dt'ds' \\
=& \mathrm{hol}^C_{a'^{s=0}_{1,0}} \circ \int_0^1 \int_0^1 \mathrm{hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathrm{hol}^E_{a^s_{t,0}}\, dtds \\
&+ \Big(\int_0^1 \int_0^1 \mathrm{hol}^C_{a'^{s'}_{1,t'}} \circ \omega(a',b')_{\gamma^{s'}_{t'}} \circ \mathrm{hol}^E_{a'^{s'}_{t',0}}\, dt'ds' \Big) \circ \mathrm{hol}^E_{a^{s=1}_{1,0}} \\
&- \int_0^1 \int_0^1 \mathrm{hol}^C_{a'^{s'}_{1,t'}} \circ \omega(a',b')_{\gamma^{s'}_{t'}} \circ \mathrm{hol}^E_{a'^{s'}_{t',0}} \circ \mathrm{hol}^E_{a^{s=s'}_{1,0}}\, dt'ds' \\
=& \mathrm{hol}^C(a'_0) \circ \mathrm{hol}(\sigma) + \mathrm{hol}(\sigma') \circ \mathrm{hol}(a_1) - B.
\end{aligned}$$

We conclude that $\mathrm{hol}(\sigma' \underset{\mathbf{H}}{\bullet} \sigma) = \mathrm{hol}^C(a'_0) \circ \mathrm{hol}(\sigma) + \mathrm{hol}(\sigma') \circ \mathrm{hol}^E(a_1) = \mathrm{hol}(\sigma') \underset{\mathbf{H}}{\bullet} \mathrm{hol}(\sigma)$. We leave it to the reader to check that hol commutes with the horizontal inverses. □

Lemma A.8. $\mathrm{hol}(\sigma)$ *is independent of the* 3*-homotopy class of* σ.

Proof. Given a homotopy $H = adt + bds + cdu : TI^3 \to A$, we denote by $\gamma^{s,u}_t$ rather than $\gamma(t,s,u)$ the base path. Then σ_u will denote, respectively, the A-homotopy $\sigma_u := (adt + bds)|_{I^2 \times \{u\}}$ so that H is a homotopy $H : \sigma_0 \Rightarrow \sigma_1$. Also, $a^{s,u}_{t',t}$ will refer to the path $a|_{[t,t'] \times \{s\} \times \{u\}} dt$, while

$$\begin{aligned}
\mathrm{hol}^E_{a^{s,u}_{t',t}} &: E_{\gamma^{s,u}_t} \to E_{\gamma^{s,u}_{t'}}, \\
\mathrm{hol}^C_{a^{s,u}_{t,0}} &: C_{\gamma^{s,u}_t} \to C_{\gamma^{s,u}_{t'}}
\end{aligned}$$

will denote the corresponding holonomies. We will show that $\mathrm{hol}(\sigma_u)$ is independent of u, which is an immediate consequence of (2.4). Indeed, we compute

$$\begin{aligned}
\frac{d}{du} \mathrm{hol}(\sigma_u) &= \int_0^1 \int_0^1 \frac{d}{du} \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \omega(a,b)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds \\
&= \int_0^1 \int_0^1 \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \nabla_u\, \omega(a,b)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds
\end{aligned}$$

$$= -\underbrace{\int_0^1\int_0^1 \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \nabla_a\, \omega(b,c)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds}_{=:A}$$
$$- \underbrace{\int_0^1\int_0^1 \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \nabla_b\, \omega(c,a)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds.}_{=:B}$$

On the right-hand side, we have

$$\begin{aligned} B &= \int_0^1\int_0^1 \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \nabla_b\, \omega(c,a)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds \\ &= \int_0^1\int_0^1 \frac{d}{ds} \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \omega(c,a)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds \\ &= \int_0^1 \Big(\mathrm{hol}^C_{a^{1,u}_{1,t}} \circ \omega(c,a)_{\gamma^{1,u}_t} \circ \mathrm{hol}^E_{a^{1,u}_{t,0}} - \mathrm{hol}^C_{a^{0,u}_{1,t}} \circ \omega(c,a)_{\gamma^{0,u}_t} \circ \mathrm{hol}^E_{a^{0,u}_{t,0}}\Big) dt = 0. \end{aligned}$$

Also,

$$\begin{aligned} A &= \int_0^1\int_0^1 \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \mathrm{hol}^C_{b^{u,t}_{s,1}} \circ \Big(\mathrm{hol}^C_{b^{u,t}_{1,s}} \circ \nabla_a\, \omega(b,c)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{b^{u,t}_{s,0}}\Big) \circ \mathrm{hol}^E_{b^{u,t}_{0,s}} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds \\ &= \int_0^1\int_0^1 \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \mathrm{hol}^C_{b^{u,t}_{s,1}} \circ \Big(\frac{d}{dt} \mathrm{hol}^C_{b^{u,t}_{1,s}} \circ \omega(b,c)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{b^{u,t}_{s,0}}\Big) \circ \mathrm{hol}^E_{b^{u,t}_{0,s}} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds \\ &= \int_0^1\int_0^1 \frac{d}{dt} \mathrm{hol}^C_{a^{s,u}_{1,t}} \circ \omega(b,c)_{\gamma^{s,u}_t} \circ \mathrm{hol}^E_{a^{s,u}_{t,0}}\, dtds \\ &= \int_0^1 \Big(\omega(b,c)_{\gamma^{s,u}_1} \circ \mathrm{hol}^E_{a^{s,u}_{1,0}} - \mathrm{hol}^C_{a^{s,u}_{1,0}} \circ \omega(b,c)_{\gamma^{s,u}_0}\Big) ds = 0, \end{aligned}$$

which completes the proof. □

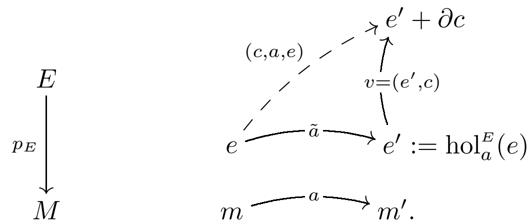
Appendix B. A geometric description of the semidirect product

It is possible to illustrate geometrically the construction of the transformation 2-groupoid $\mathcal{P}_2(A) \ltimes \mathcal{E}$ by a series of explicit diagrams. Although essentially informal, these diagrams usually offer guidance in the understanding of the algebraic construction.

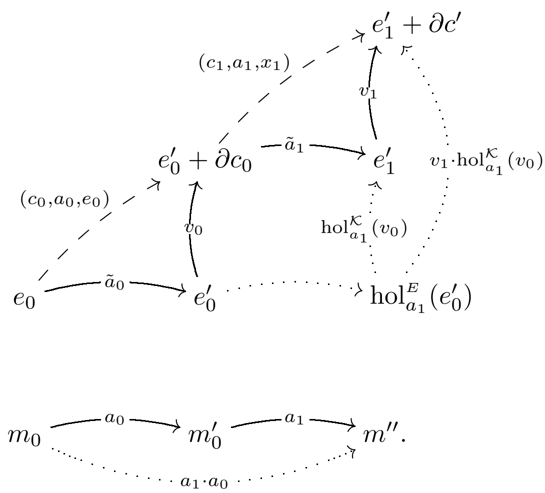
In order to explain this, we shall use systematically the following notation: given a 1-morphism $(c, a, e) \in \mathcal{P}_1(A) \ltimes \mathcal{E}$, we denote by e' and v the following elements: $e' := \mathrm{hol}^E_a(e) \in E$ and $v := (c, e') \in \mathcal{G}(\mathcal{K}) = E \oplus C$ (and we proceed similarly when indices are involved).

The geometric illustration goes as follows: in a 1-morphism (c, a, e), the couple (a, e) should be thought of as the horizontal lift $\tilde{a}$ of a with source e. Hence, (a, e) has target e'. Then (c, a, e) behaves like the composition of $\tilde{a}$ with v. In other words, one may think of (c, a, e) as a "formal concatenation" $(c, a, e) = v \cdot \tilde{a}$. This can be pictured as follows:

(B.1)



Using the above diagram as a guidance, we easily recover the source and target maps on the space of 1-morphisms. One can also recover the composition of 1-morphisms by a concatenation process, as follows:



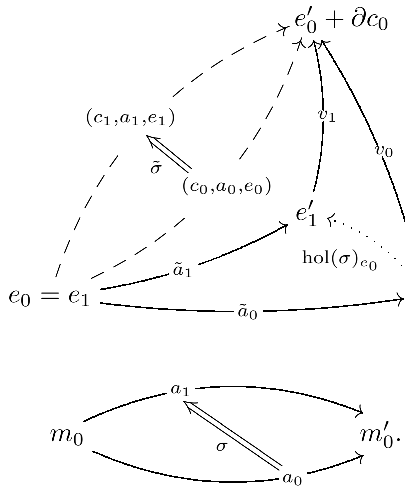
Then the inversion of 1-morphisms can be recovered in an easy manner.

In order to illustrate the space of 2-morphisms, given a 2-morphism $(c_0, \sigma, e_0) \in P_2(A) \ltimes \mathcal{E} = C \ {}_{p_C}\!\times_{\mathbf{t_H}} P_2(A) \ {}_{\mathbf{s_H}}\!\times_{p_E} E$, we shall use the following notation:

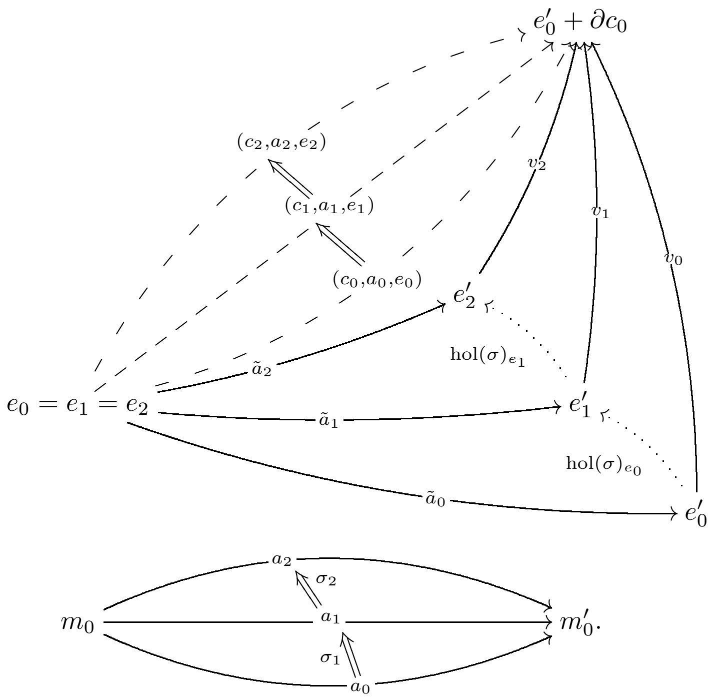
- $a_0 := \mathbf{s_V}(\sigma) \in P_1(A)$ denotes the source of σ;
- $v_0 := (e'_0, c_0) \in E \oplus C$, where $e'_0 := \mathrm{hol}^E_a(e_0)$;
- $a_1 := \mathbf{t_V}(\sigma) \in P_1(A)$ denotes the target of σ;
- $v_1 := (e'_1, c_1) \in E \oplus C$, where $e'_1 := \mathrm{hol}^E_a(e_1)$ and $e_1 := e_0$.

The idea lying behind 2-morphisms is the following. First, we lift a_0, a_1 into D-paths $\tilde{a}_0, \tilde{a}_1$, starting at a same point $e_0 \in E$. Although a_0 and a_1 are homotopic A-paths, because of the presence of curvature, their horizontal lifts $\tilde{a}_0, \tilde{a}_1$ are not homotopic in D. Since this is precisely what $\mathrm{hol}(\sigma)$ measures, we obtain $(c_0, h, e_0) =: \tilde{\sigma}$ as a formal homotopy between $v_0 \cdot a_0 = (c_0, a_0, e_0)$ and $v_1 \cdot a_1 = (c_1, a_1, e_0)$ provided that we set $c_1 = c_0 - \mathrm{hol}(\sigma)_{e_0}$, as suggested by the following diagram:

(B.2)



In this way, we easily recover the source and target maps on the space of 2-morphisms. Then the vertical composition of 2-morphisms can be depicted by a concatenation process:



We leave it to the reader to figure out the horizontal composition.

Acknowledgments

The authors would like to thank Matias del Hoyo and Pedro Frejlich for their comments in the early stages of this work.

References

[1] C. A. Abad and M. Crainic, *Representations up to homotopy of Lie algebroids*, J. Reine Angew. Math. **663** (2012), 91–126, DOI 10.1515/CRELLE.2011.095. MR2889707

[2] C. Arias Abad and M. Crainic, *Representations up to homotopy and Bott's spectral sequence for Lie groupoids*, Adv. Math. **248** (2013), 416–452, DOI 10.1016/j.aim.2012.12.022. MR3107517

[3] C. Arias Abad and F. Schätz, *Deformations of Lie brackets and representations up to homotopy*, Indag. Math. (N.S.) **22** (2011), no. 1-2, 27–54, DOI 10.1016/j.indag.2011.07.003. MR2853613

[4] C. Arias Abad and F. Schätz, *The A_∞ de Rham theorem and integration of representations up to homotopy*, Int. Math. Res. Not. IMRN **16** (2013), 3790–3855, DOI 10.1093/imrn/rns166. MR3090711

[5] C. Arias Abad and F. Schätz, *Higher holonomies: Comparing two constructions*, Differential Geom. Appl. **40** (2015), 14–42, DOI 10.1016/j.difgeo.2015.02.003. MR3333093

[6] J. C. Baez and A. S. Crans, *Higher-dimensional algebra. VI. Lie 2-algebras*, Theory Appl. Categ. **12** (2004), 492–538. MR2068522

[7] M. Berger, *A panoramic view of Riemannian geometry*, Springer-Verlag, Berlin, 2003. MR2002701

[8] O. Brahic, *Extensions of Lie brackets*, J. Geom. Phys. **60** (2010), no. 2, 352–374, DOI 10.1016/j.geomphys.2009.10.006. MR2587399

[9] O. Brahic and C. Zhu, *Lie algebroid fibrations*, Adv. Math. **226** (2011), no. 4, 3105–3135, DOI 10.1016/j.aim.2010.10.006. MR2764883

[10] O. Brahic, A. Cabrera, and C. Ortiz, *Obstructions to the integrability of $\mathcal{VB}$-algebroids*, arXiv:1403.1990 (2014).

[11] H. Bursztyn, A. Cabrera, and M. del Hoyo, *Vector bundles over Lie groupoids and algebroids*, Adv. Math. **290** (2016), 163–207, DOI 10.1016/j.aim.2015.11.044. MR3451921

[12] A. Coste, P. Dazord, and A. Weinstein, *Groupoïdes symplectiques* (French), Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2, Publ. Dép. Math. Nouvelle Sér. A, vol. 87, Univ. Claude-Bernard, Lyon, 1987, pp. i-ii, 1–62. MR996653

[13] M. Crainic, *Prequantization and Lie brackets*, J. Symplectic Geom. **2** (2004), no. 4, 579–602. MR2197220

[14] M. Crainic and R. L. Fernandes, *Integrability of Lie brackets*, Ann. of Math. (2) **157** (2003), no. 2, 575–620, DOI 10.4007/annals.2003.157.575. MR1973056

[15] M. Crainic and C. Zhu, *Integrability of Jacobi and Poisson structures* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **57** (2007), no. 4, 1181–1216. MR2339329

[16] T. Drummond, M. Jotz Lean, and C. Ortiz, *$\mathcal{VB}$-algebroid morphisms and representations up to homotopy*, Differential Geom. Appl. **40** (2015), 332–357, DOI 10.1016/j.difgeo.2015.03.005. MR3333112

[17] A. Gracia-Saz and R. A. Mehta, *Lie algebroid structures on double vector bundles and representation theory of Lie algebroids*, Adv. Math. **223** (2010), no. 4, 1236–1275, DOI 10.1016/j.aim.2009.09.010. MR2581370

[18] A. Gracia-Saz and R. A. Mehta, *$\mathcal{VB}$-groupoids and representation theory of Lie groupoids*, J. Symplectic Geom. **15** (2017), no. 3, 741–783, DOI 10.4310/JSG.2017.v15.n3.a5. MR3696590

[19] K. A. Hardie, K. H. Kamps, and R. W. Kieboom, *A homotopy 2-groupoid of a Hausdorff space*, Appl. Categ. Structures **8** (2000), no. 1-2, 209–234, DOI 10.1023/A:1008758412196. Papers in honour of Bernhard Banaschewski (Cape Town, 1996). MR1785844

[20] K. Konieczna and P. Urbański, *Double vector bundles and duality*, Arch. Math. (Brno) **35** (1999), no. 1, 59–95. MR1684522

[21] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, England, 2005. MR2157566

[22] J. F. Martins and R. F. Picken, *On two-dimensional holonomy*, Trans. Amer. Math. Soc. **362** (2010), 5657–5695.

[23] A. Nijenhuis, *On the holonomy groups of linear connections. IA, IB. General properties of affine connections*, Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15** (1953), 233–240, 241–249. MR0063737

[24] J. Pradines, *Représentation des jets non holonomes par des morphismes vectoriels doubles soudés* (French), C. R. Acad. Sci. Paris Sér. A **278** (1974), 1523–1526. MR0388432

[25] Y. Sheng and C. Zhu, *Semidirect products of representations up to homotopy*, Pacific J. Math. **249** (2011), no. 1, 211–236, DOI 10.2140/pjm.2011.249.211. MR2764948

[26] Y. Sheng and C. Zhu, *Integration of semidirect product Lie* 2*-algebras*, Int. J. Geom. Methods Mod. Phys. **9** (2012), no. 5, 1250043, 31, DOI 10.1142/S0219887812500430. MR2948862

[27] Y. Sheng and C. Zhu, *Integration of Lie* 2*-algebras and their morphisms*, Lett. Math. Phys. **102** (2012), no. 2, 223–244, DOI 10.1007/s11005-012-0578-1. MR2984165

[28] U. Schreiber and K. Waldorf, *Smooth functors vs. differential forms*, Homology Homotopy Appl. **13** (2011), no. 1, 143–203, DOI 10.4310/HHA.2011.v13.n1.a7. MR2803871

[29] U. Schreiber and K. Waldorf, *Local theory for* 2*-functors on path* 2*-groupoids*, J. Homotopy Relat. Struct. **12** (2017), no. 3, 617–658, DOI 10.1007/s40062-016-0140-4. MR3691299

Departamento de Matemática, Universidade Federal do Paraná, Setor de Ciências Exatas, Centro Politêcnico, 81531-990 Curitiba, Brasil

Email address: `olivier@ufpr.br`

Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, Cidade Universitária, 05508-090 São Paulo, Brasil

Email address: `cortiz@ime.usp.br`