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COMPACTNESS OF FOURIER INTEGRAL OPERATORS ON WEIGHTED MODULATION SPACES

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ABSTRACT. Using the matrix representation of Fourier integral operators with respect to a Gabor frame, we study their compactness on weighted modulation spaces. As a consequence, we recover and improve some compactness results for pseudodifferential operators.

1. Introduction

The aim of this paper is to investigate compactness for *Fourier integral operators* (FIOs) when acting on weighted modulation spaces. The boundedness and Schatten class properties of FIOs have been studied by several authors under various assumptions on the phase and the symbol. See, for instance, [1–4, 6, 7, 17, 18]. However, no characterization seems to be known of those FIOs which are compact. Our approach to the study of the compactness of the FIOs follows the point of view of [7], which means that our results strongly depend on the matrix representation of an FIO with respect to a Gabor frame.

For a function f on $\mathbb{R}^d$ the FIO T with symbol $\sigma \in L^\infty(\mathbb{R}^{2d})$ and phase Φ on $\mathbb{R}^{2d}$ can be formally defined by

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i}\Phi(x,\eta)}\sigma(x,\eta)\hat{f}(\eta)d\eta.$$

The phase $\Phi(x,\eta)$ is *tame*, which means that it is smooth on $\mathbb{R}^{2d}$ and fulfills the estimates

$$|\partial_z^\alpha \Phi(z)| \leq C_\alpha, \quad |\alpha| \geq 2, z \in \mathbb{R}^{2d}, \tag{1.1}$$

and the non-degeneracy condition

$$|\det \partial^2_{x,\eta}\Phi(x,\eta)| \geq \delta > 0, \quad (x,\eta) \in \mathbb{R}^{2d}. \tag{1.2}$$

The symbol σ on $\mathbb{R}^{2d}$ satisfies

$$|\partial_z^\alpha \sigma(z)| \leq C_\alpha, \quad \text{a.e. } z \in \mathbb{R}^{2d}, \ |\alpha| \leq 2N \tag{1.3}$$

for a fixed $N \in \mathbb{N}$. Here ∂_z^α denotes the distributional derivative. When $\Phi(x,\eta) = x\eta$ we recover the pseudodifferential operators (PSDOs) in the Kohn-Nirenberg form.

Let T_x and M_ω be the translation and modulation operators

$$(T_x f)(t) = f(x-t), \quad (M_\omega f)(t) = e^{2\pi i\omega t}f(t).$$

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We fix $\alpha > 0, \beta > 0$ and consider the regular lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$. Then, for $\lambda = (\alpha n, \beta m) \in \Lambda$, the time-frequency shift $\pi(\lambda)$ is defined by $\pi(\lambda)f = M_{\alpha n}T_{\beta m}f$. The set of time-frequency shifts $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ for a non-zero $g \in L^2(\mathbb{R}^d)$ is called a Gabor system.

The key result in [7] shows that the matrix representation of an FIO with respect to a Gabor frame $\mathcal{G}(g,\Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$ is well organized. In fact, for a tame phase function Φ and a symbol σ satisfying condition (1.3) there exists a constant $C_N > 0$ such that

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C_N \langle \chi(\lambda) - \mu\rangle^{-2N} \tag{1.4}$$

for every $\lambda, \mu \in \Lambda$ where χ is the *canonical transformation* of the phase Φ. We recall that $(x,\xi) = \chi(y,\eta)$ is a bilipschitz map $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ defined through the system

$$\begin{cases} y = \nabla_\eta \Phi(x,\eta), \\ \xi = \nabla_x \Phi(x,\eta). \end{cases}$$

The estimate (1.4) is an extension of previous results of Gröchenig [13] concerning almost diagonalization of PSDOs. See also [15]. The condition (1.3) on the symbol can be relaxed. In fact, if $\mathcal{G}(g,\Lambda)$ is a Parseval frame, then the estimate (1.4) also holds under the weaker assumption that σ belongs to an appropriate modulation space (see [5]).

We will use the decay estimate (1.4) to discuss the compactness of the FIOs when acting on weighted modulation spaces. More precisely, we prove that the FIO is compact when acting on some modulation space of the form $M^p_m(\mathbb{R}^d)$ if and only if the sequences

$$(\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \mu)g\rangle)_{\lambda\in\Lambda}$$

converge to zero for all $\mu \in \Lambda$, where χ' denotes a discrete version of the canonical transformation χ. This is the content of Theorem 3.10. In particular, it follows that compactness does not depend either on p or on m. To achieve our goal we need to focus our attention on a class of matrices $A = (a_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda}$ with the property that the decay of the coefficient $a_{\gamma,\gamma'}$ is determined by the distance of (γ,γ') to the graph of $\gamma = \chi(\gamma')$. We characterize when such a matrix defines a compact operator when acting on weighted ℓ^p spaces of sequences. For a quadratic phase Φ we completely characterize in Theorem 3.14 the symbols σ satisfying condition (1.3) for which the corresponding FIO is compact. The operators we are considering may fail to be bounded on mixed modulation spaces as was shown in [7]. To overcome this obstacle, an extra condition on the phase was introduced in [7]. Under this additional condition, the obtained results are extended to weighted mixed modulation spaces. As a consequence, we recover and improve some compactness results for PSDOs obtained in [9–11].

2. Preliminaries

2.1. **Modulation spaces.** The short time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R}^d)$ with respect to a non-zero window $g \in L^2(\mathbb{R}^d)$ is

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2i\pi\omega t}dt.$$

Clearly, we may also write $V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle$, where M_ω and T_x are the modulation and translation operators. Hence $V_g f$ can be defined for $f \in \mathcal{S}'(\mathbb{R}^d)$

and $g \in \mathcal{S}(\mathbb{R}^d)$. Modulation space norms are measures of the time-frequency concentration of a function or distribution. In order to quantify the decay properties of the STFT of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ we will use weight functions. A function $v : \mathbb{R}^N \to (0, \infty)$ is said to be a submultiplicative weight if it is continuous, symmetric on each coordinate and

$$v(r + k) \leq v(r)v(k).$$

The polynomial weights are the submultiplicative weights of the form

$$v_s(r) = \langle r \rangle^s = (1 + |r|^2)^{\frac{s}{2}}, \ s > 0.$$

A map $m : \mathbb{R}^N \to (0, \infty)$ is said to be v-moderate, with constant C_m, when $m(r + k) \leq C_m m(r)v(k)$ for every $r, k \in \mathbb{R}^N$. If m is v-moderate, then $1/m$ is also v-moderate. Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a v_s-moderate weight $(s > 0)$ m, and $1 \leq p, q \leq \infty$, the *modulation space* $M^{p,q}_m(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g(x, \omega)|^p \, m(x, \omega)^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} < \infty,$$

with obvious changes when $p = \infty$ or $q = \infty$. If $p = q$ we write $M^p_m(\mathbb{R}^d)$ instead of $M^{p,p}_m(\mathbb{R}^d)$. Then $M^{p,q}_m(\mathbb{R}^d)$ is a Banach space whose definition is independent of the window g. For $1 \leq p, q < \infty$, $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}_m(\mathbb{R}^d)$.

The closure of $\mathcal{S}(\mathbb{R}^d)$ in $M^{\infty,q}_m(\mathbb{R}^d)$ is denoted $M^{0,q}_m(\mathbb{R}^d)$ and $M^{p,0}_m(\mathbb{R}^d)$ is defined similarly. In particular, the closure of $\mathcal{S}(\mathbb{R}^d)$ in $M^\infty(\mathbb{R}^d)$ is denoted by $M^0(\mathbb{R}^d)$ and consists of those tempered distributions whose STFT vanishes at infinity.

For p, $q \in [1, \infty) \cup \{0\}$ the dual of $M^{p,q}_m(\mathbb{R}^d)$ can be identified with $M^{p',q'}_{1/m}(\mathbb{R}^d)$, p' and q' being the conjugate exponents of p and q. As usual, we agree that the conjugate exponent of 0 is 1. We refer to [12] for background on modulation spaces.

2.2. **Sequence spaces.** Given I and J countable sets of indices, a sequence of positive numbers $m = (m_{i,j})_{(i,j) \in I \times J}$ and $1 \leq p, q < \infty$, we consider the sequence space $\ell^{p,q}_m(I \times J)$ consisting of those sequences $x = (x_{i,j})_{(i,j) \in I \times J}$ such that

$$\|x\|_{\ell^{p,q}_m} := \left(\sum_{j \in J} \left(\sum_{i \in I} |x_{i,j} m_{i,j}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty.$$

In the case that $p = \infty$ or $q = \infty$, the previous norm is modified in the obvious way. If $p = q$ we have the weighted ℓ^p-spaces.

We denote by $\ell^{0,q}_m$ the closed subspace of $\ell^{\infty,q}_m$ consisting of those sequences $x \in \ell^{\infty,q}_m$ such that $\lim_{i \in I} |x_{i,j} m_{i,j}| = 0$ for every $j \in J$. $\ell^{p,0}_m$ is defined analogously. It turns out that $\ell^{0,q}_m(I \times J)$ (resp., $\ell^{p,0}_m(I \times J)$) coincides with the closure in $\ell^{\infty,q}_m(I \times J)$ (resp., $\ell^{p,\infty}_m(I \times J)$) of the set of those sequences with finitely many non-zero coordinates, denoted $\mathbb{C}^{(I \times J)}$.

Finally, $\ell^{0,0}_m(I \times J)$ coincides with the Banach space $c_{0,m}(I \times J)$ of all sequences $x = (x_{i,j})$ whose product with m converges to 0.

$\ell^{p,q}_m(I \times J)$ $(p, q \in [1, \infty] \cup \{0\})$ is a Banach space. For p, $q \in [1, \infty) \cup \{0\}$ the dual of $\ell^{p,q}_m(I \times J)$ can be identified with $\ell^{p',q'}_{1/m}(I \times J)$, p' and q' being the conjugate exponents of p and q. As usual, we agree that the conjugate exponent of 0 is 1. The duality is given by

$$\ell_m^{p,q}(I \times J) \times \ell_{1/m}^{p',q'}(I \times J) \to \mathbb{C},\ (x,y) \mapsto \sum_{i,j} x_{i,j} y_{i,j}.$$

Given a sequence $a = (a_{i,j})_{(i,j)\in I\times J}$ of complex numbers, we denote by D_a *the diagonal operator*

$$D_a : \mathbb{C}^{I\times J} \to \mathbb{C}^{I\times J},\ x = (x_{i,j})_{(i,j)\in I\times J} \mapsto (a_{i,j}x_{i,j})_{(i,j)\in I\times J}.$$

It is well known that D_a is a bounded operator on $\ell_m^{p,q}(I \times J)$ if $a \in \ell^\infty(I \times J)$, and moreover $\|D_a\| = \|a\|_\infty$ for all p, $q \in [1,\infty] \cup \{0\}$ and every m. As a consequence, since each $a \in c_0(I \times J)$ is the $\| \|_\infty$-limit of its finite sections, the diagonal operator D_a is compact on $\ell_m^{p,q}(I \times J)$ when $a \in c_0(I \times J)$.

A lattice on $\mathbb{R}^N$ is a set of the form $\Lambda = A\mathbb{Z}^N$, where A is an invertible $N \times N$ matrix. Given a submultiplicative weight v, a sequence of positive numbers $m = (m_\gamma)_{\gamma\in\Lambda}$ is v-moderate, with constant C_m, if $m_{\gamma+\gamma'} \le C_m m_\gamma v(\gamma')$ for all $\gamma, \gamma' \in \Lambda$. If m is v-moderate, then $1/m$ is also v-moderate. For a lattice Λ in $\mathbb{R}^N$, the translation operator $T_\gamma : \mathbb{C}^\Lambda \to \mathbb{C}^\Lambda$ is defined by

$$T_\gamma\,(x_\lambda)_{\lambda\in\Lambda} = (x_{\lambda-\gamma})_{\lambda\in\Lambda}\,.$$

If I and J are lattices in $\mathbb{R}^d$ and $\mathbb{R}^\ell$, respectively, we write $\Lambda := I \times J$, which is a lattice in $\mathbb{R}^N$ ($N = d+\ell$). Given $m = (m_\gamma)_{\gamma\in\Lambda}$, v-moderate with constant C_m, the translation operator T_γ is bounded on $\ell_m^{p,q}(\Lambda)$ for every $\gamma \in \Lambda$, and $\|T_\gamma\| \le C_m v(\gamma)$.

2.3. **Gabor frames.** We fix a function $g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ for $\alpha, \beta > 0$. The Gabor system $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g :\ \lambda \in \Lambda\}$ is said to be a *Gabor frame* if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le B\|f\|_2^2 \quad \forall f \in L^2(\mathbb{R}^d).$$

If $A = B = 1$, then the Gabor frame is said to be a *Parseval frame.* Associated to the Gabor frame $\mathcal{G}(g,\Lambda)$ we consider the analysis operator

$$C_g : L^2(\mathbb{R}^d) \to \ell^2(\Lambda), \quad f \mapsto (\langle f, \pi(\lambda)g\rangle)_{\lambda\in\Lambda}\,,$$

and its adjoint $D_g = C_g^*$, which is the synthesis operator

$$D_g : \ell^2(\Lambda) \to L^2(\mathbb{R}^d), \quad (c_\lambda)_{\lambda\in\Lambda} \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g.$$

Then $S_g = D_g \circ C_g$ is a bounded and invertible operator on $L^2(\mathbb{R}^d)$ called a frame operator. The canonical dual window of g is defined as $h = S_g^{-1}g$. It turns out that $\mathcal{G}(h,\Lambda)$ is also a Gabor frame and

$$D_g \circ C_h = D_h \circ C_g = Id_{L^2(\mathbb{R}^d)}.$$

If the Gabor frame is a Parseval frame, then $S_g = Id_{L^2(\mathbb{R}^d)}$ and $h = g$.

In the case that $\mathcal{G}(g,\Lambda)$ is a Gabor frame and $g \in \mathcal{S}(\mathbb{R}^d)$ then, as proved by Janssen (see [16] or [12, 13.5.4]), also $h = S_g^{-1}(g) \in \mathcal{S}(\mathbb{R}^d)$. Gröchenig and Leinert [14, 4.5] showed the existence of Parseval frames $\mathcal{G}(g,\Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$. Moreover, for every polynomially moderate weight m and for every $1 \le p, q \le \infty$,

$$C_g : M_m^{p,q}(\mathbb{R}^d) \to \ell_m^{p,q}(\Lambda) \text{ and } D_g : \ell_m^{p,q}(\Lambda) \to M_m^{p,q}(\mathbb{R}^d)$$

are bounded operators, weak* continuous, and $D_g \circ C_h = D_h \circ C_g = Id_{M^{p,q}_m(\mathbb{R}^d)}$. Here D_g is the transposed map of $C_g : M^{p',q'}_{1/m}(\mathbb{R}^d) \to \ell^{p',q'}_{1/m}(\Lambda)$. For $p = 1$ or $q = 1$ we take $p' = 0$ or $q' = 0$, respectively.

If $c = (c_\lambda)_{\lambda\in\Lambda}$ and $1 \le p, q < \infty$, then $D_g(c) = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g$. In the limit cases $p = \infty$ or $q = \infty$ the series in the right hand side converges to $D_g(c)$ in the weak* topology. See, for instance, [8] or [12, 12.2.3,12.2.4].

2.4. **Matrix representation of operators.** Cordero, Nicola, and Rodino [7] obtained a result on almost diagonalization for FIOs with respect to a Gabor frame which permitted the study of boundedness of Fourier integral operators on weighted modulation spaces. Our aim is to use the almost diagonalization technique to study the compactness of FIOs. To this end we need to establish a clear relationship between operators acting on modulation spaces and operators acting on appropriate sequence spaces.

From now on we assume that $\mathcal{G}(g, \Lambda)$ is a Gabor frame and $g \in \mathcal{S}(\mathbb{R}^d)$. Then $h = S_g^{-1}(g) \in \mathcal{S}(\mathbb{R}^d)$ and $D_g \circ C_h = D_h \circ C_g = Id_{M^{p,q}_m(\mathbb{R}^d)}$ for all $p, q \in [1, \infty]$ and for every v-moderate weight m. The (topological) identities $\mathcal{S}'(\mathbb{R}^d) = \bigcup\{M^2_{1/v_s} : s > 0\}$ and $\mathcal{S}(\mathbb{R}^d) = \bigcap\{M^2_{v_s} : s > 0\}$ permit us to conclude that

$$C_g, C_h : \mathcal{S}(\mathbb{R}^d) \to s(\Lambda)$$

and

$$C_g, C_h : \mathcal{S}'(\mathbb{R}^d) \to s'(\Lambda)$$

are topological isomorphisms into their ranges, where $s(\Lambda)$ is the space of rapidly decreasing sequences and $s'(\Lambda)$, its dual space, is endowed with the inductive topology. Moreover, every $f \in \mathcal{S}(\mathbb{R}^d)$ admits a decomposition

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle \pi(\lambda)g,$$

where the series converges in $\mathcal{S}(\mathbb{R}^d)$.

Definition 2.1. The *Gabor matrix* associated to a continuous and linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is defined as

$$M(T) = \left(\langle T(\pi(\lambda)g), \pi(\mu)g\rangle\right)_{(\mu,\lambda)\in\Lambda\times\Lambda}.$$

If T is an FIO with symbol σ and phase Φ we write $M(\sigma, \Phi)$ instead of $M(T)$.

Theorem 2.2. *Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a continuous and linear operator and let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with $g \in \mathcal{S}(\mathbb{R}^d)$. Then*

(1) *For $1 \le p, q < \infty$, T can be (uniquely) extended as a bounded operator from $M^{p,q}_{m_1}(\mathbb{R}^d)$ into $M^{p,q}_{m_2}(\mathbb{R}^d)$ if and only if $M(T)$ defines a bounded operator from $\ell^{p,q}_{m_1}(\Lambda)$ into $\ell^{p,q}_{m_2}(\Lambda)$.*
(2) *For $1 \le p, q \le \infty$, T can be extended as a weak* continuous operator from $M^{p,q}_{m_1}(\mathbb{R}^d)$ into $M^{p,q}_{m_2}(\mathbb{R}^d)$ if and only if $M(T)$ defines a weak* continuous operator from $\ell^{p,q}_{m_1}(\Lambda)$ into $\ell^{p,q}_{m_2}(\Lambda)$.*
(3) *Let $1 \le p, q \le \infty$ and assume that $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ is weak* continuous. Then $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ is compact if and only if $M(T) : \ell^{p,q}_{m_1}(\Lambda) \to \ell^{p,q}_{m_2}(\Lambda)$ is.*

Proof. Let h be the canonical dual window of g. Then we have

$$C_g \circ T = M(T) \circ C_h \text{ on } \mathcal{S}(\mathbb{R}^d).$$

Clearly, $M(T)$ defines a continuous operator from the range $C_h(\mathcal{S}(\mathbb{R}^d))$, which is a closed subspace of $s(\Lambda)$, into $s'(\Lambda)$. We now check that $M(T)$ defines a continuous operator from $\mathbb{C}^{(\Lambda)}$ into $s'(\Lambda)$, when $\mathbb{C}^{(\Lambda)}$ is endowed with the topology inherited by $s(\Lambda)$. To this end, we fix $x \in \mathbb{C}^{(\Lambda)}$ and observe that $D_g(x) \in \mathcal{S}(\mathbb{R}^d)$, hence $M(T) \circ C_h \circ D_g(x) = C_g \circ T \circ D_g(x)$. That is,

$$(M(T)(C_h \circ D_g)(x))_\mu = \langle T\left(D_g(x)\right), \pi(\mu)g\rangle = \sum_{\lambda\in\Lambda} \langle T\left(\pi(\lambda)g\right), \pi(\mu)g\rangle \cdot x_\lambda.$$

Consequently, for every finite sequence x we have

$$M(T)(x) = M(T)(C_h \circ D_g)(x).$$

Therefore, $M(T)$ is continuous on $\mathbb{C}^{(\Lambda)}$ when this space is considered as a subspace of $s(\Lambda)$. By density, $M(T)$ defines a continuous operator from the space $s(\Lambda)$ into $s'(\Lambda)$,

$$M(T) : s(\Lambda) \xrightarrow{D_g} S(\mathbb{R}^d) \xrightarrow{C_h} C_h(\mathcal{S}(\mathbb{R}^d)) \xrightarrow{M(T)} s'(\Lambda).$$

Then we have

$$T = D_h \circ M(T) \circ C_h \text{ on } \mathcal{S}(\mathbb{R}^d) \tag{2.1}$$

and

$$M(T) = M(T) \circ C_h \circ D_g = C_g \circ T \circ D_g \text{ on } s(\Lambda). \tag{2.2}$$

To prove (1) and (2) we only need to use density or weak* density arguments and the fact that $C_g, C_h : M^{p,q}_m(\mathbb{R}^d) \to \ell^{p,q}_m(\Lambda)$ and $D_g, D_h : \ell^{p,q}_m(\Lambda) \to M^{p,q}_m(\mathbb{R}^d)$ are bounded for $1 \le p, q < \infty$ and weak* continuous for $1 \le p, q \le \infty$.

To finish we prove (3). From the hypothesis we deduce that the identities (2.1) and (2.2) hold on $M^{p,q}_{m_1}(\mathbb{R}^d)$ and $\ell^{p,q}_{m_1}(\Lambda)$, respectively, and the conclusion follows. ☐

In the applications to the FIOs we will always consider $m_1 = m \circ \chi$ and $m_2 = m$. In the special case of PSDOs we will have $m_1 = m_2 = m$.

3. Compactness of FIOs

3.1. **FIOs on M^p_m.** Our aim is to discuss compactness properties for an FIO T whose phase is tame and with symbol $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ for some $s_0 > 2d$. Throughout this section we fix a lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ and a Parseval frame $\mathcal{G}(g, \Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$. As proved in [5], we have an estimate

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \le C\langle \chi(\lambda) - \mu\rangle^{-s_0} \quad \forall \lambda, \mu \in \Lambda. \tag{3.1}$$

Observe that any symbol satisfying condition (1.3) belongs to $M^\infty_{1\otimes v_{2N}}$.

The estimate (3.1) together with the results of subsection 2.4 suggest that we should consider operators on sequence spaces defined in terms of a matrix $A = (a_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda}$ with the property that the decay of the coefficient $a_{\gamma,\gamma'}$ is determined by the distance of (γ, γ') to the graph of $\gamma = \chi(\gamma')$. For convenience we will replace the canonical transformation χ by an appropriate discrete version $\chi' : \Lambda \to \Lambda$, defined as follows. We fix a symmetric relatively compact fundamental domain Q of Λ and, for every $\lambda \in \Lambda$, decompose any $\chi(\lambda) = r_\lambda + \chi'(\lambda)$ where $\chi'(\lambda) \in \Lambda$ and

$r_\lambda \in Q$. Since χ^{-1} is Lipschitz continuous there is $L > 0$ such that $\chi'(\lambda) = \chi'(\mu)$ implies

$$a := 2 \sup_{u \in Q} \|u\| \geq \|\chi(\lambda) - \chi(\mu)\| \geq L\|\lambda - \mu\|.$$

Hence

$$\chi'^{-1}(\{\chi'(\lambda)\}) = \{\mu \in \Lambda : \ \chi'(\mu) = \chi'(\lambda)\}$$

is contained in $\overline{B\left(\lambda, \frac{a}{L}\right)} \cap \Lambda$, which is a finite set whose cardinal does not depend on λ. This suggests the following definition.

Definition 3.1. Let v be a submultiplicative weight on $\mathbb{R}^{2d}$ and assume that $\psi : \Lambda \to \Lambda$ satisfies

$$M = \sup_{\lambda \in \Lambda} \text{card } \psi^{-1}(\{\lambda\}) < \infty.$$

We define $\mathcal{C}_{v,\psi}(\Lambda)$ as the set of all matrices $A = (a_{\gamma,\gamma'})_{\gamma,\gamma' \in \Lambda}$ such that

$$\|A\|_{\mathcal{C}_{v,\psi}} = \sum_{\gamma \in \Lambda} v(\gamma) \cdot \sup_{\lambda \in \Lambda} \left|a_{\psi(\lambda)+\gamma,\lambda}\right| < \infty.$$

Proposition 3.2. *Let T be an FIO whose phase Φ is tame and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. Then for every $0 \leq s < s_0 - 2d$ we have*

$$M(\sigma, \Phi) \in \mathcal{C}_{v_s,\chi'}.$$

Proof. We put $a_{\mu,\lambda} = \langle T\pi(\lambda)g, \pi(\mu)g\rangle$. We have to show that

$$\sum_{\gamma \in \Lambda} v_s(\gamma) \cdot \sup_{\lambda \in \Lambda} \left|a_{\chi'(\lambda)+\gamma,\lambda}\right| < \infty.$$

According to [5, Theorem 3.3],

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C\langle \chi(\lambda) - \mu\rangle^{-s_0} = C(v_{s_0}(\chi(\lambda) - \mu))^{-1}$$

for some constant C. Since there is $r_\lambda \in Q$ such that $\chi(\lambda) = \chi'(\lambda) + r_\lambda$, we obtain

$$\begin{aligned} |a_{\chi'(\lambda)+\gamma,\lambda}| &= |\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\gamma)g\rangle| \\ &\leq C(v_{s_0}(\chi(\lambda) - \chi'(\lambda) - \gamma))^{-1} \\ &= \tfrac{C}{v_{s_0}(r_\lambda - \gamma)} \leq \tfrac{C v_{s_0}(r_\lambda)}{v_{s_0}(\gamma)} \leq \tfrac{CR}{v_{s_0}(\gamma)}, \end{aligned}$$

where $R = \max\{v_{s_0}(r) : r \in Q\}$. Finally, using that $2d < s_0 - s$,

$$\sum_{\gamma \in \Lambda} v_s(\gamma) \cdot \sup_{\lambda \in \Lambda} \left|a_{\chi'(\lambda)+\gamma,\lambda}\right| \leq CR \sum_{\gamma \in \Lambda} \frac{v_s(\gamma)}{v_{s_0}(\gamma)} < \infty.$$

□

The following almost diagonal map will play an important role when discussing compactness properties of operators defined in terms of matrices in $\mathcal{C}_{v_s,\psi}$.

Definition 3.3. Let $\psi : \Lambda \to \Lambda$ be as in Definition 3.1 and $a \in \mathbb{C}^\Lambda$. Then

$$D_{a,\psi} : \mathbb{C}^\Lambda \to \mathbb{C}^\Lambda$$

is defined by $D_{a,\psi}(x) = y$ where

$$y_\gamma = \begin{cases} 0 & \text{if} \quad \gamma \notin \psi(\Lambda), \\ \displaystyle\sum_{\psi(\lambda)=\gamma} a_\lambda x_\lambda & \text{if} \quad \gamma \in \psi(\Lambda). \end{cases}$$

In particular, $D_{a,\psi}(e_\gamma) = a_\gamma e_{\psi(\gamma)}$. Moreover, $D_{a,\psi}\left(\mathbb{C}^{(\Lambda)}\right) \subset \mathbb{C}^{(\Lambda)}$.

We observe that the transposed map

$$D^t_{a,\psi} : \mathbb{C}^{(\Lambda)} \to \mathbb{C}^{(\Lambda)}$$

is given by

$$(D^t_{a,\psi}(x))_\lambda = \left(D^t_{a,\psi}(x), e_\lambda\right) = \left(x, a_\lambda e_{\psi(\lambda)}\right) = a_\lambda x_{\psi(\lambda)}.$$

In fact, $D^t_{a,\psi}$ can be extended as a map from $\mathbb{C}^\Lambda$ into itself. In the case that a is the constant sequence equal to 1 the map $D_{a,\psi}$ is denoted by I_ψ. Then for an arbitrary $a \in \mathbb{C}^\Lambda$ we have

$$D_{a,\psi} = I_\psi \circ D_a.$$

When ψ is the identity, $D_{a,\psi}$ is just the diagonal operator D_a.

Lemma 3.4. *Let*

$$M = \sup_{\gamma\in\Lambda} \operatorname{card}\left(\psi^{-1}(\{\gamma\})\right).$$

Then, there is a partition $\Lambda = \bigcup_{j=1}^M \Lambda_j$ with the property that ψ is injective when restricted to each Λ_j.

Let $m = (m_\lambda)_{\lambda\in\Lambda}$ be a positive sequence. For any $\psi : \Lambda \to \Lambda$ as in Definition 3.1 we denote by $m \circ \psi$ the sequence

$$m \circ \psi = \left(m_{\psi(\lambda)}\right)_{\lambda\in\Lambda}.$$

Proposition 3.5. *Let $\psi : \Lambda \to \Lambda$ be as in Definition* 3.1*, let $a = (a_\lambda)_{\lambda\in\Lambda}$ be a sequence of complex numbers, let $m = (m_\lambda)_{\lambda\in\Lambda}$ be a positive sequence, and $p \in [1, \infty]$. The following conditions are equivalent:*

(1) *$D_{a,\psi}$ is continuous on $\ell^2(\Lambda)$.*
(2) *$D_{a,\psi}$ is continuous from $\ell^p_{m\circ\psi}(\Lambda)$ to $\ell^p_m(\Lambda)$.*
(3) *$a \in \ell^\infty(\Lambda)$.*

Proof. It suffices to show the equivalence between conditions (2) and (3). Let us assume that condition (2) is satisfied. As $D_{a,\psi}(e_\lambda) = a_\lambda e_{\psi(\lambda)}$ then

$$\|D_{a,\psi}\| \geq \|D_{a,\psi}(\frac{e_\lambda}{m_{\psi(\lambda)}})\|_{\ell^p_m} = |a_\lambda|,$$

from where we get (3).

To check that (3) implies (2) let us first assume that $a \in \ell^\infty(\Lambda)$ and the restriction of ψ to the support of a is injective. Then

$$\|D_{a,\psi}(x)\|_{\ell^p_m} = \|(a_\lambda x_\lambda m_{\psi(\lambda)})_\lambda\|_{\ell^p} = \|(a_\lambda x_\lambda)_\lambda\|_{\ell^p_{m\circ\psi}} \leq \|a\|_{\ell^\infty} \|x\|_{\ell^p_{m\circ\psi}}.$$

In the case that condition (3) is satisfied but ψ is not injective on the support of a we apply Lemma 3.4 and decompose

$$a = \sum_{j=1}^M a^j,$$

in such a way that the support of a^j is contained in Λ_j. Then

$$D_{a,\psi} = \sum_{j=1}^{M} D_{a^j,\psi}$$

is continuous from $\ell^p_{m\circ\psi}(\Lambda)$ to $\ell^p_m(\Lambda)$ and

$$\|D_{a,\psi}\|_{\ell^p_{m\circ\psi}\to\ell^p_m} \le \sum_{j=1}^{M} \|a^j\|_{\ell^\infty} \le M\|a\|_{\ell^\infty}.$$

Hence (3) implies (2) is proved. □

The same argument shows that condition (3) in Proposition 3.5 is equivalent to $D_{a,\psi}$ being a bounded operator from $c_{0,m\circ\psi}(\Lambda)$ into $c_{0,m}(\Lambda)$.

In particular, $I_\psi : \ell^p_{m\circ\psi}(\Lambda) \to \ell^p_m(\Lambda)$ is continuous. We observe that if $p \neq q$ the map I_ψ need not be bounded on spaces $\ell^{p,q}_m(\Lambda)$.

Proposition 3.6. *Let $m = (m_\lambda)_{\lambda\in\Lambda}$ be a v-moderate positive sequence, $A = (a_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$ and $1 \le p \le \infty$ be given. Then*

(1) *$A : \ell^p_{m\circ\psi}(\Lambda) \to \ell^p_m(\Lambda)$ is a bounded operator, which is also weak* continuous.*
(2) *$A = \sum_{\gamma\in\Lambda}(T_\gamma \circ D_{a^\gamma,\psi})$ where $a^\gamma := (a_{\psi(\lambda)+\gamma,\lambda})_{\lambda\in\Lambda}$. The series converges absolutely.*

Proof. (1) It is easier to deal with the transposed map, so we first consider $b_{\gamma,\gamma'} = a_{\gamma',\gamma}$ and *claim* that $B = (b_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda}$ defines a bounded operator $B : \ell^q_{\frac{1}{m}}(\Lambda) \to \ell^q_{\frac{1}{m}\circ\psi}(\Lambda)$ for every $1 \le q \le \infty$. We should remark here that the class $\mathcal{C}_{v,\psi}(\Lambda)$ need not be closed under taking transposed. Instead we have

$$\sum_{\gamma\in\Lambda} v(\gamma)\cdot \sup_{\lambda\in\Lambda} \left|b_{\lambda,\psi(\lambda)+\gamma}\right| < \infty. \tag{3.2}$$

Using that for every $\lambda \in \Lambda$ one has $\Lambda = \psi(\lambda) + \Lambda$, we may write

$$\sum_{\gamma\in\Lambda} |b_{\lambda,\gamma} x_\gamma| = \sum_{\gamma\in\Lambda} \left|b_{\lambda,\psi(\lambda)+\gamma} x_{\psi(\lambda)+\gamma}\right|.$$

From (3.2) and inequality

$$1 \le C_m \frac{m_{\psi(\lambda)}}{m_{\psi(\lambda)+\gamma}} v(\gamma) \tag{3.3}$$

we conclude that

$$B : \ell^q_{\frac{1}{m}}(\Lambda) \to \mathbb{C}^\Lambda$$

is a well-defined operator. To prove that $Bx \in \ell^q_{\frac{1}{m}\circ\psi}(\Lambda)$ it is enough to check that

$$\sum_{\lambda\in\Lambda} |(Bx)_\lambda\, y_\lambda| < \infty$$

for every $y \in \ell^{q'}_{m\circ\psi}$. Here q' is the usual conjugate exponent. To this end we denote $\phi(\gamma) = v(\gamma)\sup_\lambda \left|b_{\lambda,\psi(\lambda)+\gamma}\right|$. Using again the inequality (3.3) we obtain

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |(Bx)_\lambda\, y_\lambda| &\leq \sum_{\lambda\in\Lambda}\sum_{\gamma\in\Lambda} |b_{\lambda,\gamma}x_\gamma y_\lambda| = \sum_{\lambda\in\Lambda}\sum_{\gamma\in\Lambda} \left|b_{\lambda,\psi(\lambda)+\gamma}x_{\psi(\lambda)+\gamma}y_{\lambda\in\Lambda}\right| \\
&\leq C_m \sum_{\lambda\in\Lambda}\sum_{\gamma\in\Lambda} \left|b_{\lambda,\psi(\lambda)+\gamma}\right| v(\gamma)\frac{\left|x_{\psi(\lambda)+\gamma}\right|}{m_{\psi(\lambda)+\gamma}}|y_\lambda|\, m_{\psi(\lambda)} \\
&\leq C_m \sum_{\gamma\in\Lambda}\phi(\gamma)\sum_{\lambda\in\Lambda}\frac{\left|x_{\psi(\lambda)+\gamma}\right|}{m_{\psi(\lambda)+\gamma}}|y_\lambda|\, m_{\psi(\lambda)} \\
&\leq MC_m\|x\|_{\ell^q_{\frac{1}{m}}}\cdot\|y\|_{\ell^{q'}_{m\circ\psi}}\cdot\|A\|_{\mathcal{C}_{v,\psi}},
\end{aligned}$$

where M is the constant in Definition 3.1. The claim is proved. Moreover, B also defines a bounded operator from $c_{0,\frac{1}{m}}$ to $c_{0,\frac{1}{m}\circ\psi}$. In fact, $B:\ell^\infty_{\frac{1}{m}}\to\ell^\infty_{\frac{1}{m}\circ\psi}$ is continuous, $B\left(\mathbb{C}^{(\Lambda)}\right)\subset\ell^1_{\frac{1}{m}\circ\psi}\subset c_{0,\frac{1}{m}\circ\psi}$, and $c_{0,\frac{1}{m}}$ is the closure of $\mathbb{C}^{(\Lambda)}$ on $\ell^\infty_{\frac{1}{m}}$. Consequently, for every $1\leq p\leq\infty$, the transposed map defines a bounded operator $A = B^t : \ell^p_{m\circ\psi}(\Lambda)\to\ell^p_m(\Lambda)$ which is also weak* continuous. The proof of (1) is complete.

(2) Since m is v-moderate with constant C_m we have

$$\|T_\gamma : \ell^p_m(\Lambda)\to\ell^p_m(\Lambda)\| \leq C_m v(\gamma).$$

Also

$$\|D_{a^\gamma,\psi} : \ell^p_{m\circ\psi}(\Lambda)\to\ell^p_m(\Lambda)\| \leq M\sup_\lambda\left|a_{\psi(\lambda)+\gamma,\lambda}\right|,$$

where M is the constant in Definition 3.1. Hence

$$\sum_{\gamma\in\Lambda}\|T_\gamma\circ D_{a^\gamma,\psi}\| \leq M\sum_{\gamma\in\Lambda}C_m v(\gamma)\sup_{\lambda\in\Lambda}|a_{\psi(\gamma)+\lambda,\lambda}| < \infty.$$

Consequently,

$$S := \sum_{\gamma\in\Lambda}(T_\gamma\circ D_{a^\gamma,\psi})$$

defines a bounded operator from $\ell^p_{m\circ\psi}(\Lambda)$ into $\ell^p_m(\Lambda)$. With a similar argument we can decompose the transposed map in terms of operators $D^t_{a^\gamma,\psi} = I^t_\psi\circ D_{a^\gamma}$, from where we conclude that S is also weak* continuous. Moreover, A and S coincide on $\{e_\lambda : \lambda\in\Lambda\}$, from where the result follows. In fact,

$$\begin{aligned}
\langle S(e_\lambda), e_\mu\rangle &= \left\langle\sum_{\gamma\in\Lambda}a_{\psi(\lambda)+\gamma,\lambda}e_{\psi(\lambda)+\gamma}, e_\mu\right\rangle = \left\langle\sum_{t\in\Lambda}a_{t,\lambda}e_t, e_\mu\right\rangle \\
&= \langle A(e_\lambda), e_\mu\rangle.
\end{aligned}$$

□

The following abstract result will be useful to obtain necessary conditions for the compactness of FIOs. We include a proof for the convenience of the reader.

Proposition 3.7. *Let E and F be Banach spaces and let $T : E \to F$ be a compact operator. We assume that $E = G'$ and $F = R'$ are dual Banach spaces, $T^t(R) \subseteq G$ and $\{x_i\}_{i\in I}$ is a sequence that converges to x in the weak* topology $\sigma(E, G)$. Then $\{(T(x_i))\}_{i\in I}$ converges to $T(x)$.*

Proof. We first check that $\{x_i\}_{i\in I}$ is a bounded sequence in E. In fact, $\{x_i\}_{i\in I}$ is a bounded sequence in $\sigma(E, G)$. If we consider the sequence of linear operators $\{\langle x_i, \cdot\rangle\}_{i\in I}$, then for every $g \in G$, $\{\langle x_i, g\rangle\}_{i\in I}$ is a bounded sequence. By Banach-Steinhaus' theorem, we obtain that $\{\langle x_i, \cdot\rangle\}_{i\in I}$ is uniformly bounded and we conclude that $\{x_i\}_{i\in I}$ is a bounded sequence in E. We assume that $\{T(x_i)\}_{i\in I}$ does not converge to $T(x)$ in norm. Then there are $\varepsilon > 0$ and a sequence of indices $(i_k)_{k=1}^{\infty} \subset I$ such that, for every k,

$$\|T(x_{i_k}) - T(x)\| > \varepsilon.$$

Since T is a compact operator, there exists a subsequence $\{T(x_{i_{k_t}})\}_t$ converging to some $y \in F$. Since $\{x_{i_{k_t}}\}_t$ $\sigma(E, G)$-converges to x we conclude that $\{T(x_{i_{k_t}})\}_t$ $\sigma(F, R)$-converges to $T(x)$. Since the norm convergence implies the $\sigma(F, R)$-convergence in F,we finally obtain that $y = T(x)$. Consequently, $\{T(x_{i_{k_t}})\}_t$ converges to $T(x)$ in norm, which is a contradiction. □

Theorem 3.8. *Let $A = (a_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$ and $1 \le p \le \infty$ be given. Then $A : \ell^p_{m\circ\psi}(\Lambda) \to \ell^p_m(\Lambda)$ is a compact operator if and only if*

$$a^\gamma := \left(a_{\psi(\lambda)+\gamma,\lambda}\right)_{\lambda\in\Lambda} \in c_0(\Lambda) \quad \forall\gamma \in \Lambda.$$

Proof. If $a^\gamma \in c_0(\Lambda)$ for every $\gamma \in \Lambda$, then $D_{a^\gamma,\psi} = I_\psi \circ D_{a^\gamma}$ is compact for each $\gamma \in \Lambda$. Hence, we can apply Proposition 3.6 to conclude that A is a compact operator.

Let us now assume that A is compact. Since $\left(\frac{e_\lambda}{m_{\psi(\lambda)}}\right)_{\lambda\in\Lambda}$ converges to zero in the weak* topology of $\ell^p_{m\circ\psi}(\Lambda)$, we can apply Proposition 3.7 to conclude that $\left(A(\frac{e_\lambda}{m_{\psi(\lambda)}})\right)_{\lambda\in\Lambda}$ converges to 0. Now, we fix $\gamma \in \Lambda$ and use that

$$\frac{m_{\psi(\lambda)+\gamma}}{m_{\psi(\lambda)}}|a_{\psi(\lambda)+\gamma,\lambda}| \le \|A(\frac{e_\lambda}{m_{\psi(\lambda)}})\|_{\ell^p_m(\Lambda)}.$$

Since m is v-moderate we obtain

$$\left|a_{\psi(\lambda)+\gamma,\lambda}\right| \le C_m v(\gamma)\|A(\frac{e_\lambda}{m_{\psi(\lambda)}})\|_{\ell^p_m(\Lambda)},$$

which finishes the proof. □

We will apply Theorem 3.8 to the study of compactness of FIOs

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i}\Phi(x,\eta)}\sigma(x, \eta)\hat{f}(\eta)d\eta$$

whose phase is tame and with symbol $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. As usual, χ is the canonical transformation of the phase Φ and $\chi' : \Lambda \to \Lambda$ is its discrete version.

Theorem 3.9. *Let T be an FIO whose phase Φ is tame and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. The following conditions are equivalent:*

(1) *$T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator.*
(2) *$M(\sigma, \Phi) : \ell^2(\Lambda) \to \ell^2(\Lambda)$ is compact.*

(3) $(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle)_\lambda \in c_0(\Lambda)$ *for every* $\mu \in \Lambda$.

Proof. Since $M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d}) \subset M^{\infty,1}(\mathbb{R}^{2d})$ we can apply [7, Theorem 6.1] to obtain that $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a bounded operator. From Theorem 2.2 we get the equivalence of conditions (1) and (2). Now it suffices to apply Proposition 3.2 and Theorem 3.8 to conclude. □

We observe that, for any positive and v_s-moderate weight m,

$$\ell^p_{m\circ\chi}(\Lambda) = \ell^p_{m\circ\chi'}(\Lambda)$$

with equivalent norms and that $m \circ \chi$ is v_s-moderate whenever m is.

Theorem 3.10. *Let T be an FIO whose phase Φ is tame and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. Then, for every $0 \le s < s_0 - 2d$, the following conditions are equivalent:*

(1) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is a compact operator.*
(2) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ *is a compact operator for some* $1 \le p < \infty$ *and for some v_s-moderate weight m.*
(3) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ *is a compact operator for every* $1 \le p < \infty$ *and for every v_s-moderate weight m.*

Proof. From [5, Corollary 5.5] and Propositions 3.2 and 3.6 we have that

$$T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d) \text{ and } M(\sigma,\Phi) : \ell^p_{m\circ\chi'}(\Lambda) \to \ell^p_m(\Lambda)$$

are bounded operators for every $1 \le p < \infty$ and for every v_s-moderate weight m. It suffices to show (2) ⇒ (3). According to Theorems 2.2 and 3.8, condition (2) is equivalent to the fact that

$$(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle)_\lambda \in c_0(\Lambda)$$

for every $\mu \in \Lambda$ and this condition does not depend on p nor on m. □

We next discuss the case $p = \infty$.

Theorem 3.11. *Let T be an FIO whose phase Φ is tame and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ and let $0 \le s < s_0 - 2d$ and m be a v_s-moderate weight. Then*

(1) *T admits a unique extension as a bounded operator*

$$T : M^\infty_{m\circ\chi}(\mathbb{R}^d) \to M^\infty_m(\mathbb{R}^d)$$

which is also weak continuous.*

(2) $T : M^\infty_{m\circ\chi}(\mathbb{R}^d) \to M^\infty_m(\mathbb{R}^d)$ *is compact if and only if*

$$(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle)_\lambda \in c_0(\Lambda)$$

for every $\mu \in \Lambda$.

Proof. (1) In fact, we consider the composition

$$T : M^\infty_{m\circ\chi}(\mathbb{R}^d) \xrightarrow{C_g} \ell^\infty_{m\circ\chi}(\Lambda) \xrightarrow{M(\sigma,\Phi)} \ell^\infty_m(\Lambda) \xrightarrow{S^*} M^\infty_m(\mathbb{R}^d),$$

where $S = C_g : M^1_{1/m}(\mathbb{R}^d) \to \ell^1_{1/m}(\Lambda)$. We observe that all the involved maps are weak* continuous. Since $\mathcal{S}(\mathbb{R}^d)$ is weak* dense in $M^\infty_{m\circ\chi}(\mathbb{R}^d)$ the extension is unique.

(2) By Theorem 2.2, T is compact if and only if $M(\sigma,\Phi) : \ell^\infty_{m\circ\chi} \to \ell^\infty_m(\Lambda)$ is. Now it suffices to apply Theorem 3.8. □

For the proof of the next result we recall that the canonical transformation $(x,\xi) = \chi(y,\eta)$ is defined through the system

$$\begin{cases} y = \nabla_\eta \Phi(x,\eta), \\ \xi = \nabla_x \Phi(x,\eta). \end{cases}$$

Theorem 3.12. *Let T be an FIO whose phase Φ is tame and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ and let $0 \le s < s_0 - 2d$. If $\sigma \in M^0(\mathbb{R}^{2d})$, then $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \le p \le \infty$ and for every v_s-moderate weight m.*

Proof. It suffices to show that $M(\sigma,\Phi)_{\mu,\lambda}$ goes to zero as $|(\lambda,\mu)|$ goes to infinity. To this end we first recall the relation between the Gabor matrix of T and the STFT of σ. We denote $\lambda = (\lambda_1,\lambda_2), \mu = (\mu_1,\mu_2) \in \mathbb{R}^{2d}$. From [7, (39)] we have

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| = |V_{\Psi_{\mu_1,\lambda_2}}\sigma(z_{\lambda,\mu})|, \tag{3.4}$$

where

$$z_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)} = (\mu_1, \lambda_2, (\mu_2 - \nabla_x\Phi(\mu_1,\lambda_2)), (\lambda_1 - \nabla_\eta\Phi(\mu_1,\lambda_2))),$$

$$\Psi_{(\mu_1,\lambda_2)}(w) = e^{2\pi \mathbf{i}\Phi_{2,(\mu_1,\lambda_2)}(w)}(\overline{g}\otimes\hat{g})(w),$$

and $\Phi_{2,(\mu_1,\lambda_2)}$ denotes the remainder of order two of the Taylor series of Φ, that is,

$$\Phi_{2,(\mu_1,\lambda_2)}(w) = 2\sum_{|\alpha|=2}\int_0^1 (1-t)\partial^\alpha\Phi((\mu_1,\lambda_2)+tw)dt\frac{w^\alpha}{\alpha!}$$

with $(\mu_1,\lambda_2), w \in \mathbb{R}^{2d}$. By the proof of [7, 6.1] we obtain that

$$D = \{\Psi_{(\mu_1,\lambda_2)} : (\mu_1,\lambda_2) \in \mathbb{Z}^{2d}\}$$

is a relatively compact set in $S(\mathbb{R}^{2d})$. Since $\sigma \in M^0(\mathbb{R}^{2d})$,

$$S(\mathbb{R}^{2d}) \to C_0(\mathbb{R}^{2d}), \quad \Psi \mapsto V_\Psi\sigma,$$

is a continuous map, hence

$$\widetilde{D} = \{V_{\Psi_{(\mu_1,\lambda_2)}}\sigma : (\mu_1,\lambda_2) \in \mathbb{Z}^{2d}\} = \{V_\Psi\sigma : \Psi \in D\}$$

is a relatively compact set in $C_0(\mathbb{R}^{2d})$. We conclude that $|V_{\Psi_{\mu_1,\lambda_2}}\sigma(z_{\lambda,\mu})|$ goes uniformly to zero as $|z_{\lambda,\mu}|$ goes to infinity.

Finally, we check that $M(\sigma,\Phi)_{\lambda,\mu}$ goes to zero as $|(\lambda,\mu)|$ goes to infinity. We can distinguish two cases:

- μ_1 or λ_2 goes to infinity. Then also $|z_{\lambda,\mu}|$ goes to infinity.
- Neither μ_1 nor λ_2 goes to infinity. We can assume that there exist $C > 0$ such that $|(\mu_1,\lambda_2)| \le C$, from where it follows that $\nabla_x\Phi(\mu_1,\lambda_2)$ and $\nabla_\eta\Phi(\mu_1,\lambda_2)$ are bounded. As $|(\lambda,\mu)|$ goes to infinity then μ_2 or λ_1 goes to infinity. From the fact that $\nabla_x\Phi(\mu_1,\lambda_2)$ and $\nabla_\eta\Phi(\mu_1,\lambda_2)$ are bounded, we conclude that $|z_{\lambda,\mu}|$ goes to infinity.

From (3.4) we deduce that the Gabor matrix $M(\sigma,\Phi)_{\lambda,\mu}$ goes to 0 as $|(\lambda,\mu)|$ goes to infinity and the proof is complete. □

We now prove that the converse is true in the particular case of quadratic phases.

Definition 3.13. The map $\Phi : \mathbb{R}^{2d} \to \mathbb{R}$ is said to be a quadratic phase if

$$\Phi(x,\eta) = \frac{1}{2}Ax\cdot x + Bx\cdot\eta + \frac{1}{2}C\eta\cdot\eta + \eta_0\cdot x - x_0\cdot\eta,$$

where $x_0, \eta_0 \in \mathbb{R}^d$, A, B, C are symmetric real matrices and B is non-degenerate.

Theorem 3.14. *Let T be an FIO with quadratic phase Φ and $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ and let $0 \leq s < s_0 - 2d$. Then the following statements are equivalent:*

(1) $\sigma \in M^0(\mathbb{R}^{2d})$.
(2) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ *is a compact operator for every* $1 \leq p \leq \infty$ *and for every* v_s*-moderate weight* m.

Proof. We need to check that (2) $\Rightarrow$ (1). We use the same notation as in the proof of Theorem 3.12. Since the phase Φ is quadratic then all its second partial derivatives are constant. Hence

$$\Phi_{2,(0,0)}(w) = \Phi_{2,(\mu_1,\lambda_2)}(w) \text{ and } \Psi_{(\mu_1,\lambda_2)}(w) = \Psi_{(0,0)}(w) = \Psi(w)$$

for every $(\mu_1, \lambda_2) \in \mathbb{R}^{2d}$. Consequently,

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| = |V_{\Psi_{(0,0)}}\sigma(z_{\lambda,\mu})|. \tag{3.5}$$

We now proceed in several steps.

We first prove that

$$\left(\langle T\pi(\lambda)g, \pi(\mu)g\rangle\right)_{\lambda,\mu\in\Lambda} \in c_0(\Lambda\times\Lambda). \tag{3.6}$$

As $M(\sigma,\Phi) \in \mathcal{C}_{v_s,\psi}$ we have

$$\sum_{\gamma\in\Lambda} v_s(\gamma)\cdot \sup_{\lambda\in\Lambda} |M(\sigma,\Phi)_{\chi'(\lambda)+\gamma,\lambda}| < \infty.$$

In particular,

$$\lim_{|\gamma|\to\infty} \sup_{\lambda\in\Lambda} |M(\sigma,\Phi)_{\chi'(\lambda)+\gamma,\lambda}| = 0. \tag{3.7}$$

Since T is a compact operator we can apply Theorems 3.9 and 3.10 to get

$$\left(M(\sigma,\Phi)_{\chi'(\lambda)+\gamma,\lambda}\right)_{\lambda} \in c_0(\Lambda) \tag{3.8}$$

for every $\gamma \in \Lambda$. Statement (3.6) is a consequence of conditions (3.7) and (3.8).

Secondly, we check that $G(z,w) = \langle T\pi(z)g, \pi(w)g\rangle$ goes to zero as $|(z,w)|$ goes to infinity on $\mathbb{R}^{4d}$. We have

$$\pi(u)g = \sum_{\nu\in\Lambda} \langle \pi(u)g, \pi(\nu)g\rangle \pi(\nu)g. \tag{3.9}$$

As $g \in S(\mathbb{R}^d) \subseteq M^1(\mathbb{R}^d)$, for every relatively compact subset $K \subset \mathbb{R}^{2d}$ there is $B > 0$ such that

$$\sum_{\nu\in\Lambda} \sup_{u\in K} |V_g g(\nu+u)| v_s(\nu) \leq B\|g\|_{M^1(\mathbb{R}^d)}$$

(see [12, 12.2.1]). In particular, we take K as a symmetric and relatively compact fundamental domain of Λ and define

$$\alpha(\nu) = \sup_{u\in K} |V_g g(\nu+u)| = \sup_{u\in K} |\langle \pi(-u)g, \pi(\nu)g\rangle|.$$

Then $\alpha \in \ell^1(\Lambda)$. Given $z, w \in \mathbb{R}^{2d}$ we can decompose $z = \mu + u$ and $w = \lambda + u'$, with $\mu, \lambda \in \Lambda$ and $u, u' \in K$. From (3.9) we obtain

$$\begin{aligned} |\langle T\pi(z)g, \pi(w)g\rangle| &= |\langle T\pi(\mu)\pi(u)g, \pi(\lambda)\pi(u')g\rangle| \\ &\leq \sum_{\nu,\nu'\in\Lambda} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle| \alpha(\nu)\alpha(\nu'). \end{aligned}$$

Let $\varepsilon > 0$ be given, take $A = \sup_{\lambda,\mu\in\Lambda} |\langle T\pi(\lambda)g, \pi(\mu)g\rangle|$, and find $M > 0$ such that

$$\sum_{|\nu|>M} \alpha(\nu) < \frac{\varepsilon}{3A\|\alpha\|_{\ell^1}}.$$

For every $\mu, \lambda \in \Lambda$ we have that

$$\sum_{|\nu|\geq M, \nu'\in\Lambda} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle| \alpha(\nu)\alpha(\nu')$$

is less than or equal to

$$A \sum_{|\nu|\geq M} \alpha(\nu) \sum_{\nu'\in\Lambda} \alpha(\nu') \leq \frac{\varepsilon}{3}$$

and also

$$\sum_{\nu\in\Lambda, |\nu'|\geq M} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle| \alpha(\nu)\alpha(\nu') \leq \frac{\varepsilon}{3}.$$

Finally, an application of (3.6) gives

$$\begin{aligned}
&|\langle T\pi(z)g, \pi(w)g\rangle| \\
&\leq \frac{2\varepsilon}{3} + \sum_{|\nu|,|\nu'|<M} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle| \alpha(\nu)\alpha(\nu') \\
&\leq \varepsilon
\end{aligned}$$

for $|z| + |w|$ large enough. The proof that $|\langle T\pi(z)g, \pi(w)g\rangle| \in C_0(\mathbb{R}^{4d})$ is complete.

We can now finish the proof that $\sigma \in M^0(\mathbb{R}^{2d})$. We recall that

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| = |V_\Psi\sigma(z_{\lambda,\mu})| \tag{3.10}$$

for every $\lambda, \mu \in \mathbb{R}^2$ and consider $V_\Psi\sigma(a, b, c, d)$ with $(a, b, c, d) \in \mathbb{R}^{4d}$. There are unique $e, f \in \mathbb{R}^d$ such that

$$(a, b, c, d) = (a, b, e - \nabla_x\Phi(a, b), f - \nabla_\eta\Phi(a, b)) = z_{f,b,a,e}.$$

Then $|V_\Psi\sigma(a, b, c, d)| = |G(f, b, a, e)|$. If $|(a, b, c, d)|$ goes to infinity we have two possibilities:

- a or b goes to infinity. Then $|(f, b, a, e)|$ goes to infinity.
- Neither a nor b goes to infinity. We can assume that there is $A > 0$ such that $|(a, b)| \leq A$, from where it follows that $\nabla_x\Phi(a, b)$ and $\nabla_\eta\Phi(a, b)$ are bounded. As $|(a, b, e - \nabla_x\Phi(a, b), f - \nabla_\eta\Phi(a, b))|$ goes to infinity and $a, b, \nabla_x\Phi(a, b), \nabla_\eta\Phi(a, b)$ are bounded, we conclude that either e or f goes to infinity. Hence $|(f, b, a, e)|$ goes to infinity.

Since $|\langle T\pi(z)g, \pi(w)g\rangle| \in C_0(\mathbb{R}^{4d})$ we can use (3.5) to conclude that $\sigma \in M^0(\mathbb{R}^{2d})$. □

3.2. **FIOs on $M^{p,q}_m$.** FIOs we are considering may fail to be bounded on mixed modulation spaces as was shown in [7]. The example was an FIO with phase $\Phi(x, \eta) = x\eta + \frac{|x|^2}{2}$, whose canonical transformation is $\chi(y, \eta) = (y, y + \eta)$. It is easy to check that $I_\chi(\ell^{2,1})$ is not contained in $\ell^{2,1}$.

To overcome this obstacle, an extra condition on the phase was introduced in [7], namely

$$\sup_{x',x,\eta} |\nabla_x\Phi(x, \eta) - \nabla_x\Phi(x', \eta)| < \infty. \tag{3.11}$$

If $\chi = (\chi_1, \chi_2)$ is the corresponding canonical transformation, from condition (3.11),

$$\chi_2(y, \eta) = \nabla_x \Phi(\chi_1(y, \eta), \eta) = \nabla_x \Phi(0, \eta) + a(y, \eta),$$

$a(y, \eta)$ being a bounded function. From now on, $\mathcal{G}(g, \Lambda)$ is a Parseval frame with $g \in \mathcal{S}(\mathbb{R}^d)$, $\Lambda_1 = \alpha\mathbb{Z}^d$, $\Lambda_2 = \beta\mathbb{Z}^d$, and $\Lambda = \Lambda_1 \times \Lambda_2$. If Q denotes a symmetric relatively compact fundamental domain of the lattice Λ, then there are $K \subseteq \Lambda_2$, finite, and a unique decomposition

$$\chi_1(\lambda_1, \lambda_2) = r_1(\lambda_1, \lambda_2) + \psi_1(\lambda_1, \lambda_2),$$
$$\chi_2(\lambda_1, \lambda_2) = r_2(\lambda_1, \lambda_2) + \psi_2(\lambda_2) + a(\lambda_1, \lambda_2)$$

for all $(\lambda_1, \lambda_2) \in \Lambda$, where $(r_1(\lambda_1, \lambda_2), r_2(\lambda_1, \lambda_2)) \in Q$, $\psi_1(\lambda_1, \lambda_2) \in \Lambda_1$, $\psi_2(\lambda_2) \in \Lambda_2$, and $a(\lambda_1, \lambda_2) \in K$. Moreover, from conditions (1.1) and (1.2) it follows that the map

$$\mathbb{R}^d \to \mathbb{R}^d,\ \eta \mapsto \nabla_x \Phi(0, \eta),$$

is a bilipschitz global diffeomorphism, which implies that

$$\sup_{\lambda_2 \in \Lambda_2} \operatorname{card} \psi_2^{-1}(\{\lambda_2\}) < \infty.$$

This motivates the following definition.

Definition 3.15. Let $\psi : \Lambda_1 \times \Lambda_2 \to \Lambda_1 \times \Lambda_2$, $\psi(i, j) = (\psi_1(i, j), \psi_2(i, j))$, be as in Definition 3.1. We say that ψ is admissible if there exist a map $\tilde{\psi}_2 : \Lambda_2 \to \Lambda_2$ as in Definition 3.1 and a finite set $K \subset \Lambda_2$ such that

$$\psi_2(i, j) = \tilde{\psi}_2(j) + a(i, j) \text{ for all } (i, j)$$

where $a(i, j) \in K$.

The discrete versión $\chi' : \Lambda \to \Lambda$ of the canonical transformation associated to a phase function satisfying conditions (1.1), (1.2), (3.11) is admissible. From now on, given ψ admissible, to simplify the notation, we will write $\psi_2(j)$ instead of $\tilde{\psi}_2(j)$.

Given an admissible $\psi : \Lambda_1 \times \Lambda_2 \to \Lambda_1 \times \Lambda_2$, let C be the cardinal of the finite set K and let $M > 0$ be such that for each $(i, j) \in \Lambda_1 \times \Lambda_2$, $\psi^{-1}(\{(i, j)\})$ has at most M elements and $\psi_2^{-1}(\{j\})$ has at most M elements for every $j \in \Lambda_2$.

Lemma 3.16. *Let ψ be admissible, and $M_1 = C \cdot M$. For each $j \in \Lambda_2$, we define $\psi_{1,j} : \Lambda_1 \to \Lambda_1$, as $\psi_{1,j}(i) := \psi_1(i, j)$. Then for each $i \in \Lambda_1$ the set $\psi_{1,j}^{-1}(\{i\})$ has at most M_1 elements.*

Proof. We fix $j \in \Lambda_2$ and $i_0 \in \Lambda_1$. If $\psi_1(i, j) = \psi_1(i_0, j)$, then

$$\psi(i, j) = (\psi_1(i_0, j), \psi_2(j) + a(i, j))$$

can take C different values. Hence, there are only $C \cdot M$ possibilities for i. □

We start by analyzing the action of the basic operators $D_{a,\psi}$ on weighted sequence spaces with mixed norm $\ell_m^{p,q}$. Since $D_{a,\psi} = I_\psi \circ D_a$, we will study the continuity of I_ψ on these spaces. To this aim, we consider the transposed map $J_\psi := I_\psi^t$, with ψ admissible. We observe that for every $\lambda \in \Lambda$,

$$J_\psi(x) = (x_{\psi(\lambda)})_\lambda.$$

Proposition 3.17. *Let ψ be admissible, let $m = (m_{i,j})_{(i,j)\in\Lambda}$ be a positive sequence, and $p, q \in [1, \infty] \cup \{0\}$. Then, J_ψ is continuous from $\ell_m^{p,q}(\Lambda)$ to $\ell_{m\circ\psi}^{p,q}(\Lambda)$.*

Proof. Let $x \in \ell^{p,q}_m(\Lambda)$ and put $y = x \cdot m$ and $\gamma = (i,j)$. Then

$$|y_{\psi(i,j)}| \le \sum_{k\in\Lambda_2} \left|y_{\psi_1(i,j),\psi_2(j)+k}\right|,$$

hence

$$\begin{aligned}\left(\sum_{i\in\Lambda_1} |y_{\psi(i,j)}|^p\right)^{\frac{1}{p}} &\le \sum_{k\in\Lambda_2}\left(\sum_{i\in\Lambda_1}\left|y_{\psi_1(i,j),\psi_2(j)+k}\right|^p\right)^{\frac{1}{p}}\\ &\le \sum_{k\in\Lambda_2}\left(M_1\sum_{\ell\in\Lambda_1}\left|y_{\ell,\psi_2(j)+k}\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\left(\sum_{j\in\Lambda_2}\left(\sum_{i\in\Lambda_1}|y_{\psi(i,j)}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} &\le \sum_k\left(\sum_{j\in\Lambda_2}\left(M_1\sum_{\ell\in\Lambda_1}\left|y_{\ell,\psi_2(j)+k}\right|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\ &\le C\left(M\sum_{h\in\Lambda_2}\left(M_1\sum_{\ell\in\Lambda_1}|y_{\ell,h}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\ &= CM^{\frac{1}{q}}M_1^{\frac{1}{p}}\|y\|_{p,q},\end{aligned}$$

where we have applied triangular inequality for the norms in ℓ^p and ℓ^q, and the facts that, for each $j \in \Lambda_2$, $\psi_2(j)$ can be repeated at most M times and $\psi_1(i,j) = \psi_{1,j}(i)$ can be repeated at most M_1 times (Lemma 3.16).

As J_ψ maps finite supported sequences into finite supported sequences, the cases $p = 0$ or $q = 0$ follow immediately. □

For $a \in \ell^\infty(\Lambda)$ we obtain, from the decomposition $D_{a,\psi} = J^t_\psi \circ D_a$, the estimate

$$\|D_{a,\psi} : \ell^{p,q}_{m\circ\psi}(\Lambda) \to \ell^{p,q}_m(\Lambda)\| \le CM^{\frac{1}{q}}M_1^{\frac{1}{p}} \cdot \|a\|_\infty.$$

Proposition 3.18. *Let $A = (a_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$, with ψ admissible and let $1 \le p,q \le \infty$ be given. Then*

(1) *A defines a bounded operator$A : \ell^{p,q}_{m\circ\psi}(\Lambda) \to \ell^{p,q}_m(\Lambda)$, which is also weak* continuous.*
(2) *$A = \sum_{\gamma\in\Lambda}(T_\gamma \circ D_{a^\gamma,\psi})$ where $a^\gamma := (a_{\psi(\lambda)+\gamma,\lambda})_{\lambda\in\Lambda}$. The convergence of the series is absolute.*

Proof. It is easier to deal with the transposed map, so we first consider $b_{\gamma,\gamma'} = a_{\gamma',\gamma}$ and *claim* that $B = (b_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda}$ defines a bounded operator

$$B : \ell^{p,q}_{\frac{1}{m}}(\Lambda) \to \ell^{p,q}_{\frac{1}{m}\circ\psi}(\Lambda)$$

for all $p,q \in [1,\infty] \cup \{0\}$. We will assume that $p,q \in [1,\infty]$. Then the case that $p = 0$ or $q = 0$ can be obtained as in the proof of Proposition 3.6.

As $\ell^{p,q}_{\frac{1}{m}}(\Lambda) \subset \ell^\infty_{\frac{1}{m}}(\Lambda)$, by Proposition 3.6, we obtain that

$$B : \ell^{p,q}_{\frac{1}{m}}(\Lambda) \to \mathbb{C}^\Lambda$$

is a well-defined operator. To prove that $Bx \in \ell^{p,q}_{\frac{1}{m}\circ\psi}(\Lambda)$ it is enough to check that

$$\sum_{\gamma\in\Lambda} \left|(Bx)_\gamma\, y_\gamma\right| < \infty$$

for every $y \in \ell^{p',q'}_{m\circ\psi}(\Lambda)$. We proceed as in Proposition 3.6. We denote $\phi(\lambda) = v(\lambda)\sup_\gamma \left|b_{\gamma,\psi(\gamma)+\lambda}\right|$. We obtain, using that translations are isometries on the spaces $\ell^{p,q}$,

$$\begin{aligned}\sum_{\gamma\in\Lambda} \left|(Bx)_\gamma\, y_\gamma\right| &\leq C_m \sum_{\lambda\in\Lambda}\phi(\lambda)\sum_{\gamma\in\Lambda}\frac{x_{\psi(\gamma)+\lambda}}{m_{\psi(\gamma)+\lambda}}|y_\gamma| m_{\psi(\gamma)} \\ &\leq C_m \sum_{\lambda\in\Lambda}\phi(\lambda)\cdot \|J_\psi(x)\|_{\ell^{p,q}_{\frac{1}{m}\circ\psi}}\cdot\|y\|_{\ell^{p',q'}_{m\circ\psi}}.\end{aligned}$$

(2) follows as in Proposition 3.6 once continuities and the estimates for the norms of the operators $D_{a^\gamma,\psi}$ are obtained. ☐

The characterization of compactness obtained in Theorem 3.8 extends to mixed spaces when ψ is admissible.

Proposition 3.19. *Let $A = (a_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$, let ψ be admissible, and $1 \leq p, q \leq \infty$ be given. Then A defines a compact operator*

$$A : \ell^{p,q}_{m\circ\psi}(\Lambda) \to \ell^{p,q}_m(\Lambda)$$

if and only if $a^\gamma := \left(a_{\psi(\lambda)+\gamma,\lambda}\right)_{\lambda\in\Lambda} \in c_0(\Lambda)\ \ \forall\gamma\in\Lambda$.

The next result extends [7, Theorem 5.2] to weighted modulation spaces and also includes the cases $p = \infty$ or $q = \infty$.

Theorem 3.20. *Let T be an FIO whose phase Φ is tame and satisfies condition (3.11), and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ with $0 \leq s < s_0 - 2d$. Then, $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ is a continuous operator for every $1 \leq p, q \leq \infty$ and for every v_s-moderate weight m.*

Theorem 3.21. *Let T be an FIO whose phase Φ is tame and satisfies condition (3.11), and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ with $0 \leq s < s_0 - 2d$. The following conditions are equivalent:*

1. *$T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator.*
2. *$T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ is a compact operator for some $1 \leq p, q \leq \infty$ and for some v_s-moderate weight m.*
3. *$T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ is a compact operator for every $1 \leq p, q \leq \infty$ and for every v_s-moderate weight m.*

3.3. **PSDOs on $M^{p,q}_m$.** Finally, we are going to consider compactness of pseudo-differential operators in Kohn-Nirenberg form. They are a particular case of FIOs when $\Phi(x, y) = x \cdot y$, and hence $\chi(y, \eta) = (y, \eta)$. If Λ is a regular lattice with symmetric relatively compact fundamental domain Q, the map χ' is the identity, therefore, it is admissible. The class of matrices $C_{v,\chi'}$ is denoted by $\mathcal{C}_v = \mathcal{C}_v(\Lambda)$ and consists of all matrices $A = (a_{\gamma,\gamma'})_{\gamma,\gamma'\in\Lambda}$ such that

$$\|A\|_{\mathcal{C}_v} = \sum_{\gamma\in\Lambda} v(\gamma)\cdot\sup_{\lambda\in\Lambda}|a_{\lambda,\gamma+\lambda}| < \infty.$$

According to [13, Lemma 3.5], $\mathcal{C}_v$ is an algebra. Since the weight v is symmetric, it follows that

$$\sum_{\gamma\in\Lambda} v(\gamma)\cdot\sup_{\lambda\in\Lambda}|a_{\lambda,\gamma+\lambda}| = \sum_{\gamma\in\Lambda} v(\gamma)\cdot\sup_{\lambda\in\Lambda}|a_{\gamma+\lambda,\lambda}|.$$

This means that $A\in\mathcal{C}_v$ if and only if $A^t\in\mathcal{C}_v$. Each $A\in\mathcal{C}_v$ defines a bounded operator

$$A:\ell^{p,q}_m(\Lambda)\to\ell^{p,q}_m(\Lambda)$$

for $p,q\in[1,\infty]\cup\{0\}$ and each v-moderate sequence m. The compactness of the map is independent on p, q and on m. This allows us to improve results obtained in [10] and [11].

For convenience, we state the results for Weyl pseudodifferential operators. We recall that every operator from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ can be represented as a pseudodifferential operator L_σ with Weyl symbol σ and as a pseudodifferential operator in Kohn-Nirenberg form with symbol τ. We refer to [12, Chapter 14] where the relation between σ and τ is established. In particular, for $s\geq 0$, $\sigma\in M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$ if and only if $\tau\in M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$.

Theorem 3.22. *Let $\sigma\in M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$ be given. Then the following statements are equivalent:*

(1) $L_\sigma: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ *is compact.*
(2) $L_\sigma: M^{p,q}_m(\mathbb{R}^d)\to M^{p,q}_m(\mathbb{R}^d)$ *is compact for all* p, $q\in[1,\infty]$ *and every* v_s*-moderate weight* m.
(3) $L_\sigma: M^{p,q}_m(\mathbb{R}^d)\to M^{p,q}_m(\mathbb{R}^d)$ *is compact for some* p, $q\in[1,\infty]$ *and some* v_s*-moderate weight* m.
(4) $\sigma\in M^0(\mathbb{R}^{2d})$.

Proof. Let $\mathcal{G}(g,\Lambda)$ be a Gabor frame with $g\in\mathcal{S}(\mathbb{R}^d)$ and $\Lambda=\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$ for α, $\beta>0$. Then, according to [13, Theorem 3.2],

$$M(\sigma):=(\langle L_\sigma\pi(\lambda)g,\pi(\mu)g\rangle)_{(\mu,\lambda)\in\Lambda\times\Lambda}\in\mathcal{C}_{v_s}(\Lambda).$$

Moreover, it follows from (2.1) and (2.2) that $L_\sigma: M^{p,q}_m(\mathbb{R}^d)\to M^{p,q}_m(\mathbb{R}^d)$ is compact if and only if $M(\sigma):\ell^{p,q}_m(\Lambda)\to\ell^{p,q}_m(\Lambda)$ is. Now, the equivalences among (1), (2), and (3) follow from Proposition 3.19. Finally, since $M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})\subset M^{\infty,1}(\mathbb{R}^{2d})$ we can apply [10, Theorem 4.6] to obtain that condition (1) is equivalent to condition (4). ☐

Alternatively we could argue as follows. According to Theorem 3.19, $M(\sigma)$: $\ell^{p,q}_m(\Lambda)\to\ell^{p,q}_m(\Lambda)$ is a compact operator if and only if

$$(\langle L_\sigma\pi(\lambda)g,\pi(\lambda+\mu)g\rangle)_{\lambda\in\Lambda}\in c_0(\Lambda) \tag{3.12}$$

for every $\mu\in\Lambda$. By [13, 3.1],

$$|\langle L_\sigma\pi(\lambda)g,\pi(\lambda+\mu)g\rangle| = \left|V_\Phi\sigma(\lambda+\frac{\mu}{2},j(\mu))\right|,$$

where $\Phi=W(g,g)$, and $j:\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ is the map $j(\xi,\omega)=(\omega,-\xi)$. This permits us to prove that condition (3.12) is equivalent to the fact that $\sigma\in M^0(\mathbb{R}^{2d})$.

We want to finish with some comments regarding localization operators (see, for instance, [9, 11] and the references therein). The compact localization operators on $L^2(\mathbb{R}^d)$ were characterized in [9] in terms of the behavior of the STFT of their symbols. The condition there obtained also gives compactness for the localization operators when acting on weighted modulation spaces of Hilbert-type $M^2_m(\mathbb{R}^d)$ ([11, 5.6]). Since every localization operator can be described as a PSDO in Weyl form, Theorem 3.22 permits us to conclude that the compactness of the localization operator on a modulation class $M^p_m(\mathbb{R}^d)$ does not depend on p nor m. This conclusion could not be achieved with the techniques used in [11].

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