

NONPOLAR SINGULARITIES OF LOCAL ZETA FUNCTIONS IN SOME SMOOTH CASE

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Dedicated to Professor Takeo Ohsawa on the occasion of his retirement

ABSTRACT. It is known that local zeta functions associated with real analytic functions can be analytically continued as meromorphic functions to the whole complex plane. In this paper, the case of specific (nonreal analytic) smooth functions is precisely investigated. Indeed, asymptotic limits of the respective local zeta functions at some singularities in one direction are explicitly computed. Surprisingly, it follows from these behaviors that these local zeta functions have singularities different from poles.

1. INTRODUCTION

Let us consider the following integrals of the form

$$(1.1) \quad Z_f(\varphi)(s) = \int_{\mathbb{R}^2} |f(x, y)|^s \varphi(x, y) dx dy \quad \text{for } s \in \mathbb{C},$$

where f, φ are real-valued (C^∞) smooth functions defined on an open neighborhood U of the origin in \mathbb{R}^2 , and the support of φ is contained in U . Since the integrals $Z_f(\varphi)(s)$ converge locally uniformly on the region $\operatorname{Re}(s) > 0$, they become holomorphic functions there. It has been known in many cases that they can be holomorphically continued to wider regions. After this process, these integrals become holomorphic functions on domains containing the region $\operatorname{Re}(s) > 0$, which are sometimes called *local zeta functions*. In this paper, we are interested in the case in which f satisfies the condition

$$(1.2) \quad f(0, 0) = 0, \quad \nabla f(0, 0) = (0, 0),$$

and hereafter we always assume this condition. (Our issues in this paper are easy unless (1.2) is satisfied.)

In order to see the region where $Z_f(\varphi)$ can be holomorphically continued, the following index plays important roles:

$$(1.3) \quad c_0(f) := \sup \left\{ \mu > 0 : \begin{array}{l} \text{there exists an open neighborhood } V \text{ of} \\ \text{the origin in } U \text{ such that } |f|^{-\mu} \in L^1(V) \end{array} \right\}.$$

This $c_0(f)$ is called the *critical integrability index* of f (it is also called the *log canonical threshold* or the *singularity exponent*). The determination of the value of $c_0(f)$ is an important issue in the singularity theory, and there have been many interesting works from many points of view (this problem will be discussed later).

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In order to see a clear relationship between the index $c_0(f)$ and the region where $Z_f(\varphi)(s)$ becomes holomorphic, we assume in the Introduction that φ satisfies the condition

$$(1.4) \quad \varphi(0, 0) > 0, \quad \varphi(x, y) \geq 0 \quad \text{on } U.$$

Indeed, under condition (1.4), the relationship between the convergence of the integrals (1.1) and their holomorphy implies the equality

$$(1.5) \quad c_0(f) = \sup \left\{ \rho > 0 : \begin{array}{l} \text{the domain in which } Z_f(\varphi) \text{ can be holomorphically} \\ \text{continued contains the region } \operatorname{Re}(s) > -\rho \end{array} \right\}.$$

Without condition (1.4), the right-hand side of (1.5) may be greater than $c_0(f)$ (see [19], etc.).

The above equality (1.5) means that $Z_f(\varphi)$ is holomorphic on the region $\operatorname{Re}(s) > -c_0(f)$ and, moreover, that $Z_f(\varphi)$ has some singularities on the vertical line $\operatorname{Re}(s) = -c_0(f)$. More exactly, the following lemma implies that $Z_f(\varphi)$ must have a singularity at $s = -c_0(f)$.

Lemma 1.1. *$Z_f(\varphi)$ cannot be holomorphically continued to any open neighborhood of $s = -c_0(f)$.*

The proof of the above lemma will be given in Section 6. The purpose of this paper is to consider the following question.

Question 1. What kind of singularity does $Z_f(\varphi)$ have at $s = -c_0(f)$?

Under certain assumptions of f , the integrals $Z_f(\varphi)(s)$ have meromorphic continuation to the whole complex plane and, in particular, they have a pole at $s = -c_0(f)$. A brief history of the studies done on this phenomena is as follows. In 1954, in an invited talk at ICM Amsterdam, I. M. Gel'fand conjectured that if f is a polynomial and the support of φ is sufficiently small, then the integrals $Z_f(\varphi)(s)$ can be analytically continued as meromorphic functions to the whole complex plane. The case in which f are monomials was investigated in [12]. Gel'fand's conjecture was affirmatively solved as a stronger form by Bernstein, S. I. Gel'fand [4], and Atiyah [2] independently. They showed that if f is real analytic and the support of φ is sufficiently small, then $Z_f(\varphi)$ can be analytically continued as a meromorphic function to the whole complex plane, and their poles belong to the union of finite number of arithmetic progressions which consist of negative rational numbers. Their proofs use Hironaka's resolution of singularities [15]. The authors [18] generalize these results when f belongs to some class of smooth functions.

More precisely, let us consider an issue about the determination of the value of $c_0(f)$. We are also interested in the more detailed local behavior of $Z_f(\varphi)$ near $s = -c_0(f)$. In the real analytic case, this issue is to decide the location and order of the leading pole for $Z_f(\varphi)$. In the seminal work of Varchenko [23], when f is real analytic and satisfies some conditions, $c_0(f)$ can be expressed by using the *Newton polyhedron* of f as

$$(1.6) \quad c_0(f) = 1/d(f),$$

where $d(f)$ is the *Newton distance* of f (see [23], [1]) and the order of pole at $s = -1/d(f)$ depends on some topological information of the Newton polyhedron

of f . He uses the theory of toric varieties based on the geometry of Newton polyhedra. More detailed situation of meromorphic continuation of $Z_f(\varphi)$ is investigated in [8], [9], [7], [21], etc. A recent interesting work [5] treating the equality (1.6) is from another approach. We remark that the above-mentioned studies treat the general dimensional case. In the same paper [23], Varchenko more deeply investigated the two-dimensional case. Indeed, without any assumption, he shows that the equality (1.6) holds for real analytic f on *adapted coordinates*. Here adapted coordinates are important coordinates in the study of oscillatory integrals, and their existence is shown in two dimensions in [23], [22], [16], etc. More generally, let us consider the smooth case. In the above cited paper [18], the authors show that Varchenko’s result can be naturally generalized in a certain restricted class of smooth functions. On the other hand, Greenblatt [13] obtains a sharp result which generalizes Varchenko’s two-dimensional result.

Theorem 1.2 (Greenblatt [13]). *When f is a smooth function defined on U in \mathbb{R}^2 , the equality (1.6) holds on adapted coordinates.*

In more detail, in the same paper [13], Greenblatt explains the delicate situation about the local integrability of $|f|^{-\mu}$ around $\mu = c_0(f)$ in the smooth case by using the specific function

$$(1.7) \quad f(x, y) = x^a y^b + x^a y^{b-2} e^{-1/|x|^{1/(2b)}},$$

where a, b are nonnegative integers satisfying $a < b$ and $2 \leq b$ (see Remark 3.2 for his result). Note that the second term in (1.7) is a (nonreal analytic) flat function. The purpose of this paper is to investigate a slight generalization of the above example more deeply and to understand a detailed situation of analytic continuation of the respective local zeta functions.

This paper is organized as follows. In Section 2, we state a main theorem showing the failure of meromorphy of some local zeta functions. In order to show the main theorem, we substantially investigate similar integrals $Z(\sigma)$ in Section 3. The main theorem is shown in Section 4 by using the results in Section 3. In Section 5, we give some property of domains of convergence of the integrals $Z_f(\varphi)(s)$, which is analogous to Landau’s theorem on the Dirichlet series with positive coefficients. This result implies Lemma 1.1. Our computations in this paper are very specific and, it is to be hoped, give good observations for future studies about properties of local zeta functions in the general smooth case. From our results, many elementary (but probably not so easy) questions are naturally raised, some of which are listed in Section 6.

In this paper, we use $C, C_1, C_2, \epsilon, \delta$ for various kinds of constants without further comment.

2. MAIN RESULTS

In this section, we consider the integrals $Z_f(\varphi)(s)$ with smooth functions f of the following form:

$$(2.1) \quad f(x, y) = x^a y^b + x^a y^{b-q} e^{-1/|x|^p},$$

where a, b, p, q satisfy the following:

- a, b, q are nonnegative integers satisfying $a < b, 2 \leq b, 1 \leq q \leq b$.
- p is a positive real number.

We remark that $e^{-1/|x|^p}$ is regarded as a smooth function defined on \mathbb{R} by considering that its value at 0 takes 0. As mentioned in the Introduction, the above function f slightly generalizes the function (1.7), which is investigated by Greenblatt [13] (he considers the case in which $q = 2$ and $p = 1/(2b)$). Note that the coordinate (x, y) satisfies the adapted coordinates conditions in [16], and that the Newton distance of f is b (i.e., $d(f) = b$).

Let us consider the case in which the second term, which is flat, does not appear in (2.1) (i.e., $f(x, y) = x^a y^b$). From [12], it is easy to see that $Z_f(\varphi)$ can be regarded as a meromorphic function on \mathbb{C} and that the poles of $Z_f(\varphi)$ are contained in the set $\{-j/a, -k/b : j, k \in \mathbb{N}\}$. When φ satisfies (1.4), the leading pole exists at $s = -1/b$, whose order is 1.

Now, when f is as in (2.1), it follows from Theorem 1.2 and Lemma 1.1 in the Introduction that $Z_f(\varphi)$ is holomorphic on the region $\text{Re}(s) > -1/b$ and that $Z_f(\varphi)$ has a singularity at $s = -1/b$. (Note that $c_0(f) = 1/b$.) More precisely, we see the behavior at $-1/b$ of the restriction of $Z_f(\varphi)$ to the real axis as follows. In this paper, we use the symbol $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, which is traditionally used in the analysis of the Riemann zeta function.

Theorem 2.1. *Let f be as in (2.1), and let φ be as in (1.1). We assume that q is even. Then the following hold:*

(i) *If $p > 1 - a/b$, then*

$$(2.2) \quad \lim_{\sigma \rightarrow -1/b+0} (b\sigma + 1)^{1 - \frac{1-a/b}{p}} \cdot Z_f(\varphi)(\sigma) = 4A \cdot \varphi(0, 0),$$

where A is the positive constant defined by

$$(2.3) \quad A = \int_0^\infty x^{-a/b} (1 - e^{-1/(qx^p)}) dx.$$

Note that the above improper integral converges.

(ii) *If $p = 1 - a/b$, then*

$$(2.4) \quad \lim_{\sigma \rightarrow -1/b+0} |\log(b\sigma + 1)|^{-1} \cdot Z_f(\varphi)(\sigma) = \frac{4}{pq} \cdot \varphi(0, 0).$$

(iii) *If $0 < p < 1 - a/b$, then there exists a constant $B(\varphi)$ which depends on a, b, p, q, φ but is independent of σ such that*

$$(2.5) \quad \lim_{\sigma \rightarrow -1/b+0} Z_f(\varphi)(\sigma) = B(\varphi),$$

where $B(\varphi)$ is positive if φ satisfies the condition (1.4).

Of course, if $Z_f(\varphi)$ had a pole of order m at $s = -1/b$, then $\lim_{\sigma \rightarrow -1/b+0} (b\sigma + 1)^m \cdot Z_f(\varphi)(\sigma)$ must be a positive value. Noticing that $0 < 1 - \frac{1-a/b}{p} < 1$, we can see the following from the above theorem with Lemma 1.1 in the Introduction.

Corollary 2.2. *Under the assumption in Theorem 2.1 with the condition (1.4) on φ , $Z_f(\varphi)$ cannot be meromorphically continued to any open neighborhood of $s = -1/b$ in \mathbb{C} . In other words, the singularity of $Z_f(\varphi)(s)$ at $s = -1/b$ is different from a pole.*

If $\text{Re}(s) > -c_0(f)$, then $|f|^s$ can be regarded as a *distribution* by considering the map from $C_0^\infty(U)$ to \mathbb{C} defined by

$$\varphi \mapsto \langle |f|^s, \varphi \rangle = \int_{\mathbb{R}^2} |f|^s \varphi dx dy = Z_f(\varphi)(s).$$

Furthermore, the equalities (2.2), (2.4) in (i), (ii) can be interpreted as in the following:

$$(2.6) \quad \begin{aligned} \lim_{\sigma \rightarrow -1/b+0} (b\sigma + 1)^{1-\frac{1-a/b}{p}} |f|^\sigma &= 4A\delta, \\ \lim_{\sigma \rightarrow -1/b+0} |\log(b\sigma + 1)|^{-1} |f|^\sigma &= \frac{1}{pq}\delta, \end{aligned}$$

where $\delta \in \mathcal{D}'(U)$ is Dirac's delta function. The limits in the left-hand sides of (2.6) are taken in the topology of $\mathcal{D}'(U)$. On the other hand, the map B from $C_0^\infty(U)$ to \mathbb{R} , defined by $\varphi \mapsto B(\varphi) (= \langle B, \varphi \rangle)$ from (iii), can also be interpreted as a distribution:

$$\lim_{\sigma \rightarrow -1/b+0} |f|^\sigma = B.$$

The continuity of the above map B will be explained in Remark 4.1.

3. ASYMPTOTIC LIMITS OF ASSOCIATED INTEGRALS

For a set U in \mathbb{R}^2 , let us define the integral of the form

$$(3.1) \quad Z_U(\sigma) := \int_U \left| x^a y^b + x^a y^{b-q} e^{-1/|x|^p} \right|^\sigma dx dy \quad \text{for } \sigma < 0,$$

where a, b, p, q satisfy the conditions in the previous section.

In this section, we consider the case in which $U = U_+(r_1, r_2)$ with $r_1, r_2 \in (0, 1)$, where

$$U_+(r_1, r_2) = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq r_1, 0 \leq y \leq r_2\},$$

and simply denote

$$(3.2) \quad Z(\sigma) := Z_{U_+(r_1, r_2)}(\sigma).$$

Let $e(x)$ be the smooth function defined by

$$(3.3) \quad e(x) := \exp\left(\frac{-1}{qx^p}\right) \quad \text{for } x > 0$$

and $e(0) = 0$ which frequently appears in the computation below. Since the function e is monotonically increasing, we can define the function $\rho : [0, \infty) \rightarrow [0, r_1]$ by

$$(3.4) \quad \rho(y) = \begin{cases} e^{-1}(y) = \left(\frac{-1}{q \log y}\right)^{1/p} & \text{if } 0 \leq y < e(r_1), \\ r_1 & \text{if } y \geq e(r_1). \end{cases}$$

Since the Newton distance of f equals b , Theorem 1.2, due to Greenblatt, implies that the integral $Z(\sigma)$ converges if $\sigma > -1/b$ and diverges if $\sigma < -1/b$. The purpose of this section is to compute exact asymptotic limits of $Z(\sigma)$ as $\sigma \rightarrow -1/b + 0$.

Theorem 3.1. *The integral $Z(\sigma)$ satisfies the following.*

(i) *If $p > 1 - a/b$, then*

$$\lim_{\sigma \rightarrow -1/b+0} (b\sigma + 1)^{1-\frac{1-a/b}{p}} \cdot Z(\sigma) = A,$$

where A is as in (2.3).

(ii) *If $p = 1 - a/b$, then*

$$\lim_{\sigma \rightarrow -1/b+0} |\log(b\sigma + 1)|^{-1} \cdot Z(\sigma) = \frac{1}{pq}.$$

(iii) If $0 < p < 1 - a/b$, then the limit of $Z(\sigma)$ as $\sigma \rightarrow -1/b + 0$ exists and it satisfies

$$\begin{aligned} & \max \left\{ \frac{L(\lambda)}{(1 + \lambda^q)^{1/b}} + \frac{M(\lambda)}{(1 + \lambda^{-q})^{1/b}} : \lambda > 0 \right\} \\ & \leq \lim_{\sigma \rightarrow -1/b+0} Z(\sigma) \leq \min \{L(\lambda) + M(\lambda) : \lambda > 0\}, \end{aligned}$$

where $L(\lambda), M(\lambda)$ are positive constants depending on λ as in (3.13), (3.28) in the proof below.

Remark 3.2. In [13], Greenblatt shows the boundedness of $Z(\sigma)$ near $\sigma = -1/b$ in the case of $p = 1/(2b)$ and $q = 2$, which is contained in the above case (iii).

3.1. Auxiliary function with a parameter. Let λ be a positive number. The set $U_+(r_1, r_2)$ is decomposed as $U_1(\lambda) \cup U_2(\lambda)$ with

$$\begin{aligned} U_1(\lambda) &= \{(x, y) \in U_+(r_1, r_2) : \lambda y \geq e(x)\}, \\ U_2(\lambda) &= \{(x, y) \in U_+(r_1, r_2) : \lambda y < e(x)\}. \end{aligned}$$

The integral $Z(\sigma)$ is expressed as

$$(3.5) \quad Z(\sigma) = Z_1^{(\lambda)}(\sigma) + Z_2^{(\lambda)}(\sigma),$$

where

$$(3.6) \quad Z_j^{(\lambda)}(\sigma) = \int_{U_j(\lambda)} (x^a y^b + x^a y^{b-q} e^{-1/x^p})^\sigma dx dy \quad \text{for } j = 1, 2.$$

Note that

$$(3.7) \quad \begin{aligned} x^a y^b (1 + y^{-q} e^{-1/x^p}) &\leq (1 + \lambda^q) x^a y^b && \text{for } (x, y) \in U_1(\lambda), \\ x^a y^{b-q} e^{-1/x^p} (y^q e^{1/x^p} + 1) &\leq (1 + \lambda^{-q}) x^a y^{b-q} e^{-1/x^p} && \text{for } (x, y) \in U_2(\lambda). \end{aligned}$$

Thus, each $Z_j^{(\lambda)}(\sigma)$ can be estimated by using the following integrals:

$$(3.8) \quad \tilde{Z}_1^{(\lambda)}(\sigma) = \int_{U_1(\lambda)} x^{a\sigma} y^{b\sigma} dx dy,$$

$$(3.9) \quad \tilde{Z}_2^{(\lambda)}(\sigma) = \int_{U_2(\lambda)} x^{a\sigma} y^{(b-q)\sigma} e^{-\sigma/x^p} dx dy.$$

Indeed, $Z_1^{(\lambda)}(\sigma)$ and $Z_2^{(\lambda)}(\sigma)$ are convergent if and only if so are $\tilde{Z}_1^{(\lambda)}(\sigma)$ and $\tilde{Z}_2^{(\lambda)}(\sigma)$. Moreover, they satisfy

$$(3.10) \quad \begin{aligned} (1 + \lambda^q)^\sigma \tilde{Z}_1^{(\lambda)}(\sigma) &< Z_1^{(\lambda)}(\sigma) < \tilde{Z}_1^{(\lambda)}(\sigma), \\ (1 + \lambda^{-q})^\sigma \tilde{Z}_2^{(\lambda)}(\sigma) &< Z_2^{(\lambda)}(\sigma) < \tilde{Z}_2^{(\lambda)}(\sigma) \end{aligned}$$

for $\sigma < 0$. In order to prove Theorem 3.1, let us investigate the behaviors of $\tilde{Z}_1^{(\lambda)}(\sigma)$ and $\tilde{Z}_2^{(\lambda)}(\sigma)$ as $\sigma \rightarrow -1/b + 0$.

3.2. Preliminary lemma. For $\alpha > 0$, let ψ_α be the smooth function defined by

$$(3.11) \quad \psi_\alpha(x) := \frac{1 - \alpha^x}{x} \quad \text{for } x > 0.$$

The following properties of ψ_α play important roles in the computation below.

Lemma 3.3. *The function ψ_α satisfies the following properties.*

(i) *There exist positive constants C_1, C_2 such that*

$$-\log \alpha - C_1x < \psi_\alpha(x) < -\log \alpha - C_2x$$

for $x \in (0, 1)$. In particular, $\lim_{x \rightarrow +0} \psi_\alpha(x) = -\log \alpha$.

(ii) *If $\alpha \in (0, 1)$, then $\lim_{x \rightarrow \infty} \psi_\alpha(x) = 0$.*

(iii) *If $\alpha \in (0, 1)$, then ψ_α is monotonically decreasing, in particular, $0 < \psi_\alpha(x) < -\log \alpha$ for $x > 0$.*

Proof. The above properties of ψ_α can be easily seen by using Taylor’s formula. \square

Remark 3.4. In the computation below, the function $1 - e(x)$ often appears. This function can be expressed by using ψ_α with $\alpha = \exp(-q^{-1})$ as follows:

$$1 - e(x) = \frac{1 - \exp(-q^{-1}x^{-p})}{x^{-p}} \cdot x^{-p} = \psi_\alpha(x^{-p})x^{-p}.$$

From Lemma 3.3, we can see that

$$(3.12) \quad q^{-1}x^{-p} - C_1x^{-2p} \leq 1 - e(x) \leq q^{-1}x^{-p} - C_2x^{-2p} \quad \text{for } x \geq 1,$$

where C_1, C_2 are positive constants.

3.3. Asymptotics of $\tilde{Z}_1^{(\lambda)}(\sigma)$. Let us investigate the behavior of $\tilde{Z}_1^{(\lambda)}(\sigma)$, which is essentially important.

Lemma 3.5.

(i) *If $p > 1 - a/b$, then*

$$\lim_{\sigma \rightarrow -1/b+0} (b\sigma + 1)^{1 - \frac{1-a/b}{p}} \cdot \tilde{Z}_1^{(\lambda)}(\sigma) = A,$$

where A is as in (2.3).

(ii) *If $p = 1 - a/b$, then*

$$\lim_{\sigma \rightarrow -1/b+0} |\log(b\sigma + 1)|^{-1} \cdot \tilde{Z}_1^{(\lambda)}(\sigma) = \frac{1}{pq}.$$

(iii) *If $0 < p < 1 - a/b$, then $\lim_{\sigma \rightarrow -1/b+0} \tilde{Z}_1^{(\lambda)}(\sigma) = L(\lambda)$, with*

$$(3.13) \quad L(\lambda) = \frac{\rho(\lambda r_2)^{1-a/b}}{1 - a/b} \cdot \log(\lambda r_2) + \frac{\rho(\lambda r_2)^{1-a/b-p}}{q(1 - a/b - p)}.$$

Proof. In the proof, we introduce the variable

$$X = b\sigma + 1,$$

which will be convenient for many kinds of limit processes later. Note that

$$\sigma \rightarrow -1/b + 0 \iff X \rightarrow +0.$$

Now, applying an iterated integral to (3.8), we have

$$(3.14) \quad \tilde{Z}_1^{(\lambda)}(\sigma) = \frac{1}{\lambda^X} \frac{1}{X} \int_0^{\rho(\tilde{r}_2)} x^{a\sigma} (\tilde{r}_2^X - e(x)^X) dx,$$

where $\tilde{r}_2 := \lambda r_2$.

Case (i): $p > 1 - a/b$. Changing the integral variable in (3.14) by

$$x = X^{1/p}u \iff u = X^{-1/p}x \quad (dx = X^{1/p}du),$$

we have

$$(3.15) \quad \tilde{Z}_1^{(\lambda)}(\sigma) = \frac{1}{\lambda X} \frac{1}{X^{1-\frac{1+a\sigma}{p}}} \int_0^{\rho(\tilde{r}_2)/X^{1/p}} u^{a\sigma}(\tilde{r}_2^X - e(u))du.$$

We decompose the integral in (3.15) as $G_1(\sigma) + G_2(\sigma) + G_3(\sigma)$, with

$$(3.16) \quad \begin{aligned} G_1(\sigma) &= \int_0^1 u^{a\sigma}(\tilde{r}_2^X - e(u))du, \\ G_2(\sigma) &= \int_1^{\rho(\tilde{r}_2)/X^{1/p}} u^{a\sigma}(1 - e(u))du, \\ G_3(\sigma) &= (\tilde{r}_2^X - 1) \int_1^{\rho(\tilde{r}_2)/X^{1/p}} u^{a\sigma} du. \end{aligned}$$

The limit of $G_1(\sigma)$. Since $\tilde{r}_2^X \leq \max\{1, \lambda\}$, the integrand can be estimated as

$$u^{a\sigma}(\tilde{r}_2^X - e(u)) < u^{-a/b} \max\{1, \lambda\}$$

for $(X, u) \in (0, 1] \times (0, 1]$. Since $-a/b > -1$, the Lebesgue convergence theorem implies that

$$(3.17) \quad \lim_{\sigma \rightarrow -1/b+0} G_1(\sigma) = \int_0^1 u^{-a/b}(1 - e(u))du.$$

The limit of $G_2(\sigma)$. Let $\epsilon_0 := a/b + p - 1 > 0$. Then

$$(3.18) \quad a\sigma - p + 1 = \frac{a}{b}(b\sigma + 1) - \frac{a}{b} - p + 1 = \frac{a}{b}X - \epsilon_0.$$

Since $u < \rho(\tilde{r}_2)/X^{1/p}$ ($\Leftrightarrow X < \rho(\tilde{r}_2)^p u^{-p}$) and $e^x \leq 1 + (e - 1)x$ for $x \in (0, 1)$, there exists a $\delta > 0$ such that

$$(3.19) \quad 1 < u^{\frac{a}{b}X} = \exp\left(\frac{a}{b}X \cdot \log u\right) < 1 + Du^{-p} \log u$$

for $(u, X) \in [1, \rho(\tilde{r}_2)/X^{1/p}) \times (0, \delta)$, where $D := (e - 1)\frac{a}{b}\rho(\tilde{r}_2)^p$. From (3.18), (3.12), (3.19), we have

$$\begin{aligned} u^{a\sigma}(1 - e(u)) &< q^{-1}u^{a\sigma-p} = q^{-1}u^{\frac{a}{b}X-1-\epsilon_0} \\ &< q^{-1}(1 + Du^{-p} \log u)u^{-1-\epsilon_0} \leq Cu^{-1-\epsilon_0} \end{aligned}$$

for $(u, X) \in [1, \rho(\tilde{r}_2)/X^{1/p}) \times (0, \delta)$, where $C > 0$ is a constant independent of u and σ . Thus, the Lebesgue convergence theorem implies that

$$(3.20) \quad \lim_{\sigma \rightarrow -1/b+0} G_2(\sigma) = \int_1^\infty u^{-a/b}(1 - e(u))du.$$

The limit of $G_3(\sigma)$. Since $1 - \tilde{r}_2^X = X\psi_{\tilde{r}_2}(X)$, $G_3(\sigma)$ can be computed as follows:

$$G_3(\sigma) = X^{1-\frac{a\sigma+1}{p}}\psi_{\tilde{r}_2}(X) \cdot \frac{\rho(\tilde{r}_2)^{a\sigma+1} - X^{(a\sigma+1)/p}}{a\sigma + 1}.$$

From Lemma 3.3(i), we can see the following:

$$(3.21) \quad \lim_{\sigma \rightarrow -1/b} X^{-1+\frac{a\sigma+1}{p}}G_3(\sigma) = (-\log \tilde{r}_2) \cdot \frac{\rho(\tilde{r}_2)^{1-a/b}}{1-a/b}.$$

Therefore, (3.17), (3.20), (3.21) imply that

$$\lim_{\sigma \rightarrow -1/b+0} X^{1-\frac{1-a/b}{p}}\tilde{Z}_1^{(\lambda)}(\sigma) = \lim_{\sigma \rightarrow -1/b+0} G_1(\sigma) + \lim_{\sigma \rightarrow -1/b+0} G_2(\sigma) = A.$$

Here we used the fact that $\lim_{X \rightarrow +0} X^{cX} = 1$ ($c \in \mathbb{R}$).

Case (ii) : $p = 1 - a/b$. Since the equalities (3.17) and (3.21) always hold for any $p > 0$, it suffices to consider the behavior of $G_2(\sigma)$ in the case in which $p = 1 - a/b$.

The limit of $G_2(\sigma)$. Since $\epsilon_0 = 0$ in (3.18), $\alpha\sigma = p - 1 + \frac{a}{b}X$ holds. Thus, the estimates (3.12), (3.19) imply that there exist $C_1, C_2, \delta > 0$ such that

$$(3.22) \quad q^{-1}u^{-1} - C_1u^{-p-1} < u^{a\sigma}(1 - e(u)) < q^{-1}u^{-1} + C_2u^{-p-1} \log u,$$

for $(u, X) \in [1, \rho(\tilde{r}_2)/X^{1/p}] \times (0, \delta)$.

Now, we rewrite (3.16) as follows:

$$\begin{aligned} G_2(\sigma) &= q^{-1} \int_1^{\rho(\tilde{r}_2)/X^{1/p}} \frac{1}{u} du - \int_1^{\rho(\tilde{r}_2)/X^{1/p}} \left(\frac{q^{-1}}{u} - u^{a\sigma}(1 - e(u)) \right) du \\ &=: H_1(\sigma) - H_2(\sigma). \end{aligned}$$

A direct computation gives

$$(3.23) \quad H_1(\sigma) = q^{-1}(\log \rho(\tilde{r}_2) - p^{-1} \log X).$$

From (3.22), there exist $\epsilon, C > 0$ such that

$$(3.24) \quad |H_2(\sigma)| \leq C \int_1^{\rho(\tilde{r}_2)/X^{1/p}} u^{-p+\epsilon-1} du \leq \frac{C}{p-\epsilon}.$$

From (3.17), (3.21), (3.23), (3.24), we have

$$\begin{aligned} \lim_{\sigma \rightarrow -1/b} |\log X|^{-1}\tilde{Z}_1^{(\lambda)}(\sigma) &= \lim_{\sigma \rightarrow -1/b} |\log X|^{-1}G_2(\sigma) \\ &= \lim_{\sigma \rightarrow -1/b} |\log X|^{-1}H_1(\sigma) = \frac{1}{pq}. \end{aligned}$$

Case (iii) : $0 < p < 1 - a/b$. From (3.14), $\tilde{Z}_1^{(\lambda)}(\sigma)$ can be decomposed as $J_1(\sigma) + J_2(\sigma)$, with

$$\begin{aligned} J_1(\sigma) &= \frac{1}{\lambda^X} \int_0^{\rho(\tilde{r}_2)} x^{a\sigma} \frac{1 - e(x)^X}{X} dx, \\ J_2(\sigma) &= \frac{1}{\lambda^X} \cdot \frac{\tilde{r}_2^X - 1}{X} \int_0^{\rho(\tilde{r}_2)} x^{a\sigma} dx. \end{aligned}$$

The limit of $J_1(\sigma)$. Using the function ψ_α with $\alpha := \exp(-q^{-1})$ and Lemma 3.3(i), we have

$$(3.25) \quad \frac{1 - e(x)^X}{X} = \frac{\psi_\alpha(Xx^{-p})}{x^p} \longrightarrow q^{-1}x^{-p} \quad \text{as } \sigma \rightarrow -1/b + 0.$$

On the other hand, Lemma 3.3(iii) implies that

$$\left| x^{a\sigma} \cdot \frac{1 - e(x)^X}{X} \right| \leq Cx^{-a/b-p}$$

for $(x, X) \in (0, \rho(\tilde{r}_2)] \times (0, 1)$, where $C > 0$ is a constant depending only on \tilde{r}_2 . Since $-a/b - p > -1$, the Lebesgue convergence theorem implies that

$$(3.26) \quad \lim_{\sigma \rightarrow -1/b+0} J_1(\sigma) = \frac{1}{q} \int_0^{\rho(\tilde{r}_2)} x^{-a/b-p} dx = \frac{\rho(\tilde{r}_2)^{1-a/b-p}}{q(1-a/b-p)}.$$

The limit of $J_2(\sigma)$. A direct computation gives

$$J_2(\sigma) = -\frac{1}{\lambda^X} \cdot \psi_{\tilde{r}_2}(X) \cdot \frac{\rho(\tilde{r}_2)^{a\sigma+1}}{a\sigma + 1}.$$

Therefore, we have

$$(3.27) \quad \lim_{\sigma \rightarrow -1/b+0} J_2(\sigma) = \frac{\rho(\tilde{r}_2)^{1-a/b}}{1-a/b} \cdot \log(\tilde{r}_2).$$

From (3.26), (3.27), we obtain the limit of $\tilde{Z}_1^{(\lambda)}(\sigma)$ in (iii). □

3.4. Asymptotics of $\tilde{Z}_2^{(\lambda)}(\sigma)$. The behavior of $\tilde{Z}_2^{(\lambda)}(\sigma)$ can easily be seen with a direct computation.

Lemma 3.6. $\lim_{\sigma \rightarrow -1/b} \tilde{Z}_2^{(\lambda)}(\sigma) = M(\lambda)$ with

$$(3.28) \quad M(\lambda) = \frac{b^2}{q(b-a)} \cdot \frac{1}{\lambda^{q/b}} \cdot \rho(\lambda r_2)^{1-a/b} + \frac{b}{q} r_2^{q/b} \int_{\rho(\lambda r_2)}^{r_1} x^{-a/b} \exp(1/(bx^p)) dx.$$

Proof. By decomposing the integral region $U_2(\lambda)$ into the following two sets,

$$\{(x, y) \in U_2(\lambda) : x \leq \rho(\tilde{r}_2)\}, \quad \{(x, y) \in U_2(\lambda) : x > \rho(\tilde{r}_2)\},$$

the integral $\tilde{Z}_2^{(\lambda)}(\sigma)$ can be computed as

$$\tilde{Z}_2^{(\lambda)}(\sigma) = \frac{1}{X - q\sigma} \cdot \frac{1}{\lambda^{X-q\sigma}} \int_0^{\rho(\tilde{r}_2)} x^{a\sigma} e(x)^X dx + \frac{r_2^{X-q\sigma}}{X - q\sigma} \int_{\rho(\tilde{r}_2)}^{r_1} x^{a\sigma} e^{-\sigma/x^p} dx.$$

It can easily be computed that the limit of the right-hand side of the above equation as $\sigma \rightarrow -1/b + 0$ is $M(\lambda)$ in (3.28). □

3.5. Proof of Theorem 3.1. From (3.5), (3.10), when the integrals $\tilde{Z}_1^{(\lambda)}(\sigma)$ and $\tilde{Z}_2^{(\lambda)}(\sigma)$ converge, $Z(\sigma)$ can be estimated as

$$(3.29) \quad \begin{aligned} & (1 + \lambda^q)^\sigma \cdot \tilde{Z}_1^{(\lambda)}(\sigma) + (1 + \lambda^{-q})^\sigma \cdot \tilde{Z}_2^{(\lambda)}(\sigma) \\ & < Z(\sigma) < \tilde{Z}_1^{(\lambda)}(\sigma) + \tilde{Z}_2^{(\lambda)}(\sigma). \end{aligned}$$

First, let us consider case (i). Since Lemmas 3.5(i) and 3.7 imply that

$$\lim_{\sigma \rightarrow -1/b+0} X^{1-\frac{1-a/b}{p}} \cdot \tilde{Z}_1^{(\lambda)}(\sigma) = A, \quad \lim_{\sigma \rightarrow -1/b+0} X^{1-\frac{1-a/b}{p}} \cdot \tilde{Z}_2^{(\lambda)}(\sigma) = 0,$$

the estimates (3.29) give

$$\begin{aligned}
 (3.30) \quad (1 + \lambda^q)^{-1/b} \cdot A &\leq \varliminf_{\sigma \rightarrow -1/b+0} X^{1-\frac{1-a/b}{p}} \cdot Z(\sigma) \\
 &\leq \varlimsup_{\sigma \rightarrow -1/b+0} X^{1-\frac{1-a/b}{p}} \cdot Z(\sigma) \leq A.
 \end{aligned}$$

Note that $Z(\sigma)$ is independent of λ . Considering the limit as $\lambda \rightarrow 0$ in (3.30), we obtain (i). Case (ii) can be similarly shown.

Let us consider case (iii). Since the integral $Z(\sigma)$ is a monotone decreasing function as $\sigma \in (-1/b, 0)$, the boundedness of $Z(\sigma)$ from (3.29) implies the existence of the limit $\lim_{\sigma \rightarrow -1/b} Z(\sigma)$. Moreover, since $Z(\sigma)$ is independent of λ , the following inequalities can be obtained from (3.29), (3.13), (3.28):

$$\begin{aligned}
 (3.31) \quad \sup \left\{ \frac{L(\lambda)}{(1 + \lambda^q)^{1/b}} + \frac{M(\lambda)}{(1 + \lambda^{-q})^{1/b}} : \lambda > 0 \right\} \\
 \leq \lim_{\sigma \rightarrow -1/b+0} Z(\sigma) \leq \inf \{L(\lambda) + M(\lambda) : \lambda > 0\}.
 \end{aligned}$$

Furthermore, the supremum and infimum in (3.31) can be respectively replaced by the maximum and minimum by using the lemma below. As a result, the inequalities in (iii) in the theorem are obtained.

Lemma 3.7.

$$\begin{aligned}
 \lim_{\lambda \rightarrow 0} L(\lambda) = 0, \quad \lim_{\lambda \rightarrow 0} M(\lambda) = \infty, \quad \lim_{\lambda \rightarrow \infty} L(\lambda) = \infty, \quad \lim_{\lambda \rightarrow \infty} M(\lambda) = 0, \\
 \lim_{\lambda \rightarrow 0} (1 + \lambda^q)^{-1/b} \cdot L(\lambda) = 0, \quad \lim_{\lambda \rightarrow 0} (1 + \lambda^{-q})^{-1/b} \cdot M(\lambda) = 0, \\
 \lim_{\lambda \rightarrow \infty} (1 + \lambda^q)^{-1/b} \cdot L(\lambda) = 0, \quad \lim_{\lambda \rightarrow \infty} (1 + \lambda^{-q})^{-1/b} \cdot M(\lambda) = 0.
 \end{aligned}$$

The proof of the above lemma is easy, so it is left to the reader.

4. PROOF OF THEOREM 2.1

For $r_1, r_2 \in (0, 1)$, let

$$U(r_1, r_2) := \{(x, y) \in \mathbb{R}^n : |x| < r_1, |y| < r_2\}.$$

The behavior of $Z_f(\varphi)$ can be appropriately approximated by that of a more simple function $Z_{U(r_1, r_2)}(\sigma)$ (see (3.1)). Furthermore, under the assumption that q is even, $f(x, y) = f(|x|, |y|)$ for any $(x, y) \in \mathbb{R}^2$, which implies that

$$(4.1) \quad Z_{U(r_1, r_2)}(\sigma) = 4Z_{U_+(r_1, r_2)}(\sigma) = 4Z(\sigma).$$

Therefore, Theorem 2.1 can be proved by using Theorem 3.1 as follows.

Cases (i), (ii). For any $\epsilon > 0$, there exist $r_1, r_2 \in (0, 1)$ such that

$$\varphi(0, 0) - \epsilon \leq \varphi(x, y) \leq \varphi(0, 0) + \epsilon \quad \text{for } (x, y) \in U(r_1, r_2).$$

These inequalities imply that

$$\begin{aligned}
 (4.2) \quad (\varphi(0, 0) - \epsilon) \cdot Z_{U(r_1, r_2)}(\sigma) &\leq Z_f(\varphi)(\sigma) - \int_{U \setminus U(r_1, r_2)} |f(x, y)|^\sigma \varphi(x, y) dx dy \\
 &\leq (\varphi(0, 0) + \epsilon) \cdot Z_{U(r_1, r_2)}(\sigma).
 \end{aligned}$$

We remark that the integral in (4.2) converges and is bounded by a positive constant which is independent of σ since f does not vanish on $U \setminus U(r_1, r_2)$.

As a result, the inequalities (4.2) and Theorem 3.1 easily imply (i), (ii) in Theorem 2.1.

Cases (iii). We define

$$\varphi_+(x, y) = \max\{\varphi(x, y), 0\}$$

and

$$\varphi_-(x, y) = \max\{-\varphi(x, y), 0\}.$$

Of course, $\varphi(x, y) = \varphi_+(x, y) - \varphi_-(x, y)$ holds.

Now, let R_1, R_2 be positive constants such that the support of φ is contained in $U(R_1, R_2)$. Then Theorem 3.1(iii) implies that there exist $\delta > 0$ and $C_{\pm}(R_1, R_2) > 0$, which depends on R_1, R_2 and is independent of σ , such that

$$\begin{aligned} Z_f(\varphi_{\pm})(\sigma) &\leq \max_{(x,y) \in U} \varphi_{\pm}(x, y) \cdot Z_{\text{Supp}(\varphi_{\pm})}(\sigma) \\ (4.3) \qquad &\leq \max_{(x,y) \in U} \varphi_{\pm}(x, y) \cdot Z_{U(R_1, R_2)}(\sigma) \\ &\leq C_{\pm}(R_1, R_2) \cdot \max_{(x,y) \in U} \varphi_{\pm}(x, y) \end{aligned}$$

for $\sigma \in (-1/b, -1/b + \delta)$. Since $Z_f(\varphi_{\pm})(\sigma)$ are monotone decreasing functions of $\sigma \in (-1/b, 0)$, the above estimates easily imply that there exist nonnegative constants $B_+(\varphi)$ and $B_-(\varphi)$ such that

$$\lim_{\sigma \rightarrow -1/b+0} Z_f(\varphi_{\pm})(\sigma) = B_{\pm}(\varphi).$$

Let $B(\varphi) := B_+(\varphi) - B_-(\varphi)$. Then we can get the limit in (iii). Note that when φ satisfies (1.4), $B(\varphi) = B(\varphi_+)$ and $B(\varphi_-) = 0$ hold, so $B(\varphi)$ is positive.

Remark 4.1. The inequalities in (4.3) imply the continuous property for the distribution defined by the map: $\varphi \mapsto B(\varphi) = \lim_{\sigma \rightarrow -1/b} Z_f(\varphi)(\sigma)$.

5. LANDAU TYPE THEOREM FOR LOCAL ZETA FUNCTIONS

In this section, we deal with $Z_f(\varphi)(s)$ in the general dimensional case, i.e.,

$$(5.1) \qquad Z_f(\varphi)(s) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) dx \qquad \text{for } s \in \mathbb{C},$$

where f, φ are real-valued smooth functions defined on an open neighborhood U of the origin in \mathbb{R}^n and the support of φ is contained in U .

Lemma 1.1 in the Introduction follows from the following theorem.

Theorem 5.1. *Suppose that $f(0) = 0$, $|f(x)| < 1$, and $\varphi(x) \geq 0$ on U . Let ρ be a nonpositive number such that the integral $Z_f(\varphi)(s)$ in (5.1) converges if $\text{Re}(s) > \rho$. If $Z_f(\varphi)$ can be analytically continued as a holomorphic function to some open neighborhood of $s = \rho$, then there exists a positive number δ such that the integral $Z_f(\varphi)(\rho - \delta)$ converges.*

Indeed, if $Z_f(\varphi)(s)$ can be holomorphically continued across the point $s = -c_0(f)$, then the above theorem implies that $Z_f(\varphi)(s)$ becomes a holomorphic function on the region, $\text{Re}(s) > -c_0(f) - \delta$ with some positive δ , which is a contradiction of (1.5).

The above property of local zeta functions itself is interesting and is analogous to Landau's theorem on the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$, where a_n are nonnegative numbers (see [24]).

Proof. From the assumption, $Z_f(\varphi)$ can be considered a holomorphic function on the region $\text{Re}(s) > \rho$, and, moreover, there exist an open neighborhood V of $s = \rho$ and a holomorphic function \tilde{Z} defined on the set $V \cup \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$ such that $\tilde{Z} = Z_f(\varphi)$ on the region $\text{Re}(s) > \rho$.

Now, there exists a positive number δ such that the disc

$$D := \{z \in \mathbb{C} : |z - (\rho + 1)| < 1 + 2\delta\}$$

is contained in the region $V \cup \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$. Since \tilde{Z} is holomorphic on D , its Taylor series converges to the value of $\tilde{Z}(s)$ at any point of D , i.e.,

$$(5.2) \quad \tilde{Z}(s) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d^j \tilde{Z}}{ds^j}(\rho + 1)(s - (\rho + 1))^j \quad \text{for } s \in D.$$

Since the point $s = \rho - \delta$ is contained in D , we have

$$(5.3) \quad \tilde{Z}(\rho - \delta) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d^j \tilde{Z}}{ds^j}(\rho + 1)(-\delta - 1)^j,$$

and this series converges. On the other hand, we have

$$(5.4) \quad \frac{d^j \tilde{Z}}{ds^j}(s) = \int_{\mathbb{R}^n} |f(x)|^s (\log |f(x)|)^j \varphi(x) dx \quad \text{for } j \in \mathbb{N}$$

if s satisfies $\text{Re}(s) > \rho$. Indeed, it is easy to show the possibility of the exchange of integral and derivatives. Substituting (5.4) into (5.3), we have

$$(5.5) \quad \begin{aligned} \tilde{Z}(\rho - \delta) &= \sum_{j=0}^{\infty} \frac{1}{j!} (-\delta - 1)^j \int_{\mathbb{R}^n} |f(x)|^{\rho+1} (\log |f(x)|)^j \varphi(x) dx \\ &= \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} |f(x)|^{\rho+1} \frac{1}{j!} ((-\delta - 1) \log |f(x)|)^j \varphi(x) dx. \end{aligned}$$

Since all terms of the series in (5.5) are positive and the series converges, the order of the summation and the integral can be exchanged. Therefore,

$$\begin{aligned} \tilde{Z}(\rho - \delta) &= \int_{\mathbb{R}^n} |f(x)|^{\rho+1} \left(\sum_{j=0}^{\infty} \frac{1}{j!} ((-\delta - 1) \log |f(x)|)^j \right) \varphi(x) dx \\ &= \int_{\mathbb{R}^n} |f(x)|^{\rho+1} e^{(-\delta-1) \log |f(x)|} \varphi(x) dx \\ &= \int_{\mathbb{R}^n} |f(x)|^{\rho-\delta} \varphi(x) dx. \end{aligned}$$

The above equalities imply that the last integral converges. □

6. DISCUSSION AND OPEN QUESTIONS

6.1. Singularities of $Z_f(\varphi)(s)$. First, let us consider the case in which f is as in (2.1) and φ satisfies condition (1.4). As mentioned in Corollary 2.2, the singularity of $Z_f(\varphi)(s)$ at $s = -1/b$ is different from a pole. To be more elementary, the following question is naturally raised.

Question 2. Is the singularity of $Z_f(\varphi)$ at $s = -1/b$ isolated or not?

If this singularity was isolated, then it must be an essential singularity. At present, it seems impossible to answer this question with the information from Theorem 2.1 only.

Next, let us consider a more global property of $Z_f(\varphi)(s)$. It should be expected that $Z_f(\varphi)(s)$ can be holomorphically extended to a wider domain containing the region $\text{Re}(s) > -1/b$. The following question is considered a first step to addressing this problem.

Question 3. Does $Z_f(\varphi)$ have another singularity on the vertical line $\text{Re}(s) = -c_0(f)$?

6.2. Openness problem. Let f be a smooth function with the condition (1.2). Let us consider the following subset in \mathbb{R} :

$$H(f) := \left\{ \mu > 0 : \begin{array}{l} \text{there exists an open neighborhood } V \text{ of } \\ \text{the origin in } U \text{ such that } |f|^{-\mu} \in L^1(V) \end{array} \right\}.$$

(Of course, $c_0(f) = \sup H(f)$.) When f is real analytic, the set $H(f)$ is *open* in \mathbb{R} from the fact that $s = -c_0(f)$ is a pole for $Z_f(\varphi)$ for φ satisfying (1.4). Without the real analyticity assumption, our observation implies that $H(f)$ is not always open. More precisely, in the case of (2.1), the openness of $H(f)$ depends on the relationship among the parameters a, b, p . Generally, when f is a smooth function, the following question seems interesting.

Question 4. Which condition on f gives the openness (or closedness) of $H(f)$?

An analogous problem has been deeply investigated in the case of complex variables from the viewpoint of complex geometry. The *openness conjecture*, raised by Demailly and Kollár [6], is the following: “If ϕ is plurisubharmonic, then the set $H(e^{-\phi})$ is always open.” This conjecture has been affirmatively solved in [10], [11], [3]. Our observation indicates that the openness of $H(f)$ needs some kind of good property of f .

6.3. Oscillatory integrals. Let us consider an oscillatory integral of the form

$$I_f(\varphi)(\tau) := \int_{\mathbb{R}^2} e^{i\tau f(x,y)} \varphi(x,y) dx dy \quad \text{for } \tau > 0,$$

where f, φ are as in (1.1) and they satisfy the conditions (1.2), (1.4).

It is known (see [23], [1], etc.) that there is a deep relationship between the behavior of oscillatory integrals at infinity and the distribution of poles of local zeta functions. Indeed, the Mellin transformation gives a clear relationship between oscillatory integrals and some functions similar to local zeta functions.

First, let us consider the case in which f is real analytic. As mentioned in the Introduction, the integral $Z_f(\varphi)(s)$ can be analytically continued as a meromorphic function to the whole complex plane. Furthermore, under some assumption, its leading pole exists at $s = -1/d(f)$ and its order is $m(f)$. (Note that $m(f)$ is the positive integer determined by some topological information of the Newton polyhedron of f .) Using this fact, we have

$$(6.1) \quad \lim_{\tau \rightarrow +\infty} \tau^{1/d(f)} (\log \tau)^{-m(f)+1} I_f(\varphi)(\tau) = C_f(\varphi),$$

where $C_f(\varphi)$ is a positive constant independent of τ .

Next, let us consider the smooth case. Although formula (6.1) can be directly generalized in many smooth cases [14], [17], [18], there exist examples for which the

behavior of $I_f(\varphi)$ is different from that of (6.1). In [20], the authors investigate the case in which

$$(6.2) \quad f(x, y) = y^b + e^{-1/|x|^p},$$

where $p > 0$ and $b \in \mathbb{Z}$ with $b \geq 2$, and obtain the following:

$$\lim_{\tau \rightarrow +\infty} \tau^{1/b} (\log \tau)^{1/p} \cdot I_f(\varphi)(\tau) = C_b \varphi(0, 0),$$

where C_b is a nonzero constant depending only on b and is explicitly computed. We remark that $d(f) = b$ and $m(f) = 1$ in this case. Since (6.2) is a special case of (2.1) ($a = 0, b = q$), Theorem 2.1 implies that if $p > 1$, then

$$\lim_{\sigma \rightarrow -1/b+0} (b\sigma + 1)^{-1/p+1} \cdot Z_f(\varphi)(\sigma) = C\varphi(0, 0),$$

where C is a positive constant.

Now, we are interested in how flat functions affect the behavior of $I_f(\varphi)(\tau)$. Observing the case in which f is real analytic or f is as in (6.2), one can easily recognize some correspondence between the behaviors of $Z_f(\varphi)(\sigma)$ and $I_f(\varphi)(\tau)$. It will be valuable to affirmatively answer the following question.

Question 5. When a positive limit of $(\sigma + c_0(f))^\alpha Z_f(\varphi)(\sigma)$ as $\sigma \rightarrow -c_0(f) + 0$ exists, does a nonvanishing limit of $\tau^{c_0(f)} (\log \tau)^{-\alpha+1} \cdot I_f(\varphi)(\tau)$ as $\tau \rightarrow +\infty$ exist?

In particular, let f be as in (2.1) with $p > 1 - a/b$; then does the following,

$$\lim_{\tau \rightarrow +\infty} \tau^{1/b} (\log \tau)^{\frac{1-a/b}{p}} \cdot I_f(\varphi)(\tau) = C\varphi(0, 0),$$

where C is a positive constant which is independent of τ , hold?

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