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EQUIDISTRIBUTION THEOREMS ON STRONGLY PSEUDOCONVEX DOMAINS

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ABSTRACT. This work consists of two parts. In the first part, we consider a compact connected strongly pseudoconvex CR manifold X with a transversal CR S^1 action. We establish an equidistribution theorem on zeros of CR functions. The main techniques involve a uniform estimate of Szegő kernel on X. In the second part, we consider a general complex manifold M with a strongly pseudoconvex boundary X. By using classical result of Boutet de Monvel–Sjöstrand about Bergman kernel asymptotics, we establish an equidistribution theorem on zeros of holomorphic functions on $\overline{M}$.

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1. Introduction and statement of the main results

The study of equidistribution of zeros of holomorphic sections has become intensively active in recent years. Shiffman–Zelditch [19] established an equidistribution property for high powers of a positive line bundle. Dinh–Sibony [10] extended the equidistribution with estimate of convergence speed and applied it to general measures. More results about equidistribution of zeros of holomorphic sections in different cases, such as line bundles with singular metrics, general base spaces, and general measures, were obtained in [3–5, 8, 9, 17, 18]. Important methods to study equidistribution include uniform estimates for Bergman kernel functions [16, 20] and techniques for complex dynamics in higher dimensions [11]. Our article is the first to

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study equidistribution on CR manifolds and on complex manifolds with boundary. In the first part, we establish an equidistribution theorem on zeros of CR functions. The proof involves uniform estimates for Szegő kernel functions [13]. In the second part, we consider a general complex manifold M with a strongly pseudoconvex boundary X, and we establish an equidistribution theorem on zeros of holomorphic functions on $\overline{M}$ by using classical result of Boutet de Monvel–Sjöstrand [2].

We now state our main results. We refer to section 2 for some notations and terminology used here. Let $(X, T^{1,0}X)$ be a compact connected strongly pseudoconvex CR manifold with a transversal CR S^1 action $e^{i\theta}$ (cf. section 2), where $T^{1,0}X$ is a CR structure of X. The dimension of X is $2n+1$, $n \geq 1$. Denote by $T \in C^\infty(X, TX)$ the real vector field induced by the S^1 action. Take an S^1 invariant Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ such that there is an orthogonal decomposition $\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T$. Then there exists a natural global L^2 inner product $(\cdot | \cdot)$ on $C^\infty(X)$ induced by $\langle\cdot|\cdot\rangle$.

For every $q \in \mathbb{N}$, put

$$X_q := \{x \in X : e^{i\theta} \circ x \neq x, \forall \theta \in (0, \frac{2\pi}{q}),\ e^{i\frac{2\pi}{q}} \circ x = x\}.$$

Set $p := \min\{q \in \mathbb{N} : X_q \neq \emptyset\}$. Put $X_{\rm reg} = X_p$. For simplicity, we assume that $p = 1$. Since X is connected, X_1 is open and dense in X. Assume that $X = \cup_{j=0}^{t-1} X_{p_j}$, $1 = p_0 < p_1 < \cdots < p_{t-1}$, and put $X_{\rm sing} := \cup_{j=1}^{t-1} X_{p_j}$.

Let $\bar\partial_b : C^\infty(X) \to \Omega^{0,1}(X)$ be the tangential Cauchy–Riemann operator. For each $m \in \mathbb{Z}$, put

$$H^0_{b,m}(X) := \{u \in C^\infty(X) : Tu = imu, \bar\partial_b u = 0\}. \tag{1.1}$$

It is well known that $\dim H^0_{b,m}(X) < \infty$ (see [15]). Let $f_1 \in H^0_{b,m}(X), \ldots, f_{d_m} \in H^0_{b,m}(X)$ be an orthonormal basis for $H^0_{b,m}(X)$. The Szegő kernel function associated to $H^0_{b,m}(X)$ is given by:

$$S_m(x) := \sum_{j=1}^{d_m} |f_j(x)|^2 .$$

When the S^1 action is globally free, it is well known that $S_m(x) \approx m^n$ uniformly on X. When X is locally free, we only have $S_m(x) \approx m^n$ locally uniformly on $X_{\rm reg}$ in general (see Theorem 3.1). Moreover, $S_m(x)$ can be zero at some point of $X_{\rm sing}$ even for m large (see [15] and [12]). Let

$$\alpha = [p_1, \ldots, p_{t-1}]; \tag{1.2}$$

that is, α is the least common multiple of $p_1, \ldots, p_{t-1}$. In Theorem 3.5, we will show that there exist positive integers $1 = k_0 < k_1 < \cdots < k_{t-1}$ independent of m such that

$$cm^n \leq S_{\alpha m}(x) + S_{k_1 \alpha m}(x) + \cdots + S_{k_{t-1}\alpha m}(x) \leq \frac{1}{c} m^n \ \text{ on } X$$

for all $m \gg 1$, where $0 < c < 1$ is a constant independent of m. For each $m \in \mathbb{N}$, put

$$A_m(X) := \bigcup_{j=0}^{t-1} H^0_{b,k_j\alpha m}(X). \tag{1.3}$$

We write $d\mu_m$ to denote the normalized Haar measure on the unit sphere, defined in the natural way by using a fixed orthonormal basis,

$$SA_m(X) := \{g \in A_m(X);\ (\,g\,|\,g\,) = 1\}\,.$$

Let $a_m = \dim A_m(X)$. We fix an orthonormal basis $\left\{g_j^{(m)}\right\}_{j=1}^{a_m}$ of $A_m(X)$ with respect to $(\,\cdot\,|\,\cdot\,)$, then we can identify the sphere S^{2a_m-1} to $SA_m(X)$ by

$$(z_1, \ldots, z_{a_m}) \in S^{2a_m-1} \to \sum_{j=1}^{k} z_j g_j^{(m)} \in SA_m(X),$$

and we have

$$d\mu_m = \frac{dS^{2a_m-1}}{\mathrm{vol}\,(S^{2a_m-1})}, \tag{1.4}$$

where dS^{2a_m-1} denotes the standard Haar measure on S^{2a_m-1}. We consider the probability space $\Omega(X) := \prod_{m=1}^{\infty} SA_m(X)$ with the probability measure $d\mu := \prod_{m=1}^{\infty} d\mu_m$. We denote $u = \{u_m\} \in \Omega(X)$.

Since the S^1 action is transversal and CR, $X \times \mathbb{R}$ is a complex manifold with the following holomorphic tangent bundle and complex structure J,

$$\begin{aligned} &T^{1,0}X \oplus \{\mathbb{C}(T - i\frac{\partial}{\partial\eta})\}, \\ &JT = \frac{\partial}{\partial\eta},\ Ju = iu \text{ for } u \in T^{1,0}X. \end{aligned} \tag{1.5}$$

Let $v(z,\theta,\eta)$ be a nontrivial holomorphic function on $X \times \mathbb{R}$. We write $[v = 0]$ to denote the current of integration with multiplicities over the analytic hypersurface $\{v = 0\}$ determined by the nontrivial holomorphic function v on $X \times \mathbb{R}$. That is, for a smooth $2n$-form $g \in \Omega_0^{2n}(X \times \mathbb{R})$ with compact support in $X \times \mathbb{R}$, we have

$$\langle [v = 0], g\rangle = \int_{\{v=0\}} g. \tag{1.6}$$

Denote by $\tilde{\partial}$ (resp., $\bar{\tilde{\partial}}$) the ∂-operator (resp., $\overline{\partial}$-operator) with respect to the complex structure in (1.5). By the Lelong–Poincaré formula [7, III-2.15] and [16, Theorem 2.3.3] (see Proposition 4.1), we have

$$\langle [v = 0], g\rangle = \frac{i}{2\pi}\int \tilde{\partial}\bar{\tilde{\partial}} \log|v|^2 \wedge g. \tag{1.7}$$

For $u \in A_m(X)$, it is easy to see that there exists a unique function $v(x,\eta) \in C^\infty(X \times \mathbb{R})$, which is holomorphic in $X \times \mathbb{R}$ such that $v\big|_{\eta=0} = u$ (see Lemma 2.6). For all $g \in \Omega^{p,q}(X)$, we extend g trivially in the variable η on $X \times \mathbb{R}$. Then, for $f \in \Omega_0^{n,n}(X)$ and every $\chi(\eta) \in C_0^\infty(\mathbb{R})$, $f \wedge \omega_0 \wedge \chi(\eta)d\eta$ is a smooth $2n$-form on $X \times \mathbb{R}$ with compact support in $X \times \mathbb{R}$. We then define $\langle [v = 0], f \wedge \omega_0 \wedge \chi(\eta)d\eta\rangle$ as in (1.6). The main result of the first part is the following.

Theorem 1.1. *With the above notations and assumptions, fix $\chi(\eta) \in C_0^\infty(\mathbb{R})$ with $\int \chi(\eta)d\eta = 1$, and let ε_m be a sequence with $\lim_{m\to\infty} m\varepsilon_m = 0$. Then for $d\mu$-almost every $u = \{u_m\} \in \Omega(X)$, we have*

$$\lim_{m\to\infty} \frac{1}{m}\langle\, [v_m = 0], f\wedge\omega_0\wedge\frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta\,\rangle = \alpha\frac{1 + k_1^{n+1} + \cdots + k_{t-1}^{n+1}}{1 + k_1^{n} + \cdots + k_{t-1}^{n}}\frac{i}{\pi}\int_X \mathcal{L}_X\wedge f\wedge\omega_0 \tag{1.8}$$

for all $f \in \Omega^{n-1,n-1}(X)$*, where* $v_m(x,\eta)$ *is the unique holomorphic function on* $X \times \mathbb{R}$ *such that* $v_m\big|_{\eta=0} = u_m(x)$*,* $\alpha = [p_1, \ldots, p_{t-1}]$*,* $f \wedge \omega_0 \wedge \frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta$ *is a smooth* (n,n) *form on* $X \times \mathbb{R}$*, the duality* $\langle \cdot , \cdot \rangle$ *in* (1.8) *is given by* (1.6)*,* η *denotes the coordinate on* $\mathbb{R}$*,* ω_0 *is the Reeb one form on* X *(see the discussion in the beginning of section* 2.2*), and* $\mathcal{L}_X$ *denotes the Levi form of* X *with respect to the Reeb one form* ω_0 *(see Definition* 2.1*).*

Remark 1.2. We explain the role ε_m in Theorem 1.1. For simplicity, assume that $t = 2$ and $m_1 := \alpha m$, $m_2 := \alpha k_1 m$. Let $u \in H^0_{b,m_1}(X) \oplus H^0_{b,m_2}(X)$. Let (z, θ, φ) be BRT coordinates on an open set D of X (see Theorem 2.5). On D, we write

$$u = u_1 + u_2 = \tilde{u}_1(z)e^{im_1\theta} + \tilde{u}_2(z)e^{im_2\theta} \in H^0_{b,m_1}(X) \oplus H^0_{b,m_2}(X).$$

Then the unique holomorphic function $v(z, \theta, \eta) \in C^\infty(X \times \mathbb{R})$ with $v\big|_{\eta=0} = u$ is given by

$$v(z, \theta, \eta) = \tilde{u}_1(z)e^{im_1(\theta+i\eta)} + \tilde{u}_2(z)e^{im_2(\theta+i\eta)}.$$

Then, formally

$$\begin{aligned} &\langle\, [v(z,\theta,\eta)=0], f(z,\theta) \wedge \omega_0 \wedge \frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta \,\rangle \\ &= \langle\, [\tilde{u}_1(z)e^{im_1(\theta+i\eta)} + \tilde{u}_2(z)e^{im_2(\theta+i\eta)} = 0], f(z,\theta) \wedge \omega_0 \wedge \frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta \,\rangle \\ &= \langle\, [\tilde{u}_1(z)e^{im_1(\theta+i\varepsilon_m\eta)} + \tilde{u}_2(z)e^{im_2(\theta+i\varepsilon_m\eta)} = 0], f(z,\theta) \wedge \omega_0 \wedge \chi(\eta)d\eta \,\rangle. \end{aligned} \tag{1.9}$$

From the last equation of (1.9), intuitively speaking, when $m\varepsilon_m \to 0$, the integral

$$\langle\, [v(z,\theta,\eta)=0], f(z,\theta) \wedge \omega_0 \wedge \frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta \,\rangle$$

will converge to the integration of "CR" current $\langle\, [u(z,\theta)=0], f(z,\theta) \wedge \omega_0 \,\rangle$.

Remark 1.3. Assume that the S^1-action is globally free. Let $u \in H^0_{b,m}(X)$. Let (z, θ, φ) be BRT coordinates on an open set D of X (see Theorem 2.5). On D, we write $u = \tilde{u}(z)e^{im\theta}$ and the unique holomorphic function $v(z, \theta, \eta) \in C^\infty(X \times \mathbb{R})$ with $v\big|_{\eta=0} = u$ is given by $v(z, \theta, \eta) = u(z)e^{im(\theta+i\eta)}$. Then $\{v = 0\} = \{u = 0\} \times \mathbb{R}$ and for every $\varepsilon_m > 0$, we have

$$\langle\, [v(z,\theta,\eta)=0], f(z,\theta) \wedge \omega_0 \wedge \frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta \,\rangle = \langle\, [v(z,\theta,\eta)=0], f(z,\theta) \wedge \omega_0 \wedge \chi(\eta)d\eta \,\rangle.$$

For the globally free case, we do not need ε_m in Theorem 1.1.

When the S^1-action is globally free, then $t = 1$, $\alpha = 1$, $A_m(X) = H^0_{b,m}(X)$, and $\Omega(X) = \prod_{m=1}^\infty SA_m(X) = \prod_{m=1}^\infty SH^0_{b,m}(X)$. From Remark 1.3, we deduce the following.

Corollary 1.4. *With the same notations and assumptions in Theorem* 1.1*, if the* S^1*-action is globally free, then for* $d\mu$*-almost every* $u = \{u_m\} \in \Omega(X)$*, we have*

$$\lim_{m\to\infty} \frac{1}{m}\langle\, [v_m = 0], f \wedge \omega_0 \wedge \chi(\eta)d\eta \,\rangle = \frac{i}{\pi}\int_X \mathcal{L}_X \wedge f \wedge \omega_0 \tag{1.10}$$

for all $f \in \Omega^{n-1,n-1}(X)$*, where* v_m *is the unique holomorphic function on* $X \times \mathbb{R}$ *such that* $v_m\big|_{\eta=0} = u_m$*.*

Let Y be a compact Kähler manifold with $\dim_{\mathbb{C}} Y = n$, and let L be a line bundle over Y with a smooth Hermitian metric h such that the induced curvature R^L is positive on Y. Let e_L be a local frame of L. We write $|e_L(y)|_h = e^{-\phi}$. Then $R^L = 2\partial\overline{\partial}\phi$. Take $\omega := \frac{i}{2\pi}R^L$ to be the Kähler form of Y. Denote by $H^0(Y, L^m)$ the space of all holomorphic sections of L^m. Set $\Omega(Y, L) := \prod_{m=1}^{\infty} SH^0(Y, L^m)$ with the probability measure $d\mu := \prod_{m=1}^{\infty} d\mu_m$ (cf. (1.4)) As an application of Theorem 1.1, we obtain the classical equidistribution theorem on line bundles (see e.g. [19, Theorem 1.1] and [16, Theorem 5.3.3]).

Corollary 1.5. *With the above notations and assumptions, for $d\mu$-almost every $s = \{s_m\} \in \Omega(Y, L)$, we have*

$$\lim_{m\to\infty} \frac{1}{m}[s_m = 0] = \omega \tag{1.11}$$

in the sense of currents.

Now we formulate the main result of the second part. Let M be a relatively compact open subset with C^∞ boundary X of a complex manifold M' of dimension $n+1$ with a smooth Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on its holomorphic tangent bundle $T^{1,0}M'$. The Hermitian metric on holomorphic tangent bundle induces a Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\oplus_{k=1}^{2n+2}\Lambda^k(\mathbb{C}T^*M')$. Let $r \in C^\infty(M', \mathbb{R})$ be a defining function of X; that is, $X = \{z \in M';\, r(z) = 0\}$, $M = \{z \in M';\, r(z) < 0\}$. We take r so that $\|dr\|^2 = \langle\, dr \,|\, dr\,\rangle = 1$ on X. In this work, we assume that X is strongly pseudoconvex; that is, $\partial\overline{\partial}r|_{T^{1,0}X}$ is positive definite at each point of X, where $T^{1,0}X := T^{1,0}M' \bigcap \mathbb{C}TX$ is the standard CR structure on X. Let dv_M be the volume form on M induced by $\langle\,\cdot\,|\,\cdot\,\rangle$, and let $(\,\cdot\,|\,\cdot\,)_M$ be the L^2 inner product on $C_0^\infty(M)$ induced by dv_M and let $L^2(M)$ be the completion of $C_0^\infty(M)$ with respect to $(\,\cdot\,|\,\cdot\,)_M$. Let $H^0_{(2)}(M) = \left\{u \in L^2(M);\, \overline{\partial}u = 0\right\}$. By using classical result of Boutet de Monvel–Sjöstrand [2, Theorem 1.5], we see that $C^\infty(\overline{M}) \bigcap H^0_{(2)}(M)$ is dense in $H^0_{(2)}(M)$ in the $L^2(M)$ space, and we can find $g_j \in C^\infty(\overline{M}) \bigcap H^0_{(2)}(M)$ with $(\, g_j \,|\, g_k\,)_M = \delta_{j,k}$, $j, k = 1, 2, \ldots$, such that the set

$$A(M) := \text{span}\,\{\mathrm{g}_1, \mathrm{g}_2, \ldots\} \tag{1.12}$$

is dense in $H^0_{(2)}(M)$. That is, for every $h \in H^0_{(2)}(M)$, we can find $h_\ell \in A(M)$, $\ell = 1, 2, \ldots$, such that $\lim_{\ell\to\infty} h_\ell = h$ in $L^2(M)$ space.

To state our equidistribution theorem, we need to introduce some notations. For every $m \in \mathbb{N}$, let $A_m(M) = \text{span}\,\{g_1, \ldots, g_m\}$, where $g_j \in H^0_{(2)}(M) \bigcap C^\infty(\overline{M})$, $j = 1, \ldots, m$, are as (1.12). Let $d\mu_m$ be the equidistribution probability measure on the unit sphere

$$SA_m(M) := \{g \in A_m(M);\, (\, g \,|\, g\,)_M = 1\}\,.$$

Let $\beta := \{b_j\}_{j=1}^{\infty}$ with $b_1 < b_2 < \cdots$ and $b_j \in \mathbb{N}$, for every $j = 1, 2, \ldots$. We consider the probability space

$$\Omega(M, \beta) := \prod_{j=1}^{\infty} SA_{b_j}(M) \tag{1.13}$$

with the probability measure

$$d\mu(\beta) := \prod_{j=1}^{\infty} d\mu_{b_j}\,. \tag{1.14}$$

We denote $u = \{u_k\} \in \Omega(M, \beta)$. For $g \in H^0_{(2)}(M) \bigcap C^\infty(\overline{M})$, we let $[g = 0]$ denote the zero current in M.

Let $B^{*0,1}M' = \{u \in T^{*0,1}M';\, \langle\, u \,|\, \overline{\partial} r\, \rangle = 0\}$, where $T^{*0,1}M'$ denotes the bundle of $(0,1)$ forms on M'. Let $B^{*1,0}M' := \overline{B^{*0,1}M'}$, and let $B^{*p,q}M' := \Lambda^p(B^{*1,0}M') \wedge \Lambda^q(B^{*0,1}M')$, $p, q = 1, \ldots, n$. Let $\omega_0 = J(dr)$, where J is the standard complex structure map on T^*M', and let $\mathcal{L}_X \in C^\infty(X, T^{*1,1}X)$ be the Levi form induced by ω_0 (see Definition 2.1). Our second main result is the following.

Theorem 1.6. *With the notations and assumptions above, fix $\psi \in C^\infty_0([-1, -\frac{1}{2}])$. There exists a sequence $\beta = \{b_j\}_{j=1}^\infty$ independent of ψ with $b_1 < b_2 < \cdots$, $b_j \in \mathbb{N}$, $j = 1, 2, \ldots$, such that for $d\mu(\beta)$-almost every $u = \{u_k\} \in \Omega(M, \beta)$, we have*

$$\lim_{k\to\infty} \langle\, [u_k = 0], (2i)kr\psi(kr)\phi \wedge \partial r \wedge \overline{\partial} r\, \rangle = -(n+2)\frac{i}{2\pi} c_0 \int_X \mathcal{L}_X \wedge \omega_0 \wedge \phi \tag{1.15}$$

for all $\phi \in C^\infty(\overline{M}, B^{*n-1,n-1}M')$, where $c_0 = \int_{\mathbb{R}} \psi(x)dx$, $\Omega(M, \beta)$ and $d\mu(\beta)$ are as in (1.13) *and* (1.14)*, respectively.*

Remark 1.7. By the result of Boutet de Monvel–Sjöstrand [2, Theorem 1.5], we have

$$\textstyle\sum_{j=1}^\infty |g_j(x)|^2 \sim \left|r^{-(n+2)}(x)\right| \text{ in } \overline{M}.$$

The numbers b_j are chosen so that the function $\sum_{s=1}^{b_j} |g_s(x)|^2 \sim \left|r^{-(n+2)}(x)\right|$ on $\{x \in M : -\frac{1}{k} \leq r \leq -\frac{1}{2k}\}$ (see Theorem 5.6). In general, $\sum_{s=1}^{j} |g_s(x)|^2$ could not be asymptotically $\left|r^{-(n+2)}(x)\right|$, and we cannot not take b_j to be j. It is an interesting question to determine the subsequence b_j.

Remark 1.8. Note that for any smooth (n, n) form on M, we can write $\phi \wedge \partial r \wedge \overline{\partial} r$ near the boundary X, where $\phi \in C^\infty(\overline{M}, B^{*n-1,n-1}M')$. From the proof of Theorem 1.6, we actually prove that for $d\mu(\beta)$-almost every $u = \{u_k\} \in \Omega(M, \beta)$, we have

$$\lim_{k\to\infty} \Big(\frac{1}{k}\langle\, [u_k = 0], g_k\, \rangle + i\frac{n+2}{2\pi}\frac{1}{k}\int_M \partial\overline{\partial}\log(-r) \wedge g_k\Big) = 0, \tag{1.16}$$

for all k-uniformly test form $g_k \in \Omega^{n,n}_0(M)$. Here k-uniformly test form $g_k \in \Omega^{n,n}_0(M)$ means that for any smooth $(1,1)$ form ψ, the integral $\int g_k \wedge \psi$ is uniformly bounded in k. For example, $k^2 r\psi(kr)\phi \wedge \partial r \wedge \overline{\partial} r$ is a k-uniformly test form, where $\psi \in C^\infty_0([-1, -\frac{1}{2}])$ and $\phi \in C^\infty(\overline{M}, B^{*n-1,n-1}M')$. In Theorem 1.6, we take special test form $r\psi(kr)\phi \wedge \partial r \wedge \overline{\partial} r$ since Theorem 1.6 aims to show the asymptotic behavior of the currents $\{[u_k = 0]\}$ when the supports of test forms tend to approach the boundary X.

The paper is organized as follows. In section 2 we collect some notations we use throughout and we recall the basic knowledge about CR manifolds. In section 3 we recall a theorem about Szegő kernel asymptotics and give a uniform estimate of Szegő kernel functions. Section 4 is devoted to proving Theorem 1.1. In section 5, we first construct holomorphic functions with specific rate near the boundary, and we prove Theorem 1.6.

2. Preliminaries

2.1. **Standard notations.** We shall use the following notations: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R};\, x \geq 0\}$. For a multi-index

$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we denote by $|\alpha| = \alpha_1 + \ldots + \alpha_n$ its norm and by $l(\alpha) = n$ its length. For $m \in \mathbb{N}$, write $\alpha \in \{1, \ldots, m\}^n$ if $\alpha_j \in \{1, \ldots, m\}$, $j = 1, \ldots, n$. α is strictly increasing if $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. For $x = (x_1, \ldots, x_n)$, we write

$$\begin{aligned} & x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \\ & \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \\ & D_{x_j} = \frac{1}{i}\partial_{x_j}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}, \quad D_x = \frac{1}{i}\partial_x . \end{aligned}$$

Let $z = (z_1, \ldots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, be coordinates of $\mathbb{C}^n$. We write

$$\begin{aligned} & z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}, \quad \overline{z}^\alpha = \overline{z}_1^{\alpha_1} \ldots \overline{z}_n^{\alpha_n}, \\ & \partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\Big), \quad \partial_{\overline{z}_j} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\Big), \\ & \partial_z^\alpha = \partial_{z_1}^{\alpha_1} \ldots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}, \quad \partial_{\overline{z}}^\alpha = \partial_{\overline{z}_1}^{\alpha_1} \ldots \partial_{\overline{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \overline{z}^\alpha} . \end{aligned}$$

For $j, s \in \mathbb{Z}$, set $\delta_{j,s} = 1$ if $j = s$, $\delta_{j,s} = 0$ if $j \neq s$.

Let W be a C^∞ paracompact manifold. We let TW and T^*W denote the tangent bundle of W and the cotangent bundle of W, respectively. The complexified tangent bundle of W and the complexified cotangent bundle of W will be denoted by $\mathbb{C}TW$ and $\mathbb{C}T^*W$, respectively. Write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TW and T^*W. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}TW \times \mathbb{C}T^*W$. Let G be a C^∞ vector bundle over W. The fiber of G at $x \in W$ will be denoted by G_x. Let E be a vector bundle over a C^∞ paracompact manifold W_1. We write $G \boxtimes E^*$ to denote the vector bundle over $W \times W_1$ with fiber over $(x, y) \in W \times W_1$ consisting of the linear maps from E_y to G_x. Let $Y \subset W$ be an open set. From now on, the spaces of distribution sections of G over Y and smooth sections of G over Y will be denoted by $D'(Y, G)$ and $C^\infty(Y, G)$, respectively. Let $E'(Y, G)$ be the subspace of $D'(Y, G)$ whose elements have compact support in Y. Put $C_0^\infty(Y, G) := C^\infty(Y, G) \bigcap E'(Y, G)$.

Let G and E be C^∞ vector bundles over paracompact orientable C^∞ manifolds W and W_1, respectively, equipped with smooth densities of integration. If $A : C_0^\infty(W_1, E) \to D'(W, G)$ is continuous, we write $K_A(x, y)$ or $A(x, y)$ to denote the distribution kernel of A.

Let $H(x, y) \in D'(W \times W_1, G \boxtimes E^*)$. We write H to denote the unique continuous operator $C_0^\infty(W_1, E) \to D'(W, G)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

Let M be a relatively compact open subset with C^∞ boundary X of a complex manifold M'. Let F be a C^∞ vector bundle over M'. Let $C^\infty(\overline{M}, F)$, $D'(\overline{M}, F)$ denote the spaces of restrictions to M of elements in spaces $C^\infty(M', F)$, $D'(M', F)$, respectively.

2.2. **CR manifolds.** Let $(X, T^{1,0}X)$ be a compact, orientable CR manifold of dimension $2n+1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of X, that is, $T^{1,0}X$ is a subbundle of rank n of the complexified tangent bundle $\mathbb{C}TX$, satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$. We fix a real nonvanishing 1 form $\omega_0 \in C(X, T^*X)$ so that $\langle \omega_0(x)\,,\, u\,\rangle = 0$, for every $u \in T_x^{1,0}X \oplus T_x^{0,1}X$, for every $x \in X$. We call ω_0 Reeb one form on X.

Definition 2.1. For $p \in X$, the Levi form $\mathcal{L}_{X,p}$ of X at p is the Hermitian quadratic form on $T^{1,0}_pX$ given by $\mathcal{L}_{X,p}(U,V) = -\frac{1}{2i}\langle\, d\omega_0(p)\,,\, U\wedge\overline{V}\,\rangle$, $U,V\in T^{1,0}_pX$.

Denote by $\mathcal{L}_X$ the Levi form on X.

Fix a global nonvanishing vector field $T \in C^\infty(X,TX)$ such that $\omega_0(T) = -1$, and T is transversal to $T^{1,0}X \oplus T^{0,1}X$. We call T Reeb vector field on X. Take a smooth Hermitian metric $\langle\cdot \mid \cdot\rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$, $\langle u \mid v\rangle$ is real if u,v are real tangent vectors, $\langle\, T\,|\,T\,\rangle = 1$ and T is orthogonal to $T^{1,0}X\oplus T^{0,1}X$. For $u\in\mathbb{C}TX$, we write $|u|^2 := \langle u|u\rangle$. Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with subbundles of the complexified cotangent bundle $\mathbb{C}T^*X$. Define the vector bundle of (p,q)-forms by $T^{*p,q}X := (\wedge^pT^{*1,0}X)\wedge(\wedge^qT^{*0,1}X)$. The Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of (p,q) forms $T^{*p,q}X, p,q = 0,1,\cdots,n$. We shall also denote all these induced metrics by $\langle\cdot|\cdot\rangle$. Note that we have the pointwise orthogonal decompositions:

$$\begin{gathered}\mathbb{C}T^*X = T^{*1,0}X\oplus T^{*0,1}X\oplus\{\lambda\omega_0 : \lambda\in\mathbb{C}\}\,,\\ \mathbb{C}TX = T^{1,0}X\oplus T^{0,1}X\oplus\{\lambda T : \lambda\in\mathbb{C}\}\,.\end{gathered}\tag{2.1}$$

Let D be an open set of X. Let $\Omega^{p,q}(D)$ denote the space of smooth sections of $T^{*p,q}X$ over D, and let $\Omega^{p,q}_0(D)$ be the subspace of $\Omega^{p,q}(D)$ whose elements have compact support in D. For each point $x\in X$, in this paper, we will identify $\mathcal{L}_{X,x}$ as a $(1,1)$ form at x. Hence, $\mathcal{L}_X\in\Omega^{1,1}(X)$.

Now, we assume that X admits an S^1-action: $S^1\times X\to X, (e^{i\theta},x)\to e^{i\theta}\circ x$. Here we use $e^{i\theta}$ to denote the S^1-action. Let $\widetilde{T}\in C^\infty(X,TX)$ be the global real vector field induced by the S^1-action given as follows:

$$(\widetilde{T}u)(x) = \frac{\partial}{\partial\theta}\left(u(e^{i\theta}\circ x)\right)\Big|_{\theta=0},\ u\in C^\infty(X).\tag{2.2}$$

Definition 2.2. We say that the S^1-action $e^{i\theta}$ $(0\le\theta<2\pi)$ is CR if

$$[\widetilde{T}, C^\infty(X,T^{1,0}X)]\subset C^\infty(X,T^{1,0}X),$$

where $[\ ,\ ]$ is the Lie bracket between the smooth vector fields on X. Furthermore, the S^1-action is called transversal if for each $x\in X$ one has

$$\mathbb{C}\widetilde{T}(x)\oplus T^{1,0}_x(X)\oplus T^{0,1}_xX = \mathbb{C}T_xX.$$

If the S^1 action is transversal and CR, we will always take the Reeb one form on X to be the global real one form determined by $\langle\,\omega_0\,,\,u\,\rangle = 0$, for every $u\in T^{1,0}X\oplus T^{0,1}X$ and $\langle\,\omega_0\,,\,\widetilde{T}\,\rangle = -1$, and we will always take the Reeb vector field on X to be $\widetilde{T}$. Hence, we will also write T to denote the global real vector field induced by the S^1-action.

Until further notice, we assume that $(X,T^{1,0}X)$ is a compact connected strongly pseudoconvex CR manifold with a transversal CR S^1-action $e^{i\theta}$. For every $q\in\mathbb{N}$, put

$$X_q := \{x\in X : e^{i\theta}\circ x\neq x, \forall\theta\in(0,\frac{2\pi}{q}),\ e^{i\frac{2\pi}{q}}\circ x = x\}.\tag{2.3}$$

Set $p := \min\{q\in\mathbb{N} : X_q\neq\emptyset\}$. Thus, $X_{\rm reg} = X_p$. Note that one can renormalize the S^1-action by lifting such that the new S^1-action satisfies $X_1\neq\emptyset$ (see [6]). For simplicity, we assume that $p = 1$. If X is connected, then X_1 is open and dense in X. Assume that

$$X = \cup_{j=0}^{t-1} X_{p_j}, \quad 1 =: p_0 < p_1 < \cdots < p_{t-1}.$$

Put $X_{sing} := X_{sing}^1 = \cup_{j=1}^{t-1} X_{p_j}$, and $X_{sing}^r := \cup_{j=r}^{t-1} X_{p_j}$ for $2 \leq r \leq t-1$. Take the convention that $X_{sing}^t = \emptyset$. The proposition below follows from [6].

Proposition 2.3. *Here,* X_{sing}^r *is a closed subset of* X*, for* $1 \leq r \leq t$.

Fix $\theta_0 \in [0, 2\pi)$. Let

$$de^{i\theta_0} : \mathbb{C}T_x X \to \mathbb{C}T_{e^{i\theta_0}x} X$$

denote the differential map of $e^{i\theta_0} : X \to X$. By the properties of transversal CR S^1-actions, we can check that

$$\begin{aligned} de^{i\theta_0} &: T_x^{1,0} X \to T_{e^{i\theta_0}x}^{1,0} X, \\ de^{i\theta_0} &: T_x^{0,1} X \to T_{e^{i\theta_0}x}^{0,1} X, \\ de^{i\theta_0}&(T(x)) = T(e^{i\theta_0}x). \end{aligned} \tag{2.4}$$

Let $(e^{i\theta_0})^* : \Lambda^q(\mathbb{C}T^*X) \to \Lambda^q(\mathbb{C}T^*X)$ be the pull back of $e^{i\theta_0}, q = 0, 1 \cdots, 2n+1$. From (2.4), we can check that for every $q = 0, 1, \cdots, n$

$$(e^{i\theta_0})^* : T_{e^{i\theta_0}x}^{*0,q} X \to T_x^{*0,q} X.$$

Let $u \in \Omega^{0,q}(X)$. The Lie derivative of u along the direction T is denoted by Tu. We have $Tu \in \Omega^{0,q}(X)$ for all $u \in \Omega^{0,q}(X)$.

Let $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy–Riemann operator. From (2.4), it is straightforward to check that

$$T\overline{\partial}_b = \overline{\partial}_b T \text{ on } \Omega^{0,q}(X). \tag{2.5}$$

For every $m \in \mathbb{Z}$, put $\Omega_m^{0,q}(X) := \{u \in \Omega^{0,q}(X) : Tu = imu\}$. For $q = 0$, we write $C_m^\infty(X) := \Omega_m^{0,0}(X)$. We denote by $\overline{\partial}_{b,m}$ the restriction of $\overline{\partial}_b$ to $\Omega_m^{0,q}(X)$. From (2.5) we have the $\overline{\partial}_{b,m}$-complex for every $m \in \mathbb{Z}$:

$$\overline{\partial}_{b,m} : \cdots \to \Omega_m^{0,q-1}(X) \to \Omega_m^{0,q}(X) \to \Omega_m^{0,q+1}(X) \to \cdots .$$

For $m \in \mathbb{Z}$, the qth $\overline{\partial}_{b,m}$-cohomology is given by

$$H_{b,m}^q(X) := \frac{\operatorname{Ker} \overline{\partial}_b : \Omega_m^{0,q}(X) \to \Omega_m^{0,q+1}(X)}{\operatorname{Im} \overline{\partial}_b : \Omega_m^{0,q-1}(X) \to \Omega_m^{0,q}(X)}. \tag{2.6}$$

Moreover, we have [15]

$$\dim H_{b,m}^q(X) < \infty \text{ for all } q = 0, \ldots, n.$$

Definition 2.4. A function $u \in C^\infty(X)$ is a Cauchy–Riemann function (CR function for short) if $\overline{\partial}_b u = 0$; that is, $\overline{Z}u = 0$ for all $Z \in C^\infty(X, T^{1,0}X)$. For $m \in \mathbb{N}$, $H_{b,m}^0(X)$ is called the mth positive Fourier component of the space of CR functions.

We recall the canonical local coordinates (BRT coordinates) due to Baouendi–Rothschild–Treves (see [1]).

Theorem 2.5. *With the notations and assumptions above, fix $x_0 \in X$. There exist local coordinates $(x_1, \cdots, x_{2n+1}) = (z, \theta) = (z_1, \cdots, z_n, \theta), z_j = x_{2j-1} + ix_{2j}, 1 \leq j \leq n, x_{2n+1} = \theta$, centered at x_0, defined on $D = \{(z,\theta) \in \mathbb{C}^n \times \mathbb{R} : |z| < \varepsilon, |\theta| < \delta\}$, such that*

$$\begin{aligned} T &= \frac{\partial}{\partial \theta} \\ Z_j &= \frac{\partial}{\partial z_j} + i\frac{\partial \varphi(z)}{\partial z_j}\frac{\partial}{\partial \theta}, j = 1, \cdots, n, \end{aligned} \tag{2.7}$$

where $\{Z_j(x)\}_{j=1}^n$ form a basis of $T_x^{1,0}X$ for each $x \in D$, and $\varphi(z) \in C^\infty(D, \mathbb{R})$ is independent of θ. We call D a canonical local patch and (z, θ, φ) canonical coordinates (BRT coordinates) centered at x_0.

Note that Theorem 2.5 holds if X is not strongly pseudoconvex.

On the BRT coordinate D, the action of the partial Cauchy–Riemann operator is the following

$$\bar{\partial}_b u = \sum_{j=1}^n (\frac{\partial u}{\partial \bar{z}_j} - i\frac{\partial \varphi}{\partial \bar{z}_j}\frac{\partial u}{\partial \theta})d\bar{z}_j.$$

We can check that

$$\omega_0 = -d\theta + i\sum_{j=1}^n \frac{\partial \varphi}{\partial z_j}dz_j - i\sum_{j=1}^n \frac{\partial \varphi}{\partial \bar{z}_j}d\bar{z}_j.$$

Hence the Levi form is

$$\mathcal{L}_X = -\frac{1}{2i}d\omega_0\big|_{T^{1,0}X} = \partial\bar{\partial}\varphi. \tag{2.8}$$

If $u \in H^0_{b,m}(X)$, then $\bar{\partial}_b u = 0$. It is equivalent to

$$\frac{\partial u}{\partial \bar{z}_j} - i\frac{\partial \varphi}{\partial \bar{z}_j}\frac{\partial u}{\partial \theta} = 0, \ \forall j.$$

Moreover, since $Tu = imu$, u can be written locally as

$$u\big|_D = e^{im\theta}\tilde{u}(z).$$

Then

$$\begin{aligned} &\frac{\partial \tilde{u}}{\partial \bar{z}_j} + m\frac{\partial \varphi}{\partial \bar{z}_j}\tilde{u} \\ &= \frac{\partial}{\partial \bar{z}_j}(\tilde{u}e^{m\varphi}) = 0, \ \forall j. \end{aligned} \tag{2.9}$$

That is to say, $\tilde{u}e^{m\varphi}$ is holomorphic with respect to the $(z_1, ..., z_n)$-coordinate.

Let $X \times \mathbb{R}$ be the complex manifold with the following holomorphic tangent bundle and complex structure J,

$$\begin{aligned} &T^{1,0}X \oplus \{\mathbb{C}(T - i\frac{\partial}{\partial \eta})\}, \\ &JT = \frac{\partial}{\partial \eta}, \ Ju = iu \text{ for } u \in T^{1,0}X. \end{aligned} \tag{2.10}$$

Lemma 2.6. *Let $u \in \oplus_{m \in \mathbb{Z}, |m| \leq N} C^\infty_m(X)$ with $\bar{\partial}_b u = 0$, where $N \in \mathbb{N}$. Then there exists a unique function v, which is holomorphic in $X \times \mathbb{R}$ such that $v\big|_{\eta=0} = u$.*

Proof. Let D be a canonical local coordinate patch with canonical local coordinates $x=(z,\theta)$. On D, we write $u=\sum_{m\in\mathbb{Z},|m|\le N}u_m(z)e^{im\theta}$. Note that in canonical local coordinates $x=(z,\theta)$, we have $T=\frac{\partial}{\partial\theta}$. Set

$$v:=\sum_{m\in\mathbb{Z},|m|\le N}u_m(z)e^{im(\theta+i\eta)}.$$

From $\overline{\partial}_b u=0$, it is easy to check that v is holomorphic on $D\times\mathbb{R}$ with respect to the complex structure (2.10) and $v|_{\eta=0}=u$. If there exists another function $\tilde v$ satisfies the same properties. Then $\tilde v-v$ is holomorphic, $(\tilde v-v)\big|_{\eta=0}=0$. So $\tilde v=v$. Thus, we can define v as a global CR function on $X\times\mathbb{R}$, and we have $v|_{\eta=0}=u$. The proof is completed. □

3. Uniform estimate of Szegő kernel functions

In this section, we will give a uniform estimate of Szegő kernel function on X. We keep the notations and assumptions the same as in the previous sections. We first recall a recent result about Szegő kernel asymptotic expansion on CR manifolds with S^1 action from Herrmann–Hsiao–Li [13].

For $x,y\in X$, let $d(x,y)$ denote the Riemannian distance between x and y induced by $\langle\cdot|\cdot\rangle$. Let A be a closed subset of X. Put $d(x,A):=\inf\{d(x,y);\,y\in A\}$.

Theorem 3.1. *Recall that we work with the assumptions that X is a compact connected strongly pseudoconvex CR manifold of dimension $2n+1$, $n\ge1$, with a transversal CR S^1-action. With the above notations for $X_{p_r}, 0\le r\le t-1$, there are $b_j(x)\in C^\infty(X)$, $j=0,1,2,\dots$, such that for any $r=0,1,\dots,t-1$, any differential operator $P_\ell:C^\infty(X)\to C^\infty(X)$ of order $\ell\in\mathbb{N}_0$ and every $N\in\mathbb{N}$, there are $\varepsilon_0>0$ and C_N independent of m with the following estimate:*

$$\begin{aligned}&\Big|P_\ell\Big(S_m(x)-\sum_{s=1}^{p_r}e^{\frac{2\pi(s-1)}{p_r}mi}\sum_{j=0}^{N-1}m^{n-j}b_j(x)\Big)\Big|\\&\le C_N\Big(m^{n-N}+m^{n+\frac{\ell}{2}}e^{-m\varepsilon_0 d(x,X^{r+1}_{sing})^2}\Big),\ \forall m\ge1,\ \forall x\in X_{p_r},\end{aligned}$$

where $b_0(x)\ge\epsilon>0$ on X for some universal constant ϵ.

Note that when m is a multiple of p_r, then $\sum_{s=1}^{p_r}e^{\frac{2\pi(s-1)}{p_r}mi}$ is equal to p_r. When m is not a multiple of p_r, then $\sum_{s=1}^{p_r}e^{\frac{2\pi(s-1)}{p_r}mi}$ is equal to 0.

Corollary 3.2. *With the above notations and assumptions, we have*

$$S_m(x)\le Cm^n,\ \forall m\ge1,\ x\in X,$$

where $C>0$ is a constant independent of m.

Fix $r=0,1,\dots,t-1$. There is a $m_0>0$ such that for every $m\ge m_0$, $p_r|m$ we have

$$S_m(x)\ge m^n(p_rb_0(x)-c_1e^{-m\varepsilon_0 d(x,X^{r+1}_{\rm sing})^2}-c_1\frac{1}{m})$$

for any $x\in X_{p_r}$, where $c_1>0$ is a constant independent of m.

Corollary 3.3. *With the above notations and assumptions, let $r=0$. We have*

$$\lim_{m\to\infty}\frac{S_m(x)}{m^n}=b_0(x),\ \forall x\in X_{\rm reg}.$$

Let $x, x_1 \in X$. We have

$$S_m(x) = S_m(x_1) + R_m(x, x_1),$$
$$R_m(x, x_1) = \int_0^1 \frac{\partial}{\partial t}\Big(S_m(tx + (1-t)x_1)\Big)dt. \tag{3.1}$$

By Theorem 3.1 with $l = 1$, we have the following corollary.

Corollary 3.4. *We have*

$$|R_m(x, x_1)| \leq c_2 m^{n+\frac{1}{2}} d(x, x_1), \quad \forall (x, x_1) \in X \times X,$$

where $c_2 > 0$ is a constant independent of m.

The main result in this section is the following.

Theorem 3.5. *There exist positive integers $k_1 < \cdots < k_{t-1}$ independent of m and $m_0 > 0$, such that for all $m \geq m_0$ with $p_j | m$, $j = 0, 1, \ldots, t-1$, we have*

$$\frac{1}{C} m^n \leq S_m(x) + S_{k_1 m}(x) + \cdots + S_{k_{t-1} m}(x) \leq C m^n, \forall x \in X,$$

where $S_{k_j m}(x)$ is the Szegő kernel function associated with $H^0_{b,k_j m}(X)$, and $C > 1$ is a constant independent of m.

Proof. Put $X^0_{\text{sing}} := X_{\text{reg}}$. We claim that for every $j \in \{0, 1, \ldots, t-1\}$, we can find $k_0 := 1 < k_1 < \cdots < k_{t-1-j}$ and $m_0 > 0$ such that for all $m \geq m_0$ with $p_s | m$, $s = j, j+1, \ldots, t-1$, we have

$$\frac{1}{C} m^n \leq S_m(x) + S_{k_1 m}(x) + \cdots + S_{k_{t-1-j} m}(x) \leq C m^n, \quad \forall x \in X^j_{\text{sing}}, \tag{3.2}$$

where $C > 1$ is a constant independent of m.

We prove the claim (3.2) by induction over j. Let $j = t-1$. Since $X^t_{\text{sing}} = \emptyset$, by Theorem 3.1, we see that for all $m \gg 1$ with $p_{t-1} | m$, we have

$$S_m(x) \approx m^n \ \text{ on } X^{t-1}_{\text{sing}}.$$

The claim (3.2) holds for $j = t-1$. Assume that the claim (3.2) holds for some $0 < j_0 \leq t-1$. We are going to prove the claim (3.2) holds for $j_0 - 1$. By the induction assumption, there exist positive integers $k_0 := 1 < k_1 < \cdots < k_{t-1-j_0}$ independent of m and $m_0 > 0$ such that for all $m \geq m_0$ with $p_s | m$, $s = j_0, j_0+1, \ldots, t-1$, we have

$$\frac{1}{C} m^n \leq A_m(x) := S_m(x) + S_{k_1 m}(x) + \cdots + S_{k_{t-1-j_0} m}(x) \leq C m^n, \ \forall x \in X^{j_0}_{\text{sing}}, \tag{3.3}$$

where $C > 1$ is a constant independent of m. In view of Corollary 3.2, we see that there is a large constant $C_0 > 1$ and $m_1 > 0$ such that for all $m \geq m_1$ with $p_{j_0 - 1} | m$ and all $x \in X_{p_{j_0-1}}$ with $d(x, X^{j_0}_{\text{sing}}) \geq \frac{C_0}{\sqrt{m}}$, we have

$$S_m(x) \geq c m^n, \tag{3.4}$$

where $c > 0$ is a constant independent of m. Fix $C_0 > 0$, where C_0 is as in the discussion before (3.4), and let $k \in \mathbb{N}$ and $m \gg 1$ with $p_s | m$, $s = j_0, j_0+1, \ldots, t-1$. Consider the set

$$S_{k,m} := \left\{ x \in X_{p_{j_0-1}};\, d(x, X^{j_0}_{\text{sing}}) \leq \frac{C_0}{\sqrt{km}} \right\}.$$

Let $x \in S_{k,m}$. Since $X^{j_0}_{\rm sing}$ is a closed subset of X by Proposition 2.3, there is a point $x_2 \in X^{j_0}_{\rm sing}$ such that $d(x, x_2) = d(x, X^{j_0}_{\rm sing})$. By (3.1), we write

$$\begin{aligned} A_m(x) &= S_m(x) + S_{k_1 m}(x) + \cdots + S_{k_{t-1-j_0} m}(x) \\ &= \Big(S_m(x_2) + S_{k_1 m}(x_2) + \cdots + S_{k_{t-1-j_0} m}(x_2)\Big) \\ &\quad + \Big(R_m(x, x_2) + R_{k_1 m}(x, x_2) + \cdots + R_{k_{t-1-j_0} m}(x, x_2)\Big) \\ &= A_m(x_2)(1 + v_m(x, x_2)), \end{aligned} \tag{3.5}$$

where

$$v_m(x, x_2) := (A_m(x_2))^{-1}\Big(R_m(x, x_2) + R_{k_1 m}(x, x_2) + \cdots + R_{k_{t-1-j_0} m}(x, x_2)\Big).$$

Then with Corollary 3.4,

$$|v_m| \lesssim \frac{C_0}{\sqrt{km}} m^{-n} m^{n+\frac{1}{2}} \lesssim \frac{C_0}{\sqrt{k}}. \tag{3.6}$$

From (3.5) and (3.6), we see that there is a large constant k_{t-j_0} and $m_2 > 0$ such that for all $m \geq m_2$ with $p_s | m$, $s = j_0, j_0 + 1, \ldots, t - 1$, we have

$$A_m(x) \geq \hat{c} m^n, \quad \forall x \in S_{k_{t-j_0} m} := \left\{ x \in X_{p_{j_0-1}};\, d(x, X^{j_0}_{\rm sing}) \leq \frac{C_0}{\sqrt{k_{t-j_0} m}} \right\}, \tag{3.7}$$

where $\hat{c} > 0$ is a constant independent of m. In view of (3.4), we see that for all $m \geq \max\{m_1, m_2\}$ with $p_{j_0-1} | m$, we have

$$S_{k_{t-j_0} m}(x) \geq \widetilde{c} m^n, \quad \forall x \in X_{p_{j_0-1}} \quad \text{with } d(x, X^{j_0}_{\rm sing}) \geq \tfrac{C_0}{\sqrt{k_{t-j_0} m}}, \tag{3.8}$$

where $\widetilde{c} > 0$ is a constant independent of m. From (3.8) and (3.7), we get the claim (3.2) for $j = j_0 - 1$. By the induction assumption, we get the claim (3.2) and the theorem follows then. □

4. Equidistribution on CR manifolds

This section is devoted to proving Theorem 1.1. For simplicity, we assume that $X = X_{p_0} \bigcup X_{p_1}$, $p_0 = 1$. The proof of general case is similar. Let k_1 be as in Theorem 3.5. Let $\alpha = [1, p_1] = p_1$. We recall notations used in section 1. For each $m \in \mathbb{N}$, put $A_m(X) := H^0_{b,\alpha m}(X) \bigcup H^0_{b,\alpha k_1 m}(X)$, $SA_m(X) := \{g \in A_m(X);\, (\, g \,|\, g\,) = 1\}$, and let $d\mu_m$ denote the normalized Haar measure on the unit sphere $SA_m(X)$. We consider the probability space $\Omega(X) := \prod_{m=1}^{\infty} SA_m(X)$ with the probability measure $d\mu := \prod_{m=1}^{\infty} d\mu_m$.

We first recall briefly the Lelong–Poincaré formula (see [7, III-2.15] and [16, Theorem 2.3.3]).

Proposition 4.1. *Let Y be a complex manifold, and h be a meromorphic function on Y, which does not vanish identically on any connected component of Y. Then h is locally integrable on Y and satisfies the following:*

$$\langle [h = 0], w \rangle = \int_{\{h=0\}} w = \frac{i}{\pi} \int \partial\overline{\partial} \log |h| \wedge w, \tag{4.1}$$

where w is any test form on Y.

Let $u \in SA_m(X)$, and let $v(z,\theta,\eta)$ be holomorphic function on $X \times \mathbb{R}$ with $v|_{\eta=0} = u$. For simplicity, let $m_1 := \alpha m$, $m_2 := \alpha k_1 m$. On D, we write

$$u = u_1 + u_2 = \tilde{u}_1(z)e^{im_1\theta} + \tilde{u}_2(z)e^{im_2\theta} \in H^0_{b,m_1}(X) \oplus H^0_{b,m_2}(X).$$

Then,

$$v = \tilde{u}_1(z)e^{im_1\theta - m_1\eta} + \tilde{u}_2(z)e^{im_2\theta - m_2\eta}.$$

Let $g \in \Omega_0^{2n}(X \times \mathbb{R})$, $\langle [v=0], g\rangle$ is defined in (1.6). Denote by $\tilde{\partial}$ (resp., $\bar{\tilde{\partial}}$) the ∂-operator (resp., $\overline{\partial}$-operator) with respect to the complex structure in (1.5). Since v is holomorphic with the complex structure, then by the Lelong–Poincaré formula, we have

$$\langle [v=0], g\rangle = \frac{i}{2\pi}\int \tilde{\partial}\bar{\tilde{\partial}} \log |v|^2 \wedge g, \tag{4.2}$$

This is globally defined which is independent of the choice of BRT coordinates in the following local calculation.

Let D be a local BRT canonical coordinate patch with canonical local coordinates (z,θ,φ). Let $x = (x_1,\ldots,x_{2n+1}) = (z,\theta)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1,\ldots,n$. We choose a partition of unity $\{\psi_\ell\}$ on X, and consider $\psi_\ell g$ with $\mathrm{supp}\psi_\ell \subset D$. So we can assume $\mathrm{supp} g \subset D \times \mathbb{R}$. On $D \times \mathbb{R}$, we have

$$\begin{aligned}\tilde{\partial} &= \sum_{j=1}^n \Big(\frac{\partial}{\partial z_j} + i\frac{\partial\varphi(z)}{\partial z_j}\frac{\partial}{\partial\theta}\Big)dz_j + \frac{1}{2}\Big(\frac{\partial}{\partial\theta} - i\frac{\partial}{\partial\eta}\Big)(-\omega_0 + id\eta)\\ \bar{\tilde{\partial}} &= \sum_{j=1}^n \Big(\frac{\partial}{\partial \bar{z}_j} - i\frac{\partial\varphi(z)}{\partial \bar{z}_j}\frac{\partial}{\partial\theta}\Big)d\bar{z}_j + \frac{1}{2}\Big(\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\eta}\Big)(-\omega_0 - id\eta).\end{aligned} \tag{4.3}$$

Recall that

$$\omega_0 = -d\theta + i\sum_{j=1}^n \frac{\partial\varphi}{\partial z_j}dz_j - i\sum_{j=1}^n \frac{\partial\varphi}{\partial \bar{z}_j}d\bar{z}_j.$$

Then

$$\begin{aligned}-\omega_0 + id(\eta-\varphi) &= d\theta - i\sum_{j=1}^n \frac{\partial\varphi}{\partial z_j}dz_j + i\sum_{j=1}^n \frac{\partial\varphi}{\partial \bar{z}_j}d\bar{z}_j + id\eta - i\sum_{j=1}^n \frac{\partial\varphi}{\partial z_j}dz_j - i\sum_{j=1}^n \frac{\partial\varphi}{\partial \bar{z}_j}d\bar{z}_j\\ &= d\theta - 2i\sum_{j=1}^n \frac{\partial\varphi}{\partial z_j}dz_j + id\eta.\end{aligned}$$

Denote by ∂ and $\overline{\partial}$ the standard ∂-operator and $\overline{\partial}$-operator on $(z,\theta+i\eta)$-coordinates. For simplicity, let h be a function (or form) on $X \times \mathbb{R}$, and we have:

$$\begin{aligned}(\tilde{\partial}h)(z,\theta,\eta-\varphi) &= \sum_{j=1}^n \Big(\frac{\partial h}{\partial z_j}(z,\theta,\eta-\varphi) + i\frac{\partial\varphi(z)}{\partial z_j}\frac{\partial h}{\partial\theta}(z,\theta,\eta-\varphi)\Big)dz_j\\ &\quad + \frac{1}{2}\Big(\frac{\partial h}{\partial\theta}(z,\theta,\eta-\varphi) - i\frac{\partial h}{\partial\eta}(z,\theta,\eta-\varphi)\Big)(-\omega_0 + id(\eta-\varphi))\end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}(\frac{\partial h}{\partial z_j}(z,\theta,\eta-\varphi) + i\frac{\partial\varphi(z)}{\partial z_j}\frac{\partial h}{\partial\theta}(z,\theta,\eta-\varphi))dz_j \\
&+ \frac{1}{2}(\frac{\partial h}{\partial\theta}(z,\theta,\eta-\varphi) - i\frac{\partial h}{\partial\eta}(z,\theta,\eta-\varphi))(d\theta - 2i\sum_{j=1}^{n}\frac{\partial\varphi}{\partial z_j}dz_j + id\eta) \\
&= \sum_{j=1}^{n}(\frac{\partial h}{\partial z_j}(z,\theta,\eta-\varphi) - \frac{\partial\varphi(z)}{\partial z_j}\frac{\partial h}{\partial\eta}(z,\theta,\eta-\varphi))dz_j \\
&+ \frac{1}{2}(\frac{\partial h}{\partial\theta}(z,\theta,\eta-\varphi) - i\frac{\partial h}{\partial\eta}(z,\theta,\eta-\varphi))(d\theta + id\eta) \\
&= \partial(h(z,\theta,\eta-\varphi)).
\end{aligned}
$$

Similarly we have

$$
(4.5)\quad (\tilde{\overline{\partial}}h)(z,\theta,\eta-\varphi) = \overline{\partial}(h(z,\theta,\eta-\varphi)),\quad (\tilde{\partial}\tilde{\overline{\partial}}h)(z,\theta,\eta-\varphi) = \partial\overline{\partial}(h(z,\theta,\eta-\varphi)).
$$

Fix $\chi(\eta) \in C_0^\infty(\mathbb{R})$ with $\int\chi(\eta)d\eta = 1$. Let $f \in \Omega_0^{n-1,n-1}(D)$. Note that $\frac{\partial}{\partial\overline{z}_j}v(z,\theta,\eta-\varphi(z)) = 0$, $j = 1,\ldots,n$, $(\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\eta})v(z,\theta,\eta-\varphi(z)) = 0$. From this observation, (4.2), (4.4), (4.5) and the Lelong–Poincaré formula, we have

$$
\begin{aligned}
&\langle[v(z,\theta,\eta)=0], f(z,\theta)\wedge\omega_0(z,\theta)\wedge\chi(\eta)d\eta\rangle \\
&= \frac{i}{2\pi}\int\partial\overline{\partial}\log|v(z,\theta,\eta-\varphi)|^2\wedge f(z,\theta)\wedge\omega_0(z,\theta)\wedge\chi(\eta-\varphi)d(\eta-\varphi).
\end{aligned}\tag{4.6}
$$

To prove Theorem 1.1, we only need to show that for $d\mu$-almost every $\{u_m\} \in \Omega(X)$, we have

$$
(4.7)\quad \lim_{m\to\infty}\frac{1}{m}\langle\,[v_m=0], f\wedge\omega_0\wedge\frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta\,\rangle = \alpha\frac{1+k_1^{n+1}}{1+k_1^n}\frac{i}{\pi}\int_X\mathcal{L}_X\wedge f\wedge\omega_0,
$$

where $v_m(x,\eta) \in C^\infty(X\times\mathbb{R})$ is the unique holomorphic function on $X\times\mathbb{R}$ with $v_m(x,\eta)|_{\eta=0} = u_m(x)$.

It follows from (4.6) that

$$
\begin{aligned}
&\langle[v=0], f\wedge\omega_0\wedge\frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta\rangle \\
&= \frac{i}{2\pi}\int\partial\overline{\partial}\log|\tilde{u}_1(z)e^{im_1\theta+m_1(\varphi-\eta)} + \tilde{u}_2(z)e^{im_2\theta+m_2(\varphi-\eta)}|^2 \\
&\quad\wedge f(z,\theta)\wedge\omega_0\wedge\chi(\frac{\eta-\varphi}{\varepsilon_m})\frac{1}{\varepsilon_m}(d\eta-d\varphi).
\end{aligned}\tag{4.8}
$$

Let S_{m_1} (resp., S_{m_2}) be the Szegő kernel functions of $H^0_{b,m_1}(X)$ (resp., $H^0_{b,m_2}(X)$). By using the same arguments as found in Shiffman–Zelditch [19, Section 3] and Ma–Marinescu [16, Section 5.3] and (4.8), we deduce that for $d\mu$-almost every $\{u_m\} \in \Omega(X)$, we have

(4.9)

$$
\begin{aligned}
&\lim_{m\to\infty}\Big(\frac{1}{m}\langle\,[v_m=0], f\wedge\omega_0\wedge\frac{1}{\varepsilon_m}\chi(\frac{\eta}{\varepsilon_m})d\eta\,\rangle - \frac{i}{2m\pi} \\
&\int\partial\overline{\partial}\log(e^{2m_1(\varphi-\eta)}S_{m_1}+e^{2m_2(\varphi-\eta)}S_{m_2})\wedge f\wedge\omega_0\wedge\chi(\frac{\eta-\varphi}{\varepsilon_m})\frac{1}{\varepsilon_m}(d\eta-d\varphi)\Big)=0.
\end{aligned}
$$

Let $F_m = e^{2m_1(\varphi-\eta)}S_{m_1} + e^{2m_2(\varphi-\eta)}S_{m_2}$.

Proof of Theorem 1.1. In view of (4.9), to prove Theorem 1.1, it suffices to compute

$$\lim_{m\to\infty}\frac{i}{2m\pi}\int\partial\bar\partial\log F_m\wedge f\wedge\omega_0\wedge\chi(\frac{\eta-\varphi}{\varepsilon_m})\frac{1}{\varepsilon_m}(d\eta-d\varphi). \tag{4.10}$$

Recall that $S_{m_1}+S_{m_2}\approx m^n$ on X (see Theorem 3.5). We write $F=F_m, a_1=S_{m_1}, a_2=S_{m_2}$ for short. We have

$$\partial\bar\partial\log F=\frac{\partial\bar\partial F}{F}-\frac{\partial F\wedge\bar\partial F}{F^2}. \tag{4.11}$$

We can check that

$$\begin{aligned}\partial F&=\partial(e^{2m_1(\varphi-\eta)}a_1+e^{2m_2(\varphi-\eta)}a_2)\\&=e^{2m_1(\varphi-\eta)}\partial a_1+e^{2m_2(\varphi-\eta)}\partial a_2\\&\quad+2m_1a_1e^{2m_1(\varphi-\eta)}\partial(\varphi-\eta)+2m_2a_2e^{2m_2(\varphi-\eta)}\partial(\varphi-\eta),\end{aligned} \tag{4.12}$$

$$\begin{aligned}\bar\partial F&=\bar\partial(e^{2m_1(\varphi-\eta)}a_1+e^{2m_2(\varphi-\eta)}a_2)\\&=e^{2m_1(\varphi-\eta)}\bar\partial a_1+e^{2m_2(\varphi-\eta)}\bar\partial a_2\\&\quad+2m_1a_1e^{2m_1(\varphi-\eta)}\bar\partial(\varphi-\eta)+2m_2a_2e^{2m_2(\varphi-\eta)}\bar\partial(\varphi-\eta).\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\partial F\wedge\bar\partial F=&\,(e^{2m_1(\varphi-\eta)}\partial a_1+e^{2m_2(\varphi-\eta)}\partial a_2\\&+2m_1a_1e^{2m_1(\varphi-\eta)}\partial(\varphi-\eta)+2m_2a_2e^{2m_2(\varphi-\eta)}\partial(\varphi-\eta))\\&\wedge(e^{2m_1(\varphi-\eta)}\bar\partial a_1+e^{2m_2(\varphi-\eta)}\bar\partial a_2\\&+2m_1a_1e^{2m_1(\varphi-\eta)}\bar\partial(\varphi-\eta)+2m_2a_2e^{2m_2(\varphi-\eta)}\bar\partial(\varphi-\eta)).\end{aligned} \tag{4.14}$$

Moreover, we have

$$\begin{aligned}\partial\bar\partial F=&\,\partial(e^{2m_1(\varphi-\eta)}\bar\partial a_1+e^{2m_2(\varphi-\eta)}\bar\partial a_2\\&+2m_1a_1e^{2m_1(\varphi-\eta)}\bar\partial(\varphi-\eta)+2m_2a_2e^{2m_2(\varphi-\eta)}\bar\partial(\varphi-\eta))\\=&\,e^{2m_1(\varphi-\eta)}\partial\bar\partial a_1+2m_1e^{2m_1(\varphi-\eta)}\partial(\varphi-\eta)\wedge\bar\partial a_1\\&+e^{2m_2(\varphi-\eta)}\partial\bar\partial a_2+2m_2e^{2m_2(\varphi-\eta)}\partial(\varphi-\eta)\wedge\bar\partial a_2\\&+2m_1a_1e^{2m_1(\varphi-\eta)}\partial\bar\partial(\varphi-\eta)+2m_1\partial(a_1e^{2m_1(\varphi-\eta)})\wedge\bar\partial(\varphi-\eta)\\&+2m_2a_2e^{2m_2(\varphi-\eta)}\partial\bar\partial(\varphi-\eta)+2m_2\partial(a_2e^{2m_2(\varphi-\eta)})\wedge\bar\partial(\varphi-\eta),\end{aligned} \tag{4.15}$$

and furthermore, we have

$$\begin{aligned}&2m_1\partial(a_1e^{2m_1(\varphi-\eta)})\wedge\bar\partial(\varphi-\eta)\\&=2m_1(e^{2m_1(\varphi-\eta)}\partial a_1\wedge\bar\partial(\varphi-\eta)+2m_1a_1e^{2m_1(\varphi-\eta)}\partial(\varphi-\eta)\wedge\bar\partial(\varphi-\eta))\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}&2m_2\partial(a_2e^{2m_2(\varphi-\eta)})\wedge\bar\partial(\varphi-\eta)\\&=2m_2(e^{2m_2(\varphi-\eta)}\partial a_2\wedge\bar\partial(\varphi-\eta)+2m_2a_2e^{2m_2(\varphi-\eta)}\partial(\varphi-\eta)\wedge\bar\partial(\varphi-\eta)).\end{aligned} \tag{4.17}$$

We first compute the following kinds of terms in (4.10):

$$\int e^{2m_j(\varphi-\eta)}\partial(\varphi-\eta)\wedge\bar\partial a_j/F\wedge f\wedge\omega_0\wedge\chi(\frac{\eta-\varphi}{\varepsilon_m})\frac{1}{\varepsilon_m}(d\eta-d\varphi),\quad j\in\{1,2\}. \tag{4.18}$$

$$\int e^{2m_j(\varphi-\eta)}\bar\partial(\varphi-\eta)\wedge\partial a_j/F\wedge f\wedge\omega_0\wedge\chi(\frac{\eta-\varphi}{\varepsilon_m})\frac{1}{\varepsilon_m}(d\eta-d\varphi),\quad j\in\{1,2\}. \tag{4.19}$$

$$\int \frac{1}{m} e^{2m_j(\varphi-\eta)} \partial\bar{\partial} a_j / F \wedge f \wedge \omega_0 \wedge \chi(\frac{\eta-\varphi}{\varepsilon_m}) \frac{1}{\varepsilon_m}(d\eta - d\varphi), \quad j \in \{1,2\}. \tag{4.20}$$

$$\int a_j e^{2m_j(\varphi-\eta)} e^{2m_k(\varphi-\eta)} \partial a_k \wedge \bar{\partial}(\varphi-\eta)/F^2 \wedge f \wedge \omega_0 \wedge \chi(\frac{\eta-\varphi}{\varepsilon_m}) \frac{1}{\varepsilon_m}(d\eta-d\varphi), \quad j,k \in \{1,2\}. \tag{4.21}$$

$$\int a_j e^{2m_j(\varphi-\eta)} e^{2m_k(\varphi-\eta)} \overline{\partial} a_k \wedge \partial(\varphi-\eta)/F^2 \wedge f \wedge \omega_0 \wedge \chi(\frac{\eta-\varphi_m}{\varepsilon_m}) \frac{1}{\varepsilon_m}(d\eta-d\varphi), \quad j,k \in \{1,2\}. \tag{4.22}$$

$$\int \frac{1}{m} e^{2m_j(\varphi-\eta)} e^{2m_k(\varphi-\eta)} \partial a_j \wedge \bar{\partial} a_k / F^2 \wedge f \wedge \omega_0 \wedge \chi(\frac{\eta-\varphi}{\varepsilon_m}) \frac{1}{\varepsilon_m}(d\eta-d\varphi), \quad j,k \in \{1,2\}. \tag{4.23}$$

It is straightforward to check that

$$\partial(\varphi-\eta) \wedge \omega_0 \wedge (d\eta - d\varphi) = 0, \quad \overline{\partial}(\varphi-\eta) \wedge \omega_0 \wedge (d\eta - d\varphi) = 0.$$

From this observation, we see that terms (4.18), (4.19),(4.21), and (4.22) are zero.

For (4.20) and (4.23), note that $\lim_{m\to\infty} m\varepsilon_m = 0$, then $\lim_{m\to\infty} e^{2m_j(\varphi-\eta)} = 1$ in the support of $\chi(\frac{\eta-\varphi}{\varepsilon_m})$. From Theorem 3.1 and Lebesgue dominate theorem, we have

$$\begin{aligned} &\left| \int \frac{1}{m} e^{2m_j(\varphi-\eta)} \partial\bar{\partial} a_j / F \wedge f \wedge \omega_0 \wedge \chi(\frac{\eta-\varphi}{\varepsilon_m}) \frac{1}{\varepsilon_m}(d\eta - d\varphi) \right| \\ &\lesssim \frac{1}{m} \int_X \frac{m_1^n + m_1^{n+1} e^{-m_1\varepsilon_0 d^2(x, X_{\mathrm{sing}})}}{m^n} \to 0 \quad \text{as } m \to \infty, \quad \forall j \in \{1,2\}, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} &\left| \int \frac{1}{m} e^{2m_j(\varphi-\eta)} e^{2m_k(\varphi-\eta)} \partial a_j \wedge \bar{\partial} a_k / F^2 \wedge f \wedge \omega_0 \wedge \chi(\frac{\eta-\varphi}{\varepsilon_m}) \frac{1}{\varepsilon_m}(d\eta - d\varphi) \right| \\ &\lesssim \frac{1}{m} \int_X \frac{m_1^n + m_1^{n+1} e^{-m_1\varepsilon_0 d^2(x, X_{\mathrm{sing}})}}{m^n} \to 0 \quad \text{as } m \to \infty, \quad \forall j,k \in \{1,2\}. \end{aligned} \tag{4.25}$$

From (4.14), (4.15), (4.16), (4.17), and the discussion above, we conclude that the only contribution terms in (4.10) are those involving $\partial\bar{\partial}\varphi$, which is exactly the Levi form $\mathcal{L}_X$ of X. Then for $d\mu$-almost every $\{u_m\} \in \Omega(X)$, we have

$$\begin{aligned} &\lim_{m\to\infty} \frac{1}{m} \langle\, [v_m = 0], f \wedge \omega_0 \wedge \frac{1}{\varepsilon_m} \chi(\frac{\eta}{\varepsilon_m}) d\eta \,\rangle \\ &= \lim_{m\to\infty} \frac{i}{2m\pi} \int \partial\bar{\partial} \log F_m \wedge f \wedge \omega_0 \wedge \chi(\frac{\eta-\varphi}{\varepsilon_m}) \frac{1}{\varepsilon_m}(d\eta - d\varphi) \\ &= \lim_{m\to\infty} \frac{i}{2m\pi} \int (2m_1 a_1 e^{2m_1(\varphi-\eta)} + 2m_2 a_2 e^{2m_2(\varphi-\eta)}) \\ &\qquad\qquad / F_m \cdot \partial\bar{\partial}\varphi \wedge f \wedge \omega_0 \wedge \chi(\frac{\eta-\varphi}{\varepsilon_m}) \frac{1}{\varepsilon_m}(d\eta - d\varphi) \\ &= \lim_{m\to\infty} \frac{i}{\pi} \int \frac{\alpha S_{\alpha m}(x) + k_1 \alpha S_{\alpha k_1 m}(x)}{S_{\alpha m}(x) + S_{\alpha k_1 m}(x)} \partial\bar{\partial}\varphi \wedge f \wedge \omega_0. \end{aligned} \tag{4.26}$$

From Corollary 3.3, Theorem 3.1, Lebesgue dominate theorem, and (4.26), we deduce (4.7). Theorem 1.1 follows. ∎

5. Equidistribution on complex manifolds with strongly pseudoconvex boundary

In this section, we will prove Theorem 1.6. Let M be a relatively compact open subset with C^∞ boundary X of a complex manifold M' of dimension $n+1$ with a smooth Hermitian metric $\langle\cdot|\cdot\rangle$ on its holomorphic tangent bundle $T^{1,0}M'$. From now on, we will use the same notations and assumptions as in the discussion before Theorem 1.6. We will first recall the classical results of Boutet de Monvel–Sjöstrand [2] (see also second part in [14]). We then construct holomorphic functions with specific rate near the boundary. We first recall the Hörmander symbol spaces

Definition 5.1. Let $m\in\mathbb{R}$; $S^m_{1,0}(M'\times M'\times]0,\infty[)$ is the space of all $a(x,y,t)\in C^\infty(M'\times M'\times]0,\infty[)$ such that for all local coordinate patch U with local coordinates $x=(x_1,\ldots,x_{2n+2})$ and all compact sets $K\subset U$ and all $\alpha\in\mathbb{N}_0^{2n+2}$, $\beta\in\mathbb{N}_0^{2n+2}$, $\gamma\in\mathbb{N}_0$, there is a constant $c>0$ such that $\left|\partial_x^\alpha\partial_y^\beta\partial_t^\gamma a(x,y,t)\right|\leq c(1+|t|)^{m-|\gamma|}$, $(x,y,t)\in K\times]0,\infty[$. Here $S^m_{1,0}$ is called the space of symbols of order m type $(1,0)$. We write $S^{-\infty}_{1,0}=\bigcap S^m_{1,0}$.

Let $S^m_{1,0}(\overline{M}\times\overline{M}\times]0,\infty[)$ denote the space of restrictions to $M\times M\times]0,\infty[$ of elements in $S^m_{1,0}(M'\times M'\times]0,\infty[)$.

Let $a_j\in S^{m_j}_{1,0}(\overline{M}\times\overline{M}\times]0,\infty[)$, $j=0,1,2,\ldots$, with $m_j\searrow-\infty$, $j\to\infty$. Then there exists $a\in S^{m_0}_{1,0}(\overline{M}\times\overline{M}\times]0,\infty[)$ such that

$$a-\sum_{0\leq j<k}a_j\in S^{m_k}_{1,0}(\overline{M}\times\overline{M}\times]0,\infty[)$$

for every $k\in\mathbb{N}$. If a and a_j have the properties above, we write

$$a\sim\sum_{j=0}^{\infty}a_j \text{ in } S^{m_0}_{1,0}(\overline{M}\times\overline{M}\times[0,\infty[).$$

Let dv_M be the volume form on M induced by $\langle\cdot|\cdot\rangle$, and let $(\cdot|\cdot)_M$ be the L^2 inner product on $C_0^\infty(M)$ induced by dv_M, and let $L^2(M)$ be the completion of $C_0^\infty(M)$ with respect to $(\cdot|\cdot)_M$. Let $H^0_{(2)}(M)=\left\{u\in L^2(M);\,\overline{\partial}u=0\right\}$. Let $B:L^2(M)\to H^0(M)$ be the orthogonal projection with respect to $(\cdot|\cdot)_M$, and let $B(z,w)\in D'(M\times M)$ be the distribution kernel of B. We recall classical result of Boutet de Monvel–Sjoštrand [2].

Theorem 5.2. *With the notations and assumptions above, we have*

$$B(z,w)=\int_0^\infty e^{i\phi(z,w)t}b(z,w,t)dt+H(z,w) \tag{5.1}$$

(*for the precise meaning of the oscillatory integral* $\int_0^\infty e^{i\phi(z,w)t}b(z,w,t)dt$, *see Remark* 5.3 *below*), *where* $H(z,w)\in C^\infty(\overline{M}\times\overline{M})$,

$$\begin{aligned}
&b(z,w,t)\in S^{n+1}_{1,0}(\overline{M}\times\overline{M}\times]0,\infty[),\\
&b(z,w,t)\sim\textstyle\sum_{j=0}^\infty b_j(z,w)t^{n+1-j} \text{ in the space } S^{n+1}_{1,0}(\overline{M}\times\overline{M}\times]0,\infty[),\\
&b_j(z,w)\in C^\infty(\overline{M}\times\overline{M}),\quad j=0,1,\ldots,\\
&b_0(z,z)\neq0,\ z\in X,
\end{aligned} \tag{5.2}$$

and

(5.3)

$$\begin{aligned}&\phi(z,w) \in C^\infty(\overline{M} \times \overline{M}),\\ &\phi(z,z) = 0, \quad z \in X, \quad \phi(z,w) \neq 0 \quad \textit{if} \quad (z,w) \notin \operatorname{diag}(X \times X),\\ &\operatorname{Im}\phi(z,w) > 0 \quad \textit{if} \quad (z,w) \notin X \times X,\\ &\phi(z,z) = r(z)g(z) \textit{ on } \overline{M},\ g(z) \in C^\infty(\overline{M}) \textit{ with } |g(z)| > c \textit{ on } \overline{M},\ c > 0 \textit{ is a constant.}\end{aligned}$$

Moreover, there is a content $C > 1$ such that

(5.4)

$$\frac{1}{C}(\operatorname{dist}(x,y))^2 \leq |d_y\phi(x,y)|^2 + |\operatorname{Im}\phi(x,y)| \leq C(\operatorname{dist}(x,y))^2, \quad \forall (x,y) \in X \times X,$$

where d_y denotes the exterior derivative on X and $\operatorname{dist}(x,y)$ denotes the distance between x and y with respect to the give Hermitian metric $\langle \cdot | \cdot \rangle$ on X.

Remark 5.3. Let ϕ and $b(z,w,t)$ be as in Theorem 5.2. Let $y = (y_1, \ldots, y_{2n+1})$ be local coordinates on X, and extend $y_1, \ldots, y_{2n+1}$ to real smooth functions in some neighborhood of X. We work with local coordinates $w = (y_1, \ldots, y_{2n+1}, r)$ defined on some neighborhood U of $p \in X$. Let $u \in C_0^\infty(U)$. Choose a cutoff function $\chi(t) \in C^\infty(\mathbb{R})$ so that $\chi(t) = 1$ when $|t| < 1$, and $\chi(t) = 0$ when $|t| > 2$. Set

$$(B_\epsilon u)(z) = \int_0^\infty \int_{\overline{M}} e^{i\phi(z,w)t} b(z,w,t)\chi(\epsilon t)u(w)dv_M(w)dt.$$

Since $d_y\phi \neq 0$ where $\operatorname{Im}\phi = 0$ (see (5.4)), we can integrate by parts in y and t, and obtain $\lim_{\epsilon\to 0}(B_\epsilon u)(z) \in C^\infty(\overline{M})$. This means that $B = \lim_{\epsilon\to 0} B_\epsilon : C^\infty(\overline{M}) \to C^\infty(\overline{M})$ is continuous.

We have the following corollary of Theorem 5.2.

Corollary 5.4. *Under the notations and assumptions above, we have*

$$B(z,z) = F(z)(-r(z))^{-n-2} + G(z)\log(-r(z)) \quad \textit{on } \overline{M}, \tag{5.5}$$

where $F, G \in C^\infty(\overline{M})$ and $|F(z)| > c$ on X, $c > 0$ is a constant.

Since $C^\infty(\overline{M}) \bigcap H^0_{(2)}(M)$ is dense in $H^0_{(2)}(M)$ in $L^2(M)$, we can find $g_j \in C^\infty(\overline{M}) \bigcap H^0_{(2)}(M)$ with $(\, g_j \,|\, g_k \,)_M = \delta_{j,k}$, $j,k = 1,2,\ldots$, such that the set

$$A(M) := \operatorname{span}\{\mathrm{g}_1, \mathrm{g}_2, \ldots\} \tag{5.6}$$

is dense in $H^0_{(2)}(M)$. Moreover, for every $u \in L^2(M)$, we have

$$\textstyle\sum_{j=1}^N g_j(\, u \,|\, g_j \,)_M \to Bu \text{ in } L^2(M) \quad \text{as } N \to \infty. \tag{5.7}$$

Fix $k \in \mathbb{N}$, k large. Fix $x_0 \in M$ with $\frac{1}{2k} \leq |r(x_0)| \leq \frac{1}{k}$. Let $x = (x_1, \ldots, x_{2n+2})$ be local coordinates of M defined in a small neighborhood of x_0 with $x(x_0) = 0$. Let $\chi \in C_0^\infty(\mathbb{R}^{2n+2})$ with $\chi \equiv 1$ near $0 \in \mathbb{R}^{2n+2}$. For $\varepsilon > 0$, put $\chi_\varepsilon(x) = \varepsilon^{-(2n+2)}\chi(\frac{x}{\varepsilon})$. From (5.7), for every $\varepsilon > 0$, ε small, we have

$$\sum_{j=0}^\infty |(\, g_j \,|\, \chi_\varepsilon \,)_M|^2 = (\, B\chi_\varepsilon \,|\, \chi_\varepsilon \,)_M. \tag{5.8}$$

Since $B(z,w) \in C^\infty(M \times M)$, we have

$$\lim_{\varepsilon \to 0}\Big(\sum_{j=1}^{\infty} |(\, g_j \,|\, \chi_\varepsilon \,)_M|^2\Big) = B(x_0,x_0)m(x_0), \tag{5.9}$$

where $m(x)dx_1 \cdots dx_{2n+2} = dv_M$.

From (5.7), for every $\varepsilon_1, \varepsilon_2 > 0$, $\varepsilon_1, \varepsilon_2$ small, we have

$$\begin{aligned}&\sum_{j=0}^{\infty} |(\, g_j \,|\, \chi_{\varepsilon_1} \,)_M - (\, g_j \,|\, \chi_{\varepsilon_2} \,)_M|^2 \\ &= (\, B\chi_{\varepsilon_1} \,|\, \chi_{\varepsilon_1} \,)_M - (\, B\chi_{\varepsilon_1} \,|\, \chi_{\varepsilon_2} \,)_M - (\, B\chi_{\varepsilon_2} \,|\, \chi_{\varepsilon_1} \,)_M + (\, B\chi_{\varepsilon_2} \,|\, \chi_{\varepsilon_2} \,)_M.\end{aligned} \tag{5.10}$$

Since $B(z,w) \in C^\infty(M \times M)$, we deduce that for every $\delta > 0$, there is a $C_\delta > 0$ such that for all $0 < \varepsilon_1, \varepsilon_2 < C_\delta$, we have

$$\sum_{j=0}^{\infty} |(\, g_j \,|\, \chi_{\varepsilon_1} \,)_M - (\, g_j \,|\, \chi_{\varepsilon_2} \,)_M|^2 < \delta. \tag{5.11}$$

Now, we can prove the following.

Theorem 5.5. *We have* $\sum_{j=1}^{\infty} |g_j(x_0)|^2 = B(x_0,x_0)m(x_0)$.

Proof. From (5.9), it is easy to see that

$$\sum_{j=1}^{\infty} |g_j(x_0)|^2 \leq B(x_0,x_0)m(x_0). \tag{5.12}$$

Let $\delta > 0$, and fix $0 < \varepsilon_0 < C_\delta$, where C_δ is as in (5.11). Since $\sum_{j=1}^{\infty} |(\, g_j \,|\, \chi_{\varepsilon_0} \,)_M|^2 < \infty$, there is a $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty} |(\, g_j \,|\, \chi_{\varepsilon_0} \,)_M|^2 < \delta. \tag{5.13}$$

Now, for every $0 < \varepsilon < \varepsilon_0$, from (5.11) and (5.13), we have

$$\begin{aligned}\sum_{j=N+1}^{\infty} |(\, g_j \,|\, \chi_\varepsilon \,)_M|^2 &\leq 2 \sum_{j=N+1}^{\infty} |(\, g_j \,|\, \chi_\varepsilon \,)_M - (\, g_j \,|\, \chi_{\varepsilon_0} \,)_M|^2 + 2 \sum_{j=N+1}^{\infty} |(\, g_j \,|\, \chi_{\varepsilon_0} \,)_M|^2 \\ &\leq 2 \sum_{j=1}^{\infty} |(\, g_j \,|\, \chi_\varepsilon \,)_M - (\, g_j \,|\, \chi_{\varepsilon_0} \,)_M|^2 + 2 \sum_{j=N+1}^{\infty} |(\, g_j \,|\, \chi_{\varepsilon_0} \,)_M|^2 \\ &\leq 4\delta.\end{aligned} \tag{5.14}$$

From (5.14), we deduce that

$$\limsup_{\varepsilon \to 0} \sum_{j=N+1}^{\infty} |(\, g_j \,|\, \chi_\varepsilon \,)_M|^2 \leq 4\delta. \tag{5.15}$$

Now,

$$
\begin{aligned}
&\sum_{j=1}^{\infty}|g_j(x_0)|^2 \geq \sum_{j=1}^{N}|g_j(x_0)|^2 = \lim_{\varepsilon\to 0}\sum_{j=1}^{N}|(\,g_j\,|\,\chi_\varepsilon\,)_M|^2\\
&\geq \liminf_{\varepsilon\to 0}\Big(\sum_{j=1}^{\infty}|(\,g_j\,|\,\chi_\varepsilon\,)_M|^2-\sum_{N+1}^{\infty}|(\,g_j\,|\,\chi_\varepsilon\,)_M|^2\Big)\\
&\geq \liminf_{\varepsilon\to 0}\sum_{j=1}^{\infty}|(\,g_j\,|\,\chi_\varepsilon\,)_M|^2-\limsup_{\varepsilon\to 0}\sum_{N+1}^{\infty}|(\,g_j\,|\,\chi_\varepsilon\,)_M|^2.
\end{aligned}
\tag{5.16}
$$

From (5.9), (5.15), and (5.16), we deduce that

$$\sum_{j=1}^{\infty}|g_j(x_0)|^2 \geq B(x_0,x_0)m(x_0)-4\delta.$$

Since δ is arbitrary, we conclude that

$$\sum_{j=1}^{\infty}|g_j(x_0)|^2 \geq B(x_0,x_0)m(x_0). \tag{5.17}$$

From (5.17) and (5.12), the theorem follows. □

From Theorem 5.5 and (5.5), we deduce that there is an $N_{x_0}\in\mathbb{N}$ such that

$$\left|r^{n+2}(x_0)\sum_{j=1}^{N_{x_0}}|g_j(x_0)|^2\right| \geq \frac{1}{2}\,|F(x_0)|\,, \tag{5.18}$$

where F is as in (5.5). Let

$$h_{x_0} := \frac{1}{\sum_{j=1}^{N_{x_0}}|g_j(x_0)|^2}\sum_{j=1}^{N_{x_0}}g_j(x)\,|\overline{g}_j(x_0)|\,. \tag{5.19}$$

Then, $h_{x_0}\in H^0_{(2)}(M)\cap C^\infty(\overline{M})$ with $(\,h_{x_0}\,|\,h_{x_0}\,)_M=1$, and there is a small neighborhood U_{x_0} of x_0 in M such that

$$|h_{x_0}(x)| \geq \frac{1}{4}\,|F(x)|\,. \tag{5.20}$$

Assume that $\{x\in M, \frac{1}{2k}\leq|r(x)|\leq\frac{1}{k}\}\subset U_{x_0}\bigcup U_{x_1}\bigcup\cdots\bigcup U_{x_{a_k}}$, and let h_{x_j} be as in (5.19), $j=0,1,\ldots,a_k$. Take $\beta_k\in\mathbb{N}$ be a large number so that

$$\Big\{h_{x_0},h_{x_1},\ldots,h_{x_{a_k}}\Big\}\subset \operatorname{span}\{g_1,g_2,\ldots,g_{\beta_k}\}\,.$$

From (5.20), it is easy to see that

$$\left|r^{n+2}(x)\sum_{j=1}^{\beta_k}|g_j(x)|^2\right| \geq \frac{1}{4}\,|F(x)| \quad \text{on } \{x\in M, \tfrac{1}{2k}\leq|r(x)|\leq\tfrac{1}{k}\}. \tag{5.21}$$

Note that $|F(x)| > c$ on X, where $c > 0$ is a constant. From this observation and (5.21), we get

Theorem 5.6. *There is a $k_0 \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, $k \geq k_0$, we can find $\beta_k \in \mathbb{N}$ such that*

$$\left| r^{n+2}(x) \sum_{j=1}^{\beta_k} |g_j(x)|^2 \right| \geq c_0 \quad on\ \left\{x \in M, \tfrac{1}{2k} \leq |r(x)| \leq \tfrac{1}{k}\right\}, \tag{5.22}$$

where $c_0 > 0$ is a constant independent of k.

Let $b_j = \beta_{k_0+j} \in \mathbb{N}$, $j = 1, 2, \ldots$, where β_j and k_0 are as in Theorem 5.6. For every $m \in \mathbb{N}$, let $A_m(M)$, $SA_m(M)$, and $d\mu_m$ be as in the discussion before (1.13). Let $\beta := \{b_j\}_{j=1}^{\infty}$, and let $\Omega(M, \beta)$ and $d\mu(\beta)$ be as in (1.13) and (1.14), respectively. For each $k = 1, 2, 3, \ldots$, let

$$P_k(x) := \sum_{j=1}^{b_k} |g_j(x)|^2 .$$

Let $u_k \in SA_{b_k}(M)$. Then, u_k can be written as $u_k = \sum_{j=1}^{b_k} \lambda_j g_j$ with $\sum_{j=1}^{b_k} |\lambda_j|^2 = 1$. We have the following.

Theorem 5.7. *With the notations and assumptions above, fix $\psi \in C_0^\infty([-1, -\frac{1}{2}])$. Then, for $d\mu(\beta)$-almost every $u = \{u_k\} \in \Omega(M, \beta)$, we have*

$$\lim_{k\to\infty}\Big(\langle\, [u_k=0], (2i)kr\psi(kr)\phi\wedge\partial r\wedge\overline{\partial}r\,\rangle + \frac{1}{\pi}\int_{\overline{M}}\Big(\log P_k(x)\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\partial r\wedge\overline{\partial}r\Big) = 0, \tag{5.23}$$

for all $\phi \in C^\infty(\overline{M}, B^{*n-1,n-1}M')$.

Proof. The proof essentially follows from Shifffman–Zelditch [19], we only sketch the proof here. By using density argument, we only need to prove that for any $\phi \in C^\infty(\overline{M}, B^{*n-1,n-1}T^*M')$, there exist $d\mu(\beta)$-almost every $u = \{u_k\} \in \Omega(M, \beta)$, such that

$$\lim_{k\to\infty}\Big(\langle\, [u_k=0], (2i)kr\psi(kr)\phi\wedge\partial r\wedge\overline{\partial}r\,\rangle + \frac{1}{\pi}\int_{\overline{M}}\Big(\log P_k(x)\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\partial r\wedge\overline{\partial}r\Big) = 0. \tag{5.24}$$

We claim that

$$\begin{aligned} R_k := \int_{S^{2b_k-1}} \Big| \langle\, [\sum_{j=1}^{b_k} \lambda_j g_j = 0], (2i)kr\psi(kr)\phi \wedge \partial r \wedge \overline{\partial} r\,\rangle \\ + \frac{1}{\pi}\int_{\overline{M}} (\log P_k(x)) kr\psi(kr)\partial\overline{\partial}\phi \wedge \partial r \wedge \overline{\partial} r \Big|^2 d\mu_{b_k}(\lambda) = O(\frac{1}{k^2}). \end{aligned} \tag{5.25}$$

From (5.25), we see that $\sum_{k=1}^{\infty} R_k < +\infty$, and by Lebesgue measure theory, we get (5.24). Hence, we only need to prove (5.25).

For $(x, y) \in \overline{M} \times \overline{M}$, put

$$Q_k(x, y) := \int_{S^{2b_k-1}} \log\Big(\frac{\left|\sum_{j=1}^{b_k} \lambda_j g_j(x)\right|^2}{P_k(x)}\Big) \log\Big(\frac{\left|\sum_{j=1}^{b_k} \lambda_j g_j(y)\right|^2}{P_k(y)}\Big) d\mu_{b_k}(\lambda),$$

$$f_k := -\frac{1}{\pi} kr\psi(kr)\partial\overline{\partial}\phi(y) \wedge \partial r \wedge \overline{\partial} r \in C_0^\infty(M, T^{*n+1,n+1}M').$$

By using the same argument in [19] (see also Theorem 5.3.3 in [16]), we can check that

$$R_k = \int_{\overline{M}\times\overline{M}} Q_k(x,y) f_k(x)\wedge f_k(y). \tag{5.26}$$

Moreover, from Lemma 5.3.2 in [16], there is a constant $C_k > 0$ independent of $(x,y)\in\overline{M}\times\overline{M}$ such that

$$|Q_k(x,y) - C_k| \le C, \quad \forall (x,y)\in M\times M, \tag{5.27}$$

where $C>0$ is a constant independent of k. From (5.27), it is easy to check that

$$\left|\int_{\overline{M}\times\overline{M}}\Big(Q_k(x,y)-C_k\Big) f_k(x)\wedge f_k(y)\right| = O(\frac{1}{k^2}). \tag{5.28}$$

By using integration by parts, we see that $\int_{\overline{M}\times\overline{M}}\Big(Q_k(x,y)-C_k\Big)f_k(x)\wedge f_k(y) = R_k$. From this observation and (5.28), the claim (5.25) follows. □

Proof of Theorem 1.6. In view of Theorem 5.7, we only need to show that

$$\lim_{k\to\infty} -\frac{1}{\pi}\int_{\overline{M}}(\log P_k(x))kr\psi(kr)\partial\overline{\partial}\phi\wedge\partial r\wedge\overline{\partial}r = -(n+2)\frac{i}{2\pi}c_0\int_X \mathcal{L}_X\wedge\omega_0\wedge\phi,$$

where $c_0 = \int_{\mathbb{R}}\psi(x)dx$. Now,

$$\begin{aligned}
&-\frac{1}{\pi}\int_{\overline{M}}\Big(\log P_k(x)\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\partial r\wedge\overline{\partial}r\\
&=-\frac{1}{\pi}\int_{\overline{M}}\Big(\log(P_k(x)(-r)^{n+2}(x))\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\partial r\wedge\overline{\partial}r\\
&\quad+\frac{n+2}{\pi}\int_{\overline{M}}\Big(\log(-r)(x)\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\partial r\wedge\overline{\partial}r\\
&=-\frac{1}{\pi}\int_{\overline{M}}\Big(\log(P_k(x)(-r)^{n+2}(x))\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\partial r\wedge\overline{\partial}r\\
&\quad-i\frac{n+2}{2\pi}\int_{\overline{M}}\Big(\log(-r)(x)\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\omega_0\wedge dr,
\end{aligned}\tag{5.29}$$

where $\omega_0 = J(dr)$, J is the standard complex structure map on T^*M'.

From Theorem 5.6, it is easy to see that

$$\lim_{k\to+\infty} -\frac{1}{\pi}\int_{\overline{M}}\Big(\log(P_k(x)(-r)^{n+2}(x))\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\partial r\wedge\overline{\partial}r = 0. \tag{5.30}$$

By using integration by parts, we have

$$\begin{aligned}
&-i\frac{n+2}{2\pi}\int_{\overline{M}}\Big(\log(-r)(x)\Big)kr\psi(kr)\partial\overline{\partial}\phi\wedge\omega_0\wedge dr\\
&=-i\frac{n+2}{2\pi}\int_{\overline{M}}\Big((\partial\overline{\partial}\log(-r))(x)\Big)kr\psi(kr)\phi\wedge\omega_0\wedge dr\\
&=-i\frac{n+2}{2\pi}\int_{\overline{M}}\partial\overline{\partial}r(x)k\psi(kr)\phi\wedge\omega_0\wedge dr\\
&\to -(n+2)\frac{i}{2\pi}c_0\int_X\mathcal{L}_X\wedge\omega_0\wedge\phi \quad\text{as } k\to\infty,
\end{aligned}\tag{5.31}$$

where $c_0 = \int_{\mathbb{R}}\psi(x)dx$. From (5.29), (5.30), and (5.31), the theorem follows. □

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