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COHEN–MACAULAYNESS AND CANONICAL MODULE OF RESIDUAL INTERSECTIONS

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Abstract. We show the Cohen–Macaulayness and describe the canonical module of residual intersections $J = \mathfrak{a}\colon_R I$ in a Cohen–Macaulay local ring R, under sliding depth type hypotheses. For this purpose, we construct and study, using a recent article of Hassanzadeh and the second author, a family of complexes that contains important information on a residual intersection and its canonical module. We also determine several invariants of residual intersections as the graded canonical module, the Hilbert series, the Castelnuovo–Mumford regularity and the type. Finally, whenever I is strongly Cohen–Macaulay, we show duality results for residual intersections that are closely connected to results by Eisenbud and Ulrich. It establishes some tight relations between the Hilbert series of some symmetric powers of $I/\mathfrak{a}$. We also provide closed formulas for the types and for the Bass numbers of some symmetric powers of $I/\mathfrak{a}$.

Contents

1. Introduction

The concept of residual intersection was introduced by Artin and Nagata in [1], as a generalization of linkage; it is more ubiquitous, but also harder to understand. Geometrically, let X and Y be two irreducible closed subschemes of a scheme Z with $\operatorname{codim}_Z(X) \leq \operatorname{codim}_Z(Y) = s$ and $Y \nsubseteq X$. Then Y is called a residual intersection

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of X if the number of equations needed to define $X \cup Y$ as a subscheme of Z is the smallest possible, i.e., s. For a ring R and a finitely generated R-module M, let $\mu_R(M)$ denote the minimum number of generators of M.

The precise definition of a residual intersection is the following.

Definition 1.1. Let R be a Noetherian ring, let I be an ideal of height g, and let $s \geq g$ be an integer.

1. An *s-residual intersection* of I is a proper ideal J of R such that $\mathrm{ht}(J) \geq s$ and $J = (\mathfrak{a} \colon {}_R I)$ for some ideal $\mathfrak{a} \subset I$ which is generated by s elements.
2. An *arithmetic s-residual intersection* of I is an s-residual intersection J of I such that $\mu_{R_\mathfrak{p}}((I/\mathfrak{a})_\mathfrak{p}) \leq 1$ for all prime ideals $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) \leq s$.
3. A *geometric s-residual intersection* of I is an s-residual intersection J of I such that $\mathrm{ht}(I + J) \geq s + 1$.

Notice that an s-residual intersection is a direct link if I is unmixed and $s = \mathrm{ht}(I)$. Also any geometric s-residual intersection is arithmetic.

The theory of residual intersections has been a center of interest since the 1980s, when Huneke repaired in [16] an argument of Artin and Nagata in [1], introducing the notion of a strongly Cohen–Macaulay ideal: an ideal such that all of its Koszul homology is Cohen–Macaulay. The notion of strong Cohen–Macaulayness is stable under even linkage, in particular ideals linked to a complete intersection satisfy this property.

In [16], Huneke showed that if R is a Cohen–Macaulay local ring, J is a s-residual intersection of a strongly Cohen–Macaulay ideal I of R satisfying G_s, then R/J is Cohen–Macaulay of codimension s. Following [1], one says that I satisfies G_s if the number of generators $\mu_{R_\mathfrak{p}}(I_\mathfrak{p})$ is at most $\dim(R_\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ with $I \subset \mathfrak{p}$ and $\dim(R_\mathfrak{p}) \leq s - 1$, and that I satisfies G_∞ if I satisfies G_s for all s. Later, Herzog, Vasconcelos, and Villarreal in [17] replaced the assumption of strong Cohen–Macaulayness by the weaker sliding depth condition, for geometric residuals, but they also showed that this assumption cannot be weakened any further. On the other hand, Huneke and Ulrich proved in [15] that the condition G_s is superfluous for ideals in the linkage class of a complete intersection. More precisely, they show the following.

Theorem ([15]). *Let R be a Gorenstein local ring, and let I be an ideal of height g that is evenly linked to a strongly Cohen–Macaulay ideal satisfying G_∞. If $J = \mathfrak{a} \colon {}_R I$ is an s-residual intersection of I, then R/J is Cohen–Macaulay of codimension s and the canonical module of R/J is the $(s - g + 1)$th symmetric power of $I/\mathfrak{a}$.*

Let us notice that, in the proof of this statement, it is important to keep track of the canonical module of the residual along the deformation argument that they are using.

A natural question is then to know whether the G_s assumption is at all needed to assert that residuals of ideals that are strongly Cohen–Macaulay, or satisfy the weaker sliding depth condition, are always Cohen–Macaulay, and to describe the canonical module of the residual. In this direction, Hassanzadeh and the second author remarked in [11] that the following long-standing assertions were, explicitly or implicitly, conjectured.

Conjecture ([4,15,22])**.** *Let R be a Cohen–Macaulay local* (*or *local*) *ring, and let I be strongly Cohen–Macaulay, or even let it just satisfy sliding depth. Then, for any s-residual intersection $J = (\mathfrak{a}\colon {}_R I)$ of I, the following hold:*

(1) *R/J is Cohen–Macaulay.*
(2) *The canonical module of R/J is the $(s-g+1)$th symmetric power of $I/\mathfrak{a}$, if R is Gorenstein, with $g = \operatorname{ht}(I) \leq s$.*
(3) *$\mathfrak{a}$ is minimally generated by s elements.*
(4) *J is unmixed.*
(5) *When R is positively graded over a field, the Hilbert series of R/J depends only upon I and the degrees of the generators of $\mathfrak{a}$.*

The first item in the conjecture was shown by Hassanzadeh [8] for arithmetic residual intersections, and thus in particular for geometric residual intersections, under the sliding depth condition. In a recent article [11], Hassanzadeh and the second author proved that the second and fifth items in the conjecture hold for the arithmetic residual intersections of strongly Cohen–Macaulay ideals, and that the third and fourth items in the conjecture are true if $\operatorname{depth}(R/I) \geq \dim(R) - s$, with I satisfying the sliding depth condition.

In the text, we will complete the picture by showing that the first and fifth items in the conjecture hold whenever I satisfies $\mathcal{SD}_1$, and that the second item in the conjecture is true if I satisfies $\mathcal{SD}_2$ ($\mathcal{SD}_0$ is the sliding depth condition, and $\mathcal{SD}_\infty$ is strong Cohen–Macaulayness; see Definition 3.7 for the definition of the intermediate $\mathcal{SD}_k$ conditions).

In particular, all items in the conjecture hold for strongly Cohen–Macaulay ideals. The following puts together part of these results (see 4.5, 4.8, and 6.2).

Theorem. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with canonical module ω. Assume that $J = (\mathfrak{a}\colon {}_R I)$ is an s-residual intersection of I with $\mathfrak{a} \subset I$ and that $\operatorname{ht}(I) = g \leq s = \mu_R(\mathfrak{a})$. Then the following hold:*

(i) *R/J is Cohen–Macaulay of codimension s if I satisfies $\mathcal{SD}_1$.*

If furthermore I is strongly Cohen–Macaulay and $\operatorname{Tor}_1^R(R/I, \omega) = 0$,

(ii) $\omega_{R/J} \simeq \operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega$,
(iii) $\omega_{\operatorname{Sym}_R^k(I/\mathfrak{a})} \simeq \operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R \omega$ *for* $1 \leq k \leq s-g$.

Notice that $\operatorname{Tor}_1^R(R/I, \omega) = 0$ if R is Gorenstein or I has finite projective dimension.

A key ingredient of our proofs is a duality result between some of the first symmetric powers of $I/\mathfrak{a}$ together with a description of the canonical module of the residual as in items (ii) and (iii) above. This could be compared to recent results of Eisenbud and Ulrich that obtained similar dualities under slightly different hypotheses in [6]. In their work, conditions on the local number of generators are needed and depth conditions are asked for some of the first powers of the ideal I, along the lines of [23], and the duality occurs between powers $I^t/\mathfrak{a}I^{t-1}$ in place of symmetric powers $\operatorname{Sym}^t(I/\mathfrak{a})$. Although their results and ours coincide in an important range of situations, like for geometric residuals of strongly Cohen–Macaulay ideals satisfying G_s, the domains of validity are quite distinct. We prove the following (Theorem 6.7).

Theorem. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring, and let $\mathfrak{a} \subset I$ be two ideals of R, with $\operatorname{ht}(I) = g$. Suppose that $J = (\mathfrak{a}\colon {}_R I)$ is an s-residual intersection of I. If*

I is strongly Cohen–Macaulay, then $\omega_{R/J} \simeq \mathrm{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$ and, for all $0 \leq k \leq s-g+1$, the following hold:

(i) *The R/J-module $\mathrm{Sym}^{k}_{R/J}(I/\mathfrak{a})$ is faithful and Cohen–Macaulay.*

(ii) *The multiplication*

$$\mathrm{Sym}^{k}_{R/J}(I/\mathfrak{a}) \otimes_{R/J} \mathrm{Sym}^{s-g+1-k}_{R/J}(I/\mathfrak{a}) \longrightarrow \mathrm{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$$

is a perfect pairing.

(iii) *Setting $A := \mathrm{Sym}_{R/J}(I/\mathfrak{a})$, the graded R/J-algebra*

$$\overline{A} := A/A_{>s-g+1} = \bigoplus_{i=0}^{s-g+1} \mathrm{Sym}^{i}_{R/J}(I/\mathfrak{a})$$

is Gorenstein.

The paper is organized as follows.

In Section 2, we collect the notations and general facts about Koszul complexes. We prove duality results for Koszul cycles in Propositions 2.2 and 2.4. We also describe the structure of the homology modules of the approximation complexes in Propositions 2.5 and 2.6.

In Section 3, we construct a family of residual approximation complex, all of the same finite size, $\{{}^{M}_{k}\mathcal{Z}^{+}_{\bullet}\}_{k\in\mathbb{Z}}$. This family is a generalization of the family $\{{}_{k}\mathcal{Z}^{+}_{\bullet}\}_{k\in\mathbb{Z}}$ that is built in a recent article [11] by Hassanzadeh and the second author. We study the properties of these complexes, in particular complexes ${}^{\omega}_{k}\mathcal{Z}^{+}_{\bullet}$, where ω is the canonical module of R. The main results of this section are Propositions 3.2, 3.3, and 3.5.

In Section 4, we prove one of the main results of this paper: the Cohen–Macaulayness and the description of the canonical module of residual intersections. Recall that in [8], Hassanzadeh proved that, under the sliding depth condition, $H_0({}_{0}\mathcal{Z}^{+}_{\bullet}) = R/K$ is Cohen–Macaulay of codimension s, with $K \subset J$, $\sqrt{K} = \sqrt{J}$, and further $K = J$ whenever the residual is arithmetic. First, we consider the height 2 case and show that under the $\mathcal{SD}_1$ condition, there exists an epimorphism $\varphi : H_0({}^{\omega}_{s-1}\mathcal{Z}^{+}_{\bullet}) \twoheadrightarrow \omega_{R/K}$ which is an isomorphism if I satisfies $\mathcal{SD}_2$ (Proposition 4.4). By exploring these complexes, we show that, under the $\mathcal{SD}_1$ condition, $K = J$; and therefore, under the $\mathcal{SD}_2$ condition, the canonical module of R/J is $H_0({}^{\omega}_{s-1}\mathcal{Z}^{+}_{\bullet})$. In a second step, we reduce the general case to the height 2 case. Our main results in this section are Theorems 4.5 and 4.8.

In Section 5, we study the stability of Hilbert functions and Castelnuovo–Mumford regularity of residual intersections. Using the acyclicity of ${}_{0}\mathcal{Z}^{+}_{\bullet}$, Proposition 5.1 says that the Hilbert function of R/J depends only on the degrees of the generators of $\mathfrak{a}$ and the Koszul homologies of I. The graded structure of the canonical module of R/J in Proposition 5.3 is the key to deriving the Castelnuovo–Mumford regularity of residual intersection in Corollary 5.4.

Finally, in Section 6, we consider the case in which I is strongly Cohen–Macaulay. In Theorem 6.2, we prove that, for $1 \leq k \leq s-g$,

$$\omega_{\mathrm{Sym}^{k}_{R}(I/\mathfrak{a})} \simeq \mathrm{Sym}^{s-g+1-k}_{R}(I/\mathfrak{a}) \otimes_{R} \omega$$

whenever $\operatorname{Tor}_1^R(R/I,\omega)=0$. In Theorem 6.7, we deduce from this and Lemma 6.6 that, whenever R is Gorenstein, the pairing

$$\operatorname{Sym}^k_{R/J}(I/\mathfrak{a})\otimes_{R/J}\operatorname{Sym}^{s-g+1-k}_{R/J}(I/\mathfrak{a})\longrightarrow\operatorname{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$$

given by multiplication is a perfect pairing. We derive some tight relations between the Hilbert series of the symmetric powers of $I/\mathfrak{a}$ in Corollary 6.8 and give the closed formulas for the types and for the Bass number of some symmetric powers of $I/\mathfrak{a}$ in Corollaries 6.9 and 6.10, respectively.

2. Koszul cycles and approximation complexes

In this section, we collect the notations and general facts about Koszul complexes and approximation complexes. The reader can consult, for instance, [2, Chapter 1] and [12, 13, 14, 21]. We give some results on the duality for Koszul cycles and describe the 0th homology modules of approximation complexes with coefficients in a module.

Assume that R is a Noetherian ring, and assume that $I=(f_1,\dots,f_r)$ is an ideal of R. Let M be a finitely generated R-module. The symmetric algebra of M is denoted by $\operatorname{Sym}_R(M)$, and the kth symmetric power of M is denoted by $\operatorname{Sym}^k_R(M)$. We consider $S=R[T_1,\dots,T_r]$ as a standard graded algebra over $S_0=R$. For a graded S-module N, the kth graded component of N is denoted by $N_{[k]}$. We make $\operatorname{Sym}_R(I)$ an S-algebra via the graded ring homomorphism $S\longrightarrow\operatorname{Sym}_R(I)$, sending T_i to f_i as an element of $\operatorname{Sym}_R(I)_{[1]}=I$, and we write $\operatorname{Sym}_R(I)=S/\mathfrak{L}$.

For a sequence of elements $\mathbf{x}$ in R, we denote the Koszul complex by $K_\bullet(\mathbf{x};M)$, its cycles by $Z_i(\mathbf{x};M)$, its boundaries by $B_i(\mathbf{x};M)$, and its homologies by $H_i(\mathbf{x};M)$. If $M=R$, then we denote, for simplicity, K_i,Z_i,B_i,H_i. To set more notation, when we draw the picture of a double complex obtained from a tensor product of two complexes (in the sense of [25, 2.7.1]) in which at least one of them is finite, say, $A\otimes B$, where B is finite, we always put A in the vertical one and B in the horizontal one. We also label the module which is in the upright corner by $(0,0)$ and consider the labels for the rest, as the points in the third quadrant.

Lemma 2.1. *Let R be a ring, and let $I=(f_1,\dots,f_r)$ be an ideal of R. If $I=R$, then $Z_i\simeq\bigwedge^i R^{r-1}$.*

Proof. Since $I=R$, $H_i=0$ for all i by [2, Proposition 1.6.5(c)]. The result follows from the fact that the Koszul complex is split exact in this case. □

Let us recall the conditions $\mathcal{S}_k$ of Serre. Let R be a Noetherian ring, and let k be a nonnegative integer. A finitely generated R-module M satisfies *Serre's condition $\mathcal{S}_k$* if

$$\operatorname{depth}(M_\mathfrak{p})\geq\min\{k,\dim M_\mathfrak{p}\}$$

for every prime ideal $\mathfrak{p}$ of R.

Let $(R,\mathfrak{m})$ be local. The local cohomology modules of an R-module M are denoted by $H^i_\mathfrak{m}(M)$. These can be computed with the Čech complex $C^\bullet_\mathfrak{m}$ constructed on a parameter system of R: $H^i_\mathfrak{m}(M)=H^i(M\otimes_R C^\bullet_\mathfrak{m})$.

Duality results for Koszul homology modules over Gorenstein rings have been obtained by several authors, for instance in [5,9,18]. For Koszul cycles, the following holds.

Proposition 2.2. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, and let $I=(f_1,\dots,f_r)$ be an ideal of R. Suppose that R satisfies $\mathcal{S}_2$ and that $\operatorname{ht}(I)\geq 2$. Then, for all $0\leq i\leq r-1$,*

$$Z_i \simeq \operatorname{Hom}_R(Z_{r-1-i},R).$$

Proof. The inclusions $Z_i \hookrightarrow K_i = \bigwedge^i R^r$ and $Z_{r-1-i} \hookrightarrow K_{r-1-i} = \bigwedge^{r-1-i} R^r$ induce a map

$$\varphi_i : Z_i \times Z_{r-1-i} \longrightarrow K_i \times K_{r-1-i} \longrightarrow K_{r-1},$$

where the last map is the multiplication of the Koszul complex, which is a differential graded algebra, and $\operatorname{Im}(\varphi_i)\subset Z_{r-1}\simeq K_r \simeq R$. It follows that φ_i induces a map

$$\psi_i : Z_i \longrightarrow \operatorname{Hom}_R(Z_{r-1-i},R).$$

We induct on the height to show that, for every $\mathfrak{p}\in\operatorname{Spec}(R)$, $(\psi_i)_\mathfrak{p}$ is an isomorphism. If $\operatorname{ht}(\mathfrak{p})<2$, then $I_\mathfrak{p}=R_\mathfrak{p}$, by Lemma 2.1,

$$(Z_i)_\mathfrak{p} \simeq \bigwedge^i R_\mathfrak{p}^{r-1}$$

and

$$(Z_{r-1-i})_\mathfrak{p} \simeq \bigwedge^{r-1-i} R_\mathfrak{p}^{r-1},$$

and [2, Proposition 1.6.10(b)] shows that $(\psi_i)_\mathfrak{p}$ is an isomorphism.

Suppose that $\operatorname{ht}(\mathfrak{p})\geq 2$ and $(\psi_i)_\mathfrak{q}$ is an isomorphism for all primes contained properly in $\mathfrak{p}$. Replacing R with $R_\mathfrak{p}$ and $\mathfrak{m}$ with $\mathfrak{p}R_\mathfrak{p}$, we can suppose that ψ_i is an isomorphism on the punctured spectrum: the kernel and the cokernel of ψ_i are annihilated by a power of $\mathfrak{m}$. It follows that $H^j_\mathfrak{m}(\operatorname{Ker}(\psi_i))=H^j_\mathfrak{m}(\operatorname{Coker}(\psi_i))=0$ for $j>0$. Since R satisfies $\mathcal{S}_2$, $\operatorname{depth}(Z_i)\geq\min\{2,\operatorname{depth}(R)\}=2$. The exact sequence

$$0\longrightarrow \operatorname{Ker}(\psi_i)\longrightarrow Z_i \longrightarrow \operatorname{Im}(\psi_i)\longrightarrow 0$$

implies that $\operatorname{Ker}(\psi_i)=H^0_\mathfrak{m}(\operatorname{Ker}(\psi_i))=0$. Observing that

$$\operatorname{depth}(\operatorname{Hom}_R(Z_{r-1-i},R))\geq\min\{2,\operatorname{depth}(R)\}=2,$$

the exact sequence

$$0\longrightarrow Z_i\longrightarrow \operatorname{Hom}_r(Z_{r-1-i},R)\longrightarrow \operatorname{Coker}(\psi_i)\longrightarrow 0$$

implies that $\operatorname{Coker}(\psi_i)=H^0_\mathfrak{m}(\operatorname{Coker}(\psi_i))=0$. ☐

To fix the terminology we will use, we recall some notations and definitions. Let $(R,\mathfrak{m})$ be a Noetherian local ring. The injective envelope of the residue field $R/\mathfrak{m}$ is denoted by $E(R/\mathfrak{m})$ (or by E when the ring is clearly identified by the context). The Matlis dual of an R-module M is the module $M^\vee=\operatorname{Hom}_R(M,E(R/\mathfrak{m}))$. The Matlis duality functor is exact, sends Noetherian modules to Artinian modules and Artinian modules to Noetherian modules, and preserves annihilators.

When the module M is finitely generated, we have $M^{\vee\vee}\simeq\widehat{M}$, the $\mathfrak{m}$-adic completion of M, while $X\simeq X^{\vee\vee}$ when the module X is of finite length.

When R is the homomorphic image of a Gorenstein local ring A, the canonical module of a finitely generated R-module M, denoted by ω_M, is defined by

$$\omega_M := \operatorname{Ext}_A^{m-n}(M, A),$$

where $m = \dim(A)$ and $n = \dim(M) = \dim(R/\operatorname{ann}_R(M))$. This module does not depend on A. By the local duality theorem

$$H^n_{\mathfrak{m}}(M) \simeq \omega_M^\vee.$$

We are particularly interested in the case in which R admits a canonical module; hence, in the sequel, we assume that R is the quotient of a Gorenstein ring and write ω for the canonical module of R. Whenever R is Cohen–Macaulay, ω is a canonical module of R in the sense of [2, Definition 3.3.1].

If R is a Gorenstein local ring, $\omega \simeq R$, therefore, by Proposition 2.2,

$$\omega_{Z_p} \simeq Z_{r-1-p}$$

for all $0 \leq p \leq r-1$. To generalize this result, we will use a result of Herzog and Kunz.

Lemma 2.3 ([10, Lemma 5.8]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let M, N be two finitely generated R-modules. If $\widehat{M} \simeq \widehat{N}$, then $M \simeq N$.*

We will denote by $Z_i^\omega := Z_i(\mathbf{f}; \omega)$ the module of ith Koszul cycles, with $\mathbf{f} = f_1, \ldots, f_r$.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d which is an epimorphic image of a Gorenstein ring. Suppose that $I = (f_1, \ldots, f_r)$ is an ideal of R, with $\operatorname{ht}(I) \geq 2$. Then, for all $0 \leq p \leq r-1$,*

$$\omega_{Z_p} \simeq Z_{r-1-p}^{\omega_R}.$$

Moreover, if R satisfies $\mathcal{S}_2$, then

$$\omega_{Z_p^{\omega_R}} \simeq Z_{r-1-p}.$$

Proof. For simplicity, set $\omega := \omega_R$. First, we consider the truncated complexes

$$\mathbf{K}_\bullet^{>p} : 0 \longrightarrow K_r \longrightarrow \cdots \longrightarrow K_{p+1} \longrightarrow Z_p \longrightarrow 0.$$

The double complex $C^\bullet_{\mathfrak{m}}(\mathbf{K}_\bullet^{>p})$ gives rise to two spectral sequences. The second terms of the horizontal spectral are

$${}^2\mathbf{E}_{\mathrm{hor}}^{-i,-j} = H^j_{\mathfrak{m}}(H_{i+p}),$$

and the first terms of the vertical spectral are

$$\begin{tikzcd}
0 \arrow[r] & H^0_{\mathfrak{m}}(K_r) \arrow[r] & \cdots \arrow[r] & H^0_{\mathfrak{m}}(K_{p+1}) \arrow[r] & H^0_{\mathfrak{m}}(Z_p) \arrow[r] & 0 \\
 & \vdots & \cdots & \vdots & \vdots & \\
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(K_r) \arrow[r] & \cdots \arrow[r] & H^{d-1}_{\mathfrak{m}}(K_{p+1}) \arrow[r] & H^{d-1}_{\mathfrak{m}}(Z_p) \arrow[r] & 0 \\
0 \arrow[r] & H^d_{\mathfrak{m}}(K_r) \arrow[r] & \cdots \arrow[r] & H^d_{\mathfrak{m}}(K_{p+1}) \arrow[r] & H^d_{\mathfrak{m}}(Z_p) \arrow[r] & 0.
\end{tikzcd}$$

Since I annihilates H_i, $\dim(H_i) = \dim(R/I) \leq \dim(R) - \operatorname{ht}(I) \leq d-2$ if $H_i \neq 0$. Therefore, ${}^2\mathbf{E}_{\text{hor}}^{-i,-j} = H^j_{\mathfrak{m}}(H_{i+p}) = 0$ for all $j > d-2$. The comparison of two spectral sequences gives a short exact sequence

$$H^d_{\mathfrak{m}}(K_{p+2}) \longrightarrow H^d_{\mathfrak{m}}(K_{p+1}) \longrightarrow H^d_{\mathfrak{m}}(Z_p) \longrightarrow 0. \tag{2.1}$$

By local duality,

$$H^d_{\mathfrak{m}}(K_i) \simeq \operatorname{Hom}_R(K_i,\omega)^\vee \simeq (\operatorname{Hom}_R(K_i,R)\otimes_R\omega)^\vee \simeq (K_{r-i}\otimes_R\omega)^\vee = K_{r-i}(\mathbf{f};\omega)^\vee.$$

Thus the exact sequence (2.1) provides an exact sequence

$$K_{r-p-2}(\mathbf{f};\omega)^\vee \longrightarrow K_{r-p-1}(\mathbf{f};\omega)^\vee \longrightarrow H^d_{\mathfrak{m}}(Z_p) \longrightarrow 0$$

that gives $H^d_{\mathfrak{m}}(Z_p) \simeq {Z^{\omega}_{r-1-p}}^\vee$. Then the first isomorphism follows from this isomorphism, the local duality, and Lemma 2.3.

The second assertion is proved similarly, by considering the truncated complexes

$$\mathbf{K}^{\omega}{}^{>p}_{\bullet} : 0 \longrightarrow K_r(\mathbf{f};\omega) \longrightarrow \cdots \longrightarrow K_{p+1}(\mathbf{f};\omega) \longrightarrow Z^\omega_p \longrightarrow 0$$

and the double complex $C^\bullet_{\mathfrak{m}}(\mathbf{K}^{\omega}{}^{>p}_{\bullet})$.

Since I annihilates $H_i(\mathbf{f};\omega)$, $\dim(H_i(\mathbf{f};\omega)) \leq \dim(R) - \operatorname{ht}(I) \leq d-2$ for all $0 \leq i \leq r-2$. Thus $H^j_{\mathfrak{m}}(H_i(\mathbf{f};\omega)) = 0$ for all $j > d-2$ and $0 \leq i \leq r-2$. By comparing two spectral sequences, we also obtain a short exact sequence

$$H^d_{\mathfrak{m}}(K_{p+2}(\mathbf{f};\omega)) \longrightarrow H^d_{\mathfrak{m}}(K_{p+1}(\mathbf{f};\omega)) \longrightarrow H^d_{\mathfrak{m}}(Z^\omega_p) \longrightarrow 0. \tag{2.2}$$

By local duality,

$$\begin{aligned} H^d_{\mathfrak{m}}(K_i(\mathbf{f};\omega)) &\simeq H^d_{\mathfrak{m}}(K_i\otimes_R\omega) \simeq \operatorname{Hom}_R(K_i\otimes_R\omega,\omega)^\vee \\ &\simeq \operatorname{Hom}_R(K_i,\operatorname{Hom}_R(\omega,\omega))^\vee \simeq \operatorname{Hom}_R(K_i,R)^\vee \simeq K^\vee_{r-i} \end{aligned}$$

as $\operatorname{Hom}_R(\omega,\omega) \simeq R$ since R satisfies $\mathcal{S}_2$.

The exact sequence (2.2) provides an exact sequence

$$K^\vee_{r-p-2} \longrightarrow K^\vee_{r-p-1} \longrightarrow H^d_{\mathfrak{m}}(Z^\omega_p) \longrightarrow 0,$$

which shows that $H^d_{\mathfrak{m}}(Z^\omega_p) \simeq Z^\vee_{r-1-p}$. □

Now we describe the 0th homology module of *approximation complexes*. These complexes were introduced in [21] and systematically developed in [12, 13]. Recall that the approximation complex $\mathcal{Z}_\bullet(\mathbf{f};M)$ is

$$0 \longrightarrow Z^M_r \otimes_R S(-r) \longrightarrow \cdots \longrightarrow Z^M_1 \otimes_R S(-1) \xrightarrow{\partial^{\mathbf{T}}_M} Z^M_0 \otimes_R S \longrightarrow 0,$$

which can be written

$$0 \longrightarrow Z^M_r[\mathbf{T}](-r) \longrightarrow \cdots \longrightarrow Z^M_1[\mathbf{T}](-1) \xrightarrow{\partial^{\mathbf{T}}_M} M[\mathbf{T}] \longrightarrow 0,$$

where $\mathbf{T} = T_1,\dots,T_r$, and where $Z^M_i = Z_i(\mathbf{f};M)$ is the ith Koszul cycle of $K_\bullet(\mathbf{f};M)$. By definition,

$$H_0(\mathcal{Z}_\bullet(\mathbf{f};M)) \simeq M[T_1,\dots,T_r]/\mathfrak{L}_M, \tag{2.3}$$

where $\mathfrak{L}_M$ is the submodule of $M[T_1,\dots,T_r]$ generated by the linear forms $c_1T_1 + \cdots + c_rT_r$ with $(c_1,\dots,c_r) \in Z^M_1$.

Let $\mathcal{F}_\bullet$ be a free resolution of R/I of the form

$$\cdots \longrightarrow F_1 \xrightarrow{\delta} R^r \longrightarrow R \longrightarrow 0,$$

where F_1 is the free R-module indexed by a generating set of Z_1. By definition,

$$\operatorname{Tor}_1^R(R/I, M) = Z_1^M/\operatorname{Im}(\delta \otimes 1_M) \hookrightarrow M^r/\operatorname{Im}(\delta \otimes 1_M),$$

where 1_M denotes the identity morphism on M. Note that δ is induced by the inclusion $i : Z_1 \hookrightarrow R^r$. Therefore, $\operatorname{Im}(\delta \otimes 1_M) = (i \otimes 1_M)(Z_1 \otimes_R M)$, and we obtain an exact sequence

$$Z_1 \otimes_R M \xrightarrow{i\otimes 1_M} Z_1^M \longrightarrow \operatorname{Tor}_1^R(R/I, M) \longrightarrow 0. \tag{2.4}$$

Let $\mathfrak{L}$ be the submodule of $S = R[T_1, \ldots, T_r]$ generated by the linear forms $c_1T_1 + \cdots + c_rT_r$ with $(c_1, \ldots, c_r) \in Z_1$. Then the exact sequence

$$0 \longrightarrow \mathfrak{L} \xrightarrow{\theta} S \longrightarrow \operatorname{Sym}_R(I) \longrightarrow 0$$

provides an exact sequence

$$\mathfrak{L} \otimes_R M \xrightarrow{\theta\otimes 1_M} M[T_1, \ldots, T_r] \longrightarrow \operatorname{Sym}_R(I) \otimes_R M \longrightarrow 0.$$

The image of $\theta \otimes 1_M$ is denoted by $\mathfrak{L}M$. It follows that

$$\operatorname{Sym}_R(I) \otimes_R M \simeq M[T_1, \ldots, T_r]/\mathfrak{L}M. \tag{2.5}$$

Notice that $\mathfrak{L}M$ is the submodule of $M[T_1, \ldots, T_r]$ generated by the linear forms $c_1T_1 + \cdots + c_rT_r$ with $(c_1, \ldots, c_r) \in \operatorname{Im}(\delta \otimes 1_M) \subset Z_1^M$; thus $\mathfrak{L}M \subset \mathfrak{L}_M$.

Let $\mathfrak{L}'$ be the submodule of $M[T_1, \ldots, T_r]/\mathfrak{L}M$ generated by the linear forms $c_1T_1 + \cdots + c_rT_r + \mathfrak{L}M$ with $(c_1, \ldots, c_r) + \operatorname{Im}(\delta \otimes 1_M) \in \operatorname{Tor}_1^R(R/I, M)$. Then

$$\mathfrak{L}' = \mathfrak{L}_M/\mathfrak{L}M.$$

It follows that $\mathfrak{L}_M = \mathfrak{L}M + \mathfrak{L}'$. Thus we have already proved the following.

Proposition 2.5. *Let R be a Noetherian ring, and let $I = (f_1, \ldots, f_r)$ be an ideal of R. Assume that M is a finitely generated R-module. Then*

$$H_0(\mathcal{Z}_\bullet(\mathbf{f}; M)) \simeq M[T_1, \ldots, T_r]/(\mathfrak{L}M + \mathfrak{L}'),$$

where $\mathfrak{L} \subset S$ is the defining ideal of $\operatorname{Sym}_R(I)$ *and $\mathfrak{L}'$ is spanned by generators of* $\operatorname{Tor}_1^R(R/I, M)$.

Proposition 2.6. *Let R be a Noetherian ring, and let $I = (f_1, \ldots, f_r)$ be an ideal of R. Assume that M is a finitely generated R-module. Then there exists a natural epimorphism*

$$\varphi : \operatorname{Sym}_R(I) \otimes_R M \longrightarrow\!\!\!\!\!\rightarrow H_0(\mathcal{Z}_\bullet(\mathbf{f}; M))$$

that equals $H_0(\mathcal{Z}_\bullet(\mathbf{f}; R)) \simeq \operatorname{Sym}_R(I)$ when $M = R$. Furthermore, φ is an isomorphism if and only if $\operatorname{Tor}_1^R(R/I, M) = 0$.

Proof. As $\mathfrak{L}M \subset \mathfrak{L}_M$, we can define an epimorphism

$$\varphi : \operatorname{Sym}_R(I) \otimes_R M \longrightarrow\!\!\!\!\!\rightarrow H_0(\mathcal{Z}_\bullet(\mathbf{f}; M))$$

by (2.3) and (2.5). Moreover, the kernel of φ is isomorphic to $\mathfrak{L}_M/\mathfrak{L}M$. Thus $\operatorname{Tor}_1^R(R/I, M) = 0$ if and only if φ is an isomorphism. □

3. RESIDUAL APPROXIMATION COMPLEXES

Assume that R is a Noetherian ring of dimension d, and that $I = (\mathbf{f}) = (f_1, \dots, f_r)$ is an ideal of height g. Let $\mathfrak{a} = (a_1, \dots, a_s)$ be an ideal contained in I with $s \geq g$. Set $J = \mathfrak{a} \colon_R I$, set $S = R[T_1, \dots, T_r]$, and set $\mathfrak{g} := (T_1, \dots, T_r)$. We write $a_i = \sum_{j=1}^{r} c_{ji} f_j$ and $\gamma_i = \sum_{j=1}^{r} c_{ji} T_j$. Notice that the γ_i's depend on how one expresses the a_i's as a linear combination of the f_i's. Set $\gamma = \gamma_1, \dots, \gamma_s$. Finally, for a graded module N, we define $\operatorname{end}(N) := \sup\{\mu \mid N_\mu \neq 0\}$ and $\operatorname{indeg}(N) := \inf\{\mu \mid N_\mu \neq 0\}$.

Let M be a finitely generated R-module. We denote by $\mathcal{Z}_\bullet(\mathbf{f}; M)$ the approximation complex associated with $\mathbf{f}$ with coefficients in M, and by $K_\bullet(\gamma; S)$ the Koszul complex associated with γ with coefficients in S. Let $\mathcal{D}_\bullet^M = \operatorname{Tot}(\mathcal{Z}_\bullet(\mathbf{f}; M) \otimes_S K_\bullet(\gamma; S))$. Then

$$\mathcal{D}_i^M = \bigoplus_{j=i-s}^{i} (Z_j^M \otimes_R S)^{\binom{s}{i-j}}(-i),$$

with $Z_j^M = 0$ for $j < 0$ or $j > r$, and for $j = r$ unless $\operatorname{depth}_M(I) = 0$.

In what follows, we assume that $\operatorname{depth}_M(I) > 0$ (hence, $Z_r^M = 0$) in order that the complexes we construct have length s.

We recall that the kth graded component of a graded S-module N is denoted by $N_{[k]}$. We have $(\mathcal{D}_i^M)_{[k]} = 0$ for all $k < i$. Consequently, the complex $(\mathcal{D}_\bullet^M)_{[k]}$ is

$$0 \longrightarrow (\mathcal{D}_k^M)_{[k]} \longrightarrow (\mathcal{D}_{k-1}^M)_{[k]} \longrightarrow \cdots \longrightarrow (\mathcal{D}_0^M)_{[k]} \longrightarrow 0.$$

The Čech complex of S with respect to the ideal $\mathfrak{g} = (T_1, \dots, T_r)$ is denoted by $C_\mathfrak{g}^\bullet = C_\mathfrak{g}^\bullet(S)$.

We now consider the double complex $C_\mathfrak{g}^\bullet \otimes_S \mathcal{D}_\bullet^M$ that gives rise to two spectral sequences. The second terms of the horizontal spectral are

$${}^2\mathbf{E}_{\text{hor}}^{-i,-j} = H_\mathfrak{g}^j(H_i(\mathcal{D}_\bullet^M)),$$

and the first terms of the vertical spectral are

$${}^1\mathbf{E}_{\text{ver}}^{-\bullet,-j} = \begin{cases} 0 \longrightarrow H_\mathfrak{g}^r(\mathcal{D}_{r+s-1}^M) \longrightarrow \cdots \longrightarrow H_\mathfrak{g}^r(\mathcal{D}_1^M) \longrightarrow H_\mathfrak{g}^r(\mathcal{D}_0^M) \longrightarrow 0 & \text{if } j = r \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H_\mathfrak{g}^r(\mathcal{D}_i^M) \simeq \bigoplus_{j=i-s}^{i} (Z_j^M \otimes_R H_\mathfrak{g}^r(S))^{\binom{s}{i-j}}(-i)$$

by [8, Lemma 2.1]. Since $\operatorname{end}(H_\mathfrak{g}^r(S)) = -r$, it follows that $\operatorname{end}(H_\mathfrak{g}^r(\mathcal{D}_i^M)) = i - r$ if $\mathcal{D}_i^M \neq 0$, and thus $H_\mathfrak{g}^r(\mathcal{D}_i^M)_{[i-r+j]} = 0$ for all $j \geq 1$. Hence, the kth graded component of ${}^1\mathbf{E}_{\text{ver}}^{-\bullet,-r}$ is the complex

$$0 \longrightarrow H_\mathfrak{g}^r(\mathcal{D}_{r+s-1}^M)_{[k]} \longrightarrow \cdots \longrightarrow H_\mathfrak{g}^r(\mathcal{D}_{r+k+1}^M)_{[k]} \longrightarrow H_\mathfrak{g}^r(\mathcal{D}_{r+k}^M)_{[k]} \longrightarrow 0.$$

Comparison of the spectral sequences for the two filtrations leads to the definition of the complex of length s:

$${}_k^M\mathcal{Z}_\bullet^+ : 0 \longrightarrow {}_k^M\mathcal{Z}_s^+ \longrightarrow \cdots \longrightarrow {}_k^M\mathcal{Z}_{k+1}^+ \xrightarrow{\tau_k} {}_k^M\mathcal{Z}_k^+ \longrightarrow \cdots \longrightarrow {}_k^M\mathcal{Z}_0^+ \longrightarrow 0,$$

wherein

$$ {}_k^M\mathcal{Z}_i^+ = \begin{cases} (\mathcal{D}_i^M)_{[k]}, & i \leq \min\{k, s\}, \\ H_{\mathfrak{g}}^r(\mathcal{D}_{r-1+i}^M)_{[k]}, & i > k, \end{cases} $$

and the morphism τ_k is defined through the transgression. Notice that ${}_k^M\mathcal{Z}_\bullet^+$ is a direct generalization of the complex ${}_k\mathcal{Z}_\bullet^+$ in [11, Section 2.1].

Since $H_{\mathfrak{g}}^r(M \otimes_R S) \simeq M \otimes_R H_{\mathfrak{g}}^r(S)$, for any R-module M, ${}_k^M\mathcal{Z}_\bullet^+$ have, like graded strands of $\mathcal{D}_\bullet^M$, components that are direct sums of Koszul cycles of $K_\bullet(\mathbf{f}; M)$.

The structure of ${}_k^M\mathcal{Z}_\bullet^+$ depends upon the generating sets of I, on the expression of the generators of $\mathfrak{a}$ in terms of the generators of I, and on M. The complex ${}_k^R\mathcal{Z}_\bullet^+$ considered by Hassanzadeh and the second author in [11] will be denoted by ${}_k\mathcal{Z}_\bullet^+$ instead of ${}_k^R\mathcal{Z}_\bullet^+$.

Definition 3.1. The complex ${}_k^M\mathcal{Z}_\bullet^+$ is called the kth *residual approximation complex* of $J = \mathfrak{a}\colon{}_R I$ with coefficients in M.

We consider the morphism

$$ M[T_1, \dots, T_r]^s(-1) \simeq M \otimes_R S^s(-1) \xrightarrow{1_M \otimes \partial_1^\gamma} M \otimes_R S \simeq M[T_1, \dots, T_r], $$

where ∂_1^γ is the first differential of $K_\bullet(\gamma; S)$, and we denote by γM the image of $1_M \otimes \partial_1^\gamma$. It is the submodule of $M[T_1, \dots, T_r]$ generated by the linear forms $\gamma_1, \dots, \gamma_s$. Recall from Section 2 that we set $\mathfrak{L}$ for the defining ideal of $\mathrm{Sym}_R(I)$ in S, and we set $\mathfrak{L}'$ for the module spanned by the linear forms corresponding to generators of $\mathrm{Tor}_1^R(R/I, M)$.

Proposition 3.2. *Let R be a Noetherian ring, and let $\mathfrak{a} \subset I$ be two ideals of R. Suppose that M is a finitely generated R-module. Then*

$$ H_0(\mathcal{D}_\bullet^M) \simeq M[T_1, \dots, T_r]/(\mathfrak{L}M + \mathfrak{L}' + \gamma M) $$

and, for all $k \geq 1$,

$$ H_0({}_k^M\mathcal{Z}_\bullet^+) \simeq M[T_1, \dots, T_r]_{[k]}/(\mathfrak{L}M + \mathfrak{L}' + \gamma M)_{[k]}. $$

Proof. The first isomorphism follows from the definition of $\mathcal{D}_\bullet^M$ and Proposition 2.5. The last isomorphism is a consequence of the fact that, for all $k \geq 1$, $H_0({}_k^M\mathcal{Z}_\bullet^+) = H_0(\mathcal{D}_\bullet^M)_{[k]}$ is the kth graded component of $H_0(\mathcal{D}_\bullet^M)$. □

Proposition 3.3. *Let R be a Noetherian ring, and let $\mathfrak{a} \subset I$ be two ideals of R. Assume that M is a finitely generated R-module. Then, for all $k \geq 1$, there exists a natural epimorphism*

$$ \psi : \mathrm{Sym}_R^k(I/\mathfrak{a}) \otimes_R M \twoheadrightarrow H_0({}_k^M\mathcal{Z}_\bullet^+). $$

Furthermore, ψ is an isomorphism if $\mathrm{Tor}_1^R(R/I, M) = 0$.

Proof. As $\mathrm{Sym}_R(I/\mathfrak{a}) \simeq \mathrm{Sym}_R(I)/\mathfrak{a}\mathrm{Sym}_R(I) \simeq S/(\mathfrak{L} + (\gamma))$, we have an exact sequence

$$ \mathfrak{L} \oplus (\gamma) \xrightarrow{\alpha} S \longrightarrow \mathrm{Sym}_R(I/\mathfrak{a}) \longrightarrow 0 $$

which provides a commutative diagram with exact rows

$$\begin{tikzcd}
(\mathfrak{L} \oplus (\gamma)) \otimes_R M \arrow[r, "\alpha\otimes 1_M"] \arrow[d, "\simeq"'] & M[\mathbf{T}] \arrow[r] \arrow[d, "="] & \mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R M \arrow[r] \arrow[d, "="] & 0 \\
\mathfrak{L} \otimes_R M \oplus (\gamma) \otimes_R M \arrow[r, "\theta\otimes 1_M \oplus \beta\otimes 1_M"] & M[\mathbf{T}] \arrow[r] & \mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R M \arrow[r] & 0,
\end{tikzcd}$$

where β is the inclusion $(\gamma) \hookrightarrow S$, and hence $\mathrm{Im}(\beta \otimes 1_M) = \gamma M$. It follows that

$$\mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R M \simeq M[\mathbf{T}]/\mathrm{Im}(\alpha \otimes 1_M) \simeq M[T_1, \dots, T_r]/(\mathfrak{L}M + \gamma M).$$

The natural onto map

$$M[T_1, \dots, T_r]/(\mathfrak{L}M + \gamma M) \twoheadrightarrow M[T_1, \dots, T_r]/(\mathfrak{L}M + \mathfrak{L}' + \gamma M)$$

provides an epimorphism, for all $k \geq 1$,

$$\psi : \mathrm{Sym}_R^k(I/\mathfrak{a}) \otimes_R M \twoheadrightarrow H_0({}^M_k\mathcal{Z}^+_\bullet)$$

by Proposition 3.2. Moreover, $\mathrm{Tor}_1^R(R/I, M) = 0$ is equivalent to $\mathfrak{L}_M = \mathfrak{L}M$. Thus ψ is an isomorphism if $\mathrm{Tor}_1^R(R/I, M) = 0$. ☐

The following remark will be used in the proof of the next proposition.

Remark 3.4. Let M be a module over a ring R. Suppose that N is a quotient of $M[T_1, \dots, T_r]$, with T_i indeterminates of degree 1, by a graded submodule. Then, for all $k \geq 1$,

$$\mathrm{ann}_R(N_k) \subset \mathrm{ann}_R(N_{k+1}).$$

Proposition 3.5. *Let R be a Noetherian ring, and let $\mathfrak{a} \subset I$ be two ideals of R. Assume that M is a finitely generated R-module. Then $J = \mathfrak{a} :_R I$ annihilates $H_0({}^M_k\mathcal{Z}^+_\bullet)$ for all $k \geq 1$.*

Proof. Fix $k \geq 1$. The epimorphism ψ in Proposition 3.3 implies that

$$\mathrm{ann}_R(\mathrm{Sym}_R^k(I/\mathfrak{a}) \otimes_R M) \subset \mathrm{ann}_R(H_0({}^M_k\mathcal{Z}^+_\bullet)). \tag{3.1}$$

On the other hand, one always has

$$\mathrm{ann}_R(\mathrm{Sym}_R^k(I/\mathfrak{a})) \subset \mathrm{ann}_R(\mathrm{Sym}_R^k(I/\mathfrak{a}) \otimes_R M). \tag{3.2}$$

Notice that $\mathrm{Sym}_R(I/\mathfrak{a}) \simeq \mathrm{Sym}_R(I)/(\gamma)\mathrm{Sym}_R(I) \simeq S/(\mathfrak{L} + (\gamma))$. By Lemma 3.4,

$$J = \mathrm{ann}_R(I/\mathfrak{a}) \subset \mathrm{ann}_R(\mathrm{Sym}_R^k(I/\mathfrak{a})). \tag{3.3}$$

By (3.1), (3.2), and (3.3), $J \subset \mathrm{ann}_R(H_0({}^M_k\mathcal{Z}^+_\bullet))$. ☐

However, the structure of $H_0({}^M_0\mathcal{Z}^+_\bullet)$ is difficult to determine. We recall a definition of Hassanzadeh and the second author in [11, Definition 2.1].

Definition 3.6. Let R be a Noetherian ring, and let $\mathfrak{a} \subset I$ be two ideals of R. The *disguised s-residual intersection of I w.r.t. $\mathfrak{a}$* is the unique ideal K such that $H_0({}_0\mathcal{Z}^+_\bullet) = R/K$.

To make use of the acyclicity of the ${}_k\mathcal{Z}^+_\bullet$ complexes, we recall the definition of classes of ideals that meet these requirements.

Definition 3.7. Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d, and let $I=(f_1,\dots,f_r)$ be an ideal of height g. Let $k\geq 0$ be an integer. Then the following hold:

(i) I satisfies the *sliding depth condition*, $\mathcal{SD}_k$, if

$$\operatorname{depth}(H_i(\mathbf{f};R))\geq \min\{d-g,d-r+i+k\}\qquad \forall i;$$

also $\mathcal{SD}$ stands for $\mathcal{SD}_0$.

(ii) I satisfies the *sliding depth condition on cycles*, $\mathcal{SDC}_k$, if

$$\operatorname{depth}(Z_i(\mathbf{f};R))\geq \min\{d-r+i+k,d-g+2,d\}\qquad \forall i\leq r-g.$$

(iii) I is *strongly Cohen–Macaulay* if $H_i(\mathbf{f};R)$ is Cohen–Macaulay for all i.

Clearly I is strongly Cohen–Macaulay if and only if I satisfies $\mathcal{SD}_t$ for all $t\geq r-g$. Some of the basic properties and relations between such conditions, $\mathcal{SD}_k$ and $\mathcal{SDC}_k$, are given in [8, Remark 2.4 and Proposition 2.5], [11, Proposition 2.4]; also see [14,17,24]. It will be of importance to us that $\mathcal{SD}_k$ implies $\mathcal{SDC}_{k+1}$ whenever R is a Cohen–Macaulay local ring by [8, Proposition 2.5].

Remark 3.8. Notice that, adding an indeterminate x to the ring and to ideals I and $\mathfrak{a}$, one has $(\mathfrak{a}+(x))\colon(I+(x))=(\mathfrak{a}\colon I)+(x)$ in $R[x]$ and in its localization at $\mathfrak{m}+(x)$. Hence, for most statements, one may reduce to the case in which the height of I is big enough, if needed.

In a recent article [11, Theorem 2.6], Hassanzadeh and the second author proved the following results. The Cohen–Macaulay hypothesis in this theorem is needed to show that if, for an R-module M, $\operatorname{depth}(M)\geq d-t$, then, for any prime $\mathfrak{p}$, $\operatorname{depth}(M_{\mathfrak{p}})\geq \operatorname{ht}(\mathfrak{p})-t$; see [24, Section 3.3].

Theorem 3.9. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, and let $I=(f_1,\dots,f_r)$ be an ideal of height g. Let $s\geq g$, and fix $0\leq k\leq\min\{s,s-g+2\}$. Suppose that one of the following hypotheses holds:*

(i) *$r+k\leq s$ and I satisfies $\mathcal{SD}$.*

(ii) *$r+k\geq s+1$, I satisfies $\mathcal{SD}$, and $\operatorname{depth}(Z_i)\geq d-s+k$ for $0\leq i\leq k$.*

(iii) *I is strongly Cohen–Macaulay.*

Then, for any s-residual intersection $J=(\mathfrak{a}\colon{}_RI)$, the complex ${}_k\mathcal{Z}^+_\bullet$ is acyclic. Furthermore, $\operatorname{Sym}^k_R(I/\mathfrak{a})$, for $1\leq k\leq s-g+2$, and the disguised residual intersection R/K are Cohen–Macaulay of codimension s.

Notice that condition (iii) is stronger than (i) and (ii). In [8, Theorem 2.11], Hassanzadeh showed that, under the sliding depth condition $\mathcal{SD}$, $K\subset J$ and $\sqrt{K}=\sqrt{J}$, and further that $K=J$, whenever the residual is arithmetic.

4. Cohen–Macaulayness and canonical module of residual intersections

In this section, we will prove two important conjectures in the theory of residual intersections: the Cohen–Macaulayness of the residual intersections and the description of their canonical module.

In order to make a reduction to a lower height case and prove the Cohen–Macaulayness when $s=g$, we first state the following proposition, which is a trivial generalization of [17, Lemma 3.5] that only treated the sliding depth condition $\mathcal{SD}$. The proof goes along the same lines.

Proposition 4.1. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring, let I be an ideal of height g, and let $k \geq 0$ be an integer. Let $x_1,\dots,x_\ell$ be a regular sequence in I. Let the prime denote the canonical epimorphism $R \longrightarrow R' = R/(x_1,\dots,x_\ell)$. Then I satisfies $\mathcal{SD}_k$ if and only if I' satisfies $\mathcal{SD}_k$ (in R'). In particular, I is strongly Cohen–Macaulay if and only if I' is strongly Cohen–Macaulay.*

Proposition 4.2. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, and let I be an ideal of height g. Let $\mathbf{x} = x_1,\dots,x_g$ be a regular sequence contained in I, and let $J = ((\mathbf{x})\colon{}_R I)$. Suppose that R/I is Cohen–Macaulay, and that I satisfies $\mathcal{SD}$. Then R/J is Cohen–Macaulay of codimension g.*

Proof. The proof goes along the same lines as in [17] (where the result is stated in a weaker form). ☐

To study the Cohen–Macaulayness of residual intersections in the general case, we will use the following lemma.

Lemma 4.3. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with canonical module ω. Suppose that $S = R[T_1,\dots,T_r]$ is the standard graded polynomial ring over R and that $\mathfrak{g} := S_+$. Let $\mathfrak{a} \subset I = (f_1,\dots,f_r)$ be two ideals of R, with $\operatorname{ht}(I) = g$. If $J = (\mathfrak{a}\colon{}_R I)$ is an s-residual intersection of I, then the following hold:*

(i) *There is a natural graded isomorphism*

$$H^r_{\mathfrak{g}}(S) \simeq {}^*\mathrm{Homgr}_S(S(-r),R).$$

In particular, for all $\mu \in \mathbb{Z}$,

$$H^r_{\mathfrak{g}}(S)_\mu \simeq S^*_{-\mu-r} = \mathrm{Hom}_R(S_{-\mu-r},R).$$

(ii) *If $g \geq 2$, then* $\operatorname{depth}({}_k\mathcal{Z}^+_0) = \operatorname{depth}({}_k\mathcal{Z}^+_s) = d$ *for all* $0 \leq k \leq s-1$.

(iii) *If $g = 2$ and I satisfies $\mathcal{SD}_\ell$, then*

$$\operatorname{depth}({}_0\mathcal{Z}^+_i) \geq \min\{d, d-s+i+\ell\}$$

for all $1 \leq i \leq s-1$.

(iv) *If $g \geq 2$, then the following diagram, where the vertical isomorphisms are induced by the identifications $H^d_{\mathfrak{m}}(Z_*) \simeq {Z^\omega_{r-1-*}}^\vee$ in Proposition 2.4, is commutative for all $0 \leq k \leq s-2$:*

$$\begin{tikzcd}
H^d_{\mathfrak{m}}({}_k\mathcal{Z}^+_s) \arrow[r] \arrow[d, "\simeq"'] & H^d_{\mathfrak{m}}({}_k\mathcal{Z}^+_{s-1}) \arrow[d, "\simeq"] \\
({}_{s-k-1}^{\ \ \omega}\mathcal{Z}^+_0)^\vee \arrow[r] & ({}_{s-k-1}^{\ \ \omega}\mathcal{Z}^+_1)^\vee.
\end{tikzcd}$$

Proof.

(i) This is the graded local duality theorem.

(ii) Since $Z_{r-1} \simeq Z_0 = R$, $\operatorname{depth}({}_k\mathcal{Z}^+_0) = \operatorname{depth}({}_k\mathcal{Z}^+_s) = d$.

(iii) By [8, Proposition 2.5], I satisfies $\mathcal{SDC}_{\ell+1}$; that is,

$$\operatorname{depth}(Z_j) \geq \min\{d-r+j+\ell+1, d\}$$

for all $0 \leq j \leq r-2$.

For any $1 \leq i \leq s-1$,

$${}_0\mathcal{Z}^+_i = H^r_{\mathfrak{g}}(\mathcal{D}_{r-1+i})_{[0]} = \bigoplus_{j=r-1+i-s}^{r-1} \left(Z_j \otimes_R H^r_{\mathfrak{g}}(S)\right)^{\binom{s}{r-1+i-j}}_{[-r+1-i]}.$$

Thus ${}_0\mathcal{Z}_i^+$ is a direct sum of copies of modules $Z_\delta, \dots, Z_{r-1}$, where $\delta = \max\{0, r-1+i-s\}$. Notice that $0 \le \delta \le r-2$. It follows that

$$\begin{aligned}\operatorname{depth}({}_0\mathcal{Z}_i^+) &= \min_{\delta \le j \le r-1}\{\operatorname{depth}(Z_j)\} = \min\{\min_{\delta \le j \le r-2}\{\operatorname{depth}(Z_j)\}, d\} \\ &\ge \min\{d, d-r+\delta+\ell+1\} \ge \min\{d, d-s+i+\ell\}.\end{aligned}$$

(iv) We have the following commutative diagrams, for all $0 \le k \le s-2$,

$$\begin{tikzcd}
H^d_{\mathfrak{m}}({}_k\mathcal{Z}_s^+) = H^d_{\mathfrak{m}}(H^r_{\mathfrak{g}}(\mathcal{D}_{r+s-1}))_{[k]} \arrow[r] \arrow[d, "\simeq"'] & H^d_{\mathfrak{m}}(H^r_{\mathfrak{g}}(\mathcal{D}_{r+s-2}))_{[k]} = H^d_{\mathfrak{m}}({}_k\mathcal{Z}_{s-1}^+) \arrow[d, "\simeq"] \\
H^d_{\mathfrak{m}}(Z_{r-1}) \otimes_R H^r_{\mathfrak{g}}(S)_{[k-r-s+1]} \arrow[r] \arrow[d, "\simeq"'] & H^d_{\mathfrak{m}}(Z_{r-1}) \otimes_R H^r_{\mathfrak{g}}(S)^s_{[k-r-s+2]} \oplus H^d_{\mathfrak{m}}(Z_{r-2}) \otimes_R H^r_{\mathfrak{g}}(S)_{[k-r-s+2]} \arrow[d, "\simeq"] \\
Z_0^{\omega\vee} \otimes_R S^*_{[s-k-1]} \arrow[r] \arrow[d, "\simeq"'] & Z_0^{\omega\vee} \otimes_R (S^s_{[s-k-2]})^* \oplus Z_1^{\omega\vee} \otimes_R S^*_{[s-k-2]} \arrow[d, "\simeq"] \\
(Z_0^\omega \otimes_R S_{[s-k-1]})^\vee \arrow[r] \arrow[d, "\simeq"'] & (Z_0^\omega \otimes_R S^s_{[s-k-2]} \oplus Z_1^\omega \otimes_R S_{[s-k-2]})^\vee \arrow[d, "\simeq"] \\
({}_{s-k-1}^{\;\;\omega}\mathcal{Z}_0^+)^\vee = ((\mathcal{D}_0^\omega)_{[s-k-1]})^\vee \arrow[r] & ((\mathcal{D}_1^\omega)_{[s-k-1]})^\vee = ({}_{s-k-1}^{\;\;\omega}\mathcal{Z}_1^+)^\vee,
\end{tikzcd}$$

where the first and the last diagrams are commutative by the definitions, the second diagram is commutative by the natural isomorphisms in item (i) and Proposition 2.4, and the third diagram is commutative by the natural isomorphism

$$Z_i^{\omega\vee} \otimes_R S^*_{[\ell]} \simeq (Z_i^\omega \otimes_R S_{[\ell]})^\vee$$

for all i, ℓ; see [3, II, Section 4, no. 4, Proposition 4]. ☐

Proposition 4.4. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with canonical module ω, and let $I = (f_1, \dots, f_r)$ be an ideal of height 2. Suppose that $J = (\mathfrak{a} \colon_R I)$ is an s-residual intersection of I, and that K is the disguised s-residual intersection of I w.r.t. $\mathfrak{a}$. If I satisfies $\mathcal{SD}_1$, then there exists an epimorphism of R-modules*

$$\phi : H_0({}_{s-1}^{\;\omega}\mathcal{Z}_\bullet^+) \twoheadrightarrow \omega_{R/K},$$

and ϕ is an isomorphism if I satisfies $\mathcal{SD}_2$.

Proof. Since I satisfies $\mathcal{SD}_1$, ${}_0\mathcal{Z}_\bullet^+$ is acyclic and R/K is Cohen–Macaulay of dimension $d-s$ by Theorem 3.9. By local duality,

$$H^{d-s}_{\mathfrak{m}}(R/K) \simeq \omega_{R/K}^\vee. \tag{4.1}$$

Now the double complex $C^\bullet_{\mathfrak{m}}({}_0\mathcal{Z}_\bullet^+)$ gives rise to two spectral sequences. The second terms of the horizontal spectral are

$${}^2\mathbf{E}_{\mathrm{hor}}^{-i,-j} = \begin{cases} H^{d-s}_{\mathfrak{m}}(R/K) & \text{if } j = d-s \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the first terms of the vertical spectral are

$$\begin{tikzcd}
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & H^{d-s+2}_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & H^{d-1}_{\mathfrak{m}}({}_0\mathcal{Z}^+_{s-2}) \arrow[r] & \cdots \arrow[r] & H^{d-1}_{\mathfrak{m}}({}_0\mathcal{Z}^+_2) \arrow[r] & H^{d-1}_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) & 0 \\
H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_s) \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_{s-1}) \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_{s-2}) \arrow[r] & \cdots \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_2) \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) \arrow[r] & H^d_{\mathfrak{m}}(R)
\arrow[from=6-1, to=1-7, dash, dashed]
\end{tikzcd}$$

since $\operatorname{depth}({}_0\mathcal{Z}^+_i) \geq d-s+i+1$, for all $1 \leq i \leq s-1$, by Lemma 4.3(iii).

By the convergence of the spectral sequences, we obtain

$$H^{d-s}_{\mathfrak{m}}(R/K) \simeq {}^{\infty}\mathbf{E}^{-s,-d}_{\text{ver}} \subset {}^{2}\mathbf{E}^{-s,-d}_{\text{ver}}. \tag{4.2}$$

By Lemma 4.3(iv), we have the following commutative diagram:

$$\begin{tikzcd}
H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_s) \arrow[r] \arrow[d, "\simeq"'] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_{s-1}) \arrow[d, "\simeq"] \\
({}_{s-1}^{\ \omega}\mathcal{Z}^+_0)^\vee \arrow[r] & ({}_{s-1}^{\ \omega}\mathcal{Z}^+_1)^\vee .
\end{tikzcd}$$

Therefore,

$${}^{2}\mathbf{E}^{-s,-d}_{\text{ver}} \simeq H_0({}_{s-1}^{\ \omega}\mathcal{Z}^+_\bullet)^\vee. \tag{4.3}$$

By (4.1), (4.2), and (4.3), we can define a monomorphism of R-modules by the compositions

$$\omega^\vee_{R/K} \xrightarrow{\simeq} H^{d-s}_{\mathfrak{m}}(R/K) \xrightarrow{\simeq} {}^{\infty}\mathbf{E}^{-s,-d}_{\text{ver}} \hookrightarrow {}^{2}\mathbf{E}^{-s,-d}_{\text{ver}} \xrightarrow{\simeq} H_0({}_{s-1}^{\ \omega}\mathcal{Z}^+_\bullet)^\vee$$

which provides an epimorphism

$$\phi : H_0({}_{s-1}^{\ \omega}\mathcal{Z}^+_\bullet) \longrightarrow \omega_{R/K}.$$

If I satisfies $\mathcal{SD}_2$, then $\operatorname{depth}({}_0\mathcal{Z}^+_i) \geq \min\{d, d-s+i+2\}$, for all $1 \leq i \leq s-1$, by Lemma 4.3(iii). It follows that

$$H^{d-s}_{\mathfrak{m}}(R/K) \simeq {}^{\infty}\mathbf{E}^{-s,-d}_{\text{ver}} = {}^{2}\mathbf{E}^{-s,-d}_{\text{ver}},$$

and thus ϕ is an isomorphism. □

Now we state our main result, which answers the question of Huneke and Ulrich in [15, Question 5.7] and also answers the conjecture of Hassanzadeh and the second author in [11, Conjecture 5.9].

Theorem 4.5. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, with canonical module ω, and let $\mathfrak{a} \subset I$ be two ideals of R with $\operatorname{ht}(I) = g \leq s$. Suppose that I satisfies $\mathcal{SD}_1$ and $J = (\mathfrak{a}\colon_R I)$ is an s-residual intersection of I. Then R/J is Cohen–Macaulay of dimension $d - s$.*

Proof. Let K be the disguised s-residual intersection of I w.r.t. $\mathfrak{a}$. Since I satisfies $\mathcal{SD}_1$, and hence $\mathcal{SD}$, R/K is Cohen–Macaulay of dimension $d-s$ by Theorem 3.9, and $K \subset J$ by [8, Theorem 2.11]. The proof will be completed by showing that $J \subset K$.

We first consider the case in which $g = 2$. By Proposition 4.4, there is the epimorphism

$$\phi : H_0({}_{s-1}^{\ \omega}\mathcal{Z}^+_\bullet) \longrightarrow \omega_{R/K}.$$

As R/K is Cohen–Macaulay, $\operatorname{ann}_R(\omega_{R/K}) = \operatorname{ann}_R(R/K) = K$. The epimorphism ϕ implies that

$$\operatorname{ann}_R(H_0({}_{s-1}^{\ \omega}\mathcal{Z}^+_\bullet)) \subset \operatorname{ann}_R(\omega_{R/K}) = K.$$

By Proposition 3.5, $J \subset \operatorname{ann}_R(H_0({}_{s-1}^{\ \omega}\mathcal{Z}^+_\bullet)) \subset K$.

We may always reduce to the case $g \geq 2$ by Remark 3.8. If $g > 2$, then we can choose a regular sequence $\mathbf{a}$ of length $g-2$ inside $\mathfrak{a}$ which is a part of a minimal generating set of $\mathfrak{a}$. Since R is Cohen–Macaulay, by [2, Theorem 2.1.3], $R/\mathbf{a}$ is a Cohen–Macaulay local ring of dimension $d-g+2$. Moreover, $J/\mathbf{a} = \mathfrak{a}/\mathbf{a}\colon I/\mathbf{a}$ and $\mu(\mathfrak{a}/\mathbf{a}) = \mu(\mathfrak{a}) - g + 2$; therefore, $J/\mathbf{a}$ is an $(s-g+2)$-residual intersection of $I/\mathbf{a}$ which is of height 2. Furthermore, $I/\mathbf{a}$ satisfies $\mathcal{SD}_1$ by Proposition 4.1. Hence, it follows from the height 2 case that $R/J \simeq (R/\mathbf{a})/(J/\mathbf{a})$ is Cohen–Macaulay of dimension $d-s$. ☐

It follows from the proof of Proposition 3.5 that $J \subset \operatorname{ann}_R(\operatorname{Sym}^k_R(I/\mathfrak{a}))$, for all $k \geq 1$. Then a natural question is, under what conditions does one have

$$\operatorname{ann}_R(\operatorname{Sym}^k_R(I/\mathfrak{a})) = J?$$

It is known that $\operatorname{ann}_R(\operatorname{Sym}^k_R(I/\mathfrak{a})) = J$, for all $k \geq 1$, whenever J is arithmetic in [11, Corollary 2.8(iv)]. The next result answers this question.

Corollary 4.6. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with canonical module ω, and let $\mathfrak{a} \subset I$ be two ideals of R with $\operatorname{ht}(I) = g$. Suppose that J is an s-residual intersection of I, and let $1 \leq k \leq s-g+1$.*

(i) *If I satisfies $\mathcal{SD}_1$, then $\operatorname{Sym}^k_R(I/\mathfrak{a})$ is a faithful R/J-module.*
(ii) *If I satisfies strongly Cohen–Macaulay, then $\operatorname{Sym}^k_R(I/\mathfrak{a})$ is a maximal Cohen–Macaulay faithful R/J-module.*

Proof.

(i) The proof will be completed by showing that $\operatorname{ann}_R(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})) \subset J$. As in the proof of Theorem 4.5, it suffices to prove that $\operatorname{ann}_R(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})) \subset J$ in the case $g = 2$. The inclusions $\operatorname{ann}_R(\operatorname{Sym}^{s-1}_R(I/\mathfrak{a})) \subset \operatorname{ann}_R(H_0({}_{s-1}^{\ \omega}\mathcal{Z}^+_\bullet)) \subset K = J$ are demonstrated in the proofs of Proposition 3.5 and Theorem 4.5.
(ii) This follows immediately from Theorems 3.9 and 4.5 and the first item. ☐

The following example shows that the above corollary does not hold for the $(s-g+2)$th symmetric power of $I/\mathfrak{a}$.

Example 4.7 ([11, Example 2.10])**.** Let $R = \mathbb{Q}[x,y]$, let $I = (x,y)$, and let $\mathfrak{a} = (x^2, y^2)$. We set $J = \mathfrak{a} \colon {}_R I$. Using `Macaulay2` [7], we see that $J = (x^2, xy, y^2)$ is a 2-residual intersection (a link, in this case) of I and that

$$\operatorname{Sym}_R(I/\mathfrak{a}) \simeq R[T_1,T_2]/\big(xT_1, yT_2, -yT_1 + xT_2\big).$$

Thus a free resolution of $\operatorname{Sym}^2_R(I/\mathfrak{a})$ is

$$\begin{tikzcd} 0 \arrow[r] & R^3 \arrow[r, "N"] & R^6 \arrow[r, "M"] & R^3 \arrow[r] & \operatorname{Sym}^2_R(I/\mathfrak{a}) \arrow[r] & 0, \end{tikzcd}$$

where

$$M = \begin{pmatrix} x & 0 & 0 & y & 0 & 0 \\ 0 & x & 0 & 0 & y & 0 \\ 0 & 0 & x & 0 & 0 & y \end{pmatrix}$$

and

$$N = \begin{pmatrix} -y & 0 & 0 \\ 0 & -y & 0 \\ 0 & 0 & -y \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}.$$

It follows that

$$\operatorname{ann}_R(\operatorname{Sym}^2_R(I/\mathfrak{a})) = (x,y) \supsetneq J.$$

We now give a description of the canonical module of residual intersections.

Theorem 4.8. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with canonical module ω, and let $\mathfrak{a} \subset I$ be two ideals of R with $\operatorname{ht}(I) = g$. Suppose that I satisfies $\mathcal{SD}_2$, that $\operatorname{Tor}^R_1(R/I, \omega) = 0$, and that $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Then the canonical module of R/J is $\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega$.*

Proof. We first consider the case in which $g = 2$. By Proposition 4.4 and Theorem 4.5,

$$\omega_{R/J} \simeq H_0({}_{s-1}^{\ \omega}\mathcal{Z}^+_\bullet) \simeq \operatorname{Sym}^{s-1}_R(I/\mathfrak{a}) \otimes_R \omega.$$

The last isomorphism is by Proposition 3.3.

We may always reduce to the case $g \geq 2$ by Remark 3.8. If $g > 2$, then we can choose a regular sequence $\mathbf{a}$ of length $g-2$ inside $\mathfrak{a}$ which is part of a minimal generating set of $\mathfrak{a}$, as in the proof of Theorem 4.5. As $\mathbf{a} \subset I$ is regular on ω,

$$\operatorname{Tor}^R_1(R/I, \omega) \simeq \operatorname{Tor}^{R/\mathbf{a}}_1(R/I, \omega/\mathbf{a}\omega) = 0.$$

Furthermore, observing that the canonical module of $R/\mathbf{a}$ is $\omega/\mathbf{a}\omega$, it follows from the height 2 case that

$$\omega_{R/J} \simeq \operatorname{Sym}^{(s-g+2)-1}_{R/\mathbf{a}}\left(\frac{I/\mathbf{a}}{\mathfrak{a}/\mathbf{a}}\right) \otimes_{R/\mathbf{a}} \omega_{R/\mathbf{a}} \simeq \operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega. \qquad \square$$

Notice that the hypothesis $\operatorname{Tor}^R_1(R/I, \omega) = 0$ is always satisfied for ideals of finite projective dimension. In particular, if R is Gorenstein, then $\omega \simeq R$, and hence $\operatorname{Tor}^R_1(R/I, \omega) \simeq \operatorname{Tor}^R_1(R/I, R) = 0$; therefore, the canonical module of R/J is the $(s-g+1)$th symmetric power of $I/\mathfrak{a}$. As a consequence, the second conjecture in the introduction is proved under the $\mathcal{SD}_2$ condition.

Remark 4.9. (i) This is under the assumptions of Theorem 4.8, but I satisfies only $\mathcal{SD}_1$ and not $\mathcal{SD}_2$. Then there exists an epimorphism of R-modules

$$\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega \twoheadrightarrow \omega_{R/J}.$$

(ii) In the height 2 case, by using Proposition 4.4, we could omit the assumption $\operatorname{Tor}_1^R(R/I,\omega) = 0$ in Theorem 4.8. In this case, the canonical module of R/J is the $(s-1)$th graded component of

$$\omega[T_1,\dots,T_r]/(\mathfrak{L}\omega + \mathfrak{L}' + \gamma\omega)$$

by Proposition 3.2 and Theorem 4.5.

The following example shows that Theorem 4.8 does not hold if I satisfies only the $\mathcal{SD}$ condition.

Example 4.10 ([6, Example 2.9])**.** Let $R = k[[x_1,\dots,x_5]]$, and let I be the ideal of 2×2 minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

Then I is of height 3. If we take $\mathfrak{a}$ to be the ideal generated by four sufficiently general cubic forms in I, then $J = \mathfrak{a}\colon_R I$ is a 4-residual intersection. Using `Macaulay2` [7], it is easy to see that I satisfies $\mathcal{SD}$. Moreover, we see that $I^2/\mathfrak{a}I$ requires 20 generators, whereas $\omega_{R/J}$ requires only 16. Thus there is no surjection $\omega_{R/J} \twoheadrightarrow I^2/\mathfrak{a}I$; therefore, $\omega_{R/J}$ is not isomorphic to $\operatorname{Sym}_R^2(I/\mathfrak{a})$.

Computation of the initial degree of $\operatorname{Sym}_R^2(I/\mathfrak{a})$ and $\omega_{R/J}$ shows that there can be no surjection $\operatorname{Sym}_R^2(I/\mathfrak{a}) \twoheadrightarrow \omega_{R/J}$. This shows that the $\mathcal{SD}_1$ condition in Remark 4.9(i) is necessary.

Recall that, in a Noetherian local ring $(R,\mathfrak{m})$, the *type* of finitely generated R-module M is the dimension of the $R/\mathfrak{m}$-vector space $\operatorname{Ext}_R^{\operatorname{depth}(M)}(R/\mathfrak{m}, M)$, and it is denoted by $r_R(M)$, or just $r(M)$. The minimal number of generators of the R-module M is the dimension of the $R/\mathfrak{m}$-vector space $R/\mathfrak{m} \otimes_R M$, and it is denoted by $\mu(M)$. Notice that if M, N are two finitely generated R-modules, then

$$\mu(M \otimes_R N) = \mu(M)\mu(N).$$

From Theorem 4.8, computing the type of the residual is then a simple computation that gives the following corollary.

Corollary 4.11. *Under the assumptions of Theorem* 4.8*,*

$$r(R/J) = \binom{\mu(I/\mathfrak{a}) + s - g}{\mu(I/\mathfrak{a}) - 1} r(R).$$

Thus R/J is Gorenstein if and only if R is Gorenstein and $\mu(I/\mathfrak{a}) = 1$.

5. STABILITY OF HILBERT FUNCTIONS AND CASTELNUOVO–MUMFORD REGULARITY OF RESIDUAL INTERSECTIONS

We now use the resolution ${}_0\mathcal{Z}_\bullet^+$ of a residual intersection to provide much information concerning R/J, like the stability of Hilbert functions and the Castelnuovo–Mumford regularity of residual intersections.

First, we study the stability of the Hilbert functions of residual intersections. We recall the definitions of the Hilbert function, Hilbert polynomial, and Hilbert series; the reader can consult, for instance, [2, Chapter 4]. Let M be a graded R-module whose graded components M_n have finite length for all n. The numerical function $H(M,-):\mathbb{Z}\longrightarrow\mathbb{Z}$ with $H(M,n)=\operatorname{length}(M_n)$, for all $n\in\mathbb{Z}$, is the Hilbert function, and $H_M(t):=\sum_{n\in\mathbb{Z}}H(M,n)t^n$ is the Hilbert series of M.

If R is assumed to be generated over R_0 by elements of degree 1, that is, $R=R_0[R_1]$, and M is a finitely generated graded R-module of dimension $m\geq 1$, then there exists a polynomial $P_M(X)\in\mathbb{Q}[X]$ of degree $m-1$ such that $H(M,n)=P_M(n)$ for all $n\gg 0$. This polynomial is called the Hilbert polynomial of M. We can write

$$P_M(X)=\sum_{i=0}^{m-1}(-1)^{m-1-i}e_{m-1-i}\binom{X+i}{i}.$$

Then the *multiplicity* of M is defined to be

$$e(M)=\begin{cases}e_0 & \text{if } m>0,\\ \operatorname{length}(M) & \text{if } m=0.\end{cases}$$

In [4], Eisenbud, Ulrich, and the first author restated an old question of Stanley in [20] asking for which open sets of ideals $\mathfrak{a}$ the Hilbert function of $R/\mathfrak{a}$ depends only on the degrees of the generators $\mathfrak{a}$. More precisely, they consider the following two conditions:

(A1) Is the Hilbert function of $R/\mathfrak{a}$ constant on the open set of ideals $\mathfrak{a}$ generated by s forms of the given degrees such that $\operatorname{ht}(\mathfrak{a}\colon_R I)\geq s$?

(A2) Is the Hilbert function of $R/(\mathfrak{a}\colon_R I)$ constant on this set?

It is shown in [4, Theorem 2.1] that ideals with some sliding depth conditions in conjunction with G_{s-1} or G_s satisfy these two conditions. In [11, Proposition 3.1], Hassanzadeh and the second author proved that if $(R,\mathfrak{m})$ is a Cohen–Macaulay graded local ring of dimension d over an Artinian local ring R_0, and if $\mathfrak{a}\subset I$ are two homogeneous ideals, I satisfies $\mathcal{SD}$, and $\operatorname{depth}(R/I)\geq d-s$, then the above condition (A1) is satisfied for any s-residual intersection $J=(\mathfrak{a}\colon_R I)$. It follows directly from [8, Theorem 2.11], [11, Proposition 3.1] that if I satisfies $\mathcal{SD}$, then, for any arithmetic s-residual intersection $J=(\mathfrak{a}\colon_R I)$, the above condition (A2) is satisfied.

In the next proposition, we will show that the above condition (A2) is satisfied for any residual intersection under the $\mathcal{SD}_1$ condition.

Proposition 5.1. *Let $(R,\mathfrak{m})$ be a graded Cohen–Macaulay local ring over an Artinian local ring R_0, and let $\mathfrak{a}\subset I$ be two homogeneous ideals with $\operatorname{ht}(I)=g$. Suppose that I satisfies $\mathcal{SD}_1$, and that $J=(\mathfrak{a}\colon_R I)$ is an s-residual intersection of I. Then the Hilbert function of R/J satisfies the above condition (A2).*

Proof. By Theorems 3.9 and 4.5, the complex ${}_0\mathcal{Z}_\bullet^+$ is a resolution of R/J. Hence, the Hilbert function of R/J can be written in terms of the Hilbert functions of the components of the complex ${}_0\mathcal{Z}_\bullet^+$, which, according to the definition of ${}_0\mathcal{Z}_\bullet^+$, are just some direct sums of Koszul cycles of I shifted by the twists appearing in the Koszul complex $K_\bullet(\gamma;S)$. Since the Hilbert functions of Koszul cycles are inductively calculated in terms of those of the Koszul homology modules, the Hilbert function of R/J depends only on the Koszul homology modules of I and on the degrees of the generators of $\mathfrak{a}$. $\square$

Next, the important numerical invariant associated with an algebraic or geometric object is the Castelnuovo–Mumford regularity. Assume that $R = \bigoplus_{n\geq 0} R_n$ is a positively graded Noetherian *local ring of dimension d over a Noetherian local ring $(R_0, \mathfrak{m}_0)$. Set $\mathfrak{m} = \mathfrak{m}_0 + R_+$. Suppose that I and $\mathfrak{a}$ are two homogeneous ideals of R generated by homogeneous elements $f_1, \ldots, f_r$ and $a_1, \ldots, a_s$, respectively. For a homogeneous ideal $\mathfrak{b}$, the sum of the degrees of a minimal generating set of $\mathfrak{b}$ is denoted by $\sigma(\mathfrak{b})$. For a finitely generated graded R-module M, the Castelnuovo–Mumford regularity of M is defined as $\operatorname{reg}(M) := \max\{\operatorname{end}(H^i_{R_+}(M)) + i\}$. In [8], Hassanzadeh defined the regularity with respect to the maximal ideal $\mathfrak{m}$ as $\operatorname{reg}_{\mathfrak{m}}(M) := \max\{\operatorname{end}(H^i_{\mathfrak{m}}(M)) + i\}$. He proved that

$$\operatorname{reg}(M) \leq \operatorname{reg}_{\mathfrak{m}}(M) \leq \operatorname{reg}(M) + \dim(R_0),$$

for any a finitely generated graded R-module M, whenever R is a Cohen–Macaulay *local ring; see [8, Proposition 3.4].

The next proposition improves [8, Theorem 3.6] by removing the arithmetic hypothesis of residual intersections.

Proposition 5.2. *Let $(R, \mathfrak{m})$ be a positively graded Cohen–Macaulay *local ring over a Noetherian local ring $(R_0, \mathfrak{m}_0)$, and let $\mathfrak{a} \subset I$ be two homogeneous ideals with* $\operatorname{ht}(I) = g$. *Suppose that I satisfies $\mathcal{SD}_1$. Then, for any s-residual intersection $J = (\mathfrak{a} \colon {}_R I)$,*

$$\operatorname{reg}(R/J) \leq \operatorname{reg}(R) + \dim(R_0) + \sigma(\mathfrak{a}) - (s - g + 1)\operatorname{indeg}(I/\mathfrak{a}) - s.$$

Proof. The proof of this result goes along the same lines as that in [8, Theorem 3.6]. Indeed, Theorem 4.5 implies that R/J is Cohen–Macaulay and is resolved by ${}_0\mathcal{Z}^+_\bullet$. $\square$

The next proposition improves on the result of Hassanzadeh and the second author in [11, Proposition 3.3].

Proposition 5.3. *Let $(R, \mathfrak{m})$ be a positively graded Cohen–Macaulay *local ring over a Noetherian local ring $(R_0, \mathfrak{m}_0)$ with canonical module ω. Let $\mathfrak{a} \subset I$ be two homogeneous ideals with* $\operatorname{ht}(I) = g$, *and let $J = (\mathfrak{a} \colon {}_R I)$ be an s-residual intersection of I. Suppose that I satisfies $\mathcal{SD}_2$ and that* $\operatorname{Tor}^R_1(R/I, \omega) = 0$. *Then*

$$\omega_{R/J} = \operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega(\sigma(\mathfrak{a})).$$

Proof. The proof proceeds along the same lines as in the local case. $\square$

The following result is already an improvement of [8, Proposition 3.15], and also of [11, Proposition 3.3]. We show the equality of the proposed upper bound for Castelnuovo–Mumford regularity of residual intersections in Proposition 5.2. This equality is showed by Hassanzadeh for perfect ideals of height 2 [8, Theorem 3.16(iii)].

Corollary 5.4. *Under the assumptions of Proposition* 5.3*,*

$$\operatorname{reg}_{\mathfrak{m}}(R/J) = \operatorname{reg}_{\mathfrak{m}}(R) + \sigma(\mathfrak{a}) - (s - g + 1)\operatorname{indeg}(I/\mathfrak{a}) - s.$$

In particular, if $\dim(R_0) = 0$*, then*

$$\operatorname{reg}(R/J) = \operatorname{reg}(R) + \sigma(\mathfrak{a}) - (s - g + 1)\operatorname{indeg}(I/\mathfrak{a}) - s.$$

Proof. By Theorem 4.5, R/J is Cohen–Macaulay of dimension $d-s$. By using the local duality theorem and Proposition 5.3,

$$\begin{aligned}\operatorname{reg}_{\mathfrak{m}}(R/J) &= \operatorname{end}(H^{d-s}_{\mathfrak{m}}(R/J)) + d - s = -\operatorname{indeg}(\omega_{R/J}) + d - s\\ &= \sigma(\mathfrak{a}) - \operatorname{indeg}(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega) + d - s\\ &= \sigma(\mathfrak{a}) - \operatorname{indeg}(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})) - \operatorname{indeg}(\omega) + d - s\\ &= \operatorname{reg}_{\mathfrak{m}}(R) + \sigma(\mathfrak{a}) - \operatorname{indeg}(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})) - s\end{aligned}$$

since $\operatorname{reg}_{\mathfrak{m}}(R) = \operatorname{end}(H^d_{\mathfrak{m}}(R)) + d = -\operatorname{indeg}(\omega) + d$.

It remains to prove that $\operatorname{indeg}(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})) = (s-g+1)\operatorname{indeg}(I/\mathfrak{a})$.

Let $g_1, \dots, g_\ell$ be a minimal set of generators of $I/\mathfrak{a}$. We have

$$\operatorname{Sym}_R(I/\mathfrak{a}) \otimes_R R/\mathfrak{m} \simeq (R/\mathfrak{m})[Y_1, \dots, Y_\ell],$$

where Y_i is the class of g_i in $\operatorname{Sym}_R(I/\mathfrak{a}) \otimes_R R/\mathfrak{m}$.

Suppose that $\deg(g_1) = \operatorname{indeg}(I/\mathfrak{a})$. Since

$$\operatorname{Sym}_R(I/\mathfrak{a}) \otimes_R R/\mathfrak{m} \simeq (R/\mathfrak{m})[Y_1, \dots, Y_\ell]$$

is a polynomial ring, we see that $Y_1^{s-g+1} \neq 0$, and hence $g_1^{s-g+1} \neq 0$ (this product in $\operatorname{Sym}_R(I/\mathfrak{a})$) and $g_1^{s-g+1} \in \operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})$. Thus

$$\begin{aligned}\operatorname{indeg}(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})) &\leq \deg(g_1^{s-g+1})\\ &= (s-g+1)\deg(g_1) = (s-g+1)\operatorname{indeg}(I/\mathfrak{a}).\end{aligned}$$

On the other hand, $\operatorname{indeg}(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})) \geq (s-g+1)\operatorname{indeg}(I/\mathfrak{a})$. Thus

$$\operatorname{indeg}(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a})) = (s-g+1)\operatorname{indeg}(I/\mathfrak{a}).$$

The remaining part follows from $\operatorname{reg}(M) \leq \operatorname{reg}_m(M) \leq \operatorname{reg}(M) + \dim(R_0)$ for any finitely generated graded R-module M. ☐

Finally, we close this section by giving some tight relations between the Hilbert series of a residual intersection and the $(s-g+1)$th symmetric power of I/a.

Corollary 5.5. *Let $(R, \mathfrak{m})$ be a positively graded Cohen–Macaulay *local algebra of dimension d over an Artinian local ring R_0 with canonical module ω. Suppose that $\mathfrak{a} \subset I$ are two homogeneous ideals of R with* $\operatorname{ht}(I) = g$, *and that $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Write*

$$H_{R/J}(t) = \frac{P(t)}{(1-t^a)^{d-s}}, \qquad H_{\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega}(t) = \frac{Q(t)}{(1-t^a)^{d-s}},$$

with a being the least common multiple of the degrees of the generators of the algebra R over R_0 and $P(t), Q(t) \in \mathbb{Z}[t, t^{-1}]$, with $P(1), Q(1) > 0$. If I satisfies $\mathcal{SD}_2$ and $\operatorname{Tor}^R_1(R/I, \omega) = 0$, *then*

$$P(t) = t^{\sigma(\mathfrak{a}) + a(d-s)} Q(t^{-1}).$$

In particular, if R is generated over R_0 by elements of degree 1—that is, $R = R_0[R_1]$—then

$$e(R/J) = e(\operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega).$$

Proof. By Proposition 5.3,

$$\omega_{R/J} \simeq \mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega(\sigma(\mathfrak{a})).$$

It follows from [2, Corollary 4.4.6] that

$$H_{\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})\otimes_R \omega(\sigma(\mathfrak{a}))}(t) = (-1)^{d-s} H_{R/J}(t^{-1})$$

is equivalent to

$$H_{\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})\otimes_R \omega}(t) = (-1)^{d-s} t^{\sigma(\mathfrak{a})} H_{R/J}(t^{-1}).$$

Thus

$$Q(t) = t^{\sigma(\mathfrak{a})+a(d-s)} P(t^{-1})$$

gives

$$P(t) = t^{\sigma(\mathfrak{a})+a(d-s)} Q(t^{-1}).$$

In particular,

$$e(R/J) = P(1) = Q(1) = e(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega)$$

by [2, Proposition 4.1.9]. ☐

6. Duality for residual intersections of strongly Cohen–Macaulay ideals

Duality for residual intersections traces back to Peskine and Szpiro and the theory of liaison in [19]. A full picture is given in the recent work of Eisenbud and Ulrich [6], motivated by previous results of Huneke, Ulrich, and van Straten. The duality results in [6] are proved under depth hypotheses on powers of the ideal I and the hypothesis G_s on the local number of generators.

In this section, we show that duality for residual intersections in the case I is a strongly Cohen–Macaulay ideal, with no hypothesis on the local number of generators. In this case, the structure of the canonical module of some symmetric powers of $I/\mathfrak{a}$ is given. Therefore, we can establish some tight relations between the Hilbert series of the symmetric powers of $I/\mathfrak{a}$, and we give the closed formulas for the type and for the Bass number of $\mathrm{Sym}_R^k(I/\mathfrak{a})$.

First, we prove the duality of residual approximation complexes in the height 2 case.

Proposition 6.1. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with canonical module ω, and let $\mathfrak{a} \subset I$ be two ideals of R. Suppose that I is a strongly Cohen–Macaulay ideal of height 2, and that $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Then, for all $0 \le k \le s-2$,*

$$\omega_{H_0({}_k\mathcal{Z}_\bullet^+)} \simeq H_0({}_{s-k-1}^{\;\omega}\mathcal{Z}_\bullet^+).$$

Proof. By Theorem 3.9, the complex ${}_k\mathcal{Z}_\bullet^+$ is acyclic and $H_0({}_k\mathcal{Z}_\bullet^+)$ is Cohen–Macaulay of dimension $d-s$. Therefore, by local duality,

$$\omega^\vee_{H_0({}_k\mathcal{Z}_\bullet^+)} \simeq H_\mathfrak{m}^{d-s}(H_0({}_k\mathcal{Z}_\bullet^+)). \tag{6.1}$$

As I is strongly Cohen–Macaulay of height 2, we have $\mathrm{depth}(Z_i) = d$ for all $0 \le i \le r-1$. By the definition of ${}_k\mathcal{Z}_\bullet^+$, for all $0 \le i \le s$, ${}_k\mathcal{Z}_i^+$ is a direct sum of copies of modules $Z_0, Z_1, \ldots, Z_{r-1}$; therefore, $\mathrm{depth}({}_k\mathcal{Z}_i^+) = d$. We now consider the double complex $C^\bullet_\mathfrak{m}({}_k\mathcal{Z}^+_\bullet)$ that gives rise to two sequences. The second terms of the horizontal spectral are

$$^2\mathbf{E}_{\text{hor}}^{-i,-j} = \begin{cases} H^{d-s}_\mathfrak{m}(H_0({}_k\mathcal{Z}^+_\bullet)) & \text{if } j = d-s \text{ and } i = 0,\\ 0 & \text{otherwise,}\end{cases}$$

and the first terms of the vertical spectral are

$$^1\mathbf{E}_{\text{ver}}^{-i,-j} = \begin{cases} 0 \longrightarrow H^d_\mathfrak{m}({}_k\mathcal{Z}^+_s) \longrightarrow \cdots \longrightarrow H^d_\mathfrak{m}({}_k\mathcal{Z}^+_1) \longrightarrow H^d_\mathfrak{m}({}_k\mathcal{Z}^+_0) \longrightarrow 0 & \text{if } j = d,\\ 0 \quad \text{otherwise.} \end{cases}$$

By the convergence of the spectral sequences, we obtain

$$H^{d-s}_\mathfrak{m}(H_0({}_k\mathcal{Z}^+_\bullet)) \simeq {}^\infty\mathbf{E}_{\text{ver}}^{-s,-d} = {}^2\mathbf{E}_{\text{ver}}^{-s,-d}. \tag{6.2}$$

By Lemma 4.3(iv), we have a commutative diagram for all $0 \leq k \leq s-2$,

$$\begin{tikzcd} H^d_\mathfrak{m}({}_k\mathcal{Z}^+_s) \arrow[r] \arrow[d, "\simeq"'] & H^d_\mathfrak{m}({}_k\mathcal{Z}^+_{s-1}) \arrow[d, "\simeq"] \\ ({}_{s-k-1}^{\ \ \omega}\mathcal{Z}^+_0)^\vee \arrow[r] & ({}_{s-k-1}^{\ \ \omega}\mathcal{Z}^+_1)^\vee. \end{tikzcd}$$

Therefore,

$$^2\mathbf{E}_{\text{ver}}^{-s,-d} \simeq H_0({}_{s-k-1}^{\ \ \omega}\mathcal{Z}^+_\bullet)^\vee. \tag{6.3}$$

By (6.1), (6.2), (6.3), and Lemma 2.3,

$$\omega_{H_0({}_k\mathcal{Z}^+_\bullet)} \simeq H_0({}_{s-k-1}^{\ \ \omega}\mathcal{Z}^+_\bullet). \qquad \square$$

We now state the main result of this section. Let us recall that if M, N, L are three R-modules, then a morphism $\varphi : M \otimes_R N \longrightarrow L$ is a perfect pairing if $\psi_1 : M \longrightarrow \operatorname{Hom}_R(N, L)$, sending m to $\psi_1(m) : n \mapsto \varphi(m \otimes n)$, and $\psi_2 : N \longrightarrow \operatorname{Hom}_R(M, L)$, sending n to $\psi_2(n) : m \mapsto \varphi(m \otimes n)$, are two isomorphisms.

Theorem 6.2. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, with canonical module ω, and let $\mathfrak{a} \subset I$ be two ideals of R with $\operatorname{ht}(I) = g$. Suppose that $J = (\mathfrak{a}\colon {}_R I)$ is an s-residual intersection of I. If I is strongly Cohen–Macaulay and $\operatorname{Tor}_1^R(R/I, \omega) = 0$, then, for all $1 \leq k \leq s-g$, the following hold:*

(i) *The canonical module of $\operatorname{Sym}^k_R(I/\mathfrak{a})$ is $\operatorname{Sym}^{s-g+1-k}_R(I/\mathfrak{a}) \otimes_R \omega$.*

(ii) *There is a perfect pairing*

$$(\operatorname{Sym}^k_R(I/\mathfrak{a}) \otimes_R \omega) \otimes_R \operatorname{Sym}^{s-g+1-k}_R(I/\mathfrak{a}) \longrightarrow \operatorname{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega.$$

Proof.

(i) First, we treat the case $g = 2$. By Proposition 6.1, for all $1 \leq k \leq s-2$,

$$\begin{aligned} \omega_{\operatorname{Sym}^k_R(I/\mathfrak{a})} &\simeq H_0({}_{s-k-1}^{\ \ \omega}\mathcal{Z}^+_\bullet) \\ &\simeq \operatorname{Sym}^{s-k-1}_R(I/\mathfrak{a}) \otimes_R \omega. \end{aligned}$$

The last isomorphism follows from Proposition 3.3.

Now we may suppose that $g \geq 2$ by Remark 3.8. If $g > 2$, then we choose a regular sequence $\mathbf{a}$ of length $g-2$ inside $\mathfrak{a}$ which is a part of a minimal generating set of $\mathfrak{a}$, as in the proof of Theorem 4.8. As $I/\mathbf{a}$ is strongly Cohen–Macaulay by Proposition 4.1, it follows from the height 2 case that

$$\begin{aligned}\omega_{\mathrm{Sym}^k_R(I/\mathfrak{a})} \simeq \omega_{\mathrm{Sym}^k_{R/\mathbf{a}}(\frac{I/\mathbf{a}}{\mathfrak{a}/\mathbf{a}})} &\simeq \mathrm{Sym}^{(s-g+2)-k-1}_{R/\mathbf{a}}\Big(\frac{I/\mathbf{a}}{\mathfrak{a}/\mathbf{a}}\Big) \otimes_{R/\mathbf{a}} (\omega/\mathbf{a}\omega)\\ &\simeq \mathrm{Sym}^{s-g+1-k}_R(I/\mathfrak{a}) \otimes_R \omega.\end{aligned}$$

(ii) It suffices to prove that, for all $1 \le k \le s-g$,

$$\mathrm{Sym}^k_R(I/\mathfrak{a}) \otimes_R \omega \simeq \mathrm{Hom}_R\big(\mathrm{Sym}^{s-g+1-k}_R(I/\mathfrak{a}), \mathrm{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega\big).$$

As $\mathrm{Sym}^{s-g+1-k}_R(I/\mathfrak{a})$ is a maximal Cohen–Macaulay R/J-module by Corollary 4.6(ii) and $\mathrm{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega$ is the canonical module of R/J by Theorem 4.8,

$$\omega_{\mathrm{Sym}^{s-g+1-k}_R(I/\mathfrak{a})} \simeq \mathrm{Hom}_R\big(\mathrm{Sym}^{s-g+1-k}_R(I/\mathfrak{a}), \mathrm{Sym}^{s-g+1}_R(I/\mathfrak{a}) \otimes_R \omega\big).$$

The conclusion follows from (i). ☐

In particular, if the residual intersections are geometric, we obtain the following results, which could be compared to those of [6, Theorem 2.2].

Corollary 6.3. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d, and let $\mathfrak{a} \subset I$ be two ideals of R. Assume that I is a strongly Cohen–Macaulay ideal of height g, and that $J = (\mathfrak{a} \colon_R I)$ is a geometric s-residual intersection of I. Then, for all $1 \le k \le s-g$,*

- (i) *the canonical module of $I^k/\mathfrak{a}I^{k-1}$ is $I^{s-g+1-k}/\mathfrak{a}I^{s-g-k}$, and*
- (ii) *there is a perfect pairing*

$$I^k/\mathfrak{a}I^{k-1} \otimes_R I^{s-g+1-k}/\mathfrak{a}I^{s-g-k} \longrightarrow I^{s-g+1}/\mathfrak{a}I^{s-g}.$$

Proof. It is an immediate translation from Theorem 6.2, in view of the facts that $\mathrm{Sym}^k_R(I/\mathfrak{a}) \simeq I^k/\mathfrak{a}I^{k-1}$ by [11, Corollary 2.11] and $\omega_R \simeq R$. ☐

Notice that the pairing in this corollary, and in the main theorem above need not be given by multiplication. However, Eisenbud and Ulrich proved that, in many situations in which our results apply, the multiplication indeed produces a perfect pairing. In this regard, an example they provide is interesting.

Example 6.4 ([6, Example 2.8])**.** Let $R = k[[x,y,z]]$, where k is an infinite field and $I = (x,y)^2$. If $\mathfrak{a}$ is generated by three sufficiently general elements of degree 3 in I, then $J = \mathfrak{a} \colon_R I$ is a 3-residual intersection. Using `Macaulay2` [7], they verified that I is strongly Cohen–Macaulay; hence, $\omega_{R/J} \simeq \mathrm{Sym}^2_R(I/\mathfrak{a})$. Moreover, $\omega_{I/\mathfrak{a}} \simeq I/\mathfrak{a}$.

Computation shows that there is a unique (up to scalars) nonzero homogeneous map $I/\mathfrak{a} \otimes_R I/\mathfrak{a} \longrightarrow \omega_{R/J}$ of lowest degree, and this is a perfect pairing. But they notice that there can be no perfect pairing $I/\mathfrak{a} \otimes_R I/\mathfrak{a} \longrightarrow I^2/\mathfrak{a}I$ because the target is annihilated by $(x,y,z)^2$, while $I/\mathfrak{a}$ is not. This implies that $\omega_{R/J} \ne I^2/\mathfrak{a}I$ and J is not geometric.

However, the multiplication with value in the symmetric square $I/\mathfrak{a} \otimes_R I/\mathfrak{a} \longrightarrow \mathrm{Sym}^2_R(I/\mathfrak{a})$ is a perfect pairing.

Next, we will show that the perfect pairing in Theorem 6.2, and also in Corollary 6.3, could be chosen by multiplication. First, we need the following lemmas.

Lemma 6.5. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring, and let S be a Noetherian standard graded R-algebra. For any $s \geq t$, we consider*

$$\psi : S_t \longrightarrow \mathrm{Hom}_R(S_{s-t}, S_s),$$

the natural map given by the algebra structure of S. If $H^0_{S_+}(S \otimes_R k)_t = 0$, then $\psi \otimes k$ is into.

Proof. Let $L \in S_t$ be such that $0 \neq \overline{L} \in S_t \otimes_R k = (S \otimes_R k)_t$. The element L is sent to the class of the homomorphism $\times L$. We have to prove that this class is not zero. As

$$\mathfrak{m}\mathrm{Hom}_R(S_{s-t}, S_s) \subseteq \mathrm{Hom}_R(S_{s-t}, \mathfrak{m}S_s),$$

it suffices to show that the image of $\times L$ is not contained in $\mathfrak{m}S_s$. The assertion is obvious if $s = t$. If $s > t$, as $\overline{L} \notin H^0_{S_+}(S \otimes_R k)_t$ and $S_{s-t} = (S_+)^{s-t}$, there exist $u \in S_{s-t}$ such that $\overline{L}.\overline{u} \neq 0$. Hence, the image of $\times L$ contains $L.u \notin \mathfrak{m}S_s$. ☐

Lemma 6.6. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring, and let M be a finitely generated R-module. For any $s \geq t$, if there exists a R-module isomorphism*

$$\varphi : \mathrm{Hom}_R(\mathrm{Sym}^{s-t}_R(M), \mathrm{Sym}^s_R(M)) \longrightarrow \mathrm{Sym}^t_R(M),$$

then the natural map given by the algebra structure of $\mathrm{Sym}_R(M)$,

$$\psi : \mathrm{Sym}^t_R(M) \longrightarrow \mathrm{Hom}_R(\mathrm{Sym}^{s-t}_R(M), \mathrm{Sym}^s_R(M)),$$

is an isomorphism.

Proof. The assertion of the lemma is equivalent to showing that $\varphi \circ \psi$ is onto, which in turn is equivalent to $\psi \otimes_R k$ being into (or equivalently onto).

Choose $\tau_1 : R^n \longrightarrow M$ onto with n minimal (equivalently such that $R^n \otimes_R k \simeq M \otimes_R k$ via τ_1). Then $\tau := \mathrm{Sym}_R(\tau_1) : \mathrm{Sym}_R(R^n) \longrightarrow \mathrm{Sym}_R(M)$ is onto and $\tau \otimes_R k$ is an isomorphism identifying $\mathrm{Sym}_R(M) \otimes_R k$ with a polynomial ring in n variables. It follows that $S = \mathrm{Sym}_R(M)$ satisfies the condition of Lemma 6.5; hence, $\psi \otimes k$ is into. ☐

Note that $\mathrm{Sym}^0_{R/J}(I/\mathfrak{a}) = R/J$ and $\mathrm{Sym}^k_{R/J}(I/\mathfrak{a}) = \mathrm{Sym}^k_R(I/\mathfrak{a})$ for $k > 0$. We have the following results.

Theorem 6.7. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring, and let $\mathfrak{a} \subset I$ be two ideals of R with $\mathrm{ht}(I) = g$. Suppose that $J = (\mathfrak{a} \colon_R I)$ is an s-residual intersection of I. If I is strongly Cohen–Macaulay, then $\omega_{R/J} \simeq \mathrm{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$ and, for all $0 \leq k \leq s - g + 1$, the following hold:*

- (i) *The R/J-module $\mathrm{Sym}^k_{R/J}(I/\mathfrak{a})$ is faithful and Cohen–Macaulay.*
- (ii) *The multiplication*

$$\mathrm{Sym}^k_{R/J}(I/\mathfrak{a}) \otimes_{R/J} \mathrm{Sym}^{s-g+1-k}_{R/J}(I/\mathfrak{a}) \longrightarrow \mathrm{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$$

  is a perfect pairing.
- (iii) *Setting $A := \mathrm{Sym}_{R/J}(I/\mathfrak{a})$, the graded R/J-algebra*

$$\overline{A} := A/A_{>s-g+1} = \bigoplus_{i=0}^{s-g+1} \mathrm{Sym}^i_{R/J}(I/\mathfrak{a})$$

  is Gorenstein.

Proof. The first item is Corollary 4.6(ii). The second and last items directly follow from Lemma 6.6 together with Theorem 6.2(ii) and (i), respectively. ☐

Corollary 6.8. *Let* $(R, \mathfrak{m})$ *be a positively graded Cohen–Macaulay* **local algebra of dimension* d *over an Artinian local ring* R_0 *with canonical module* ω*. Suppose that* $\mathfrak{a} \subset I$ *are two homogeneous ideals of* R *with* $\mathrm{ht}(I) = g$*, and that* $J = (\mathfrak{a} \colon {}_R I)$ *is an* s*-residual intersection of* I*. Write*

$$H_{\mathrm{Sym}_R^k(I/\mathfrak{a})}(t) = \frac{P_k(t)}{(1-t^a)^{d-s}}, \qquad H_{\mathrm{Sym}_R^k(I/\mathfrak{a})\otimes_R \omega}(t) = \frac{Q_k(t)}{(1-t^a)^{d-s}},$$

with a *the least common multiple of the degrees of the generators of the algebra* R *over* R_0 *and* $P_k(t), Q_k(t) \in \mathbb{Z}[t, t^{-1}]$*, with* $P_k(1), Q_k(1) > 0$*, for each* $1 \le k \le s-g$*. If* I *is strongly Cohen–Macaulay and* $\mathrm{Tor}_1^R(R/I, \omega) = 0$*, then*

$$P_k(t) = t^{\sigma(\mathfrak{a})+a(d-s)} Q_{s-g+1-k}(t^{-1}).$$

In particular, if R *is generated over* R_0 *by elements of degree* 1*, that is,* $R = R_0[R_1]$*, then*

$$e(\mathrm{Sym}_R^k(I/\mathfrak{a})) = e(\mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R \omega).$$

Proof. The proof is analogous to that of Corollary 5.5. It follows from the fact that

$$H_{\mathrm{Sym}_R^k(I/\mathfrak{a})\otimes_R \omega}(t) = (-1)^{d-s} t^{\sigma(\mathfrak{a})} H_{\mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a})}(t^{-1}). \qquad ☐$$

The next corollary enables us to calculate the type of some symmetric powers of $I/\mathfrak{a}$. This is comparable with the results of Hassanzadeh and the second author in [11, Theorem 2.12].

Corollary 6.9. *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* d *with canonical module* ω*, and let* $\mathfrak{a} \subset I$ *be two ideals of* R *with* $\mathrm{ht}(I) = g$*. Suppose that* $J = (\mathfrak{a} \colon {}_R I)$ *is an* s*-residual intersection of* I*. If* I *is strongly Cohen–Macaulay and* $\mathrm{Tor}_1^R(R/I, \omega) = 0$*, then, for each* $1 \le k \le s-g$*,*

$$r(\mathrm{Sym}_R^k(I/\mathfrak{a})) = \binom{\mu(I/\mathfrak{a}) + s - g - k}{\mu(I/\mathfrak{a}) - 1} r(R).$$

Proof. The proof is totally similar to that of Corollary 4.11. For all $1 \le k \le s-g$,

$$r(\mathrm{Sym}_R^k(I/\mathfrak{a})) = \mu(\mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R \omega)$$

by Theorem 6.2(i) and [2, Proposition 3.3.11]. ☐

Let R be a Noetherian ring, let M be a finitely generated R-module, and let $\mathfrak{p} \in \mathrm{Spec}(R)$. The finite number

$$\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})}(\mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})) = \dim_{k(\mathfrak{p})}(\mathrm{Ext}^i_R(R/\mathfrak{p}, M)_\mathfrak{p})$$

is called the *ith Bass number* of M with respect to $\mathfrak{p}$, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. If R is local, then $r(M) = \mu_{\mathrm{depth}(M)}(\mathfrak{m}, M)$. These numbers have an interpretation in terms of the minimal injective resolution of M (see [2, Proposition 3.2.9]). The next corollary enables us to calculate the Bass numbers of some symmetric powers of $I/\mathfrak{a}$.

Corollary 6.10. *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* d *with canonical module* ω*, and let* $\mathfrak{a} \subset I$ *be two ideals of* R *with* $\mathrm{ht}(I) = g$*. Suppose that* $J = (\mathfrak{a} \colon {}_R I)$ *is an* s*-residual intersection of* I*. Let* $\mathfrak{p}$ *be a prime ideal containing* J

of R with $\operatorname{ht}(\mathfrak{p}) = i$. *If I is strongly Cohen–Macaulay and* $\operatorname{Tor}_1^R(R/I, \omega) = 0$, *then, for every* $1 \leq k \leq s - g + 1$,

$$\mu_{i-s}(\mathfrak{p}, \operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega) = \binom{\mu((I/\mathfrak{a})_\mathfrak{p}) + s - g - k}{\mu((I/\mathfrak{a})_\mathfrak{p}) - 1}.$$

Proof. By Theorem 4.5, R/J is Cohen–Macaulay of dimension $d - s$, and by Corollary 4.6(ii), $\operatorname{Sym}_R^k(I/\mathfrak{a})$ is a maximal Cohen–Macaulay faithful R/J-module for all $1 \leq k \leq s - g + 1$. Furthermore, by Theorem 6.2(i) and Theorem 4.8, $\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega$ is a maximal Cohen–Macaulay faithful R/J-module for all $1 \leq k \leq s - g + 1$.

First, $\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega \simeq \omega_{R/J}$ by Theorem 4.8, $\operatorname{ht}(\mathfrak{p}) = i - s$ in R/J, and $\mu_{i-s}(\mathfrak{p}, \omega_{R/J}) = 1$ by [2, Theorem 3.3.10]. This proves the case $k = s - g + 1$.

Suppose that $J \subset \mathfrak{p}_s \subsetneq \mathfrak{p}_{s+1} \subsetneq \cdots \subsetneq \mathfrak{p}_i = \mathfrak{p}$ is a maximal chain of primes of $\operatorname{Spec}(R/J)$ contained in $\mathfrak{p}$. Let $b_j \in \mathfrak{p}_j - \mathfrak{p}_{j-1}$ for all $s + 1 \leq j \leq i$. Then $\mathfrak{b} = (b_{s+1}, \ldots, b_i)$ is a regular sequence over R/J, and therefore also over $\operatorname{Sym}_R^k(I/\mathfrak{a})$ and $\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega$, for all $1 \leq k \leq s - g + 1$.

Let $1 \leq k \leq s - g$. Then $(b_{s+1}, \ldots, b_i)$ is a regular sequence over $(\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p}$ and annihilates $k(\mathfrak{p})$; hence, [2, Lemma 1.2.4] gives

$$\begin{aligned}\operatorname{Ext}_{R_\mathfrak{p}}^{i-s}(k(\mathfrak{p}), (\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p}) &\simeq \operatorname{Hom}_{R_\mathfrak{p}}\big(k(\mathfrak{p}), (\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\big)\\ &\simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}), \operatorname{Hom}_R(\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}), \omega_{R/J}) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big).\end{aligned}$$

The last isomorphism follows from Theorems 4.8 and 6.2(ii). By [2, Proposition 3.3.3],

$$\begin{aligned}&\operatorname{Hom}_R\big(\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}), \omega_{R/J}\big) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\\ &\qquad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}, \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big).\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\operatorname{Ext}_{R_\mathfrak{p}}^{i-s}(k(\mathfrak{p}), (\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p})\\ &\quad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}), \operatorname{Hom}_{R_\mathfrak{p}}\big(\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}, \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\big)\Big)\\ &\quad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} \operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}, \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big)\\ &\quad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}) \otimes_R \operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}), \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big)\\ &\quad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(\operatorname{Sym}_{k(\mathfrak{p})}^{s-g+1-k}(k(\mathfrak{p}) \otimes_R I/\mathfrak{a}), \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big).\end{aligned}$$

Since $k(\mathfrak{p}) \otimes_R I/\mathfrak{a} \simeq k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} (I/\mathfrak{a})_\mathfrak{p}$ is a $k(\mathfrak{p})$-vector space of dimension $\mu_\mathfrak{p} := \mu((I/\mathfrak{a})_\mathfrak{p})$,

$$\operatorname{Sym}_{k(\mathfrak{p})}(k(\mathfrak{p}) \otimes_R I/\mathfrak{a}) \simeq k(\mathfrak{p})[Y_1, \ldots, Y_{\mu_\mathfrak{p}}].$$

It follows that

$$\begin{aligned}\operatorname{Ext}_{R_\mathfrak{p}}^{i-s}(k(\mathfrak{p}), (\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p}) &\simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p})^{\binom{\mu_\mathfrak{p}+s-g-k}{\mu_\mathfrak{p}-1}}, \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big)\\ &\simeq \operatorname{Hom}_{R_\mathfrak{p}}\big(k(\mathfrak{p}), \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\big)^{\binom{\mu_\mathfrak{p}+s-g-k}{\mu_\mathfrak{p}-1}}\\ &\simeq \operatorname{Ext}_{R_\mathfrak{p}}^{i-s}\big(k(\mathfrak{p}), (\omega_{R/J})_\mathfrak{p}\big)^{\binom{\mu_\mathfrak{p}+s-g-k}{\mu_\mathfrak{p}-1}}.\end{aligned}$$

The last isomorphism follows from the fact that $\mathfrak{b}R_{\mathfrak{p}}$ is regular over $(\omega_{R/J})_{\mathfrak{p}}$ and annihilates $k(\mathfrak{p})$. Therefore,

$$\begin{aligned}\mu_{i-s}(\mathfrak{p}, \operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega) &= \binom{\mu_{\mathfrak{p}} + s - g - k}{\mu_{\mathfrak{p}} - 1}\mu_{i-s}(\mathfrak{p}, \omega_{R/J}) \\ &= \binom{\mu_{\mathfrak{p}} + s - g - k}{\mu_{\mathfrak{p}} - 1}. \qquad \square\end{aligned}$$

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