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$\mathfrak{osp}(1,2)$ AND GENERALIZED BANNAI–ITO ALGEBRAS

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ABSTRACT. Generalizations of the (rank-1) Bannai–Ito algebra are obtained from a refinement of the grade involution of the Lie superalgebra $\mathfrak{osp}(1,2)$. A hyperoctahedral extension is derived by using a realization of $\mathfrak{osp}(1,2)$ in terms of Dunkl operators associated with the Weyl group B_3.

1. INTRODUCTION

The Bannai–Ito algebra $\mathcal{BI}_3$ is the associative algebra over $\mathbb{C}$ with three generators K_{12}, K_{23}, K_{13} that satisfy the relations

(1.1)
$$\{K_{12}, K_{23}\} = K_{13} + \omega_{13}, \quad \{K_{12}, K_{13}\} = K_{23} + \omega_{23}, \quad \{K_{13}, K_{23}\} = K_{12} + \omega_{12},$$

where $\{A, B\} = AB + BA$ is the anticommutator and where ω_{12}, ω_{23}, and ω_{13} are structure constants. It is readily verified that the Casimir element

$$Q = K_{12}^2 + K_{13}^2 + K_{23}^2 \tag{1.2}$$

belongs to the center of $\mathcal{BI}_3$.

The Bannai–Ito algebra was first presented in [1] as the algebra encoding the bispectral properties of the Bannai–Ito polynomials [2]. Its connection to the Lie superalgebra $\mathfrak{osp}(1,2)$ was understood soon after when (a central extension of) $\mathcal{BI}_3$ was shown to be the centralizer of the coproduct embedding of $\mathfrak{osp}(1,2)$ in the enveloping algebra of the threefold product of this superalgebra [3–5]. As a consequence, the Bannai–Ito polynomials were seen [3] to be essentially the Racah coefficients of $\mathfrak{osp}(1,2)$. In the following, we shall keep the notation $\mathcal{BI}_3$ when the symbols ω_{ij}, $ij = 12, 13, 23$, are central elements instead of constants.

The Bannai–Ito algebra was further shown [6] to be isomorphic to the degenerate double affine Hecke algebra of type $(C_1^\vee, C_1)$.

This Bannai–Ito algebra proves relevant in a variety of contexts where realizations of $\mathfrak{osp}(1,2)$ arise. It intervenes in Dunkl harmonic analysis on the 2-sphere [7] and is the symmetry algebra in three dimensions of a superintegrable model with reflections [8] and of the Dirac–Dunkl equation [9]. These models are actually quite useful in the identification of the higher rank extensions of $\mathcal{BI}_3$ that have been obtained [10].

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In view of the fundamental features and various applications of the Bannai–Ito algebra, it should be of interest to explore possible generalizations. When the Bannai–Ito algebra is identified as the algebra of the intermediate Casimir operators in the triple product of $\mathfrak{osp}(1,2)$ algebras, the reflection group $\mathbb{Z}_2^3$ formed by the grade involutions of each factor in $\mathfrak{osp}(1,2)^{\otimes 3}$ is present. The purpose of this paper is to show that the Bannai–Ito algebra admits the following generalization that lies outside the setting of coproduct homomorphisms. For $i, j, k \in \{1,2,3\}$ and all distinct, the elements C_i and C_{ij} generate the extended Bannai–Ito algebra defined by the relations

$$\{C_{ij}, C_{jk}\} = C_{ik} + \{C_j, C_{ijk}\} + \{C_i, C_k\} \tag{1.3}$$

and

$$[C_{ij}, C_k] + [C_{jk}, C_i] + [C_{ik}, C_j] = 0, \tag{1.4}$$

and where C_{ijk} are central elements. Note that when C_i are constant multiples of the identity, (1.4) is trivially satisfied, while (1.3) corresponds to a central extension of (1.1). We should also stress that the algebra closes quadratically in terms of its generators. We will show that the extended Bannai–Ito algebra appears when the $\mathfrak{osp}(1,2)$ grade involution decomposes in a product of supplementary involutions that leave the even part of the algebra invariant. The extended Bannai–Ito algebra will be generated by the elements centralizing $\mathfrak{osp}(1,2)$ that can be found in this situation. The results bear a connection to recent work on the symmetries of the Dirac–Dunkl equation [11].

The paper will proceed as follows. The definition and basic features of the Lie superalgebra $\mathfrak{osp}(1,2)$ will be recalled in Section 2, where the supplementary involutions and their properties will be introduced. Only three such involutions will be needed to extend the rank-1 Bannai–Ito algebra. The centralizing elements will be constructed in Section 3, and the proof that they satisfy relations (1.3) and (1.4) will be outlined.

Section 4 will explain how the standard Bannai–Ito algebra $\mathcal{BI}_3$ is recovered when the framework is specialized to the threefold product of $\mathfrak{osp}(1,2)$. Section 5 will be dedicated to an interesting hyperoctahedral extension of the Bannai–Ito algebra that results when $\mathfrak{osp}(1,2)$ is realized in terms of Dunkl operators associated with the B_3-Weyl group. Explicit expressions will be provided and an extension of (1.1) involving elements of the signed permutation group on three objects will be obtained. Section 6 will offer a discussion of the generalized Bannai–Ito algebra that is found when the $\mathfrak{osp}(1,2)$ realization involves a Clifford algebra. The paper will end with concluding remarks. Some details of the derivation of the structure relations of the generalized algebra will be given in the appendix.

2. $\mathfrak{osp}(1,2)$ and involutions

The Lie superalgebra $\mathfrak{osp}(1,2)$ can be presented by adjoining to the three generators A_0, A_+ A_- the grade involution P that accounts for the $\mathbb{Z}_2$-grading of the algebra. That A_0 and $A_\pm$ are, respectively, even and odd generators is enforced by taking

$$P^2 = 1, \quad [P, A_0] = 0, \quad \{P, A_\pm\} = 0, \tag{2.1}$$

with $[A, B] = AB - BA$. This is then supplemented by the relations

$$[A_0, A_\pm] = \pm A_\pm, \qquad \{A_+, A_-\} = 2A_0 \tag{2.2}$$

to complete the definition of $\mathfrak{osp}(1,2)$. The Casimir element

$$S = \frac{1}{2}\big([A_-, A_+] - 1\big) \tag{2.3}$$

enjoys the same relations that P does with A_0 and $A_\pm$, and thus $\Gamma = SP$ commutes with all generators:

$$[\Gamma, A_\pm] = [\Gamma, A_0] = [\Gamma, P] = 0. \tag{2.4}$$

Recalling that $[A, BC] = \{A, B\}C - B\{A, C\}$, it is useful to record that

$$[A_\mp, A_\pm^2] = 2[A_0, A_\pm] = \pm 2A_\pm. \tag{2.5}$$

The three even generators

$$B_\pm = A_\pm^2 \quad \text{and} \quad A_0 \tag{2.6}$$

are readily seen to close onto the $su(1,1)$ commutation relations and to commute with P:

$$[B_+, B_-] = -4A_0, \quad [A_0, B_\pm] = \pm 2B_\pm, \quad [P, B_\pm] = [P, A_0] = 0. \tag{2.7}$$

The $su(1,1)$ Casimir element

$$C = \frac{1}{4}\big(A_0^2 - B_+B_- - 2A_0\big) \tag{2.8}$$

is related as follows with the Casimir element $S = \Gamma P$:

$$\Gamma^2 - \Gamma P = 4C + \frac{3}{2}. \tag{2.9}$$

Let us now introduce supplementary involutions P_i, $i = 1, \dots, n$ such that

$$P = P_1 P_2 \cdots P_n \tag{2.10}$$

and

$$P_i^2 = 1, \quad [P_i, P_j] = 0, \quad i \neq j. \tag{2.11}$$

We shall make two additional assumptions on the relations that these involutions have with the $\mathfrak{osp}(1,2)$ generators:

i. We will suppose that they all commute with the $su(1,1)$ generators:

$$[P_i, A_0] = [P_i, B_\pm] = 0 \qquad \forall i. \tag{2.12}$$

ii. We shall impose an additivity or decomposition property of the form

$$\big[P_i, [P_j, A_\pm]\big] = 0, \qquad i \neq j, \tag{2.13}$$

stating that the commutator of the odd generators with any involution is even with respect to all the others.

The last property entails the following lemma that will prove quite useful.

Lemma 2.1. *Subject to the above definitions and hypotheses, together with the $\mathfrak{osp}(1,2)$ generators A_+ and A_-, the involutions P_i, $i = 1, \dots, n$ satisfy the following relations:*

$$P_i A_\pm P_j + P_j A_\pm P_i = P_i P_j A_\pm + A_\pm P_i P_j, \qquad i \neq j, \tag{2.14}$$

and

$$P_i A_\pm A_\mp A_\pm P_j + P_j A_\pm A_\mp A_\pm P_i = P_i P_j A_\pm A_\mp A_\pm + A_\pm A_\mp A_\pm P_i P_j, \qquad i \neq j. \tag{2.15}$$

Proof. Formula (2.14) straightforwardly results from expanding (2.13). To prove (2.15), consider first the upper signs. From the commutation relation (2.2), one has

$$A_+A_-A_+ = -A_-A_+^2 + 2A_0A_+. \tag{2.16}$$

It hence follows that

$$\begin{aligned} P_iA_+A_-A_+P_j + (i \leftrightarrow j) &= P_i(-A_-A_+^2 + 2A_0A_+)P_j + (i \leftrightarrow j) \\ &= -P_iA_-P_jA_+^2 + 2A_0P_iA_+P_j + (i \leftrightarrow j) \\ &= -(P_iP_jA_- + A_-P_iP_j)A_+^2 + 2A_0(P_iP_jA_+ + A_+P_iP_j) \\ &= P_iP_j(-A_-A_+^2 + 2A_0A_+) + (-A_-A_+^2 + 2A_0A_+)P_iP_j \\ &= P_iP_jA_+A_-A_+ + A_+A_-A_+P_iP_j, \end{aligned} \tag{2.17}$$

where we have used $[P_i, A_+^2] = [P_i, A_0] = 0$, (2.14), and (2.16).

The proof of (2.15) with lower signs proceeds in the same way. □

3. Centralizing elements and a realization of the generalized Bannai–Ito algebra

Denote by $[n] = \{1, \ldots, n\}$ the set of the first n positive integers. Let $S = \{s_1, \ldots, s_k\}$ be an ordered k-subset of $[n]$. Write

$$P_S = P_{s_1}P_{s_2}\cdots P_{s_k}. \tag{3.1}$$

We shall now introduce elements involving the supplementary involutions that will form a centralizer of $\mathfrak{osp}(1,2)$. In the special cases of cardinality 1, 2, and 3, they will provide a realization of the extended Bannai–Ito algebras (1.3) and (1.4). Note that we will use the same notation to denote the realization that follows and the abstract definition.

Proposition 3.1. *The elements*

$$C_S = \frac{1}{4}\{A_-, [A_+, P_S]\} - \frac{1}{2}P_S \tag{3.2a}$$

or equivalently

$$C_S = \frac{1}{4}\{[P_S, A_-], A_+\} - \frac{1}{2}P_S \tag{3.2b}$$

associated with the set S centralize $\mathfrak{osp}(1,2)$; that is, they satisfy

$$[C_S, A_0] = [C_S, A_\pm] = [C_S, P] = 0. \tag{3.3}$$

Proof. The equality of the formulas (3.2a) and (3.2b) for C_S is immediately obtained by expanding both expressions and using $\{A_+, A_-\} = 2A_0$ and $[P_S, A_0] = 0$. The relations $[C_S, A_0] = 0$ and $[C_S, P] = 0$ follow from the fact that C_S is bilinear in A_+ and A_-, that A_0 commutes with P_S, and that $\{P, A_\pm\} = 0$. Let us now show that $[C_S, A_-] = 0$. First, one observes that

$$\begin{aligned} [C_S, A_-] &= \left[\frac{1}{4}A_-[A_+, P_S] + \frac{1}{4}[A_+, P_S]A_- - \frac{1}{2}P_S, A_-\right] \\ &= \frac{1}{4}\big[[A_+, P_S], A_-^2\big] - \frac{1}{2}[P_S, A_-]. \end{aligned} \tag{3.4}$$

Using the Jacobi identity and the fact that the involutions commute with the $su(1,1)$ generators $B_\pm = A_\pm^2$, one completes the proof by observing that

$$[C_S, A_-] = -\frac{1}{4}\big[[A_-^2, A_+], P_S\big] - \frac{1}{2}[P_S, A_-] = 0, \tag{3.5}$$

with the help of (2.5). The demonstration that $[C_S, A_+] = 0$ proceeds in exactly the same way when the expression (3.2b) for C_S is used. □

Remark 3.2. Owing to the fact that the involutions commute, C_S is symmetric under the permutations of the elements of S. For example, in the case of subsets S of cardinality 2, the 2-indices C_{ij}, $i, j = 1, \dots, n$, satisfy

$$C_{ij} = C_{ji}, \qquad i \neq j. \tag{3.6}$$

Remark 3.3. For $S = [n]$, $P_{[n]} = P$ by hypothesis and since $PA_\pm = -A_\pm P$,

$$C_{[n]} = \frac{1}{4}\big\{A_-, [A_+, P]\big\} - \frac{1}{2}P = \frac{1}{2}\big([A_-, A_+] - 1\big)P = \Gamma. \tag{3.7}$$

The following corollary is immediate given that $[C_S, A_\pm] = 0$ and $[C_S, P] = 0$ by Proposition 3.1.

Corollary 3.4. *The elements C_S defined in Proposition* 3.1 *also have the property of commuting with $C_{[n]} = \Gamma$:*

$$[C_S, C_{[n]}] = 0. \tag{3.8}$$

The next two lemmas provide relations between the 1-index C_i, $i = 1, \dots, n$, and the involutions.

Lemma 3.5. *The* 1*-index elements*

$$C_i = \frac{1}{4}\big\{A_-, [A_+, P_i]\big\} - \frac{1}{2}P_i \tag{3.8a}$$

or equivalently

$$C_i = \frac{1}{4}\big\{[P_i, A_-], A_+\big\} - \frac{1}{2}P_i \tag{3.8b}$$

satisfy

$$C_i P_j + C_j P_i = P_i C_j + P_j C_i \tag{3.9}$$

for all $i \neq j$.

Proof. Take $i \neq j$. Beginning with (3.8a) and using (2.13), one finds that

$$\begin{aligned}[C_i, P_j] &= \frac{1}{4}\Big[A_-[A_+, P_i] + [A_+, P_i]A_-, P_j\Big] \\ &= \frac{1}{4}\Big([A_-, P_j][A_+, P_i] + [A_+, P_i][A_-, P_j]\Big).\end{aligned} \tag{3.10}$$

Similarly, starting with (3.8b), one obtains

$$\begin{aligned}[C_j, P_i] &= \frac{1}{4}\Big[[P_j, A_-]A_+ + A_+[P_j, A_-], P_i\Big] \\ &= \frac{1}{4}\Big([P_j, A_-][A_+, P_i] + [A_+, P_i][P_j, A_-]\Big).\end{aligned} \tag{3.11}$$

One then sees that

$$[C_i, P_j] = -[C_j, P_i], \qquad i \neq j, \tag{3.12}$$

from which the relation (3.9) follows. □

In view of the expression (3.10) (or (3.11)) and given (2.13), it follows that

$$\big[P_i,[P_j,C_k]\big]=0 \qquad \text{for } i,j,k \text{ distinct.} \tag{3.13}$$

Lemma 3.5 admits a multi-index generalization.

Lemma 3.6. *Let $S=\{s_1,\ldots,s_k\}$ be an ordered k-subset of $[n]$, and write*

$$P_S=\prod_{\ell=1}^{k}P_{s_\ell}. \tag{3.14}$$

One has

$$P_S\left(\sum_{i=1}^{k}P_{s_i}C_{s_i}\right)=\left(\sum_{i=1}^{k}C_{s_i}P_{s_i}\right)P_S. \tag{3.15}$$

Proof. The above relation can be written in the form

$$\sum_{i=1}^{k}[P_{S\setminus\{s_i\}},C_{s_i}]=0. \tag{3.16}$$

Use induction. We know that (3.15) is true for $k=2$, from Lemma 3.5. Assume that (3.15), or equivalently (3.16), is valid if the cardinality of S is $k-1$. This means that for some j between 1 and k,

$$\sum_{i=1;i\neq j}^{k}[P_{S\setminus\{s_i,s_j\}},C_{s_i}]=0. \tag{3.17}$$

Note that

$$P_{S\setminus\{s_i\}}=P_{S\setminus\{s_i,s_j\}}P_{s_j}, \qquad j\neq i. \tag{3.18}$$

Use (3.18) to write

$$\sum_{i=1}^{k}[P_{S\setminus\{s_i\}},C_{s_i}]=\frac{1}{k}\sum_{\ell=1}^{k}\left([P_{S\setminus\{s_\ell\}},C_{s_\ell}]+\sum_{j=1;\,j\neq\ell}^{k}[P_{S\setminus\{s_\ell,s_j\}}P_{s_j},C_{s_\ell}]\right). \tag{3.19}$$

Some commutator algebra thus gives

$$\left(\frac{k-1}{k}\right)\sum_{i=1}^{k}[P_{S\setminus\{s_i\}},C_{s_i}]=\frac{1}{k}\sum_{\substack{j,\ell=1\\ j\neq\ell}}^{k}\Big([P_{S\setminus\{s_\ell,s_j\}},C_{s_\ell}]P_{s_j}+P_{S\setminus\{s_\ell,s_j\}}[P_{s_j},C_{s_\ell}]\Big). \tag{3.20}$$

Since $[P_{s_j},C_{s_\ell}]=-[P_{s_\ell},C_{s_j}]$ per (3.12), the terms $P_{S\setminus\{s_\ell,s_j\}}[P_{s_j},C_{s_\ell}]$ cancel out 2 by 2 in the sums, and we are left with

$$\sum_{i=1}^{k}[P_{S\setminus\{s_i\}},C_{s_i}]=\frac{1}{k-1}\sum_{j=1}^{k}\left(\sum_{\ell=1;\,\ell\neq j}^{k}[P_{S\setminus\{s_\ell,s_j\}},C_{s_\ell}]\right)P_{s_j}, \tag{3.21}$$

which is 0 according to the induction hypothesis (3.17). □

Proposition 3.1 shows that when the supplementary involutions P_i, $i=1,\ldots,n$, are present, a centralizer of $\mathfrak{osp}(1,2)$ is generated by the elements C_S corresponding to all subsets of $[n]$ with k ordered elements for $k=1,\ldots,n-1$. Since we wish to extend the rank-1 Bannai–Ito algebra, we shall take $n=3$. A centralizer of $\mathfrak{osp}(1,2)$ will then be generated by the 1- and 2-index elements C_1, C_2, C_3 and C_{12},

C_{23}, C_{13} to which the Casimir element C_{123} can be added. Remember that C_{123} commutes with C_i, C_{ij}, $i,j = 1,2,3$, but does not commute with the involutions P_i, $i = 1,2,3$. It shall be seen that the elements C_i, $i = 1,2,3$, together with C_{ijk} take the place of the structure constants and complement the elements C_{ij}, $ij = 12, 13, 23$, that generalize the generators K_{ij} of the Bannai–Ito algebra. The task is now to obtain the relations that these elements obey.

It is useful to record the special form that some relations take when the indices are restricted to the set $\{1,2,3\}$. Take the indices $i,j,k \in \{1,2,3\}$ to all be distinct, thus forming a permutation of $(1,2,3)$. Clearly,

$$P = P_iP_jP_k, \qquad i,j,k \in \{1,2,3\} \text{ all distinct.} \tag{3.22}$$

Given that $PA_\pm = -A_\pm P$, formulas of the type

$$P_iP_jA_\pm P_k = -P_kA_\pm P_iP_j \tag{3.23}$$

follow immediately. The result of Lemma 3.6 will specialize to

$$[P_iP_j, C_k] + [P_jP_k, C_i] + [P_iP_k, C_j] = 0. \tag{3.24}$$

Moreover,

$$C_{ijk} = C_{123} = \Gamma. \tag{3.25}$$

It is in fact possible to show that this Casimir operator can be written as a combination of the 1- and 2-index elements C_i and C_{ij} together with the involutions P_i.

Proposition 3.7. *When $n = 3$ and $i,j,k \in \{1,2,3\}$ are all distinct, the Casimir element C_{ijk} has the following expression:*

$$C_{ijk} = C_{ij}P_k + C_{jk}P_i + C_{ik}P_j - C_kP_iP_j - C_iP_jP_k - C_jP_iP_k - \frac{1}{2}P_iP_jP_k. \tag{3.26}$$

Proof. Using the definition (3.2a) and (3.23), one sees that

$$C_{ij}P_k - C_kP_iP_j = \frac{1}{2}\big(A_-P_kA_+P_iP_j - A_+P_kA_-P_iP_j\big). \tag{3.27}$$

Now

$$\begin{aligned}\frac{1}{2}A_-P_kA_+P_iP_j + \text{ cyclic permutations} &= \frac{1}{4}\big(A_-P_kA_+P_iP_j + A_-P_iA_+P_jP_k\big)\\ &+ \frac{1}{4}\big(A_-P_iA_+P_jP_k + A_-P_jA_+P_iP_k\big)\\ &+ \frac{1}{4}\big(A_-P_jA_+P_iP_k + A_-P_kA_+P_iP_j\big).\end{aligned} \tag{3.28}$$

With the help of (2.14) and of (3.23) again, one finds that

$$\begin{aligned}&\frac{1}{2}A_-P_kA_+P_iP_j + \text{ cyclic permutations}\\ &\quad = \frac{3}{4}A_-A_+P - \left(\frac{1}{4}A_-P_kA_+P_iP_j + \text{ cyclic permutations}\right),\end{aligned} \tag{3.29}$$

from which one obtains that

$$A_-P_kA_+P_iP_j + \text{ cyclic permutations} = A_-A_+P. \tag{3.30}$$

Similarly, one gets

$$A_+P_kA_-P_iP_j + \text{ cyclic permutations} = A_+A_-P. \tag{3.31}$$

One thus sees that

(3.32)
$$\Big(C_{ij}P_k - C_kP_{ij} + \text{ cyclic permutations}\Big) - \frac{1}{2}P_iP_jP_k = \frac{1}{2}\big(A_-A_+ - A_+A_- - 1\big)P = \Gamma,$$

which completes the proof. □

We are now ready to state our main result, which shows that the elements C_i and C_{ij}, $i, j = 1, 2, 3$, satisfy the relations (1.3) and (1.4) of the extended Bannai–Ito algebra.

Theorem 3.8. *For $i, j, k \in \{1, 2, 3\}$ and all distinct, the elements C_i and C_{ij} satisfy the relations of the extended Bannai–Ito algebra:*

$$\{C_{ij}, C_{jk}\} = C_{ik} + \{C_j, C_{ijk}\} + \{C_i, C_k\} \tag{3.33}$$

and

$$[C_{ij}, C_k] + [C_{jk}, C_i] + [C_{ik}, C_j] = 0. \tag{3.34}$$

Although a bit involved, the proofs proceed straightforwardly. The basic strategy is to separately expand and simplify the expressions on the left- and right-hand sides to show that they coincide. From the definitions of C_i, C_{ij}, and C_{ijk}, we see that (3.33) and (3.34) will amount to equalities between expressions that involve terms that are quartic, quadratic, and of degree 0 in the $\mathfrak{osp}(1,2)$ generators A_- and A_+, which always occur in pairs (so as to commute with A_0). Expand those expressions in full. It is practical to focus on each of these categories of terms and to verify that they are identically equal on their own upon comparing the left- and right-hand sides. Furthermore, when dealing with the quartic component, it is also possible to concentrate in turn on terms of definite type with respect to the ordering, e.g., terms having the structure $A_+A_-^2A_+$, $A_-A_+^2A_-$ or $A_+A_-A_+A_-$ and $A_-A_+A_-A_+$ together, for instance. In order to not overly clutter the flow of the presentation, we shall relegate further discussion of the proofs to the appendix, where examples of the computations that are needed will be given.

Remark 3.9. In view of Corollary 3.4, we have in addition

$$[C_i, C_{ijk}] = [C_{ij}, C_{ijk}] = 0, \qquad i, j, k \text{ distinct.} \tag{3.35}$$

It follows that the term $\{C_j, C_{ijk}\}$ in (3.33) can also be written as $2\,C_jC_{ijk}$. The relations (3.33) and (3.34) thus bear a kinship to those of the Bannai–Ito algebra. In a situation where the C_i are constants, (3.34) trivializes and one recovers the relations (1.1) with ω_{ij} elements of the center of the algebra generated by C_{ij}. We shall discuss in the next section an instance in which this happens.

4. Embeddings of $\mathfrak{osp}(1,2)$ into $\mathfrak{osp}(1,2)^{\otimes 3}$

A situation in which three involutions naturally arise is when one considers the coproduct embedding of $\mathfrak{osp}(1,2)$ into its threefold tensor product $\mathfrak{osp}(1,2)^{\otimes 3}$. We shall now indicate how this fits within the framework that has been developed and how previously known results appear. Let us recall the definition of the $\mathfrak{osp}(1,2)$ coproduct: it is a coassociative homomorphism

$$\Delta : \mathfrak{osp}(1,2) \to \mathfrak{osp}(1,2) \otimes \mathfrak{osp}(1,2) \tag{4.1}$$

such that

$$\begin{aligned}\Delta(A_\pm) &= A_\pm \otimes P + 1 \otimes A_\pm,\\ \Delta(A_0) &= A_0 \otimes 1 + 1 \otimes A_0,\\ \Delta(P) &= P \otimes P.\end{aligned} \tag{4.2}$$

Upon letting

$$\Delta^{(2)} = (\Delta \otimes 1) \circ \Delta, \tag{4.3}$$

one has a homomorphism $\Delta^{(2)} : \mathfrak{osp}(1,2) \to \mathfrak{osp}(1,2)^{\otimes 3}$. Introduce the three involutions

$$P^{(1)} = P \otimes 1 \otimes 1, \quad P^{(2)} = 1 \otimes P \otimes 1, \quad P^{(3)} = 1 \otimes 1 \otimes P. \tag{4.4}$$

We have

$$\begin{aligned}\mathcal{A}_0 &= \Delta^{(2)}(A_0) = A_0^{(1)} + A_0^{(2)} + A_0^{(3)},\\ \mathcal{A}_\pm &= \Delta^{(2)}(A_\pm) = A_\pm^{(1)} P^{(2)} P^{(3)} + A_\pm^{(2)} P^{(3)} + A_\pm^{(3)},\\ \mathcal{P} &= \Delta^{(2)}(P) = P^{(1)} P^{(2)} P^{(3)},\end{aligned} \tag{4.5}$$

where the suffix designates to which factor the element belongs. Since $\Delta^{(2)}$ is a homomorphism, we have a version of $\mathfrak{osp}(1,2)$ generated by $\mathcal{A}_0$, $\mathcal{A}_\pm$, and $\mathcal{P}$ with three involutions $P^{(i)}$, $i = 1,2,3$, that satisfy the requirements of our setup, namely,

$$P^{(1)} P^{(2)} P^{(3)} = \mathcal{P} \tag{4.6}$$

and

$$[P^{(i)}, \mathcal{A}_0] = 0, \quad \Big[P^{(i)}, [P^{(j)}, \mathcal{A}_\pm]\Big] = 0, \qquad i,j = 1,2,3,\ i \neq j. \tag{4.7}$$

The second relation of (4.7) is verified since

$$\begin{aligned}[\mathcal{A}_\pm, P^{(1)}] &= 2A_\pm^{(1)} P^{(1)} P^{(2)} P^{(3)},\\ [\mathcal{A}_\pm, P^{(2)}] &= 2A_\pm^{(2)} P^{(2)} P^{(3)},\\ [\mathcal{A}_\pm, P^{(3)}] &= 2A_\pm^{(3)} P^{(3)}.\end{aligned} \tag{4.8}$$

Let us then examine what our construction entails in this case. Consider first the 1-index elements C_i, and take $i = 1$; for instance,

$$\begin{aligned}C_1 &= \frac{1}{4}\Big\{\mathcal{A}_-, [\mathcal{A}_+, P^{(1)}]\Big\} - \frac{1}{2}P^{(1)} = \frac{1}{2}[\mathcal{A}_-, A_+^{(1)}]P - \frac{1}{2}P^{(1)}\\ &= \frac{1}{2}\Big([A_-^{(1)}, A_+^{(1)}] - 1\Big)P^{(1)} = \Gamma^{(1)},\end{aligned} \tag{4.9}$$

where $\Gamma^{(1)}$ stands for the Casimir operator of the first factor in $\mathfrak{osp}(1,2)^{\otimes 3}$. Similarly, we find that in fact

$$C_i = \Gamma^{(i)}, \qquad i = 1,2,3. \tag{4.10}$$

The various C_i manifestly all commute. In this case, one refers to the simple embeddings of $\mathfrak{osp}(1,2)$ into $\mathfrak{osp}(1,2)^{\otimes 3}$ as one of the factors. In addition to those and to $\Delta^{(2)}$, there are other embeddings labeled by two indices for which the generators are

(4.11)
$$\mathcal{A}_0^{(ij)} = A_0^{(i)} + A_0^{(j)}, \qquad i,j = 1,2,3$$
$$\mathcal{A}_\pm^{(12)} = A_\pm^{(1)} P^{(2)} + A_\pm^{(2)}, \quad \mathcal{A}_\pm^{(23)} = A_\pm^{(2)} P^{(3)} + A_\pm^{(3)}, \quad \mathcal{A}_\pm^{(13)} = A_\pm^{(1)} P^{(2)} P^{(3)} + A_\pm^{(3)}$$
$$\mathcal{P}^{(12)} = P^{(1)} P^{(2)}, \quad \mathcal{P}^{(23)} = P^{(2)} P^{(3)}, \quad \mathcal{P}^{(13)} = P^{(1)} P^{(3)}.$$

To these embeddings correspond the Casimirs

$$\Gamma^{(ij)} = \frac{1}{2}\Big([\mathcal{A}_-^{(ij)}, \mathcal{A}_+^{(ij)}] - 1\Big)\mathcal{P}^{(ij)}. \tag{4.12}$$

It is immediate to observe that the Casimir operators $C_i = \Gamma^{(i)}$ of each of the factors commute with all of the generators in (4.11), and hence with the intermediate Casimirs, namely,

$$[\Gamma^{(i)}, \Gamma^{(jk)}] = 0 \qquad \forall\, i,j,k \ (j \neq k). \tag{4.13}$$

Let us now come to the 2-index elements and consider, for instance,

$$C_{13} = \frac{1}{4}\Big\{\mathcal{A}_-, [\mathcal{A}_+, P^{(1)} P^{(3)}]\Big\} - \frac{1}{2}\mathcal{P}^{(13)}. \tag{4.14}$$

With the help of (4.8), we see that

$$\begin{aligned} C_{13} &= \frac{1}{4}\Big\{\mathcal{A}_-, [\mathcal{A}_+, P^{(1)}] P^{(3)} + P^{(1)} [\mathcal{A}_+, P^{(3)}]\Big\} - \frac{1}{2}\mathcal{P}^{(13)} \\ &= \frac{1}{2}\Big\{\mathcal{A}_-, A_+^{(1)} P^{(1)} P^{(2)} + A_+^{(3)} P^{(1)} P^{(3)}\Big\} - \frac{1}{2}\mathcal{P}^{(13)} \\ &= \frac{1}{2}\Big\{\mathcal{A}_-, \mathcal{A}_+^{(13)} P^{(1)} P^{(3)}\Big\} - \frac{1}{2}\mathcal{P}^{(13)}. \end{aligned} \tag{4.15}$$

Now $\mathcal{A}_- = \mathcal{A}_-^{(13)} + A_-^{(2)} P^{(3)}$ and

$$\{A_-^{(2)} P^{(3)}, \mathcal{A}_+^{(13)} P^{(1)} P^{(3)}\} = \{A_-^{(2)} P^{(3)}, A_+^{(1)} P^{(1)} P^{(2)} + A_+^{(3)} P^{(1)} P^{(3)}\} = 0. \tag{4.16}$$

Hence

$$C_{13} = \frac{1}{2}\Big([\mathcal{A}_-^{(13)}, \mathcal{A}_+^{(13)}] - 1\Big)\mathcal{P}^{(13)} = \Gamma^{(13)}, \tag{4.17}$$

and we observe that C_{13} is the intermediate Casimir element associated with the embedding $(ij) = (13)$ of (4.11). In general, we find that

$$C_{ij} = \Gamma^{(ij)}. \tag{4.18}$$

As always,

$$C_{ijk} = \frac{1}{2}\Big([\mathcal{A}_-, \mathcal{A}_+] - 1\Big)\mathcal{P} \tag{4.19}$$

is the Casimir element of the main $\mathfrak{osp}(1,2)$ of the construction, here $\Delta^{(2)}\big(\mathfrak{osp}(1,2)\big)$.

The centralizer of $\Delta^{(2)}\big(\mathfrak{osp}(1,2)\big)$ is therefore the algebra generated by the various Casimir operators: $C_i = \Gamma^{(i)}$, $C_{ij} = \Gamma^{(ij)}$, and $C_{ijk} = \Gamma$. Equation (3.34) is hence trivially satisfied and $\{C_j, C_{ijk}\} = 2\Gamma^{(j)}\Gamma$, while $\{C_i, C_k\} = 2\Gamma^{(i)}\Gamma^{(k)}$. It thus follows that the relations of Theorem 3.8 reduce to those of the Bannai–Ito algebra, with ω_{ij} central in this embedding situation. Furthermore, if we were to consider products of three irreducible representations, the Casimirs $\Gamma^{(i)}$ would be constants. This confirms that the defining relations (3.33) and (3.34) generalize those of the

Bannai–Ito algebra, whose appearance in the Racah problem for $\mathfrak{osp}(1,2)$ is here seen as a special case in our framework.

5. A hyperoctahedral extension of the Bannai–Ito algebra

We shall obtain in this section a generalization of the Bannai–Ito algebra that involves the signed symmetric group in three objects. This will be achieved by considering a generalization of $\mathfrak{osp}(1,2)$ built from Dunkl operators [12]. Let x_i, $i = 1, 2, \dots, n$, be the variables; we shall keep their number arbitrary for the moment. We wish to realize the involutions P_i by the reflections R_i in the plane $x_i = 0$, i.e.,

$$R_i f(\dots, x_i, \dots) = f(\dots, -x_i, \dots). \tag{5.1}$$

It is hence natural to consider the Dunkl operators associated with the Weyl group of type B_n which contains the reflections R_i and the permutations π_{ij}:

$$\pi_{ij} f(\dots, x_i, \dots, x_j, \dots) = f(\dots, x_j, \dots, x_i, \dots). \tag{5.2}$$

These B_n-Dunkl operators depend on two parameters, a and b, and are defined by [13, 14]

$$D_i = \frac{\partial}{\partial x_i} + \frac{b}{x_i}(1 - R_i) + a \sum_{j \neq i} \left(\frac{1}{x_i - x_j}(1 - \pi_{ij}) + \frac{1}{x_i + x_j}(1 - R_i R_j \pi_{ij}) \right). \tag{5.3}$$

Remarkably, they form a commuting set:

$$[D_i, D_j] = 0 \qquad \forall i, j. \tag{5.4}$$

Their commutation relations with the coordinates are found to be

$$[D_i, x_j] = \delta_{ij} \left(1 + a \sum_{k \neq i} (1 + R_i R_k) \pi_{ik} + 2b R_i \right) - (1 - \delta_{ij}) a (1 - R_i R_j) \pi_{ij}. \tag{5.5}$$

A special feature of the Dunkl operators of type B_n is their behavior under reflections. Indeed, using $R_i \pi_{ij} = \pi_{ij} R_j$, it is easy to see that

$$\{R_i, D_i\} = 0, \quad [R_i, D_j] = 0, \quad i \neq j. \tag{5.6}$$

These relations are also valid of course when D_i is replaced by the coordinate x_i. Property (5.6) is key. Let us now restrict ourselves to the situation with three variables x_1, x_2, x_3 and the corresponding B_3-Dunkl operators D_1, D_2, D_3. Take

$$\begin{aligned} \hat{A}_- &= D_1 R_2 R_3 + D_2 R_3 + D_3, \\ \hat{A}_+ &= x_1 R_2 R_3 + x_2 R_3 + x_3. \end{aligned} \tag{5.7}$$

In view of the reflection properties (5.6) and given (5.4), one sees that

$$\begin{aligned} \hat{B}_- &= A_-^2 = D_1^2 + D_2^2 + D_3^2, \\ \hat{B}_+ &= A_+^2 = x_1^2 + x_2^2 + x_3^2. \end{aligned} \tag{5.8}$$

It is known [15, 16] that such operators $\hat{B}_\pm$ provide a realization of the $su(1,1)$ commutation relations (2.7), with the Euler operator

$$\hat{A}_0 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + 3 \left(2a + b + \frac{1}{2} \right) \tag{5.9}$$

playing the role of A_0. It is in fact not too difficult to see moreover that $\hat{A}_\pm$ close onto the $\mathfrak{osp}(1,2)$ relations:

$$\{\hat{A}_-, \hat{A}_+\} = 2\hat{A}_0, \qquad [\hat{A}_0, \hat{A}_\pm] = \pm\hat{A}_\pm. \tag{5.10}$$

Let us remark that the form of $\hat{A}_-$ and $\hat{A}_+$ in (5.7) is reminiscent of the expressions we had for $\mathcal{A}_-$ and $\mathcal{A}_+$ in Section 4 as a result of the coproduct action. The key difference though is that we no longer have the analogy of $[A_-^{(i)}, A_+^{(j)}] = 0$, $i \neq j$, as is seen from (5.5).

The reader might also think that the Dunkl operators

$$\nabla_i = \frac{\partial}{\partial x_i} + \frac{\mu_i}{x_i}(1 - R_i), \qquad i = 1, 2, 3, \tag{5.11}$$

associated with the reflection group $\mathbb{Z}_2^3$ and with μ_1, μ_2, μ_3 constants, could prove pertinent to introduce the reflections R_i as involutions. This is true and has in fact been considered already [7,8]; note however that

$$[\nabla_i, x_j] = 0, \qquad i \neq j, \tag{5.12}$$

in this case, which leads to a realization of the embedding situation of Section 4. The uniform specialization $\mu_1 = \mu_2 = \mu_3 = b$ arises with the B_3-operators when $a = 0$.

Before going further, we should confirm that the requirements on the involutions are satisfied. This is so because P is realized as $R = R_1R_2R_3$; indeed,

$$[R, \hat{A}_0] = 0, \qquad \{R, \hat{A}_\pm\} = 0. \tag{5.13}$$

Moreover, owing to (5.7), we see that

$$\Big[R_i, [R_j, \hat{A}_\pm]\Big] = 0, \qquad i \neq j. \tag{5.14}$$

We can thus apply our formalism to get the explicit expressions of the centralizing elements in this concrete realization of $\mathfrak{osp}(1,2)$ and to determine as well the defining relations of the particular generalized Bannai–Ito algebra that emerges.

Let

$$S_{ij} = [D_i, x_j]. \tag{5.15}$$

We see from (5.5) that $S_{ij} = S_{ji}$, namely, that $[D_i, x_j] = [D_j, x_i]$. It is useful to record for reference the explicit expressions for S_{ij} for $i \leq j$, $i, j = 1, 2, 3$:

$$\begin{aligned}
S_{11} &= 1 + a(1 + R_1R_2)\pi_{12} + a(1 + R_1R_3)\pi_{13} + 2bR_1,\\
S_{22} &= 1 + a(1 + R_1R_2)\pi_{12} + a(1 + R_2R_3)\pi_{23} + 2bR_2,\\
S_{33} &= 1 + a(1 + R_1R_3)\pi_{13} + a(1 + R_2R_3)\pi_{23} + 2bR_3,\\
S_{12} &= -a(1 - R_1R_2)\pi_{12},\\
S_{13} &= -a(1 - R_1R_3)\pi_{13},\\
S_{23} &= -a(1 - R_2R_3)\pi_{23}.
\end{aligned} \tag{5.16}$$

Let

$$M_{ij} = x_iD_j - x_jD_i, \qquad i, j = 1, 2, 3. \tag{5.17}$$

The commutation relation of these Dunkl angular momentum operators have been given in [17] and read

$$\begin{aligned}[M_{ij}, M_{k\ell}] &= M_{i\ell}S_{jk} + M_{jk}S_{i\ell} - M_{ik}S_{\ell j} - M_{j\ell}S_{ik} \\ &= S_{jk}M_{i\ell} + S_{i\ell}M_{jk} - S_{\ell j}M_{ik} - S_{ik}M_{j\ell}.\end{aligned} \tag{5.18}$$

Using $C_{ij} = \{\hat{A}_-, [\hat{A}_+, R_iR_j]\}/4 - R_iR_j/2$, it is straightforward to arrive at the following result.

Proposition 5.1. *The* 2*-index elements that commute with* $\hat{A}_-$ *and* $\hat{A}_+$ *are*

$$\begin{aligned} C_{12} &= M_{12}R_1 + \frac{1}{2}(S_{11} + S_{22} - 1)R_1R_2 - \frac{1}{2}(S_{13} + S_{23}R_1), \\ C_{23} &= M_{23}R_2 + \frac{1}{2}(S_{22} + S_{33} - 1)R_2R_3 - \frac{1}{2}(S_{12}R_3 + S_{13}), \\ C_{13} &= M_{13}R_1R_2 + \frac{1}{2}(S_{11} + S_{33} - 1)R_1R_3 - \frac{1}{2}(S_{12}R_3 + S_{23}R_1). \end{aligned} \tag{5.19}$$

We note that in obtaining the final form of the formulas (5.19), one uses

$$S_{ij}R_iR_j = -S_{ij}, \qquad i \neq j. \tag{5.20}$$

Introduce now the quantities $Q_{ij} = Q_{ji}$ in the group algebra of B_3, which are defined as follows:

$$\begin{aligned} Q_{12} &= \frac{1}{2}(1 + R_1 + R_2 - R_1R_2)\pi_{12}, \\ Q_{13} &= \frac{1}{2}(R_1 + R_2 + R_3 - R_1R_2R_3)\pi_{13}, \\ Q_{23} &= \frac{1}{2}(1 + R_2 + R_3 - R_2R_3)\pi_{23}. \end{aligned} \tag{5.21}$$

One first observes that these Q_{ij} are involutions:

$$Q_{ij}^2 = 1. \tag{5.22}$$

Furthermore, one finds that the algebra they form is isomorphic to the algebra of the permutation operators π_{12}, π_{13}, and π_{23} of the symmetric group S_3. Indeed, one checks that

$$Q_{12}Q_{13} = Q_{23}Q_{12} = Q_{13}Q_{23}, \qquad Q_{12}Q_{23} = Q_{23}Q_{13} = Q_{13}Q_{12} \tag{5.23}$$

together with $Q_{ij}R_j = R_iQ_{ij}$. One discovers also that

$$[Q_{ij}, C_{ij}] = 0, \tag{5.24}$$

and that

$$\begin{aligned} &Q_{12}C_{13} = C_{23}Q_{12}, \quad Q_{12}C_{23} = C_{13}Q_{12}, \quad Q_{13}C_{12} = C_{23}Q_{13}, \\ &Q_{13}C_{23} = C_{12}Q_{13}, \quad Q_{23}C_{12} = C_{13}Q_{23}, \quad Q_{23}C_{13} = C_{12}Q_{23}. \end{aligned} \tag{5.25}$$

Consider now the 1-index elements $C_i = \{\hat{A}_-, [\hat{A}_+, R_i]\}/4 - R_i/2$. One readily finds that

$$\begin{aligned} C_1 &= \frac{1}{2}(S_{11}R_1 + S_{12}R_1R_2 + S_{13}R_1R_2R_3 - R_1), \\ C_2 &= \frac{1}{2}(-S_{12} + S_{22}R_2 - S_{23} - R_2), \\ C_3 &= \frac{1}{2}(-S_{13}R_2 - S_{23} + S_{33}R_3 - R_3). \end{aligned} \tag{5.26}$$

These expressions can be rewritten in terms of the involutions Q_{ij}, $i,j = 1,2,3$. The results read as follows.

Proposition 5.2. *The centralizing* 1*-index elements* C_i, $i = 1,2,3$, *are given by*

$$C_i = a(Q_{ij} + Q_{ik}) + b, \tag{5.27}$$

with $i,j,k \in \{1,2,3\}$ *and all distinct.*

Remark 5.3. Since

$$C_i + C_j - C_k = 2aQ_{ij} + b, \tag{5.28}$$

it follows that the quantities Q_{ij} commute also with $\hat{A}_{\pm}$. We thus see that these involutions enlarge the set of symmetries formed by the operators C_{ij}.

Remark 5.4. One can now verify that

$$C_{ij} = M_{ij}R_iR_{j-1} + C_iR_j + C_jR_i + \frac{1}{2}R_iR_j, \tag{5.29}$$

where $R_iR_{j-1} = R_i$ if $i = j-1$.

Proposition 5.5. *In the realization* (5.7), (5.9) *of* $\mathfrak{osp}(1,2)$ *with the* B_3*-Dunkl operators, the Casimir element* $\Gamma = \frac{1}{2}\big([\hat{A}_-, \hat{A}_+] - 1\big)R$, $R = R_1R_2R_3$, *is given by*

$$\Gamma = M_{12}R_1R_3 + M_{13}R_1 + M_{23}R_1R_2 + \frac{1}{2}(S_{11} + S_{22} + S_{33} - 1)R. \tag{5.30}$$

Proof. This is obtained by a direct computation. □

Remark 5.6. Using (5.19) and (5.26) and invoking property (5.20) again, one can see that (5.30) matches with formula (3.26). Given (3.26), if one simplifies the part $C_iR_jR_k + C_jR_iR_k + C_kR_iR_j = \frac{1}{2}(S_{11} + S_{22} + S_{33} - 3)R$, one may also write Γ in the form

$$\begin{aligned}\Gamma &= C_{12}R_3 + C_{13}R_2 + C_{23}R_1\\ &\quad - \Big(a\Big((1 + R_1R_2)\pi_{12} + (1 + R_1R_3)\pi_{13} + (1 + R_2R_3)\pi_{23}\Big)\\ &\qquad + b(R_1 + R_2 + R_3) + \frac{1}{2}\Big)R.\end{aligned} \tag{5.31}$$

The Jucys–Murphy elements [18,19] of B_n have been given in [20]. For B_3, they are

$$R_1, \quad R_2, \quad R_3, \quad m_2 = (1 + R_1R_2)\pi_{12}, \quad m_3 = (1 + R_1R_3)\pi_{13} + (1 + R_2R_3)\pi_{23}. \tag{5.32}$$

It can be checked that all of these elements commute with each other. The symmetric polynomials in these entities are known to generate the center of the group algebra of B_3. It is thus not surprising that the above Jucys–Murphy elements appear in the expression (5.31) of the central Casimir operator Γ.

We are now ready to determine the specific form that the commutation relations (3.33) and (3.34) take.

Proposition 5.7. *Let* $i,j,k \in \{1,2,3\}$ *be distinct. The defining relation* (3.33) *of the algebra generated by the operators* C_i, C_{ij}, *and* C_{ijk} *respectively given by* (5.27), (5.29), *and* (5.31), *for instance, takes the form*

$$\begin{aligned}\{C_{ij}, C_{jk}\} &= C_{ik} + 2\Gamma\Big(a(Q_{ij} + Q_{jk}) + b\Big) + a^2\big(3\{Q_{ij}, Q_{jk}\} + 2\big)\\ &\quad + 2ab(Q_{ij} + Q_{jk} + 2Q_{ik}) + 2b^2.\end{aligned} \tag{5.33}$$

Proof. This simply follows from (3.33) and the expression of the 1-index elements C_i. Given (5.31), one may directly verify that $\{\Gamma, C_j\} = 2\Gamma C_j$. One shall note that

$$\{Q_{ij} + Q_{ik}, Q_{ik} + Q_{jk}\} = 3\{Q_{ij}, Q_{jk}\} + 2, \tag{5.34}$$

a relation easily derived from (5.23). □

Remark 5.8. The relation (3.34) is readily seen to be identically satisfied in the present realization. Indeed, we have

$$[C_{12}, C_3]+[C_{23}, C_1]+[C_{13}, C_2] = [C_{12}, Q_{13}+Q_{23}]+[C_{23}, Q_{12}+Q_{13}]+[C_{13}, Q_{12}+Q_{23}], \tag{5.35}$$

and one shows that these terms add up to 0 with the help of relations (5.25).

Remark 5.9. We observe that when $a = 0$, the relations (5.33) correspond to the relations (1.1) of the Bannai–Ito algebra, with central structure constants given by $\omega_{12} = \omega_{13} = \omega_{23} = 2\Gamma b + 2b^2$. When $a \neq 0$, we have a remarkable generalization that involves the generators π_{ij} and R_i, $i, j \in \{1, 2, 3\}$ of the signed permutation group on three objects. This hyperoctahedral extension of the Bannai–Ito algebra has C_{ij} and Q_{ij} as generators, with relations given by (5.22)–(5.25) and (5.33), where C_{ijk} is central.

Proposition 5.10. *The generalized Bannai–Ito algebra defined by* (5.33) *admits a Casimir operator C such that*

$$[C, C_{ij}] = [C, C_i] = 0, \tag{5.36}$$

which is given by

$$C = C_{12}^2 + C_{13}^2 + C_{23}^2 - a^2 Q^2 - 4abQ, \tag{5.37}$$

with

$$Q = Q_{12} + Q_{13} + Q_{23}. \tag{5.38}$$

Proof. A direct calculation shows on the one hand that

$$[C_{12}^2 + C_{13}^2 + C_{23}^2, C_{12}] = 3a^2\big[\{Q_{12}, Q_{23}\}, C_{12}\big] + 4ab[Q, C_{12}]. \tag{5.39}$$

It is seen on the other hand that

$$\{Q_{12}, Q_{13}\} = \{Q_{13}, Q_{23}\} = \{Q_{12}, Q_{23}\}, \tag{5.40}$$

and therefore that

$$Q^2 = 3 + 3\{Q_{12}, Q_{13}\}. \tag{5.41}$$

This confirms that C commutes with C_{12}. The same goes for C_{13} and C_{23}. Consider now the commutator $[C, C_1]$. Note that

$$\begin{aligned}[C_{12}^2 + C_{13}^2 + C_{23}^2, C_1] &= a[C_{12}^2 + C_{13}^2 + C_{23}^2, Q_{12} + Q_{13}] \\ &= a[C_{13}^2 + C_{23}^2, Q_{12}] + a[C_{12}^2 + C_{23}^2, Q_{13}],\end{aligned} \tag{5.42}$$

and check that each commutator in the last line vanishes because of (5.25). Observe moreover that

$$[Q, C_1] = a[Q_{13}, Q_{12} + Q_{23}] = 0. \tag{5.43}$$

It follows that $[C, C_1] = 0$. One shows similarly that $[C, C_2] = [C, C_3] = 0$. □

Remark 5.11. We may expect the invariant C to be related to the Casimir operator of $\mathfrak{osp}(1,2)$, and indeed one finds that

$$C = \Gamma^2 + 3(a^2 + b^2) - \frac{1}{4}. \tag{5.44}$$

6. $\mathfrak{osp}(1,2)$ realizations with Clifford algebras

It is known that $\mathfrak{osp}(1,2)$ can be realized using Dunkl operators and generators of Clifford algebras [16]. The centralizer of the superalgebra is then identified with the symmetries of the corresponding Dirac–Dunkl (massless) equation. We now wish to indicate how the formalism developed in this paper applies in the Clifford algebra context and to give the expressions of the operators that then form the generalized Bannai–Ito algebra.

Consider a set of n generators e_i, $i = 1, \ldots, n$, of a Clifford algebra which verify

$$\{e_i, e_j\} = 2\delta_{ij}. \tag{6.1}$$

These e_i are taken to commute with operators acting on functions of the coordinates x_i, $i = 1, \ldots, n$, and in particular with permutation operators. Let $\mathcal{D}_i$ be Dunkl operators associated with some arbitrary reflection group, and take

$$\tilde{A}_- = \sum_{i=1}^{n} \mathcal{D}_i e_i \tag{6.2}$$

and

$$\tilde{A}_+ = \sum_{i=1}^{n} x_i e_i. \tag{6.3}$$

Given that $[\mathcal{D}_i, \mathcal{D}_j] = 0$, it is manifest that

$$\tilde{B}_- = \tilde{A}_-^2 = \sum_{i=1}^{n} \mathcal{D}_i^2 \tag{6.4}$$

and

$$\tilde{B}_+ = \tilde{A}_+^2 = \sum_{i=1}^{n} x_i^2. \tag{6.5}$$

$\tilde{B}_-^2$ and $\tilde{A}_-$ are, respectively, the Laplace–Dunkl and Dirac–Dunkl operators. It can be checked [16] that the operators $\tilde{A}_\pm$ realize the $\mathfrak{osp}(1,2)$ commutation relations $\{\tilde{A}_-, \tilde{A}_+\} = 2\tilde{A}_0$, $[\tilde{A}_0, \tilde{A}_\pm] = \pm\tilde{A}_\pm$ when $\tilde{A}_0$ is taken to be the Euler operator,

$$\tilde{A}_0 = \frac{1}{2}\sum_{i=1}^{n} \{x_i, \mathcal{D}_i\}. \tag{6.6}$$

Provided that $\{R_i, \mathcal{D}_i\} = 0$, $[R_i, \mathcal{D}_j] = 0$, $i \neq j$, the grade involution can again be realized by $R = \prod_{i=1}^{n} R_i$. As already observed, the last equation is enforced for the Dunkl operators associated with the reflection groups $\mathbb{Z}_2^n$ and B_n. In the case of $\mathbb{Z}_2^n$, albeit in a model with Clifford generators, the coproduct embedding of $\mathfrak{osp}(1,2)$ into $\mathfrak{osp}(1,2)^{\otimes 3}$ is realized, and the centralizer will in general be the higher rank Bannai–Ito algebra [10]. We shall therefore focus anew on the B_n-Dunkl operators D_i given in (5.3) and take $n = 3$. Replacing $\mathcal{D}_i$ with D_i in (6.2) and (6.3), we now observe that

$$[\tilde{A}_-, R_i] = 2D_i e_i R_i, \qquad [\tilde{A}_+, R_i] = 2x_i e_i R_i. \tag{6.7}$$

The requirement $\Big[R_i, [R_j, \tilde{A}_\pm]\Big] = 0$, $i \neq j$, is thus satisfied. We can therefore apply our formalism here also to compute the generators of the corresponding centralizer in this model of $\mathfrak{osp}(1,2)$.

Proposition 6.1. *Given the realization* (6.2), (6.3), *and* (6.6), *for* $n = 3$, *the elements* C_i, C_{ij} *and the Casimir operator* $C_{ijk} = \Gamma$ *that satisfy the relations* (3.33) *and* (3.34) *of the generalized Bannai–Ito algebra are given by*

$$C_i = \frac{1}{2}\big(S_{ii} - S_{ij}e_ie_j - S_{ik}e_ie_k - 1\big)R_i, \tag{6.8}$$

$$C_{ij} = -M_{ij}e_ie_jR_iR_j + \frac{1}{2}\big(S_{ii} + S_{jj} - S_{ik}e_ie_k - S_{jk}e_je_k - 1\big)R_iR_j, \tag{6.9}$$

and

$$\Gamma = \left(-M_{12}e_1e_2 - M_{13}e_1e_3 - M_{23}e_2e_3 + \frac{1}{2}\big(S_{11} + S_{22} + S_{33} - 1\big)\right)R, \tag{6.10}$$

with $R = R_1R_2R_3$, *and* $i, j, k \in \{1, 2, 3\}$ *all distinct.*

Proof. These formulas are obtained by an explicit evaluation of

$$C_S = \frac{1}{4}\Big\{\tilde{A}_-, [\tilde{A}_+, R_S]\Big\} - \frac{1}{2}R_S$$

for $S = \{i\}, \{i, j\}, \{i, j, k\}$. □

Remark 6.2. It is readily seen that the expression for $\Gamma = C_{123}$ is in keeping with (3.26) and that we have

$$\Gamma = C_{12}R_3 + C_{13}R_2 + C_{23}R_1 - C_1R_2R_3 - C_2R_1R_3 - C_3R_1R_2 - \frac{1}{2}R_1R_2R_3. \tag{6.11}$$

Remark 6.3. When $a = 0$, the generators become

$$C_{ij} = \Big(-M_{ij}e_ie_j + b(R_i + R_j) + \frac{1}{2}\Big)R_iR_j, \quad C_i = b, \quad D_i = \frac{\partial}{\partial x_i} + \frac{b}{x_i}(1 - R_i). \tag{6.12}$$

We then recover the symmetries of the Dirac–Dunkl equation found in [9] with the parameters of the $\mathbb{Z}_2^3$-Dunkl operators all equal to b.

Let us record another expression for C_i.

Proposition 6.4. *The* 1*-index elements* C_i *can be given as follows:*

$$C_i = a(W_{ij} + W_{ik})e_iR_i + b, \qquad i, j, k \in \{1, 2, 3\} \text{ all distinct}, \tag{6.13}$$

with

$$W_{ij} = \frac{1}{2}\Big((e_i - e_j)\pi_{ij} + (e_i + e_j)R_iR_j\pi_{ij}\Big). \tag{6.14}$$

Proof. This result follows from (6.8) and the definition (5.16). It is the analogue of (5.27). Notice that W_{ij} has mixed symmetry. □

We now wish to discuss the relation that the results of this section have with the study of the symmetries of the Dirac–Dunkl equations carried out in [11]. Let us stress that the grade involution P and its decomposition into $P = \prod_{i=1}^{n} P_i$ are central in our abstract framework. In contradistinction, working exclusively in the realm of Clifford algebras, De Bie, Oste, and Van der Jeugt [11] have designed an alternative method to obtain symmetries of Dirac–Dunkl equations specifically. Let us briefly review their approach, which has the merit of applying to any reflection group.

Take $[n] = \{1, \dots, n\}$ and $S = \{s_1, \dots, s_k\} \subseteq [n]$. In the notation of Section 3, let $x_S = \sum_{i=1}^k x_{s_i} e_{s_i}$, $\mathcal{D}_S = \sum_{i=1}^k \mathcal{D}_{s_i} e_{s_i}$, and $e_S = \prod_{i=1}^k e_{s_i}$. Script letters will be used in the following to identify quantities that pertain to the generic Dunkl operator $\mathcal{D}_i$. As shown in [11], the quantities

$$\mathcal{O}_S = \frac{1}{2}\Big(\mathcal{D}_{[n]} x_S e_S - e_S x_S \mathcal{D}_{[n]} - e_S\Big) \tag{6.15}$$

either commute or anticommute with $\mathcal{D}_{[n]}$ and $x_{[n]}$ for all S, depending on the cardinality $|S|$ of S; namely,

$$\mathcal{D}_{[n]}\mathcal{O}_S = (-1)^{|S|}\mathcal{O}_S\mathcal{D}_{[n]} \quad \text{and} \quad x_{[n]}\mathcal{O}_S = (-1)^{|S|}\mathcal{O}_S\, x_{[n]}. \tag{6.16}$$

The quantities $\mathcal{O}_S$ and the algebra they form are the objects of consideration in [11]. Our C_S and their algebraic properties are a priori distinct.

Now, for $n = 3$, $i, j, k \in \{1, 2, 3\}$ and unequal, a simple application of formula (6.16) gives

$$\mathcal{O}_{ij} = \mathcal{M}_{ij} + \frac{1}{2}\big(\mathcal{S}_{ii} + \mathcal{S}_{jj}\big)e_i e_j - \frac{1}{2}\mathcal{S}_{ik} e_j e_k + \frac{1}{2}\mathcal{S}_{jk} e_i e_k - \frac{1}{2} e_i e_j \tag{6.17}$$

with

$$\mathcal{M}_{ij} = x_i \mathcal{D}_j - x_j \mathcal{D}_i \quad \text{and} \quad \mathcal{S}_{ij} = [x_i, \mathcal{D}_j]. \tag{6.18}$$

This operator $\mathcal{O}_{ij}$, according to (6.16), will commute with $\tilde{A}_- = \mathcal{D}_{[n]}$ and $\tilde{A}_+ = x_{[n]}$. As it happens, when the $\mathcal{D}_i$ are the B_3-Dunkl operators D_i, this would seemingly give centralizing elements that differ from those that we have already found and have given in Proposition 6.1. This is reconciled by remarking that there are additional centralizing elements or symmetries when the Dunkl operators at hand transform like derivatives under the reflections, a case in point for the B_n-Dunkl operators. When this is so, it is readily verified [9] that the operators

$$Z_i = e_i R_i \tag{6.19}$$

will obey

$$\{\tilde{A}_\pm, Z_i\} = 0. \tag{6.20}$$

Products $Z_i Z_j$ will hence be symmetries. Focus now on quantities $\mathcal{O}_S$ associated with the B_3-Dunkl operators D_i. In light of the previous observation, upon comparing (6.9) and (6.17), we find that

$$C_{ij} = \mathcal{O}_{ij} e_i e_j R_i R_j; \tag{6.21}$$

in other words, we see that C_{ij} and $\mathcal{O}_{ij}$ differ by a factor which is itself a symmetry given that the B_3-Dunkl operators satisfy (5.6).

Similarly, we can show that

$$C_i = \mathcal{O}_i e_i R_i. \tag{6.22}$$

Therefore, by multiplying two quantities $\mathcal{O}_i$ and $e_i R_i$ that anticommute with $\tilde{A}_\pm$, we find the centralizing C_i. This explains the relation between the two approaches when our formalism is applied to $\mathfrak{osp}(1,2)$ realizations with Clifford algebras.

7. Conclusion

This paper has introduced generalizations of the Bannai–Ito algebra that are of rank 1. They are intimately connected to the Lie superalgebra $\mathfrak{osp}(1,2)$. It has been shown that whenever the grade involution P of $\mathfrak{osp}(1,2)$ factors into a product of appropriate involutions, elements that centralize $\mathfrak{osp}(1,2)$ appear.

When P admits three such factors, $P = P_1P_2P_2$, the centralizer thus formed generalizes the Bannai–Ito algebra. Realizations of $\mathfrak{osp}(1,2)$ in terms of Dunkl operators have been considered to obtain concrete models of generalized Bannai–Ito algebras. This has produced in particular an hyperoctahedral extension of the Bannai–Ito algebra that exhibits an interesting structure.

This study brings up many questions. Determining which generalized Bannai–Ito algebras other realizations of $\mathfrak{osp}(1,2)$ would entail naturally comes to mind. Also among those questions is, what would the approach bring when applied to other superalgebras or more complicated structures? The number of supplementary involutions has been restricted to 3 in order to obtain centralizers of rank 1. Higher rank cases will result upon lifting this constraint. As a matter of fact, the hyperoctahedral generalization of arbitrary rank will be presented in a forthcoming publication [21] from the perspective of the rational B_n-Calogero model for nonidentical particles. Developing the representation theory of this B_n-extended Bannai–Ito algebra would be instructive. One expects relations with special functions in one and many variables extending the connection that the Bannai–Ito polynomials have with the eponym algebra.

We look forward to exploring these avenues in the near future.

Appendix A

We here give more details on how Theorem 3.8 is proven along with the strategy described in Section 3 after the statement of the theorem. Generally, as one proceeds with the proofs, it is appropriate to use the following expanded form,

$$C_S = \frac{1}{4}\Big(A_-A_+P_S - A_-P_SA_+ + A_+P_SA_- - P_SA_+A_-\Big) - \frac{1}{2}P_S, \tag{A.1}$$

for the centralizing elements.

Sketch of proof of (3.33). Consider the left-hand side (l.h.s.) of (3.33). In expanding $\{C_{ij}, C_{jk}\}$, the first thing to do is to eliminate the involutions P_j with index j by using formulas such as (3.23) to bring the two P_j factors together and make them disappear, given that $P_j^2 = 1$. It is immediate to check that in (3.33), the terms without $A_\pm$ involving only P_iP_k match on both sides.

Let us now focus on the terms that are bilinear in A_+ and A_-. In the anticommutators, these occur in the cross terms with the pure involution part of the relevant C_S as a factor. Start with the right-hand side (r.h.s.). Using $\{P, A_\pm\} = 0$ and formula (3.23) repeatedly, one finds that the bilinear terms in $C_{ik} + \{C_j, C_{ijk}\} + \{C_i, C_k\}$ simplify to

$$\begin{aligned}\text{bilinears on r.h.s.} &= \frac{3}{8}\big[P_iP_k, A_-A_+\big] - \frac{1}{8}\big[P_iP_k, A_+A_-\big] + \frac{1}{4}A_+P_iP_kA_- \\ &\quad - \frac{1}{4}A_-P_iP_kA_+ + \frac{1}{8}P_i[A_+, A_-]P_k + \frac{1}{8}P_k[A_+, A_-]P_i.\end{aligned} \tag{A.2}$$

This is seen to coincide with what is found after reducing with the same tools the sum of the bilinears in $\{C_{ij}, C_{jk}\}$, the l.h.s. Remaining is the part that is quartic in the odd generators. Let us describe for example how one deals with the terms of the form $A_+A_-^2A_+$. Recall that $[P_i, A_-^2] = [P_j, A_-^2] = [P_k, A_-^2] = 0$. After the elimination of P_j, the terms featuring A_-^2 in $\{C_{ij}, C_{jk}\}$ are found to be

$$-\frac{1}{16}\Big(A_+A_-^2P_iA_+P_k + A_+A_-^2P_iP_kA_+ + P_iA_+A_-^2A_+P_k + P_iA_+A_-^2P_kA_+ + (i \leftrightarrow j)\Big). \tag{A.3}$$

The terms in A_-^2 in $\{C_i, C_k\}$ are easily read off and found to coincide with those of $\{C_{ij}, C_{jk}\}$ given in (A.3) except that the first and last have opposite signs. This is nicely compensated for by the corresponding terms in $\{C_j, C_{ijk}\}$. Indeed, collecting the terms in A_-^2 in

$$\begin{aligned}\{C_j, C_{ijk}\} &= \Big\{\frac{1}{4}(A_-A_+P_j - A_-P_jA_+ + A_+P_jA_- - P_jA_+A_-) \\ &\quad -\frac{1}{2}P_j, \frac{1}{2}(A_-A_+ - A_+A_-)P - \frac{1}{2}P\Big\}\end{aligned} \tag{A.4}$$

yields

$$\begin{aligned}&\frac{1}{8}\Big(-A_+P_iP_kA_-^2A_+ - P_iP_kA_+A_-^2A_+ - A_+A_-^2A_+P_iP_k - A_+A_-^2P_iP_kA_+\Big) \\ &\quad = -\frac{1}{8}\Big(A_+A_-^2P_iA_+P_k + P_iA_+P_kA_-^2A_+ + (i \leftrightarrow j)\Big).\end{aligned} \tag{A.5}$$

It thus follows that the terms in A_-^2 in $\{C_j, C_{ijk}\} + \{C_i, C_k\}$ are equal to those of $\{C_{ij}, C_{jk}\}$.

One then follows the same approach with the other types in the quartic part to complete the proof of relation (3.33). Note that when dealing with the $(A_-A_+)^2$ and $(A_+A_-)^2$ terms jointly, the ones found in $\{C_j, C_{ijk}\}$ read

$$\begin{aligned}&\frac{1}{8}\Big(A_-A_+P_iP_kA_-A_+ + A_-A_+A_-P_iP_kA_+ + A_-P_iP_kA_+A_-A_+ + A_-A_+A_-A_+P_iP_k\Big) \\ &\quad + \frac{1}{8}\left(A_+P_iP_kA_-A_+A_- + A_+A_-P_iP_kA_+A_- + P_iP_kA_+A_-A_+A_- \right. \\ &\qquad \left. + A_+A_-A_+P_iP_kA_-\right).\end{aligned} \tag{A.6}$$

The identity (2.15), as well as (2.14), is here needed to rewrite those terms in the form

$$\begin{aligned}&\frac{1}{8}\left(A_-A_+P_iA_-P_kA_+ + A_-P_iA_+A_-A_+P_k + A_+P_iA_-P_kA_+A_- \right. \\ &\qquad \left. + P_iA_+A_-A_+P_kA_- + (i \leftrightarrow k)\right).\end{aligned} \tag{A.7}$$

They are then seen to perfectly combine with the analogous terms in $\{C_i, C_k\}$ to equal what is found in $\{C_{ij}, C_{jk}\}$. □

Sketch of proof of (3.34). It is immediate to see that the terms involving only the involutions trivially vanish. Recall that $C_{ij} = \frac{1}{4}\{A_-, [A_+, P_iP_j]\} - \frac{1}{2}P_iP_j$. From

Lemma 3.6 and its specialization as formula (3.24) for $n = 3$, we see that the bilinear part in A_- and A_+ of (3.34) reduces to

(A.8)

$$-\frac{1}{8}\Big[A_-A_+P_iP_j - A_-P_iP_jA_+ + A_+P_iP_jA_- - P_iP_jA_+A_-, P_k\Big] + \text{cyclic permutations.}$$

Expanding over the cyclic permutations and combining terms to use (2.14), one arrives at

(A.9)

$$-\frac{1}{8}\Big[3A_-A_+P + 3PA_+A_- - P_iP_j\{A_+, A_-\}P_k - P_jP_k\{A_+, A_-\}P_i - P_iP_k\{A_+, A_-\}P_j\Big],$$

which adds up to 0 since the involutions commute with $\{A_+, A_-\} = 2A_0$. □

The quartic part is dealt with as in the proof of (3.33) by looking separately at the various orderings. One regroups terms arising from the different permutations and uses formula (2.14) to shift the involutions and observe that all terms cancel.

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