

THE ASCENDING CENTRAL SERIES OF NILPOTENT LIE ALGEBRAS WITH COMPLEX STRUCTURE

ADELA LATORRE, LUIS UGARTE, AND RAQUEL VILLACAMPA

ABSTRACT. We obtain several restrictions on the terms of the ascending central series of a nilpotent Lie algebra \mathfrak{g} under the presence of a complex structure J . In particular, we find a bound for the dimension of the center of \mathfrak{g} when it does not contain any non-trivial J -invariant ideal. Thanks to these results, we provide a structural theorem describing the ascending central series of 8-dimensional nilpotent Lie algebras \mathfrak{g} admitting this particular type of complex structure J . Since our method is constructive, it allows us to describe the complex structure equations that parametrize all such pairs (\mathfrak{g}, J) .

CONTENTS

1. Introduction	3867
2. Classes of complex structures on a nilpotent Lie algebra \mathfrak{g}	3869
3. Strongly non-nilpotent complex structures	3872
4. The 8-dimensional case	3885
Acknowledgments	3901
References	3901

1. INTRODUCTION

Let \mathfrak{g} be an even-dimensional real Lie algebra. A complex structure on \mathfrak{g} is an endomorphism $J: \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $J^2 = -\text{Id}$ and the “Nijenhuis condition”

$$(1) \quad N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0$$

for all $X, Y \in \mathfrak{g}$. Finding Lie algebras endowed with such structures constitutes an interesting problem with important algebraic and geometrical applications. In the last few years several results on classifications of Lie algebras \mathfrak{g} admitting complex structures J have been published, mainly dealing with low dimensions. In the 4-dimensional case, the solvable Lie algebras admitting a complex structure were classified by Ovando in [19]. Concerning dimension 6, Andrada, Barberis, and Dotti classified in [1] the pairs (\mathfrak{g}, J) , where \mathfrak{g} is any Lie algebra and J is an abelian complex structure; i.e., J satisfies $[JX, JY] = [X, Y]$ for every $X, Y \in \mathfrak{g}$. The classification of 6-dimensional nilpotent Lie algebras that admit a complex structure

Received by the editors September 21, 2017.

2010 *Mathematics Subject Classification*. Primary 17B30; Secondary 53C30, 53C15.

This work was partially supported by the projects MINECO (Spain) MTM2014-58616-P and Gobierno de Aragón/Fondo Social Europeo–Grupo Consolidado E15 Geometría.

(not necessarily abelian) was achieved by Salamon in [20]. Later, the different complex structures on each of these algebras were classified by Ceballos, Otal, Ugarte, and Villacampa in [6]. In the 6-dimensional solvable case, complex structures of certain specific types are studied in [3] and [13]. The existence of complex structures on 6-dimensional product Lie algebras has been recently investigated in [10]. However, apart from some classifications of complex parallelizable structures, i.e., those coming from complex Lie algebras (see e.g. [15, 18]), little is yet known in real dimensions higher than six.

In this paper we focus on nilpotent Lie algebras. Fixing an even real dimension $2n$, it seems clear that the knowledge of a classification of $2n$ -dimensional nilpotent Lie algebras could be of great help in the study of the existence of complex structures. Nonetheless, complete classifications are only known up to dimension 7 (see [15] and the references therein). Hence, in order to investigate the pairs (\mathfrak{g}, J) in dimensions $2n \geq 8$, other techniques are needed. In fact, when dealing with high dimensional cases, most of the efforts have been directed to obtain algebraic constraints to the existence of complex structures; see for instance the papers [5, 14, 16] for quasi-filiform Lie algebras and the results in [17] bounding the nilpotency step of nilpotent Lie algebras admitting complex structures. Although some partial classifications are achieved in eight dimensions, they generally require additional conditions, such as the existence of hypercomplex structures [11], SKT metrics [12], or balanced metrics compatible with abelian complex structures [2]. Some other partial results on the construction of complex structures have been recently obtained in [4].

Our main objective here is to develop a method for constructing all complex structures on even-dimensional nilpotent Lie algebras \mathfrak{g} . This method will allow us to obtain several restrictions on the terms of the ascending central series of \mathfrak{g} imposed by the existence of a complex structure. Among them, we prove an upper bound on the dimension of the center of \mathfrak{g} when this subspace does not contain any non-trivial J -invariant ideal. In such case, a structural theorem in eight dimensions is provided, together with a parametrization of the space of complex structures. We next explain in detail the contents of the paper.

In Section 2 we consider a partition of the space of complex structures J on a nilpotent Lie algebra \mathfrak{g} into *quasi-nilpotent* and *strongly non-nilpotent* structures (see Definition 2.2 and Figure 1). The first class is given by those J 's for which there exists a non-trivial J -invariant ideal in the center of \mathfrak{g} . We notice that this class contains those complex structures of nilpotent type [9]. The second class appeared for the first time in [8], and we will simply refer to it as the class of S_nN complex structures. After showing that quasi-nilpotent complex structures can be constructed as a certain extension of lower dimensional structures, we notice that the essentially new complex structures that arise in each even real dimension are the S_nN ones. For this reason, in the rest of the paper we mainly focus on these complex structures.

In Section 3 we obtain algebraic constraints to the existence of strongly non-nilpotent complex structures J on a nilpotent Lie algebra \mathfrak{g} in terms of its ascending central series $\{\mathfrak{g}_k\}_k$. One can think of S_nN complex structures as those J 's for which the spaces \mathfrak{g}_k are far from being J -invariant. This will be reflected in the fact that the center \mathfrak{g}_1 of \mathfrak{g} is rather small, whereas the nilpotency step of \mathfrak{g} can be rather large. Indeed, in Section 3.1 we prove, among other restrictions for the terms \mathfrak{g}_k ,

that the nilpotency step s of \mathfrak{g} satisfies $s \geq 3$, and in Section 3.2 we obtain an upper bound for the dimension of \mathfrak{g}_1 . An application to the study of existence of complex structures on products of nilpotent Lie algebras is given in Section 3.3.

Section 4 is devoted to dimension 8. The main result is Theorem 4.1, which gives the structure of the ascending central series of 8-dimensional nilpotent Lie algebras \mathfrak{g} admitting an S_n -complex structure J . For the proof of this result we construct a *doubly adapted* basis for each pair (\mathfrak{g}, J) (see Definition 4.11) which allows us to explicitly describe the terms \mathfrak{g}_k in the ascending central series of \mathfrak{g} . Indeed, in eight dimensions one has that $\dim \mathfrak{g}_1 = 1$ and $\mathfrak{g}_5 = \mathfrak{g}$ (since $s \leq 5$). In Section 4.1 we focus on the space \mathfrak{g}_2 and find its possible dimensions, together with its description in terms of appropriate generators. A similar analysis is made in Section 4.2 for \mathfrak{g}_3 and \mathfrak{g}_4 . Finally, since our method is constructive, it allows us to describe in Section 4.3 the complex structure equations that parametrize all the strongly non-nilpotent complex structures on nilpotent Lie algebras of dimension 8.

2. CLASSES OF COMPLEX STRUCTURES ON A NILPOTENT LIE ALGEBRA \mathfrak{g}

In this section we introduce different classes of complex structures on a nilpotent Lie algebra \mathfrak{g} , and we show that those complex structures J for which the center of \mathfrak{g} contains a non-trivial J -invariant ideal are a certain extension of lower dimensional pairs $(\tilde{\mathfrak{g}}, \tilde{J})$. As a consequence, we arrive at the fact that the essentially new complex structures that arise in each even real dimension are the so-called strongly non-nilpotent complex structures.

The *ascending central series* of a Lie algebra \mathfrak{g} is given by $\{\mathfrak{g}_k\}_k$, where

$$(2) \quad \begin{cases} \mathfrak{g}_0 = \{0\} \text{ and} \\ \mathfrak{g}_k = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{g}_{k-1}\} \text{ for } k \geq 1. \end{cases}$$

Observe that $\mathfrak{g}_1 = Z(\mathfrak{g})$ is the center of \mathfrak{g} . The Lie algebra \mathfrak{g} is said to be *nilpotent* if there is an integer $s \geq 1$ such that $\mathfrak{g}_k = \mathfrak{g}$ for every $k \geq s$. In this case, the smallest integer s satisfying the latter condition is called the *nilpotency step* of \mathfrak{g} , and the Lie algebra is said to be *s-step nilpotent*.

Let \mathfrak{g} be a nilpotent Lie algebra (NLA for short) of real dimension $2n$. We introduce the following terminology.

Definition 2.1. An NLA \mathfrak{g} has ascending type (m_1, \dots, m_s) if s is the nilpotency step of \mathfrak{g} and $m_k = \dim \mathfrak{g}_k$ for each $1 \leq k \leq s$ being \mathfrak{g}_k the terms in the ascending central series $\{\mathfrak{g}_k\}_k$ of \mathfrak{g} .

Thus, any NLA \mathfrak{g} has an associated s -tuple

$$(m_1, \dots, m_{s-1}, m_s) := (\dim \mathfrak{g}_1, \dots, \dim \mathfrak{g}_{s-1}, \dim \mathfrak{g}_s)$$

which strictly increases, i.e., $0 < m_1 < \dots < m_{s-1} < m_s = 2n$. Notice that m_1 is precisely the dimension of the center of \mathfrak{g} . Moreover, since \mathfrak{g} is nilpotent one has that $\dim \mathfrak{g}_{s-1} \leq 2(n-1)$, and the nilpotency step verifies $s \leq 2n-1$.

Obviously, NLAs with different ascending type are non-isomorphic. However, the converse is only true up to real dimension 4. Indeed, there are three non-isomorphic 4-dimensional NLAs whose ascending types are (4), (2, 4), and (1, 2, 4). In contrast, there exist four non-isomorphic 6-dimensional NLAs with the same ascending type (2, 6) (see for instance [7], where the ascending type of every 6-dimensional NLA is given).

We observe that the existence of a complex structure J on an NLA \mathfrak{g} imposes restrictions on the ascending type of \mathfrak{g} . These restrictions are completely known up to dimension 6 (see [7, 20, 21]), but they are still to be understood in higher dimensions. With this goal in mind, we will first reduce the problem to a specific class of complex structures that we will next introduce.

Let J be a complex structure on \mathfrak{g} , that is, an endomorphism $J: \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $J^2 = -\text{Id}$ and the integrability condition (1). As was observed in [9], the terms \mathfrak{g}_k in the series (2) may not be invariant under J . For this reason, a new series $\{\mathfrak{a}_k(J)\}_k$ adapted to the complex structure J is introduced in [9], namely:

$$(3) \quad \begin{cases} \mathfrak{a}_0(J) = \{0\} & \text{and} \\ \mathfrak{a}_k(J) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{a}_{k-1}(J) \text{ and } [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{k-1}(J)\} & \text{for } k \geq 1. \end{cases}$$

We will refer to the series $\{\mathfrak{a}_k(J)\}_k$ as the *ascending J -compatible series of \mathfrak{g}* .

Note that every $\mathfrak{a}_k(J) \subseteq \mathfrak{g}_k$ is an even-dimensional J -invariant ideal of \mathfrak{g} satisfying $0 \leq \dim \mathfrak{a}_k(J) \leq m_k$ for every $k \geq 1$. In particular, $\mathfrak{a}_1(J)$ is the largest subspace of the center \mathfrak{g}_1 which is J -invariant.

Definition 2.2. A complex structure J on an NLA \mathfrak{g} is said to be:

- (i) *strongly non-nilpotent*, or *S_nN* for short, if $\mathfrak{a}_1(J) = \{0\}$;
- (ii) *quasi-nilpotent* if it satisfies $\mathfrak{a}_1(J) \neq \{0\}$; moreover, J will be called
 - (ii.1) *nilpotent* if there exists an integer $t > 0$ such that $\mathfrak{a}_t(J) = \mathfrak{g}$,
 - (ii.2) *weakly non-nilpotent* if there is an integer $t > 0$ such that $\mathfrak{a}_t(J) = \mathfrak{a}_l(J)$ for every $l \geq t$ and $\mathfrak{a}_t(J) \neq \mathfrak{g}$.

The notion of nilpotent complex structure was first introduced and studied in [9], whereas that of strongly non-nilpotent appeared for the first time in [8].

Let us remark that the first division above is based on whether the ascending J -compatible series $\{\mathfrak{a}_k(J)\}_k$ of \mathfrak{g} is trivially zero or not. Also notice that non-nilpotent structures are those satisfying $\mathfrak{a}_k(J) \neq \mathfrak{g}$, for every $k \geq 1$, and they can be either weakly or strongly non-nilpotent.

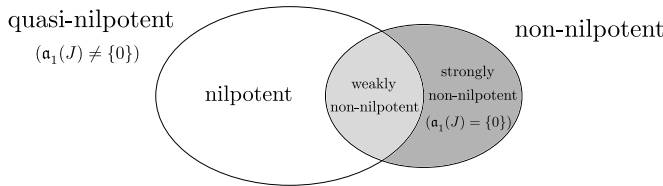


FIGURE 1. Partition of the space of complex structures.

It is well known that in dimension 4, any complex structure on an NLA is nilpotent. In dimension 6, any complex structure is either nilpotent or strongly non-nilpotent. Moreover, these two classes cannot coexist on the same 6-dimensional NLA (see [7, 20, 21]).

We next show that weakly non-nilpotent complex structures appear for the first time in dimension 8. Furthermore, we illustrate that different classes of complex structures can coexist on the same NLA when the dimension is greater than 6.

Example 2.3. Let \mathfrak{g} be the 8-dimensional NLA given by the basis $\{X_1, \dots, X_8\}$, in terms of which the non-zero brackets are

$$[X_1, X_3] = [X_2, X_4] = X_6, \quad [X_3, X_5] = -X_1, \quad [X_4, X_5] = -X_2.$$

The ascending central series (2) of \mathfrak{g} is the following one:

$$\mathfrak{g}_1 = \langle X_6, X_7, X_8 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_6, X_7, X_8 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g},$$

which implies that \mathfrak{g} has ascending type (3, 5, 8).

Let J be the almost complex structure defined by

$$JX_1 = X_2, \quad JX_3 = X_4, \quad JX_5 = X_6, \quad JX_7 = X_8.$$

It is easy to see that $N_J \equiv 0$, i.e., J is integrable, and the ascending J -compatible series (3) of \mathfrak{g} satisfies $\mathfrak{a}_l(J) = \mathfrak{a}_l(J) = \langle X_7, X_8 \rangle$ for every $l \geq 1$. Hence, the complex structure J is weakly non-nilpotent.

Let us observe that we can also define a nilpotent complex structure \hat{J} on \mathfrak{g} as follows:

$$\hat{J}X_1 = X_2, \quad \hat{J}X_3 = X_4, \quad \hat{J}X_5 = X_8, \quad \hat{J}X_6 = -X_7.$$

Indeed, $N_{\hat{J}} \equiv 0$ and $\mathfrak{a}_1(\hat{J}) = \langle X_6, X_7 \rangle$, $\mathfrak{a}_2(\hat{J}) = \langle X_1, X_2, X_6, X_7 \rangle$, and $\mathfrak{a}_3(\hat{J}) = \mathfrak{g}$.

The following example shows that even nilpotent and strongly non-nilpotent complex structures can coexist on the same NLA. It should be noted that the dimension of the Lie algebra is 10 (see Corollary 3.13, where this is proved to be the smallest dimension where this phenomenon can occur).

Example 2.4 ([9]). Let \mathfrak{g} be the nilpotent Lie algebra of dimension 10 defined by a basis $\{X_1, \dots, X_{10}\}$ with non-zero brackets:

$$[X_3, X_9] = [X_4, X_{10}] = X_1, \quad [X_5, X_9] = [X_6, X_{10}] = X_2,$$

$$[X_7, X_9] = X_3, \quad [X_7, X_{10}] = X_4, \quad [X_8, X_9] = X_5, \quad [X_8, X_{10}] = X_6.$$

It is easy to see that $\mathfrak{g}_1 = \langle X_1, X_2 \rangle$, $\mathfrak{g}_2 = \langle X_1, X_2, X_3, X_4, X_5, X_6 \rangle$, and $\mathfrak{g}_3 = \mathfrak{g}$. Hence, \mathfrak{g} has ascending type (2, 6, 10).

The almost complex structure J given by

$$JX_1 = -X_7, \quad JX_2 = -X_8, \quad JX_3 = X_4, \quad JX_5 = X_6, \quad JX_9 = X_{10}$$

is integrable. Moreover, J is strongly non-nilpotent since $\mathfrak{a}_1(J) = \{0\}$.

On the other hand, \hat{J} defined by

$$\hat{J}X_1 = X_2, \quad \hat{J}X_3 = X_4, \quad \hat{J}X_5 = X_6, \quad \hat{J}X_7 = X_8, \quad \hat{J}X_9 = X_{10}$$

is also a complex structure on \mathfrak{g} , and it satisfies $\mathfrak{a}_1(\hat{J}) = \mathfrak{g}_1$, $\mathfrak{a}_2(\hat{J}) = \mathfrak{g}_2$, and $\mathfrak{a}_3(\hat{J}) = \mathfrak{g}$. Therefore, \hat{J} is a nilpotent complex structure on \mathfrak{g} .

One can see from Definition 2.2 that complex structures J of quasi-nilpotent type on an NLA \mathfrak{g} are, up to a certain point, compatible with the nilpotent structure of \mathfrak{g} . One can take advantage of this fact, as we next show following the ideas in [8, 9].

Let J be a complex structure on an NLA \mathfrak{g} such that $\mathfrak{a}_1(J) \neq \{0\}$. The ascending J -compatible series of \mathfrak{g} satisfies

$$\{0\} = \mathfrak{a}_0(J) \subsetneq \mathfrak{a}_1(J) \subsetneq \dots \subsetneq \mathfrak{a}_{t-1}(J) \subsetneq \mathfrak{a}_t(J) = \mathfrak{a}_l(J), \quad \text{for } l \geq t,$$

t being the smallest integer for which the series stabilizes. It is worth noting that t does not necessarily coincide with the nilpotency step s of the Lie algebra \mathfrak{g} (see Example 2.3). Nonetheless, one always has $t \leq s$.

We consider the sequence of quotient Lie algebras

$$\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}_1(J) \longrightarrow \cdots \longrightarrow \mathfrak{g}/\mathfrak{a}_q(J) \xrightarrow{\pi_{q+1}} \mathfrak{g}/\mathfrak{a}_{q+1}(J) \longrightarrow \cdots \longrightarrow \mathfrak{g}/\mathfrak{a}_t(J),$$

where π_{q+1} is the natural projection onto $\mathfrak{g}/\mathfrak{a}_{q+1}(J)$, with $\ker \pi_{q+1} = \mathfrak{a}_{q+1}(J)/\mathfrak{a}_q(J)$. For the sake of simplicity, we will denote $\tilde{\mathfrak{g}}_q = \mathfrak{g}/\mathfrak{a}_q(J)$ for each $1 \leq q \leq t$. Observe that the Lie algebras $\tilde{\mathfrak{g}}_q$ are nilpotent. Moreover, one has that the last term in the sequence above satisfies $\tilde{\mathfrak{g}}_t = \{0\}$ when J is a complex structure of nilpotent type, and $\tilde{\mathfrak{g}}_t \neq \{0\}$ otherwise.

It is easy to see that the complex structure J defined on \mathfrak{g} induces in a natural way a complex structure \tilde{J}_q on $\tilde{\mathfrak{g}}_q$, for $1 \leq q \leq t$, as follows:

$$\tilde{J}_q(\tilde{X}) := \widetilde{JX}, \quad \text{for } \tilde{X} \in \tilde{\mathfrak{g}}_q,$$

where \tilde{X} and \widetilde{JX} denote the classes of X and JX , respectively, in the quotient $\tilde{\mathfrak{g}}_q$. Therefore,

- if J is nilpotent, then $(\tilde{\mathfrak{g}}_t, \tilde{J}_t)$ is a (complex) abelian Lie algebra;
- if J is weakly non-nilpotent, then $(\tilde{\mathfrak{g}}_t, \tilde{J}_t)$ is an NLA endowed with a strongly non-nilpotent complex structure.

Hence, in both cases, the given pair (\mathfrak{g}, J) can be recovered as a finite sequence of complex extensions starting from the lower dimensional pair $(\tilde{\mathfrak{g}}_t, \tilde{J}_t)$. That is to say, quasi-nilpotent complex structures can be constructed from lower dimensions. Hence, the essentially new complex structures that arise in each even real dimension are those of strongly non-nilpotent type.

As we have previously said, SnN complex structures appear in dimensions ≥ 6 . In fact, in six dimensions one has the following structural result.

Theorem 2.5 ([21, 22]). *Let \mathfrak{g} be an NLA of dimension 6. If \mathfrak{g} admits an SnN complex structure, then \mathfrak{g} has ascending type $(1, 3, 6)$ or $(1, 3, 4, 6)$.*

This allows us to immediately assert that an NLA \mathfrak{g} with ascending type, for instance $(1, 4, 6)$, does not admit any complex structure. Indeed, if we assume the converse, then the complex structure J on \mathfrak{g} must be SnN, as the center of \mathfrak{g} is 1-dimensional, and consequently $\mathfrak{a}_1(J) = \{0\}$. However, this is not possible by Theorem 2.5.

Our goal in the next sections is to find, in any even dimension, new restrictions on the ascending type (m_1, \dots, m_s) of an NLA \mathfrak{g} endowed with a strongly non-nilpotent complex structure J . One of the main consequences of our study is a structural result for 8-dimensional NLAs admitting an SnN complex structure (see Theorem 4.1), thus providing an analogous result to Theorem 2.5 in eight dimensions.

We finish this section by noting that the existence of a complex structure on an NLA is characterized in [20] in terms of the existence of a certain complex $(1,0)$ -basis for $\mathfrak{g}_{\mathbb{C}}^*$. However, we will not make use of this point of view here, and we will instead develop a more constructive approach to the problem.

3. STRONGLY NON-NILPOTENT COMPLEX STRUCTURES

Let us focus our attention on those complex structures J on \mathfrak{g} satisfying $\mathfrak{a}_1(J) = \{0\}$. In this case, the construction of the pairs (\mathfrak{g}, J) cannot be based on lower dimensional classifications and another approach is needed. In this section, we will show that these (\mathfrak{g}, J) can be found using the ascending central series and

constructing an appropriate J -adapted basis of \mathfrak{g} . In the first part of the section, we provide some general results about the ascending central series $\{\mathfrak{g}_k\}_k$ of those NLAs \mathfrak{g} admitting SnN complex structures. In particular, we see that the nilpotency step of \mathfrak{g} must be at least 3. Then, we focus on the term \mathfrak{g}_1 , which corresponds to the center of \mathfrak{g} , and find an upper bound for its dimension. In the last part, our results are applied to the study of existence of complex structures on products of NLAs.

3.1. Restrictions on the ascending central series $\{\mathfrak{g}_k\}_k$. We introduce here a collection of results showing that the ascending central series of an NLA \mathfrak{g} is restricted under the existence of a complex structure J on \mathfrak{g} . We also obtain several algebraic obstructions for an NLA \mathfrak{g} to admit complex structures.

Let us start with some results where J can be a complex structure of any type (recall Definition 2.2).

Lemma 3.1. *Let (\mathfrak{g}, J) be a $2n$ -dimensional NLA endowed with a complex structure. Suppose that there exists a subspace $\mathcal{W} \subset \mathfrak{g}_k$, where $k > 1$, such that $\dim \mathcal{W} = n$ and $\mathcal{W} \cap J\mathcal{W} = \{0\}$. The following hold:*

- (i) *if $\mathfrak{g}_{k-1} = J\mathfrak{g}_{k-1}$, then $\mathfrak{g}_k = \mathfrak{g}$;*
- (ii) *if there is $X \in \mathcal{W}$ such that $JX \in \mathfrak{g}_{k+1}$ and $JX \notin \mathfrak{g}_k$, then there exists $Y \in \mathfrak{g}_{k-1}$ such that $JY \in \mathfrak{g}_k$ and $JY \notin \mathfrak{g}_{k-1}$.*

Proof. Let $\{X_i\}_{i=1}^n$ be a basis for \mathcal{W} . Observe that $X_i \notin J\mathcal{W}$, for each $1 \leq i \leq n$, and $\{X_i, JX_i\}_{i=1}^n$ gives a basis of \mathfrak{g} . First, let us note that $X_i \in \mathfrak{g}_k$, for every $1 \leq i \leq n$, implies that

$$[X_i, X_j], [X_i, JX_j] \in \mathfrak{g}_{k-1} \quad \forall j = 1, \dots, n.$$

For the brackets $[JX_i, JX_j]$, considering the Nijenhuis condition (1) and the hypothesis in (i), we obtain

$$[JX_i, JX_j] = [X_i, X_j] + J[JX_i, X_j] + J[X_i, JX_j] \in \mathfrak{g}_{k-1} + J\mathfrak{g}_{k-1} = \mathfrak{g}_{k-1}$$

for all $1 \leq i, j \leq n$. Thus, we can easily conclude that $JX_1, \dots, JX_n \in \mathfrak{g}_k$ and $\mathfrak{g}_k = \mathfrak{g}$.

For the second part, let us suppose that there exists $X \in \mathcal{W} \subset \mathfrak{g}_k$ such that $JX \in \mathfrak{g}_{k+1}$ and $JX \notin \mathfrak{g}_k$. Then, it is possible to find $V \in \mathfrak{g}$ such that $0 \neq [JX, V] \in \mathfrak{g}_k$ but $[JX, V] \notin \mathfrak{g}_{k-1}$. As vector spaces, $\mathfrak{g} = \mathcal{W} \oplus J\mathcal{W}$ so we can express $V = V_1 + V_2$, where $V_1 \in \mathcal{W}$ and $V_2 \in J\mathcal{W}$. Hence, $[JX, V] = [JX, V_1] + [JX, V_2]$, with $[JX, V_1] \in \mathfrak{g}_{k-1}$, and consequently $[JX, V_2]$ must verify the same property as V . Writing $V_2 = JZ$, with $Z \in \mathcal{W}$, one has that $0 \neq [JX, JZ] \in \mathfrak{g}_k$, but $[JX, JZ] \notin \mathfrak{g}_{k-1}$. Applying the Nijenhuis condition, we get

$$[JX, JZ] - [X, Z] = J([JX, Z] + [X, JZ]).$$

Observe that the left-hand side belongs to \mathfrak{g}_k , but it is not contained in \mathfrak{g}_{k-1} . Hence, the element $0 \neq Y = [JX, Z] + [X, JZ] \in \mathfrak{g}_{k-1}$ and $JY \in \mathfrak{g}_k$, but it does not belong to \mathfrak{g}_{k-1} . □

Lemma 3.2. *Let (\mathfrak{g}, J) be a $2n$ -dimensional NLA endowed with a complex structure. Suppose there exist a 2-dimensional J -invariant subspace $\mathcal{V} \subset \mathfrak{g}$ and a non-zero element $X \in \mathfrak{g}_1$ such that $\mathfrak{g} = \mathfrak{g}_k \oplus \mathcal{V} \oplus \langle JX \rangle$ as vector spaces for some $k \geq 1$. Then, $JX \in \mathfrak{g}_{k+1}$.*

Proof. Let $\{X_i, JX_i\}_{i=1}^n$ be a J -adapted basis of \mathfrak{g} . Then, any non-zero element $W \in \mathfrak{g}$ can be written as $W = \sum_{i=1}^n a_i X_i + b_i JX_i$, where $a_k \neq 0$ for some $1 \leq k \leq n$ (indeed, if $a_k = 0$ for every k , then $b_k \neq 0$ for some k and it suffices to interchange the roles of X_k and JX_k in the J -adapted basis).

We consider $X \in \mathfrak{g}_1$ and $\mathcal{V} = \langle Y, JY \rangle$. We have

$$X = \sum_{i=1}^n a_i X_i + b_i JX_i, \quad Y = \sum_{i=1}^n \tilde{a}_i X_i + \tilde{b}_i JX_i, \quad \text{with } a_r, \tilde{a}_s \neq 0 \text{ for some } r \neq s,$$

since X, Y are non-zero linearly independent elements. The arrangement

$$X'_1 = X, \quad X'_r = X_1, \quad X'_s = X_n, \quad X'_n = Y, \quad X'_i = X_i, \quad \text{for } i \neq 1, k, s, n,$$

can be made in such a way that $\{X'_i, JX'_i\}$ is a new J -adapted basis of \mathfrak{g} . Renaming $X'_k \equiv X_k$ for $1 \leq k \leq n$, we have that $X_1 = X$ and $\mathcal{V} = \langle X_n, JX_n \rangle$. Hence, it is clear that

$$\mathfrak{g}_k = \langle X_1, \dots, X_{n-1}, JX_2, \dots, JX_{n-1} \rangle.$$

Since the algebra is nilpotent and $\mathfrak{g}_k \neq \mathfrak{g}$, there exists $0 \neq Z \in \mathfrak{g}_{k+1}$ such that $Z \notin \mathfrak{g}_k$. We can consider, without loss of generality, that

$$Z = a X_n + b JX_n + c JX_1,$$

where $a, b, c \in \mathbb{R}$ (otherwise, apply a similar argument to that contained in the proof of Lemma 3.1).

If $a = b = 0$, then $c \neq 0$ and the result holds immediately.

If a or b are non-zero, then we can define a new basis for \mathcal{V} as follows:

$$X'_n = b X_n - a JX_n, \quad JX'_n = b JX_n + a X_n.$$

In order to prove that $JX_1 \in \mathfrak{g}_{k+1}$ it suffices to see that $[X'_n, JX_1], [JX'_n, JX_1] \in \mathfrak{g}_k$.

On the one hand, it is clear that $[Z, JX_1] \in \mathfrak{g}_k$, i.e.,

$$[Z, JX_1] = a_1 X_1 + \dots + a_{n-1} X_{n-1} + b_2 JX_2 + \dots + b_{n-1} JX_{n-1},$$

where $a_i, b_i \in \mathbb{R}$. On the other hand, note that we can write $Z = J(X'_n + c X_1) \in \mathfrak{g}_{k+1}$, so using the Nijenhuis condition we obtain

$$[Z, JX_1] = J[c X_1 + X'_n, JX_1] = J[X'_n, JX_1].$$

Therefore,

$$J[X'_n, JX_1] = a_1 X_1 + \dots + a_{n-1} X_{n-1} + b_2 JX_2 + \dots + b_{n-1} JX_{n-1},$$

and applying J to the previous expression, we have

$$-[X'_n, JX_1] = a_1 JX_1 + a_2 JX_2 + \dots + a_{n-1} JX_{n-1} - (b_2 X_2 + \dots + b_{n-1} X_{n-1}).$$

Due to the nilpotency of \mathfrak{g} , we get $a_1 = 0$ and thus $[X'_n, JX_1] \in \mathfrak{g}_k$.

Finally, since $[JX'_n, JX_1] = J[X'_n, JX_1]$, we can also ensure that $[JX'_n, JX_1] \in \mathfrak{g}_k$. □

Remark 3.3. In the conditions of Lemma 3.2, we notice that if $\mathfrak{g}_{k+1} \cap \mathcal{V} \neq \{0\}$, then $\mathfrak{g}_{k+1} = \mathfrak{g}$, whereas if $\mathfrak{g}_{k+1} \cap \mathcal{V} = \{0\}$, then $\mathfrak{g}_{k+1} = \mathfrak{g}_k \oplus \langle JX \rangle$ and $\mathfrak{g}_{k+2} = \mathfrak{g}$.

In the case of SnN complex structures, the ascending J -compatible series $\{a_k(J)\}_k$ is identically zero. For this reason, in order to study this type of structure we will focus on the way J interacts with the terms \mathfrak{g}_k of the ascending central series of \mathfrak{g} .

Lemma 3.4. *Let (\mathfrak{g}, J) be a $2n$ -dimensional NLA endowed with a complex structure, and assume that $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$ for some integer $k \geq 1$. Then one has:*

- (i) $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_k = \{0\}$;
- (ii) *if $r > 1$ is the smallest integer such that $\mathfrak{g}_{k+r} \cap J\mathfrak{g}_k \neq \{0\}$, then $\mathfrak{g}_{k+r-1} \cap J\mathfrak{g}_{k+r-1} \neq \{0\}$.*

Proof. For case (i) we argue by contradiction. Suppose that there exists $X \in \mathfrak{g}_k$ such that $JX \in \mathfrak{g}_{k+1}$ and $JX \notin \mathfrak{g}_k$. Then, it is possible to find $Y \in \mathfrak{g}$ satisfying $0 \neq [JX, Y] \in \mathfrak{g}_k$ but not contained in the subspace \mathfrak{g}_{k-1} . Let us denote by T the non-zero element given by $T = [JX, Y] + [X, JY] \in \mathfrak{g}_k$. By the Nijenhuis condition, we have

$$JT = J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y] \in \mathfrak{g}_k.$$

Therefore, both the element T and its image by J belong to \mathfrak{g}_k , which contradicts the hypothesis $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$.

For case (ii), take an element $X \in \mathfrak{g}_k$ such that $JX \in \mathfrak{g}_{k+r}$ and $JX \notin \mathfrak{g}_{k+r-1}$. Following the same argument as before, we can find an element $Y \in \mathfrak{g}$ satisfying $0 \neq [JX, Y] \in \mathfrak{g}_{k+r-1}$ but not contained in the subspace \mathfrak{g}_{k+r-2} . Now, the non-zero elements T and JT belong to \mathfrak{g}_{k+r-1} , being $T = [JX, Y] + [X, JY]$. □

Some interesting consequences arise from this result.

Proposition 3.5. *Let (\mathfrak{g}, J) be a $2n$ -dimensional NLA endowed with a complex structure and suppose that $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$ for some integer $k \geq 1$. Then:*

- (i) *the nilpotency step of \mathfrak{g} satisfies $s \geq k + 2$; moreover, if $s = k + 2$, then $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1} \neq \{0\}$;*
- (ii) *\mathfrak{g}_{k+1} cannot contain a subspace \mathcal{W} satisfying $\dim \mathcal{W} = n$ and $\mathcal{W} \cap J\mathcal{W} = \{0\}$.*

Proof. For part (i), it suffices to apply Lemma 3.4(i) in order to see that $\mathfrak{g}_{k+1} \neq \mathfrak{g}$. Hence, $s \geq k + 2$. Furthermore, if $s = k + 2$, then one has $\mathfrak{g}_{k+2} = \mathfrak{g}$ and also $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_k = \{0\}$ by Lemma 3.4(i). To obtain our result, it suffices to make use of Lemma 3.4(ii) with $r = 2$.

For (ii), let us assume the opposite; i.e., suppose that there exists $\mathcal{W} \subset \mathfrak{g}_{k+1}$ such that $\dim \mathcal{W} = n$ and $\mathcal{W} \cap J\mathcal{W} = \{0\}$. By part (i) it is clear that $\mathfrak{g}_{k+1} \neq \mathfrak{g}$. Necessarily, there is $Z \in \mathfrak{g}_{k+2} \subseteq \mathfrak{g}$ and not belonging to \mathfrak{g}_{k+1} . As vector spaces, $\mathfrak{g} = \mathfrak{g}_{k+1} \oplus \mathcal{U}$, where \mathcal{U} is a certain subspace of $J\mathcal{W}$, so we can write $Z = Z_1 + JX$, being $Z_1 \in \mathfrak{g}_{k+1}$ and $0 \neq X \in \mathcal{W}$. Moreover, since $Z \notin \mathfrak{g}_{k+1}$, then $JX \in \mathfrak{g}_{k+2}$, but $JX \notin \mathfrak{g}_{k+1}$. Applying Lemma 3.1(ii) for $r = k + 1$, it would be possible to find a non-zero element $Y \in \mathfrak{g}_k$ such that $JY \in \mathfrak{g}_{k+1}$. However, this contradicts Lemma 3.4(i). □

Thanks to the previous proposition with $k = 1$, one obtains the following result.

Corollary 3.6. *Any nilpotent Lie algebra endowed with a strongly non-nilpotent complex structure is at least 3-step.*

We next give some bounds for the dimensions of the terms of the ascending central series of \mathfrak{g} under the presence of a strongly non-nilpotent complex structure.

Proposition 3.7. *Let (\mathfrak{g}, J) be a $2n$ -dimensional NLA endowed with a complex structure satisfying $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$ for some $k \geq 1$. Then:*

- (i) $k \leq \dim \mathfrak{g}_k \leq n - 2$;

- (ii) $1 + \dim \mathfrak{g}_k \leq \dim \mathfrak{g}_{k+1} \leq 2n - 3$; furthermore, if in addition $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1} \neq \{0\}$, then one indeed has $2 + \dim \mathfrak{g}_k \leq \dim \mathfrak{g}_{k+1} \leq 2n - 3$;
- (iii) if $\dim \mathfrak{g}_k = n - r$, for some $r \geq 1$, then $\mathfrak{g}_{k+r-1} \cap J\mathfrak{g}_{k+r-1} \neq \{0\}$; moreover, $n \leq \dim \mathfrak{g}_{k+r-1}$.

Proof. For part (i) observe that the lower bound is clear, because \mathfrak{g} is a nilpotent Lie algebra and its ascending central series strictly increases. For the upper bound, first note that the hypothesis $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$ leads to $\dim \mathfrak{g}_k \leq n$. Thus, we just need to discard the cases $\dim \mathfrak{g}_k = n$ and $\dim \mathfrak{g}_k = n - 1$.

Let us first suppose that $\dim \mathfrak{g}_k = n$. By Lemma 3.4(i), one has $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_k = \{0\}$, so the dimension of the space $\mathfrak{h} = \mathfrak{g}_{k+1} \oplus J\mathfrak{g}_k \subseteq \mathfrak{g}$ must satisfy $(n + 1) + n \leq \dim \mathfrak{g}_{k+1} + \dim J\mathfrak{g}_k = \dim \mathfrak{h} \leq \dim \mathfrak{g} = 2n$. As we can see, this is a contradiction.

Let us now consider $\dim \mathfrak{g}_k = n - 1$. The nilpotency of \mathfrak{g} guarantees the existence of some $Y \in \mathfrak{g}_{k+1}$ not contained in \mathfrak{g}_k . Furthermore, as a consequence of Lemma 3.4(i) we have that $Y \neq JX$ for any $X \in \mathfrak{g}_k$. Hence, one has $\mathfrak{g}_k \cap \langle Y \rangle = \{0\}$, and $\mathcal{W} = \mathfrak{g}_k \oplus \langle Y \rangle \subseteq \mathfrak{g}_{k+1}$ is an n -dimensional subspace satisfying $\mathcal{W} \cap J\mathcal{W} = \{0\}$. However, this contradicts Proposition 3.5(ii). Therefore, we conclude that $k \leq \dim \mathfrak{g}_k \leq n - 2$.

In part (ii), the lower bound is straightforward. For the upper bound, we first observe that Proposition 3.5(i) and the nilpotency of \mathfrak{g} imply that $\dim \mathfrak{g}_{k+1} \leq 2(n - 1)$. Consequently, it suffices to discard the case $\dim \mathfrak{g}_{k+1} = 2(n - 1)$. Let us then assume that $\dim \mathfrak{g}_{k+1} = 2(n - 1)$. Due to Proposition 3.5(ii), notice that \mathfrak{g}_{k+1} is J -invariant and thus $J\mathfrak{g}_{k+1} = \mathfrak{g}_{k+1}$. Therefore,

$$\{0\} = \mathfrak{g}_{k+1} \cap J\mathfrak{g}_k = J\mathfrak{g}_{k+1} \cap \mathfrak{g}_k = \mathfrak{g}_{k+1} \cap \mathfrak{g}_k = \mathfrak{g}_k,$$

which is a contradiction.

If we now assume the additional hypothesis $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1} \neq \{0\}$, then there exists $0 \neq X \in \mathfrak{g}$ such that $X \in \mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1}$. Moreover, since $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1}$ is a J -invariant space, also $0 \neq JX \in \mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1}$. In particular, one has a 2-dimensional subspace $\mathcal{V} = \langle X, JX \rangle \subset \mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1}$. In addition, since $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_k = \{0\}$ by Lemma 3.4(i), we can ensure that $\mathcal{V} \cap \mathfrak{g}_k = \{0\}$, and the result holds.

Finally, let us prove part (iii). Observe that if $\mathfrak{g}_{k+r-2} \cap J\mathfrak{g}_{k+r-2} \neq \{0\}$, then the result is trivial. Hence, let us focus on the case $\mathfrak{g}_{k+r-2} \cap J\mathfrak{g}_{k+r-2} = \{0\}$. On the one hand, by part (i), one has $\dim \mathfrak{g}_{k+r-2} \leq n - 2$. On the other hand, we know that $n - r = \dim \mathfrak{g}_k < \dim \mathfrak{g}_{k+1} < \dots < \dim \mathfrak{g}_{k+r-2}$. Since these inequalities are strict, it is easy to see that $\dim \mathfrak{g}_{k+r-2} \geq n - 2$. Joining these two facts, we can conclude that $\dim \mathfrak{g}_{k+r-2} = n - 2$ and thus $\dim \mathfrak{g}_{k+r-1} \geq n - 1$. Using (i) again, the result is straightforward. For the second part of the statement, use part (ii), bearing in mind that the given hypothesis implies that there should be an integer $k \leq t \leq k + r - 2$ such that $\mathfrak{g}_t \cap J\mathfrak{g}_t = \{0\}$ and $\mathfrak{g}_{t+1} \cap J\mathfrak{g}_{t+1} \neq \{0\}$. □

As an immediate consequence of part (ii) we have

Corollary 3.8. *Let (\mathfrak{g}, J) be a $2n$ -dimensional NLA endowed with a complex structure, where $n \geq 3$. If $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$ and $\dim \mathfrak{g}_{k+1} = 1 + \dim \mathfrak{g}_k$, for some $k \geq 1$, then $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1} = \{0\}$.*

Using the result above, we reach an algebraic obstruction to the existence of complex structures.

Corollary 3.9. *Let \mathfrak{g} be a $2n$ -dimensional NLA. If $\dim \mathfrak{g}_{n-1} = n - 1$, then \mathfrak{g} does not admit any complex structure.*

Proof. First note that the nilpotency of \mathfrak{g} implies that $\mathfrak{g}_k \subsetneq \mathfrak{g}_{k+1}$, for each $1 \leq k \leq s - 1$, s being the nilpotency step of \mathfrak{g} . In fact, using the hypothesis we have that $\dim \mathfrak{g}_k = k$, for every $1 \leq k \leq n - 1$ or, equivalently, $\dim \mathfrak{g}_{k+1} = 1 + \dim \mathfrak{g}_k$.

Suppose that there exists a complex structure J on \mathfrak{g} . Since $\dim \mathfrak{g}_1 = 1$, clearly $\mathfrak{g}_1 \cap J\mathfrak{g}_1 = \{0\}$, so J must be of SnN type. For $n = 2$, it is well known that there are no SnN complex structures on NLAs. For $n \geq 3$, we can apply Corollary 3.8 with $k = 1$ to get $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$, but we are again in the conditions of Corollary 3.8, this time with $k = 2$. Repeating the process, one finally gets $\mathfrak{g}_{n-1} \cap J\mathfrak{g}_{n-1} = \{0\}$. However, this contradicts Proposition 3.7(i). \square

We notice that this result provides a restriction to the existence of complex structures based on the ascending type of the NLA: if a $2n$ -dimensional Lie algebra \mathfrak{g} has ascending type $(1, 2, \dots, n - 1, m_n, \dots, m_s)$, then \mathfrak{g} does not admit any complex structure.

As a consequence, we recover the following result proven by Goze and Remm in [16]. Recall that an m -dimensional filiform Lie algebra is an $(m - 1)$ -step nilpotent Lie algebra.

Corollary 3.10. *Filiform Lie algebras do not admit complex structures.*

Proof. Any $2n$ -dimensional filiform Lie algebra \mathfrak{g} has ascending type $(1, 2, \dots, n - 1, \dots, 2n - 2, 2n)$. Hence, $\dim \mathfrak{g}_{n-1} = n - 1$, and the result is a direct consequence of Corollary 3.9. \square

3.2. Dimension of the center. The goal of this section is to obtain an upper bound for the dimension of the center of a $2n$ -dimensional NLA \mathfrak{g} endowed with an SnN complex structure J . More precisely, we improve the bound given in Proposition 3.7(i) for $k = 1$ when $n \geq 4$.

Theorem 3.11. *Let (\mathfrak{g}, J) be a $2n$ -dimensional nilpotent Lie algebra with $n \geq 4$ endowed with a strongly non-nilpotent complex structure J . Then, $1 \leq \dim \mathfrak{g}_1 \leq n - 3$.*

Proof. As a consequence of Proposition 3.7(i), it suffices to discard the case $\dim \mathfrak{g}_1 = n - 2$. We will prove the theorem by contradiction, using the following idea. As a starting point, we consider a $2n$ -dimensional vector space \mathfrak{g} with an almost-complex structure J . We will try to endow \mathfrak{g} with the structure of a nilpotent Lie algebra with $\dim \mathfrak{g}_1 = n - 2$, assuming that J must be integrable and of SnN type. In particular, we will first impose that \mathfrak{g} has an $(n - 2)$ -dimensional center and then find all possible combinations for the remaining terms in the ascending central series $\{\mathfrak{g}_k\}_k$. This will be done defining the Lie brackets of \mathfrak{g} in terms of the elements of a basis \mathcal{B} that will be constructed along the process, attending to the nilpotency of \mathfrak{g} and the Jacobi identity

$$(4) \quad \text{Jac}(X, Y, Z) := [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0,$$

for any $X, Y, Z \in \mathfrak{g}$, the Nijenhuis condition (1), and the strongly non-nilpotency of J . These four conditions are checked at every stage of the method, discarding the cases in which any of them fails. In the end, we will see that all the possible cases can be rejected.

Let us start with the proof. Suppose that $\dim \mathfrak{g}_1 = n - 2$ and consider

$$\mathfrak{g}_1 = \langle X_1, \dots, X_{n-2} \rangle.$$

In this way, we can choose X_1, \dots, X_{n-2} as generators in \mathcal{B} . By the Nijenhuis condition, we have ¹

$$[X_k, \cdot]_{1 \leq k \leq n-2} = 0, \quad [JX_k, JX_r]_{1 \leq k < r \leq n-2} = 0,$$

since X_1, \dots, X_{n-2} are in the center of \mathfrak{g} .

We want to complete $\{X_k, JX_k\}_{k=1}^{n-2}$ up to a J -adapted basis \mathcal{B} of \mathfrak{g} , for which it suffices to find two vectors linearly independent of the previous ones that will be called X_{n-1} and X_n . Observe that the yet completely undetermined brackets for \mathfrak{g} are exactly those involving at least one of the elements $X_{n-1}, X_n, JX_{n-1}, JX_n \in \mathcal{B}$.

We should focus on \mathfrak{g}_2 . Our first observation is that $\mathfrak{g}_2 \cap J\mathfrak{g}_1 = \{0\}$, due to Lemma 3.4(i). Moreover, we know that $\dim \mathfrak{g}_2 \geq n - 1$, so applying Proposition 3.7(i) with $k = 2$, one immediately has $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$, and thus $\dim \mathfrak{g}_2 \geq n$ (see Proposition 3.7(ii)). Hence, by Proposition 3.5(ii) for $k = 1$, we conclude that $\dim \mathfrak{g}_2 = n$. As a consequence $\mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathcal{V}$, \mathcal{V} being a J -invariant 2-dimensional space. We can take a basis $\{X, JX\}$ of \mathcal{V} and then choose $X_{n-1} = X$. Then,

$$\mathfrak{g}_1 = \langle X_1, \dots, X_{n-2} \rangle, \quad \mathfrak{g}_2 = \langle X_1, \dots, X_{n-1}, JX_{n-1} \rangle.$$

Since the elements X_{n-1}, JX_{n-1} of the basis \mathcal{B} determined above belong to \mathfrak{g}_2 , we are able to set the following real brackets:

$$[X_{n-1}, JX_k]_{1 \leq k \leq n-2} = \sum_{i=1}^{n-2} \mu_i^k X_i, \quad [X_{n-1}, JX_{n-1}] = \sum_{i=1}^{n-2} \beta_i X_i.$$

Applying the Nijenhuis condition,

$$[JX_k, JX_{n-1}]_{1 \leq k \leq n-2} = -\sum_{i=1}^{n-2} \mu_i^k JX_i.$$

These brackets belong to \mathfrak{g}_1 , so necessarily $\mu_i^k = 0$ for every $1 \leq i, k \leq n - 2$.

Up to this moment, we have

$$(5) \quad \begin{aligned} [X_k, \cdot]_{1 \leq k \leq n-2} &= 0, & [X_{n-1}, JX_{n-1}] &= \sum_{i=1}^{n-2} \beta_i X_i, \\ [X_{n-1}, JX_k]_{1 \leq k \leq n-2} &= 0, & [JX_k, JX_r]_{1 \leq k < r \leq n-1} &= 0. \end{aligned}$$

Since \mathfrak{g}_1 and \mathfrak{g}_2 are already determined, let us move to \mathfrak{g}_3 . We know that $\dim \mathfrak{g}_3 \geq n + 1$ due to the nilpotency of \mathfrak{g} . First suppose that $\mathfrak{g}_3 \cap J\mathfrak{g}_1 = \{0\}$. Then, there should be a vector in \mathfrak{g}_3 linearly independent with $\{X_i, JX_i\}_{i=1}^{n-1}$. We select this vector as an element of \mathcal{B} and denote it by X_n . Moreover, due to the nilpotency of \mathfrak{g} one could find $Z \in \mathcal{W} = \langle X_1, \dots, X_n \rangle \subset \mathfrak{g}_3$ such that $JZ \in \mathfrak{g}_4$ and $JZ \notin \mathfrak{g}_3$. Applying Lemma 3.1(ii) with $r = 3$, there exists an element $Y \in \mathfrak{g}_2$ such that $JY \in \mathfrak{g}_3$ and $JY \notin \mathfrak{g}_2$, but this is not possible by construction.

Therefore, we may assume that $\mathfrak{g}_3 \cap J\mathfrak{g}_1 \neq \{0\}$; i.e., there is a vector $X \in \mathfrak{g}_1$ such that $JX \in \mathfrak{g}_3$. We next see that it is possible to set $JX_1 \in \mathfrak{g}_3$ without loss of generality. Notice that a generic non-zero $X \in \mathfrak{g}_1$ can be expressed as $X = \sum_{i=1}^{n-2} s_i X_i$, with $(s_1, \dots, s_{n-2}) \neq (0, \dots, 0)$. Since JX is a generator of \mathfrak{g}_3 , we would like to include it in \mathcal{B} . However, as we want \mathcal{B} to be J -adapted, also X should belong to \mathcal{B} , so we must arrange its generators in order to fulfill this requirement. We can do this as follows:

- if $s_1 \neq 0$, consider $X'_1 = X$ and $X'_k = X_k$ for every $2 \leq k \leq n - 1$;

¹Notation: By $[X_k, \cdot]_{1 \leq k \leq n-2}$ we denote all the brackets $[X_k, \cdot]$, where $1 \leq k \leq n - 2$.

- if $s_1 = \dots = s_r = 0$, for some $1 \leq r \leq n - 3$, and $s_{r+1} \neq 0$, then choose $X'_1 = X$, $X'_{r+1} = X_1$, and $X'_k = X_k$ for every $2 \leq k \leq n - 1$ such that $k \neq r + 1$.

Let us notice that this type of change does not affect the structure of the ascending central series that has been adjusted up to this moment. In fact, the brackets of the elements $\{X'_k, JX'_k\}_{k=1}^{n-1}$ still follow (5), maybe modifying the coefficients β_i if necessary (which were anyway free). Furthermore, we get $JX'_1 \in \mathfrak{g}_3$.

This fact allows us to conclude that our assumption is equivalent to $JX_1 \in \mathfrak{g}_3$, up to arrangement of generators. In this way, we have

$$(6) \quad \begin{aligned} \mathfrak{g}_1 &= \langle X_1, \dots, X_{n-2} \rangle, & \mathfrak{g}_2 &= \langle X_1, \dots, X_{n-1}, JX_{n-1} \rangle, \\ \mathfrak{g}_3 &\supseteq \langle X_1, \dots, X_{n-1}, JX_1, JX_{n-1} \rangle. \end{aligned}$$

Since $JX_1 \in \mathfrak{g}_3$ but $JX_1 \notin \mathfrak{g}_2$, there must exist an element $Y \in \mathfrak{g}$ such that $[JX_1, Y] \in \mathfrak{g}_2$ and satisfying $[JX_1, Y] \notin \mathfrak{g}_1$. In view of the brackets (5), this element Y should be linearly independent of $\{X_i, JX_i\}_{i=1}^{n-1}$. Hence, we can choose $X_n = Y$ as a new generator in \mathcal{B} . Observe that $X_n \in \mathfrak{g}_k$ and $X_n \notin \mathfrak{g}_2$ for some $k \geq 3$. Using the fact that $JX_1 \in \mathfrak{g}_3$, we can take

$$[X_n, JX_1] = \sum_{i=1}^{n-2} b_i^1 X_i + b_{n-1}^1 X_{n-1} + B_{n-1}^1 JX_{n-1},$$

where $b_i^1, B_{n-1}^1 \in \mathbb{R}$, for $1 \leq i \leq n - 1$. Let us remark that one needs $(b_{n-1}^1, B_{n-1}^1) \neq (0, 0)$ in order to ensure that $X_n \notin \mathfrak{g}_2$. Furthermore, the Nijenhuis condition yields

$$[JX_1, JX_n] = - \sum_{i=1}^{n-2} b_i^1 JX_i - b_{n-1}^1 JX_{n-1} + B_{n-1}^1 X_{n-1} \in \mathfrak{g}_2,$$

and thus $b_i^1 = 0$, for every $1 \leq i \leq n - 2$. Moreover, $X_{n-1}, JX_{n-1} \in \mathfrak{g}_2$, so

$$\begin{aligned} [X_{n-1}, X_n] &= \sum_{i=1}^{n-2} \alpha_i X_i, & [X_{n-1}, JX_n] &= \sum_{i=1}^{n-2} \gamma_i X_i, \\ [X_n, JX_{n-1}] &= \sum_{i=1}^{n-2} a_i X_i, \end{aligned}$$

where $\alpha_i, a_i, \gamma_i \in \mathbb{R}$, for $1 \leq i, k \leq n - 2$. If we now apply the Nijenhuis condition, we get

$$[JX_{n-1}, JX_n] = \sum_{i=1}^{n-2} \alpha_i X_i + \sum_{i=1}^{n-2} (\gamma_i - a_i) JX_i \in \mathfrak{g}_1.$$

Necessarily $a_i = \gamma_i$ for all $1 \leq i, k \leq n - 2$.

Our collection of Lie brackets is now:

$$(7) \quad \begin{aligned} [X_k, \cdot]_{1 \leq k \leq n-2} &= 0, & [X_n, JX_{n-1}] &= \sum_{i=1}^{n-2} \gamma_i X_i, \\ [X_{n-1}, X_n] &= \sum_{i=1}^{n-2} \alpha_i X_i, & [X_n, JX_n] &\text{unknown}, \\ [X_{n-1}, JX_k]_{1 \leq k \leq n-2} &= 0, & [JX_k, JX_r]_{1 \leq k < r \leq n-1} &= 0, \\ [X_{n-1}, JX_{n-1}] &= \sum_{i=1}^{n-2} \beta_i X_i, & [JX_1, JX_n] &= B_{n-1}^1 X_{n-1} \\ & & &\quad - b_{n-1}^1 JX_{n-1}, \\ [X_{n-1}, JX_n] &= \sum_{i=1}^{n-2} \gamma_i X_i, & [JX_k, JX_n]_{2 \leq k \leq n-2} &\text{unknown}, \\ [X_n, JX_1] &= b_{n-1}^1 X_{n-1} & [JX_{n-1}, JX_n] &= \sum_{i=1}^{n-2} \alpha_i X_i, \\ & & &\quad + B_{n-1}^1 JX_{n-1}, \\ & & [X_n, JX_k]_{2 \leq k \leq n-2} &\text{unknown}, \end{aligned}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ for $1 \leq i \leq n - 2$. Let us remark that only the values preserving the dimension of the ascending central series (6) are valid for our purposes.

Recall that we also have to ensure that the Jacobi identity (4) holds for any triplet (X, Y, Z) . Computing $Jac(X_n, JX_1, Z)$, for $Z = X_{n-1}, JX_{n-1}$, we obtain

$$b_{n-1}^1 \beta_i = B_{n-1}^1 \beta_i = 0, \quad 1 \leq i \leq n - 2.$$

Since $(b_{n-1}^1, B_{n-1}^1) \neq (0, 0)$, then $\beta_i = 0$ for $1 \leq i \leq n - 2$. Using these conditions, (7) becomes

$$\begin{aligned}
 (8) \quad & [X_k, \cdot]_{1 \leq k \leq n-2} = 0, & [X_n, JX_{n-1}] &= \sum_{i=1}^{n-2} \gamma_i X_i, \\
 & [X_{n-1}, X_n] = \sum_{i=1}^{n-2} \alpha_i X_i, & [X_n, JX_n] & \text{unknown}, \\
 & [X_{n-1}, JX_k]_{1 \leq k \leq n-1} = 0, & [JX_k, JX_r]_{1 \leq k < r \leq n-1} &= 0, \\
 & [X_{n-1}, JX_n] = \sum_{i=1}^{n-2} \gamma_i X_i, & [JX_1, JX_n] &= B_{n-1}^1 X_{n-1} \\
 & & & \quad - b_{n-1}^1 JX_{n-1}, \\
 & [X_n, JX_1] = b_{n-1}^1 X_{n-1} & [JX_k, JX_n]_{2 \leq k \leq n-2} & \text{unknown}, \\
 & & & \quad + B_{n-1}^1 JX_{n-1}, \\
 & [X_n, JX_k]_{2 \leq k \leq n-2} \text{ unknown}, & [JX_{n-1}, JX_n] &= \sum_{i=1}^{n-2} \alpha_i X_i,
 \end{aligned}$$

where the coefficients are real numbers preserving the ascending type and satisfying the Jacobi identity.

Let us now consider $Jac(X_n, JX_n, JX_1)$. Using (8), we get

$$[[X_n, JX_n], JX_1] = 2 \sum_{i=1}^{n-2} (b_{n-1}^1 \gamma_i + B_{n-1}^1 \alpha_i) X_i.$$

At this point, we ignore the value of $[X_n, JX_n]$. However, it is clear that such bracket will depend, at most, on all the other elements of the basis $\{X_i, JX_i\}_{i=1}^{n-1}$. Since the bracket of each one of these elements with JX_1 equals zero, as (8) shows, we can conclude that $[[X_n, JX_n], JX_1] = 0$. Hence,

$$(9) \quad b_{n-1}^1 \gamma_i + B_{n-1}^1 \alpha_i = 0 \quad \forall i = 1, \dots, n - 2.$$

We will make use of these equations below.

Let us now go back to the series (6). To finish the proof we will study two different cases, namely, $\dim(\mathfrak{g}_3 \cap J\mathfrak{g}_1) \geq 2$ and $\dim(\mathfrak{g}_3 \cap J\mathfrak{g}_1) = 1$. We will see that none of them are valid and in this way condition $\mathfrak{g}_3 \cap J\mathfrak{g}_1 \neq \{0\}$ will be contradicted. Since we have already refused the condition $\mathfrak{g}_3 \cap J\mathfrak{g}_1 = \{0\}$ (see p. 3878) the proof of the theorem will be completed.

Study of $\dim(\mathfrak{g}_3 \cap J\mathfrak{g}_1) \geq 2$. Assume there is another element in \mathfrak{g}_1 whose image by J belongs to \mathfrak{g}_3 . Arranging generators if necessary, we can suppose that $JX_2 \in \mathfrak{g}_3$:

$$\begin{aligned}
 \mathfrak{g}_1 &= \langle X_1, \dots, X_{n-2} \rangle, & \mathfrak{g}_2 &= \langle X_1, \dots, X_{n-1}, JX_{n-1} \rangle, \\
 \mathfrak{g}_3 &\supseteq \langle X_1, \dots, X_{n-1}, JX_1, JX_2, JX_{n-1} \rangle.
 \end{aligned}$$

Then, it should be possible to find an element $Y \in \mathfrak{g}$ such that $[JX_2, Y] \in \mathfrak{g}_2$ and $[JX_2, Y] \notin \mathfrak{g}_1$. From the brackets (8), one can see that Y must depend on X_n or JX_n . Therefore, repeating a similar argument to that for JX_1 we have

$$[X_n, JX_2] = b_{n-1}^2 X_{n-1} + B_{n-1}^2 JX_{n-1}, \quad [JX_2, JX_n] = B_{n-1}^2 X_{n-1} - b_{n-1}^2 JX_{n-1},$$

where $b_{n-1}^2, B_{n-1}^2 \in \mathbb{R}$. In particular $(b_{n-1}^2, B_{n-1}^2) \neq (0, 0)$ since JX_2 does not belong to \mathfrak{g}_2 . The Jacobi identity (4) for the triplet (X_n, JX_n, JX_2) together with (8) leads to

$$[[X_n, JX_n], JX_2] = 2 \sum_{i=1}^{n-2} (b_{n-1}^2 \gamma_i + B_{n-1}^2 \alpha_i) X_i.$$

By analogous reasoning to that used for (9), we get

$$(10) \quad b_{n-1}^2 \gamma_i + B_{n-1}^2 \alpha_i = 0 \quad \forall i = 1, \dots, n - 2.$$

Let us solve the system of equations given by (9) and (10).

We first suppose that $b_{n-1}^1 = 0$. Then $B_{n-1}^1 \neq 0$, and we get $\alpha_i = 0$ for $1 \leq i \leq n - 2$ from (9). Observe that $(\gamma_1, \dots, \gamma_{n-2}) \neq (0, \dots, 0)$, or otherwise $X_{n-1}, JX_{n-1} \in \mathfrak{g}_1$, which is a contradiction. Therefore, from (10) we conclude that $b_{n-1}^2 = 0$. Now, we take $0 \neq X = B_{n-1}^2 JX_1 - B_{n-1}^1 JX_2 \in J\mathfrak{g}_1$. Note that the bracket of X with every element of the basis vanishes. Thus $X \in \mathfrak{g}_1 \cap J\mathfrak{g}_1$, but this is not possible as J is of strongly non-nilpotent type.

Suppose now that $b_{n-1}^1 \neq 0$. Then we can solve γ_i from (9), and replacing these values in (10), we get

$$\alpha_i \left(B_{n-1}^2 - \frac{B_{n-1}^1 b_{n-1}^2}{b_{n-1}^1} \right) = 0, \quad 1 \leq i \leq n - 2.$$

Notice that there exists some $\alpha_i \neq 0$ (otherwise, $\gamma_i = 0$ for every $1 \leq i \leq n - 2$), so the other factor in the equations above should vanish. In particular, this implies that $b_{n-1}^2 \neq 0$ (or otherwise $(b_{n-1}^2, B_{n-1}^2) = (0, 0)$, which is not allowed). Take $0 \neq X = b_{n-1}^2 JX_1 - b_{n-1}^1 JX_2$. Again, $X \in \mathfrak{g}_1 \cap J\mathfrak{g}_1$, and a contradiction is attained.

This concludes the analysis of this case and shows that it is not a valid one.

Study of $\dim(\mathfrak{g}_3 \cap J\mathfrak{g}_1) = 1$. Under this hypothesis, we have that $JX_k \notin \mathfrak{g}_3$ for every $2 \leq k \leq n - 2$. We next divide the study according to the dimension of \mathfrak{g}_3 . By (6), simply recall that $\dim \mathfrak{g}_3 \geq n + 1$.

- Let $\dim \mathfrak{g}_3 > n + 1$. In this case, it is clear that $X_n \in \mathfrak{g}_3$, so it is possible to fix the unknown real brackets in (8):

$$[X_n, JX_k] = \sum_{i=1}^{n-2} c_i^k X_i + c_{n-1}^k X_{n-1} + C_{n-1}^k JX_{n-1}, \quad k = 2, \dots, n - 2, n,$$

$$[JX_k, JX_n] = - \sum_{i=1}^{n-2} c_i^k JX_i + C_{n-1}^k X_{n-1} - c_{n-1}^k JX_{n-1}, \quad k = 2, \dots, n - 2.$$

If there is some $k \in \{2, \dots, n - 2\}$ such that $c_i^k = 0$, for every $1 \leq i \leq n - 2$, then $JX_k \in \mathfrak{g}_3$. However, this is not possible by the hypothesis of this case. Thus, we can ensure that $(c_1^k, \dots, c_{n-2}^k) \neq (0, \dots, 0)$ for every $2 \leq k \leq n - 2$. Let us note that this implies that $JX_n \notin \mathfrak{g}_3$. Hence,

$$\begin{aligned} \mathfrak{g}_1 &= \langle X_1, \dots, X_{n-2} \rangle, & \mathfrak{g}_2 &= \langle X_1, \dots, X_{n-1}, JX_{n-1} \rangle, \\ \mathfrak{g}_3 &= \langle X_1, \dots, X_n, JX_1, JX_{n-1} \rangle, \end{aligned}$$

so in particular, $\dim \mathfrak{g}_3 = n + 2$. We move to the study of \mathfrak{g}_4 for this case.

Due to the nilpotency of the Lie algebra, at least JX_2 (up to arrangement of generators) or JX_n should belong to \mathfrak{g}_4 . If $JX_2 \in \mathfrak{g}_4$, then in particular $[JX_2, JX_n] \in \mathfrak{g}_3$, so $c_i^2 = 0$ for $2 \leq i \leq n - 2$. The Jacobi identity $Jac(X_n, JX_n, JX_2)$ implies that

$$0 = -c_1^2 (b_{n-1}^1 X_{n-1} + B_{n-1}^1 JX_{n-1}) - 2 \sum_{i=1}^{n-2} (c_{n-1}^2 \gamma_i + C_{n-1}^2 \alpha_i) X_i.$$

Since $(b_{n-1}^1, B_{n-1}^1) \neq (0, 0)$, necessarily $c_1^2 = 0$. Nonetheless, this yields $JX_2 \in \mathfrak{g}_3$, which is not allowed. As a consequence, we are led to consider

$$\mathfrak{g}_4 = \langle X_1, \dots, X_n, JX_1, JX_{n-1}, JX_n \rangle.$$

However, as $[JX_k, JX_n] \in \mathfrak{g}_3$, one has that $c_i^k = 0$, for every $2 \leq i, k \leq n-2$, but this implies that $JX_k \in \mathfrak{g}_4$, for $2 \leq k \leq n-2$, which is a contradiction. Hence, $\dim \mathfrak{g}_3$ cannot be greater than $n + 1$.

- Let $\dim \mathfrak{g}_3 = n + 1$, i.e.,

$$\begin{aligned} \mathfrak{g}_1 &= \langle X_1, \dots, X_{n-2} \rangle, & \mathfrak{g}_2 &= \langle X_1, \dots, X_{n-1}, JX_{n-1} \rangle, \\ \mathfrak{g}_3 &= \langle X_1, \dots, X_{n-1}, JX_1, JX_{n-1} \rangle. \end{aligned}$$

We will see that this is also an invalid case by carefully studying \mathfrak{g}_4 . Notice that due to the construction of the ascending central series, $\mathfrak{g}_4 \cap J\mathfrak{g}_1 \neq \{0\}$. We are going to consider two different cases according to the dimension of $\mathfrak{g}_4 \cap J\mathfrak{g}_1$.

- ▷ Assume $\dim(\mathfrak{g}_4 \cap J\mathfrak{g}_1) = 1$. Since $JX_1 \in \mathfrak{g}_3 \subset \mathfrak{g}_4$, this implies that $JX_k \notin \mathfrak{g}_4$ for every $2 \leq k \leq n-2$. Then, $\mathfrak{g}_4 \cap \langle X_n, JX_n \rangle \neq \{0\}$. Observe that, without loss of generality, we can suppose $X_n \in \mathfrak{g}_4$ because the role of the elements X_n and JX_n in (8) is interchangeable. Moreover, it is easy to see that $JX_n \notin \mathfrak{g}_4$ (otherwise, $\mathfrak{g}_4 = \mathfrak{g}$, which is a contradiction) and thus

$$\mathfrak{g}_4 = \langle X_1, \dots, X_n, JX_1, JX_{n-1} \rangle.$$

Recall that the nilpotency of the algebra implies that the ascending central series finishes in \mathfrak{g} , so there is $Y \in \mathfrak{g}_5$ such that $Y \notin \mathfrak{g}_4$. Necessarily, Y is linearly dependent with some $JX_2, \dots, JX_{n-2}, JX_n$. In fact, we can assume that, up to a change of basis, $Y = JX$ for certain $X \in \mathcal{W} = \langle X_1, \dots, X_n \rangle \subset \mathfrak{g}_4$. Let us note that we are in the conditions of Lemma 3.1(ii) with $r = 4$. Thus, there exists $Z \in \mathfrak{g}_3$ such that $JZ \in \mathfrak{g}_4$ and $JZ \notin \mathfrak{g}_3$. However, if we take an element $Z = \sum_{i=1}^{n-1} s_i X_i + S_1 JX_1 + S_{n-1} JX_{n-1} \in \mathfrak{g}_3$, then we can see that $JZ \in \mathfrak{g}_4$ if and only if $JZ \in \mathfrak{g}_3$, which is a contradiction.

- ▷ We next study $\dim(\mathfrak{g}_4 \cap J\mathfrak{g}_1) > 1$. Up to arrangement of generators one can assume that $JX_2 \in \mathfrak{g}_4$, so

$$\mathfrak{g}_4 \supseteq \langle X_1, \dots, X_{n-1}, JX_1, JX_2, JX_{n-1} \rangle.$$

Then, there should be an element $Y \in \mathfrak{g}$ such that $[JX_2, Y] \in \mathfrak{g}_3$ and $[JX_2, Y] \notin \mathfrak{g}_2$. Necessarily, Y depends on X_n or JX_n , and it is possible to set the real brackets

$$\begin{aligned} [X_n, JX_2] &= \sum_{i=1}^{n-2} c_i^2 X_i + c_{n-1}^2 X_{n-1} + C_1^2 JX_1 + C_{n-1}^2 JX_{n-1}, \\ [JX_2, JX_n] &= -\sum_{i=1}^{n-2} c_i^2 JX_i - c_{n-1}^2 JX_{n-1} + C_1^2 X_1 + C_{n-1}^2 X_{n-1}. \end{aligned}$$

Observe that $c_i^2 = 0$, for every $2 \leq i \leq n-2$, because both brackets belong to \mathfrak{g}_3 . Furthermore, they cannot lie in \mathfrak{g}_2 , so one has $(c_1^2, C_1^2) \neq (0, 0)$. Now, imposing $Jac(X_n, JX_n, JX_2)$ the following system of equations must be solved:

$$\begin{cases} 0 &= C_1^2 B_{n-1}^1 + c_1^2 b_{n-1}^1, \\ 0 &= C_1^2 b_{n-1}^1 - c_1^2 B_{n-1}^1. \end{cases}$$

Since $(c_1^2, C_1^2) \neq (0, 0)$ and $(b_{n-1}^1, B_{n-1}^1) \neq (0, 0)$, the system has no solution.

This finishes the study of this case and the proof of Theorem 3.11. □

3.3. Some consequences and applications. In this section we point out some consequences of the results obtained in the previous sections, as well as some applications to the study of existence of complex structures on certain products of nilpotent Lie algebras.

In six dimensions, if an NLA \mathfrak{g} has a strongly non-nilpotent complex structure, then its center has dimension 1 (see Theorem 2.5). Alternatively, this result can also be obtained as a consequence of Proposition 3.7(i). In dimension 8, by Theorem 3.11 we directly have:

Corollary 3.12. *Nilpotent Lie algebras of dimension 8 admitting strongly non-nilpotent complex structures have 1-dimensional centers.*

Note that in dimension 10, there are NLAs with 2-dimensional centers admitting SnN complex structures, as Example 2.4 illustrates. Thus, in ten dimensions the upper bound in Theorem 3.11 is attained.

In [21, Corollary 7] it is proved that a given 6-dimensional NLA can only admit complex structures of either nilpotent or non-nilpotent type (recall Figure 1). Moreover, in the second case the structure is necessarily SnN. The next result shows that something similar happens in eight dimensions, in the sense that quasi-nilpotent and strongly non-nilpotent complex structures cannot coexist on the same NLA. Notice that a similar statement does not hold in dimension 10, as Example 2.4 illustrates.

Corollary 3.13. *Let \mathfrak{g} be an NLA of dimension 8 admitting complex structures. Then, all of them are either quasi-nilpotent or strongly non-nilpotent.*

Proof. On the one hand, if \mathfrak{g} admits an SnN complex structure J , then by Theorem 3.11 the center \mathfrak{g}_1 has dimension 1. On the other hand, if J' is a quasi-nilpotent complex structure on \mathfrak{g} , then by Definition 2.2 we have $\{0\} \neq \mathfrak{a}_1(J') \subseteq \mathfrak{g}_1$. Since $\mathfrak{a}_1(J')$ has dimension at least 2, the center \mathfrak{g}_1 must have dimension ≥ 2 . This is a contradiction, so J and J' cannot coexist on \mathfrak{g} . □

It is a difficult problem to decide if complex structures exist or not on a given product Lie algebra (see for example the recent paper [10] about complex structures on 6-dimensional product Lie algebras). As a consequence of Theorem 3.11 we have the following result for complex structures on certain products of NLAs.

Corollary 3.14. *Let \mathfrak{g} be an NLA of dimension $2n$, with $n \geq 4$, admitting a complex structure J . If \mathfrak{g} is the product of $n-2$ Lie algebras, i.e., $\mathfrak{g} \cong \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_{n-2}$, then J is quasi-nilpotent.*

Proof. Since the center of \mathfrak{h}_j , $1 \leq j \leq n-2$, has dimension at least 1, we have that $\dim \mathfrak{g}_1 \geq n-2$. By Theorem 3.11, we know that if J is SnN, then $\dim \mathfrak{g}_1 \leq n-3$. Consequently, J can only be of quasi-nilpotent type. □

The previous corollary for $n = 4$ gives a restriction on the type of complex structures that an 8-dimensional product nilpotent Lie algebra can admit. In the following examples we illustrate how to make use of this result to study the existence of complex structures on product Lie algebras.

Example 3.15. Let us consider the 8-dimensional Lie algebra $\mathfrak{g} = \mathfrak{h} \times \mathbb{R}$, where \mathfrak{h} is the nilpotent Lie algebra generated by $\{X_1, \dots, X_7\}$ satisfying

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_1, X_4] = X_5, \quad [X_2, X_3] = X_6, \\ [X_1, X_5] = [X_2, X_6] = X_7.$$

Let us denote by X_8 a generator of \mathbb{R} . It is easy to see that the NLA \mathfrak{g} has ascending type $(2, 4, 5, 6, 8)$.

Suppose that \mathfrak{g} admits a complex structure J . By Corollary 3.14 we have that J must be quasi-nilpotent. Moreover, since the center of \mathfrak{g} is given by

$$\mathfrak{g}_1 = \langle X_7, X_8 \rangle,$$

it is clear that the first term in the ascending series of \mathfrak{g} compatible with J is $\mathfrak{a}_1(J) = \mathfrak{g}_1$. Now, the argument at the end of Section 2 implies that the quotient Lie algebra $\tilde{\mathfrak{g}}_1 = \mathfrak{g}/\mathfrak{a}_1(J) = \mathfrak{g}/\langle X_7, X_8 \rangle$ has a complex structure \tilde{J}_1 . However, $\tilde{\mathfrak{g}}_1$ is the 6-dimensional NLA generated by $\{\tilde{X}_1, \dots, \tilde{X}_6\}$ with

$$[\tilde{X}_1, \tilde{X}_2] = \tilde{X}_3, \quad [\tilde{X}_1, \tilde{X}_3] = \tilde{X}_4, \quad [\tilde{X}_1, \tilde{X}_4] = \tilde{X}_5, \quad [\tilde{X}_2, \tilde{X}_3] = \tilde{X}_6,$$

and this NLA does not admit any complex structure, as proved in [20].

In conclusion, the 8-dimensional NLA $\mathfrak{g} = \mathfrak{h} \times \mathbb{R}$ cannot admit any complex structure.

Example 3.16. Let us consider the 8-dimensional Lie algebra $\mathfrak{g} = \mathfrak{h}_1 \times \mathfrak{h}_2$, where \mathfrak{h}_1 is the nilpotent Lie algebra of dimension 5 generated by $\{X_1, \dots, X_5\}$ satisfying

$$[X_1, X_2] = X_4, \quad [X_1, X_4] = [X_2, X_3] = X_5,$$

and \mathfrak{h}_2 is the nilpotent Lie algebra of dimension 3 given by $\{X_6, X_7, X_8\}$ with $[X_6, X_7] = X_8$.

It is easy to see that \mathfrak{g} has ascending type $(2, 6, 8)$ and the center of \mathfrak{g} is exactly

$$\mathfrak{g}_1 = \langle X_5, X_8 \rangle.$$

If the NLA \mathfrak{g} admits a complex structure J , then J is necessarily quasi-nilpotent by Corollary 3.14. Moreover, since $\dim \mathfrak{g}_1 = 2$ we have that the first term in the ascending series of \mathfrak{g} compatible with J is $\mathfrak{a}_1(J) = \mathfrak{g}_1$.

By the last part of Section 2, the quotient Lie algebra $\tilde{\mathfrak{g}}_1 = \mathfrak{g}/\mathfrak{a}_1(J) = \mathfrak{g}/\langle X_5, X_8 \rangle$ has a complex structure \tilde{J}_1 . Let us observe that the 6-dimensional NLA $\tilde{\mathfrak{g}}_1$ is generated by $\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4, \tilde{X}_6, \tilde{X}_7\}$ satisfying

$$[\tilde{X}_1, \tilde{X}_2] = \tilde{X}_4.$$

This Lie algebra is precisely the product of the 3-dimensional Heisenberg Lie algebra and \mathbb{R}^3 , so it has complex structures. Indeed, it follows from [6] that, up to isomorphism, the complex structure \tilde{J}_1 on $\tilde{\mathfrak{g}}_1$ is determined by

$$\tilde{J}_1 \tilde{X}_1 = -\tilde{X}_2, \quad \tilde{J}_1 \tilde{X}_3 = \tilde{X}_4, \quad \tilde{J}_1 \tilde{X}_6 = -\tilde{X}_7.$$

Now, we can extend this complex structure to \mathfrak{g} as follows. Define the almost complex structure J on \mathfrak{g} as

$$JX_1 = -X_2, \quad JX_3 = X_4, \quad JX_6 = -X_7, \quad JX_5 = -X_8.$$

A direct calculation shows that J is integrable. Hence, the product Lie algebra $\mathfrak{g} = \mathfrak{h}_1 \times \mathfrak{h}_2$ has a complex structure.

4. THE 8-DIMENSIONAL CASE

In this section we determine the structure of the ascending central series for those 8-dimensional NLAs admitting strongly non-nilpotent complex structures. More concretely, we prove the following structural theorem, which is the analogous result in eight dimensions to Theorem 2.5.

Theorem 4.1. *Let \mathfrak{g} be an NLA of dimension 8. If \mathfrak{g} admits an SnN complex structure, then \mathfrak{g} has ascending type $(1, 3, 8)$, $(1, 3, 5, 8)$, $(1, 3, 6, 8)$, $(1, 3, 5, 6, 8)$, $(1, 4, 8)$, $(1, 4, 6, 8)$, $(1, 5, 8)$, or $(1, 5, 6, 8)$.*

For the proof of this theorem, in Section 4.1 we start studying the term \mathfrak{g}_2 of the ascending central series $\{\mathfrak{g}_k\}_k$, and we devote Section 4.2 to the remaining terms \mathfrak{g}_k , $k \geq 3$. It is important to remark that, since our method is constructive, we will be able to describe the complex structure equations that parametrize all the SnN complex structures in dimension 8. This is detailed in Section 4.3.

Before beginning the proof, we would like to explain the main ideas on which the method is based.

We first observe that there are some general results in dimension 8 that come straight from Section 3. More precisely, if \mathfrak{g} is an 8-dimensional NLA with an SnN complex structure J , then:

$$(11) \quad \left\{ \begin{array}{l} \bullet \dim \mathfrak{g}_1 = 1 \text{ (see Theorem 3.11 and Corollary 3.12);} \\ \bullet 2 \leq \dim \mathfrak{g}_2 \leq 5 \text{ (see Proposition 3.7(ii));} \\ \bullet \mathfrak{g}_2 \cap J\mathfrak{g}_1 = \{0\} \text{ (see Lemma 3.4);} \\ \bullet \mathfrak{g}_3 \cap J\mathfrak{g}_3 \neq \{0\} \text{ and } \dim \mathfrak{g}_3 \geq 4 \text{ (see Proposition 3.7(iii))} \\ \hspace{15em} \text{with } k = 1, r = 3). \end{array} \right.$$

Furthermore, thanks to García Vergnolle and Remm [14], we know that if an 8-dimensional NLA is quasi-filiform, i.e., its nilpotency step is 6, then it does not admit any complex structure. This fact together with Corollary 3.6 allows us to conclude that an 8-dimensional NLA \mathfrak{g} endowed with a strongly non-nilpotent complex structure has nilpotency step $3 \leq s \leq 5$. Consequently, $\dim \mathfrak{g}_1 = 1$ and $\mathfrak{g}_5 = \mathfrak{g}$, so the study of the ascending central series is reduced to the terms \mathfrak{g}_2 , \mathfrak{g}_3 , and \mathfrak{g}_4 . For this aim we will use the same ideas contained in the proof of Theorem 3.11. That is, we will construct a J -adapted basis \mathcal{B} of \mathfrak{g} that will also be adapted to the nilpotent structure of \mathfrak{g} . Let us begin the construction.

Since $\dim \mathfrak{g}_1 = 1$, we can establish $\mathfrak{g}_1 = \langle X_1 \rangle$. In this way, X_1 and JX_1 can be considered two elements in the J -adapted basis \mathcal{B} . Moreover, it is clear that $[X_1, \cdot] = 0$ and $[JX_1, J\cdot] = J[JX_1, \cdot]$, as X_1 is a central element and J is integrable, so it satisfies the Nijenhuis condition (1).

We now focus on \mathfrak{g}_2 . By the nilpotency of \mathfrak{g} and conditions (11), there is a vector $Y \in \mathfrak{g}_2$ linearly independent with X_1 and JX_1 . Let us denote $X_2 = Y$. Then,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 \supseteq \langle X_1, X_2 \rangle.$$

Observe that X_2 and its image by J can be taken as new elements of \mathcal{B} . By construction, $[X_2, \cdot] \in \langle X_1 \rangle$. Furthermore, $[X_2, JX_1] = \lambda X_1$ and thus $[JX_1, JX_2] = -\lambda JX_1$, for some $\lambda \in \mathbb{R}$, due to (1). The nilpotency of the Lie algebra requires that $\lambda = 0$. Hence,

$$(12) \quad [X_1, \cdot] = 0, \quad [X_2, JX_1] = 0, \quad [X_2, JX_2] = b_{22}^1 X_1, \quad [JX_1, JX_2] = 0,$$

where $b_{22}^1 \in \mathbb{R}$. This is the starting point of our construction. We also introduce the following notation, which will be of great help in the next sections.

Notation 4.2. Let $\{X_i, JX_i\}_{i=1}^n$ be a J -adapted basis of \mathfrak{g} . For each $1 \leq i \leq n$, we denote $\mathcal{V}_i = \langle X_i, JX_i \rangle$. In this way, as a vector space, \mathfrak{g} can be decomposed as a direct sum of the 2-dimensional J -invariant spaces \mathcal{V}_i ; that is, $\mathfrak{g} = \bigoplus_{i=1}^n \mathcal{V}_i$.

4.1. The term \mathfrak{g}_2 . In this section we provide the possible dimensions for the term \mathfrak{g}_2 . Moreover, we give a description of this space in terms of elements of the J -adapted basis \mathcal{B} .

Proposition 4.3. *Let (\mathfrak{g}, J) be an 8-dimensional NLA endowed with a strongly non-nilpotent complex structure. Then, $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$. As a consequence, $3 \leq \dim \mathfrak{g}_2 \leq 5$.*

Proof. We proceed by contradiction. Let us suppose that $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$. From Proposition 3.7(i) we have $\dim \mathfrak{g}_2 = 2$; thus

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle.$$

Moreover, by part (i) we know that the nilpotency step s of \mathfrak{g} satisfies $s = 4$ or $s = 5$.

Let us now focus on \mathfrak{g}_3 . By (11) we know that $\mathfrak{g}_3 \cap J\mathfrak{g}_3 \neq \{0\}$. As a consequence of Lemma 3.4(i) with $k = 2$, we also have that $\mathfrak{g}_3 \cap J\mathfrak{g}_2 = \{0\}$, so we can ensure the existence of a 2-dimensional subspace $\mathcal{V}_3 = \langle X_3, JX_3 \rangle$ contained in \mathfrak{g}_3 . Furthermore, due to Proposition 3.5(ii) it is not possible to find another element in \mathfrak{g}_3 linearly independent of $\{X_i, JX_i\}_{i=1}^3$. Therefore, the ascending central series is as follows:

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle.$$

Observe that one can find a vector in \mathfrak{g} , which we will call X_4 , such that $\mathcal{B} = \{X_i, JX_i\}_{i=1}^4$ and $X_4, JX_4 \notin \mathfrak{g}_3$. With this information, it is possible to add some new Lie brackets to those in (12):

$$[X_2, X_k]_{k=3,4} = a_{2k}^1 X_1, \quad [X_2, JX_k]_{k=3,4} = b_{2k}^1 X_1, \quad [X_3, X_4] = a_{34}^1 X_1 + a_{34}^2 X_2,$$

$$[X_3, JX_k]_{1 \leq k \leq 4} = b_{3k}^1 X_1 + b_{3k}^2 X_2, \quad [X_4, JX_3] = b_{43}^1 X_1 + b_{43}^2 X_2,$$

where the coefficients are real numbers. Furthermore,

$$[JX_1, JX_3] = J[JX_1, X_3] = -b_{31}^1 JX_1 - b_{31}^2 JX_2 \in \mathfrak{g}_2,$$

so we conclude that $b_{31}^1 = b_{31}^2 = 0$. In a similar way, imposing the equations $N_J(X_2, X_3)$ and $N_J(X_3, X_4)$ given by (1), we obtain

$$\begin{aligned} b_{32}^1 &= b_{23}^1, & b_{32}^2 &= 0, & [JX_2, JX_3] &= a_{23}^1 X_1, \\ b_{43}^1 &= b_{34}^1, & b_{43}^2 &= b_{34}^2, & [JX_3, JX_4] &= a_{34}^1 X_1 + a_{34}^2 X_2. \end{aligned}$$

Up to this moment, we have the following real brackets:

$$\begin{aligned}
 (13) \quad & [X_1, \cdot] = 0, & [X_4, JX_k]_{k=1,2} & \text{unknown}, \\
 & [X_2, X_k]_{k=3,4} = a_{2k}^1 X_1, & [X_4, JX_3] & = b_{34}^1 X_1 + b_{34}^2 X_2, \\
 & [X_2, JX_1] = 0, & [X_4, JX_4] & \text{unknown}, \\
 & [X_2, JX_k]_{k=2,3,4} = b_{2k}^1 X_1, & [JX_1, JX_k]_{k=2,3} & = 0, \\
 & [X_3, X_4] = a_{34}^1 X_1 + a_{34}^2 X_2, & [JX_1, JX_4] & \text{unknown}, \\
 & [X_3, JX_1] = 0, & [JX_2, JX_3] & = a_{23}^1 X_1, \\
 & [X_3, JX_2] = b_{23}^1 X_1, & [JX_2, JX_4] & \text{unknown}, \\
 & [X_3, JX_k]_{k=3,4} = b_{3k}^1 X_1 + b_{3k}^2 X_2, & [JX_3, JX_4] & = a_{34}^1 X_1 + a_{34}^2 X_2.
 \end{aligned}$$

Moreover, for those brackets above involving X_4 , we can write

$$[X_4, JX_k] = \sum_{i=1}^3 b_{4k}^i X_i + \sum_{j=1, j \neq k}^4 c_{4k}^j JX_j, \quad k = 1, 2, 4.$$

Now, using the Nijenhuis condition and the nilpotency of \mathfrak{g} , we have

$$\begin{aligned}
 b_{41}^1 &= 0, & [JX_1, JX_4] &= -\sum_{i=2}^3 b_{41}^i JX_i + \sum_{j=2}^4 c_{41}^j JX_j, \\
 b_{42}^2 &= 0, & [JX_2, JX_4] &= (a_{24}^1 + c_{42}^1)X_1 + c_{42}^3 X_3 + c_{42}^4 X_4 + (b_{24}^1 - b_{42}^1)JX_1 \\
 & & & - b_{42}^3 JX_3.
 \end{aligned}$$

Hence, the unknown brackets in (13) can be expressed as

$$\begin{aligned}
 (14) \quad & [X_4, JX_1] = \sum_{i=2}^3 b_{41}^i X_i + \sum_{j=2}^4 c_{41}^j JX_j, & [X_4, JX_4] &= \sum_{i=1}^3 b_{44}^i X_i + \sum_{j=1}^3 c_{44}^j JX_j, \\
 & [X_4, JX_2] = \sum_{i=1, i \neq 2}^3 b_{42}^i X_i + \sum_{j=1, j \neq 2}^4 c_{42}^j JX_j, \\
 & [JX_1, JX_4] = -\sum_{i=2}^3 b_{41}^i JX_i + \sum_{j=2}^4 c_{41}^j X_j, \\
 & [JX_2, JX_4] = (a_{24}^1 + c_{42}^1)X_1 + c_{42}^3 X_3 + c_{42}^4 X_4 + (b_{24}^1 - b_{42}^1)JX_1 - b_{42}^3 JX_3.
 \end{aligned}$$

Observe that the position of JX_k ($k = 1, 2, 4$) in the ascending central series will immediately make some of the coefficients above equal to zero. Indeed, recall that \mathfrak{g} can be either 4-step or 5-step nilpotent. We will next analyze these two possibilities.

The 4-step case. To ensure that $\mathfrak{g}_4 = \mathfrak{g}$, the following values in (14) are needed:

$$b_{24}^1 = b_{42}^1, \quad b_{41}^2 = c_{41}^2 = c_{41}^4 = c_{42}^1 = c_{42}^4 = c_{44}^1 = c_{44}^2 = 0.$$

Note that the only (possibly) non-zero brackets involving JX_1 are $[X_4, JX_1] = b_{41}^3 X_3 + c_{41}^3 JX_3$ and $[JX_1, JX_4] = c_{41}^3 X_3 - b_{41}^3 JX_3$. Since $JX_1 \notin \mathfrak{g}_1$, necessarily $(b_{41}^3, c_{41}^3) \neq (0, 0)$.

Let us study the Jacobi identity to contradict this case. Considering

$$Jac(X_3, JX_1, Y) \text{ for } Y = X_4, JX_4,$$

we have

$$\left. \begin{aligned}
 c_{41}^3 (b_{33}^1 X_1 + b_{33}^2 X_2) &= 0, \\
 b_{41}^3 (b_{33}^1 X_1 + b_{33}^2 X_2) &= 0,
 \end{aligned} \right\} \text{ so we conclude that } b_{33}^1 = b_{33}^2 = 0.$$

From $Jac(X_2, JX_1, Y)$, with $Y = X_4, JX_4$, we obtain

$$\begin{pmatrix} a_{23}^1 & b_{23}^1 \\ b_{23}^1 & -a_{23}^1 \end{pmatrix} \begin{pmatrix} b_{41}^3 \\ c_{41}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and thus } a_{23}^1 = b_{23}^1 = 0.$$

Now, we observe that due to the previous choices we need $(a_{34}^2, b_{34}^2) \neq (0, 0)$, or otherwise $X_3 \in \mathfrak{g}_2$. If we compute $Jac(X_3, JX_2, Y)$, for $Y = X_4, JX_4$, we get

$$b_{22}^1 a_{34}^2 = b_{22}^1 b_{34}^2 = 0, \quad \text{which implies that } b_{22}^1 = 0.$$

Finally, using $Jac(X_4, JX_4, Y)$ with $Y = X_3, JX_3$ one has

$$\begin{pmatrix} b_{24}^1 & -a_{24}^1 \\ a_{24}^1 & b_{24}^1 \end{pmatrix} \begin{pmatrix} a_{34}^2 \\ b_{34}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and necessarily } a_{24}^1 = b_{24}^1 = 0.$$

Nonetheless, replacing all these zero values in (13), one then realizes that X_2 belongs to the center of \mathfrak{g} , which is a contradiction. Therefore, this case does not hold and the algebra must be 5-step nilpotent.

The 5-step case. First, we observe that $\mathfrak{g}_4 \cap J\mathfrak{g}_2 \neq \{0\}$. In fact, if we assume $\mathfrak{g}_4 \cap J\mathfrak{g}_2 = \{0\}$, then $X_4 \in \mathfrak{g}_4$ and we are in the conditions of Lemma 3.1(ii) for $r = 4$. This means that there should be a vector $W \in \mathfrak{g}_3$ such that $JW \in \mathfrak{g}_4$ but $JW \notin \mathfrak{g}_3$. Due to the arrangement of the ascending central series, the only possibility is having $W \in \mathfrak{g}_2$, which contradicts our assumption.

In addition, we remark that the nilpotency of \mathfrak{g} yields $\dim \mathfrak{g}_4 = 5$ or 6 . We will next see that none of these cases are valid, thus finishing the proof.

- Let $\dim \mathfrak{g}_4 = 5$. Then,

$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_3, JY \rangle, \quad \text{where } Y \in \mathfrak{g}_2.$$

Let us express $Y = \alpha_1 X_1 + \alpha_2 X_2$. After a suitable arrangement of generators, we can assume that either $(\alpha_1, \alpha_2) = (1, 0)$ or $(\alpha_1, \alpha_2) = (0, 1)$. We analyze these two possibilities in the next lines.

On the one hand, if $Y = X_1$, then

$$\begin{aligned} \mathfrak{g}_1 &= \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle, \\ \mathfrak{g}_4 &= \langle X_1, X_2, X_3, JX_1, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}. \end{aligned}$$

This structure of the ascending central series forces the Lie brackets to follow (13) and (14) with

$$b_{41}^2 = c_{41}^2 = c_{41}^4 = c_{42}^4 = c_{44}^2 = 0.$$

The contradiction is reached repeating the same calculations for the Jacobi identity performed in the case $\mathfrak{g}_4 = \mathfrak{g}$.

On the other hand, if

$$\begin{aligned} \mathfrak{g}_1 &= \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle, \\ \mathfrak{g}_4 &= \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}, \end{aligned}$$

a similar argument can be applied to obtain a contradiction after studying the Jacobi identities. More concretely, imposing the vanishing of $Jac(X_2, JX_2, Y)$ for $Y = X_4, JX_4$, $Jac(X_3, JX_2, Y)$ for $Y = X_4, JX_4$, and $Jac(X_4, JX_4, JX_1)$, we get $JX_1 \in \mathfrak{g}_4$, which is not possible.

Due to the previous discussion, the case $\dim \mathfrak{g}_4 = 5$ is not valid.

- Assume $\dim \mathfrak{g}_4 = 6$. Repeating a similar argument, one can show that this case cannot hold either. Indeed, it suffices to consider the cases $J\mathfrak{g}_2 \subset \mathfrak{g}_4$ and its opposite and impose the vanishing of the Jacobi identities in order to obtain contradictions with the assumed arrangement of the ascending central series. \square

As a consequence of this result, we obtain the possible structure of the term \mathfrak{g}_2 .

Corollary 4.4. *Let (\mathfrak{g}, J) be an 8-dimensional NLA admitting a strongly non-nilpotent complex structure. Then,*

$$\dim \mathfrak{g}_1 = 1 \quad \text{and} \quad \mathfrak{g}_2 = \begin{cases} \mathfrak{g}_1 \oplus \mathcal{V}_2, \\ \mathfrak{g}_1 \oplus \mathcal{V}_2 \oplus \langle Y \rangle, \\ \mathfrak{g}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3, \end{cases}$$

where $Y \notin J\mathfrak{g}_1$ and $\mathcal{V}_2, \mathcal{V}_3$ are J -invariant subspaces of dimension 2 (see Notation 4.2).

4.2. The terms \mathfrak{g}_3 and \mathfrak{g}_4 . Once we have obtained the structure of \mathfrak{g}_2 , we next describe the spaces \mathfrak{g}_3 and \mathfrak{g}_4 depending on the dimension of \mathfrak{g}_2 . As a final result, we obtain a structural theorem for the ascending central series (see Theorem 4.1).

Let us remark that every case of Corollary 4.4 implies $\mathfrak{g}_1 \oplus \mathcal{V}_2 \subseteq \mathfrak{g}_2$. Hence, we can set

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathcal{V}_2 = \langle X_2, JX_2 \rangle.$$

This determines a family $\{X_i, JX_i\}_{i=1}^2$ that will be completed up to a J -adapted basis \mathcal{B} of the Lie algebra \mathfrak{g} . In addition, the Lie brackets within the previous family satisfy

$$[X_1, \cdot] = 0, \quad [X_2, \cdot], [JX_2, \cdot] \in \langle X_1 \rangle.$$

4.2.1. Study for $\dim \mathfrak{g}_2 = 3$. In this case, one clearly has $\mathfrak{g}_1 \oplus \mathcal{V}_2 = \mathfrak{g}_2$. As a first observation for \mathfrak{g}_3 , we prove the following result.

Lemma 4.5. *Let (\mathfrak{g}, J) be an 8-dimensional NLA endowed with an S_nN complex structure. If $\dim \mathfrak{g}_2 = 3$, then $\mathfrak{g}_3 \neq \mathfrak{g}_2 \oplus J\mathfrak{g}_1$.*

Proof. We argue by contradiction. Let us suppose that $\mathfrak{g}_3 = \mathfrak{g}_2 \oplus J\mathfrak{g}_1 = \mathcal{V}_1 \oplus \mathcal{V}_2$. Then, we can assume that $\mathcal{V}_3 \cap \mathfrak{g}_4 \neq \{0\}$; i.e., there exists an element X_3 in \mathcal{B} belonging to \mathfrak{g}_4 . The ascending central series starts as follows:

$$\begin{aligned} \mathfrak{g}_1 &= \langle X_1 \rangle, & \mathfrak{g}_2 &= \langle X_1, X_2, JX_2 \rangle, & \mathfrak{g}_3 &= \langle X_1, X_2, JX_1, JX_2 \rangle, \\ \mathfrak{g}_4 &\supseteq \langle X_1, X_2, X_3, JX_1, JX_2 \rangle. \end{aligned}$$

Up to this moment, we have fixed six elements in \mathcal{B} , namely, $\{X_i, JX_i\}_{i=1}^3$. Moreover, $\mathfrak{g} \setminus (\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3)$ is a J -invariant space of dimension 2 that we will call \mathcal{V}_4 . We will consider two different situations: $\mathcal{V}_4 \cap \mathfrak{g}_4 \neq \{0\}$ and $\mathcal{V}_4 \cap \mathfrak{g}_4 = \{0\}$.

Study of $\mathcal{V}_4 \cap \mathfrak{g}_4 \neq \{0\}$. Due to the previous hypothesis, there exists a new element in \mathfrak{g}_4 linearly independent of $\{X_i, JX_i\}_{i=1}^3$ that we will denote X_4 . In this way,

$$\mathfrak{g}_4 \supseteq \langle X_1, X_2, X_3, X_4, JX_1, JX_2 \rangle.$$

By Lemma 3.1(i) for $k = 4$, we have $\mathfrak{g}_4 = \mathfrak{g}$. Hence, it is possible to express the Lie brackets as follows, making use of the Nihenjuis condition and the nilpotency of \mathfrak{g} :
(15)

$$\begin{aligned} [X_1, \cdot] &= 0, & [X_3, JX_k]_{k=3,4} &= \sum_{i=1}^2 (b_{3k}^i X_i + c_{3k}^i JX_i), \\ [X_2, X_k]_{k=3,4} &= a_{2k}^1 X_1, & [X_4, JX_1] &= b_{41}^2 X_2 + c_{41}^2 JX_2, \\ [X_2, JX_1] &= 0, & [X_4, JX_2] &= b_{24}^1 X_1, \\ [X_2, JX_k]_{k=2,3,4} &= b_{2k}^1 X_1, & [X_4, JX_k]_{k=3,4} &= \sum_{i=1}^2 (b_{4k}^i X_i + c_{4k}^i JX_i), \\ [X_3, X_4] &= \sum_{i=1}^2 (a_{34}^i X_i + \alpha_{34}^i JX_i), & [JX_1, JX_2] &= 0, \\ [X_3, JX_1] &= b_{31}^2 X_2 + c_{31}^2 JX_2, & [JX_1, JX_k]_{k=3,4} &= c_{k1}^2 X_2 - b_{k1}^2 JX_2, \\ [X_3, JX_2] &= b_{23}^1 X_1, & [JX_2, JX_k]_{k=3,4} &= a_{2k}^1 X_1, \\ [JX_3, JX_4] &= \sum_{i=1}^2 (a_{34}^i - c_{34}^i + c_{43}^i) X_i + \sum_{i=1}^2 (\alpha_{34}^i + b_{34}^i - b_{43}^i) JX_i. \end{aligned}$$

The idea now is to impose the vanishing of the Jacobi identity (4) in order to get a contradiction. We sketch how to proceed. Consider $Jac(X_3, JX_3, Y)$ for $Y = X_4, JX_4$ and $Jac(X_4, JX_4, Y)$ for $Y = X_3, JX_3$ to obtain the following homogeneous systems of equations:

$$A \begin{pmatrix} b_{31}^2 \\ c_{31}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} b_{41}^2 \\ c_{41}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} c_{43}^1 - c_{34}^1 & b_{34}^1 - b_{43}^1 \\ -(b_{34}^1 - b_{43}^1) & c_{43}^1 - c_{34}^1 \end{pmatrix}.$$

In order to preserve the arrangement of the ascending central series, the determinant of A must be zero, i.e., $c_{43}^1 = c_{34}^1$ and $b_{43}^1 = b_{34}^1$. With these values of the parameters, we go back to the previous four Jacobi identities and obtain

$$B \begin{pmatrix} b_{31}^2 \\ c_{31}^2 \end{pmatrix} = c_{33}^1 \begin{pmatrix} c_{41}^2 \\ b_{41}^2 \end{pmatrix}, \quad B \begin{pmatrix} c_{41}^2 \\ b_{41}^2 \end{pmatrix} = c_{44}^1 \begin{pmatrix} b_{31}^2 \\ c_{31}^2 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} \alpha_{34}^1 & c_{34}^1 \\ c_{34}^1 & -\alpha_{34}^1 \end{pmatrix}.$$

The contradiction appears here, since it is not possible to have the determinant of B either equal to zero or different from zero if we want to preserve the desired ascending central series.

Study of $\mathcal{V}_4 \cap \mathfrak{g}_4 = \{0\}$. First, let us note that $\mathfrak{g}_5 = \mathfrak{g}$. The proof will be finished after analyzing two cases: $\mathcal{V}_3 \subset \mathfrak{g}_4$ and $\mathcal{V}_3 \not\subset \mathfrak{g}_4$.

- Let us consider $\mathcal{V}_3 \subset \mathfrak{g}_4$. This is equivalent to $\mathfrak{g}_4 = \mathfrak{g}_3 \oplus \mathcal{V}_3$, so in particular we observe that \mathfrak{g}_4 is J -invariant. The Lie brackets are the same as in (15) with the only exception being

$$[X_4, JX_4] = b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2 + c_{44}^3 JX_3.$$

Let us remark that one needs $(b_{44}^3, c_{44}^3) \neq (0, 0)$ in order to ensure that $\mathfrak{g}_4 \neq \mathfrak{g}$. We again compute different Jacobi identities to get a contradiction. More concretely, focusing on the triplets (X_2, JX_1, Y) , where $Y = X_3, X_4, JX_3, JX_4$, we can conclude that $b_{22}^1 = 0$. Furthermore, from (X_4, JX_4, Y) with $Y = X_2, JX_1, JX_2$, we obtain $a_{23}^1 = b_{23}^1 = b_{31}^2 = c_{31}^2 = 0$. Now, look at $Jac(X_3, JX_3, Y)$, for $Y = X_4, JX_4$, to get $b_{33}^2 = c_{33}^2 = 0$. Finally, working again with $Jac(X_4, JX_4, Y)$, this time for $Y = X_3, JX_3$, we reach the desired contradiction.

- We now focus on the case $\mathcal{V}_3 \not\subset \mathfrak{g}_4$, i.e., $\mathfrak{g}_4 = \mathfrak{g}_3 \oplus \langle X_3 \rangle$. We remark that the Lie brackets agree with (15), except for $[X_4, JX_4]$ (which coincides with

that defined a few lines above) and $[X_4, JX_3], [JX_3, JX_4]$ (which belong to \mathfrak{g}_4). More precisely, let

$$[X_4, JX_3] = b_{43}^1 X_1 + b_{43}^2 X_2 + b_{43}^3 X_3 + c_{43}^1 JX_1 + c_{43}^2 JX_2,$$

where $b_{43}^i, c_{43}^i \in \mathbb{R}$, for $i = 1, 2$. Applying the Nijenhuis condition, one obtains

$$[JX_3, JX_4] = \sum_{i=1}^2 (a_{34}^i - c_{34}^i + c_{43}^i) X_i + \sum_{i=1}^2 (a_{34}^i + b_{34}^i - b_{43}^i) JX_i - b_{43}^3 JX_3.$$

Since our Lie algebra \mathfrak{g} is nilpotent, we necessarily have $b_{43}^3 = 0$. Nevertheless, this choice makes $JX_3 \in \mathfrak{g}_4$, which is not allowed.

This completes the proof of the lemma. □

Lemma 4.6. *Let (\mathfrak{g}, J) be an 8-dimensional NLA endowed with an SnN complex structure. If $\dim \mathfrak{g}_2 = 3$ and $\mathfrak{g}_3 \cap J\mathfrak{g}_1 \neq \{0\}$, then the ascending central series of \mathfrak{g} satisfies*

$$\dim \mathfrak{g}_1 = 1, \quad \mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathcal{V}_2, \quad \mathfrak{g}_3 \text{ is } J\text{-invariant with } \dim \mathfrak{g}_3 = 6 \text{ or } 8,$$

\mathcal{V}_2 being a 2-dimensional J -invariant space.

Proof. Making use of the hypothesis and Lemma 4.5, we can find $X_1, X_2, X_3 \in \mathfrak{g}$ such that

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, JX_1, JX_2 \rangle.$$

We consider $\{X_i, JX_i\}_{i=1}^3$ belonging to the J -adapted basis \mathcal{B} and X_4, JX_4 generators of the J -invariant subspace $\mathcal{V}_4 = \mathfrak{g} \setminus (\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3)$. We can set the following brackets, where we have already applied the Nijenhuis condition and the nilpotency of \mathfrak{g} :

$$(16) \quad \begin{aligned} [X_1, \cdot] &= 0, & [X_4, JX_1] &= b_{41}^2 X_2 + c_{41}^2 JX_2, \\ [X_2, X_k]_{k=3,4} &= a_{2k}^1 X_1, & [X_4, JX_2] &= b_{24}^1 X_1, \\ [X_2, JX_1] &= 0, & [X_4, JX_k]_{k=3,4} & \text{ unknown,} \\ [X_2, JX_k]_{k=2,3,4} &= b_{2k}^1 X_1, & [JX_1, JX_2] &= 0, \\ [X_3, X_4] &= a_{34}^1 X_1 + a_{34}^2 X_2 & [JX_1, JX_k]_{k=3,4} &= c_{k1}^2 X_2 - b_{k1}^2 JX_2, \\ & \quad + a_{34}^3 JX_2, \\ [X_3, JX_1] &= b_{31}^2 X_2 + c_{31}^2 JX_2, & [JX_2, JX_k]_{k=3,4} &= a_{2k}^1 X_1, \\ [X_3, JX_2] &= b_{23}^1 X_1, & [JX_3, JX_4] & \text{ unknown,} \\ [X_3, JX_k]_{k=3,4} &= b_{3k}^1 X_1 + b_{3k}^2 X_2 & & \\ & \quad + c_{3k}^2 JX_2, \end{aligned}$$

where all the coefficients are real numbers.

We should accomplish the study of \mathfrak{g}_3 . The idea is the following one: first, we will suppose that \mathfrak{g}_3 is not J -invariant and then contradict the assumption. Afterwards, we will show that valid solutions exist when \mathfrak{g}_3 is J -invariant of dimension both 6 and 8.

Study when \mathfrak{g}_3 is not J -invariant. The possible dimensions for this space are 5 and 6. Let us separately analyze each case.

- Let $\dim \mathfrak{g}_3 = 6$. This implies that $\mathcal{V}_4 \cap \mathfrak{g}_3 \neq \{0\}$, and we can set

$$\mathfrak{g}_3 = \langle X_1, X_2, X_3, X_4, JX_1, JX_2 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

Hence, the unknown brackets in (16) can be determined:

$$\begin{aligned} [X_4, JX_k]_{k=3,4} &= b_{4k}^1 X_1 + b_{4k}^2 X_2 + c_{4k}^2 JX_2, \\ [JX_3, JX_4] &= a_{34}^1 X_1 + (a_{34}^2 - c_{34}^2 + c_{43}^2) X_2 + (b_{34}^1 - b_{43}^1) JX_1 \\ &\quad + (\alpha_{34}^2 + b_{34}^2 - b_{43}^2) JX_2. \end{aligned}$$

Observe that $b_{43}^1 \neq b_{34}^1$, or otherwise $\mathfrak{g}_3 = \mathfrak{g}$. Computing the Jacobi identities $Jac(JX_3, JX_4, Y)$ with $Y = X_3, X_4$, we are led to consider $b_{31}^2 = b_{41}^2 = c_{31}^2 = c_{41}^2 = 0$, but then one observes that indeed $JX_1 \in \mathfrak{g}_1$, which is a contradiction.

- We now assume that $\dim \mathfrak{g}_3 = 5$. As $J\mathfrak{g}_3 \neq \mathfrak{g}_3$, we have

$$\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle,$$

and we need to turn our attention to \mathfrak{g}_4 . Two different situations arise: $\mathcal{V}_4 \cap \mathfrak{g}_4 \neq \{0\}$ and $\mathcal{V}_4 \cap \mathfrak{g}_4 = \{0\}$.

▷ If $\mathcal{V}_4 \cap \mathfrak{g}_4 \neq \{0\}$, the unknown real Lie brackets can be set as follows:

$$\begin{aligned} [X_4, JX_3] &= b_{43}^1 X_1 + b_{43}^2 X_2 + c_{43}^1 JX_1 + c_{43}^2 JX_2, \\ [X_4, JX_4] &= b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2, \\ [JX_3, JX_4] &= (a_{34}^1 + c_{43}^1) X_1 + (a_{34}^2 - c_{34}^2 + c_{43}^2) X_2 + (b_{34}^1 - b_{43}^1) JX_1 \\ &\quad + (\alpha_{34}^2 + b_{34}^2 - b_{43}^2) JX_2. \end{aligned}$$

Observe that one necessarily has $\mathfrak{g}_4 = \mathfrak{g}$. The contradiction arises from the Jacobi identity. More concretely, imposing the vanishing of $Jac(X_2, JX_1, Y)$, for $Y = X_3, JX_3, X_4, JX_4$, one obtains $b_{22}^1 = 0$ in order to ensure that $JX_1 \notin \mathfrak{g}_1$. Next, if we compute $Jac(X_3, JX_3, Y)$ for $Y = X_4, JX_4$ and $Jac(X_4, JX_4, Z)$ for $Z = X_3, JX_1$, we must assume that $b_{31}^2 = c_{31}^2 = 0$ if we want to preserve the arrangement of the series. In a similar way, $Jac(X_3, JX_1, Y)$ with $Y = X_4, JX_4$ imply $a_{23}^1 = b_{23}^1 = 0$. Coming back to $Jac(X_3, JX_3, X_4)$ and $Jac(X_3, JX_3, JX_4)$, one gets $b_{33}^2 = c_{33}^2 = 0$. Finally, taking into account the previous fixed values, the contradiction comes from $Jac(X_4, JX_4, JX_3)$.

▷ If $\mathcal{V}_4 \cap \mathfrak{g}_4 = \{0\}$, the structure of the ascending central series is

$$\begin{aligned} \mathfrak{g}_1 &= \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle, \\ \mathfrak{g}_4 &= \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}. \end{aligned}$$

We note that the only different bracket between this case and the previous one is

$$[X_4, JX_4] = b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2 + c_{44}^3 JX_3,$$

where $b_{44}^i, c_{44}^i \in \mathbb{R}$, for $i = 1, 2, 3$. For this reason, one needs $c_{44}^3 \neq 0$ in order to effectively separate the two situations. Repeating the same Jacobi computations as above, we conclude that this case is not valid.

This proves that \mathfrak{g}_3 must be J -invariant.

Study when \mathfrak{g}_3 is J -invariant. There are two possibilities for \mathfrak{g}_3 : either $\dim \mathfrak{g}_3 = 6$ or $\dim \mathfrak{g}_3 = 8$. In what follows, we will see that in both cases one can find an appropriate choice of the parameters that fulfill the Jacobi identity and preserve the desired ascending central series.

- If $\dim \mathfrak{g}_3 = 6$, then

$$\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

The unknown brackets in (16) can be expressed as:

$$\begin{aligned} [X_4, JX_3] &= b_{34}^1 X_1 + b_{43}^2 X_2 + c_{43}^2 JX_2, \\ [X_4, JX_4] &= b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2 + c_{44}^3 JX_3, \\ [JX_3, JX_4] &= a_{34}^1 X_1 + (a_{34}^2 - c_{34}^2 + c_{43}^2) X_2 + (\alpha_{34}^2 + b_{34}^2 - b_{43}^2) JX_2. \end{aligned}$$

From $Jac(X_2, JX_1, Y)$ for $Y = X_3, X_4, JX_3, JX_4$, we are able to conclude that $b_{22}^1 = 0$, since $(b_{31}^2, c_{31}^2, b_{41}^2, c_{41}^2) \neq (0, 0, 0, 0)$ to ensure $JX_1 \notin \mathfrak{g}_1$. Then, calculating $Jac(X_4, JX_4, Y)$ with $Y = X_2, JX_1, JX_2$, two systems of equations arise:

$$C \begin{pmatrix} a_{23}^1 \\ b_{23}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C \begin{pmatrix} c_{31}^2 \\ b_{31}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{where } C = \begin{pmatrix} b_{44}^3 & c_{44}^3 \\ c_{44}^3 & -b_{44}^3 \end{pmatrix}.$$

Using $Jac(X_4, JX_4, Y)$ for $Y = X_3, JX_3$ when $(b_{44}^3, c_{44}^3) \neq (0, 0)$, and $Jac(X_4, JX_4, X_3)$, $Jac(X_3, JX_1, Y)$, $Jac(X_3, JX_3, Y)$, for $Y = X_4, JX_4$, when $(b_{44}^3, c_{44}^3) = (0, 0)$, one obtains

$$a_{23}^1 = b_{23}^1 = b_{31}^2 = b_{33}^2 = c_{31}^2 = c_{33}^2 = 0.$$

We note that 19 structure constants are still undetermined. They must be chosen in such a way that the desired ascending central series is preserved and the 3 equations given by the non-trivial Jacobi identities are fulfilled. Here, we simply note that particular solutions satisfying the two previous requirements can be found.

- If $\dim \mathfrak{g}_3 = 8$, then $\mathfrak{g}_3 = \mathfrak{g}$ and the unknown brackets in (16) are given by:

$$\begin{aligned} [X_4, JX_3] &= b_{34}^1 X_1 + b_{43}^2 X_2 + c_{43}^2 JX_2, \\ [X_4, JX_4] &= b_{44}^1 X_1 + b_{44}^2 X_2 + c_{44}^2 JX_2, \\ [JX_3, JX_4] &= a_{34}^1 X_1 + (a_{34}^2 - c_{34}^2 + c_{43}^2) X_2 + (\alpha_{34}^2 + b_{34}^2 - b_{43}^2) JX_2, \end{aligned}$$

where $b_{44}^1, b_{4k}^2, c_{4k}^2 \in \mathbb{R}$ for $k = 1, 3, 4$. Let us study the Jacobi identity (4). From $Jac(X_2, JX_1, Y)$, for $Y = X_3, X_4, JX_3, JX_4$, we get

$$b_{22}^1 c_{31}^2 = b_{22}^1 c_{41}^2 = b_{22}^1 b_{31}^2 = b_{22}^1 b_{41}^2 = 0.$$

Since $(b_{31}^2, c_{31}^2, b_{41}^2, c_{41}^2) \neq (0, 0, 0, 0)$ is needed to ensure that $JX_1 \notin \mathfrak{g}_1$, one immediately concludes that $b_{22}^1 = 0$. The remaining Jacobi identities give a quadratic system with 8 equations and 22 unknowns that admits particular solutions fulfilling all the prerequisites.

This concludes the proof of the lemma. □

Lemma 4.7. *Let (\mathfrak{g}, J) be an 8-dimensional NLA endowed with an $S_n N$ complex structure. If $\dim \mathfrak{g}_2 = 3$ and $\mathfrak{g}_3 \cap J\mathfrak{g}_1 = \{0\}$, then the ascending central series of \mathfrak{g} satisfies*

$$\dim \mathfrak{g}_1 = 1, \quad \mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathcal{V}_2, \quad \mathfrak{g}_3 = \mathfrak{g}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3, \quad \mathfrak{g}_4 \text{ is } J\text{-invariant,}$$

where $\mathcal{V}_2, \mathcal{V}_3$ are 2-dimensional J -invariant spaces.

Proof. By hypothesis, one can find $X_1, X_2, X_3 \in \mathfrak{g}$ such that

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, JX_2 \rangle.$$

Furthermore, we know that $JX_1 \notin \mathfrak{g}_3$, so \mathfrak{g} must be at least 4-step nilpotent. Let us take $\{X_i, JX_i\}_{i=1}^3$ in the J -adapted basis \mathcal{B} and X_4, JX_4 generators of the J -invariant subspace $\mathcal{V}_4 = \mathfrak{g} \setminus (\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3)$. The following real brackets can be set, bearing in mind the Nijenhuis condition and the nilpotency of \mathfrak{g} :

$[X_1, \cdot] = 0,$	$[X_4, JX_1]$ unknown,
$[X_2, X_k]_{k=3,4} = a_{2k}^1 X_1,$	$[X_4, JX_2] = b_{24}^1 X_1,$
$[X_2, JX_1] = 0,$	$[X_4, JX_k]_{k=3,4}$ unknown,
$[X_2, JX_k]_{k=2,3,4} = b_{2k}^1 X_1,$	$[JX_1, JX_2] = 0,$
$[X_3, X_4] = a_{34}^1 X_1 + a_{34}^2 X_2 + \alpha_{34}^2 JX_2,$	$[JX_1, JX_3] = c_{31}^2 X_2 - b_{31}^2 JX_2,$
$[X_3, JX_1] = b_{31}^2 X_2 + c_{31}^2 JX_2,$	$[JX_1, JX_4]$ unknown,
$[X_3, JX_2] = b_{23}^1 X_1,$	$[JX_2, JX_k]_{k=3,4} = a_{2k}^1 X_1$
$[X_3, JX_k]_{k=3,4} = b_{3k}^1 X_1 + b_{3k}^2 X_2 + c_{3k}^2 JX_2,$	$[JX_3, JX_4]$ unknown.

To prove the result it suffices to follow the next scheme. First, one checks by contradiction that $\mathcal{V}_4 \cap \mathfrak{g}_3 = \{0\}$. As a consequence, one has $4 \leq \dim \mathfrak{g}_3 \leq 5$. Then, the 4-dimensional case is rejected (studying $\mathcal{V}_4 \cap \mathfrak{g}_4 \neq \{0\}$ and $\mathcal{V}_4 \cap \mathfrak{g}_4 = \{0\}$), and finally one shows that solutions exist for the 5-dimensional case. This completes the study of every possibility, and thus our result is attained. \square

As a consequence of Lemmas 4.5, 4.6, and 4.7, we reach the following structure result.

Proposition 4.8. *Let (\mathfrak{g}, J) be an 8-dimensional NLA endowed with an $S_n N$ complex structure. If $\dim \mathfrak{g}_2 = 3$, then the ascending central series of \mathfrak{g} satisfies*

$$\dim \mathfrak{g}_1 = 1, \quad \mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathcal{V}_2,$$

and one of the following possibilities:

- (i) $\mathfrak{g}_3 = \mathfrak{g};$
- (ii) $\mathfrak{g}_3 = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3, \quad \mathfrak{g}_4 = \mathfrak{g};$
- (iii) $\mathfrak{g}_3 = \mathfrak{g}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3, \quad \mathfrak{g}_4 = \mathfrak{g};$
- (iv) $\mathfrak{g}_3 = \mathfrak{g}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3, \quad \mathfrak{g}_4 = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3, \quad \mathfrak{g}_5 = \mathfrak{g},$

where \mathcal{V}_i are 2-dimensional J -invariant spaces.

4.2.2. *Study for $\dim \mathfrak{g}_2 = 4$.* After the analysis of $\dim \mathfrak{g}_2 = 3$, let us suppose that $\dim \mathfrak{g}_2 = 4$. As a consequence of Corollary 4.4, we can find $X_1, X_2, X_3 \in \mathfrak{g}$ such that

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle.$$

We consider $\{X_i, JX_i\}_{i=1}^3$ in the J -adapted basis \mathcal{B} and denote by X_4, JX_4 the generators of the J -invariant subspace $\mathcal{V}_4 = \mathfrak{g} \setminus (\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3)$. The Lie brackets of

\mathfrak{g} have the following form:

$$\begin{aligned}
 (17) \quad & [X_1, \cdot] = 0, & [X_4, JX_1] \text{ unknown}, \\
 & [X_2, X_k]_{k=3,4} = a_{2k}^1 X_1, & [X_4, JX_2] = b_{24}^1 X_1, \\
 & [X_2, JX_1] = 0, & [X_4, JX_k]_{k=3,4} \text{ unknown}, \\
 & [X_2, JX_k]_{k=2,3,4} = b_{2k}^1 X_1, & [JX_1, JX_k]_{k=2,3} = 0, \\
 & [X_3, X_4] = a_{34}^1 X_1, & [JX_1, JX_4] \text{ unknown}, \\
 & [X_3, JX_1] = 0, & [JX_2, JX_k]_{k=3,4} = a_{2k}^1 X_1, \\
 & [X_3, JX_2] = b_{23}^1 X_1, & [JX_3, JX_4] \text{ unknown}, \\
 & [X_3, JX_k]_{k=3,4} = b_{3k}^1 X_1,
 \end{aligned}$$

where all the coefficients are real numbers.

Proposition 4.9. *Let (\mathfrak{g}, J) be an 8-dimensional NLA endowed with an $S_n N$ complex structure. If $\dim \mathfrak{g}_2 = 4$, then the ascending central series of \mathfrak{g} satisfies*

$$\dim \mathfrak{g}_1 = 1, \quad \mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathcal{V}_2 \oplus \langle Y \rangle, \quad \mathfrak{g}_3 \text{ is } J\text{-invariant with } \dim \mathfrak{g}_3 = 6 \text{ or } 8,$$

where $Y \notin J\mathfrak{g}_1$ and \mathcal{V}_2 is a 2-dimensional J -invariant space.

Proof. As a consequence of the previous lines we directly focus on \mathfrak{g}_3 . We will first see by contradiction that \mathfrak{g}_3 must be J -invariant and then provide two valid arrangements of the ascending central series.

Study when \mathfrak{g}_3 is not J -invariant. Due to the nilpotency of \mathfrak{g} , one has $5 \leq \dim \mathfrak{g}_3 \leq 6$. We next analyze these two possibilities.

- Let $\dim \mathfrak{g}_3 = 6$. As a consequence of our assumptions, one directly has $\mathcal{V}_4 \cap \mathfrak{g}_3 \neq \{0\}$, i.e.,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \quad \mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, X_4, JX_2 \rangle.$$

This arrangement together with the Nijenhuis condition allows us to fix the unknown real brackets in (17):

$$\begin{aligned}
 (18) \quad & [X_4, JX_1] = b_{41}^2 X_2 + b_{41}^3 X_3 + c_{41}^2 JX_2, \\
 & [X_4, JX_3] = b_{43}^1 X_1 + b_{43}^2 X_2 + c_{43}^2 JX_2, \\
 & [X_4, JX_4] = b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^2 JX_2, \\
 & [JX_1, JX_4] = c_{41}^2 X_2 - b_{41}^2 JX_2 - b_{41}^3 JX_3, \\
 & [JX_3, JX_4] = a_{34}^1 X_1 + c_{43}^2 X_2 + (b_{34}^1 - b_{43}^1) JX_1 - b_{43}^2 JX_2.
 \end{aligned}$$

Moreover, \mathfrak{g}_3 must contain a 4-dimensional J -invariant subspace. After an arrangement of generators, we can ensure that one of the spaces \mathcal{V}_i , where $i = 1, 3, 4$, lies in \mathfrak{g}_3 . We separately study each possibility, discarding all of them:

- ▷ If $\mathcal{V}_4 \subset \mathfrak{g}_3$, then we see from (18) that $\mathfrak{g}_3 = \mathfrak{g}$, which contradicts our assumption.
- ▷ Let $\mathcal{V}_3 \subset \mathfrak{g}_3$. The Lie brackets in (18) tell us that $b_{43}^1 = b_{34}^1$ and $b_{41}^3 \neq 0$ (to ensure $\mathfrak{g}_3 \neq \mathfrak{g}$). Moreover, from $Jac(X_4, JX_4, JX_1)$ we need $b_{43}^2 = c_{43}^2 = 0$, but this choice makes $JX_3 \in \mathfrak{g}_2$, which is a contradiction.
- ▷ Suppose that $\mathcal{V}_1 \subset \mathfrak{g}_3$. Then, from (18) we find $b_{41}^3 = 0$ and $b_{43}^1 \neq b_{34}^1$. Using $Jac(X_4, JX_4, JX_3)$ we also get $b_{41}^2 = c_{41}^2 = 0$. However, these values lead to $JX_1 \in \mathfrak{g}_1$, which is not possible.

- Let $\dim \mathfrak{g}_3 = 5$. Notice that $\mathcal{V}_4 \cap \mathfrak{g}_3 = \{0\}$ (otherwise, one would apply Lemma 3.1(ii) with $r = 3$ and reach a contradiction), so \mathfrak{g}_3 must contain a 4-dimensional J -invariant subspace. Similarly to the previous case, we can assume that one of the spaces \mathcal{V}_1 or \mathcal{V}_3 is contained in \mathfrak{g}_3 . Again, none of the situations are possible.

▷ Let us suppose that $\mathcal{V}_1 \subset \mathfrak{g}_3$. The first terms of the ascending central series are

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle.$$

The unknown brackets in (17) can be set as follows. The integrability of J and the nilpotency of \mathfrak{g} imply that

$$\begin{aligned} [X_4, JX_1] &= b_{41}^2 X_2 + c_{41}^2 JX_2, \\ [JX_1, JX_4] &= c_{41}^2 X_2 - b_{41}^2 JX_2. \end{aligned}$$

In addition, if either $\mathcal{V}_4 \cap \mathfrak{g}_4 \neq \{0\}$ or $\mathcal{V}_4 \cap \mathfrak{g}_4 = \{0\}$, it is easy to see that

$$\begin{aligned} [X_4, JX_3] &= b_{43}^1 X_1 + b_{43}^2 X_2 + c_{43}^1 JX_1 + c_{43}^2 JX_2, \\ [X_4, JX_4] &= b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2 + c_{44}^3 JX_3, \\ [JX_3, JX_4] &= (a_{34}^1 + c_{43}^1) X_1 + c_{43}^2 X_2 + (b_{34}^1 - b_{43}^1) JX_1 - b_{43}^2 JX_2, \end{aligned}$$

where \mathfrak{g} is 4-step nilpotent if $c_{44}^3 = 0$ and 5-step nilpotent otherwise. Imposing the vanishing of $Jac(X_4, JX_4, JX_3)$, a contradiction is obtained.

▷ Let us consider $\mathcal{V}_3 \subset \mathfrak{g}_3$. In this case, the arrangement of the ascending central series starts as follows:

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle.$$

The unknown brackets in (17) can be determined as follows. Due to the Nijenhuis condition and the nilpotency of the algebra, one has

$$\begin{aligned} [X_4, JX_3] &= b_{34}^1 X_1 + b_{43}^2 X_2 + c_{43}^2 JX_2, \\ [JX_3, JX_4] &= a_{34}^1 X_1 + c_{43}^2 X_2 - b_{43}^2 JX_2. \end{aligned}$$

Furthermore, for both $\mathcal{V}_4 \cap \mathfrak{g}_4 \neq \{0\}$ and $\mathcal{V}_4 \cap \mathfrak{g}_4 = \{0\}$, it is possible to set

$$\begin{aligned} [X_4, JX_1] &= b_{41}^2 X_2 + b_{41}^3 X_3 + c_{41}^2 JX_2 + c_{41}^3 JX_3, \\ [X_4, JX_4] &= b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2 + c_{44}^3 JX_3, \\ [JX_1, JX_4] &= c_{41}^2 X_2 + c_{41}^3 X_3 - b_{41}^2 JX_2 - b_{41}^3 JX_3, \end{aligned}$$

where \mathfrak{g} is 4-step nilpotent if $c_{44}^1 = 0$ and 5-step nilpotent otherwise. The vanishing of $Jac(X_4, JX_4, JX_1)$ generates a new contradiction.

Observe that we have rejected the case in which \mathfrak{g}_3 is not J -invariant. Hence, we are led to consider the opposite situation.

Study when \mathfrak{g}_3 is J -invariant. One necessarily has $\mathfrak{g}_3 \supseteq \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ and

$$\begin{aligned} [X_4, JX_1] &= b_{41}^2 X_2 + c_{41}^2 JX_2, \\ [X_4, JX_3] &= b_{34}^1 X_1 + b_{43}^2 X_2 + c_{43}^2 JX_2, \\ (19) \quad [X_4, JX_4] &= b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2 + c_{44}^3 JX_3, \\ [JX_1, JX_4] &= c_{41}^2 X_2 - b_{41}^2 JX_2, \\ [JX_3, JX_4] &= a_{34}^1 X_1 + c_{43}^2 X_2 - b_{43}^2 JX_2, \end{aligned}$$

where \mathfrak{g} is 4-step nilpotent if $c_{44}^1 = c_{44}^3 = 0$ and 5-step nilpotent otherwise. From $Jac(X_2, JX_3, Y)$ and $Jac(X_3, JX_1, Y)$ for $Y = X_4, JX_4$, one obtains $a_{23}^1 = b_{22}^1 = b_{23}^1 = 0$ in order to preserve the desired arrangement of the series. Furthermore, it is possible to give particular solutions to the remaining Jacobi identities, showing that our two cases are valid. \square

4.2.3. *Study for $\dim \mathfrak{g}_2 = 5$.* We finally focus our attention on $\dim \mathfrak{g}_2 = 5$. In this case it is possible to find $X_1, X_2, X_3 \in \mathfrak{g}$ such that

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle.$$

Let $\{X_i, JX_i\}_{i=1}^3$ belong to the J -adapted basis \mathcal{B} and consider X_4, JX_4 generators of the J -invariant subspace $\mathcal{V}_4 = \mathfrak{g} \setminus (\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3)$. The Lie brackets of \mathfrak{g} are given by

$$(20) \quad \begin{aligned} [X_1, \cdot] &= 0, & [X_4, JX_1] & \text{unknown}, \\ [X_2, X_k]_{k=3,4} &= a_{2k}^1 X_1, & [X_4, JX_2] &= b_{24}^1 X_1, \\ [X_2, JX_1] &= 0, & [X_4, JX_3] &= b_{34}^1 X_1, \\ [X_2, JX_k]_{k=2,3,4} &= b_{2k}^1 X_1, & [X_4, JX_4] & \text{unknown}, \\ [X_3, X_4] &= a_{34}^1 X_1, & [JX_1, JX_k]_{k=2,3} &= 0, \\ [X_3, JX_1] &= 0, & [JX_1, JX_4] & \text{unknown}, \\ [X_3, JX_2] &= b_{23}^1 X_1, & [JX_2, JX_k]_{k=3,4} &= a_{2k}^1 X_1, \\ [X_3, JX_k]_{k=3,4} &= b_{3k}^1 X_1, & [JX_3, JX_4] &= a_{34}^1 X_1, \end{aligned}$$

where all the coefficients are real numbers.

Proposition 4.10. *Let (\mathfrak{g}, J) be an 8-dimensional NLA endowed with an SnN complex structure. If $\dim \mathfrak{g}_2 = 5$, then the ascending central series of \mathfrak{g} satisfies*

$$\dim \mathfrak{g}_1 = 1, \quad \mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3, \quad \mathfrak{g}_3 \text{ is } J\text{-invariant},$$

where \mathcal{V}_i are 2-dimensional J -invariant spaces.

Proof. We observe that $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathcal{V}_4 \oplus J\mathfrak{g}_1$, so we are in the conditions of Lemma 3.2 with $k = 2$. As a consequence, we can directly conclude that either $\mathfrak{g}_3 = \mathfrak{g}$ if $\mathcal{V}_4 \cap \mathfrak{g}_3 \neq \{0\}$ or $\mathfrak{g}_4 = \mathfrak{g}$ if $\mathcal{V}_4 \cap \mathfrak{g}_3 = \{0\}$. In the second case, we notice that $\mathcal{V}_1 \subset \mathfrak{g}_3$, which means $\mathfrak{g}_3 = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$. Hence, in both cases \mathfrak{g}_3 is J -invariant. To finish the proof, it suffices to show that the two situations are valid.

Let us first assume that $\mathcal{V}_4 \cap \mathfrak{g}_3 \neq \{0\}$. Then,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g},$$

and we can set all the unknown real brackets in (20):

$$(21) \quad \begin{aligned} [X_4, JX_1] &= b_{41}^2 X_2 + b_{41}^3 X_3 + c_{41}^2 JX_2 + c_{41}^3 JX_3, \\ [X_4, JX_4] &= b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^2 JX_2 + c_{44}^3 JX_3, \\ [JX_1, JX_4] &= c_{41}^2 X_2 + c_{41}^3 X_3 - b_{41}^2 JX_2 - b_{41}^3 JX_3. \end{aligned}$$

The Jacobi identity gives rise to a quadratic system with 9 equations and 17 (real) unknowns that admit appropriate solutions. Hence, this case leads to a first positive result.

We now study the opposite situation, namely, $\mathcal{V}_4 \cap \mathfrak{g}_3 = \{0\}$. Then,

$$\begin{aligned} \mathfrak{g}_1 &= \langle X_1 \rangle, & \mathfrak{g}_2 &= \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \\ \mathfrak{g}_3 &= \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, & \mathfrak{g}_4 &= \mathfrak{g}, \end{aligned}$$

and we have

$$\begin{aligned} [X_4, JX_1] &= b_{41}^2 X_2 + b_{41}^3 X_3 + c_{41}^2 JX_2 + c_{41}^3 JX_3, \\ [X_4, JX_4] &= b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2 + c_{44}^3 JX_3, \\ [JX_1, JX_4] &= c_{41}^2 X_2 + c_{41}^3 X_3 - b_{41}^2 JX_2 - b_{41}^3 JX_3. \end{aligned}$$

In particular, notice that $c_{44}^1 \neq 0$, or otherwise we would go back to the first case. Also here it is possible to find a solution to the Jacobi identities preserving the arrangement of the ascending central series. \square

As a consequence of the previous results, we have found every 8-dimensional NLA \mathfrak{g} admitting an SnN complex structure J . Indeed, we have provided a J -adapted basis of each \mathfrak{g} additionally adapted to the structure of the ascending central series $\{\mathfrak{g}_k\}_k$. This fact motivates the following definition.

Definition 4.11. Let \mathfrak{g} be an s -step nilpotent Lie algebra of dimension $2n$ endowed with an almost-complex structure J . A J -adapted basis $\mathcal{B} = \{X_k, JX_k\}_{k=1}^n$ of \mathfrak{g} will be called *doubly adapted* if there is a permutation $\mathcal{B}^\sigma = \{V_1, \dots, V_{2n}\}$ of the elements of \mathcal{B} such that \mathcal{B}^σ is adapted to the ascending central series $\{\mathfrak{g}_k\}_k$, i.e.,

$$\begin{aligned} \mathfrak{g}_1 &= \langle V_1, \dots, V_{m_1} \rangle \subset \mathfrak{g}_2 = \langle V_1, \dots, V_{m_1}, V_{m_1+1}, \dots, V_{m_2} \rangle \subset \dots \subset \mathfrak{g}_s = \mathfrak{g} \\ &= \langle V_1, \dots, V_{2n} \rangle, \end{aligned}$$

being $m_k = \dim \mathfrak{g}_k$.

4.3. Complex structure equations. In the previous section, we have completely determined those NLAs \mathfrak{g} of dimension 8 admitting an SnN complex structure J . Here, we describe the complex structure equations for every such pair (\mathfrak{g}, J) .

Let \mathfrak{g} be an NLA of dimension 8 endowed with a strongly non-nilpotent complex structure. Let $\mathcal{B} = \{X_1, X_2, X_3, X_4, JX_1, JX_2, JX_3, JX_4\}$ be the doubly adapted basis that we have constructed in each case of Section 4.2, in this specific order. Consider its dual basis $\mathcal{B}^* = \{e^1, \dots, e^8\}$. We recall that the differential of any element $e \in \mathfrak{g}^*$ can be computed using the Lie brackets of \mathfrak{g} by means of the well-known formula

$$de(X, Y) = -e([X, Y]),$$

where $X, Y \in \mathfrak{g}$. Since J is a complex structure, one can then construct the $(1, 0)$ -basis

$$(22) \quad \omega^1 = e^4 - ie^8, \quad \omega^2 = e^3 - ie^7, \quad \omega^3 = e^2 - ie^6, \quad \omega^4 = e^1 - ie^5,$$

and calculate the (complex) structure equations of (\mathfrak{g}, J) . We present them in different propositions according to the dimension of \mathfrak{g}_2 .

Proposition 4.12. *Let J be a strongly non-nilpotent complex structure on an 8-dimensional nilpotent Lie algebra \mathfrak{g} such that $\dim \mathfrak{g}_1 = 1$ and $\dim \mathfrak{g}_2 = 3$. The pair (\mathfrak{g}, J) is parametrized by*

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = A\omega^{1\bar{1}} - B(\omega^{14} - \omega^{1\bar{4}}), \\ d\omega^3 = (C - D)\omega^{12} - E(\omega^{14} - \omega^{1\bar{4}}) + F\omega^{1\bar{1}} + (G + D)\omega^{1\bar{2}} - H(\omega^{24} - \omega^{2\bar{4}}) \\ \quad + (C - G)\omega^{2\bar{1}} + K\omega^{2\bar{2}}, \\ d\omega^4 = L\omega^{1\bar{1}} + M\omega^{1\bar{2}} + N\omega^{1\bar{3}} - \bar{M}\omega^{2\bar{1}} + i s\omega^{2\bar{2}} + P\omega^{2\bar{3}} - \bar{N}\omega^{3\bar{1}} - \bar{P}\omega^{3\bar{2}}, \end{cases}$$

where the coefficients are complex numbers with $s \in \mathbb{R}$ and satisfy the Jacobi identity. Moreover:

- (i) if $(\dim \mathfrak{g}_k)_k = (1, 3, 8)$ then $A = B = 0$ and $\Re L = 0$,
- (ii) if $(\dim \mathfrak{g}_k)_k = (1, 3, 6, 8)$ then $B = H = K = P = 0$,
- (iii) if $(\dim \mathfrak{g}_k)_k = (1, 3, 5, 8)$ then $K = P = 0$ and $\Re L = 0$,
- (iv) if $(\dim \mathfrak{g}_k)_k = (1, 3, 5, 6, 8)$ then $H = K = P = s = 0$ and $\Re L \neq 0$.

Proof. On the one hand, from the Lie brackets obtained in the proof of Lemma 4.6 one has real structure equations:

$$\left\{ \begin{array}{l} de^1 = -a_{23}^1 e^{23} - a_{24}^1 e^{24} - b_{23}^1 e^{27} - b_{24}^1 e^{28} - a_{34}^1 e^{34} - b_{23}^1 e^{36} - b_{33}^1 e^{37} \\ \quad - b_{34}^1 e^{38} - b_{24}^1 e^{46} - b_{34}^1 e^{47} - b_{44}^1 e^{48} - a_{23}^1 e^{67} - a_{24}^1 e^{68} - a_{34}^1 e^{78}, \\ de^2 = -a_{34}^2 e^{34} - b_{31}^2 e^{35} - b_{33}^2 e^{37} - b_{34}^2 e^{38} - b_{41}^2 e^{45} - b_{43}^2 e^{47} \\ \quad - b_{44}^2 e^{48} - c_{31}^2 e^{57} - c_{41}^2 e^{58} - (a_{34}^2 - c_{34}^2 + c_{43}^2) e^{78}, \\ de^3 = -b_{44}^3 e^{48}, \\ de^4 = 0, \\ de^5 = -c_{44}^1 e^{48}, \\ de^6 = -\alpha_{34}^2 e^{34} - c_{31}^2 e^{35} - c_{33}^2 e^{37} - c_{34}^2 e^{38} - c_{41}^2 e^{45} - c_{43}^2 e^{47} \\ \quad - c_{44}^2 e^{48} + b_{31}^2 e^{57} + b_{41}^2 e^{58} - (\alpha_{34}^2 + b_{34}^2 - b_{43}^2) e^{78}, \\ de^7 = -c_{44}^3 e^{48}, \\ de^8 = 0, \end{array} \right.$$

where the structure constants satisfy $b_{44}^3 = c_{44}^1 = c_{44}^3 = 0$ for $(\dim \mathfrak{g}_k)_k = (1, 3, 8)$ and $a_{23}^1 = b_{23}^1 = b_{31}^2 = b_{33}^2 = c_{31}^2 = c_{33}^2 = 0$ for $(\dim \mathfrak{g}_k)_k = (1, 3, 6, 8)$.

On the other hand, using the Lie brackets found in the proof of Lemma 4.7 we get

$$\left\{ \begin{array}{l} de^1 = -a_{24}^1 e^{24} - b_{24}^1 e^{28} - a_{34}^1 e^{34} - b_{33}^1 e^{37} - b_{34}^1 e^{38} \\ \quad - b_{24}^1 e^{46} - b_{34}^1 e^{47} - b_{44}^1 e^{48} - a_{24}^1 e^{68} - a_{34}^1 e^{78}, \\ de^2 = -a_{34}^2 e^{34} - b_{31}^2 e^{35} - b_{34}^2 e^{38} - b_{41}^2 e^{45} - b_{43}^2 e^{47} \\ \quad - b_{44}^2 e^{48} - c_{31}^2 e^{57} - c_{41}^2 e^{58} - (a_{34}^2 - c_{34}^2 + c_{43}^2) e^{78}, \\ de^3 = -b_{41}^3 e^{45} - b_{44}^3 e^{48} - c_{41}^3 e^{58}, \\ de^4 = 0, \\ de^5 = -c_{44}^1 e^{48}, \\ de^6 = -\alpha_{34}^2 e^{34} - c_{31}^2 e^{35} - c_{34}^2 e^{38} - c_{41}^2 e^{45} - c_{43}^2 e^{47} \\ \quad - c_{44}^2 e^{48} + b_{31}^2 e^{57} + b_{41}^2 e^{58} - (\alpha_{34}^2 + b_{34}^2 - b_{43}^2) e^{78}, \\ de^7 = -c_{41}^3 e^{45} - c_{44}^3 e^{48} + b_{41}^3 e^{58}, \\ de^8 = 0, \end{array} \right.$$

where the structure constants satisfy $c_{44}^1 = 0$ for $(\dim \mathfrak{g}_k)_k = (1, 3, 5, 8)$ and $b_{31}^2 = c_{31}^2 = b_{33}^2 = 0$ for $(\dim \mathfrak{g}_k)_k = (1, 3, 5, 6, 8)$.

Constructing the $(1, 0)$ -basis $\{\omega^k\}_{k=1}^4$ described in (22) we get the desired result, simply defining the following complex numbers:

$$\begin{aligned}
 (23) \quad & A = \frac{c_{44}^3 + i b_{44}^3}{2}, \quad F = \frac{c_{44}^2 + i b_{44}^2}{2}, \quad M = \frac{a_{34}^1 + i b_{34}^1}{2}, \\
 & B = \frac{c_{41}^3 + i b_{41}^3}{2}, \quad G = \frac{a_{34}^2 - i a_{34}^2}{2}, \quad N = \frac{a_{24}^1 + i b_{24}^1}{2}, \\
 & C = \frac{c_{34}^2 + i b_{34}^2}{2}, \quad H = \frac{c_{31}^2 + i b_{31}^2}{2}, \quad P = \frac{a_{23}^1 + i b_{23}^1}{2}, \\
 & D = \frac{c_{43}^2 + i b_{43}^2}{2}, \quad K = \frac{c_{33}^2 + i b_{33}^2}{2}, \quad s = \frac{b_{33}^1}{2}. \\
 & E = \frac{c_{41}^2 + i b_{41}^2}{2}, \quad L = \frac{c_{44}^1 + i b_{44}^1}{2},
 \end{aligned}$$

□

Proposition 4.13. *Let J be a strongly non-nilpotent complex structure on an 8-dimensional nilpotent Lie algebra \mathfrak{g} such that $\dim \mathfrak{g}_1 = 1$ and $\dim \mathfrak{g}_2 = 4$. The pair (\mathfrak{g}, J) is parametrized by the structure equations*

$$\begin{cases}
 d\omega^1 &= 0, \\
 d\omega^2 &= A\omega^{1\bar{1}}, \\
 d\omega^3 &= -D(\omega^{12} - \omega^{1\bar{2}}) - E(\omega^{14} - \omega^{1\bar{4}}) + F\omega^{1\bar{1}}, \\
 d\omega^4 &= L\omega^{1\bar{1}} + M\omega^{1\bar{2}} + N\omega^{1\bar{3}} - \bar{M}\omega^{2\bar{1}} + i s \omega^{2\bar{2}} - \bar{N}\omega^{3\bar{1}},
 \end{cases}$$

where the coefficients are complex numbers with $s \in \mathbb{R}$ and satisfy the Jacobi identity. Furthermore:

- (i) if $(\dim \mathfrak{g}_k)_k = (1, 4, 8)$, then $(\Re A, \Re L) = (0, 0)$,
- (ii) if $(\dim \mathfrak{g}_k)_k = (1, 4, 6, 8)$, then $(\Re A, \Re L) \neq (0, 0)$.

Proof. Using the Lie brackets (17) and (19) we obtain the real structure equations:

$$\begin{cases}
 de^1 &= -a_{24}^1 e^{24} - b_{24}^1 e^{28} - a_{34}^1 e^{34} - b_{33}^1 e^{37} - b_{34}^1 e^{38} - b_{24}^1 e^{46} - b_{34}^1 e^{47} \\
 &\quad - b_{44}^1 e^{48} - a_{24}^1 e^{68} - a_{34}^1 e^{78}, \\
 de^2 &= -b_{41}^2 e^{45} - b_{43}^2 e^{47} - b_{44}^2 e^{48} - c_{41}^2 e^{58} - c_{43}^2 e^{78}, \\
 de^3 &= -b_{44}^3 e^{48}, \\
 de^4 &= 0, \\
 de^5 &= -c_{44}^1 e^{48}, \\
 de^6 &= -c_{41}^2 e^{45} - c_{43}^2 e^{47} - c_{44}^2 e^{48} + b_{41}^2 e^{58} + b_{43}^2 e^{78}, \\
 de^7 &= -c_{44}^3 e^{48}, \\
 de^8 &= 0.
 \end{cases}$$

Now, construct the $(1, 0)$ -basis $\{\omega^k\}_{k=1}^4$ described in (22) to get the result, where the parameters follow (23). □

Proposition 4.14. *Let J be an SnN complex structure on an 8-dimensional NLA \mathfrak{g} such that $\dim \mathfrak{g}_1 = 1$ and $\dim \mathfrak{g}_2 = 5$. The pair (\mathfrak{g}, J) is parametrized by the*

structure equations

$$\left\{ \begin{array}{l} d\omega^1 = 0, \\ d\omega^2 = A\omega^{1\bar{1}} - B(\omega^{14} - \omega^{1\bar{4}}), \\ d\omega^3 = F\omega^{1\bar{1}} - E(\omega^{14} - \omega^{1\bar{4}}), \\ d\omega^4 = L\omega^{1\bar{1}} + M\omega^{1\bar{2}} + N\omega^{1\bar{3}} - \bar{M}\omega^{2\bar{1}} + i s\omega^{2\bar{2}} + P\omega^{2\bar{3}} - \bar{N}\omega^{3\bar{1}} - \bar{P}\omega^{3\bar{2}} \\ \quad + i t\omega^{3\bar{3}}, \end{array} \right.$$

where the coefficients are complex numbers, $s, t \in \mathbb{R}$, and they satisfy the Jacobi identity. Moreover:

- (i) if $(\dim \mathfrak{g}_k)_k = (1, 5, 8)$, then $\Re L = 0$,
- (ii) if $(\dim \mathfrak{g}_k)_k = (1, 5, 6, 8)$, then $\Re L \neq 0$.

Proof. Directly from the Lie brackets (20) and (21) we obtain the real structure equations:

$$\left\{ \begin{array}{l} de^1 = -a_{23}^1 e^{23} - a_{24}^1 e^{24} - b_{22}^1 e^{26} - b_{23}^1 e^{27} - b_{24}^1 e^{28} - a_{34}^1 e^{34} - b_{23}^1 e^{36} - b_{33}^1 e^{37}, \\ \quad - b_{34}^1 e^{38} - b_{24}^1 e^{46} - b_{34}^1 e^{47} - b_{44}^1 e^{48} - a_{23}^1 e^{67} - a_{24}^1 e^{68} - a_{34}^1 e^{78}, \\ de^2 = -b_{41}^2 e^{45} - b_{44}^2 e^{48} - c_{41}^2 e^{58}, \\ de^3 = -b_{41}^3 e^{45} - b_{44}^3 e^{48} - c_{41}^3 e^{58}, \\ de^4 = 0, \\ de^5 = -c_{44}^1 e^{48}, \\ de^6 = -c_{41}^2 e^{45} - c_{44}^2 e^{48} + b_{41}^2 e^{58}, \\ de^7 = -c_{41}^3 e^{45} - c_{44}^3 e^{48} + b_{41}^3 e^{58}, \\ de^8 = 0, \end{array} \right.$$

where $c_{44}^1 = 0$ for $(\dim \mathfrak{g}_k)_k = (1, 5, 8)$ and $c_{44}^1 \neq 0$ for $(\dim \mathfrak{g}_k)_k = (1, 4, 6, 8)$. Constructing the $(1, 0)$ -basis $\{\omega^k\}_{k=1}^4$ given by (22), we get our complex equations above, where $t = \frac{b_{22}^1}{2}$ and the rest of the parameters follow (23). □

ACKNOWLEDGMENTS

Some results of this paper overlap with the first author’s Ph.D. thesis at the University of Zaragoza, under the supervision of the second and third authors. We would like to thank D. Angella, A. Fino, S. Ivanov, and M. T. Lozano for their interest and useful comments and suggestions on the subject. The final version of this paper was prepared during the stay of the first and third authors at the Fields Institute. They would like to thank this institution for its kind hospitality and support.

REFERENCES

[1] A. Andrada, M. L. Barberis, and I. Dotti, *Corrigendum: Classification of abelian complex structures on six-dimensional Lie algebras [MR2763953]*, J. Lond. Math. Soc. (2) **87** (2013), no. 1, 319–320, DOI 10.1112/jlms/jds044. MR3022719

- [2] Adrián Andrada and Raquel Villacampa, *Abelian balanced Hermitian structures on unimodular Lie algebras*, *Transform. Groups* **21** (2016), no. 4, 903–927, DOI 10.1007/s00031-015-9352-7. MR3569562
- [3] Daniele Angella, Antonio Otal, Luis Ugarte, and Raquel Villacampa, *Complex structures of splitting type*, *Rev. Mat. Iberoam.* **33** (2017), no. 4, 1309–1350, DOI 10.4171/RMI/973. MR3729601
- [4] Rutwig Campoamor Stursberg, Isolda E. Cardoso, and Gabriela P. Ovando, *Extending invariant complex structures*, *Internat. J. Math.* **26** (2015), no. 11, 1550096, 25, DOI 10.1142/S0129167X15500962. MR3413991
- [5] Gil R. Cavalcanti and Marco Gualtieri, *Generalized complex structures on nilmanifolds*, *J. Symplectic Geom.* **2** (2004), no. 3, 393–410, DOI 10.4310/JSG.2004.v2.n3.a5. MR2131642
- [6] M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa, *Invariant complex structures on 6-nilmanifolds: classification, Frölicher spectral sequence and special Hermitian metrics*, *J. Geom. Anal.* **26** (2016), no. 1, 252–286, DOI 10.1007/s12220-014-9548-4. MR3441513
- [7] Luis A. Cordero, Marisa Fernández, Alfred Gray, and Luis Ugarte, *Nilpotent complex structures on compact nilmanifolds*, *Proceedings of the Workshop on Differential Geometry and Topology (Palermo, 1996)*, *Rend. Circ. Mat. Palermo (2) Suppl.* **49** (1997), 83–100. MR1602971
- [8] Luis A. Cordero, Marisa Fernández, Alfred Gray, and Luis Ugarte, *The holomorphic fiber bundle structure of some compact complex nilmanifolds*, *Proceedings of the 1st International Meeting on Geometry and Topology (Braga, 1997)*, *Cent. Mat. Univ. Minho, Braga*, 1998, pp. 207–221. MR1694956
- [9] Luis A. Cordero, Marisa Fernández, Alfred Gray, and Luis Ugarte, *Compact nilmanifolds with nilpotent complex structures: Dolbeault cohomology*, *Trans. Amer. Math. Soc.* **352** (2000), no. 12, 5405–5433, DOI 10.1090/S0002-9947-00-02486-7. MR1665327
- [10] Andrzej Czarnecki and Marcin Sroka, *Six-dimensional product Lie algebras admitting integrable complex structures*, *J. Pure Appl. Algebra* **222** (2018), no. 5, 1111–1125, DOI 10.1016/j.jpaa.2017.06.010. MR3742221
- [11] Isabel G. Dotti and Anna Fino, *Hypercomplex eight-dimensional nilpotent Lie groups*, *J. Pure Appl. Algebra* **184** (2003), no. 1, 41–57, DOI 10.1016/S0022-4049(03)00104-X. MR1997309
- [12] Nicola Enrietti, Anna Fino, and Luigi Vezzoni, *Tamed symplectic forms and strong Kähler with torsion metrics*, *J. Symplectic Geom.* **10** (2012), no. 2, 203–223. MR2926995
- [13] Anna Fino, Antonio Otal, and Luis Ugarte, *Six-dimensional solvmanifolds with holomorphically trivial canonical bundle*, *Int. Math. Res. Not. IMRN* **24** (2015), 13757–13799, DOI 10.1093/imrn/rnv112. MR3436163
- [14] Lucia Garcia Vergnolle and Elisabeth Remm, *Complex structures on quasi-filiform Lie algebras*, *J. Lie Theory* **19** (2009), no. 2, 251–265. MR2572128
- [15] Ming-Peng Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and \mathbb{R})*, *ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)—University of Waterloo (Canada)*, 1998. MR2698220
- [16] Michel Goze and Elisabeth Remm, *Non existence of complex structures on filiform Lie algebras*, *Comm. Algebra* **30** (2002), no. 8, 3777–3788, DOI 10.1081/AGB-120005819. MR1922311
- [17] D. V. Millionshchikov, *Complex structures on nilpotent Lie algebras and descending central series*, *Rend. Semin. Mat. Univ. Politec. Torino* **74** (2016), no. 1, 163–182. MR3772585
- [18] Iku Nakamura, *Complex parallelisable manifolds and their small deformations*, *J. Differential Geometry* **10** (1975), 85–112. MR0393580
- [19] Gabriela Ovando, *Invariant complex structures on solvable real Lie groups*, *Manuscripta Math.* **103** (2000), no. 1, 19–30, DOI 10.1007/s002290070026. MR1794329
- [20] S. M. Salamon, *Complex structures on nilpotent Lie algebras*, *J. Pure Appl. Algebra* **157** (2001), no. 2-3, 311–333, DOI 10.1016/S0022-4049(00)00033-5. MR1812058
- [21] Luis Ugarte, *Hermitian structures on six-dimensional nilmanifolds*, *Transform. Groups* **12** (2007), no. 1, 175–202, DOI 10.1007/s00031-005-1134-1. MR2308035
- [22] Luis Ugarte and Raquel Villacampa, *Non-nilpotent complex geometry of nilmanifolds and heterotic supersymmetry*, *Asian J. Math.* **18** (2014), no. 2, 229–246, DOI 10.4310/AJM.2014.v18.n2.a3. MR3217635

CENTRO UNIVERSITARIO DE LA DEFENSA - I.U.M.A., ACADEMIA GENERAL MILITAR, CRTA. DE HUESCA S/N, 50090 ZARAGOZA, SPAIN

Email address: adela@unizar.es

DEPARTAMENTO DE MATEMÁTICAS - I.U.M.A., UNIVERSIDAD DE ZARAGOZA, CAMPUS PLAZA SAN FRANCISCO, 50009 ZARAGOZA, SPAIN

Email address: ugarte@unizar.es

CENTRO UNIVERSITARIO DE LA DEFENSA - I.U.M.A., ACADEMIA GENERAL MILITAR, CRTA. DE HUESCA S/N, 50090 ZARAGOZA, SPAIN

Email address: raquelvg@unizar.es