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CONJUGACY CLASSES OF COMMUTING NILPOTENTS

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Abstract. We consider the space $\mathcal{M}_{q,n}$ of regular q-tuples of commuting nilpotent endomorphisms of k^n modulo simultaneous conjugation. We show that $\mathcal{M}_{q,n}$ admits a natural homogeneous space structure, and that it is an affine space bundle over $\mathbb{P}^{q-1}$. A closer look at the homogeneous structure reveals that, over $\mathbb{C}$ and with respect to the complex topology, $\mathcal{M}_{q,n}$ is a smooth vector bundle over $\mathbb{P}^{q-1}$. We prove that, in this case, $\mathcal{M}_{q,n}$ is diffeomorphic to a direct sum of twisted tangent bundles. We also prove that $\mathcal{M}_{q,n}$ possesses a universal property and represents a functor of ideals, and we use it to identify $\mathcal{M}_{q,n}$ with an open subscheme of a punctual Hilbert scheme. Using a result of A. Iarrobino's, we show that $\mathcal{M}_{q,n} \to \mathbb{P}^{q-1}$ is not a vector bundle, hence giving a family of affine space bundles that are not vector bundles.

1. Introduction

The space of commuting matrices $\{(A, B) \in \mathfrak{gl}_n \oplus \mathfrak{gl}_n \mid [AB] = 0\}$, oft called the *commuting variety*, has received a fair amount of attention, especially in regard to the irreducibility question [MT52, MT55, Ger61, GS00]. The *nilpotent commuting variety* is the closed subvariety consisting of commuting nilpotent pairs, which has also been researched extensively by several authors: Baranovsky [Bar01] ($\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > n$), Basili [Bas03] ($\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \geq n/2$) and Premet [Pre03] (for all characteristics) showed that the space is irreducible and has dimension $n^2 - 1$. Baranovsky went on to conjecture that the corresponding variety for any complex semisimple Lie algebra $\mathfrak{g}$ is equidimensional of dimension $\dim \mathfrak{g}$. Indeed, a more general statement is proven in the aforementioned article by Premet: When the characteristic of the algebraically closed base field k is good for a connected reductive group G over k and the derived group of G is simply connected, each irreducible component of

$$\{(X, Y) \in \mathfrak{g} \mid [XY] = 0,\ X, Y \text{ nilpotent}\}$$

has dimension equal to that of the derived group $[G, G]$.

In this article, we shall be concerned with the space $\mathcal{M}_{q,n}$ of regular q-tuples of commuting nilpotents $N_1, \dots, N_q \in \mathrm{End}(V)$ up to simultaneous conjugation, where k is an algebraically closed field and V is a k-vector space of dimension n. A q-tuple $(N_1, \dots, N_q)$ is said to be *regular* if $N_i^{n-1} \neq 0$ for some i. It is rather surprising that this natural space, with a definite moduli theory flavor, has

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completely evaded any research activity. Indeed, it turns out that $\mathcal{M}_{q,n}$ exhibits a rich and interesting geometry which will be revealed throughout the paper. In the rest of the introduction, we shall give a detailed overview of our results and a road map for the paper.

We begin by associating with a q-tuple $(N_1,\dots,N_q)$ of commuting nilpotents in $\mathrm{End}(V)$, a k-algebra homomorphism

$$\begin{array}{rccc}\rho: & k[x_1,\dots,x_q] & \to & \mathrm{End}(V),\\ & x_i & \to & N_i.\end{array}$$

That is, we have an associated *representation* of $A_{q,n} = k[x_1,\dots,x_q]/\langle x_1,\dots,x_q\rangle^n$. In Section 2, we shall prove that two cyclic representations ρ,ρ' are isomorphic if and only if $\ker(\rho)=\ker(\rho')$ (Proposition 2.4). Moreover, an annihilator $\ker(\rho)$ of a regular representation is of the form $\alpha(\mathfrak{q}_1)$ for some automorphism α of $A_{q,n}$, where $\mathfrak{q}_1 = \langle x_2, x_3,\dots,x_q\rangle$ (Lemma 2.9). Hence the space $\mathcal{M}_{q,n}$ of regular q-tuples of commuting nilpotents modulo conjugation can be realized as the orbit space $\mathrm{Aut}(A_{q,n})/G_1$, where G_1 is the stabilizer of $\mathfrak{q}_1$. The subsequent sections are devoted to the study of the structure of this orbit space.

In Section 3, we define relevant algebraic groups and gather some basic properties which will be employed in the study of $\mathcal{M}_{q,n}$. The parabolic group $P_1 = \mathrm{GL}_q(k)\cap G_1$, the group $\mathcal{I}_{q,n} = \ker(\mathrm{Aut}(A_{q,n})\to \mathrm{GL}_q(k))$ of linearly trivial automorphisms, and the quotient group $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ play especially important roles. These are affine group schemes, and we compute their dimensions.

In Section 4, we show that $\mathcal{M}_{q,n}$ as a homogeneous space is isomorphic to an affine space bundle over $\mathbb{P}^{q-1}$ with fiber isomorphic to the quotient group $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ (Proposition 4.3).

Proposition. *We have an isomorphism* $\mathcal{M}_{q,n}\simeq \mathrm{GL}_q(k)\times^{P_1}\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$, *where* P_1 *acts on* $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ *by conjugation. In particular,* $\mathcal{M}_{q,n}$ *is an equivariant bundle of relative dimension* $(q-1)(n-2)$ *over* $\mathbb{P}^{q-1}_k$.

Due to this proposition, it is clear that the P_1-space structure of the quotient group $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ is the key for understanding the bundle structure of $\mathcal{M}_{q,n}$ over $\mathbb{P}^{q-1}$. To this end, we further investigate the structure of groups $\mathcal{I}_{q,n}$ and $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ in Section 5. Corollary 5.4 gives a coordinate system under which $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ is identified with an affine space, and this allows us to study the affine bundle structure of $\mathcal{M}_{q,n}$ in an explicit manner in the subsequent section.

In Section 6, we build on the results from Section 5 and study the topology of $\mathcal{M}_{q,n}$ as a complex manifold over $\mathbb{C}$. Algebraically, the P_1-space structure of $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ is not very well behaved. But once we pass to the smooth category, the P_1-structure is easy to understand due to our work in Section 5. We prove the following.

Theorem. *The moduli space* $\mathcal{M}_{q,n}$ *as a smooth fiber bundle is isomorphic to the direct sum* $\bigoplus_{j=2}^{n-1} T_{\mathbb{P}^{q-1}}(+j)$ *of twisted tangent bundles.*

In Section 7, we shall prove that $\mathcal{M}_{q,n}$ is a fine moduli scheme in the sense of algebro-geometric moduli theory. The space $\mathcal{M}_{q,n}$ is an orbit space $\mathrm{Aut}(A_{q,n})/G_1$ parametrizing length n ideals of $A_{q,n}$, and we can show that the induced sheaf $\mathcal{I}(q_1)$ on it has a universal property. The next theorem follows from this.

Theorem. *The space* $\mathcal{M}_{q,n}$ *is a fine moduli scheme for the moduli functor* $\underline{\mathcal{M}_{q,n}}$: $\mathrm{Sch}/k \to \mathrm{Sets}$ *from the category of* k*-schemes to the category of sets, defined by*

$$\underline{\mathcal{M}_{q,n}}(S) = \{\textit{Ideal sheaves } \mathcal{I} \subset \mathcal{O}_S \otimes_k A_{q,n} \textit{ ARR and flat over } S\},$$

where ARR is short for "annihilates regular representations" (*Definition* 7.1)*.*

By using the universal property, we can readily identify $\mathcal{M}_{q,n}$ with an open subscheme of a suitable Hilbert scheme. Our main theorem below will be proved in Section 8.

Theorem. $\mathcal{M}_{q,n}$ *is isomorphic to an open subscheme of the punctual Hilbert scheme* $\mathrm{Hilb}^n_{[0]}\,\mathbb{P}^q$*.*

This implies that $\mathcal{M}_{2,n}$ has $\mathrm{Hilb}^n_{[0]}\,\mathbb{P}^2$ as its completion since the latter is irreducible (cf. [Bar01, Theorem 7]). In general, the punctual Hilbert schemes are reducible and $\mathcal{M}_{q,n}$ sits inside a particular component [Iar72b].

Finally, we use a theorem by Iarrobino [Iar73] and show the following.

Theorem. $\mathcal{M}_{q,n} \to \mathbb{P}^{q-1}$ *is not a vector bundle in the algebraic category for* $n \geq 4$*.*

This is quite interesting on its own: Examples of affine space fiber bundles that are not vector bundles are quite rare.

2. The Artinian algebra $A_{q,n}$ and its representations

With a q-tuple $(N_1, \ldots, N_q)$ of commuting nilpotents in $\mathrm{End}(V)$, one can associate a k-algebra homomorphism

$$\begin{array}{rccc} \phi: & k[x_1, \ldots, x_q] & \to & \mathrm{End}(V), \\ & x_i & \to & N_i. \end{array}$$

Since N_i's are nilpotent, $\ker(\phi)$ contains the ideal $J_{q,n}$ generated by all forms of degree n in the variables $x_1, \ldots, x_q$. Of course, $J_{q,n} = \mathfrak{m}_0^n$, where $\mathfrak{m}_0$ is the maximal ideal generated by $x_1, \ldots, x_q$.

Definition 2.1. Let $A_{q,n} = k[x_1, \ldots, x_q]/J_{q,n}$, and let $\mathfrak{m} = \mathfrak{m}_0/J_{q,n}$. We shall call $A_{q,n}$ the *ring of* n*-nil polynomials.*

Clearly $A_{q,n}$ is an Artinian k-algebra of k-dimension $\binom{n+q-1}{n-1}$.

A representation of $A_{q,n}$ will mean a k-algebra homomorphism $\rho : A_{q,n} \to \mathrm{End}(V)$. Through a representation ρ, V is endowed with an $A_{q,n}$-module structure. We denote it by V_ρ. The correspondence between q-tuples of commuting n-nilpotents in $\mathrm{End}(V)$ and representations of $A_{q,n}$ is bijective. For this reason, we consider

$$\mathcal{N}_{q,n} := \left\{ (N_1, \ldots, N_q) \in \prod^q \mathbb{A}^{n^2} \;\middle|\; N_i^n = 0, [N_i, N_j] = 0 \;\; \forall i, j \right\}$$

as the variety of representations of $A_{q,n}$ in V, regarded as a subvariety of the q-fold product of the affine n^2-space with underlying vector space $\mathrm{End}(V)$.

Definition 2.2. A representation ρ is called regular if $\rho(u^{n-1}) \neq 0$ for some $u \in \mathfrak{m}$. It is said to be *cyclic* if there is a vector $v \in V$ such that $\rho(A_{q,n}) \cdot v = V$.

Definition 2.3. A q-tuple $(N_1, \ldots, N_q)$ of commuting nilpotents in $\mathrm{End}(V)$ is said to be regular (resp., cyclic) if the corresponding representation $\rho : A_{q,n} \to \mathrm{End}(V)$ determined by $\rho(x_i) = N_i$ is regular (resp., cyclic).

Evidently a regular representation is cyclic. Also, the regular (resp., cyclic) representations form a Zariski open subset $\mathcal{N}^r_{q,n}$ (resp., $\mathcal{N}^c_{q,n}$) of the variety

$$\mathcal{N}_{q,n} = \{(N_1, \ldots, N_q) \in M_n(k)^{\oplus q} \mid [N_i N_j] = 0, N_i^n = 0 \ \ \forall i, j\}$$

of q-tuples of commuting nilpotent matrices since it is the complement of the vanishing locus of a collection of suitable minors. Note that this does not mean that the nonregular locus is determinantal since the nilpotency condition defining $\mathcal{N}_{q,n}$ is in general not determinantal. We write $\mathcal{M}_{q,n}$ (resp., $\mathcal{M}^c_{q,n}$) for the set of equivalence classes of points of $\mathcal{N}^r_{q,n}$ (resp., $\mathcal{N}^c_{q,n}$) under simultaneous conjugation. Clearly $\mathcal{M}_{q,n}$ is a proper subset of $\mathcal{M}^c_{q,n}$. Note that we will drop the superscript "r" when we pass from $\mathcal{N}^r_{q,n}$ to the quotient $\mathcal{M}_{q,n}$ for notational convenience later on.

Given a representation ρ, we write $\mathcal{A}(\rho)$ for the annihilator in $A_{q,n}$ of V_ρ.

Proposition 2.4. *Let ρ and ρ' be two cyclic representations of $A_{q,n}$. Then V_ρ and $V_{\rho'}$ are isomorphic as $A_{q,n}$-modules if and only if $\mathcal{A}(\rho) = \mathcal{A}(\rho')$. The isomorphism classes of cyclic representations of $A_{q,n}$ are in bijective correspondence with ideals in $A_{q,n}$ of codimension n.*

Proof. If ρ is cyclic with cyclic generator v, then the annihilator of v is equal to the annihilator of V_ρ, and the map $a \mapsto av$ induces an isomorphism between $A_{q,n}/\mathcal{A}(\rho)$ and V_ρ. It is trivial that if two representations are isomorphic, then their annihilators are equal. All that must be shown is that if I is an ideal in $A_{q,n}$ of codimension n, then there is a cyclic representation (ρ, V) with annihilator $I = \mathcal{A}(\rho)$.

If I is of codimension n, then $A_{q,n}/I$ is a vector space of dimension n, so there is an isomorphism (of k-vector spaces) $\theta : A_{q,n}/I \to V$. Define ρ by the equation $\rho(a)v = \theta(a\theta^{-1}(v))$. This gives a representation of $A_{q,n}$ with cyclic vector $\theta(1)$. $\square$

Lemma 2.5. *Let $(Z_1, \ldots, Z_q)$ be a regular q-tuple, i.e., $Z_i^{n-1} \neq 0$ for some i. Then*

(1) *any Z_j is a polynomial in Z_i, and*
(2) *if (ρ, V) is the corresponding representation, then there is a k-algebra isomorphism $\rho(A_{q,n}) \simeq k[z]/z^n k[z]$.*

Proof. Suppose that Z_1 is of rank $n-1$, and let v be a vector not annihilated by Z_1^{n-1}. Then $\{v, Z_1 v, Z_1^2 v, \ldots, Z_1^{n-1} v\}$ form a basis of V: If $\sum_{i=0}^{n-1} a_i Z^i v = 0$ with $a_0 = a_1 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$, then we would have

$$Z^{n-m-1}(a_m Z^m v) = -Z^{n-m-1}(a_{m+1} Z^{m+1} v + \cdots + a_{n-1} Z^{n-1} v) = 0,$$

which is contradictory of our choice of v. With respect to this ordered cyclic basis, Z_1 is represented by the $n \times n$ matrix X, with 1 on the subdiagonal and 0 elsewhere. An elementary computation shows that any $n \times n$ matrix commuting with X is a polynomial in X. The first item follows, and we let f_i denote the polynomial without a constant term such that $f_i(Z_1) = Z_i$. Then the kernel of ρ is generated by $x_i - f_i(x_1)$, so $\rho(A_{q,n})$ is isomorphic to $A_{q,n}/\langle x_2 - f_2(x_1), \ldots, x_q - f_q(x_1)\rangle \simeq k[x_1]/\langle x_1^n \rangle$. $\square$

Definition 2.6. We let $\mathrm{GA}_{q,n}$ denote the algebraic group of k-algebra automorphisms of $A_{q,n}$.

Note that $\mathrm{GA}_{q,n}$ is a linear algebraic group since it is a closed subgroup of $\mathrm{GL}(A_{q,n})$, where $A_{q,n}$ is viewed as a finite-dimensional k-vector space.

Definition 2.7. A subset $\{u_1, \dots, u_r\} \subset A_{q,n}$ will be called a *system of nil parameters* of $A_{q,n}$ of length r if the $A_{q,n}/\langle u_1, \dots, u_r\rangle$ is isomorphic to $A_{q-r,n}$. The variables $x_1, \dots, x_q$ are called the *standard nil parameters.*

Remark 2.8. Throughout this paper, we shall let $\mathfrak{q}_1$ *denote the ideal of $A_{q,n}$ generated by $x_2, \dots, x_q$.*

Lemma 2.9. *Let ρ be a regular representation. Then there is a system of nil parameters, $u_1, \dots, u_q$ such that $A(\rho)$ is the ideal generated by $u_2, \dots, u_q$. Moreover, there is an automorphism $\alpha \in \mathrm{GA}_{q,n}$ such that $\alpha(x_i) = u_i$ and $\alpha(\mathfrak{q}_1) = A(\rho)$.*

Proof. Since ρ is regular, $\rho(u_1)^{n-1} \neq 0$ for some $u_1 \in \mathfrak{m}$. We have seen that this implies that the image of ρ is of the form $k[z]/z^n k[z]$, where $\rho(u_1)$ is the class of z. Let $\mathfrak{q} = \ker(\rho)$. Now ρ induces the map $\varphi : \mathfrak{m}/\mathfrak{m}^2 \to (\bar{z})/(\bar{z}^2)$ of cotangent spaces. Choose $v_2, \dots, v_q \in \mathfrak{m}$, whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis for the kernel of φ. Then there is a polynomial $f_j(z) \in z^2 k[z]$ such that $\rho(v_j) \equiv f_j(z)$ modulo $z^n k[z]$. Hence $v_j - f_j(u_1) \in \mathfrak{q}$, and it is congruent to v_j modulo $\mathfrak{m}^2$. Let $u_j = v_j - f_j(u_1)$, $j = 2, \dots, q$. The elements $u_1, u_2, \dots, u_q$ are a basis for $\mathfrak{m}/\mathfrak{m}^2$, so they generate $A_{q,n}$ as a k-algebra. That is, they are a set of nil parameters for $A_{q,n}$. The elements $u_2, \dots, u_q$ generate an ideal $\mathfrak{q}' \subseteq \mathfrak{q}$ and $A_{q,n}/\mathfrak{q}' \simeq k[z]/z^n k[z]$. Hence by dimension count, $\mathfrak{q} = \mathfrak{q}'$. It immediately follows that we may define an automorphism $\alpha \in \mathrm{GA}_{q,n}$ such that $\alpha(x_i) = u_i$ and $\alpha(\mathfrak{q}_1) = \mathfrak{q} = A(\rho)$. ☐

Lemma 2.10. *Let I be a colength n ideal of $A_{q,n}$. The following are equivalent.*

(i) *I is the annihilator of a regular representation.*
(ii) *$A_{q,n}/I \simeq k[z]/\langle z^n\rangle$, where $k[z]$ is a polynomial ring in one variable z over k.*
(iii) $\dim_k I/\mathfrak{m}^2 \cap I = q - 1$.

Proof. If I is the annihilator of a regular representation, then by Lemma 2.9, $A_{q,n}/I \simeq k[u_1]/\langle u_1^n\rangle$. Suppose that there is an isomorphism $\phi : A_{q,n}/I \to k[z]/\langle z^n\rangle$. Then there exist polynomials f_j of degree n without constant terms such that $x_j = f_j(\zeta)$, where $\zeta = \phi^{-1}(z)$ equal modulo $\mathfrak{m}^2$ to $\sum a_i x_i \neq 0$ (since otherwise $\zeta^{n-1} = 0$). Let c_j be the linear term coefficient of f_j. Modulo $\mathfrak{m}^2$, $x_j = c_j\zeta$ and $\zeta = \sum a_i x_i$, so $\zeta = (\sum a_i c_i)\zeta$ in the k vector space $\mathfrak{m}/\mathfrak{m}^2$, i.e., $\sum a_i c_i = 1$. Hence we see that $I/I \cap \mathfrak{m}^2$ is spanned by $x_1 - c_1\zeta, \dots, x_q - c_q\zeta$ with a single relation $\sum a_i(x_i - c_i\zeta) = 0$.

Assume (iii), and let $\{u_2, u_3, \dots, u_q\} \subset I$ give rise to a basis of $I/I \cap \mathfrak{m}^2$. Expand it to a basis $\{u_1, u_2, \dots, u_q\}$ of $\mathfrak{m}/\mathfrak{m}^2$ with a linear form u_1. Then $A_{q,n}/\langle u_2, \dots, u_q\rangle \simeq k[u_1]/\langle u_1^n\rangle$, so $I = \langle u_2, \dots, u_q\rangle$ by dimension reasons. Then I is the annihilator of the representation $\rho : A_{q,n} \to \mathrm{End}(A_{q,n}/I)$, $\rho(x_i) = u_i\cdot$, which is regular since $u_1^{n-1} \neq 0$. ☐

Definition 2.11 ([Kle98, She09])**.** An ideal of $A_{q,n}$ satisfying the conditions in Lemma 2.10 is called *curvilinear* or *aligned.*

Remark 2.12. Note that if I is curvilinear, then the *type* of I is

$$T_n := (\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots),$$

i.e., $\dim_k (A_{q,n})_j/I_j = 1$ for $j = 0, \dots, n-1$ and 0 otherwise, where I_j is the set of degree j initial forms of members of I. This follows since there is a *graded algebra* isomorphism $A_{q,n}/\mathrm{in}I \simeq k[z]/\langle z^n\rangle$, where $\mathrm{in}I$ is the graded ideal generated by the initial forms of I. There is a decomposition

$$\mathrm{Hilb}^n R = \bigcup_{|T|=n} Z_T$$

of the Hilbert scheme of colength n ideals of $R = k[[x_1, \dots, x_q]]$ according to the type T, and the $r = 2$ case has been extensively studied [Iar72a, Iar73, Iar77, G$\ddot{9}$0]. They study the component of the punctual Hilbert scheme that contains the stratum Z_{T_n}. This is closely related to our results in Section 8.

3. Preliminary results on the algebraic groups of automorphisms of $A_{q,n}$

We let Ω denote the k-vector subspace of $A_{q,n}$ generated by $x_1, \dots, x_q$. By abusing notation, we shall identify Ω with the cotangent space $\mathfrak{m}/\mathfrak{m}^2$. Let σ be an automorphism of $A_{q,n}$. Then $\sigma(\mathfrak{m}) = \mathfrak{m}$, so σ induces a linear automorphism of Ω. The map which assigns to σ the associated automorphism of $\Omega \simeq \coprod_{i=1}^q k\bar{x}_i$ is clearly a group morphism from the group $\mathrm{GA}_{q,n}$ of automorphisms of $A_{q,n}$ to $\mathrm{GL}(\Omega)$.

Definition 3.1. Let $\pi : \mathrm{GA}_{q,n} \to \mathrm{GL}(\Omega)$ denote the morphism which sends the automorphism σ of $A_{q,n}$ to the associated linear automorphism in $\mathrm{GL}(\Omega)$. Also, we identify $\mathrm{GL}(\Omega)$ with $\mathrm{GL}_q(k)$ by using the basis $\bar{x}_1, \dots, \bar{x}_q$.

Clearly π is surjective: Let Ω_1 be the subspace of Ω spanned by $x_2, \dots, x_q$. Any $\alpha = (\alpha_{ij}) \in \mathrm{GL}_q(k)$ naturally defines an element $\tilde{\alpha} \in \mathrm{GA}_{q,n}$ defined by $\tilde{\alpha}(x_i) = \sum_j \alpha_{ij} x_j$. This defines a section $\chi : \mathrm{GL}_q(k) \to \mathrm{GA}_{q,n}$, and we frequently identify $\mathrm{GL}_q(k)$ with its image in $\mathrm{GA}_{q,n}$ as the group of linear automorphisms of $A_{q,n}$. An automorphism $\sigma \in \mathrm{GA}_{q,n}$ will be called *linearly trivial* if it is in the kernel of π. If σ is linearly trivial, there are quadratic expressions $s_i(x_1, \dots, x_q) \in \mathfrak{m}^2,\ i = 1, \dots, q$ such that $\sigma(x_i) = x_i - s_i$. We let $\mathcal{I}_{q,n}$ denote the kernel of π:

$$1 \to \mathcal{I}_{q,n} \to \mathrm{GA}_{q,n} \to \mathrm{GL}_q(k) \to 1. \tag{3.1}$$

We also wish to describe the stabilizer G_1 of the ideal $\mathfrak{q}_1$. Note that π carries G_1 to the stabilizer P_1 in $\mathrm{GL}_q(k)$ of the codimension 1 space Ω_1 in $\Omega = \mathfrak{m}/\mathfrak{m}^2$. Then the exact sequence (3.1) restricts to the exact sequence

$$1 \to \mathcal{I}_{q,n} \cap G_1 \to G_1 \to P_1 \to 1. \tag{3.2}$$

The section χ carries P_1 to the stabilizer of Ω_1 in $\mathrm{GL}_q(k)$. As a consequence, $\mathrm{GA}_{q,n}$ and G_1 are compatibly semidirect products $\mathrm{GL}_q(k) \cdot \mathcal{I}_{q,n}$ and $P_1 \cdot (\mathcal{I}_{q,n} \cap G_1)$, respectively, so the action of an element of $\mathrm{GL}_q(k)$ on $A_{q,n}$ is determined by its linear action on Ω.

We first offer some preliminary results on the algebraic groups involved.

Proposition 3.2. *The exact sequences* (3.1) *and* (3.2) *with the section* χ *induce isomorphisms of varieties*

$$\begin{aligned} \mathrm{GA}_{q,n} &\simeq \mathrm{GL}_q(k) \times \mathcal{I}_{q,n}, \\ G_1 &\simeq P_1 \times (\mathcal{I}_{q,n} \cap G_1). \end{aligned}$$

Moreover, the following hold:

(1) *The groups $\mathcal{I}_{q,n}$ and $\mathcal{I}_{q,n} \cap G_1$ are unipotent. Let $\mathcal{I}^j_{q,n}$ denote the subgroup of $\mathcal{I}_{q,n}$ consisting of elements which reduce to the identity modulo $\mathfrak{m}^{j+2}$. Each of these groups is normal in $\mathrm{GA}_{q,n}$*

(2) *There is an isomorphism of additive group schemes $\mathcal{I}^j_{q,n}/\mathcal{I}^{j+1}_{q,n} \simeq (\mathfrak{m}^{j+2}/\mathfrak{m}^{j+3})^q$.*

(3) *There are natural isomorphisms of varieties*

$$\mathcal{I}_{q,n} \simeq (\mathfrak{m}^2)^{(q)}, \tag{3.3}$$

$$\mathcal{I}_{q,n} \cap G_1 \simeq \mathfrak{m}^2 \times (\mathfrak{q}_1\mathfrak{m})^{(q-1)}. \tag{3.4}$$

The exponents in parentheses represent iterated Cartesian products.

(4) *The groups $\mathrm{GA}_{q,n}$ and G_1 are connected affine group schemes.*

Proof. The two initial product decompositions follow from the existence of the section χ. We define a map α from $\mathcal{I}_{q,n}$ to $(\mathfrak{m}^2)^{(q)}$ as follows: For $\sigma \in \mathcal{I}_{q,n},\ \sigma(x_i) = x_i+u_i$ for some unique element $u_i \in \mathfrak{m}^2$. Let $\alpha(\sigma) = (u_1, \dots, u_q)$, the corresponding q-tuple of elements of $\mathfrak{m}^2$. Conversely, given any such q-tuple, consider the elements $x'_i = x_i + u_i$. Let $A = k[x'_1, \dots x'_q]$ be the subalgebra of $A_{q,n}$ generated by these elements. The x'_i generate $\mathfrak{m}/\mathfrak{m}^2$, so the associated graded of A and that of $A_{q,n}$ are the same. Hence the two algebras are of the same dimension and the x'_i constitute a system of nil parameters. Thus there is an automorphism of $A_{q,n}$ sending x_i to x'_i, so α is surjective as well as injective. This establishes (3.3).

The element $\sigma \in \mathcal{I}_{q,n}$ is in $\mathcal{I}_{q,n} \cap G_1$ if and only if $\sigma(x_i) \in \mathfrak{q}_1$ for $i \geq 2$. Now $\sigma(x_i) = x_i + u_i$, with $u_i \in \mathfrak{m}^2$, so $\sigma \in \mathcal{I}_{q,n} \cap G_1$ if and only if $u_i \in \mathfrak{q}_1 \cap \mathfrak{m}^2$ for each $i \geq 2$. Said otherwise, $\sigma \in \mathcal{I}_{q,n} \cap G_1$ if and only if $\alpha(\sigma) \in \mathfrak{m}^2 \times (\mathfrak{q}_1\mathfrak{m})^{(q-1)}$. Thus (3.4) is established.

For the first item, if $\mathcal{I}_{q,n}$ is unipotent, then its subgroup $\mathcal{I}_{q,n} \cap G_1$ is as well. To see that $\mathcal{I}_{q,n}$ is unipotent, we examine the filtration of item 1. The surjection $A_{q,n} \to A_{q,r},\ r < n$, induces a surjective map $\mathrm{GA}_{q,n} \to GA_{q,r}$ with kernel $\mathcal{I}^{r-2}_{q,n}$, whence each of the groups is normal in $\mathrm{GA}_{q,n}$. Hence if $\sigma \in \mathcal{I}^j_{q,n}$, it induces the identity on $A_{q,n}/\mathfrak{m}^{j+2}$. Consequently, $\sigma(x_i) = x_i + u_i(x_1, \dots, x_q),\ u_i \in \mathfrak{m}^{j+2}$. Suppose that τ is another such automorphism, and that $\tau(x_i) = x_i+v_i(x_1, \dots, x_q)$ with $v_i \in \mathfrak{m}^{j+2}$. Then $\sigma\circ\tau(x_i) = \sigma(x_i+v_i(x_1, \dots, x_q)) = x_i+u_i(x_1, \dots, x_q)+v_i(x_1+u_1, \dots, x_q+u_q)$. Now v_i is a polynomial with lowest degree terms of degree at least $j+2$, so we may write $v_i(x_1+u_i, \dots, x_q+u_q) = v_i(x_1, \dots, x_q)+\sum_{(\nu_1,\dots,\nu_q)} c_{(\nu_1,\dots,\nu_q)} u_1^{\nu_1} \dots u_q^{\nu_q}$. In this last expression, the q-tuples $(\nu_1, \dots, \nu_q)$ have nonnegative integral entries with at least one positive entry, and the coefficients $c_{(\nu_1,\dots,\nu_q)}$ are polynomials in the x_i with no constant terms. Consequently, since u_i is of degree at least $j+2$, all terms except the first are in $\mathfrak{m}^{j+3}$. In particular, $\sigma \circ \tau(x_i) \equiv x_i + u_i(x_1, \dots, x_q) + v_i(x_1, \dots, x_q)$ $\mathrm{mod}(\mathfrak{m}^{j+3})$. For $\sigma \in \mathcal{I}^j_{q,n}$, let $\alpha_j(\sigma) = (\overline{\sigma(x_1) - x_1}, \dots, \overline{\sigma(x_q) - x_q})$, where the overline denotes the class in $\mathfrak{m}^{j+2}/\mathfrak{m}^{j+3}$. The preceding calculation shows that α_j is a homomorphism to the additive group scheme $(\mathfrak{m}^{j+2}/\mathfrak{m}^{j+3})^q$. For any q-tuple $(u_1, \dots, u_q)$ with all of the $u_i \in \mathfrak{m}^{j+2}$, the elements $x_i + u_i$ are a system of nil parameters, so there is an automorphism σ sending x_i to $x_i + u_i$ for each i. It follows that α_j is surjective. Hence $\mathcal{I}_{q,n}$ admits a finite filtration so that successive quotients are group schemes of additive type isomorphic to the vector spaces $\mathfrak{m}^{j+2}/\mathfrak{m}^{j+3}$. Hence it is unipotent and connected. $\square$

We compute the dimensions of the groups $\mathcal{I}_{q,n}$ and $G_1 \cap \mathcal{I}_{q,n}$ and the relevant homogeneous spaces.

Proposition 3.3.

(1) $\dim(\mathcal{I}_{q_n}) = q(\binom{q+n-1}{n-1} - (q+1))$,
(2) $\dim(\mathcal{I}_{q,n} \cap G_1) = \binom{q+n-1}{n-1} + (q-1)(\binom{q+n-1}{n-1} - (q+1) - (n-2))$,
(3) $\dim(\mathrm{GA}_{q,n}) = q\binom{q+n-1}{n-1} - q$,
(4) $\dim(G_1) = q\binom{q+n-1}{n-1} - qn + n - 1$,
(5) $\dim(\mathrm{GA}_{q,n}/G_1) = (q-1)(n-1)$,
(6) $\dim(\mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1)) = (q-1)(n-2)$.

Proof. First, note that $A_{q,n}$ is graded and isomorphic as a vector space to the sum of the first $n-1$ symmetric powers of the vector space of dimension q. Hence its dimension is equal to the dimension of the forms of degree $n-1$ in $q+1$ variables, that is, $\binom{q+n-1}{n-1}$. Hence the dimension of $\mathfrak{m}$ is $\binom{q+n-1}{n-1} - 1$, and the dimension of $\mathfrak{m}^2$ is $\binom{q+n-1}{n-1} - q - 1$.

Now we wish to compute the dimensions of the ideals $\mathfrak{q}_1$ and $\mathfrak{q}_1\mathfrak{m}$. Now $A_{q,n}/\mathfrak{q}_1 \simeq k[\bar{z}]$, where $\bar{z}$ is the residue class of z in $k[z]/z^n k[z]$. The isomorphism sends the residue class of x_1 to $\bar{z}$. Hence $\dim(\mathfrak{q}_1) = \binom{q+n-1}{n-1} - n$.

Now note that $\mathfrak{q}_1 \cap \mathfrak{m}^2 = \mathfrak{q}_1\mathfrak{m}$. To see this, note that an element of $\mathfrak{m}^2$ can be written in the form $x_1^2 f(x_1) + u$ with $u \in \mathfrak{q}_1\mathfrak{m}$. Thus modulo $\mathfrak{q}_1$, this just becomes $(\bar{z})^2 f(\bar{z})$, so it lies in $\mathfrak{q}_1$ if and only if $x_1^2 f(x_1) = 0$. This means that $\mathfrak{q}_1 \cap \mathfrak{m}^2 = \mathfrak{q}_1\mathfrak{m}$. Hence we may compute the dimension of $\mathfrak{q}_1\mathfrak{m}$ from the exact sequence

$$0 \to \mathfrak{q}_1\mathfrak{m} \to \mathfrak{m}^2 \to (\bar{z})^2 k[\bar{z}] \to 0.$$

As a consequence, $\dim(\mathfrak{q}_1\mathfrak{m}) = \binom{q+n+1}{n-1} - (q+1) - (n-2) = \binom{q+n+1}{n-1} - q - n + 1$.

To compute the dimensions of the groups $\mathcal{I}_{q,n}$ and $\mathcal{I}_{q,n} \cap G_1$, we recall the isomorphisms (3.3) and (3.4). Hence as a variety, $\mathcal{I}_{q,n}$ is isomorphic to the q-fold Cartesian product $(\mathfrak{m}^2)^q$. That is, it is isomorphic to affine k-space of dimension $q(\binom{q+n-1}{n-1} - (q+1))$.

Applying (3.4), $\mathcal{I}_{q,n} \cap G_1$ is isomorphic to the product of the vector space $\mathfrak{m}^2$ and the $(q-1)$th Cartesian power of the affine space $\mathfrak{q}_1\mathfrak{m}$. We find that $\dim(\mathcal{I}_{q,n} \cap G_1)$ is

$$\binom{q+n-1}{n-1} - (q+1) + (q-1)(\binom{q+n-1}{n-1} - (q+1) - (n-2)).$$

Since $\dim(\mathrm{GL}_q(k)) = q^2$ and $\dim(P_1) = q^2 - q + 1$, we may use the exact sequences (3.1) and (3.2) to compute the dimensions of $\mathrm{GA}_{q,n}$ and G_1. The formulae in the theorem can now be obtained from some basic algebra computations combined with these results. $\square$

4. The quasihomogeneous structure on $\mathcal{M}_{q,n}$

Due to Lemma 2.9, we can identify the space $\mathcal{M}_{q,n}$ with the orbit $\mathrm{GA}_{q,n} \cdot \mathfrak{q}_1$, giving it a homogeneous space structure. From now on, we will use the following.

Definition 4.1. $\mathcal{M}_{q,n}$ means the homogeneous space $\mathrm{GA}_{q,n} \cdot \mathfrak{q}_1 \simeq \mathrm{GA}_{q,n}/G_1$.

Lemma 4.2. *There is a natural fibration $\pi : \mathcal{M}_{q,n} \to \mathcal{M}_{q,n-1}$ with fiber isomorphic to $\mathbb{A}^{q-1}$.*

Proof. Note that there is a natural projection $\pi : A_{q,n} \to A_{q,n-1}$ sending $I + \mathfrak{m}^n$ to $I + \mathfrak{m}^{n-1}$. This induces a homomorphism $\iota : \mathrm{GA}_{q,n} \to GA_{q,n-1}$, which in turn leads to a surjective map of orbits $\mathcal{M}_{q,n} = \mathrm{GA}_{q,n} \cdot \mathfrak{q}_1 \to GA_{q,n-1} \cdot \mathfrak{q}_1 = \mathcal{M}_{q,n-1}$. Define

$$U_i^n = \{I \in \mathcal{M}_{q,n} \,|\, x_i^{n-1} \notin I\}.$$

Let $I \in U_i^n$. By Lemma 2.5 applied to the regular q-tuple $(x_1\cdot, \dots, x_2\cdot)$ of endomorphisms $x_j \cdot (a+I) = x_j \cdot a + I$ of $A_{q,n}/I$, there exist polynomials $f_j(t) = \sum_{s=1}^{n-1} a_{js} t_i^s$ such that $x_j \cdot = f_j(x_i \cdot)$ or $x_j - f_j(x_i) \in I \;\; \forall j \neq i$. Since $\langle x_j - f_j(x_1)\rangle_{j \neq i}$ has the same k-dimension as I, it equals I; i.e., I is completely determined by the coefficients a_{js} of f_j, and it follows that $U_i^n \simeq \mathbb{A}^{(q-1)(n-1)}$. The fibration π, in terms of the coordinates (a_{js}), is simply the projection $(a_{js})_{s=1}^{n-1} \mapsto (a_{js})_{s=1}^{n-2}$, and the lemma follows. □

Proposition 4.3. *$\mathcal{M}_{q,n}$ is an equivariant bundle of relative dimension $(q-1)(n-2)$ over $\mathbb{P}_k^{q-1}$. More precisely, we have an isomorphism*

$$\mathcal{M}_{q,n} \simeq \mathrm{GL}_q(k) \times^{P_1} \mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1).$$

Here, P_1 acts on $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1)$ by conjugation; i.e., $p \cdot [\sigma] = [p \circ \sigma \circ p^{-1}]$, where $\circ$ denotes the multiplication (composition) in the automorphism group $\mathrm{GA}_{q,n}$.

Proof. Since $\mathcal{I}_{q,n}$ is normal in $\mathrm{GA}_{q,n}$, P_1 normalizes it, and we may consider the semidirect product $H = P_1 \cdot \mathcal{I}_{q,n}$. Recall the exact sequence (3.2). Since P_1 can be viewed as a subgroup of $\mathrm{GA}_{q,n}$ as above, we may write the stabilizer G_1 as the semidirect product $P_1 \cdot (\mathcal{I}_{q,n} \cap G_1)$. In particular, $G_1 \subseteq H$. By Lemma 2.9, $\mathrm{GA}_{q,n}$ operates transitively on $\mathcal{M}_{q,n}$, which can be written as the orbit $\mathrm{GA}_{q,n} \cdot \mathfrak{q}_1 = \mathrm{GA}_{q,n}/G_1$.

Since $G_1 \subseteq H$, there is a natural $\mathrm{GL}_q(k)$-equivariant fibration

$$\varpi : \mathrm{GA}_{q,n}/G_1 \mapsto \mathrm{GA}_{q,n}/H$$

with the base $\mathrm{GA}_{q,n}/H = (\mathrm{GA}_{q,n}/\mathcal{I}_{q,n})/(H/\mathcal{I}_{q,n}) = \mathrm{GL}_q(k)/P_1 = \mathbb{P}_k^{q-1}$. Since ϖ is a P_1-equivariant fibration, it equals $\mathrm{GL}_q(k) \times^{P_1} \varpi^{-1}(P_1)$. The fiber $\varpi^{-1}(P_1)$ is H/G_1, where the groups on the top and the bottom are the semidirect products $P_1 \cdot \mathcal{I}_{q,n}$ and $P_1 \cdot (\mathcal{I}_{q,n} \cap G_1)$, respectively. Hence the fiber is isomorphic to $\mathcal{I}_{q,n}/\mathcal{I}_{q,n} \cap G_1$. It has dimension $(q-1)(n-2)$ by Lemma 3.3. Note that P_1 acts on $P_1 \cdot \mathcal{I}_{q,n}/P_1 \cdot (\mathcal{I}_{q,n} \cap G_1)$ by left multiplication and on $\mathcal{I}_{q,n}/\mathcal{I}_{q,n} \cap G_1$ by conjugation, and a P_1-space isomorphism from the former to the latter is given by $[px] \mapsto [pxp^{-1}]$. □

Remark 4.4.

(1) Alternatively, one can think of the equivariant bundle map ϖ as the composition

$$\mathcal{M}_{q,n} \to \mathcal{M}_{q,n-1} \to \cdots \to \mathcal{M}_{q,2},$$

where each arrow is the fibration in Lemma 4.2. This is induced from the natural projection $A_{q,n} \to A_{q,2}$, and since $GA_{q,2} \simeq \mathrm{GL}_q$, the induced orbit map $GA_{q,n}/G_1 \to GA_{q,2}/G_1 \simeq \mathrm{GL}_q/P_1$ is indeed the fibration ϖ above.

(2) A colength 2 curvilinear ideal of $A_{q,2}$ is graded since nil-parameters of $A_{q,2}$ are linear. Conversely, generators of a graded curvilinear ideal of colength n are necessarily linear since they must have a linear term (curvilinearity) and are homogeneous at the same time. Hence the ideals parametrized by $\mathcal{M}_{q,2}$ are precisely the graded ideals of colength 2. Via this correspondence,

the fibration ϖ can be thought of as $I \mapsto \mathrm{gr}(I)$, where $\mathrm{gr}(I)$ is the associated graded ideal generated by the initial forms of members of I. And there is a "zero" section of ϖ defined simply by sending $\mathrm{gr}(I)$ to itself (forgetting the grading). This will be used in the proof of Theorem 6.3.

In order to complete our examination of the structure of $\mathcal{M}_{q,n}$, we shall determine the structure of the fiber $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ as a P_1-variety. Then by examining the P_1-space structure over $k=\mathbb{C}$, we shall see that $\mathcal{M}_{q,n}$ is diffeomorphically isomorphic to a sum of twisted tangent bundles over $\mathbb{P}^{q-1}$.

5. More on the group $\mathcal{I}_{q,n}$

We wish to understand the structure of $\mathcal{M}_{q,n}$ as a fiber bundle over $\mathbb{P}^{q-1}_k$. By Proposition 4.3, it is a homogeneous fiber space whose fiber is the P_1-space $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$. Note that $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$ has a P_1-structure through conjugation since P_1 normalizes both groups. To analyze $\mathcal{M}_{q,n}$, it is crucial to understand the P_1-space structure of $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n}\cap G_1)$.

We begin with a product decomposition for $\mathcal{I}_{q,n}$ and an invariant version of (3.3). First, we consider a group $\Gamma \subseteq \mathcal{I}_{q,n}$, which we will demonstrate to be a geometric complement to $\mathcal{I}_{q,n}\cap G_1$.

Definition 5.1. Let Γ be the subgroup of $\mathcal{I}_{q,n}$ defined

$$\Gamma = \{\sigma \in \mathrm{GA}_{q,n} : \sigma(x_1)=x_1,\ \sigma(x_i)=x_i+x_1^2 f_i(x_1),\ i\geq 2\}.$$

Proposition 5.2. *The multiplication morphism $\mu:\Gamma\times(\mathcal{I}_{q,n}\cap G_1)\to\mathcal{I}_{q,n}$ is an isomorphism of varieties.*

Proof. Since $\Gamma\cap(\mathcal{I}_{q,n}\cap G_1)=\{e\}$, it is clear that μ is an injective morphism of varieties. Since each f_i has degree $2\leq \deg(f_i)\leq n-1$, $\dim\Gamma=(q-1)(n-2)$. Hence by Proposition 3.3, the dimension of $\Gamma\times(\mathcal{I}_{q,n}\cap G_1)$ is equal to the dimension of $\mathcal{I}_{q,n}$. We may regard $\mathcal{I}_{q,n}$ as a $\Gamma\times(\mathcal{I}_{q,n}\cap G_1)$-space with the action $(u,v)\cdot g = ugv^{-1}$. Then $\mathrm{Im}(\mu)$ is the $\Gamma\times(\mathcal{I}_{q,n}\cap G_1)$-orbit of e. Hence it is open in its closure, which is $\mathcal{I}_{q,n}$. Since it is a product of affine spaces, it is affine and hence its complement is, if nonempty, of codimension 1 given as the zero set of a polynomial f on $\mathcal{I}_{q,n}$. But $f|\,\mathrm{Im}(\mu)$ is a nonconstant unit, which, by a well-known theorem of Rosenlicht, is a character. This is impossible since $\mathcal{I}_{q,n}$ is unipotent and has only trivial characters. □

Now consider the ring $k[z]$ where $z^n=0$. Any automorphism ϕ of $k[z]$ is uniquely determined by $\phi(z)$. The element $\phi(z)$ can be any element in $zk[z]$ not in $z^2k[z]$. Refer to such elements as *generating elements*. If u is a generating element, then $u^{n-1}\neq 0$ and $u^n=0$. Given any two generating elements u_1 and u_2, there is a unique automorphism ϕ such that $\phi(u_1)=u_2$. As before, we let Ω_1 denote the subspace of $A_{q,n}$ generated by $x_2,\ldots,x_q$.

Proposition 5.3. *Let $\mathfrak{q}'$ be an ideal in $A_{q,n}$ generated by a system of nil parameters such that $\mathfrak{q}'=\Omega_1$ modulo $\mathfrak{m}^2$. Then there is a unique homomorphism $\phi: A_{q,n}\mapsto k[z]$ such that $\phi(x_1)=z$ and $\mathrm{Ker}(\phi)=\mathfrak{q}'$. Moreover, there is a uniquely determined linear map $\gamma:\Omega_1\to x_1^2k[x_1]\subset A_{q,n}$ such that the elements $u-\gamma(u)$, $\forall u\in\Omega_1$, generate $\mathfrak{q}'$. In particular, the elements $x_i-\gamma(x_i)$, $i>1$, are a set of nil parameters of length $q-1$ generating $\mathfrak{q}'$.*

Proof. Since $\mathfrak{q}'$ is generated by a set of nil parameters of length $q-1$, there is an isomorphism $\beta : A_{q,n}/\mathfrak{q}' \to k[z]$. Let ϕ_0 be the composition of β with the natural surjection $A_{q,n} \to A_{q,n}/\mathfrak{q}'$. Since ϕ_0 is surjective and the image of its kernel $\mathfrak{q}'$ in $\mathfrak{m}/\mathfrak{m}^2$ is equal to the image there of Ω_1, $\phi_0(x_1)$ must be a generating element. Hence there is a unique automorphism θ of $k[z]$ carrying $\phi_0(x_1)$ to z. Replacing ϕ_0 with $\theta \circ \phi_0 = \phi$, we may assume that $\phi(x_1) = z$.

Since $\mathfrak{q}' = \Omega_1$ modulo $\mathfrak{m}^2$, it follows that, for each $u \in \Omega_1$, there is an element $u - s_u$ with $s_u \in \mathfrak{m}^2$ such that $\phi_0(u - s_u) = 0$. Now ϕ carries $\mathfrak{m}^2$ to $z^2k[z]$, so $\phi(u) = \phi(s_u)$ is a polynomial in z with neither a constant nor a linear term. Let $\phi(u) = f_u(z)$. Define a map $\gamma : \Omega_1 \mapsto x_1^2 k[x_1] \subset A_{q,n}$ by setting $\gamma(u)$ equal to $f_u(x_1)$. It is clear that γ is a linear map since it is a composition of the linear map $\phi|\Omega_1$ and the inverse of the isomorphism $k[x_1] \mapsto k[z]$. Moreover, $\phi(u - \gamma(u)) = 0$ for all u; that is, these elements are in $\mathfrak{q}'$. Consider the elements $x_i - \gamma(x_i)$, $i > 1$. These elements clearly constitute a system of nil parameters of length $q-1$. Hence they generate an ideal of codimension n which is contained in $\mathfrak{q}'$. It follows that they generate $\mathfrak{q}'$.

Since $\mathfrak{q}'$ is generated by a system of nil parameters of length $q-1$—namely, $x_i' = x_i - \gamma(x_i)$—we have $A_{q,n} = k[x_1, x_2', \ldots x_q']$, and the uniqueness of ϕ follows since ϕ is determined by $\phi(x_1) = z$ and $\phi(x_i') = 0$, $i = 2, 3, \ldots, q$. $\square$

Note that there is an obvious isomorphism between Γ and $\mathrm{Hom}_k(\Omega_1, z^2k[z])$, and we shall identify the two in the following corollary.

Corollary 5.4. *There is a natural isomorphism of varieties* $\psi : \mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1) \to \mathrm{Hom}_k(\Omega_1, z^2k[z])$.

Proof. Consider the morphism

$$\tilde{\psi} : \mathcal{I}_{q,n} \xrightarrow{\mu^{-1}} \Gamma \times (\mathcal{I}_{q,n} \cap G_1) \xrightarrow{\pi_1} \Gamma,$$

where μ is the multiplication morphism (which is an isomorphism due to Proposition 5.2) and π_1 is the projection onto the first factor. The subgroup $\mathcal{I}_{q,n} \cap G_1$ acts on $\mathcal{I}_{q,n}$ by the right multiplication, and on the product $\Gamma \times (\mathcal{I}_{q,n} \cap G_1)$ by the right multiplication on the second factor. Obviously $\tilde{\psi}$ is $\mathcal{I}_{q,n} \cap G_1$-invariant, so it descends to give the desired morphism ψ on $\mathcal{I}_{q,n}/\mathcal{I}_{q,n} \cap G_1$. The ideals $\mathfrak{q}'$ in Proposition 5.3 are precisely the ideals in the $\mathcal{I}_{q,n}$-orbit of $\mathfrak{q}_1$, and hence the statement of the proposition means that ψ is bijective. Now it follows that ψ is an isomorphism since it is a bijective morphism between affine spaces. $\square$

In order to give a complete description of $\mathcal{M}_{q,n} = \mathrm{GL}_q(k) \times^{P_1} \mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1)$, we must yet provide an explicit formula for the action of P_1 on $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1)$. To this end, we need to concretely describe the P_1-action on $\mathrm{Hom}_k(\Omega_1, z^2k[z])$ that makes ψ in Corollary 5.4 P_1-equivariant. This will be taken up in the subsequent section.

6. $\mathcal{M}_{q,n}$ AS A SMOOTH FIBER BUNDLE

In this section, we shall work over $k = \mathbb{C}$ and work in the category of smooth manifolds: We regard the algebraic groups $GA_{q,n}$, $\mathcal{I}_{q,n}$, etc., as (complex) Lie groups. We will also consider homogeneous spaces $\mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1)$ and $GA_{q,n}/G_1$

as smooth manifolds, and the isomorphism

$$\psi : \mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1) \to \mathrm{Hom}(\Omega_1, z^2k[z]) \simeq \mathbb{C}^{(q-1)(n-2)}$$

and other relevant maps as smooth maps.

Let $f \in \mathcal{I}_{q,n}$ and $s_i = \gamma(x_i)$ be the polynomials (in z) associated with $f(\mathfrak{q}_1)$ as in the Proposition 5.3. There is a natural commutative diagram

$$\begin{tikzcd} \mathcal{I}_{q,n} \arrow[d] \arrow[dr, "\eta"] & \\ \mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1) \arrow[r, "\psi"'] & \mathrm{Hom}(\Omega_1, z^2k[z]), \end{tikzcd}$$

where $\eta(f)$ maps x_i to $s_i(z)$. Then the P_1-action on $\mathrm{Hom}_k(\Omega_1, z^2k[z])$ can be written in terms of s_i as follows. Let $p = (p_{ij}) \in P_1$, and let $f(\mathfrak{q}_1) \in \mathcal{I}_{q,n}\mathfrak{q}_1$. Then by Proposition 5.3, we have $f(\mathfrak{q}_1) = (x_2 - s_2(x_1), \dots, x_q - s_q(x_1))$ for some polynomials $s_i(z) = \sum_{j=2}^{n-1} b_{ij}z^j$ with no constant and linear term, and

$$p \cdot f(\mathfrak{q}_1) = p \cdot (x_2 - s_2(x_1), \dots, x_q - s_q(x_1)) = \left(\sum_j p_{ij}x_j - s_i \left(\sum_j p_{1j}x_j \right) \right)_{i \geq 2}.$$

Setting $g = p^{-1}$ and performing a Gauss elimination, we get

$$\begin{aligned} (\textstyle\sum_j p_{ij}x_j - s_i(\sum_j p_{1j}x_j))_{i\geq 2} &= (\textstyle\sum_i g_{ki}(\sum_j p_{ij}x_j - s_i(\sum_j p_{1j}x_j)))_{i,k\geq 2} \\ &= (x_k - \textstyle\sum_i g_{ki}s_i(\sum_j p_{1j}x_j))_{k\geq 2}. \end{aligned}$$

This is a fairly complicated action and is not well understood in general. But in the special case in which $p_{1j} = 0$ for $j \geq 2$, the action is linear. Then in terms of (b_{ij}), $p \cdot f(\mathfrak{q}_1)$ corresponds to

$$((p^{-1})_{ik}b_{kj}p_{11}^j). \tag{†}$$

Restating this in terms of representations, we obtain the following.

Lemma 6.1. *Let $t \in \mathbb{C}$, and let $\varphi_t : P_1 \to P_1$ be the group homomorphism*

$$p = \begin{bmatrix} p_{11} & R \\ 0 & Q \end{bmatrix} \mapsto \begin{bmatrix} p_{11} & tR \\ 0 & Q \end{bmatrix},$$

where R, 0, Q are of size $1\times(q-1)$, $(q-1)\times 1$, and $(q-1)\times(q-1)$, respectively. Let Y_t be the P_1-space whose underlying variety is $\mathcal{I}_{q,n}/\mathcal{I}_{q,n} \cap G_1$ on which P_1 acts by

$$p \cdot [\sigma] = [\varphi_t(p)\sigma\varphi_t(p)^{-1}].$$

Then Y_0 is isomorphic to $\coprod_{j=2}^{n-1} \Omega_1^* \otimes \mathbb{C}_j$, where $\mathbb{C}_j$ is $\mathbb{C}$ acted upon by the jth power of the character $\lambda(p) = p_{11}$ $\forall p \in P_1$.

Note that for $t \neq 0$, $Y_t \simeq Y_1$ as P_1-spaces since $\varphi_t(p) = \tau p \tau^{-1}$, $\tau = \begin{bmatrix} t & 0 \\ 0 & I_2 \end{bmatrix}$, so $\Phi_t([\sigma]) = [\tau\sigma\tau^{-1}]$ is a P_1-equivariant isomorphism from Y_1 to Y_t. Let $B = \operatorname{Spec} k[t]$, and let $\mathcal{Y} = Y \times B$ be the P_1-space on which P_1 acts by $p \cdot ([\sigma], t) = ([\varphi_t(p)\sigma\varphi_t(p)^{-1}], t)$. Consider $\mathcal{E} := \mathrm{GL}_q \times^{P_1} \mathcal{Y} \to \mathrm{GL}_q / P_1 \times B \simeq \mathbb{P}^{q-1} \times B$. It is an isotrivial family of affine $(q-1)(n-2)$-bundles over $\mathbb{P}^{q-1}$: We have

$$E_t := \mathcal{E}|(\mathbb{P}^{q-1} \times \{t\}) = \mathrm{GL}_q \times^{P_1} Y_t \simeq \mathrm{GL}_q \times^{P_1} Y_1 \simeq \mathcal{M}_{q,n}.$$

Recall that two smooth manifolds X_1, X_2 are said to be *deformation equivalent* if there exists a smooth family $\mathcal{X} \to S$ over a smooth connected base S and two points $s_1, s_2 \in S$ such that the fibers $\mathcal{X}_{s_i}$ are diffeomorphic to X_i, $i = 1, 2$.

Proposition 6.2. *The moduli space $\mathcal{M}_{q,n}$ as a smooth fiber bundle over $\mathbb{P}^{q-1}$ is smooth deformation equivalent to the direct sum*

$$\bigoplus_{j=2}^{n-1} T_{\mathbb{P}^{q-1}}(+j)$$

of twisted tangent bundles over $\mathbb{P}^{q-1}$.

Proof. Consider the one-parameter family $E_t := \mathrm{GL}_q \times^{P_1} Y_t$ of affine bundles over $\mathbb{P}^{q-1}$. Then $E_0 = \bigoplus_{j=2}^{n-1} T_{\mathbb{P}^{q-1}}(+j)$ since Ω_1^* corresponds to the tangent bundle, and $\mathbb{C}_j$ to $\mathcal{O}_{\mathbb{P}^{q-1}}(+j)$. Since $E_1 = \mathcal{M}_{q,n}$, it follows that as smooth fiber bundles $\mathcal{M}_{q,n}$ and $\bigoplus_{j=2}^{n-1} T_{\mathbb{P}^{q-1}}(+j)$ are deformation equivalent. $\square$

Our moduli space $\mathcal{M}_{q,n}$ is a fiber bundle over $\mathrm{GL}_q(k)/P_1 = \mathbb{P}^{q-1}$ with fibers $F := \mathcal{I}_{q,n}/\mathcal{I}_{q,n} \cap G_1$. Since F is an affine space, the theory of microbundles can be applied to show that $\mathcal{M}_{q,n}$ as a smooth fiber bundle is isomorphic to a smooth vector bundle; i.e., its structure group reduces to the general linear group.

Theorem 6.3. *The moduli space $\mathcal{M}_{q,n}$ as a smooth fiber bundle is isomorphic to the direct sum $\bigoplus_{j=2}^{n-1} T_{\mathbb{P}^{q-1}}(+j)$ of twisted tangent bundles.*

Proof. First, note that there is a distinguished zero section $0_n : \mathbb{P}^{q-1} \to \mathcal{M}_{q,n}$ defined by sending $\bar{g} \in \mathrm{GL}_q / P_1$ to $g \cdot q_1$ or, equivalently, to

$$(g, 0) \in \mathrm{GL}_q \times^{P_1} \mathrm{Hom}(\Omega_1, z^2\mathbb{C}[z]/\langle z^n \rangle).$$

In terms of ideals, this is defined by sending a graded ideal to itself forgetting the grading (Remark 4.4). Let N denote the $\mathbb{R}$-dimension of the affine space $\mathrm{Hom}(\Omega_1, z^2\mathbb{C}[z]/\langle z^n \rangle)$. The isomorphism classes of smooth fiber bundles with fibers diffeomorphic to $\mathbb{R}^N$ is functorially in bijective correspondence with the homotopy classes of maps from $\mathbb{P}^{q-1}$ to $B\,\mathrm{Diff}(\mathbb{R}^N, 0)]$, where $B\,\mathrm{Diff}(\mathbb{R}^N, 0)$ is the classifying space of the group $\mathrm{Diff}(\mathbb{R}^N, 0)$ of diffeomorphisms of $\mathbb{R}^N$ fixing 0. By the *Alexander trick*, the natural inclusion

$$\mathrm{GL}_N(\mathbb{R}) \hookrightarrow \mathrm{Diff}(\mathbb{R}^N, 0)$$

is a homotopy equivalence, and it follows that $(\mathcal{M}_{q,n} \to \mathbb{P}^{q-1}, 0_n)$ has a reduction of structure group to $\mathrm{GL}_N(\mathbb{R})$. More specifically, up to diffeomorphism, $\mathcal{M}_{q,n} \to \mathbb{P}^{q-1}$ is isomorphic to the normal bundle of the zero section 0_n. By Proposition 6.2, the maps corresponding to the two smooth vector bundles $\mathcal{M}_{q,n}$ and $\bigoplus_{j=2}^{n-1} T_{\mathbb{P}^{q-1}}(+j)$ are homotopy equivalent. Hence they are isomorphic as smooth vector bundles over $\mathbb{P}^{q-1}$. $\square$

Corollary 6.4. *Let $\varpi : \mathcal{M}_{q,n} \to \mathcal{M}_{q,n-1}$ be the fibration in Lemma* 4.2*. Then the kernel $\varpi^{-1}(0_{n-1})$ is isomorphic to $T_{\mathbb{P}^{q-1}}(n-1)$.*

Proof. In terms of the coordinates b_{ij} above, the kernel is defined by $b_{ij} = 0$ $\forall i$, $j = 1, \ldots, n-2$. By equation (†), it is the bundle associated with the P_1-space

$$p \cdot (b_{in-1}) = ((p^-)_{ik} b_{kn-1} p_{11}^{n-1}),$$

which is $\Omega^* \otimes \mathbb{C}_{n-1}$. The corollary follows. $\square$

In fact, the corollary above holds for any algebraically closed field.

7. Universal property of $\mathcal{M}_{q,n}$

Let $G := \mathrm{GA}_{q,n}$. Recall that $\mathcal{M}_{q,n} = G/G_1$ (by definition), $G_1 = \mathrm{Stab}_G(q_1)$. We have observed in Proposition 4.3 that $\mathcal{M}_{q,n} \simeq \mathrm{GL}_q \times^{P_1} \mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1)$. From the natural sequence

$$0 \to q_1 \to A_{q,n} \to A_{q,n}/q_1 \to 0$$

of G_1-representations, we obtain the sequence of induced sheaves on G/G_1:

$$0 \to \mathcal{I}_{G/G_1}(q_1) \to \mathcal{I}_{G/G_1}(A_{q,n}) \to \mathcal{I}_{G/G_1}(A_{q,n}/q_1) \to 0.$$

Definition 7.1. An ideal sheaf $\mathcal{I} \subset A_{q,n} \otimes_k \mathcal{O}_S$ over a k-scheme S is said to *annihilate regular representations over* S (we abbreviate this as "ARR over S") if there exists a map $\rho : A_{q,n} \otimes_k \mathcal{O}_S \to V \otimes_k \mathcal{O}_S$ of $\mathcal{O}_S$-modules such that, for all closed points $b \in S$, $\rho_b := \rho \otimes 1_{k(b)} : A_{q,n} \otimes_k k(b) \to V \otimes_k k(b)$ is a regular representation and $\ker(\rho_b) = \mathcal{I} \otimes k(b)$.

We shall prove that $\mathcal{I}_{G/G_1}(q_1)$ is an ideal sheaf of $\mathcal{I}_{G/G_1}(A_{q,n}) \simeq \mathcal{O}_{\mathcal{M}_{q,n}} \otimes_k A_{q,n} = \mathcal{O}_{\mathcal{M}_{q,n}}[x_1, \ldots, x_q]/\mathfrak{m}_0^n$ such that the following hold:

(i) It annihilates regular representations over $\mathcal{M}_{q,n}$.
(ii) It is *universal* : For any k-scheme S and an ideal $\mathcal{I}$ of $\mathcal{O}_S \otimes A_{q,n}$ ARR over S, there exists a unique morphism $f_\mathcal{I} : S \to \mathcal{M}_{q,n}$ such that $(f_\mathcal{I} \times 1)^* \left(\mathcal{I}_{G/G_1}(q_1)\right) \simeq \mathcal{I}$, where 1 is the identity morphism on $\operatorname{Spec} A_{q,n}$.

Remark 7.2. Recall from Proposition 2.4 and the definition preceding it that $\ker(\rho)$ in (i) above is called the annihilator of the representation ρ and is denoted by $\mathcal{A}(\rho)$.

Proposition 7.3. *The ideal sheaf* $\mathcal{I}_{G/G_1}(q_1)$ *is ARR over* $\mathcal{M}_{q,n}$.

Proof. By Corollary 5.4 and Proposition 5.3, we may naturally identify the homogeneous space $\overline{\mathcal{I}}_{q,n} := \mathcal{I}_{q,n}/(\mathcal{I}_{q,n} \cap G_1)$ with

$$\mathrm{Hom}_k(\Omega_1, z^2k[z]) = \operatorname{Spec} k[b_{ij} \,|\, 2 \le i \le q, 2 \le j \le n-1],$$

where the coordinates b_{ij} are the obvious ones determined by

$$h(x_i) = \sum_{j=2}^{n-1} b_{ij}(h) z^j \qquad \forall h \in \mathrm{Hom}_k(\Omega_1, z^2k[z]).$$

Hence on $\overline{\mathcal{I}}_{q,n}$, we have an ideal

$$\mathcal{U}' := \left\langle x_i - \sum_{j=2}^{n-1} b_{ij} x_1^j \right\rangle_{2 \le i \le q} \subset \frac{k[b_{ij}, x_1, \ldots, x_q]}{\mathfrak{m}_0^n}.$$

The right-hand side is the global section ring $\Gamma(\mathcal{O}_{\overline{\mathcal{I}}_{q,n}}) \otimes_k A_{q,n}$. Subsequently, a versal ideal sheaf $\mathcal{U}''$ on $\mathrm{GL}_q(k) \times \overline{\mathcal{I}}_{q,n}$ is obtained by further moving $\mathcal{U}'$ around by the $\mathrm{GL}_q(k)$-action:

$$\mathcal{U}'' := \left\langle \sum_{l=1}^{q} a_{il} x_l - \sum_{j=2}^{n-1} b_{ij} \left(\sum_{l=1}^{q} a_{1l} x_l \right)^j \right\rangle_{2 \le i \le q} \subset \frac{k[a_{l\,m}, b_{ij}, x_2, \ldots, x_q]}{\mathfrak{m}_0^n}, \tag{7.1}$$

where $k[GL_q(k)] = k[a_{l\,m} \,|\, 1 \le l, m \le q]$, so $k[a_{l\,m}, b_{ij}, x_2, \ldots, x_q]/\mathfrak{m}_0^n$ is the global section ring of $\mathcal{O}_{\mathrm{GL}_q(k) \times \overline{\mathcal{I}}_{q,n}} \otimes_k A_{q,n}$.

Since P_1 acts diagonally on $\mathrm{GL}_q(k) \times \overline{\mathcal{I}}_{q,n}$ by $p \cdot (g, v) = (gp^{-1}, p \cdot v)$ and $\mathcal{U}''$ is obtained by letting $GL_q(k)$ act on $\mathcal{U}'$, $\mathcal{U}''$ is plainly invariant under the P_1-action and descends to give an ideal sheaf $\mathcal{U}$ on the quotient $\mathcal{M}_{q,n} = \left(\mathrm{GL}_q(k) \times \overline{\mathcal{I}}_{q,n}\right)/P_1$,

$$0 \to \mathcal{U} \to \mathcal{O}_{\mathcal{M}_{q,n}} \otimes_k A_{q,n} \to \mathcal{W} \to 0.$$

By construction, $\mathcal{U}$ is clearly ARR over $\mathcal{M}_{q,n}$. It is easy to see that $\mathcal{U}$ is equal to $\mathcal{I}_{G/G_1}(q_1)$: Since they are ideal subsheaves of the same sheaf $\mathcal{O}_{\mathcal{M}_{q,n}} \otimes_k A_{q,n}$ of algebras, it suffices to show that their fibers are exactly the same as the fibers of $\mathcal{O}_{\mathcal{M}_{q,n}} \otimes_k A_{q,n}$. Let $\sigma \in G$. The fiber of $\mathcal{I}_{G/G_1}(q_1)$ at σG_1 is by definition $\sigma(q_1)$. By our analysis in the previous sections, especially Proposition 5.3,

$$\sigma(q_1) = (a_{ij}) \cdot \left\langle \sum x_i + s_i(x_1) \right\rangle_{i=2,\ldots,q} \tag{7.2}$$

for a unique $((a_{ij}), (b_{ij})) \in \mathrm{GL}_q \times^{P_1} \overline{\mathcal{I}}_{q,n}$, where b_{ij} are determined by $s_i(x_1) = \sum_{j=2}^{n-1} b_{ij} x_1^j$. Comparing (7.2) with equation (7.1), we see immediately that $\sigma(q_1)$ is precisely the fiber of $\mathcal{U}$ at σG_1. $\square$

Proposition 7.4. *The ideal sheaf $\mathcal{I}_{G/G_1}(q_1) = \mathcal{U}$ is universal over $\mathcal{M}_{q,n}$.*

Proof. Let S be a k-scheme, and let $\mathcal{I}$ be an ideal of $\mathcal{O}_S \otimes A_{q,n}$ ARR over S. Let $\mathcal{V}$ denote the quotient of $\mathcal{O}_S \otimes A_{q,n}$ by $\mathcal{I}$. Let $\pi : S \times \operatorname{Spec} A_{q,n} \to S$ be the projection, and let $\mu_i \in H^0(S, \pi_* \mathrm{End}(\mathcal{V}))$ be the endomorphism of $\pi_* \mathcal{V}$ defined by multiplication by x_i^{n-1}. Since $\mathcal{I}$ is ARR, for each $s \in S$, there exists an i such that

$$\mu_i|_s \neq 0 \in \mathrm{End}(\pi_* \mathcal{V})|_s.$$

Hence there exists an open subvariety $T \ni s$ of S and a section $v \in \pi_* \mathcal{V}(T)$ such that $\mu_i|_t(v_t) \neq 0$ for all $t \in T$. At s, $\{v|_s, x_i \cdot v|_s, x_i^2 \cdot v|_s, \ldots, x_i^{n-1} \cdot v|_s\}$ is a basis for $\mathcal{V}|_s \simeq k^n$. After shrinking T if necessary, we may assume that $\{v, x_i \cdot v, x_i^2 \cdot v, \ldots, x_i^{n-1} \cdot v\}$ is a framing of $\pi_* \mathcal{V}$ over T. That is, $x_i^\ell \cdot v$'s give rise to an isomorphism

$$\mathcal{O}_T^{\oplus n} \longrightarrow \pi_* \mathcal{V}|_T$$

over T. But with respect to this framing, multiplication by x_i is represented by an $n \times n$ matrix B with 1's on the subdiagonal and zeros elsewhere. Any matrix commuting with B is easily shown to be a polynomial in B, and the μ_j's commute with each other. So, we have, over T,

$$x_j = \sum_{l=1}^{n-1} b_{jl} x_i^l \pmod{\mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{O}_T}, \quad b_{jl} \in \Gamma(\mathcal{O}_T), \quad j \neq i.$$

By dimension reasons, $x_j - \sum_{l=1}^{n-1} b_{jl} x_i^l$ generates $\mathcal{I}$ over T, and it induces a unique morphism $\psi_T : T \to \mathcal{M}_{q,n}$ such that $(\psi_T \times 1)^* \mathcal{U} = \mathcal{I}|_{T \times \operatorname{Spec} A_{q,n}}$. Indeed, we have a well-defined lifting $\tilde{\psi}_T : T \to \mathrm{GL}_n \times \mathrm{Hom}_k(\Omega_1, z^2 k[z])$ of ψ_T given by

$$t \mapsto (g, f_t),$$

where f is the homomorphism defined by $b_{jl}(t)$, and $g \in \mathrm{GL}_n$ is the permutation matrix $g \cdot x_1 = x_i$, $g \cdot x_i = x_1$, and $g \cdot x_k = x_k$ for all $k \neq 1, i$. Then $f_t(\mathfrak{q}_1)$ is the ideal defined by $x_j - \sum_{l=1}^{n-1} b_{jl}(t) x_1^l$, $j \neq 1$, and $g \cdot f_t(\mathfrak{q}_1)$ is precisely the ideal $\mathcal{I}|_{\{t\} \times \operatorname{Spec} A_{q,n}}$, which also is plainly seen to be equal to $\tilde{\psi}_T^*(\mathcal{U}'')$.

We claim that these ψ_T's glue together to give the desired morphism $S \to \mathcal{M}_{q,n}$. Indeed, consider ψ_T and $\psi_{T'}$, given by $\tilde{\psi}_T(t) = (g, f_t)$ and $\tilde{\psi}_{T'}(t') = (g', f'_{t'})$. At any point $t \in T \cap T'$ in the intersection, we have $g \cdot f_t(\mathfrak{q}_1) = \mathcal{I}_{\{t\}\times \operatorname{Spec} A_{q,n}} = g' \cdot f'_t(\mathfrak{q}_1)$, and it follows that

$$(g', f'_t) = (gp_t^{-1}, p_t \cdot f_t)$$

for some T-point p of $P_1(T)$. The dependence of p_t on t is certainly algebraic, and it follows that ψ_T and $\psi_{T'}$ agree on $T \cap T'$, defining a morphism $T \cup T' \to \mathcal{M}_{q,n}$. Since $(\psi_T \times 1_{A_{q,n}})^*\mathcal{U} = \mathcal{I}_{T\times \operatorname{Spec} A_{q,n}}$ for each T, the sheaf $\mathcal{U}$ is universal. □

Theorem 7.5. *The space* $\mathcal{M}_{q,n}$ *is a fine moduli scheme for the moduli functor* $\underline{\mathcal{M}_{q,n}} : \mathrm{Sch}/k \to \mathrm{Sets}$ *from the category of* k*-schemes to the category of sets, defined by*

$$\underline{\mathcal{M}_{q,n}}(S) = \{\textit{Ideal sheaves } \mathcal{I} \subset \mathcal{O}_S \otimes_k A_{q,n} \textit{ ARR and flat over } S\}.$$

Proof. The contents of Propositions 7.3 and 7.4 combined give the assertion of the theorem. □

There is a natural set map $\pi : \mathcal{N}^r_{q,n} \to \mathcal{M}_{q,n}$ that sends $(N_1, \dots, N_q)$ to the kernel of $\rho : k[x_1, \dots, x_q] \to \operatorname{End}(k^n)$, $\rho(x_i) = N_i$. From the universality of $\mathcal{M}_{q,n}$, it follows immediately that this is a morphism of varieties.

Corollary 7.6. *There is a* GL_n *invariant orbit map* $\pi : \mathcal{N}^r_{q,n} \to \mathcal{M}_{q,n}$.

Proof. Consider the natural map

$$\mathcal{O}_{\mathcal{N}^r_{q,n}} \otimes_k A_{q,n} \to \mathcal{O}_{\mathcal{N}^r_{q,n}} \otimes_k \operatorname{End}(V)$$

defined by sending $f(x_1, \dots, x_q) \in \mathcal{O}_{\mathcal{N}^r_{q,n}}(U)[x_1, \dots, x_q]$ to $f(N_1, \dots, N_q)$, where $(N_1, \dots, N_q)$ are, by abuse of notation, regarded as sections of $\mathcal{O}_{\mathcal{N}^r_{q,n}}$ over a given open set $U \subset \mathcal{N}^r_{q,n}$. The kernel $\mathcal{K}$ of this map is an ideal sheaf ARR over $\mathcal{N}^r_{q,n}$ giving rise to a morphism $\mathcal{N}^r_{q,n} \to \mathcal{M}_{q,n}$, which is constant on GL_n orbits due to Proposition 2.4. By the universality of $\mathcal{M}_{q,n}$ (Theorem 7.5), we conclude that π is algebraic. □

8. The moduli space as an open subscheme of a punctual Hilbert scheme

Let $X_0, \dots, X_q$ denote a set of homogeneous coordinates for $\mathbb{P}^q$, $[0] := [1, 0, \dots, 0]$, and let $x_i = X_i/X_0$ be the affine coordinates of the affine chart $\{X_0 \neq 0\}$. In this section, we shall consider the relation between $\mathcal{M}_{q,n}$ and the Hilbert scheme $\operatorname{Hilb}^n \mathbb{P}^q$ of length n zero-dimensional subschemes of $\mathbb{P}^q$. The Hilbert scheme is defined by its functor. That is, for any scheme S over k, we have

$$\operatorname{Hom}(S, \operatorname{Hilb}^n \mathbb{P}^q) = \{Z \subset \mathbb{P}^q_S \mid Z \text{ is surjective, finite flat over } S \text{ of degree } n\}.$$

Let $\mathbb{P}^{q(n)}$ be the nth symmetric product of $\mathbb{P}^q$ that parametrizes length n cycles of $\mathbb{P}^q$. Recall the Hilbert–Chow morphism $\Psi : \operatorname{Hilb}^n \mathbb{P}^q \to \mathbb{P}^{q(n)}$ that maps a zero-dimensional subscheme Z to its underlying cycle $[Z]$.

Definition 8.1. The *punctual Hilbert scheme* $\operatorname{Hilb}^n_{[0]} \mathbb{P}^q$ is the reduced fiber of the Hilbert–Chow morphism Ψ over the cycle $n \cdot [0]$.

That is, the punctual Hilbert scheme $Hilb^n_{[0]}\mathbb{P}^q$ parametrizes the zero-dimensional subschemes of length n with support at the single point $[0] \in \mathbb{P}^q$.

Theorem 8.2. $\mathcal{M}_{q,n}$ *is isomorphic to an open subscheme of the punctual Hilbert scheme* $\operatorname{Hilb}^n_{[0]} \mathbb{P}^q$.

Proof. The universal quotient $\mathcal{O}_{\mathcal{M}_{q,n}} \otimes A_{q,n} \to \mathcal{W} := \mathcal{O}_{\mathcal{M}_{q,n}} \otimes A_{q,n}/\mathcal{I}_{G/G_1}(\mathfrak{q}_1)$ gives rise to a map $\eta : \mathcal{M}_{q,n} \to \mathrm{Hilb}^n \mathbb{P}^q$ by (abusing notation and) identifying x_i with the affine coordinates x_i of $\mathbb{P}^q$ at [0]. Since $x_i^n = 0$ for all i, $\mathcal{W}$ is supported at [0]. Also, $\mathcal{M}_{q,n}$ is nonsingular (and hence reduced in particular) since it is an affine space bundle over $\mathbb{P}^{q-1}$. It follows that the map η factors through the punctual Hilbert scheme $\mathrm{Hilb}^{\mathrm{n}}_{[0]} \mathbb{P}^q$.

Consider the universal quotient sequence

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathrm{Hilb}^n_{[0]} \mathbb{P}^q}[x_1, \ldots, x_q] \to \mathcal{Q} \to 0$$

on the punctual Hilbert scheme. Multiplication by $x_i \cdot : \mathcal{Q} \to \mathcal{Q}$ defines an $\mathcal{O}_{\mathrm{Hilb}^n_{[0]} \mathbb{P}^q}$-linear map and the locus where x_i^{n-1} is identically 0 is a closed subscheme, say, Z_i, of the punctual Hilbert scheme. Then $U := \mathrm{Hilb}^{\mathrm{n}}_{[0]} \mathbb{P}^q \setminus \bigcap_i Z_i$ is an open subscheme of $\mathrm{Hilb}^{\mathrm{n}}_{[0]} \mathbb{P}^q$. Now the restriction of $\mathcal{J}$ to U is ARR by construction of U. Hence by Proposition 7.4, we have the corresponding morphism $f_{\mathcal{I}} : U \to \mathcal{M}_{q,n}$, providing an inverse to η. $\square$

Remark 8.3. Due to Lemma 2.10, $\mathcal{M}_{q,n}$ and the stratum of the punctual Hilbert scheme consisting of curvilinear ideals are the same *as sets*. They are indeed isomorphic schemes by the previous theorem.

9. $\mathcal{M}_{q,n} \to \mathbb{P}^{q-1}$ IS NOT A VECTOR BUNDLE

We shall prove in this section that, as opposed to the result (in smooth category) of the previous section, $\mathcal{M}_{q,n}$ is not a vector bundle over $\mathbb{P}^{q-1}$ in the algebraic category. This result can be regarded as an extension of the following theorem by Iarrobino [Iar73, Iar77].

Theorem 9.1 ([Iar73, Theorem 1])**.** *The variety Z parametrizing linear ideals $I \subset k[[x, y]]$ of colength 4 is locally trivial over $\mathbb{P}^1$ but is not a vector bundle.*

Here, an ideal $I \subset \langle x, y\rangle$ is said to be *linear* if I is not contained in $\langle x, y\rangle^2$. It is equivalent to I being an annihilator of a regular representation (Definition 7.1) since $f \in I$ satisfies $f^{n-1} \neq 0$ in $k[[x, y]]/\langle x, y\rangle^n$ if and only if $f \notin \langle x, y\rangle^2$. The map from Z to $\mathbb{P}^1$ is given by sending I to its associated graded ideal $\mathrm{gr}(I)$, which is linear and also of colength 4 (cf. Remark 4.4). We note that the map $Z \to \mathbb{P}^1$ is precisely our fibration $\mathcal{M}_{2,4} \to \mathcal{M}_{2,2}$ induced by the natural map $A_{2,4} \to A_{2,2}$. The key ingredient of the proof of the above theorem is that there is no section of $\mathcal{M}_{2,4} \to \mathcal{M}_{2,3} \simeq \mathcal{T}_{\mathbb{P}^1}(+1) \simeq O_{\mathbb{P}^1}(+3)$ [Iar73, Lemma 3].

In fact, the theorem above is a special case of a more general statement that the variety Z_T of *curvilinear ideals of type* T is a locally trivial affine space fibration, but not necessarily a vector bundle over a complete variety G_T of graded ideals of type T [Iar72a, Theorem 3]. The curvilinear case is when $T = T_n = (\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots)$. See Remark 4.4 for a relevant discussion.

Theorem 9.2. *$\mathcal{M}_{q,n} \to \mathbb{P}^{q-1}$ is not a vector bundle in the algebraic category for $n \geq 4$.*

Proof. Let ϖ_n denote the fibration $\mathcal{M}_{q,n} \to \mathbb{P}^{q-1}$. It is induced by the natural projection $A_{q,n} = k[x_1, \ldots, x_q]/\mathfrak{m}^n \to k[x_1, \ldots, x_q]/\mathfrak{m}^2 = A_{q,2}$. Consider the set map

$$\mathcal{M}_{2,n} \to \mathcal{M}_{q,n}$$

sending a commuting pair (N_1, N_2) to the q-tuple $(N_1, N_2, \ldots, N_2)$. It can be readily seen that this map is algebraic: Recall the universal sheaf over $\mathcal{M}_{2,n}$ from Proposition 7.4. We denote it by $\mathcal{U}$. Consider the associated ideal sheaf

$$\mathcal{U}' := \mathcal{U} + \mathcal{O}_{\mathcal{M}_{2,n}}\langle x_3 - x_2, \ldots, x_q - x_2\rangle \subset \mathcal{O}_{\mathcal{M}_{2,n}} \otimes_k A_{q,n}.$$

Note that this annihilates regular representations of $A_{q,n}$: For any $[I] \in \mathcal{M}_{2,n}$, we have

$$\begin{aligned}\mathcal{U}' \otimes \kappa([I]) &\simeq \mathcal{U} \otimes \kappa([I]) + \langle x_3 - x_2, \ldots, x_q - x_2\rangle \\ &= I + \langle x_3 - x_2, \ldots, x_q - x_2\rangle \subset k[x_1, \ldots, x_q],\end{aligned}$$

which annihilates a regular representation since $I \subset k[x_1, x_2]$ satisfies $I \not\subset \langle x, y\rangle^2$.

By the universality (Proposition 7.4), $\mathcal{U}'$ induces a morphism $\psi : \mathcal{M}_{2,n} \to \mathcal{M}_{q,n}$. Let $I \subset A_{2,n}$ be a colength n ideal annihilating a regular representation, and let (N_1, N_2) denote the commuting pair of nilpotents associated with I; i.e., I is the kernel of $A_{2,n} \to \mathrm{End}(k^n)$ given by mapping x_i to N_i. Obviously $I + \langle x_3 - x_2, \ldots, x_q - x_2\rangle$ is contained in the kernel of $A_{q,n} \to \mathrm{End}(k^n)$ given by $x_1 \mapsto N_1$ and $x_i \mapsto N_2\ \ \forall i \geq 2$. Then it must equal the kernel since they are of the same colength. It follows that ψ corresponds to the map $(N_1, N_2) \mapsto (N_1, N_2, N_2, \ldots, N_2)$.

Suppose that ϖ_n is a vector bundle. From the commutative square

$$\begin{tikzcd} \varpi_n^{-1}(\iota(\mathbb{P}^1)) = \mathcal{M}_{2,n} \arrow[r] \arrow[d] & \mathcal{M}_{q,n} \arrow[d, "\varpi_n"] \\ \mathcal{M}_{2,2} \simeq \mathbb{P}^1 \arrow[r, "\iota"'] & \mathcal{M}_{q,2} \simeq \mathbb{P}^{q-1}, \end{tikzcd}$$

we conclude that $\mathcal{M}_{2,n} \to \mathbb{P}^1$ is also a vector bundle. But then $\mathcal{M}_{2,n}$ should be isomorphic to $\bigoplus_{j=1}^{n-1} T_{\mathbb{P}^1}(+j) \simeq \bigoplus_{j=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(j+2)$ by Theorem 6.3. In this case, the projection $\pi_{n,3} : \mathcal{M}_{q,n} \to \mathcal{M}_{q,3}$ induced by $A_{q,n} \to A_{q,3}$ is simply the bundle projection $\bigoplus_{j=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(j+2) \to \mathcal{O}_{\mathbb{P}^1}(+3)$ and hence admits a section σ. Now consider the following diagram:

$$\begin{tikzcd} \mathcal{M}_{2,n} \arrow[r, "\pi_{n,4}"] \arrow[dr, "\pi_{n,3}"] & \mathcal{M}_{2,4} \arrow[d, "\pi_{4,3}"] \\ & \mathcal{M}_{2,3}, \arrow[ul, bend left, "\sigma"] \end{tikzcd}$$

which is commutative since $\pi_{n,3} = \pi_{4,3} \circ \pi_{n,4}$. Hence we have a section $\pi_{n,4} \circ \sigma$ of $\pi_{4,3}$, contradicting [Iar73, Lemma 3]. □

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