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EXTENSION OF ISOTOPIES IN THE PLANE

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Abstract. It is known that a holomorphic motion (an analytic version of an isotopy) of a set X in the complex plane $\mathbb{C}$ always extends to a holomorphic motion of the entire plane. In the topological category, it was recently shown that an isotopy $h : X \times [0,1] \to \mathbb{C}$, starting at the identity, of a plane continuum X also always extends to an isotopy of the entire plane. Easy examples show that this result does not generalize to all plane compacta. In this paper we will provide a characterization of isotopies of uniformly perfect plane compacta X which extend to an isotopy of the entire plane. Using this characterization, we prove that such an extension is always possible provided the diameters of all components of X are uniformly bounded away from zero.

1. Introduction

Denote the complex plane by $\mathbb{C}$ and the open unit disk by $\mathbb{D}$. An *isotopy* of a set $X \subset \mathbb{C}$ is a homotopy $h : X \times [0,1] \to \mathbb{C}$ such that for each $t \in [0,1]$, the function $h^t : X \to \mathbb{C}$ defined by $h^t(x) = h(x,t)$ is an embedding (i.e., a homeomorphism of X onto the range of h^t).

Suppose that $h : X \times [0,1] \to \mathbb{C}$ is an isotopy of a compactum $X \subset \mathbb{C}$ such that $h^0 = \mathrm{id}_X$. We consider the old problem of when the isotopy h can be extended to an isotopy of the entire plane.[1]

Positive classical solutions were obtained only in the case when X is a simple continuum. For example, it follows from results in [4, 5] that an isotopy of a simple closed curve can be extended over the plane (see [14] for a generalization). Much stronger results were obtained in the analytic setting. In this setting an isotopy corresponds to a holomorphic motion. If one thinks of an isotopy $h : X \times [0,1] \to \mathbb{C}$ as a continuous collection of embeddings $\{h|_{X \times \{t\}}\}_{t \in I}$ of the so-called "dynamical space" X where t is contained in the parameter space I, then, in the case of a holomorphic motion, the parameter space I is replaced by the unit disk $\mathbb{D} \subset \mathbb{C}$. Moreover, the requirement that h be continuous is replaced by the assumption that on slices the map is holomorphic. To be precise, a holomorphic motion $h : \mathbb{D} \times X \to \mathbb{C}$ is a function such that:

(i) for each $z \in \mathbb{D}$, the map $h|_{\{z\} \times X}$ is one-to-one,
(ii) $h|_{\{0\} \times X} = \mathrm{id}_X$, and
(iii) for each $x \in X$ the map $h|_{\mathbb{D} \times \{x\}}$ is holomorphic.

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[1]We are indebted to Professor R. D. Edwards, who communicated a related problem to us.

Note that in order to be consistent with standard notation, we reversed the order of the parameter space and the dynamical space X in the domain of the holomorphic motion h. Even though in this definition h is not assumed to be continuous, continuity of the map h follows from the other conditions [24]. Initial results extended the holomorphic motion over the closure of X [23, 24]. Subsequently [7, 34] it was shown that the holomorphic motion could be extended to all of $\mathbb{C}$, but only on a subdisk of the parameter space $\mathbb{D}$. These results culminated in the remarkable extension result by Slodkowski [33]: Any holomorphic motion of an arbitrary subset X of the plane extends to a holomorphic motion of the entire plane (see [3] or [11] for alternative proofs and [18] for a self-contained exposition).

Although the Slodkowski Extension Theorem holds for arbitrary plane sets, some additional restrictions are needed for the existence of an extension of an isotopy to the entire plane $\mathbb{C}$. First, it is reasonable to restrict to isotopies of plane compacta. This by itself is not enough since it is known that there exists an isotopy of a convergent sequence in the plane which cannot be extended over the plane (see [34] or [15]). On the other hand, it was shown recently in [27] that any isotopy beginning at the identity of an arbitrary plane continuum X can be extended over the plane. In this case each complementary domain U of X is simply connected, and, hence, there exists a conformal isomorphism $\varphi_U : \mathbb{D} \to U$. The proof made use of two key analytic results for these conformal isomorphisms: the Carathéodory kernel convergence theorem and the Gehring-Hayman inequality for the diameters of hyperbolic geodesics in U.

Let us now consider the case when X is a plane compactum. Since we may assume that X contains at least three points, the boundary of every complementary component U of X contains at least three points, so U is hyperbolic; i.e., there exists an analytic covering map $\varphi_U : \mathbb{D} \to U$ (see [2]).

There is an analogue of the Carathéodory kernel convergence theorem which holds for families of analytic covering maps (see Section 2.1). For an analogue of the Gehring-Hayman inequality, an additional geometric condition will be required.

Definition 1. A compact subset $X \subset \mathbb{C}$ is *uniformly perfect with constant* k provided there exists $0 < k < 1$ so that for all $r < \operatorname{diam}(X)$ and all $x \in X$,

$$\{z \in \mathbb{C} : kr \leq |z - x| \leq r\} \cap X \neq \emptyset.$$

Clearly every uniformly perfect set is perfect, and the standard "middle-third" Cantor set is uniformly perfect. It is known that the Gehring-Hayman estimate on the diameter of hyperbolic geodesics still holds for an analytic covering map $\varphi_U : \mathbb{D} \to U$ to a domain U whose boundary is uniformly perfect (see Section 2.1 for details).

The main result in this paper is a characterization of isotopies $h : X \times [0, 1] \to \mathbb{C}$ of uniformly perfect plane compacta X which can be extended over the entire plane (see Theorem 12). We use our characterization to prove that any isotopy of a plane compactum such that the diameter of every component is uniformly bounded away from zero can be extended over the plane (see Theorem 20). Along the way, we will provide simpler proofs of some of the technical results in [27].

1.1. **Notation.** By a *map* we mean a continuous function. For $z \in \mathbb{C}$, the magnitude of z is denoted $|z|$, so that the Euclidean distance between two points $z, w \in \mathbb{C}$ is $|z - w|$. Given $z_0 \in \mathbb{C}$ and $r > 0$, denote

$$B(z_0, r) = \{z \in \mathbb{C} : |z_0 - z| < r\}.$$

By a *domain* we mean a connected, open, non-empty set $U \subset \mathbb{C}$. If $X \subset \mathbb{C}$ is closed, then a *complementary domain* of X is a component of $\mathbb{C} \setminus X$. A *crosscut* of a domain U is an *open arc* Q (i.e., $Q \approx (0,1) \subset \mathbb{R}$) contained in U such that $\overline{Q}$ is a *closed arc* (i.e., $\overline{Q} \approx [0,1]$) whose endpoints are in ∂U. Note that the endpoints of $\overline{Q}$ are required to be distinct. In general, if A is an open arc whose closure $\overline{A}$ is a closed arc, we may refer to the endpoints of $\overline{A}$ as the "endpoints of A".

A *path* is a map $\gamma : [0,1] \to \mathbb{C}$. Given a domain U, we say γ is a *path in* U if $\gamma((0,1)) \subset U$. Note that *we allow the possibility that* $\gamma(0) \in \partial U$ *and/or* $\gamma(1) \in \partial U$—we still call such a path a path in U.

We will make frequent use of covering maps in this paper. Given a covering map $\varphi : V \to U$, where V and U are domains, a *lift* of a point $x \in U$ is a point $\widehat{x} \in V$ such that $\varphi(\widehat{x}) = x$. Similarly, if γ is a path with $\gamma([0,1]) \subset U$, then a lift of γ is a path $\widehat{\gamma}$ in V such that $\varphi \circ \widehat{\gamma} = \gamma$.

The *Hausdorff metric* d_H measures the distance between two compact sets $A_1, A_2 \subset \mathbb{C}$ as follows:

$$d_H(A_1, A_2) = \max\{\max_{z_1 \in A_1} \min_{z_2 \in A_2} |z_1 - z_2|,\ \max_{z_2 \in A_2} \min_{z_1 \in A_1} |z_1 - z_2|\}.$$

Equivalently, $d_H(A_1, A_2)$ is the smallest number $\varepsilon \geq 0$ such that A_1 is contained in the closed ε-neighborhood of A_2 and A_2 is contained in the closed ε-neighborhood of A_1.

Given an isotopy $h : X \times [0,1] \to \mathbb{C}$, we denote $h^t = h|_{X \times \{t\}}$ and, for $x \in X$, we denote $x^t = h^t(x)$.

2. Preliminaries

In this section we collect several tools which we use in this paper. Many of these are standard analytical results; others are less well-known.

2.1. **Bounded analytic covering maps.** It is a standard classical result (see e.g. [2]) that for any domain $U \subset \mathbb{C}$ whose complement contains at least two points and for any $z_0 \in U$, there is a complex analytic covering map $\varphi : \mathbb{D} \to U$ such that $\varphi(0) = z_0$. Moreover, this covering map φ is uniquely determined by the argument of $\varphi'(0)$.

Many of the results below hold only for analytic covering maps $\varphi : \mathbb{D} \to U$ to bounded domains U. For the remainder of this subsection, let $U \subset \mathbb{C}$ be a bounded domain, and let $\varphi_U = \varphi : \mathbb{D} \to U$ be an analytic covering map.

Theorem 2 (Fatou [16]; see e.g. [13, p. 22])**.** *The radial limits* $\lim_{r \to 1^-} \varphi(re^{i\theta})$ *exist for all points* $e^{i\theta}$ *in* $\partial\mathbb{D}$ *except possibly for a set of linear measure zero.*

From now on, we will always assume that any bounded analytic covering map $\varphi : \mathbb{D} \to U$ has been extended to be defined over all points $e^{i\theta} \in \partial\mathbb{D}$ where the radial limit exists by $\varphi(e^{i\theta}) = \lim_{r \to 1^-} \varphi(re^{i\theta})$. Note that the function φ is not necessarily continuous at these points.

For this extended map φ, we extend the notion of lifts. If γ is a path in U (recall this allows for the possibility that $\gamma(0)$ and/or $\gamma(1)$ belongs to ∂U), then a *lift* of γ is a path $\widehat{\gamma}$ in $\mathbb{D}$ such that $\varphi \circ \widehat{\gamma} = \gamma$. This means that if $\gamma(0) \in \partial U$, then $\widehat{\gamma}(0) \in \partial\mathbb{D}$ and φ is defined at the point $\widehat{\gamma}(0)$ (and $\varphi(\widehat{\gamma}(0)) = \gamma(0)$), and likewise for $\gamma(1)$ and $\widehat{\gamma}(1)$.

Theorem 3 (Riesz [30,31]; see e.g. [13, p. 22])**.** *For each $x \in \partial U$, the set of points $e^{i\theta}$ for which $\lim_{r\to 1^-} \varphi(re^{i\theta}) = x$ has linear measure zero in $\partial\mathbb{D}$.*

The next result about lifts of paths is very similar to classical results for covering maps. Since our extended map φ is not a covering map at points in $\partial\mathbb{D}$, we include a proof for completeness.

Theorem 4. *Suppose γ is a path in U such that $\gamma((0,1]) \subset U$. Let $\widehat{z} \in \mathbb{D}$ be such that $\varphi(\widehat{z}) = \gamma(1)$. Then there exists a unique lift $\widehat{\gamma}$ of γ with $\widehat{\gamma}(1) = \widehat{z}$.*

In particular, if $\gamma(0) \in \partial U$, then $\widehat{\gamma}(0) \in \partial\mathbb{D}$, φ is defined at $\widehat{\gamma}(0)$ (i.e., the radial limit of φ exists there), and $\varphi(\widehat{\gamma}(0)) = \gamma(0)$.

Proof. We may assume that $\gamma(0) \in \partial U$. Since φ is a covering map, $\gamma|_{(0,1]}$ lifts to a path with initial point $\widehat{z}$ which compactifies on a continuum $K \subset \partial\mathbb{D}$. If K is non-degenerate, then there exists by Theorem 2 a set E of positive measure in the interior of K so that for each $e^{i\theta} \in E$, the radial limit $\lim_{r\to 1^-} \gamma(re^{i\theta})$ exists. Since the graph of $\widehat{\gamma}$ compactifies on K we can choose a sequence $s_i \to 1$ so that $\widehat{\gamma}(s_i) = r_i e^{i\theta}$ with $r_i \to 1$. It follows that the radial limit $\lim_{r\to 1^-} \varphi^t(re^{i\theta}) = \gamma(1)$ for each $e^{i\theta} \in E$, a contradiction with Theorem 3. Thus K is a point $e^{i\theta}$. Hence we can extend $\widehat{\gamma}$ continuously by defining $\widehat{\gamma}(0) = e^{i\theta}$.

By a theorem of Lindelöf [22] (see e.g. [13, p. 23]), it follows from the above that the radial limit of φ at $e^{i\theta}$ exists and equals $\gamma(0)$ as required. ☐

The next result is a variant of Theorem 4, in which the base point of the path to be lifted is in the boundary of U.

In the case that the boundary of U is uniformly perfect, we prove below in Lemma 16 a stronger result about lifting a homotopy under covering maps to a domain whose boundary is changing under an isotopy. The present result can be obtained as a corollary to Lemma 16 by using the identity isotopy. We omit a proof for the non-uniformly perfect case, since we won't need it for this paper.

Theorem 5. *Suppose γ is a path in U such that $\gamma((0,1]) \subset U$ and $\gamma(0) \in \partial U$. Let $\widehat{x} \in \partial\mathbb{D}$ be such that $\varphi_U(\widehat{x}) = \gamma(0)$ and γ is homotopic to the radial path $\varphi_U|_{\{r\widehat{x}\,:\,0\le r\le 1\}}$ under a homotopy in U that fixes the point $\gamma(0)$. Then there exists a lift $\widehat{\gamma}$ of γ with $\widehat{\gamma}(0) = \widehat{x}$. Moreover, if ∂U is perfect, this lift $\widehat{\gamma}$ is unique.*

The *hyperbolic metric* on the unit disk $\mathbb{D}$ is given by the form $\frac{2|dz|}{1-|z|^2}$, meaning that the length of a smooth path $\gamma : [0,1] \to \mathbb{D}$ is $\int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}\,dt$. The important property of the hyperbolic metric for us is that (hyperbolic) geodesics in $\mathbb{D}$ are pieces of round circles or straight lines which cross the boundary $\partial\mathbb{D}$ orthogonally. Via the covering map $\varphi : \mathbb{D} \to U$, we obtain the *hyperbolic metric on U*, in which the length of a smooth path in U is equal to the length of any lift of that path under φ; this length is independent of the choice of lift. It is a standard result that the hyperbolic metric on U is independent of the choice of covering map $\varphi : \mathbb{D} \to U$.

Theorem 6 (Gehring-Hayman [28,29])**.** *Suppose ∂U is uniformly perfect with constant k. There exists a constant K such that if $\widehat{g}$ is a hyperbolic geodesic in $\mathbb{D}$ and $\widehat{\Gamma}$ is a curve with the same endpoints as $\widehat{g}$, then*

$$\operatorname{diam}(\varphi(\widehat{g})) \le K \cdot \operatorname{diam}(\varphi(\widehat{\Gamma})).$$

The constant K depends only on k, not on the domain U itself or on the choice of analytic covering map φ.

We end this subsection with a discussion of analytic covering maps of varying domains in the plane. We will make use of the notion of Carathéodory kernel convergence, which was introduced by Carathéodory for univalent analytic maps in [10], then extended by Hejhal to the case of analytic covering maps.

Let $U_1, U_2, \ldots,$ and U_∞ be domains and let $z_1, z_2, \ldots,$ and z_∞ be points with $z_n \in U_n$ for all $n = 1, 2, \ldots$ and $z_\infty \in U_\infty$. We say that $\langle U_n, z_n \rangle \to \langle U_\infty, z_\infty \rangle$ *in the sense of Carathéodory kernel convergence* provided that (i) $z_n \to z_\infty$; (ii) for any compact set $K \subset U_\infty$, $K \subset U_n$ for all but finitely many n; and (iii) for any domain U containing z_∞, if $U \subseteq U_n$ for infinitely many n, then $U \subseteq U_\infty$.

Theorem 7 ([19]; see also [12])**.** *Let $U_1, U_2, \ldots,$ and U_∞ be domains and let $z_1, z_2, \ldots,$ and z_∞ be points with $z_n \in U_n$ for all $n = 1, 2, \ldots$ and $z_\infty \in U_\infty$. Let $\varphi_\infty : \mathbb{D} \to U_\infty$ be the analytic covering map such that $\varphi(0) = z_\infty$ and $\varphi'(0) > 0$. Likewise, for each $n = 1, 2, \ldots,$ let $\varphi_n : \mathbb{D} \to U_n$ be the analytic covering map such that $\varphi_n(0) = z_n$ and $\varphi_n'(0) > 0$. Then $\langle U_n, z_n \rangle \to \langle U_\infty, z_\infty \rangle$ in the sense of Carathéodory kernel convergence if and only if $\varphi_n \to \varphi_\infty$ uniformly on compact subsets of $\mathbb{D}$.*

2.2. **Partitioning plane domains.** Let U be a bounded domain in $\mathbb{C}$. We next describe a way of partitioning U into simple sets which are either circular arcs or regions whose boundaries are unions of circular arcs.

Let $\mathcal{B}$ be the collection of all open disks $B(c, r) \subset U$ such that $|\partial B(c, r) \cap \partial U| \geq 2$. Let $\mathcal{C}$ be the collection of all centers of such disks, and for $c \in \mathcal{C}$ let $r(c)$ be the radius of the corresponding disk in $\mathcal{B}$. The set $\mathcal{C}$, called the *skeleton of U*, was studied by several authors (see for example [17]). Note that for each $c \in \mathcal{C}$, $B(c, r(c))$ is conformally equivalent with the unit disk $\mathbb{D}$ and, hence, can be endowed with the hyperbolic metric ρ_c. Let $\mathrm{Hull}(c)$ be the convex hull of the set $\partial B(c, r(c)) \cap \partial U$ in $B(c, r(c))$ *using the hyperbolic metric ρ_c on the disk $B(c, r(c))$*. The following theorem by Kulkarni and Pinkall generalizes an earlier result by Bell [6] (see [8] for a more complete description).

Theorem 8 ([21])**.** *For each $z \in U$ there exists a unique $c \in \mathcal{C}$ such that $z \in \mathrm{Hull}(c)$.*

Let $\mathcal{J}$ be the collection of all crosscuts of U which are contained in the boundaries of the sets $\mathrm{Hull}(c)$ for $c \in \mathcal{C}$. The elements of $\mathcal{J}$ are circular open arcs (called *chords*) whose endpoints are in ∂U (see Figure 1 for an illustration). Two such chords do not cross each other inside U (i.e., if $\ell \neq \ell'$ are chords in $\mathcal{J}$, then $\ell \cap \ell' = \emptyset$), and the limit of any convergent sequence of chords in $\mathcal{J}$ is either a chord in $\mathcal{J}$ or a point in ∂U. In particular, the subcollection of chords of diameter greater than or equal to ε is compact for each $\varepsilon > 0$. As such, the family $\mathcal{J}$ is close to being a *lamination* of U (see Definition 17 in Section 3 below). However, it is possible that uncountably many distinct chords in $\mathcal{J}$ have the same pair of endpoints $x, y \in \partial U$.

2.3. **Equidistant sets.** Let A_1 and A_2 be disjoint closed sets in $\mathbb{C}$. The *equidistant set* between A_1 and A_2 is the set

$$\mathrm{Equi}(A_1, A_2) = \left\{ z \in \mathbb{C} : \min_{w \in A_1} |z - w| = \min_{w \in A_2} |z - w| \right\}.$$

The equidistant set is a convenient way to define a set running "in between" A_1 and A_2. Moreover, it has a very simple local structure in the case that the sets A_1 and A_2 are not "entangled" in the sense of the following definition.

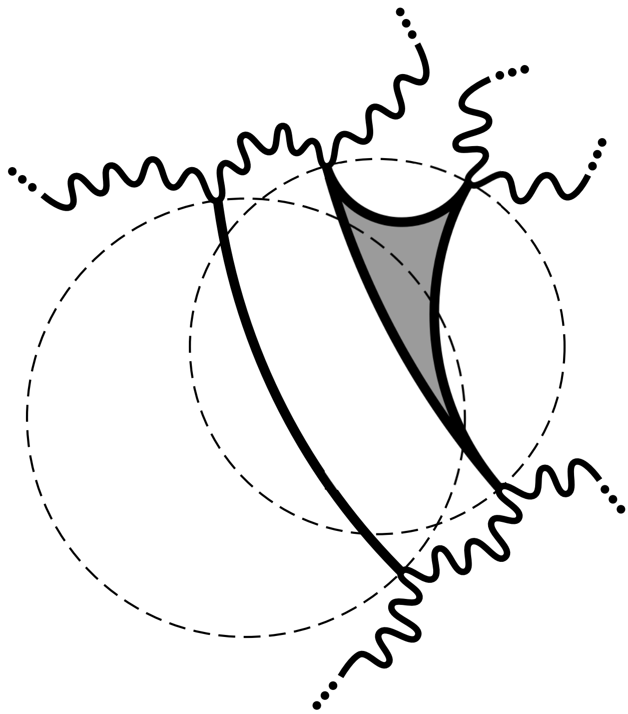

FIGURE 1. Depiction of two examples of the sets Hull(c) from the Kulkarni-Pinkall decomposition of a domain U in $\mathbb{C}$. In the picture, U is a component of the complement of the wavy lines.

Definition 9. We say that A_1 and A_2 are *non-interlaced* if whenever $B(c,r)$ is an open disk contained in the complement of $A_1 \cup A_2$, there are disjoint arcs $C_1, C_2 \subset \partial B(c,r)$ such that $A_1 \cap \partial B(c,r) \subset C_1$ and $A_2 \cap \partial B(c,r) \subset C_2$. We allow for the possibility that $C_1 = \emptyset$ in the case that $A_2 \cap \partial B(c,r) = \partial B(c,r)$, and vice versa.

By a 1-*manifold* in the plane, we mean a *closed* set $M \subset \mathbb{C}$ such that each component of M is homeomorphic either to $\mathbb{R}$ or to $\partial\mathbb{D}$, and these components are all open in M.

Theorem 10 ([1,9])**.** *Let A_1 and A_2 be disjoint closed sets in $\mathbb{C}$. If A_1 and A_2 are non-interlaced, then* $\mathrm{Equi}(A_1, A_2)$ *is a* 1*-manifold in the plane.*

2.4. **Midpoints of paths.** We identify the space of all paths in the plane $\mathbb{C}$ with the function space $\mathcal{C}([0,1],\mathbb{C})$ with the *uniform metric*; that is, the distance between two paths $\gamma_1, \gamma_2 \in \mathcal{C}([0,1],\mathbb{C})$ is equal to $\sup\{|\gamma_1(t) - \gamma_2(t)| : t \in [0,1]\}$.

The standard Euclidean length of a path is not a well-behaved function from $\mathcal{C}([0,1],\mathbb{C})$ to $[0,\infty)$. First, it is not defined (i.e., not finite) for all paths in $\mathcal{C}([0,1],\mathbb{C})$, but only for rectifiable paths. Second, paths can be arbitrarily close in the uniform metric and yet have very different Euclidean path lengths.

However, there do exist alternative "path length" functions $\mathsf{len} : \mathcal{C}([0,1],\mathbb{C}) \to [0,\infty)$ such that len is defined for *all* paths in $\mathcal{C}([0,1],\mathbb{C})$, and len is continuous with respect to the uniform metric on $\mathcal{C}([0,1],\mathbb{C})$ and the standard topology on $[0,\infty) \subset \mathbb{R}$; see [20, 25, 32]. Such an alternative path length function can be used to define a choice of "midpoint" of a path which varies continuously with the path. Specifically, the midpoint of γ is defined to be the point $\mathsf{m}(\gamma) = \gamma(t_0)$, where $t_0 \in (0,1)$ is chosen such that $\mathsf{len}(\gamma|_{[0,t_0]}) = \mathsf{len}(\gamma|_{[t_0,1]})$.

In this paper, we will not need to know any particulars about the definitions of such path length functions, but only this result about existence of such midpoints, which we state below.

Theorem 11 (See e.g. [20])**.** *There is a continuous function*

$$\mathsf{m} : \mathcal{C}([0,1], \mathbb{C}) \to \mathbb{C}$$

such that $\mathsf{m}(\gamma) \in \gamma((0,1))$ *for all* $\gamma \in \mathcal{C}([0,1], \mathbb{C})$.

Moreover, if γ_1 *and* γ_2 *are both parameterizations of a closed arc* A *(i.e., if* $\gamma_1([0,1]) = \gamma_2([0,1]) = A$ *and* γ_1 *and* γ_2 *are homeomorphisms between* $[0,1]$ *and* A*), then* $\mathsf{m}(\gamma_1) = \mathsf{m}(\gamma_2)$.

In light of the second part of Theorem 11, given an (open or closed) arc A, we define the midpoint of A to be $\mathsf{m}(A) = \mathsf{m}(\gamma)$ where γ is any path which parameterizes A ($\overline{A}$ if A is an open arc).

3. Main theorem

In this section, we state and prove the main theorem of this paper, which is a characterization of isotopies of uniformly perfect plane compacta which can be extended over the entire plane. Note that the example of an isotopy of a countable sequence which does not extend over the plane, mentioned in the Introduction, can easily be modified to obtain an isotopy $h : X \times [0,1] \to \mathbb{C}$ so that for each t, $X^t = h^t(X)$ is a uniformly perfect Cantor set with the same constant k. Thus, additional assumptions are required to ensure the extension of such an isotopy over the plane.

Theorem 12. *Suppose that* $h : X \times [0,1] \to \mathbb{C}$ *is an isotopy of a compactum* $X \subset \mathbb{C}$ *starting at the identity, such that* X^t *is uniformly perfect with the same constant* k *for each* $t \in [0,1]$*. Then the following are equivalent:*

(i) h *extends to an isotopy of the entire plane* $\mathbb{C}$.
(ii) *For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any crosscut* Q *of a complementary domain* U *of* X *with* $\mathrm{diam}(C) < \delta$*, there exists a homotopy* $h_Q : (X \cup Q) \times [0,1] \to \mathbb{C}$ *starting at the identity which extends* h *and is such that* $h^t_Q(Q) \cap X^t = \emptyset$ *and* $\mathrm{diam}(h^t_Q(Q)) < \varepsilon$ *for all* $t \in [0,1]$.

It is trivial to see that condition (i) implies condition (ii) from Theorem 12.

To obtain the converse, we will in fact prove a stronger characterization in Theorem 14 below. To state this theorem, we introduce the following simple condition.

Definition 13. Let $X \subset \mathbb{C}$ be a compact set and let $h : X \times [0,1] \to \mathbb{C}$ be an isotopy of X starting at the identity. We say that X is *encircled* if X has a component which is a large circle Σ such that $h^t|_\Sigma$ is the identity for all $t \in [0,1]$ and $X^t \setminus \Sigma$ is contained in a compact subset of the bounded complementary domain of Σ for all $t \in [0,1]$.

Note that if (ii) from Theorem 12 holds, then we may additionally assume without loss of generality (i.e., without falsifying condition (ii) from Theorem 12) that X is encircled.

3.1. **Tracking bounded complementary domains.** For the remainder of this section, we assume that $h : X \times [0,1] \to \mathbb{C}$ is an isotopy of a compact set $X \subset \mathbb{C}$ starting at the identity such that X^t is uniformly perfect for all $t \in [0,1]$ with the same constant k and X is encircled.

Clearly such an isotopy can be extended over the unbounded complementary domain of X as the identity for all $t \in [0,1]$. Hence we only need to consider bounded complementary domains for the remainder of this section.

Let U be a bounded complementary domain of X. Choose a point $z_U \in U$. Clearly the isotopy h can be extended to an isotopy $h_U : (X \cup \{z_U\}) \times [0,1] \to \mathbb{C}$ starting at the identity. Define U^t to be the complementary domain of X^t which contains the point $h_U^t(z_U) = z_U^t$. Let $\varphi_U^t : \mathbb{D} \to U^t$ be the analytic covering map such that $\varphi_U^t(0) = z_U^t$ and $(\varphi_U^t)'(0) > 0$. It is straightforward to see that if $t_n \to t_\infty$, then the pointed domains $\langle U^{t_n}, z_U^{t_n} \rangle$ converge to $\langle U^{t_\infty}, z_U^{t_\infty} \rangle$ in the sense of Carathéodory kernel convergence. Hence, by Theorem 7, the covering maps $\varphi_U^{t_n}$ converge to $\varphi_U^{t_\infty}$ uniformly on compact subsets of $\mathbb{D}$. We will always assume that the complementary domains U^t of X^t and analytic covering maps $\varphi_U^t : \mathbb{D} \to U^t$ are defined in this way. It is clear that this definition of U^t does not depend on the choices of z_U and h_U.

The following theorem is a stronger characterization of isotopies of uniformly perfect plane compacta that can be extended over the plane than the one given in Theorem 12, in the sense that condition (ii) of Theorem 14 is weaker than condition (ii) of Theorem 12. We will in fact use this stronger characterization in Section 4.

Theorem 14. *Suppose that $h : X \times [0,1] \to \mathbb{C}$ is an isotopy of a compactum $X \subset \mathbb{C}$ starting at the identity, such that X^t is uniformly perfect with the same constant k for each $t \in [0,1]$ and X is encircled. Then the following are equivalent:*

(i) *h extends to an isotopy of the entire plane $\mathbb{C}$.*

(ii) *For each bounded complementary domain U of X and each $\varepsilon > 0$ there exists $\delta > 0$ with the following property:*

For any crosscut Q in U with endpoints $a, b \in \partial U$ and with $\operatorname{diam}(Q) < \delta$, there exists a family $\{\gamma_t : t \in [0,1]\}$ such that (1) γ_t is a path in U^t joining a^t and b^t for each $t \in [0,1]$, (2) γ_0 is homotopic to Q in U with endpoints fixed, (3) $\operatorname{diam}(\gamma_t([0,1])) < \varepsilon$ for all $t \in [0,1]$, and (4) there are lifts $\widehat{\gamma}_t$ of the paths γ_t under φ_U^t such that the sets $\widehat{\gamma}_t([0,1])$ vary continuously in t with respect to the Hausdorff metric.

We have deliberately chosen to use subscripts in the notation for γ_t (instead of superscripts like γ^t) to emphasize the point that the paths γ_t are *not* required to change continuously in the sense of an isotopy or homotopy. We only require the weaker condition that the images of the lifts $\widehat{\gamma}_t$ vary continuously with respect to the Hausdorff metric. Even though condition (ii) of Theorem 14 is more cumbersome to state, we demonstrate in Section 4 that it is easier to apply.

The proofs of Theorems 12 and 14 will be completed in Section 3.4 below.

3.2. **Lifts in moving domains.** As in Section 3.1, we continue to assume that $h : X \times [0,1] \to \mathbb{C}$ is an isotopy of a compact set $X \subset \mathbb{C}$ starting at the identity, such that X^t is uniformly perfect for all $t \in [0,1]$ with the same constant k and X is encircled.

We begin by proving two statements about lifts under the covering maps φ_U^t in the spirit of the results from Section 2.1 above.

Lemma 15. *Let U be a bounded complementary domain of X. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t \in [0,1]$ if γ is a path in U^t with* $\operatorname{diam}(\gamma([0,1])) < \delta$ *and $\widehat{\gamma}$ is any lift of γ under φ_U^t, then* $\operatorname{diam}(\widehat{\gamma}([0,1])) < \varepsilon$.

Proof. Suppose the lemma fails. Then there exists $\varepsilon > 0$, a sequence γ_i of paths in U^{t_i}, and lifts $\widehat{\gamma}_i$ such that $\lim \operatorname{diam}(\gamma_i([0,1])) = 0$ and $\operatorname{diam}(\widehat{\gamma}_i([0,1])) \geq \varepsilon$ for all i. Choose two points $\widehat{a}_i, \widehat{b}_i$ in $\widehat{\gamma}_i([0,1])$ such that $|\widehat{a}_i - \widehat{b}_i| > \frac{\varepsilon}{2}$, and let $\widehat{\mathfrak{g}}_i$ be the hyperbolic geodesic with endpoints $\widehat{a}_i$ and $\widehat{b}_i$. Put $\varphi_U^{t_i}(\widehat{\mathfrak{g}}_i) = \mathfrak{g}_i$. By Theorem 6, $\operatorname{diam}(\mathfrak{g}_i) \to 0$. Since the geodesics $\widehat{\mathfrak{g}}_i$ are pieces of round circles or straight lines which cross $\partial\mathbb{D}$ perpendicularly and have diameter bigger than $\frac{\varepsilon}{2}$, there exist $\eta > 0$ and points $\widehat{x}_i \in \widehat{\mathfrak{g}}_i$ such that $|\widehat{x}_i| \leq 1 - \eta$ for all i. By choosing a subsequence we may assume that $t_i \to t_\infty$, $\widehat{x}_i \to \widehat{x}_\infty \in \mathbb{D}$, and $\lim \mathfrak{g}_i = z_\infty$ is a point in $\overline{U^{t_\infty}}$. Let K_i be the component of $\widehat{\mathfrak{g}}_i \cap B(\widehat{x}_\infty, \frac{\eta}{2})$ containing the point $\widehat{x}_i$. We may assume that $K_i \to K_\infty$, where K_∞ is a non-degenerate continuum in $\mathbb{D}$. Since $\varphi_U^{t_i} \to \varphi_U^{t_\infty}$ uniformly on compact sets in $\mathbb{D}$, $\varphi_U^{t_\infty}(K_\infty) = z_\infty$, which is a contradiction since $\varphi_U^{t_\infty}$ is a covering map. □

Given a homotopy $\Gamma : [0,1] \times [0,1] \to \mathbb{C}$ we denote for each $t \in [0,1]$, $\Gamma^t = \Gamma|_{[0,1]\times\{t\}} : [0,1] \to \mathbb{C}$.

Lemma 16. *Let U be a bounded complementary domain of X. Suppose that $\Gamma : [0,1] \times [0,1] \to \mathbb{C}$ is a homotopy with $\Gamma^t(0) = h^t(\Gamma^0(0)) \in \partial U^t$ and $\Gamma^t(s) \in U^t$ for all $s \in (0,1]$ and all $t \in [0,1]$. Let $\widehat{z} \in \mathbb{D}$ be such that $\varphi_U^0(\widehat{z}) = \Gamma^0(1)$. Then there exists a homotopy $\widehat{\Gamma} : [0,1] \times [0,1] \to \overline{\mathbb{D}}$ lifting Γ, i.e., $\varphi_U^t \circ \widehat{\Gamma}^t = \Gamma^t$ for all $t \in [0,1]$, and such that $\widehat{\Gamma}^0(1) = \widehat{z}$.*

Proof. Define $\Psi : \mathbb{D} \times [0,1] \to \bigcup_{t\in[0,1]}(U^t \times \{t\})$ by $\Psi(z,t) = (\varphi_U^t(z), t)$ for $t \in [0,1]$ and $z \in \mathbb{D}$.

Claim 16.1. Ψ is a covering map.

Proof of Claim 16.1. Let $(y_0, t_0) \in U^{t_0} \times \{t_0\}$. Choose a small simply connected neighborhood V of y_0 and $\delta > 0$ such that $\overline{V} \cap X^t = \emptyset$ and V is evenly covered by φ_U^t for all t with $|t - t_0| \leq \delta$. Hence, $V \times (t_0 - \delta, t_0 + \delta)$ is a simply connected neighborhood of (y_0, t_0) in $\bigcup_{t\in[0,1]}(U^t \times \{t\})$.

Next let $(x_0, t_0) \in \Psi^{-1}((y_0, t_0))$. Since the covering maps φ_U^t are uniformly convergent on compact sets, it is not difficult to see that there exists a map $g : (t_0 - \delta, t_0 + \delta) \to \mathbb{D} \times [0,1]$ such that $g(t_0) = (x_0, t_0)$ and $\Psi \circ g(t) = (y_0, t)$ for all t with $|t - t_0| < \delta$.

For each t with $|t - t_0| < \delta$, let $x \in U^t$ be such that $g(t) = (x, t)$, and let W^t be the component of $(\varphi_U^t)^{-1}(V)$ which contains the point x. Let

$$W = \bigcup_{t\in(t_0-\delta, t_0+\delta)} (W^t \times \{t\}).$$

Then it is not difficult to see that $\Psi|_W : W \to V \times (t_0 - \delta, t_0 + \delta)$ is a homeomorphism. Thus Ψ is a covering map. □(Claim 16.1)

Define $\alpha : [0,1] \times [0,1] \to \bigcup_{t\in[0,1]}(U^t \times \{t\})$ by $\alpha(s,t) = (\Gamma^t(s), t)$. Define the lift $\widehat{\alpha}$ of α under Ψ as follows: first lift $\alpha|_{\{1\}\times[0,1]}$, using the covering map Ψ, to define $\widehat{\alpha}|_{\{1\}\times[0,1]}$ such that $\widehat{\alpha}(1,0) = (\widehat{z}, 0)$. Next, for each $t \in [0,1]$, use Theorem 4 to lift $\alpha|_{[0,1]\times\{t\}}$ to define $\widehat{\alpha}|_{[0,1]\times\{t\}}$ so that this lift coincides with the first lift of

$\alpha|_{\{1\}\times[0,1]}$ at $(1,t)$. Finally, define $\widehat{\Gamma} = \pi_1 \circ \widehat{\alpha}$, where π_1 denotes the first coordinate projection.

Observe that for all $s \in (0,1]$, the function $\widehat{\alpha}|_{[s,1]\times[0,1]}$ is the unique lift of $\alpha|_{[s,1]\times[0,1]}$ under the covering map Ψ with $\widehat{\alpha}(1,0) = \widehat{z}$, hence is continuous by standard covering map theory. It follows that $\widehat{\alpha}$, and hence $\widehat{\Gamma}$, is continuous on $(0,1] \times [0,1]$. It remains to prove that $\widehat{\Gamma}$ is continuous at all points of the form $(0,t_0)$.

Fix $t_0 \in [0,1]$ and $\varepsilon > 0$. Choose $\delta > 0$ small enough (using Lemma 15) so that for any $t \in [0,1]$ and any open arc D in U^t of diameter less than δ, each lift $\widehat{D}$ of D under φ_U^t has diameter less than $\frac{\varepsilon}{3}$.

Choose $\eta_1, \eta_2 > 0$ small enough so that:

(i) $|\widehat{\Gamma}^{t_0}(0) - \widehat{\Gamma}^{t_0}(\eta_1)| < \frac{\varepsilon}{3}$ (this is possible since the lifted path $\widehat{\Gamma}^{t_0}$ is continuous);

(ii) $|\widehat{\Gamma}^{t}(\eta_1) - \widehat{\Gamma}^{t_0}(\eta_1)| < \frac{\varepsilon}{3}$ for each $t \in [t_0 - \eta_2, t_0 + \eta_2]$ (this is possible since we already know that $\widehat{\Gamma}$ is continuous on $(0,1] \times [0,1]$); and

(iii) $\Gamma([0,\eta_1] \times [t_0 - \eta_2, t_0 + \eta_2]) \subset B(\Gamma^{t_0}(0), \frac{\delta}{2})$ (this is possible since Γ is continuous).

Now for any $t \in [t_0 - \eta_2, t_0 + \eta_2]$, the image $\Gamma^t([0,\eta_1])$ has diameter less than δ; hence $\widehat{\Gamma}^t([0,\eta_1])$ has diameter less than $\frac{\varepsilon}{3}$. It follows that $\widehat{\Gamma}^t([0,\eta_1]) \subset B(\widehat{\Gamma}^{t_0}(0), \varepsilon)$. So $[0,\eta_1) \times (t_0 - \eta_2, t_0 + \eta_2)$ is a neighborhood of $(0,t_0)$ which is mapped by $\widehat{\Gamma}$ into $B(\widehat{\Gamma}^{t_0}(0), \varepsilon)$. Thus $\widehat{\Gamma}$ is continuous at $(0,t_0)$. □

Observe that in light of Lemma 16, condition (ii) of Theorem 12 is stronger than condition (ii) of Theorem 14. Therefore to complete the proofs of both Theorems 12 and 14, we must prove that if condition (ii) of Theorem 14 holds, then the isotopy h extends to the entire plane $\mathbb{C}$. Hence we will assume for the remainder of this section that condition (ii) of Theorem 14 holds.

Notation ($\widehat{a}^t$). Let $\widehat{a} \in \partial\mathbb{D}$ be any point at which φ_U is defined (i.e., at which the radial limit of φ_U exists). Using any sufficiently small crosscut Q in U which has one endpoint equal to $a = \varphi_U(\widehat{a})$ and which is the image of a crosscut of $\mathbb{D}$ having one endpoint equal to $\widehat{a}$, we obtain from condition (ii) of Theorem 14 a family of paths $\{\gamma_t : t \in [0,1]\}$ and lifts $\widehat{\gamma}_t$ with the properties listed there and such that $\gamma_t(0) = a^t$ for each $t \in [0,1]$ and $\widehat{\gamma}_0(0) = \widehat{a}$. Because the sets $\widehat{\gamma}_t([0,1])$ vary continuously in t with respect to the Hausdorff metric, the endpoint $\widehat{\gamma}_t(0)$ moves continuously in t. Now we define $\widehat{a}^t = \widehat{\gamma}_t(0)$ for each $t \in [0,1]$. Then $\widehat{a}^0 = \widehat{a}$ and $\varphi_U(\widehat{a}^t) = a^t$ for all $t \in [0,1]$. It is straightforward to see that this definition of $\widehat{a}^t$ is independent of the choice of crosscut Q and of the paths γ_t and lifts $\widehat{\gamma}_t$ afforded by condition (ii) of Theorem 14. Thus, in the presence of condition (ii) of Theorem 14, we can extend the superscript t notation to points in $\partial\mathbb{D}$ at which φ_U is defined. We will assume this is done for all such points $\widehat{a} \in \partial\mathbb{D}$ for the remainder of this section.

3.3. **Hyperbolic laminations.**

The following condition on a set of hyperbolic geodesics $\mathcal{L}$ is inspired by a similar notion introduced by Thurston (cf. [35]).

Definition 17. A *hyperbolic lamination* $\mathcal{L}$ in a bounded domain $U \subset \mathbb{C}$ is a closed set of pairwise disjoint hyperbolic geodesic crosscuts in U such that two distinct crosscuts in $\mathcal{L}$ are disjoint and have *at most one* common endpoint in the boundary of U and the family of crosscuts in $\mathcal{L}$ of diameter greater than or equal to ε is compact for any $\varepsilon > 0$.

We denote by $\bigcup \mathcal{L}$ the union of all the crosscuts in $\mathcal{L}$. A *gap* of $\mathcal{L}$ is the closure of a component of $U \setminus \bigcup \mathcal{L}$.

The compactness condition in Definition 17 is equivalent to the following statement: if $\langle \mathfrak{g}_n \rangle_{n=1}^{\infty}$ is a sequence of elements of $\mathcal{L}$, then either $\operatorname{diam}(\mathfrak{g}_n) \to 0$ or there is a convergent subsequence whose limit is also an element of $\mathcal{L}$.

Fix a bounded complementary domain U of X. Recall the Kulkarni-Pinkall construction described in Section 2.2: we consider the collection $\mathcal{B}$ of all open disks $B(c,r) \subset U$ such that $|\partial B(c,r) \cap \partial U| \geq 2$. For each such disk $B(c,r)$, $\mathrm{Hull}(c)$ denotes the convex hull of the set $\partial B(c, r(c)) \cap \partial U$ in $B(c, r(c))$ *using the hyperbolic metric* ρ_c *on the disk* $B(c, r(c))$. Let $\mathcal{J}$ be the collection of all crosscuts of U which are contained in the boundaries of the sets $\mathrm{Hull}(c)$ for $B(c,r) \in \mathcal{B}$.

Let

$$\widehat{\mathcal{J}} = \{\widehat{Q} : \widehat{Q} \text{ is a component of } \varphi_U^{-1}(Q) \text{ for some } Q \in \mathcal{J}\}.$$

For any $Q \in \mathcal{J}$, it is straightforward to see that each component $\widehat{Q}$ of $\varphi_U^{-1}(Q)$ is an open arc whose closure is mapped homeomorphically onto $\overline{Q}$ by φ_U.

Given an (open) arc A, we denote the set of endpoints of A by $\mathrm{Ends}(A)$; that is, $\mathrm{Ends}(A) = \{a, b\}$ means that a and b are the endpoints of (the closure of) A. Let $\mathcal{J}_{\mathrm{Ends}} = \{\mathrm{Ends}(Q) : Q \in \mathcal{J}\}$, and let $\widehat{\mathcal{J}}_{\mathrm{Ends}} = \{\mathrm{Ends}(\widehat{Q}) : \widehat{Q} \in \widehat{\mathcal{J}}\}$. These are sets of (unordered) pairs.

For each $t \in [0,1]$, let

$$\widehat{\mathcal{L}}^t = \{\widehat{\mathfrak{g}}^t : \widehat{\mathfrak{g}}^t \text{ is the hyperbolic geodesic in } \mathbb{D} \text{ joining } \widehat{a}^t, \widehat{b}^t, \text{ where } \{\widehat{a}, \widehat{b}\} \in \widehat{\mathcal{J}}_{\mathrm{Ends}}\}$$

and let

$$\mathcal{L}^t = \{\varphi_U^t(\widehat{\mathfrak{g}}^t) : \widehat{\mathfrak{g}}^t \in \widehat{\mathcal{L}}^t\}.$$

Observe that $\mathcal{L}^0$ is the collection of all hyperbolic geodesic crosscuts of $U^0 = U$ which are homotopic (with endpoints fixed) to some crosscut in $\mathcal{J}$. For $t > 0$, the collection $\mathcal{L}^t$ is obtained from $\mathcal{L}^0$ by following the motion of the endpoints of the arcs in $\mathcal{L}^0$ under the isotopy and joining the resulting points in ∂U^t by the hyperbolic geodesic crosscut $\mathfrak{g}^t = \varphi_U^t(\widehat{\mathfrak{g}}^t)$ in U^t using the hyperbolic metric induced by φ_U^t. We do *not* consider a Kulkarni-Pinkall style partition of the domain U^t for $t > 0$.

We shall prove that $\mathcal{L}^t$ is a hyperbolic lamination in U^t for each $t \in [0,1]$. We start with the following lemma.

Lemma 18. *For any* $t \in [0,1]$ *and any* $\widehat{\mathfrak{g}}^t \in \widehat{\mathcal{L}}^t$*, the map* φ_U^t *is one-to-one on* $\widehat{\mathfrak{g}}^t$*, and, hence, the corresponding element* $\mathfrak{g}^t = \varphi_U^t(\widehat{\mathfrak{g}}^t) \in \mathcal{L}^t$ *is a crosscut in* U^t*. Moreover, if* $\mathfrak{g}_1^t, \mathfrak{g}_2^t$ *are two distinct elements of* $\mathcal{L}^t$*, then* $\mathfrak{g}_1^t \cap \mathfrak{g}_2^t = \emptyset$ *(though their closures may have at most one common endpoint in* ∂U^t*).*

Proof. Let $\mathfrak{g}^0$ be an arbitrary hyperbolic crosscut of $\mathcal{L}^0$ with endpoints a and b. By the discussion at the end of Section 3.2, we can lift $\mathfrak{g}^0$ to geodesics $\widehat{\mathfrak{g}}^t$ with continuously varying endpoints. Let $\widehat{a}^t$ ($\widehat{b}^t$) be the endpoints of $\widehat{\mathfrak{g}}^t$ corresponding to a^t (b^t, respectively). Since $\mathfrak{g}^0$ is an arc, all components $\widehat{\mathfrak{g}}^0$ of $\varphi_U^{-1}(\mathfrak{g}^0)$ are pairwise disjoint geodesic crosscuts of $\mathbb{D}$. Since the endpoints of all these crosscuts move continuously in t and the points a^t and b^t are distinct, the geodesics $\widehat{\mathfrak{g}}^t$ are also pairwise disjoint open arcs for all t. Hence, φ_U^t is one-to-one on each of these crosscuts and their common image is a geodesic arc $\mathfrak{g}^t$. By a similar argument, the lifts $\widehat{\mathfrak{g}}_1^t$ and $\widehat{\mathfrak{g}}_2^t$ of two distinct geodesics $\mathfrak{g}_1^t$ and $\mathfrak{g}_2^t$ in $\mathcal{L}^t$ are pairwise disjoint in $\mathbb{D}$, and, hence, $\mathfrak{g}_1^t \cap \mathfrak{g}_2^t = \emptyset$. It follows easily from the construction that two distinct geodesics in $\mathcal{L}^0$ share at most one common endpoint, and, hence, the same is true for $\mathcal{L}^t$. ☐

To prove $\mathcal{L}^t$ is a hyperbolic lamination in U^t for each $t \in [0,1]$, it remains to show that the collection of arcs in $\mathcal{L}^t$ of diameter at least ε is compact for every $\varepsilon > 0$. This will follow from the next lemma, which states that even for varying t, the limit of a convergent sequence of elements of the corresponding $\mathcal{L}^t$ collections must belong to the limit $\mathcal{L}^t$ collection as well.

Lemma 19. *Let $\{a_1, b_1\}, \{a_2, b_2\}, \ldots$ be a sequence of pairs in $\mathcal{J}_{\text{Ends}}$ such that $a_n \to a_\infty$ and $b_n \to b_\infty$, where a_∞ and b_∞ are distinct points in ∂U. Then $\{a_\infty, b_\infty\} \in \mathcal{J}_{\text{Ends}}$.*

Furthermore, let $t_1, t_2, \ldots \in [0,1]$ be a sequence such that $t_n \to t_\infty \in [0,1]$. For each $n \in \{1,2,\ldots\} \cup \{\infty\}$ and each $t \in [0,1]$, let $\mathfrak{g}_n^t \in \mathcal{L}^t$ be the geodesic with endpoints a_n^t and b_n^t. Then $\mathfrak{g}_n^{t_n} \to \mathfrak{g}_\infty^{t_\infty}$ in the sense that there exist homeomorphisms $\theta_n : \mathfrak{g}_\infty^{t_\infty} \to \mathfrak{g}_n^{t_n}$ such that $\theta_n \to \mathrm{id}$.

Proof. Let $\widehat{\mathcal{A}} \subset \partial\mathbb{D}$ be the set of all points in $\partial\mathbb{D}$ at which φ_U^0 is defined, and let $\mathcal{A} = \{\varphi_U^0(x) : x \in \widehat{\mathcal{A}}\}$. This set $\mathcal{A}$ is the set of all accessible points in ∂U by Theorem 4. The set $\mathcal{A}$ is dense in ∂U, and the set $\widehat{\mathcal{A}}$ of lifts of points in $\mathcal{A}$ under φ_U^0 is dense in $\partial\mathbb{D}$ by Theorem 2.

Claim 19.1. For each $t \in [0,1]$, the function $\alpha^t : \partial\mathbb{D} \to \partial\mathbb{D}$ which extends the function that maps each $\widehat{y} \in \widehat{\mathcal{A}}$ to $\widehat{y}^t$ and is defined by $\alpha^t(x) = \lim_{\{\widehat{y}\to x \mid \widehat{y}\in\widehat{\mathcal{A}}\}} \widehat{y}^t$ for each $x \in \partial\mathbb{D}$ is a homeomorphism. Moreover $\alpha : \partial\mathbb{D} \times [0,1] \to \partial\mathbb{D}$, defined by $\alpha(x,t) = \alpha^t(x)$, is an isotopy starting at the identity.

Sketch of the proof of Claim 19.1. Since the restriction $\alpha^t|_{\widehat{\mathcal{A}}}$ is one-to-one and preserves circular order, it suffices to show that $\alpha^t(\widehat{\mathcal{A}})$ is dense for each t. The proof will make use of the following notion: Let $\mathbb{S}$ be the unit circle, let $\gamma : \mathbb{S} \to \mathbb{C}$ be a continuous function, and let O be a point in the unbounded complementary domain of $\gamma(\mathbb{S})$. A complementary domain U of $\gamma(\mathbb{S})$ is odd if every arc J from O to a point in the domain intersects $\gamma(\mathbb{S})$ an odd number of times, counting with multiplicity and assuming that every intersection is transverse and the total number of crossings is finite; cf. [26, Lemma 2.1].

Fix $\varepsilon > 0$. By Theorem 3, $\alpha^0(\widehat{\mathcal{A}})$ is dense. By condition (ii) of Theorem 14 and Lemma 15 there exists $\delta > 0$ so that for any crosscut C of X^0 all lifts of the paths γ_t^C (whose existence follows from condition (ii)) have diameters less than ε. Since X is uniformly perfect one can choose finitely many simple closed curves S_i which bound disjoint closed disks D_i so that $X^0 \subset \bigcup_i D_i$, $X^0 \cap \bigcup \partial D_i$ is finite, and for all i and every component C of $S_i \setminus X^0$, the diameter of C is less than δ. Moreover we can assume that for all t, $\varphi_U^t(0)$ is contained in the unbounded component of $\bigcup \gamma_t^C([0,1])$. Then all lifts $\widehat{\gamma}_t^C$ have diameter less than ε. Let $F^0 = \bigcup_i X^0 \cap S_i$ and $F^t = h^t(F^0)$. Since for all C and all t, $\gamma_t^C((0,1)) \cap X^t = \emptyset$ and $\widehat{\gamma}_t$ is continuous in the Hausdorff metric, it follows that every point of $X^t \setminus F^t$ is contained in an odd bounded complementary component of $\bigcup \gamma_t^C([0,1])$.

Every component C of $S_i \setminus X$ is a crosscut which defines a collection of paths γ_t^C by condition (ii) of Theorem 14. For all t let $\widehat{\mathcal{C}_t}$ be the collection of all lifts of all the paths γ_t^C.

Fix t. Suppose that r is a radius of the unit disk $\mathbb{D}$ so that $R = \varphi_U^t(r)$ lands on a point in $X^t \setminus F^t$. Then a terminal segment B of R must be in an odd complementary domain of $\bigcup \gamma_t^C([0,1])$. Let $A = R \setminus B$ be the initial segment of R. Then the subsegment b of r that corresponds to B is disjoint from all crosscuts in $\widehat{\mathcal{C}_t}$. Suppose that b is not contained in the shadow of one of these crosscuts. Then we may assume that the intersection of a and any member of $\widehat{\mathcal{C}_t}$ is finite and even. Since we may also assume that the intersection of a with all lifted crosscuts is finite, the intersection of a with the union of all members of $\widehat{\mathcal{C}_t}$ is a finite even number. Since φ_U^t is a local homeomorphism, the number of intersections of A with all crosscuts γ_t^C is also even, a contradiction since A terminates in an odd domain. □(Claim 19.1)

Note that by construction, $\mathcal{J}$ is almost a lamination, except that multiple arcs in $\mathcal{J}$ can share the same two endpoints. In particular, if $C(a_n b_n)$ are circular arcs in $\mathcal{J}$ joining the points a_n and b_n, then, after taking a subsequence if necessary, $\lim C(a_n b_n)$ is a circular arc in $\mathcal{J}$ joining a_∞ to b_∞. From this it follows easily that $\mathcal{L}^0$ is a lamination and if $\mathfrak{g}_n \in \mathcal{L}^0$ is the geodesic joining a_n to b_n, then $\lim \mathfrak{g}_n = \mathfrak{g}_\infty$, where $\mathfrak{g}_\infty \in \mathcal{L}^0$ is the geodesic joining a_∞ to b_∞. Choose lifts $\widehat{\mathfrak{g}}_n^t$ and $\widehat{\mathfrak{g}}_\infty^t$ under φ_U^t for each $t \in [0,1]$ as in the proof of Lemma 18, such that $\lim \widehat{\mathfrak{g}}_n^0 = \widehat{\mathfrak{g}}_\infty^0$.

Fix k. By Claim 19.1, $\lim \widehat{\mathfrak{g}}_n^{t_k} = \widehat{\mathfrak{g}}_\infty^{t_k}$. This implies immediately that $\liminf \mathfrak{g}_n^{t_k} \supset \mathfrak{g}_\infty^{t_k}$. Since the points a_n and a_∞ can be joined by a small crosscut in U, it follows from assumption (ii) of Theorem 14 that the points $a_n^{t_k}$ and $a_\infty^{t_k}$ can be joined by a small path. Hence, points $x_n^{t_k}$ in $\mathfrak{g}_n^{t_k}$ close to an endpoint (say $a_n^{t_k}$) can be joined to the endpoint $a_n^{t_k}$ by a small path (first by a small arc to a point in $\mathfrak{g}_\infty^{t_k}$ and then by a small arc in U^{t_k} to the endpoint $a_\infty^{t_k}$, followed by a small path in U^{t_k} to $a_n^{t_k}$). By Theorem 6, the subgeodesic of $\mathfrak{g}_n^{t_k}$ from $x_n^{t_k}$ to $a_n^{t_k}$ is small, and we can conclude that $\lim \mathfrak{g}_n^{t_k} = \mathfrak{g}_\infty^{t_k}$ for each k. Since the maps φ_U^t are uniformly convergent on compact subsets, $\liminf \mathfrak{g}_\infty^{t_k} \supset \mathfrak{g}_\infty^{t_\infty}$. Since by the above argument the subgeodesic from a point close to the endpoint of $\mathfrak{g}_\infty^{t_k}$ to this endpoint is small, $\lim \mathfrak{g}_\infty^{t_k} = \mathfrak{g}_\infty^{t_\infty}$. It is now easy to see that there exist homeomorphisms $\theta_n : \mathfrak{g}_\infty^{t_\infty} \to \mathfrak{g}_n^{t_n}$ such that $\theta_n \to \mathrm{id}$. □

For each $t \in [0,1]$, we conclude from Lemmas 18 and 19 (using $t_n = t$ for all n) that $\mathcal{L}^t$ is a lamination in U^t.

3.4. **Proof of Theorem 14.** In this section we will complete the proof of Theorem 14 (and hence of Theorem 12 as well).

We will employ here the path midpoint function m described in Theorem 11 of Section 2.4.

Let U be any bounded complementary domain of X, and consider the hyperbolic laminations $\mathcal{L}^t$ in U^t as constructed above in Section 3.3.

Given any element $\mathfrak{g} \in \mathcal{L}^0$, we extend the isotopy h over $\mathfrak{g}$ to $h_\mathfrak{g} : (X \cup \mathfrak{g}) \times [0,1] \to \mathbb{C}$ by defining $h_\mathfrak{g}^t(\mathsf{m}(\mathfrak{g})) = \mathsf{m}(\mathfrak{g}^t)$, and if $x \in \mathfrak{g}$ is located on the subarc with endpoints $\mathsf{m}(\mathfrak{g})$ and a (respectively, b), then $h_\mathfrak{g}^t(x)$ is the unique point on the subarc of $\mathfrak{g}^t$ with endpoints $\mathsf{m}(\mathfrak{g}^t)$ and a^t (respectively, b^t) such that $\rho^0(x, \mathsf{m}(\mathfrak{g})) = \rho^t(h_\mathfrak{g}^t(x), \mathsf{m}(\mathfrak{g}^t))$, using the hyperbolic metric ρ^t on U^t.

Now extend h to $h_{\mathcal{L}} : X \cup \bigcup \mathcal{L}^0 \to \mathbb{C}$ by defining

$$h_{\mathcal{L}}(x,t) = \begin{cases} h(x,t) & \text{if } x \in X, \\ h_{\mathfrak{g}}(x,t) & \text{if } x \in \mathfrak{g} \in \mathcal{L}^0. \end{cases}$$

Then for each $t \in [0,1]$, $h^t_{\mathcal{L}}$ is clearly a bijection from $X \cup \bigcup \mathcal{L}^0$ to $X^t \cup \bigcup \mathcal{L}^t$.

Claim 1. $h_{\mathcal{L}}$ is continuous.

Proof of Claim 1. Suppose that $(x_i, t_i) \to (x_\infty, t_\infty)$ and $x_i \in \mathfrak{g}_i \in \mathcal{L}^0$. If there exists $\varepsilon > 0$ so that $\operatorname{diam}(\mathfrak{g}_i) > \varepsilon$ for all i, then we may assume, by taking a subsequence if necessary, that $\lim \mathfrak{g}_i = \mathfrak{g}_\infty \in \mathcal{L}^0$. If x_∞ is not an endpoint of $\mathfrak{g}_\infty$, then, by uniform convergence of φ^t_U on compact sets, $\lim h_{\mathcal{L}}(x_i, t_i) = h_{\mathcal{L}}(x_\infty, t_\infty)$. If x_∞ is an endpoint of $\mathfrak{g}_\infty$ (so $x_\infty \in X$), then $\rho^0(x_i, \mathsf{m}(\mathfrak{g}_i)) \to \infty$ and again $\lim h_{\mathcal{L}}(x_i, t_i) = h_{\mathcal{L}}(x_\infty, t_\infty) = h(x_\infty, t_\infty)$. Hence we may assume that $\lim \operatorname{diam}(\mathfrak{g}_i) = 0$. Then $x_\infty \in X$ and $\lim \operatorname{diam}(h^{t_i}_{\mathcal{L}}(\mathfrak{g}_i)) = 0$. Hence, if a_i is an endpoint of $\mathfrak{g}_i$, then $\lim h_{\mathcal{L}}(x_i, t_i) = \lim h(a_i, t_i) = h(x_\infty, t_\infty)$ as desired. □(Claim 1)

Finally, we repeat the above procedure on each bounded complementary domain U of X to extend h over the hyperbolic lamination obtained from the Kulkarni-Pinkall construction as in Section 3.3 on each such U. The result is a function $H : Y \times [0,1] \to \mathbb{C}$ which is defined on the union Y of X with all the hyperbolic laminations of all bounded complementary domains of X. Note that for any $\varepsilon > 0$, there are only finitely many bounded complementary domains of X which contain a disk of diameter at least ε, and hence there are only finitely many such domains whose corresponding hyperbolic lamination contains an arc of diameter at least ε. This implies, as above, that H is continuous.

Note that each bounded complementary domain of Y is a gap of the hyperbolic lamination of one of the bounded complementary domains of X. Since all such gaps are simply connected, Y is a continuum. Hence by [27] the isotopy H of Y can be extended over the entire plane.

This completes the proof of Theorem 14. By the comments at the end of Section 3.2, this also completes the proof of Theorem 12.

In Theorem 12 we assumed that X^t is uniformly perfect for each $t \in [0,1]$. This assumption allows for the use of the powerful analytic results described in Section 2.1. It is natural to wonder if this assumption is really needed. We conjecture that this is not the case.

Conjecture 1. *Suppose that X is a plane compactum and $h : X \times [0,1] \to \mathbb{C}$ is an isotopy starting at the identity. Then the following are equivalent:*

(i) *h extends to an isotopy of the entire plane;*
(ii) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every complementary domain U of X and each crosscut Q of U with $\operatorname{diam}(Q) < \delta$, h can be extended to an isotopy $h_Q : (X \cup Q) \times [0,1] \to \mathbb{C}$ such that for all $t \in [0,1]$, $\operatorname{diam}(h^t(Q)) < \varepsilon$.*

4. Compact sets with large components

The remaining part of this paper is devoted to a proof of the following theorem.

Theorem 20. *Suppose $X \subset \mathbb{C}$ is a compact set for which there exists $\eta > 0$ such that every component of X has diameter bigger than η. Let $h : X \times [0,1] \to \mathbb{C}$ be an isotopy which starts at the identity. Then h extends to an isotopy of the entire plane which starts at the identity.*

Suppose $X \subset \mathbb{C}$ is a compact set for which there exists $\eta > 0$ such that every component of X has diameter bigger than η. Let $h : X \times [0,1] \to \mathbb{C}$ be an isotopy which starts at the identity.

Clearly in this case X^t is uniformly perfect with the same constant k for each $t \in [0,1]$ and we may assume that X is encircled. By scaling, we may also assume that for any $a \in X$ and any component C of X, there exists $c \in C$ such that $|a^t - c^t| \geq 1$ for all $t \in [0,1]$. *We will make these assumptions for the remainder of the paper.*

We will prove Theorem 20 using the characterization from Theorem 14. To this end, we fix (again for the remainder of the paper) an arbitrary bounded complementary domain U of X.

To satisfy condition (ii) of Theorem 14 we must construct, for a sufficiently small crosscut Q of U with endpoints a and b, a family of paths γ_t in U^t with endpoints a^t and b^t, which remain small during the isotopy, such that γ_0 is homotopic to Q in U with endpoints fixed, and which can be lifted under φ_U^t to paths $\widehat{\gamma}_t$ in $\mathbb{D}$ which are continuous in the Hausdorff metric. We will show first that, in the case that X has large components, it suffices to construct the family of paths γ_t to be continuous in the Hausdorff metric.

Lemma 21. *Let $a, b \in \partial U$. Suppose that $\{\gamma_t : t \in [0,1]\}$ is a family such that γ_t is a path in U^t joining a^t and b^t with $\operatorname{diam}(\gamma_t([0,1])) < \frac{1}{2}$ for each $t \in [0,1]$ and the sets $\gamma_t([0,1])$ vary continuously in t with respect to the Hausdorff metric. Then there are lifts $\widehat{\gamma}_t$ of the paths γ_t under φ_U^t such that the sets $\widehat{\gamma}_t([0,1])$ also vary continuously in t with respect to the Hausdorff metric.*

Proof. Suppose that the family γ_t is as specified in the statement. Recall that d_H denotes the Hausdorff distance. Fix $t_0 \in [0,1]$. It suffices to show that, given a lift $\widehat{\gamma}_{t_0}$ of γ_{t_0} and $0 < \varepsilon < \frac{1}{2}$, there exists $\delta > 0$ and lifts $\widehat{\gamma}_t$ of γ_t for $|t - t_0| < \delta$ such that $d_H(\widehat{\gamma}_t([0,1]), \widehat{\gamma}_{t_0}([0,1])) \leq \varepsilon$.

By Lemma 15 we can choose small disjoint open balls B_a centered at a^{t_0} and B_b centered at b^{t_0} of diameters less than $\frac{1}{4}$ such that for all t and any path λ contained in $\overline{B_a} \cap U^t$ or $\overline{B_b} \cap U^t$, the diameter of each lift of λ under φ_U^t is less than $\frac{\varepsilon}{2}$.

Let $s_a, s_b \in (0,1)$ be the numbers such that $\gamma_{t_0}(s_a) \in \partial B_a$, $\gamma_{t_0}([0, s_a)) \subset B_a$, $\gamma_{t_0}(s_b) \in \partial B_b$, and $\gamma_{t_0}((s_b, 1]) \subset B_b$. Denote $z_a = \gamma_{t_0}(s_a)$ and $z_b = \gamma_{t_0}(s_b)$. Choose an open set $O \subset \mathbb{C}$ such that $\gamma_{t_0}([s_a, s_b]) \subset O$, $\overline{O} \subset U^{t_0}$, and the diameter of $O \cup B_a \cup B_b$ is less than 1. For t sufficiently close to t_0, we have $\overline{O} \subset U^t$ and $\gamma_t([0,1]) \subset O \cup B_a \cup B_b$. Since each component of X^t has diameter greater than 1, we have that no bounded complementary component of $O \cup (B_a \cup B_b \setminus X^t)$ contains any points of X^t. It follows that there exists a simply connected open set P_t in U^t such that $\gamma_t((0,1)) \cup O \subset P_t$. This means that the covering map φ_U^t maps each component of $(\varphi_U^t)^{-1}(P_t)$ homeomorphically onto P_t.

Since the maps φ_U^t converge uniformly on compact sets as $t \to t_0$, for t sufficiently close to t_0 there exists exactly one component $\widehat{P}_t$ of $(\varphi_U^t)^{-1}(P_t)$ such that $\widehat{\gamma}_{t_0}([s_a, s_b]) \subset \widehat{P}_t$. For such t, define the lift $\widehat{\gamma}_t$ of γ_t by $\widehat{\gamma}_t = (\varphi_U^t|_{\widehat{P}_t})^{-1} \circ \gamma_t$.

To see that these lifts are Hausdorff close to $\widehat{\gamma}_{t_0}$, let $\delta > 0$ be small enough so that for all t with $|t - t_0| < \delta$ we have:

(i) There exists $\nu > 0$ such that $|(\varphi_U^t|_{\widehat{P}_t})^{-1}(x_1) - (\varphi_U^{t_0}|_{\widehat{P}_{t_0}})^{-1}(x_2)| < \frac{\varepsilon}{2}$ for all $x_1, x_2 \in \mathbb{C}$ with $|x_1 - x_2| < \nu$ and either $x_1 \in O$ or $x_2 \in O$;

(ii) $d_H(\gamma_t([0,1]), \gamma_{t_0}([0,1])) < \nu$; and

(iii) $\gamma_t([0,1]) \cap (\partial B_a \setminus O) = \emptyset$, and $\gamma_t([0,1]) \cap (\partial B_b \setminus O) = \emptyset$.

Fix t with $|t - t_0| < \delta$, and let $s \in (0,1)$. We claim that $\widehat{\gamma}_t(s)$ is within distance ε from some point in $\widehat{\gamma}_{t_0}([0,1])$.

To see this, assume first that $\gamma_t(s) \in P_t$. Let $x_1 = \gamma_t(s)$. By condition (ii) above, there exists $x_2 \in \gamma_{t_0}([0,1])$ with $|x_1 - x_2| < \nu$. By condition (i) it follows that $\widehat{\gamma}_t(s) = (\varphi_U^t|_{\widehat{P}_t})^{-1}(x_1)$ is within distance $\frac{\varepsilon}{2}$ from $(\varphi_U^{t_0}|_{\widehat{P}_{t_0}})^{-1}(x_2) \in \widehat{\gamma}_{t_0}([0,1])$.

Now suppose that $\gamma_t(s) \notin P_t$. We assume without loss of generality that $\gamma_t(s) \in B_a$. Follow the path γ_t from $\gamma_t(s)$ to a point $x_1 \in \partial B_a$ so that the section of the path γ_t in between is contained in B_a. Since this part of the path γ_t is contained in B_a, the corresponding section of the lifted path $\widehat{\gamma}_t$ from $\widehat{\gamma}_t(s)$ to $(\varphi_U^t|_{\widehat{P}_t})^{-1}(x_1)$ has diameter less than $\frac{\varepsilon}{2}$, by choice of B_a. By condition (iii) above, we have $x_1 \in O$, and by condition (ii) there exists $x_2 \in \gamma_{t_0}([0,1])$ with $|x_1 - x_2| < \nu$. By condition (i) it follows that $(\varphi_U^t|_{\widehat{P}_t})^{-1}(x_1)$ is within distance $\frac{\varepsilon}{2}$ from $(\varphi_U^{t_0}|_{\widehat{P}_{t_0}})^{-1}(x_2) \in \widehat{\gamma}_{t_0}([0,1])$. Then by the triangle inequality, we have $|\widehat{\gamma}_t(s) - (\varphi_U^{t_0}|_{\widehat{P}_{t_0}})^{-1}(x_2)| < \varepsilon$.

Thus in any case, we see that every point of $\widehat{\gamma}_t([0,1])$ is within ε of a point in $\widehat{\gamma}_{t_0}([0,1])$. By symmetry, the same argument also shows that each point of $\widehat{\gamma}_{t_0}([0,1])$ is within ε of a point in $\widehat{\gamma}_t([0,1])$. Hence $d_H(\widehat{\gamma}_t([0,1]), \widehat{\gamma}_{t_0}([0,1])) \leq \varepsilon$, as desired. □

Notation (ε, ν). For the remainder of the paper, we fix an arbitrary $\varepsilon > 0$. For later use, fix $0 < \nu < \frac{1}{3}$ small enough so that $\frac{8\nu}{1-\nu} < \frac{\varepsilon}{2}$.

To prove Theorem 20, it remains to show that there exists $\delta > 0$ such that if Q is a crosscut of U with endpoints a and b with diameter less than δ, there is a family of paths γ_t such that (1) γ_t is a path in U^t joining a^t and b^t for each $t \in [0,1]$, (2) γ_0 is homotopic to Q in U with endpoints fixed, (3) $\mathrm{diam}(\gamma_t([0,1])) < \varepsilon$ for all $t \in [0,1]$, and (4) the sets $\gamma_t([0,1])$ vary continuously in t with respect to the Hausdorff metric.

In Section 4.1, we will transform the compactum X so that the crosscut Q becomes the straight line segment $[0,1]$ in the plane, to simplify the ensuing constructions and arguments. We will refer to the transformed plane as the "normalized plane", and the image of X will be denoted by $\widetilde{X}$. In Section 4.2, we will lift the isotopy under an exponential covering map. The domain of the covering map will be called the "exponential plane", and the preimage of $\widetilde{X}$ will be denoted by $\mathbf{X}$. In Sections 4.3 and 4.4 we will replace the lift of the crosscut $[0,1]$ of $\widetilde{X}$ by an equidistant set which varies continuously in t. The projection of this equidistant set to the original plane containing X^t will be shown in Section 4.5 to be the desired path γ_t.

4.1. **The normalized plane.** In the following sections, we will make use of a covering map (which we will refer to as the "exponential map") of the plane minus the endpoints of a crosscut Q. In order to simplify the notation and work with a single exponential map below we will normalize the compactum X and the crosscut Q of X with endpoints a and b so that for all t, $a^t = 0$, $b^t = 1$, and Q becomes the straight line segment $(0,1) \subset \mathbb{R}$.

By composing with translations it is easy to see that given a crosscut Q of X with endpoints a and b we can always assume that the point a is the origin 0 and that this point remains fixed throughout the isotopy (i.e., $a^t = 0$ for all t).

Let Q be a crosscut of U with endpoints 0 and b such that $\mathrm{diam}(Q) < \frac{1}{4}$. We will impose further restrictions on the diameter of Q later.

Since all arcs in the plane are tame, there exists a homeomorphism $\Theta : \mathbb{C} \to \mathbb{C}$ such that $\Theta(Q)$ is the straight line segment joining the points 0 and b, $\Theta(0) = 0$, $\Theta(b) = b$, and $\Theta|_{\mathbb{C}\setminus B(0,2\mathrm{diam}(Q))} = \mathrm{id}_{\mathbb{C}\setminus B(0,2\mathrm{diam}(Q))}$. Let $L^t : \mathbb{C} \to \mathbb{C}$ be the linear map of the complex plane defined by $L^t(z) = \frac{1}{\Theta(b^t)}\, z$.

Notation ($\widetilde{X}$, $\widetilde{x}^t$). Define $\widetilde{X} = L^0 \circ \Theta(X)$ and define the isotopy $\widetilde{h} : \widetilde{X} \times [0,1] \to \mathbb{C}$ by
$$\widetilde{h}(\widetilde{x}, t) = L^t \circ \Theta \circ h((L^0 \circ \Theta)^{-1}(\widetilde{x}), t) = L^t \circ \Theta(x^t).$$

Here and below we adopt the notation that $\widetilde{x} = L^0 \circ \Theta(x)$ for all $x \in X$ and, hence, $\widetilde{h}^t(\widetilde{x}) = \widetilde{x}^t = L^t \circ \Theta(x^t)$. As indicated above, we will use ordinary letters to denote objects in the plane containing X and attach a tilde to the corresponding objects in the normalized plane (the plane containing $\widetilde{X}$).

In the next lemma we establish some simple properties of the induced isotopy $\widetilde{h}$.

Lemma 22. *There exists $\delta > 0$ such that if the crosscut Q of X with endpoints 0 and b has diameter $\mathrm{diam}(Q) < \delta$, then the induced isotopy $\widetilde{h} : \widetilde{X} \times [0,1] \to \mathbb{C}$ has the following properties:*

(i) $\widetilde{h}^0 = \mathrm{id}_{\widetilde{X}}$, *$\widetilde{X}$ contains the points 0 and 1, the isotopy $\widetilde{h}$ fixes these points, and the segment $(0,1) \subset \mathbb{R}$ in the complex plane is disjoint from $\widetilde{X}$;*

(ii) *if $\widetilde{x}^s \in (0,1)$ for some $s \in [0,1]$, then for each $t \in [0,1]$, $|\widetilde{x}^t| < \frac{\nu}{|\Theta(b^t)|}$; and*

(iii) *for every component $\widetilde{C}$ of $\widetilde{X}$ there exists a point $\widetilde{c} \in \widetilde{C}$ such that for all $t \in [0,1]$, $|\widetilde{c}^t| \geq \frac{1}{|\Theta(b^t)|}$.*

Proof. It follows immediately that $\widetilde{h}^0 = \mathrm{id}|_{\widetilde{X}}$, the isotopy $\widetilde{h}$ fixes the points 0 and 1, and the interval $(0,1)$ is disjoint from $\widetilde{X}$. Hence (i) holds.

Since h is uniformly continuous we can choose $0 < \delta < \frac{\nu}{4}$ so that if $x \in X$ and $|x^s| < 2\delta$ for some $s \in [0,1]$, then $|x^t| < \frac{\nu}{2}$ for all t. Suppose $\widetilde{x}^s \in (0,1)$ for some $s \in [0,1]$. Then $x^s \in Q$, and hence $|x^t| < \frac{\nu}{2}$ for all t. Then $|\widetilde{x}^{\,t}| < \frac{\nu}{2|\Theta(b^t)|} + \frac{2\delta}{|\Theta(b^t)|} \leq \frac{\nu}{|\Theta(b^t)|}$ using that $\Theta|_{B\setminus B(0,2\delta)} = \mathrm{id}$, and so (ii) holds.

By the standing assumption on X stated after Theorem 20, for every component C of X there exists a point $c \in C$ such that for all t, $|c^t| > 1$. Note that $\Theta(c^t) = c^t$ for all t. Hence, $|\widetilde{c}^{\,t}| \geq \frac{|c^t|}{\Theta(b^t)|} \geq \frac{1}{|\Theta(b^t)|}$ for all t, and (iii) holds. ☐

4.2. **The exponential plane.** Define the covering map
$$\widetilde{\exp} :\ \mathbb{C} \setminus \{(2n+1)\pi i : n \in \mathbb{Z}\} \ \to\ \mathbb{C} \setminus \{0,1\}$$
by
$$\widetilde{\exp}(z) = \frac{e^z}{e^z + 1}.$$

The function $\widetilde{\exp}$ is periodic with period $2\pi i$ and satisfies

$$\lim_{\mathbb{R}(z)\to\infty} \widetilde{\exp}(z) = 1, \quad \lim_{\mathbb{R}(z)\to-\infty} \widetilde{\exp}(z) = 0, \quad \widetilde{\exp}(\mathbb{R}) = (0,1)$$

and has poles at each point $(2n+1)\pi i$, $n \in \mathbb{Z}$.

Note that $\widetilde{\exp}$ is the composition of the maps e^z and the Möbius transformation $f(w) = \frac{w}{w+1}$. Hence the vertical line through a point $x \in \mathbb{R}$ is first mapped (by the covering map e^z) to the circle with center 0 and radius e^x and, if $x \neq 0$, then mapped by f to the circle with center $\frac{e^{2x}}{e^{2x}-1}$ and radius $\left|\frac{e^x}{e^{2x}-1}\right|$. The imaginary axis is mapped to the vertical line through the point $x = \frac{1}{2}$ with the points at the poles $(2n+1)\pi i$ mapped to infinity.

Notation ($\mathbf{X}$, $\mathbf{x}^t$, $\mathbf{E}_n(r)$). Denote by boldface $\mathbf{X}$ the preimage of $\widetilde{X}$ under the covering map $\widetilde{\exp}$, and in general we will use boldface letters to represent points and subsets of the exponential plane (the plane containing $\mathbf{X}$).

The isotopy $\widetilde{h}$ of $\widetilde{X}$ lifts to an isotopy $\mathbf{h}$ of $\mathbf{X}$; that is, $\mathbf{h} : \mathbf{X} \times [0,1] \to \mathbb{C}$ is the map satisfying $\mathbf{h}^0 = \mathrm{id}_{\mathbf{X}}$ and $\widetilde{\exp}(\mathbf{h}(\mathbf{x},t)) = \widetilde{h}(\widetilde{\exp}(\mathbf{x}),t)$ for every $\mathbf{x} \in \mathbf{X}$ and all $t \in [0,1]$. As above, given a point $\mathbf{x} \in \mathbf{X}$ (a subset $\mathbf{A} \subseteq \mathbf{X}$) and $t \in [0,1]$, denote $\mathbf{x}^t = \mathbf{h}(\mathbf{x},t)$ (respectively, $\mathbf{A}^t = \mathbf{h}(\mathbf{A},t)$).

For each $n \in \mathbb{Z}$ and each $r > 0$, let $\mathbf{E}_n(r) = B((2n+1)\pi i, r)$ be the ball of radius r centered at the point $(2n+1)\pi i$.

Lemma 23. *There exists* $0 < K < \pi$ *such that for any* $0 < r \leq K$*,*

(i) $\widetilde{\exp}\left(\bigcup_{n\in\mathbb{Z}} \mathbf{E}_n(r)\right) \subset \mathbb{C} \setminus B\left(0, \frac{1}{2r}\right)$;

(ii) $\widetilde{\exp}\left(\mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} \mathbf{E}_n(r)\right) \subset B\left(0, \frac{2}{r}\right)$.

Proof. For any $n \in \mathbb{Z}$ and sufficiently small $|z|$, we have

$$e^{(2n+1)\pi i + z} = -e^z \approx -1 - z,$$

and hence $\widetilde{\exp}((2n+1)\pi i + z) \approx \frac{1+z}{z}$. In particular, there exists $0 < K < \pi$ such that for all $|z| \leq K$,

$$\frac{1}{2|z|} \leq |\widetilde{\exp}((2n+1)\pi i + z)| \leq \frac{2}{|z|}.$$

Let $S_n = \partial B((2n+1)\pi i, r)$. Then, by the above inequality, $T = \widetilde{\exp}(\bigcup_n S_n)$ is an essential simple closed curve in the annulus centered around the origin 0 with inner radius $\frac{1}{2r}$ and outer radius $\frac{2}{r}$. Since $\widetilde{\exp}$ is periodic, all S_n have the same image T, and $\widetilde{\exp}^{-1}(T) = \bigcup_n S_n$. It follows that

$$\widetilde{\exp}(\bigcup_n B((2n+1)\pi i, r) \setminus (2n+1)\pi i)$$

is contained in the unbounded complementary domain of T, and

$$\widetilde{\exp}(\mathbb{C} \setminus \bigcup_n B((2n+1)\pi i, r))$$

is contained in the bounded complementary domain of T. Hence, $\widetilde{\exp}(\bigcup_n B((2n+1)\pi i, r)) \subset \mathbb{C} \setminus B(0, \frac{1}{2r})$ and $\widetilde{\exp}(\mathbb{C} \setminus \bigcup_n B((2n+1)\pi i, r)) \subset B(0, \frac{2}{r})$. □

4.3. **Components of $\mathbf{X}^t$.** We say a component $\mathbf{C}$ of $\mathbf{X}^t$ ($t \in [0,1]$) is *unbounded to the right* (respectively, *left*) if $\text{proj}_{\mathbb{R}}(\mathbf{C}) \subseteq \mathbb{R}$ is not bounded from above (respectively, from below).

For convenience we denote the horizontal strip $\{x+iy \in \mathbb{C} : x \in \mathbb{R},\ 2n\pi < y < 2(n+1)\pi\}$ simply by $\mathbf{HS}_n$. Observe that since $\widetilde{X} \cap (0,1) = \emptyset$ and $\widetilde{\exp}^{-1}((0,1)) = \bigcup_{n\in\mathbb{Z}}\{x+iy \in \mathbb{C} : x \in \mathbb{R},\ y = 2n\pi\}$, we have that $\mathbf{X} \subset \bigcup_{n\in\mathbb{Z}} \mathbf{HS}_n$.

Lemma 24. *There exists $\delta > 0$ such that if the crosscut Q of X with endpoints 0 and b has diameter $\text{diam}(Q) < \delta$, then the following holds for the induced isotopy $\mathbf{h}$ of $\mathbf{X}$:*

Given a component $\mathbf{C}$ of $\mathbf{X}$, let $n \in \mathbb{Z}$ be such that $\mathbf{C}$ is contained in the horizontal strip $\mathbf{HS}_n$. Let $\widetilde{D}$ be the component of $\widetilde{X}$ that contains $\widetilde{\exp}(\mathbf{C})$. Then:

(i) *if $\widetilde{D} \cap \{0,1\} = \emptyset$, then $\mathbf{C}^t \cap \mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right) \neq \emptyset$ for all $t \in [0,1]$;*

(ii) *$\mathbf{C}^t \cap \mathbf{E}_m\left(\frac{|\Theta(b^t)|}{2\nu}\right) = \emptyset$ for all $m \neq n$ and all $t \in [0,1]$; and*

(iii) *if $\widetilde{D} \cap \{0,1\} \neq \emptyset$, then $\mathbf{C}$ is unbounded to the left, to the right, or both.*

Furthermore, there exist for each $k \in \mathbb{Z}$ components $\mathbf{L}_k$ and $\mathbf{R}_k$ of $\mathbf{X}\cap\mathbf{HS}_k$ such that for all $t \in [0,1]$, $\mathbf{L}_k^t$ is unbounded to the left and $\mathbf{R}_k^t$ is unbounded to the right. Moreover, these may be chosen so that either $\mathbf{L}_k^t \cap \mathbf{E}_k\left(\frac{|\Theta(b^t)|}{2}\right) \neq \emptyset$ for all $k \in \mathbb{Z}$ or $\mathbf{R}_k^t \cap \mathbf{E}_k\left(\frac{|\Theta(b^t)|}{2}\right) \neq \emptyset$ for all $k \in \mathbb{Z}$.

See Figure 2 for an illustration.

Proof. Adopt the notation introduced in the lemma and assume $\mathbf{C}$ is contained in the horizontal strip $\mathbf{HS}_n$. Let $0 < K < \pi$ be as in Lemma 23. Choose $\delta > 0$ so small that $\frac{|\Theta(b^t)|}{\nu} < K$ for all t.

Suppose that $\widetilde{D} \cap \{0,1\} = \emptyset$. Then $\widetilde{\exp}(\mathbf{C}) = \widetilde{D}$. By Lemma 22(iii), we can choose $\widetilde{c} \in \widetilde{D}$ such that $|\widetilde{c}^t| \geq \frac{1}{|\Theta(b^t)|}$ for all t. By Lemma 23(ii),

$$\widetilde{\exp}\left(\mathbb{C} \setminus \mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right)\right) \subset B(0, \frac{1}{|\Theta(b^t)|}).$$

Hence we can choose $\mathbf{c}^0 \in \mathbf{E}_n\left(\frac{|\Theta(b^0)|}{2}\right) \cap \mathbf{C}$ such that $\widetilde{\exp}(\mathbf{c}^0) = \widetilde{c}^0$, and then $\mathbf{c}^t \in \mathbf{C}^t \cap \mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right)$ for all t. This completes the proof of (i).

Note that for all $n \in \mathbb{Z}$, $\widetilde{\exp}(\mathbb{R} \times \{2n\pi i\}) = (0,1) \subset \mathbb{R}$, and, hence, $\mathbf{X} \cap (\mathbb{R} \times \{2n\pi i\}) = \emptyset$ for all $n \in \mathbb{Z}$. To see that $\mathbf{C}^t \cap \mathbf{E}_m\left(\frac{|\Theta(b^t)|}{2\nu}\right) = \emptyset$ for $m \neq n$ and all t, note first that this is the case at $t = 0$ since $\mathbf{C}^0 = \mathbf{C} \subset \mathbf{HS}_n$. In order for a point $\mathbf{x}^s \in \mathbf{C}^s$ to enter a ball $\mathbf{E}_m\left(\frac{|\Theta(b^s)|}{2\nu}\right)$ with $n \neq m$ for some $s > 0$, it would first have to cross one of the horizontal boundary lines of $\mathbf{HS}_n$, say $\mathbf{x}^u \in \mathbb{R} \times \{2n\pi i\}$ for some $0 < u < s$. Then $\widetilde{\exp}(\mathbf{x}^u) = \widetilde{x}^u \in (0,1) \subset \mathbb{R}$. Hence by Lemma 22(ii), $|\widetilde{x}^t| < \frac{\nu}{|\Theta(b^t)|}$ for all t. Since by Lemma 23(i), $\widetilde{\exp}\left(\mathbf{E}_m\left(\frac{|\Theta(b^t)|}{2\nu}\right)\right) \subset \mathbb{C} \setminus B\left(0, \frac{\nu}{|\Theta(b^t)|}\right)$ for all t, $\mathbf{x}^s \notin \mathbf{E}_m\left(\frac{|\Theta(b^s)|}{2\nu}\right)$, a contradiction. This completes the proof of (ii).

Suppose next that $\widetilde{D}\cap\{0,1\} \neq \emptyset$. Then $\widetilde{\exp}(\mathbf{C}) = \widetilde{C}$ is a component of $\widetilde{D}\setminus\{0,1\}$ such that $\overline{\widetilde{C}}\cap\{0,1\} \neq \emptyset$. Hence $\mathbf{C}$ is unbounded to the left or to the right (or both). This completes the proof of (iii).

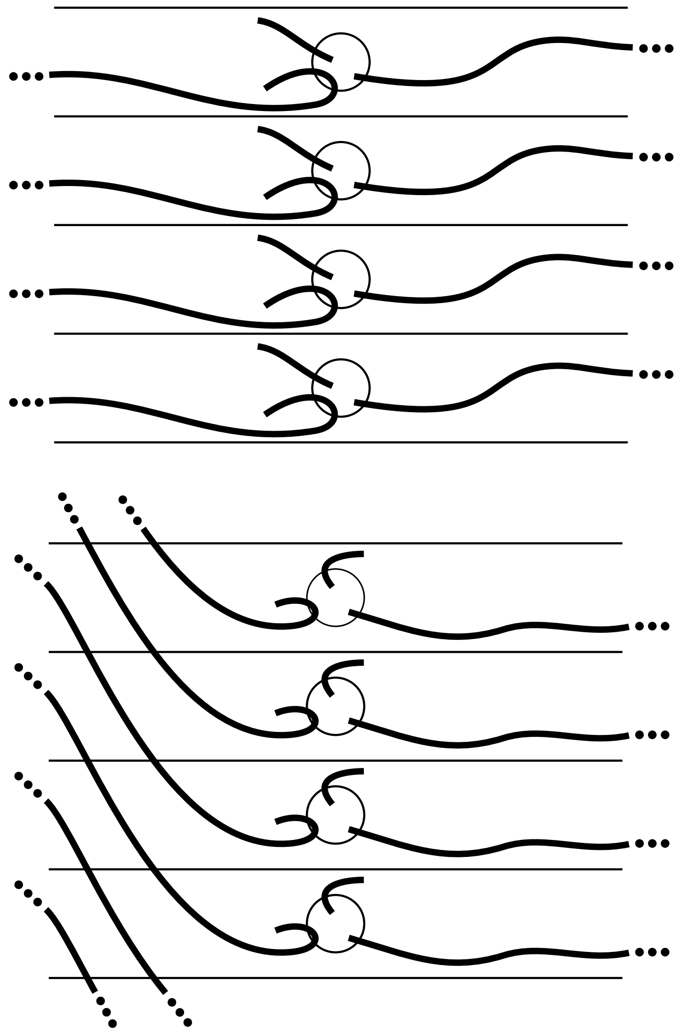

FIGURE 2. An illustration of an example of the set $\mathbf{X}^t$ at $t = 0$ (above) and at a later moment $t > 0$ (below). The horizontal lines are the preimages of $(0, 1)$ under $\widetilde{\exp}$, and the balls depicted are the sets $\mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right)$.

There must exist components $\widetilde{L}$ and $\widetilde{R}$ of $\widetilde{X} \setminus \{0, 1\}$ such that 0 is in the closure of $\widetilde{L}$ and 1 is in the closure of $\widetilde{R}$. For each $k \in \mathbb{Z}$, let $\mathbf{L}_k$ be the lift of $\widetilde{L}$ under $\widetilde{\exp}$ which is contained in the strip $\mathbf{HS}_k$, and similarly define $\mathbf{R}_k$. Then since the closure of $\widetilde{L}^t$ contains 0 and the closure of $\widetilde{R}^t$ contains 1 for all $t \in [0, 1]$, we have that for each $k \in \mathbb{Z}$, the lift $\mathbf{L}_k^t$ is unbounded to the left, and the lift $\mathbf{R}_k^t$ is unbounded to the right for all $t \in [0, 1]$.

Finally, by Lemma 22(iii), there exists a component $\widetilde{S}$ of $\widetilde{X}\setminus\{0,1\}$ whose closure contains 0 or 1, which contains a point $\widetilde{c}\in\widetilde{S}$ such that $|\widetilde{c}^{\,t}|\geq\frac{1}{|\Theta(b^t)|}$. Then, as in the proof of (ii), the component $\mathbf{S}_k^t$ of $\widetilde{\exp}^{-1}(\widetilde{S}^t)$ which contains the lift $\mathbf{c}_k^t\in\mathbf{HS}_k$ of $\widetilde{c}$ under $\widetilde{\exp}$ is unbounded to the left or to the right for all k and t and intersects $\mathbf{E}_k\left(\frac{|\Theta(b^t)|}{2}\right)$ as required. □

Notation $(\mathbf{A},\mathbf{B},\mathfrak{A},\mathfrak{B})$. Let $\mathbf{A}$ denote the set of all points of $\mathbf{X}$ above $\mathbb{R}$ and $\mathbf{B}$ the set of all points of $\mathbf{X}$ below $\mathbb{R}$. Recall that $\mathbf{X}\cap\mathbb{R}=\emptyset$, so $\mathbf{X}=\mathbf{A}\cup\mathbf{B}$. For each $t\in[0,1]$, let

$$\mathfrak{A}^t=\mathbf{A}^t\cup\bigcup_{n\geq 0}\overline{\mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right)}\quad\text{and}\quad\mathfrak{B}^t=\mathbf{B}^t\cup\bigcup_{n<0}\overline{\mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right)}.$$

Then $\mathfrak{A}^t$ and $\mathfrak{B}^t$ are disjoint closed sets, and by Lemma 24, each component of $\mathfrak{A}^t$ and of $\mathfrak{B}^t$ is either unbounded to the left or to the right.

Lemma 25. *For each $r>0$, there exists a lower bound $\ell\in\mathbb{R}$ (respectively, upper bound $u\in\mathbb{R}$) such that for all $t\in[0,1]$, if $c+di\in\mathbf{A}^t$ (respectively, $\mathbf{B}^t$) and $|c|\leq r$, then $d\geq\ell$ (respectively, $d\leq u$).*

Proof. Let $\mathbb{I}$ denote the imaginary axis, so that $[-r,r]\times\mathbb{I}$ is the strip in the plane between the vertical lines through r and $-r$.

By uniform continuity of $\widetilde{h}$ and the fact that $\widetilde{h}$ leaves 0 and 1 fixed, there must exist for each $r>0$ an $r'>r$ such that for all $\mathbf{x}\in\mathbf{X}$, if $\mathbf{x}^s\in((-\infty,-r']\cup[r',\infty))\times\mathbb{I}$ for some $s\in[0,1]$, then for all $t\in[0,1]$, $\mathbf{x}^t\notin[-r,r]\times\mathbb{I}$.

Given a point $\mathbf{x}\in\mathbf{A}\cap([-r',r']\times\mathbb{I})$, let $\widetilde{x}=\widetilde{\exp}(\mathbf{x})$ be the corresponding point of $\widetilde{X}$. Every time $\mathbf{x}$ travels vertically within the strip $[-r',r']\times\mathbb{I}$ a distance 2π, the point $\widetilde{x}$ travels around a disk of fixed radius (depending on r') centered at 0 or at 1. By uniform continuity and compactness of X, this can only happen a uniformly bounded number of times. The result follows. □

Corollary 26. *Let $\mathbf{C}$ be any component of $\mathbf{X}$. Then for any $r>0$ and any $t\in[0,1]$, the set $\mathbf{C}^t\cap\{x+yi:x\in[-r,r]\}$ is compact.*

Proof. Because the set $\mathbf{X}^t$ is periodic with period $2\pi i$, there exists an integer k such that if $\mathbf{D}$ is the copy of $\mathbf{C}$ shifted vertically by $2\pi k$, then without loss of generality $\mathbf{C}\subset\mathbf{A}$ and $\mathbf{D}\subset\mathbf{B}$. Then by Lemma 25, $\mathbf{C}^t$ is bounded below in the strip $\{x+yi:x\in[-r,r]\}$, and $\mathbf{D}^t$ is bounded above in this strip. By periodicity, it follows that $\mathbf{C}^t$ is also bounded above in this strip. □

Definition 27. Given two distinct components $\mathbf{C},\mathbf{D}$ of $\mathbf{X}$ which are both unbounded to the right (respectively, to the left), we say that $\mathbf{C}$ *lies above* $\mathbf{D}$ if there is some $R>0$ such that for all $x\in\mathbb{R}$ with $x\geq R$ (respectively, $x\leq -R$), $\max(y\in\mathbb{R}:x+iy\in\mathbf{C})>\max(y\in\mathbb{R}:x+iy\in\mathbf{D})$ and $\min(y\in\mathbb{R}:x+iy\in\mathbf{C})$ $>\min(y\in\mathbb{R}:x+iy\in\mathbf{D})$ also.

Note that it follows immediately from the definition of $\mathfrak{A}^0$ and $\mathfrak{B}^0$ that if $\mathbf{C}$ and $\mathbf{D}$ are components of $\mathfrak{A}^0$ and $\mathfrak{B}^0$, respectively, which are unbounded on the same side, then $\mathbf{C}$ lies above $\mathbf{D}$. The following lemma follows from this fact. The proof, which is left to the reader, is very similar to the proof of Lemma 2.5 in [27].

Lemma 28. *There exists $\delta > 0$ such that if the crosscut Q of X with endpoints 0 and b has diameter $\operatorname{diam}(Q) < \delta$, then the following holds for the induced isotopy $\mathbf{h}$ of $\mathbf{X}$:*

Let $\mathbf{C}$ and $\mathbf{D}$ be components of $\mathfrak{A}^0$ and $\mathfrak{B}^0$, respectively, which are both unbounded to the same side. Then $\mathbf{C}^t$ lies above $\mathbf{D}^t$ for all $t \in [0,1]$.

Consequently, if $\mathbf{E}$ and $\mathbf{F}$ are components of $\mathbf{A}$ and $\mathbf{B}$, respectively, which are both unbounded to the same side, then $\mathbf{E}^t$ lies above $\mathbf{F}^t$ for all $t \in [0,1]$.

4.4. **Equidistant set between $\mathbf{A}^t$ and $\mathbf{B}^t$.** For the remainder of this section, we assume that $\delta > 0$ is chosen so that the conclusions of Lemmas 24 and 28 hold. We also assume that the crosscut Q has diameter less than δ.

Recall that disjoint closed sets A_1 and A_2 in $\mathbb{C}$ are *non-interlaced* if whenever $B(c,r)$ is an open disk contained in the complement of $A_1 \cup A_2$, there are disjoint arcs $C_1, C_2 \subset \partial B(c,r)$ such that $A_1 \cap \partial B(c,r) \subset C_1$ and $A_2 \cap \partial B(c,r) \subset C_2$. We allow for the possibility that $C_1 = \emptyset$ in the case that $A_2 \cap \partial B(c,r) = \partial B(c,r)$, and vice versa.

Lemma 29. *$\mathbf{A}^t$ and $\mathbf{B}^t$ are non-interlaced for all $t \in [0,1]$.*

Proof. Fix $t \in [0,1]$. Let $B \subset \mathbb{C} \setminus (\mathbf{A}^t \cup \mathbf{B}^t)$ be a round open ball, and suppose for a contradiction that there exist points $\mathbf{a}_1, \mathbf{a}_2 \in \partial B \cap \mathbf{A}^t$ and $\mathbf{b}_1, \mathbf{b}_2 \in \partial B \cap \mathbf{B}^t$ such that the straight line segment $\overline{\mathbf{a}_1\mathbf{a}_2}$ separates $\mathbf{b}_1$ and $\mathbf{b}_2$ in $\overline{B}$. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be the components of $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, in $\mathfrak{A}^t$, and let $\mathbf{B}_1$ and $\mathbf{B}_2$ be the components of $\mathbf{b}_1$ and $\mathbf{b}_2$ in $\mathfrak{B}^t$. Then $[\mathbf{A}_1 \cup \mathbf{A}_2] \cap [\mathbf{B}_1 \cup \mathbf{B}_2] = \emptyset$, and by the remarks immediately following the definition of $\mathfrak{A}^t$ and $\mathfrak{B}^t$, each of these four components is either unbounded to the left or unbounded to the right. Consider an arc S in $\overline{B} \setminus (\mathbf{B}_1 \cup \mathbf{B}_2)$ joining $\mathbf{a}_1$ and $\mathbf{a}_2$. Then $\mathbf{A}_1 \cup \mathbf{A}_2 \cup S$ separates the plane into at least two components, and $\mathbf{B}_1$ and $\mathbf{B}_2$ must lie in different components of $\mathbb{C} \setminus (\mathbf{A}_1 \cup \mathbf{A}_2 \cup S)$. It is then straightforward to see by considering cases that there exist $i, j \in \{1,2\}$ such that $\mathbf{B}_i$ lies above $\mathbf{A}_j$, a contradiction with Lemma 28. ☐

For each $t \in [0,1]$, let $\mathbf{M}_t = \operatorname{Equi}(\mathbf{A}^t, \mathbf{B}^t)$. In light of Lemma 29, $\mathbf{M}_t$ is a 1-manifold by Theorem 10.

Lemma 30. *For each $t \in [0,1]$ and each $n \in \mathbb{Z}$, $\mathbf{M}_t \cap \mathbf{E}_n\left(\frac{(1-\nu)|\Theta(b^t)|}{4\nu}\right) = \emptyset$. In particular, $\mathbf{M}_t \cap \mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right) = \emptyset$.*

Proof. Let $n \in \mathbb{Z}$ and assume that $n \geq 0$ (the case $n < 0$ proceeds similarly). Since $0 < \nu < \frac{1}{3}$, $\frac{(1-\nu)|\Theta(b^t)|}{4\nu} > \frac{|\Theta(b^t)|}{2}$, so $\mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right) \subset \mathbf{E}_n\left(\frac{(1-\nu)|\Theta(b^t)|}{4\nu}\right)$.

By Lemma 24, there is a component $\mathbf{C}$ of $\mathbf{X}$ such that $\mathbf{C}^t \cap E_n\left(\frac{|\Theta(b^t)|}{2}\right) \neq \emptyset$ for all $t \in [0,1]$. Since $n \geq 0$, $\mathbf{C} \subset \mathbf{A}$.

On the other hand, given any component $\mathbf{D}$ of $\mathbf{B}$, we have by Lemma 24(i) that $\mathbf{D}^t \cap \mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2\nu}\right) = \emptyset$ for all $t \in [0,1]$. Thus $\mathbf{B}^t \cap \mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2\nu}\right) = \emptyset$ for all $t \in [0,1]$. It follows that at any point $x \in \mathbf{E}_n\left(\frac{(1-\nu)|\Theta(b^t)|}{4\nu}\right)$, the distance from x to $\mathbf{A}^t$ is less than $\frac{|\Theta(b^t)|}{2} + \frac{(1-\nu)|\Theta(b^t)|}{4\nu} = \frac{(1+\nu)|\Theta(b^t)|}{4\nu}$, while the distance from x to $\mathbf{B}^t$ is greater than $\frac{|\Theta(b^t)|}{2\nu} - \frac{(1-\nu)|\Theta(b^t)|}{4\nu} = \frac{(1+\nu)|\Theta(b^t)|}{4\nu}$. Thus $\mathbf{M}_t \cap \mathbf{E}_n\left(\frac{(1-\nu)|\Theta(b^t)|}{4\nu}\right) = \emptyset$ for all n. ☐

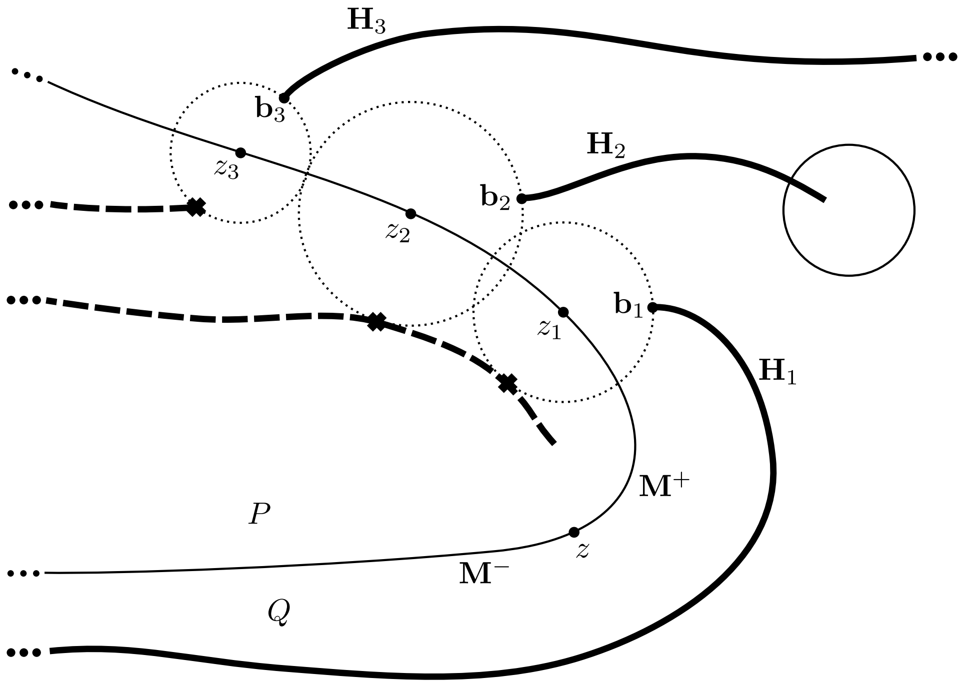

FIGURE 3. An illustration of the situation described in the proof of Lemma 31.

Lemma 31. *For each t the set $\mathbf{M_t}$ is a connected 1-manifold. Moreover, the vertical projection of $\mathbf{M}_t$ to the real axis $\mathbb{R}$ is onto.*

Proof. Since by Lemma 29 $\mathbf{A}^t$ and $\mathbf{B}^t$ are non-interlaced, by Theorem 10, $\mathbf{M}_t$ is a 1-manifold which separates $\mathbf{A}^t$ from $\mathbf{B}^t$. By Lemma 30, $\mathbf{M}_t$ is disjoint from $\bigcup_n \mathbf{E}_n\left(\frac{|\Theta(b^t)|}{2}\right)$, and, hence, $\mathbf{M}_t$ separates $\mathfrak{A}^t$ from $\mathfrak{B}^t$ (recall that $\mathfrak{A}^t$ and $\mathfrak{B}^t$ were defined above in Lemma 25). Since all components of $\mathfrak{A}^t$ and $\mathfrak{B}^t$ are unbounded, no component of $\mathbf{M}_t$ is a simple closed curve, and every component is a copy of $\mathbb{R}$ with both ends converging to infinity. By Lemma 25 each end of a component of $\mathbf{M}_t$ either converges to $-\infty$ or $+\infty$. Fix t and let $\mathbf{M}'$ be a component of $\mathbf{M}_t$. Note that for all $\mathbf{x} \in \mathbf{M}'$ there exists a set of points $\mathbf{A}^t_x \subset \mathbf{A}^t$ closest to x and $\mathbf{B}^t_x \subset \mathbf{B}^t$ closest to x and that $\bigcup_{\mathbf{x}\in\mathbf{M}'} \mathbf{A}^t_x$ and $\bigcup_{x\in\mathbf{M}'} \mathbf{B}^t_x$ are separated by the line $\mathbf{M}'$. For $x \in \mathbf{M}'$, let r_x denote the distance from x to $\mathbf{A}^t_x$ (equivalently, to $\mathbf{B}^t_x$).

If both ends of $\mathbf{M}'$ are unbounded to the same side, say on the left side, then $\mathbb{C} \setminus \mathbf{M}'$ has two complementary components P and Q, with P only unbounded to the left (see Figure 3). Assume that $\bigcup_{\mathbf{x}\in\mathbf{M}'} \mathbf{A}^t_x \subset P$ (the case $\bigcup_{\mathbf{x}\in\mathbf{M}'} \mathbf{B}^t_x \subset P$ is similar). Note that since P contains no components of $\mathbf{A}^t$ which are unbounded to the right, P must contain components of $\mathbf{A}^t$ which are unbounded to the left.

Let $z \in \mathbf{M}'$. Then $\mathbf{M}' \setminus \{z\}$ consists of two rays $\mathbf{M}^+$ and $\mathbf{M}^-$, and we may assume that $\mathbf{M}^+$ lies above $\mathbf{M}^-$. Choose $z_n \in \mathbf{M}^+$ monotonically converging to $-\infty$ and $\mathbf{b}_n \in \mathbf{B}^t_{z_n}$. Since the radii r_{z_n} are uniformly bounded, $\mathbf{b}_n$ also converges to $-\infty$. Let $\mathbf{H}_n$ be the component of $\mathbf{B}^t$ that contains $\mathbf{b}_n$.

If $\mathbf{H}_n$ is unbounded to the left, by Lemma 28 it must lie below the unbounded components of $\mathbf{A}^t$ in P and hence must "go around" $\mathbf{M}'$ as $\mathbf{H}_1$ does in Figure 3. If $\mathbf{H}_n$ is not unbounded to the left, then either it intersects some $\mathbf{E}_k(\frac{|\Theta(b^t)|}{2})$ for some $k < 0$ (as $\mathbf{H}_2$ does in Figure 3) or it is unbounded to the right (as $\mathbf{H}_3$ is in Figure 3). In any case it is clear that there exists $c \in \mathbb{R}$ such that every component $\mathbf{H}_n$ intersects the vertical line $x = c$.

For each n let d_n be such that the point $(c, d_n) \in \mathbf{H}_n$. By Lemma 25, the sequence d_n is bounded and, hence, has an accumulation point d_∞. By Corollary 26, the component of $\mathbf{B}^t$ which contains d_∞ is unbounded to the left, and clearly it lies above the unbounded components of $\mathbf{A}^t$ in P, a contradiction with Lemma 28. Hence, the vertical projection of $\mathbf{M}'$ to the real axis $\mathbb{R}$ is onto.

The proof that $\mathbf{M}_t = \mathbf{M}'$ is connected is similar and is left to the reader. □

Lemma 32. *For each $t \in [0, 1]$, the set $\widetilde{\exp}(\mathbf{M}_t) \cup \{0, 1\}$ is the image of a path $\widetilde{\gamma}_t$ in $\widetilde{U}^t$ joining 0 and 1.*

Proof. Let $\mathbb{I}$ denote the imaginary axis, so that $[-r, r] \times \mathbb{I}$ is the strip in the plane between the vertical lines through r and $-r$. By Lemma 25, for each $r > 0$, $\widetilde{\exp}(([-r, r] \times \mathbb{I}) \cap \mathbf{M}_t)$ is compact. Together with Lemma 31, this implies that we can choose a parameterization $\alpha : (0, 1) \to \mathbf{M}_t$ so that

$$\lim_{s \to 0^+} \widetilde{\exp} \circ \alpha(s) = \{0\}$$

and

$$\lim_{s \to 1^-} \widetilde{\exp} \circ \alpha(s) = \{1\}.$$

Define the path $\widetilde{\gamma}_t : [0, 1] \to \widetilde{\exp}(\mathbf{M}_t) \cup \{0, 1\}$ by $\widetilde{\gamma}_t(s) = \widetilde{\exp} \circ \alpha(s)$ for $s \in (0, 1)$, and $\widetilde{\gamma}_t(0) = 0$ and $\widetilde{\gamma}_t(1) = 1$. Then $\widetilde{\gamma}_t$ is the required path. □

4.5. **Proof of Theorem 20.** In this section we complete the proof of Theorem 20.

Recall that $\varepsilon > 0$ is a fixed arbitrary number, and $0 < \nu < \frac{1}{3}$ has been chosen so that $\frac{8\nu}{1-\nu} < \frac{\varepsilon}{2}$. Choose $0 < \delta < \frac{\varepsilon}{4}$ small enough so that the conclusions of Lemmas 24 and 28 hold (and therefore the results from Section 4.4 also hold).

For each $t \in [0, 1]$, let $\gamma_t = (L^t \circ \Theta)^{-1} \circ \widetilde{\gamma}_t$. This γ_t is a path in U^t joining 0 and b^t.

Claim 2. $\operatorname{diam}(\gamma_t([0, 1])) < \varepsilon$ for all $t \in [0, 1]$.

Proof of Claim 2. By Lemma 30, for all $t \in [0, 1]$ and $n \in \mathbb{Z}$, $\mathbf{M}_t \cap \mathbf{E}_n\left(\frac{(1-\nu)|\Theta(b^t)|}{4\nu}\right) = \emptyset$.

By Lemma 23(ii), we have $\widetilde{\exp}(\mathbf{M}_t) \subset B\left(0, \frac{8\nu}{(1-\nu)|\Theta(b^t)|}\right)$. Then $(L^t)^{-1}(\widetilde{\exp}(\mathbf{M}_t)) \subset B\left(0, \frac{8\nu}{(1-\nu)}\right)$. By the choice of ν and since Θ is a homeomorphism of $\mathbb{C}$ which is the identity outside $B(0, 2\delta) \subset B(0, \frac{\varepsilon}{2})$, it then follows that $\gamma_t([0, 1]) = (L^t \circ \Theta)^{-1}(\widetilde{\exp}(\mathbf{M}_t)) \subset B(0, \frac{\varepsilon}{2})$. □(Claim 2)

Claim 3. The sets $\gamma_t([0, 1])$ vary continuously in the Hausdorff metric, and γ_0 is homotopic to Q with endpoints fixed.

Proof of Claim 3. By Lemma 32, $\widetilde{\gamma}_t$ is a path in $\widetilde{U}^t$ with endpoints 0 and 1. To see that $\widetilde{\gamma}_0$ is homotopic to $\widetilde{Q} = (0, 1)$ note first that since $\mathbf{A}^0$ is above the real axis and $\mathbf{B}^0$ is below the real axis, for each $(x, y) \in \mathbf{M}_0$ the vertical segment from $(x, 0)$

to (x, y) is disjoint from $\mathbf{X}^0$. Hence we can construct a homotopy k between $\mathbf{M}_0$ and $\mathbb{R}$ which fixes the x-coordinate of each point in $\mathbf{M}_0$ and decreases the absolute value of the y-coordinate to zero. Then $\widetilde{\exp} \circ k$ is the required homotopy between $\widetilde{\gamma}_0$ and $\widetilde{Q}$ with endpoints fixed. Hence, $\gamma_0 = (L^0 \circ \Theta)^{-1} \circ \widetilde{\gamma}_0$ is homotopic to Q as required.

Suppose $t_i \to t_\infty$. It is easy to see that $\limsup \mathbf{M}_{t_i} \subseteq \mathbf{M}_{t_\infty}$ by the definition of the equidistant sets $\mathbf{M}_t$. Since, by Lemma 31, each $\mathbf{M}_{t_i}$ and $\mathbf{M}_{t_\infty}$ is a connected 1-manifold whose vertical projection to the real axis $\mathbb{R}$ is onto, it follows that $\liminf \mathbf{M}_{t_i} \supseteq \mathbf{M}_{t_\infty}$. Thus $\lim \mathbf{M}_{t_i} = \mathbf{M}_{t_\infty}$. It follows that $\gamma_t([0,1]) = (L^t \circ \Theta)^{-1} \circ \widetilde{\exp}(\mathbf{M}_t)$ is continuous in the Hausdorff metric. $\square$(Claim 3)

Combined with Lemma 21, Claims 2 and 3 complete the verification of condition (ii) of Theorem 14. Therefore, by Theorem 14, the isotopy h of the compactum X can be extended to the entire plane $\mathbb{C}$. This completes the proof of Theorem 20.

In Theorem 12 we have given necessary and sufficient conditions for an isotopy of a uniformly perfect compact set to extend to an isotopy of the plane. These conditions involve the existence of an extension of the isotopy over sufficiently small crosscuts while controlling the size of the image. The following problem remains open.

Problem 1. Are there intrinsic properties on X and the isotopy h of X, which do not involve the existence of extensions over small crosscuts, that characterize when an isotopy of X can be extended over the plane?

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