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FACTORING THE HIGHER DIMENSIONAL QUASICONFORMAL MAPPINGS

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Abstract. With the aid of the logarithmic spiral mapping, we construct $n(\geq 3)$-dimensional quasiconformal homeomorphisms which admit no minimal factorizations in linear, inner, or outer dilatations.

1. Introduction

Let $n(\geq 2)$ be an integer. Suppose that $f : U \to V$ is a sense-preserving homeomorphism between domains $U, V \subset \mathbb{R}^n$. Define

$$H_f(\zeta) = \limsup_{r\to 0^+} \frac{\max_{|z-\zeta|=r} |f(z) - f(\zeta)|}{\min_{|z-\zeta|=r} |f(z) - f(\zeta)|}, \quad \zeta \in U. \tag{1}$$

If $\forall\ \zeta \in U$, $H_f(\zeta) \leq H_0$ for some $H_0 \geq 1$ independent of ζ, then we call f a *quasiconformal mapping*. See [11, 13] or Theorem 34.1 in [32].

H. Grötzsch [8] first introduced plane quasiconformal mappings in 1928. For more comprehensive accounts of this area see [1–5, 14, 21, 25, 29, 30].

Higher dimensional (≥ 3) quasiconformal mappings were first considered by M. A. Lavrentieff [19], A. Markouchevitch [23], and M. Kreines [18] around 1938-1941. See [6, 12, 16, 17, 22, 24, 27, 28, 31, 32] for more complete accounts of this subject and related topics.

Note that any quasiconformal mapping $f : U \to V$ is differentiable with positive Jacobian determinant almost everywhere. At any such point ζ, we let $A = Df(\zeta)$ be the Jacobian matrix of f. Suppose that

$$\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_n^2 > 0$$

are the eigenvalues of AA^T (see Lemma 2.1 in Section 2). Define

(i) the *linear dilatation*

$$H(A) = \frac{\max\{|A\zeta| : |\zeta| = 1\}}{\min\{|A\zeta| : |\zeta| = 1\}} = \frac{\lambda_1}{\lambda_n};$$

(ii) the *inner dilatation*

$$K_I(A) = \frac{|A^\#|^n}{|\det(A)|^{n-1}} = \frac{\lambda_1\lambda_2\cdots\lambda_n}{\lambda_n^n},$$

where $A^\#$ is the adjugate matrix of A;

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(iii) the *outer dilatation*

$$K_O(A) = \frac{|A|^n}{|\det(A)|} = \frac{\lambda_1^n}{\lambda_1\lambda_2\cdots\lambda_n}.$$

The essential supremum of (i) (resp., (ii), (iii)), denoted by $H(f)$ (resp., $K_I(f)$, $K_O(f)$), is called the *linear* (resp., *inner*, *outer*) dilatation of f. When $n = 2$, we have

$$H(f) = K_I(f) = K_O(f). \tag{2}$$

For any one of the dilatations $K(f) = H(f), K_I(f)$ or $K_O(f)$, we always have trivially

$$K(f) \leq K(f_2) \cdot K(f_1), \tag{3}$$

whenever $f = f_2 \circ f_1$ (refer to (7) in Section 2 for the details). If "=" is valid and $K(f_2), K(f_1) \neq 1$ in (3), then we call $f = f_2 \circ f_1$ a *minimal* factorization in the dilatation $K(f)$.

Suppose that f is a given plane quasiconformal map with maximal dilatation K and $1 < K_1 < K$. The Measurable Riemann Mapping Theorem tells us that a minimal factorization $f = f_2 \circ f_1$ always exists with $K(f_1) = K_1$ and $K(f_2) = K/K_1$ (see Thm. 4.7 on p. 29 in [20]).

Moreover, it also follows from the Measurable Riemann Mapping Theorem that one can always present a 2-dimensional quasiconformal mapping as a composition of quasiconformal mappings with dilatation close to 1. Refer to (4.17) on p. 30 in [20]. On p. 71 of [10], Gehring says that it is an important open problem to decide if some higher dimensional form of this result holds, even for the case when $U = V = \mathbb{R}^n$. There has been very little progress made in Gehring's problem so far. Another open problem is if there are examples of higher dimensional quasiconformal maps that do admit minimal factorizations.

In this paper, we will give such an example in any dimension higher than two.

Suppose $\lambda > 0$ and let $s_\lambda(\rho, \theta) = (\rho,\ \theta + \lambda \log \rho)$ be the logarithmic spiral mapping of $\mathbb{R}^2$. Then s_λ is quasiconformal with linear dilatation

$$K = \frac{\sqrt{4+\lambda^2}+\lambda}{\sqrt{4+\lambda^2}-\lambda}.$$

Note that the logarithmic spiral mapping s_λ is also bi-Lipschitz. That is, there is $L \geq 1$ such that

$$L^{-1}\,|z - z'| \leq |s_\lambda(z) - s_\lambda(z')| \leq L\,|z - z'| \quad \forall\, z,\ z' \in \mathbb{R}^2. \tag{4}$$

The smallest $L \geq 1$ for which (4) holds is called the isometric dilatation. We can easily check that s_λ has isometric dilatation $\sqrt{K}$.

Let $f_{n,\lambda}$ ($n \geq 3$) be a quasiconformal mapping of $\mathbb{R}^n$ defined by

$$f_{n,\lambda}(z,\ t_1, \cdots,\ t_{n-2}) = (s_\lambda(z),\ t_1/\sqrt{K}, \cdots,\ t_{n-2}/\sqrt{K}), \tag{5}$$

where $z \in \mathbb{R}^2$ and $(t_1, \cdots,\ t_{n-2}) \in \mathbb{R}^{n-2}$. By using the isometric dilatation $\sqrt{K}$ of s_λ, we can deduce that

$$H(f_{n,\lambda}) = K_I(f_{n,\lambda}) = K \quad \text{and} \quad K_O(f_{n,\lambda}) = K^{n-1}.$$

Note that $f_{n,\lambda}$ is also bi-Lipschitz with isometric dilatation $\sqrt{K}$.

Then we have the following.

Theorem 1.1. *The n-dimensional quasiconformal map $f_{n,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ admits no minimal factorizations in the linear dilatation. That is,*

$$H(f_{n,\lambda}) < H(f_2) \cdot H(f_1)$$

whenever $f_{n,\lambda} = f_2 \circ f_1$ and $H(f_2),\ H(f_1) \neq 1$.

Theorem 1.2. *The n-dimensional quasiconformal map $f_{n,\lambda}$ admits no minimal factorizations in the inner dilatation.*

For the quasiconformal mapping $f_{n,\lambda}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, we have the following.

Theorem 1.3. *The map $f_{n,\lambda}^{-1}$ admits no minimal factorizations in the outer dilatation.*

Since any L-bi-Lipschitz mapping is quasiconformal with linear dilatation $\leq L^2$, Theorem 1.1 yields the following.

Corollary 1.4. *As a $\sqrt{K}$-bi-Lipschitz mapping, $f_{n,\lambda}$ cannot be written as the composition of a $\sqrt{K_1}$-bi-Lipschitz map and a $\sqrt{K/K_1}$-bi-Lipschitz map with $1 < K_1 < K$.*

Remark 1.5. In [7] M. Freedman and Z.-X. He established that it requires at least $\lambda/\sqrt{L^2-1}$ factors to write the bi-Lipschitz map s_λ into a composition of bi-Lipschitz homeomorphisms with isometric dilatations $\leq L$. On the other hand, the minimal factors of s_λ with linear dilatations $\leq L$ grows like $2\log_L \lambda$ when λ is large. Thus for large λ, the number of factors with small isometric dilatation needed to "unwind" the spiral map s_λ is much greater than the number of factors with the same linear dilatation.

Furthermore, our example of quasiconformal mapping is built from the bi-Lipschitz map of Freedman and He; but our method is quite different. It would be interesting to know if the method of Freedman and He can be extended to the case here.

Let $K(f)$ denote the linear, outer, or inner dilatation of any $n(\geq 3)$-dimensional quasiconformal mapping f. We introduce the following.

Open Problem. For any $A > 1$, is there an n-dimensional ($n \geq 3$) quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $K(f) = K/A$ such that

$$f \neq f_2 \circ f_1$$

for any quasiconformal maps $f_1,\ f_2$ with $K(f_1) \cdot K(f_1) = K$?

Notational conventions. Throughout the paper, for any matrix A we denote by A^T the transpose of A. $SO(n)$ will denote the n-dimensional special orthogonal group, i.e., $Q \in SO(n)$ if and only if $Q \cdot Q^T = I_n$ (the n-dimensional identity matrix) and with determinant $\det(Q) = 1$.

For $\lambda_i \in \mathbb{R}$ ($1 \leq i \leq n$), we denote diag $(\lambda_1,\ \lambda_2, \cdots, \lambda_n)$ to be the diagonal matrix

$$\operatorname{diag}(\lambda_1,\ \lambda_2, \cdots, \lambda_n) = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

2. Basic materials

2.1. **Qusiconformal mappings.** The following result is well known in linear algebra. Its proof, included here for completeness, is elementary.

Lemma 2.1. *If A is an $(n \times n)$-real matrix with determinant* $\det(A) > 0$*, then there exist $P,\ Q \in SO(n)$ such that*

$$P \cdot A \cdot Q = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

That is, for any orientation-preserving linear mapping $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^n$, there are orthogonal bases $\{v_1, v_2, \cdots, v_n\}$ and $\{w_1, w_2, \cdots, w_n\}$ such that the matrix A with respect to these bases is diagonal.

Proof. Since the determinant $\det(A) > 0$, the symmetric matrix AA^T is positive definite. There exists $P \in SO(n)$ such that

$$P \cdot AA^T \cdot P^T = \mathrm{diag}\,(\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Denote

$$Q = A^T \cdot P^T \cdot \mathrm{diag}\,(\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_n^{-1}).$$

Then $Q^T \cdot Q = I_n$ (the $n \times n$ identity matrix). Consequently,

$$P,\ Q \in SO(n) \ \text{ and } \ P \cdot A \cdot Q = \mathrm{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

as desired. □

For quasiconformal mappings, we have the following.

Proposition 2.2 ([32])**.** *A quasiconformal mapping $f : U \to V$ possesses the following properties:*

- *f is A. C. L. (Absolutely Continuous on Lines). Also it is differentiable with Jacobian $J_f(\zeta) > 0$ almost everywhere.*
- *$f^{-1} : V \to U$ is also quasiconformal.*
- *For any measurable set $E \subset U$, the measure $m(E) = 0$ implies that $m(f(E)) = 0$.*

Suppose that f is differentiable at $\zeta \in U$ with Jacobian $J_f(\zeta) > 0$. Take the normalized frame $\{e_1, e_2, \cdots, e_n\}$ in the tangent space $T_\zeta U \cong \mathbb{R}^n$, where

$$e_1 = (1, 0, \cdots, 0), \ \ e_2 = (0, 1, \cdots, 0), \ \ \cdots, \ \ e_n = (0, 0, \cdots, 1).$$

Then

$$\begin{aligned}
\left.\frac{\partial f}{\partial e_1}\right|_\zeta &= \quad a_{11}\, e_1 \ + a_{21}\, e_2 \ + \cdots + a_{n1}\, e_n, \\
\left.\frac{\partial f}{\partial e_2}\right|_\zeta &= \quad a_{12}\, e_1 \ + a_{22}\, e_2 \ + \cdots + a_{n2}\, e_n, \\
&\cdots \\
\left.\frac{\partial f}{\partial e_n}\right|_\zeta &= \quad a_{1n}\, e_1 \ + a_{2n}\, e_2 \ + \cdots + a_{nn}\, e_n.
\end{aligned}$$

Thus the Jacobian matrix of f at ζ is

$$Df(\zeta) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \text{with} \quad J_f(\zeta) = \det(Df(\zeta)) > 0.$$

Denote $A = Df(\zeta)$. In Section 1 we have already defined the commonly used local distortion functions $H(A), K_I(A)$, and $K_O(A)$. Obviously, using the definitions we obtain the following symmetry relations:

$$H(A) = H(A^{-1}), \ K_O(A) = K_I(A^{-1}). \tag{6}$$

Furthermore, these dilatation functions are submultiplicative. That is, for any non-degenerate $(n \times n)$-matrices A and B,

$$H(AB) \leq H(A)H(B), K_I(AB) \leq K_I(A)K_I(B), K_O(AB) \leq K_O(A)K_O(B). \tag{7}$$

For the proofs we refer the reader to (6.32) and Subsection 9.10 in [16].

From the definitions (i), (ii), (iii) in Section 1, we readily deduce that the dilatations satisfy:

$$\begin{aligned} &1 \leq H(f) \leq K_I(f), && 1 \leq H(f) \leq K_O(f), \\ &1 \leq K_O(f) \leq H(f)^{n-1}, && 1 \leq K_I(f) \leq H(f)^{n-1}. \end{aligned} \tag{8}$$

Remark 2.3. We give the following remarks:

- Väisälä [32] gave an alternative characterization of inner dilatation and outer dilatation by using the *moduli* of curve families.

  A domain $A \subset \mathbb{R}^n \cup \{\infty\}$ is called a ring if its boundary ∂A has exactly two components. For any ring $A \subset \mathbb{R}^n$, we let Γ_A denote the family of locally rectifiable curves joining the two components of ∂A. Define the modulus of A to be

  $$M(\Gamma_A) = \inf \int_A \rho^n dm, \tag{9}$$

  where the infimum is taken over all non-negative Borel functions $\rho : A \to [0, \infty]$ with

  $$\int_\gamma \rho \, ds \geq 1 \qquad \forall \gamma \in \Gamma_A.$$

  For any homeomorphism $f : U \to V$, Väisälä (Theorem 36.1 in [32]) gave

  $$K_I(f) = \sup \frac{M(\Gamma_{f(A)})}{M(\Gamma_A)}, \quad K_O(f) = \sup \frac{M(\Gamma_A)}{M(\Gamma_{f(A)})}, \tag{10}$$

  where the suprema are taken over all rings $A \subset U$ with $\overline{A} \subset U$ and $M(\Gamma_A) > 0$.

  Then, a homeomorphism $f : U \to V$ is quasiconformal if and only if one of the dilatations $K_I(f), K_O(f)$ is finite (hence both of them).
- Recall that a dilatation is lower semicontinuous if the classes $\{f : K(f) \leq K\}$ are closed under local uniform convergence. Alternatively, if $f_k : U \to V_k, \ k = 1, 2, \cdots$ is a sequence of quasiconformal mappings which converges locally uniformly to a homeomorphism $f : U \to \mathbb{V}$, then

  $$K(f) \leq \liminf_{k \to \infty} K(f_k).$$

It is well known that the inner dilatation K_I and outer dilatation K_O are lower semicontinous on the space of quasiconformal mappings.

In contrast, the linear dilatation H is not lower semicontinous on the space of higher dimension quasiconformal mappings. This failure is directly connected with the failure of rank-one convexity in the calculus of variations. We refer the reader to [15, 16] for the proofs of these facts.

Note that 1-quasiconformal homeomorphisms of plane domains are holomorphic. For $n(\geq 3)$-dimensional 1-quasiconformal homeomorphisms, we have the following Liouville Theorem due to F. Gehring [9] and Ju. Rešetnjak [26]. Note that this result involves no priori differentiability hypotheses.

Liouville Theorem ([9, 26]). *An $n(\geq 3)$-dimensional quasiconformal homeomorphism $f : U \to V$ is 1-quasiconformal if and only if f is the restriction to U of a Möbius transformation, i.e., the composition of even reflections in $(n-1)$-spheres or planes.*

2.2. **Logarithmic spiral mapping.** Recall the logarithmic spiral mapping $s_\lambda(\rho, \theta) = (\rho,\ \theta + \lambda \log \rho)$. Then s_λ is quasiconformal with maximal dilatation

$$K = \frac{\sqrt{4+\lambda^2}+\lambda}{\sqrt{4+\lambda^2}-\lambda}. \tag{11}$$

Recall the complex exponential function

$$\pi(\tau,\ \theta) = (e^\tau \cos\theta,\ e^\tau \sin\theta) : \mathbb{R}^2 \to \mathbb{R}^2\backslash\{0\}, \tag{12}$$

where $\tau + i\,\theta \in \mathbb{R}^2$. It is a covering map. Now we have the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\ l_\lambda\ } & \mathbb{R}^2 \\ \pi\big\downarrow & & \pi\big\downarrow \\ \mathbb{R}^2\backslash\{0\} & \xrightarrow{\ s_\lambda\ } & \mathbb{R}^2\backslash\{0\}, \end{array}$$

where $l_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $l_\lambda(\tau,\ \theta) = (\tau,\ \theta + \lambda\tau)$. Their Jacobian matrices satisfy

$$Ds_\lambda(z) = D\pi(l_\lambda(\pi^{-1}(z))) \cdot Dl_\lambda(\pi^{-1}(z)) \cdot D\pi^{-1}(z) \quad \forall\ z \in \mathbb{R}^2\backslash\{0\}.$$

Therefore, if $(\rho,\ \theta)$ is the polar coordinate of $z \in \mathbb{R}^2\backslash\{0\}$ and $\theta' = \theta + \lambda \log\rho$, then

$$\begin{aligned} Ds_\lambda(z) &= \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= A(\lambda, \theta') \cdot \mathrm{diag}\,(\sqrt{K},\ 1/\sqrt{K}) \cdot B(\lambda, \theta), \end{aligned} \tag{13}$$

where $K = \dfrac{\sqrt{4+\lambda^2}+\lambda}{\sqrt{4+\lambda^2}-\lambda}$ is given in (11), and

$$A(\lambda, \theta') = \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{2}{\lambda^2+4+\lambda\sqrt{\lambda^2+4}}} & -\sqrt{\frac{2}{\lambda^2+4-\lambda\sqrt{\lambda^2+4}}} \\ \sqrt{\frac{2}{\lambda^2+4-\lambda\sqrt{\lambda^2+4}}} & \sqrt{\frac{2}{\lambda^2+4+\lambda\sqrt{\lambda^2+4}}} \end{pmatrix} \in SO(2),$$

$$B(\lambda, \theta) = \begin{pmatrix} \sqrt{\frac{2}{\lambda^2+4-\lambda\sqrt{\lambda^2+4}}} & \sqrt{\frac{2}{\lambda^2+4+\lambda\sqrt{\lambda^2+4}}} \\ -\sqrt{\frac{2}{\lambda^2+4+\lambda\sqrt{\lambda^2+4}}} & \sqrt{\frac{2}{\lambda^2+4-\lambda\sqrt{\lambda^2+4}}} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2).$$

3. Proofs of the main results

Now we can begin the proof of Theorem 1.1. We will prove the theorem for $n = 3$ and the general case is similar. Recall that

$$f_{3,\lambda}(z,\ t) = (s_\lambda(z),\ t/\sqrt{K}),\ \ (z,\ t) \in \mathbb{R}^3.$$

It follows from (13) that, for almost every $\zeta = (z, t) \in \mathbb{R}^3$,

$$Df_{3,\lambda}(\zeta) = \begin{pmatrix} \mathrm{A}(\lambda,\ \theta')_{2\times 2} & \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{K} & & \\ & 1/\sqrt{K} & \\ & & 1/\sqrt{K} \end{pmatrix} \begin{pmatrix} B(\lambda,\ \theta)_{2\times 2} & \\ & 1 \end{pmatrix}.$$

We assume, by contradiction, that

$$f_{3,\lambda} = f_2 \circ f_1 \tag{14}$$

for some quasiconformal maps $f_1,\ f_2$ with linear dilatations

$$K = H(f_{3,\lambda}) = H(f_1) \cdot H(f_2)$$

and $H(f_1),\ H(f_2) \neq 1$. By replacing f_1 with $\gamma \circ f_1$ (and f_2 with $f_2 \circ \gamma^{-1}$) for some conformal affine map γ of $\mathbb{R}^3$, we may further require that

$$f_1(0,0,0) = (0,0,0), \quad f_1(0,0,1) = (0,0,1). \tag{15}$$

Now we have the following result. Its proof will be postponed to Section 4.

Lemma 3.1. *For almost every $\zeta \in \mathbb{R}^3$, there exist $P(\zeta) \in SO(3)$ and $\mu_\zeta > 0$ such that*

$$Df_1(\zeta) = P(\zeta) \cdot \begin{pmatrix} K_1 \cdot \mu_\zeta & & \\ & \mu_\zeta & \\ & & \mu_\zeta \end{pmatrix} \cdot \begin{pmatrix} B(\lambda,\ \theta)_{2\times 2} & \\ & 1 \end{pmatrix}, \tag{16}$$

where $B(\lambda,\ \theta) \in SO(2)$ is defined by (13) *and $K_1 = H(f_1)$.*

Continuation of the proof of Theorem 1.1. Denote the t-axis $\mathbb{T} = \{(z,\ t) \in \mathbb{R}^3 |\ z \equiv 0\}$. We first prove that the map f_1 in the minimal factorization (14) is unique up to a rotation around the t-axis $\mathbb{T}$.

Otherwise, we assume that $f_{3,\lambda}$ has another "minimal" factorization

$$f_{3,\lambda} = \tilde{f}_2 \circ \tilde{f}_1,$$

where $\tilde{f}_1,\ \tilde{f}_2$ are quasiconformal mappings with linear dilatations $K = H(f_{3,\lambda}) = H(\tilde{f}_1) \cdot H(\tilde{f}_2)$ and $H(\tilde{f}_1), H(\tilde{f}_2) \neq 1$. We also require that $\tilde{f}_1$ satisfies (15). Lemma 3.1 immediately implies that there exist $\widetilde{P}(\zeta) \in SO(3),\ \widetilde{\mu}_\zeta > 0$ such that

$$D\tilde{f}_1(\zeta) = \widetilde{P}(\zeta) \cdot \begin{pmatrix} \widetilde{K}_1 \cdot \widetilde{\mu}_\zeta & & \\ & \widetilde{\mu}_\zeta & \\ & & \widetilde{\mu}_\zeta \end{pmatrix} \cdot \begin{pmatrix} B(\lambda,\ \theta)_{2\times 2} & \\ & 1 \end{pmatrix}, \quad a.e.\ \zeta \in \mathbb{R}^3, \tag{17}$$

where $\widetilde{K}_1 = H(\tilde{f}_1)$. Hence, by (16) and (17),

$$D(\tilde{f}_1 \circ f_1^{-1})(\eta) = \lambda_\eta \cdot Q(\eta), \qquad \text{a.e. } \eta \in \mathbb{R}^3,$$

where $\lambda_\eta > 0$ and $Q(\eta) \in SO(3)$. From Liouville's Theorem, it follows that $\tilde{f}_1 = \gamma_1 \circ f_1$ for some 3-dimensional Möbius transformation γ_1. Combining with the assumption (15), we conclude that f_1 is unique up to a rotation around the t-axis $\mathbb{T}$.

Let R_θ be the Möbius transformation of $\mathbb{R}^3$ defined by

$$R_\theta(z,\ t) = (e^{i\theta}z,\ t) \qquad \forall\ (z,\ t) \in \mathbb{R}^3.$$

Observing that

$$f_{3,\lambda} = R_\theta \circ f_{3,\lambda} \circ R_\theta^{-1} = (R_\theta \circ f_2 \circ R_\theta^{-1}) \circ (R_\theta \circ f_1 \circ R_\theta^{-1}),$$

as a consequence of the uniqueness of f_1 we deduce that

$$R_\theta \circ f_1 \circ R_\theta^{-1} = R_{\theta'} \circ f_1.$$

Therefore

$$f_1(\mathbb{T}) = \mathbb{T}. \tag{18}$$

Let us write

$$f_1(x,y,t) = (u(x,y,t), v(x,y,t), w(x,y,t)).$$

In view of (18) we conclude that $w(0,0,t) = h(t)$, where $t \mapsto h(t)$ is a strictly monotonous function from $\mathbb{R}$ to $\mathbb{R}$. For each $a \in \mathbb{R}$, let T_a denote the homeomorphism $T_a(x,y,t) = (x,y,a-t)$. Consequently,

$$f_{3,\lambda} = T_{2a/\sqrt{K}} \circ f_{3,\lambda} \circ T_{2a} = (T_{2a/\sqrt{K}} \circ f_2 \circ T_{2h(a)}) \circ (T_{2h(a)} \circ f_1 \circ T_{2a}).$$

Thus, by the uniqueness of f_1, it follows that

$$T_{2h(a)} \circ f_1 \circ T_{2a} = M_a \circ f_1, \tag{19}$$

where M_a is a Möbius transformation. Combining (18) and (19) yields $w(x,y,t) = w(0,0,t)$. Moreover, from (19) we readily obtain that M_a is the identity map. Hence

$$T_{2h(a)} \circ f_1 \circ T_{2a} = f_1 \ \ \forall\ a \in \mathbb{R}, \tag{20}$$

which implies that $u(x,y,t),\ v(x,y,t)$ are independent of t. In addition, from (15), (20) it follows that $w(x,y,t) = t$. Therefore,

$$f_1(x,y,t) = (F_1(x,y),\ t).$$

Together with Lemma 3.1, we conclude that $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is K_1-quasiconformal, where $1 < K_1 = H(f_1) < K$. Similarly,

$$f_2(x,y,t) = (F_2(x,y),\ t/\sqrt{K}),$$

and $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is K/K_1-quasiconformal. Therefore, $s_\lambda = F_2 \circ F_1$ is a minimal factorization.

A straightforward computation shows that s_λ is quasiconformal with Beltrami differential

$$\mu_\lambda(z) = \frac{i\lambda}{2+i\lambda} \cdot \frac{z}{\bar{z}},$$

where $z = x+iy$. Since $s_\lambda = F_2 \circ F_1$ is a minimal factorization, the solvability of the Beltrami equation implies that F_1 has Beltrami differential $c \cdot \mu_\lambda$, where

$$0 < c = \frac{K_1 - 1}{K_1 + 1} \cdot \frac{\sqrt{4+\lambda^2}}{\lambda} < 1 \tag{21}$$

(see Thm. 4.7 on p. 29 in [20]).

Now consider the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{l_{c,\lambda}} & \mathbb{R}^2 \\ \pi\Big\downarrow & & \pi\Big\downarrow \\ \mathbb{R}^2\backslash\{0\} & \xrightarrow{s_{c,\lambda}} & \mathbb{R}^2\backslash\{0\}, \end{array}$$

where π is the complex exponential function defined in (12) and $l_{c,\lambda} : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism defined by

$$l_{c,\lambda}(\tau,\ \theta) = \left(\frac{1 + \frac{(1-c^2)\lambda^2}{4}}{1 + \frac{(1-c)^2\lambda^2}{4}}\,\tau,\ \theta + \frac{c\lambda}{1 + \frac{(1-c)^2\lambda^2}{4}}\,\tau \right), \quad \tau + i\,\theta \in \mathbb{R}^2.$$

The quasiconformal homeomorphism $s_{c,\lambda} = \pi \circ l_{c,\lambda} \circ \pi^{-1}$ also has Beltrami differential $c \cdot \mu_\lambda$. Therefore, $F_1 = c_0 \cdot s_{c,\lambda}$ for some $c_0 \neq 0$.

Consequently, by using the chain rule of Jacobian matrices

$$Ds_{c,\lambda} = D\pi \cdot Dl_{c,\lambda} \cdot D\pi^{-1},$$

and using (13), we deduce

$$DF_1(z) = |c_0| \cdot \rho^\sigma \cdot P_{c,\rho,\theta}(z) \cdot A_{c,\,\lambda} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{22}$$

where (ρ, θ) is the polar coordinate of $z \in \mathbb{C}\backslash\{0\}$, $P_{c,\rho,\theta}(z) \in SO(2)$, and

$$\sigma = \frac{\frac{c(1-c)\lambda^2}{2}}{1 + \frac{(1-c)^2\lambda^2}{4}} > 0, \quad A_{c,\,\lambda} = \begin{pmatrix} \frac{1+\frac{(1-c^2)\lambda^2}{4}}{1+\frac{(1-c)^2\lambda^2}{4}} & 0 \\ \frac{c\lambda}{1+\frac{(1-c)^2\lambda^2}{4}} & 1 \end{pmatrix}. \tag{23}$$

But the fact that $f_1(z,t) = (F_1(z),\ t)$ has finite linear distortion K_1 immediately implies F_1 is bi-Lipschitz, which contradicts the case $\sigma > 0$ in (22), (23).

Therefore, the quasiconformal mapping $f_{3,\lambda}$ admits no minimal factorizations in the linear dilatation, which proves the theorem. □

Proof of Theorem 1.2. We assume, by contradiction, that $f_{n,\lambda} = f_2 \circ f_1$ for some quasiconformal maps $f_1,\ f_2$ with inner dilatations

$$K = K_I(f_{n,\lambda}) = K_I(f_1) \cdot K_I(f_2)$$

and $K_1(f_1),\ K_I(f_2) \neq 1$.

Note that

$$H(f_{n,\lambda}) = K_I(f_{n,\lambda}) = K$$

and $1 \leq H(f) \leq K_I(f)$ for any quasiconformal mapping f (see (8) in Section 2). Then, together with (3), we now deduce that $f_{n,\lambda} = f_2 \circ f_1$ is also a minimal factorization in the linear dilatation, which contradicts Theorem 1.1.

The proof is complete. □

Proof of Theorem 1.3. Recall that $K_O(f) = K_I(f^{-1})$ for any quasiconformal mapping f.

Noting that $K_O(f_{n,\lambda}^{-1}) = K$, by using Theorem 1.2, we have the desired result. □

4. Proof of Lemma 3.1

This section is devoted to proving Lemma 3.1. Combining Proposition 2.2 with Lemma 2.1 allows us to show that, for almost every $\zeta,\ \eta \in \mathbb{R}^3$, there exist $P_1(\zeta),\ Q_1(\zeta),\ P_2(\eta),\ Q_2(\eta) \in SO(3)$ such that

$$\begin{aligned} Df_1(\zeta) &= P_1(\zeta)\cdot \operatorname{diag}\big(\mu_1(\zeta),\ \mu_2(\zeta),\ \mu_3(\zeta)\big)\cdot Q_1(\zeta),\\ Df_2(\eta) &= P_2(\eta)\cdot \operatorname{diag}\big(\gamma_1(\eta),\ \gamma_2(\eta),\ \gamma_3(\eta)\big)\cdot Q_2(\eta), \end{aligned}$$

where

$$\mu_1(\zeta) \ge \mu_2(\zeta) \ge \mu_3(\zeta) > 0,\ \gamma_1(\eta) \ge \gamma_2(\eta) \ge \gamma_3(\eta) > 0. \tag{24}$$

By setting $\eta = f_1(\zeta)$, from the Jacobian matrices

$$Df_{3,\lambda}(\zeta) = Df_2(f_1(\zeta))\cdot Df_1(\zeta),$$

it follows that

$$\begin{pmatrix} \mathrm{A}(\lambda,\ \theta')_{2\times 2} & \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{K} & & \\ & 1/\sqrt{K} & \\ & & 1/\sqrt{K} \end{pmatrix}\begin{pmatrix} B(\lambda,\ \theta)_{2\times 2} & \\ & 1 \end{pmatrix}$$
$$= P_2(\eta)\cdot\begin{pmatrix} \gamma_1(\eta) & & \\ & \gamma_2(\eta) & \\ & & \gamma_3(\eta) \end{pmatrix}\cdot Q_2(\eta)\cdot P_1(\zeta)\cdot\begin{pmatrix} \mu_1(\zeta) & & \\ & \mu_2(\zeta) & \\ & & \mu_3(\zeta) \end{pmatrix}\cdot Q_1(\zeta). \tag{25}$$

By rewriting (25), we obtain that, for almost every $\zeta \in \mathbb{R}^3$, $\eta = f_1(\zeta)$,

$$\begin{aligned} &\begin{pmatrix} \sqrt{K} & & \\ & 1/\sqrt{K} & \\ & & 1/\sqrt{K} \end{pmatrix}\\ =\ & O_1(\zeta)\cdot\begin{pmatrix} \gamma_1(\eta) & & \\ & \gamma_2(\eta) & \\ & & \gamma_3(\eta) \end{pmatrix}\cdot O_2(\zeta)\cdot\begin{pmatrix} \mu_1(\zeta) & & \\ & \mu_2(\zeta) & \\ & & \mu_3(\zeta) \end{pmatrix}\cdot O_3(\zeta), \end{aligned} \tag{26}$$

where

$$O_1(\zeta) = \begin{pmatrix} \mathrm{A}(\lambda,\ \theta')_{2\times 2} & \\ & 1 \end{pmatrix}^{-1}\cdot P_2(f_1(\zeta)) \in SO(3),$$

$O_2(\zeta) = Q_2(f_1(\zeta))\cdot P_1(\zeta) \in SO(3)$ and

$$O_3(\zeta) = Q_1(\zeta)\cdot\begin{pmatrix} B(\lambda,\ \theta)_{2\times 2} & \\ & 1 \end{pmatrix}^{-1} \in SO(3).$$

Now consider the actions of the left (right) matrix of (26) on the column vector $(1,0,0)^T$. By computing the Euclidean lengths of the resulting column vectors, we immediately obtain that

$$\sqrt{K} \le \gamma_1(f_1(\zeta)) \cdot \mu_1(\zeta), \quad a.e.\ \zeta \in \mathbb{R}^3. \tag{27}$$

Similarly, by considering the actions on the column vector $(0,0,1)^T$,

$$1/\sqrt{K} \ge \gamma_3(f_1(\zeta)) \cdot \mu_3(\zeta), \quad \text{a.e. } \zeta \in \mathbb{R}^3. \tag{28}$$

Note that f_1 is quasiconformal with linear dilatation $1 < H(f_1) = K_1 < K$ and f_2 is quasiconformal with linear dilatation $H(f_2) = K/K_1$. Using (27), (28), it follows from the definition of linear dilatation that, for a.e. $\zeta,\ \eta \in \mathbb{R}^3$,

$$K \le \frac{\mu_1(\zeta)\cdot\gamma_1(f_1(\zeta))}{\mu_3(\zeta)\cdot\gamma_3(f_1(\zeta))} \le \frac{\mu_1(\zeta)}{\mu_3(\zeta)}\cdot\frac{\gamma_1(\eta)}{\gamma_3(\eta)} \le K_1 \cdot K/K_1. \tag{29}$$

Hence all " $\le$ " or " $\ge$ "in (24), (27), (28), (29) must be " $=$ ". Therefore,

$$\frac{\gamma_1(f_1(\zeta))}{K/K_1} = \gamma_2(f_1(\zeta)) = \gamma_3(f_1(\zeta)), \ \ \frac{\mu_1(\zeta)}{K_1} = \mu_2(\zeta) = \mu_3(\zeta) \quad \text{a.e.} \tag{30}$$

Therefore, for almost every $\zeta \in U$,

$$\begin{pmatrix} K & & \\ & 1 & \\ & & 1 \end{pmatrix} = O_1(\zeta)\cdot\begin{pmatrix} K/K_1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\cdot O_2(\zeta)\cdot\begin{pmatrix} K_1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\cdot O_3(\zeta). \tag{31}$$

By considering the actions of both sides of (31) on the column vector $(1,0,0)^T$ again, and by computing the Euclidean lengths of the resulting column vectors, we deduce that

$$O_j(\zeta) = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \qquad j = 1,2,3. \tag{32}$$

Similarly, by considering the transposes of both sides of (31), we obtain

$$O_j(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}. \tag{33}$$

Therefore, combining (32) with (33) yields

$$O_j(\zeta) = \begin{pmatrix} 1 & \\ & R_j(\zeta)_{2\times 2} \end{pmatrix},$$

where $R_j(\zeta) \in SO(2), j = 1,2,3$ with $R_1(\zeta)\cdot R_2(\zeta)\cdot R_3(\zeta) = I_2$ (the 2×2 identity matrix). In particular, by setting

$$\mu_\varsigma \equiv \mu_2(\zeta) \ \text{ and } \ P(\zeta) \equiv P_1(\zeta)\cdot\begin{pmatrix} 1 & \\ & R_3(\zeta)_{2\times 2} \end{pmatrix},$$

we conclude that

$$\begin{aligned} Df_1(\zeta) &= P_1(\zeta)\cdot\begin{pmatrix} K_1\cdot\mu_2(\zeta) & & \\ & \mu_2(\zeta) & \\ & & \mu_2(\zeta)\end{pmatrix}\cdot Q_1(\zeta) \\ &= P(\zeta)\cdot\begin{pmatrix} K_1\cdot\mu_\zeta & & \\ & \mu_\zeta & \\ & & \mu_\zeta\end{pmatrix}\cdot\begin{pmatrix} B(\lambda,\,\theta)_{2\times 2} & \\ & 1\end{pmatrix}, \end{aligned}$$

as desired. □

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