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AN ERGODIC THEOREM FOR NONSINGULAR ACTIONS OF THE HEISENBERG GROUPS

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Abstract. We show that there is a sequence of subsets of each discrete Heisenberg group for which the nonsingular ergodic theorem holds. The sequence depends only on the group; it works for any of its nonsingular actions. To do this, we use a metric which was recently shown by Le Donne and Rigot to have the Besicovitch covering property and then apply an adaptation of Hochman's proof of the multiparameter nonsingular ergodic theorem.

1. Introduction

One of the fundamental problems in ergodic theory is to establish when an average of a function over an orbit in a dynamical system—the time average—agrees with its average over the whole space. Birkhoff's pointwise ergodic theorem states that this is the case when the dynamics are described by repeated applications of a transformation to a finite measure space, so long as the mass of each set is preserved. This theorem serves as a foundation for the class of pointwise ergodic theorems, each of which modifies or generalizes Birkhoff's result. The main result of this paper sits between two such generalizations.

The first extends the notion of time. In Birkhoff's theorem time is discrete, with a fixed map describing where each point in the space will be after each unit time step. In particular, if this map is invertible, it induces an action of the integers on the underlying space. One can change the notion of time by considering actions of groups other than the integers, such as the reals inducing continuous time. In this paper we start by considering the action of a countable group G which will eventually be taken to be the discrete Heisenberg group.

The second generalization weakens the assumption that the dynamics preserve the mass of all sets. Instead we assume only that the action is nonsingular, i.e., only that masses of the null sets (those without mass) are preserved. This means that sets with positive mass can have their mass changed by the dynamics but will not lose their mass entirely. The cost for this weakening is that time average must be weighted differently, in particular by Radon–Nikodým derivatives.

More precisely, we take G to be a countable group with a nonsingular and ergodic action on a standard probability space $(X, \mathcal{B}, \mu)$. Each $g \in G$ then induces a nonsingular map on X which we also denote by g. The measures μ and $\mu \circ g$ are

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equivalent, so the Radon–Nikodým derivatives

$$\omega_g = \frac{d\mu \circ g}{d\mu}$$

are well defined and strictly positive almost everywhere. For $f \in L^1$ and $g \in G$ we let $\hat{g}f(x) = f(g^{-1}x)\omega_{g^{-1}}(x)$ for each $x \in X$. Each of the maps $\hat{g}$ is an isometry of L^1, and the map $g \mapsto \hat{g}$ is a group homomorphism.

Given such an action and a sequence $e \in B_1 \subseteq B_2 \subseteq \cdots$ of finite subsets of G, which we refer to as a *summing sequence*, we say that the (*pointwise*) *ergodic theorem is satisfied for the sequence* (B_k) if for every integrable function f, we have

$$\lim_{k\to\infty} \frac{\sum_{g\in B_k} \hat{g}f}{\sum_{g\in B_k} \hat{g}1} = \int f \, d\mu$$

almost everywhere.

For example, in the case of Birkhoff's theorem $B_k = \{1, \dots, k\}$ all of the Radon–Nikodým derivatives are identically 1, reducing the average on the left-hand side to $\frac{1}{k}\sum_{i=1}^{k} f(ix)$—the unweighted average over the first k points in the forward orbit of x.

The main theorem of this paper is the following.

Theorem 1.1. *There is a metric d on the discrete Heisenberg group $\mathrm{H}^n = H_n(\mathbb{Z})$ for which the ergodic theorem is satisfied with $B_k = \{p \in \mathrm{H}^n : d(p, 0) \le k\}$.*

See section 3.1 for the precise definition of the metric d. We expect that similar techniques could be used to show that the corresponding result holds for the continuous Heisenberg group $\mathbb{H}^n$.

Background. The ergodic theorem was extended separately in both of the contexts mentioned earlier. In 1944 Hurewicz proved that the nonsingular version of Birkhoff's theorem [Hur44], that with $G = \mathbb{Z}$ and $B_k = \{1, \dots, k\}$, is satisfied so long as the action is conservative. The case of measure preserving actions of amenable groups was resolved far more recently by Lindenstrauss [Lin01], who proved that the theorem is satisfied whenever (B_k) is a tempered Følner sequence. A short inductive proof, given in [Lin01], shows that every Følner sequence has a tempered subsequence. It therefore follows that every probability measure preserving action of an amenable group has a summing sequence for which the ergodic theorem holds.

Knowing these results, it is natural to ask whether given a nonsingular and ergodic action of an amenable group there exists a summing sequence for which the ergodic theorem holds. In contrast with the case of integer actions, there is not a direct analogue to Lindenstrauss's result for nonsingular actions; we cite work providing a counterexample below. However, there have been extensions to actions of $\mathbb{Z}^n$.

In his paper [Fel07] Feldman used an elegant method to show that the ergodic theorem holds when the summing sets B_k are taken to be the balls $\{u \in \mathbb{Z}^n : \|u\| \le k\}$, where $\|\cdot\|$ is the sup-norm so long as the standard generators $e_1, \dots, e_n$ of $\mathbb{Z}^n$ act conservatively on X. Shortly after, Hochman [Hoc10] used a different approach to remove this additional assumption and allow $\|\cdot\|$ to be any norm on $\mathbb{R}^n$—under the relatively light assumption that the action is free. In recent work with Dooley [DJ16], the author showed that the balls of norms can be replaced with rectangles which are symmetric around the origin with side lengths growing at arbitrarily quick and distinct rates.

There are also examples for which the ergodic theorem fails. In [Hoc13] Hochman shows that for $G = \mathbb{Z}^\infty = \bigoplus_{n=1}^\infty \mathbb{Z}$ and any choice of summing sequence there is an infinite measure preserving action of $\mathbb{Z}^\infty$ for which the *ratio* ergodic theorem fails (in fact, the ratio is shown to diverge). By transferring to an equivalent probability measure, and recalling that under our assumptions the ratio ergodic theorem is a consequence of the ergodic theorem, it follows that the ergodic theorem fails for some action of $\mathbb{Z}^\infty$ regardless of the choice of summing sequence. In the same paper it is also shown that if G is taken to be the discrete Heisenberg group and $B_k = B^k$, where B is the collection of standard generators of G, then the ratio ergodic theorem fails in a similar way for every subsequence of (B_k).

The key obstacle to the validity of the ergodic theorem cited in [Hoc13] is the failure of the sequence (B_k) to satisfy a slight weakening of the Besicovitch covering property (BCP); see [Hoc10, Definition 1.4]. In the case of the Heisenberg group this property does not fail just for the sequence (B^k), as described above, it also fails for the integer balls of the Korányi distance and the Carnot–Carathéodory metric (see [Rig04, SW92]). These are two of the natural distances on the Heisenberg group.

However, Le Donne and Rigot [LDR17] have shown that there is a class of homogeneous metrics (homogeneity is defined in the paragraph preceding equation 3.1) on the Heisenberg group which satisfy the metric version of the BCP; see Definition 2.6. For groups equipped with right invariant metrics, which we will call *group metric spaces*, this property implies that the sequence of integer balls has a BCP as in [Hoc10, Definition 1.4]. We will recall the definition of one of the metrics considered by Le Donne and Rigot in section 3.1 and will take it to be the metric d in Theorem 1.1. Though the same techniques should work for any metric in the class identified in [LDR17], we treat just one for notational simplicity. Knowing that d satisfies the BCP allows us to employ a framework to prove Theorem 1.1 developed from the one employed by Hochman in his study of $\mathbb{Z}^n$.

The question underlying this and similar work is: given G, can we find a summing sequence (B_k) for which the nonsingular ergodic theorem holds regardless of the action? Hochman's work with $\mathbb{Z}^\infty$ means that the generality achieved by Lindenstrauss cannot be replicated for nonsingular actions, even in the context of abelian groups. The class of groups with for which the answer is yes is strictly smaller. On the other hand, it includes $\mathbb{Z}^n$ and H^n, both of which are "finite dimensional", unlike $\mathbb{Z}^\infty$. These groups have various useful geometrical properties (see section 2), many of which are inherited from either $\mathbb{R}^n$ or the continuous Heisenberg group $\mathbb{H}^n$. As such, it would be interesting to know whether a similar result may hold with $G = \mathbb{Q}^n$ or, more generally, countable abelian groups of finite rank.

Paper layout. In section 2 we identify a small collection of geometrical properties on a group metric space which suffice to prove a nonsingular ergodic theorem when (B_k) is its sequence of integer balls.

In section 3 we show that the discrete Heisenberg group satisfies each of these properties, and hence we deduce Theorem 1.1. Most of the work in this section will go into showing that the Heisenberg group equipped with the specified metric satisfies a condition based on the finite coarse dimension property used in [Hoc10].

2. Extending the method for proving nonsingular ergodic theorems

The main result in this section is Theorem 2.8, a version of the ergodic theorem for nonsingular actions of countable groups G equipped with right invariant metric d with a collection of useful geometrical properties. The more significant of these include satisfying the metric version of the BCP (see Definition 2.6), satisfying a property similar to having the finite coarse dimension defined by Hochman in [Hoc10] and the associated sequence of integer balls satisfying a doubling condition; see Definition 2.4. We define these and the other required properties throughout this section, and they are listed in the conditions of Theorem 2.8.

While the statement and proof of this theorem follow the lines of [Hoc10], the details are sufficiently different for us to give an outline of the main ideas and method of proof. We draw particular attention to the necessary modifications, and to new definitions. The full details of the proof can be found in the author's Ph.D. thesis [Jar18, Chapter 2].

In addition to [Hoc10], we refer to [Aar97, Theorem 2.2.1], [Fel07], which give elegant expositions of aspects of the method.

This method follows the standard approach; we first show that there is a dense subset of L^1 on which the ergodic theorem holds and then extend to the whole of L^1 by using a maximal inequality. The geometry plays a central role both in the proof of the existence of such a dense set and of the maximal inequality.

The summing sequence used will be the sequence of integer balls about the identity, i.e., the sets $B_k = B_k(e) = \{g \in G : d(g,e) \leq k\}$ for $k \in \mathbb{N}$. We require each of these sets to be finite.

The candidate for the dense subset of L^1 for which the ergodic theorem holds is

$$S = \operatorname{span}\{c + h - \hat{\sigma}h : c \in \mathbb{R}, \sigma \in G, h \in L^\infty\}.$$

This can be seen to be dense using the techniques used in [Aar97], which deals with the integer case.

Since the integral of any generator $c + h - \hat{\sigma}h$ of S is just c and

$$\frac{\sum_{g\in B_k} \hat{g}(c + h - \hat{\sigma}h)}{\sum_{g\in B_k} \hat{g}1} = c + \frac{\sum_{g\in B_k\setminus(\sigma B_k)} \hat{g}h - \sum_{g\in(B_k\sigma)\setminus B_k} \hat{g}h}{\sum_{g\in B_k} \hat{g}1},$$

as each $h \in L^\infty$, it is sufficient to prove (using the condition that $B_n = B_n^{-1}$ and replacing σ^{-1} by σ) that for all $\sigma \in G$

$$\text{(nsFC)} \qquad \frac{\sum_{g\in B_k\triangle\sigma B_k} \omega_g}{\sum_{g\in B_k} \omega_g} \to 0 \qquad \text{a.s.,}$$

which we will call the *nonsingular Følner condition.* Note that this is a condition on the action of G on (X,μ). In the measure preserving case all of the derivatives are equal to 1, and nsFC reduces down the regular Følner condition, which is a property of the group.

The nonsingular Følner condition largely follows from the metric structure we impose on the space, so let us take (M,d) to be a metric space and make some definitions.

For $r > 0$ and $x \in M$ we let $B_r(x) = \{y \in X : d(x,y) \leq r\}$, the (closed) ball of radius r about x, and we assume that each such ball carries the information about its center and radius along with it. Letting $\mathcal{V}$ be a collection of balls in M, we let $\operatorname{rmax}\mathcal{V}$ and $\operatorname{rmin}\mathcal{V}$ denote the maximum and minimum radii, respectively, of the balls in $\mathcal{V}$. We say that the collection $\mathcal{V}$ is *evenly spaced* (or well separated, in the terminology of [Hoc10]) if the Hausdorff distance between each pair of balls in $\mathcal{V}$ is at least $\operatorname{rmin}\mathcal{V}$. Given a finite set $E \subset M$, a *carpet* over E is a collection of balls $\mathcal{U} = \{B_{r(x)}(x) : x \in E\}$ centered in E. A *stack* (of height p) over E is a sequence of carpets $\mathcal{U}_1, \ldots, \mathcal{U}_p$ over E.

Definition 2.1. (M, d) is *well spaced* if there exists a $\chi \in \mathbb{N}$ such that for every finite set $E \subset M$ and every carpet $\mathcal{U}$ over E there is a subcollection $\mathcal{V}$ of $\mathcal{U}$ which covers E and which can be partitioned into χ evenly spaced subcollections.

We will take this as one of our assumptions on our group metric space, as this property is a cornerstone of Hochman's method.

Now we define a concept of boundary which is in general distinct from the one given in [Hoc10] but which coincides in the context of that paper. We use this definition because the naive extension of Hochman's definition does not behave as intended in other contexts, such as in [DJ16]. Let $(\tilde{M}, \tilde{d})$ be a metric space such that $M \subseteq \tilde{M}$ and $d = \tilde{d}|_M$. We say that $(\tilde{M}, \tilde{d})$ *extends* (M, d) and that (M, d) is a *restriction* of $(\tilde{M}, \tilde{d})$ when this occurs. For any $t \geq 0$ we define the *t-boundary* $\partial_t B_r(x)$ (with respect to $\tilde{M}$) of a ball $B_r(x)$ in M by

$$\partial_t B_r(x) = \{y \in M : \tilde{d}(y, \tilde{S}_r(x)) \leq t\},$$

where $\tilde{B}_r(x) = \{y \in \tilde{M} : \tilde{d}(x, y) \leq r\}$ and $\tilde{S}_r(x) = \{y \in \tilde{M} : \tilde{d}(x, y) = r\}$. We assume that the sphere $\tilde{S}_r(x)$ is nonempty, as this will be the case when we consider the continuous Heisenberg group. In general, the intention is that $\tilde{M}$ will be to M what $\mathbb{R}^n$ is to $\mathbb{Z}^n$. We also assume that spheres $\tilde{S}_r(x)$ retain the information about their center and radius.

Given a collection $\mathcal{V}$ of balls in M with extension $\tilde{M}$, we let $\partial\mathcal{V} = \{\tilde{S}_r(x) : B_r(x) \in \mathcal{V}\}$, a collection of spheres in $\tilde{M}$. We also call the collection $\partial\mathcal{V}$ evenly spaced if the distance between each of the spheres in $\partial\mathcal{V}$ is at least $\operatorname{rmin}\mathcal{V}$. The distinction here to the case of balls is that some spheres in $\partial\mathcal{V}$ may lie inside the balls corresponding to distinct spheres in $\mathcal{V}$.

We say that a metric space (Y, ρ) is *voidless* if for all $x \in Y$ and $r > 0$, every closed ball $B \subset Y$ such that $B \cap \{y : \rho(x, y) < r\} \neq \emptyset$ and $B \cap \{y : \rho(x, y) > r\} \neq \emptyset$ satisfies $B \cap \{y : \rho(x, y) = r\} \neq \emptyset$. In particular, note that if a metric space is such that every closed ball is path connected, then the intermediate value theorem ensures that it is voidless.

Lemma 2.2. *Let (G, d) be a group metric space, suppose that there is a voidless right invariant group metric space $(\tilde{G}, \tilde{d})$ for which G is a subgroup of $\tilde{G}$, and suppose that $(\tilde{G}, \tilde{d})$ extends (G, d). Then for any closed ball $B \subset G$ and $\sigma \in G$ we have $B \triangle \sigma B \subseteq \partial_t B$, where $t = d(\sigma, 0)$.*

Proof. Let $g \in G$. Then $d(\sigma^{-1}g, g) = d(\sigma^{-1}, 0) = d(0, \sigma)$, i.e., $\sigma^{-1}g \in B_t(g)$. Suppose that $g \notin \partial_t B$. Then $\tilde{d}(g, \partial\tilde{B}) > t$ (here $\tilde{B}$ denotes the ball in $\tilde{G}$ with the same center and radius as B). Hence, $\tilde{B}_t(g)$ does not intersect $\partial\tilde{B}$. Since $\tilde{G}$ is voidless, it follows that either $B_t(g) \subseteq B$ or $B_t(g) \subseteq B^c$. Therefore, if $g \in B$, then $\sigma^{-1}g \in B$, and if $g \in B^c$, then $\sigma^{-1}g \in B^c$, so either $g \in B \cap \sigma B$ or $g \notin B \cup \sigma B$; i.e., $g \notin \partial_t B$ implies that $g \notin B \triangle \sigma B$. ∎

So, under the conditions of the lemma to show that nsFC holds, it is enough to prove that given σ,

$$\frac{\sum_{g\in\partial_t B_k}\omega_g}{\sum_{g\in B_k}\omega_g}\to 0 \qquad \text{a.s.},$$

with $t = d(\sigma, 0)$. This can be proved under the assumptions of Theorem 2.5 by using essentially the same method as used by Hochman in [Hoc10], but with a few changes which give a more general result. Before describing these changes, we give a further definition.

For ease, when $(\tilde{M}, \tilde{d})$ extends (M, d) and we refer to points or sets being a given distance apart, this means "in the metric $\tilde{d}$".

Definition 2.3. Let (M, d) have extension $(\tilde{M}, \tilde{d})$. We say that (M, d) has *finite intersection dimension* (with respect to $\tilde{M}$) if there is a positive integer κ and an $R > 1$ such that given

(a) $t(1), \dots, t(\kappa) \geq 1$,
(b) $r(1), \dots, r(\kappa)$ such that each $r(i) \geq t(1), \dots, t(i)R$, and
(c) $x_1, \dots, x_\kappa \in M$ such that $x_i \in \bigcap_{j<i} \partial_{t(j)} B_{r(j)}(x_j)$ for all $i \leq \kappa$,

then $\bigcap_{i=1}^{\kappa} \partial_{t(i)} B_{r(i)}(x_i) = \emptyset$. In this case, we say that (M, d) has *intersection dimension κ at scale R.*

Note that if $(\tilde{M}, \tilde{d})$ has intersection dimension κ at scale R with respect to $(\tilde{M}, \tilde{d})$, then so too do all of its restrictions.

It is important to note here that the intersection dimension of a space is a minor reformulation of the coarse dimension defined by Hochman in [Hoc10], and it uses a different notion of boundary. The two quantities are in fact the same in that paper's setting. The reason for using the name "intersection dimension" is simply to avoid potential confusion with another quantity called the coarse dimension, used in coarse geometry, as it is not clear there is a connection between the two.

Our claim that

$$\frac{\sum_{g\in\partial_t B_k}\omega_g}{\sum_{g\in B_k}\omega_g}\to 0 \qquad \text{a.s.}$$

for any $t > 0$ can be proved essentially as in [Hoc10], but with a few changes. First one needs to check that the change to the definition of the boundary does not cause any issues, in particular to [Hoc10, technical theorem 4.4] and the results leading to and following from it. The key observation allowing one to change the definition of boundary here (without fundamentally changing the proofs) is that the only property of thickened boundaries used is that they contain all of the points within a given distance of the sphere being thickened, which is the definition we have taken. Note that some of the results in Hochman's paper are proved for $t = 1$ since in the setting of that paper the metric can be rescaled; however, the same argument will apply for general t with this paper's notion of boundary.

In order to allow a wider variety of summing sequences (B_k), we observe that each occurrence of the ball $B_{2k} = B_{2k}(0)$ (in the notation of that paper) can be replaced with the set $B_k^2 = B_k B_k$ (as in this paper) so that we can apply the following doubling property. Note that this is weaker than the equivalent used in [Hoc10] whenever one works with a group metric space with a (right or left) invariant metric.

Definition 2.4. Given a sequence $e \in B_1 \subset B_2 \subset \cdots$ of finite subsets of a countable group G, we say that it has the *multiplicative doubling property* (MDP) if there exist constants $D > 0$ and $K \in \mathbb{N}$ such that $|B_k B_k| \leq D|B_k|$ for all $k \geq K$.

Taken together, these changes allow one to prove the following.

Theorem 2.5. *Let (G, d) be a countable group metric space which acts nonsingularly on the probability space (X, μ), let it be well spaced, and let it be such that the sequence of integer balls (B_k) has the MDP. Suppose that there is a right invariant group metric space $(\tilde{G}, \tilde{d})$ for which $G \leq \tilde{G}$ and $(\tilde{G}, \tilde{d})$ extends (G, d). Furthermore, suppose that (G, d) has finite intersection dimension with respect to $(\tilde{G}, \tilde{d})$. Then for all $t > 0$*

$$\lim_{k\to\infty} \frac{\sum_{g\in\partial_t B_k} \omega_g}{\sum_{g\in B_k} \omega_g} = 0 \qquad a.s.$$

From this point the remaining steps in the proof of the ergodic theorem are fairly standard.

The second major ingredient in the proof, in addition to the suitable dense subset, is the maximal inequality. A geometrical assumption essential for the maximal inequality (see [Hoc10]) is the BCP.

Definition 2.6. A metric space (M, d) has the BCP if there is a constant $C > 0$ such that for any finite set $E \subset X$ and any carpet $\mathcal{U}$ over E we have a subcollection $\mathcal{V} \subseteq \mathcal{U}$ for which

$$\mathbf{1}_E \leq \sum_{U\in\mathcal{V}} \mathbf{1}_U \leq C.$$

In this situation we say that (M, d) has the BCP with constant C. A carpet satisfying the second inequality above is said to have multiplicity C.

Using the BCP, the maximal inequality can be proved using essentially the same techniques as employed by Feldman in [Fel07]. The only difference is to make changes as discussed before Theorem 2.5 in order to apply the multiplicative doubling condition.

Theorem 2.7 (The maximal inequality)**.** *Let G be a countable group, let (G, d) be a right invariant metric space which has the BCP with constant C, and suppose that the sequence of integer balls (B_k) has the MDP with constant D. Then for any $f \in L^1$ and $\epsilon > 0$*

$$\mu\left(\sup_{k\geq 1} \left| \frac{\sum_{g\in B_k} \hat{g} f}{\sum_{g\in B_k} \hat{g} 1} \right| > \epsilon \right) \leq \frac{CD}{\epsilon} \|f\|_1.$$

The following general ergodic theorem, which we will apply to the Heisenberg group later, can be proved by combining the conditions from the above results and combining them using standard techniques, as described (for example) in [Hoc10].

Theorem 2.8 (The ergodic theorem)**.** *Let G be a countable group, equipped with a metric d, which has an ergodic nonsingular action on the standard probability space (X, μ). Suppose that the following apply:*

(i) *(G, d) is well spaced.*
(ii) *The sequence of integer balls (B_k) has the MDP.*
(iii) *There is a voidless (right invariant) group metric space $(\tilde{G}, \tilde{d})$ for which G is a subgroup of $\tilde{G}$ and $(\tilde{G}, \tilde{d})$ extends (G, d).*

(iv) (G, d) *has finite intersection dimension with respect to* $(\tilde{G}, \tilde{d})$.
(v) (G, d) *has the BCP.*

Then for every $f \in L^1(\mu)$

$$\lim_{n\to\infty} \frac{\sum_{g\in B_k} \hat{g}f}{\sum_{g\in B_k} \hat{g}1} = \int f\, d\mu \qquad \textit{almost everywhere.}$$

There are indications that these conditions could be simplified, especially given additional assumptions on the group. As remarked in [Hoc10], which deals with the case where $G = \mathbb{Z}^d$ equipped with a norm metric induced from $\mathbb{R}^d$, similar approaches can be used to show that conditions (iv) and (v) are satisfied. Additionally, in both that case and the case of the Heisenberg group (which we study in the following section) the proof that condition (i) is satisfied makes use of the fact the space satisfies (v). However, in general, it is not known whether these properties are related.

3. An ergodic theorem for discrete Heisenberg actions

In this section we show that the ergodic theorem holds for the discrete Heisenberg group by checking that it satisfies the conditions of Theorem 2.8 when equipped with the metric used by Le Donne and Rigot in [LDR17]. In [LDR17, Theorem 1.14] they show that the metric d, defined below, has the BCP. Therefore, it is sufficient for us to address properties (i)–(iv) in the theorem.

3.1. **Setup.** We shall define the nth continuous Heisenberg group, $\mathbb{H}^n$, as follows. As a set, take $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ and equip it with the multiplication given by

$$(z, \tau) \cdot (w, \sigma) = \left(z + w, \tau + \sigma + \frac{1}{2}\mathrm{Im}\,\langle z, w\rangle\right),$$

where $z, w \in \mathbb{C}^n$, $\tau, \sigma \in \mathbb{R}$, and the inner product is the standard one on $\mathbb{C}^n$, given by $\langle z, w\rangle = \sum_{j=1}^n \overline{z_j} w_j$. This is essentially the same realization as that used by Le Donne and Rigot in [LDR17] except that we are using complex coordinates, and it is the exponential parametrization of the Heisenberg group. Note that $(z, \tau)^{-1} = (-z, -\tau)$ in this realization.

The nth discrete Heisenberg group, H^n, is the discrete subgroup generated by the elements of the form $(e_j, 0)$ or $(ie_j, 0)$, where e_j is a standard basis vector of $\mathbb{R}^n$. As a set,

$$\mathrm{H}^n = \{(z, \tau) \in \mathbb{H}^n : z \in \mathbb{Z}^n + i\mathbb{Z}^n, \tau \in \tfrac{1}{2}\langle \mathrm{Re}\, z, \mathrm{Im}\, z\rangle + \mathbb{Z}\}.$$

We take $G = \mathrm{H}^n$ and $\tilde{G} = \mathbb{H}^n$ in Theorem 2.8.

For each $\lambda > 0$ there is a dilation map $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ given by

$$\delta_\lambda(z, \tau) = (\lambda z, \lambda^2 \tau).$$

Each δ_λ is a group automorphism of $\mathbb{H}^n$.

To describe the range of the sums in the ergodic theorem, we use the balls of the metric given by

$$d\,(p, q) = \inf\left\{r > 0 : \delta_{1/r}(pq^{-1}) \in B_{eucl}\right\},$$

where B_{eucl} denotes the closed euclidean unit ball on $\mathbb{C}^n \times \mathbb{R}$. This is the right invariant version of the metric given in [LDR17] with $\alpha = 1$. It is *one-homogeneous* with respect to the dilation; i.e., for all $\lambda > 0$ and $p, q \in \mathbb{H}^n$ we have $d(\delta_\lambda p, \delta_\lambda q) =$

$\lambda\, d(p,q)$. By considering the case $q=0$ and using right invariance, it is not difficult to show that for $p=(z,\tau)$ and $q=(w,\sigma)$

$$d(p,q) \le r \iff \frac{\|z-w\|^2}{r^2} + \frac{\left(\tau-\sigma-\frac{1}{2}\mathrm{Im}\,\langle z,w\rangle\right)^2}{r^4} \le 1 \tag{3.1}$$

and

$$d(p,q) = r \iff \frac{\|z-w\|^2}{r^2} + \frac{\left(\tau-\sigma-\frac{1}{2}\mathrm{Im}\,\langle z,w\rangle\right)^2}{r^4} = 1, \tag{3.2}$$

where $\|\cdot\|$ is the euclidean norm on $\mathbb{C}^n$. In particular, taking $r=1$ and $q=0$ shows that the unit sphere of d is exactly the euclidean unit sphere, and the result is similar for the unit ball. This property is key to many of the arguments which follow. It also follows from (3.2) that

$$d(p,q) = \frac{1}{\sqrt{2}}\left(\|z-w\|^2 + \sqrt{\|z-w\|^4 + 4\left(\tau-\sigma-\frac{1}{2}\mathrm{Im}\,\langle z,w\rangle\right)^2}\right)^{\frac{1}{2}}. \tag{3.3}$$

This explicit expression can be used show that d is in fact a metric. In addition, as stated in [LDR17], d induces the euclidean topology. d therefore defines a (right) *homogeneous distance* on $\mathbb{H}^n$; i.e., it induces the euclidean topology and is right invariant and one-homogeneous for the dilation.

Remark 3.1 (On notation). In this section we use d to denote both the metric on $\mathbb{H}^n$ and its restriction to H^n. The notation $B_r(p)$ is exclusively used for balls in $\mathbb{H}^n$. This differs from the notation of the previous section, where the standard notation for balls is used for balls in G, but here they are balls in $\mathbb{H}^n$, which plays the role of $\tilde{G}$.

Observe that we can use the dilations and right invariance to describe any ball in $\mathbb{H}^n$. Stated explicitly: for each $r>0$ and $p\in\mathbb{H}^n$ the closed ball $B_r(p) = \delta_r(B_{eucl})\cdot p$. Since the dilation is a linear map and right multiplication by p is an affine map, it follows that each ball is a (filled) ellipsoid and in particular is convex, in the euclidean sense. From equation 3.1 each of the balls $B_r(0)$ is a (filled) ellipsoid centered on the origin with principle semiaxes along each of the real and imaginary parts of z having euclidean length r, and along τ having euclidean length r^2. The ball $B_r(p)$ is then given by translating $B_r(0)$ by p and adding a skew to the real coordinate. More precisely, if $p=(w,\sigma)$ and $(z,\tau)\in B_r(0)$, then the new real coordinate is

$$\tau+\sigma+\frac{1}{2}\mathrm{Im}\,\langle z,w\rangle,$$

and the skew is determined by

$$\mathrm{Im}\,\langle z,w\rangle = \langle \mathrm{Re}\,z, \mathrm{Im}\,w\rangle - \langle \mathrm{Im}\,z, \mathrm{Re}\,w\rangle.$$

Therefore, for example, by moving $\mathrm{Re}\,z$ in the direction of $\mathrm{Im}\,w$, we see the real coordinate increase.

It will be useful for us to note the following. Let R_θ be the $n\times n$ complex diagonal matrix with $R_\theta(j,j)=e^{i\theta_j}$, where $\theta=(\theta_j)_{j=1}^n\in\mathbb{R}^n$. Then the maps

$$(z,\tau)\mapsto(\overline{z},-\tau) \quad\text{and}\quad (z,\tau)\mapsto(R_\theta z,\tau) \tag{3.4}$$

are isometries of d.

We are now ready to start checking that the conditions of Theorem 2.8 are satisfied, to which end the following lemma will be useful.

Lemma 3.2. *Let* $(\mathbb{H}^n, d)$ *be as above, and let* $\xi > 0$. *There exists an* $N = N(\xi) \in \mathbb{N}$ *such that there are* N *open balls of radius* $\xi/2$ *centered in* $B_1(0)$ *whose union covers* $B_1(0)$. *Consequently, we have the following:*

(i) *If* $p_1, \ldots, p_l \in B_1(0)$ *and for all* $i \neq j$ $d(p_i, p_j) > \xi$, *then* $l \leq N$.
(ii) *If* $p_1, \ldots, p_l \in B_1(0)$ *with* $n \geq kN$ *for some* $k \in \mathbb{N}$, *then there is a subset* $I \subset \{1, \ldots, n\}$ *of size at least* k *with* $d(p_i, p_j) < \xi$ *for all* $i, j \in I$.

Proof. Since the metric d induces the euclidean topology, the closed unit ball $B_1(0) = B_{eucl}$ is compact, and the existence of such an N follows from this compactness. Part (i) is due to the fact that if two points lie in the same ball in the cover, then they are $< \xi$ apart. Part (ii) follows from the pigeonhole principle. $\square$

In particular, if we let $r > 0$, $p \in \mathbb{H}^n$, and $\xi = 1$ in Lemma 3.2, then $B_r(p) = \delta_r(B_{eucl}) \cdot p$ can be covered by $N(1)$ balls of radius $\frac{r}{2}$ (simply dilate and translate those used to cover B_{eucl}). This means exactly that $(\mathbb{H}^n, d)$ has the *metric* doubling property; i.e., there exists an $N \in \mathbb{N}$ such that any ball of radius r in $(\mathbb{H}^n, d)$ can be covered by N balls of radius $\frac{r}{2}$.

We can now show that property (ii) of Theorem 2.8 holds.

Corollary 3.3. *The sequence* $B_m = B_m(0) \cap \mathrm{H}^n$ *has the MDP.*

Proof. Observe that if $p = (z, \tau)$ and $q = (w, \sigma)$ are elements of H^n, then

$$d(p,q) \geq \max\left(\|z-w\|, \left| \tau - \sigma - \frac{1}{2}\mathrm{Im}\,\langle z, w\rangle \right| \right),$$

and if $p \neq q$, then either $z \neq w$, in which case $\|z - w\| \geq 1$, or $z = w$ and

$$\left| \tau - \sigma - \frac{1}{2}\mathrm{Im}\,\langle z, w\rangle \right| = |\tau - \sigma| \geq 1,$$

as $\tau, \sigma \in \frac{1}{2}\langle \mathrm{Re}\, z, \mathrm{Im}\, z\rangle + \mathbb{Z}$ and τ and σ are not equal. It follows that

$$\inf\{d(p,q) : p, q \in \mathrm{H}^n, p \neq q\} = 1$$

by considering an obvious choice of p and q. With this in mind, we fix $0 < r < \frac{1}{2}$.

Note that if $p = (z, \tau) \in \mathbb{H}^n$, we can choose $q = (w, \sigma) \in \mathrm{H}^n$ (by choosing first w, then σ) such that $|z_i - w_i| \leq \frac{1}{\sqrt{2}}$ for all $1 \leq i \leq n$ and $|\tau - \sigma - \frac{1}{2}\mathrm{Im}\,\langle z, w\rangle| \leq \frac{1}{2}$. Therefore,

$$s = \sup\{d(p, \mathrm{H}^n) : p \in \mathbb{H}^n\} \leq \frac{1}{2}\sqrt{n + \sqrt{n^2+4}}.$$

Let ν be a right invariant Haar measure on $\mathbb{H}^n$, and let $\xi = 2\frac{m-s}{2m+r}$, where $m \in \mathbb{N}$ is taken sufficiently large to ensure $\xi \geq \frac{2}{3}$ and r is as defined just above. By dilating $(2m+r)^{-1}$, then applying Lemma 3.2, and then dilating by $2m+r$, it follows that for some collection $p_1, p_2, \ldots, p_{N(\xi)} \in B_1(0)$ we have

$$B_{2m+r}(0) = \delta_{2m+r}(B_1(0)) \subseteq \bigcup_{i=1}^{N(\xi)} \delta_{2m+r}\left(B_{\xi/2}(p_i)\right) \subseteq \bigcup_{i=1}^{N(\xi)} B_{m-s}(\delta_{2m+r}(p_i)).$$

Hence, with $N = N(2/3)$, by the translation invariance of Haar measure, the fact that we took $r < \frac{1}{2} = \frac{1}{2} \inf \{d(p,q) : p, q \in \mathrm{H}^n, p \neq q\}$ and noting that $N(\xi) \leq N$,

$$\begin{aligned}|B_m^2|\nu(B_r(0)) &= \nu\left(\bigcup_{p\in B_m^2} B_r(p)\right)\\ &\leq \nu\left(B_{2m+r}(0)\right)\\ &\leq N(\xi)\,\nu(B_{m-s}(0))\\ &\leq N\nu\left(\bigcup_{p\in B_m} B_s(p)\right)\\ &\leq N\nu(B_s(0))|B_m|.\end{aligned}$$

The result follows since balls with a strictly positive radius have a positive Haar measure. □

Thus property (ii) of Theorem 2.8 is satisfied. Moreover, since d is right invariant and all the balls are euclidean convex (and so are path connected) property (iii), that the space is voidless also holds. It has already been mentioned that the central result of [LDR17] is that d satisfies the BCP on $\mathbb{H}^n$ and hence on H^n, which covers property (v). We will also use this lemma in the proof of (iv).

The remaining properties are (i), that (H^n, d) is well spaced, and (iv), that (H^n, d) has finite intersection dimension with respect to $(\mathbb{H}^n, d)$. These require a bit more work and are tackled in the next sections. From previous comments it is sufficient to show that $(\mathbb{H}^n, d)$ is well spaced and has finite intersection dimension with respect to itself.

3.2. **Intersection dimension: The separation lemmas.** We start with the intersection dimension. Recall that in order to prove that the intersection dimension is κ, we must show that given a sequence of points $p_1, \ldots, p_m$ with $m \geq \kappa$ and thickened spheres about those points, with some conditions on the thickenings and radii, the intersection of these thickened spheres is empty. We will prove this in two stages. First we show that, by increasing m, we can find a subsequence of arbitrary length with an additional property. We can then assume, without loss of generality, that this subsequence is the original sequence. We repeat this line of argument a few times, with the sequence gaining a new property with each iteration. The lemmas in this section will be used to impose these extra properties on the sequence. In the second stage we will use these to show that the resulting sequence of thickened spheres will have an empty intersection if it is sufficiently long.

Given $p \in \mathbb{H}^n \setminus \{0\}$, let $\hat{p}$ be its unique dilate on the unit sphere, i.e., $\hat{p} = \delta_{1/\lambda} p$, where $\lambda = d(p, 0) > 0$. We will call $\hat{p}$ the *projection* (of p) onto the unit sphere.

The first lemma, below, will be used to show that if radii of the earlier spheres are not too large compared to later ones and 0 is in their intersection, then the projections of their centers are separated by a distance which is at least some fixed constant (depending on the degree to which the radii are allowed to differ). This will allow us to assume that each radius is rather small compared to those preceding it.

Lemma 3.4 (Large scale separation)**.** *Let $\epsilon \in (0,1)$, let $t, \tilde{t} \geq 1$, and let $r, \tilde{r} > 0$ be such that $\tilde{r} \geq \epsilon r$. Suppose that $p, q \in \mathbb{H}^n \setminus \{0\}$ with $0 \in \partial_t B_r(p) \cap \partial_{\tilde{t}} B_{\tilde{r}}(q)$ and*

$q \in \partial_t B_r(p)$. There exists an $\bar{R} > 0$ depending only on ϵ such that if $R > \bar{R}$ and $r \geq \tilde{r} \geq t\tilde{t}R$, then

$$d(\hat{p}, \hat{q}) \geq \frac{1}{2}\left(1 - \sqrt{1 - \frac{\epsilon^2}{4}}\right) > 0.$$

Proof. Since $0 \in \partial_t B_r(p) \cap \partial_{\tilde{t}} B_{\tilde{r}}(q)$ and $q \in \partial_t B_r(p)$, the triangle inequality ensures that $d(p,q) = r + s'$ and $d(p,0) = r + s$ for some s, s' such that $|s|, |s'| \leq t$, and $d(q,0) = \tilde{r} + \tilde{s}$ for some $\tilde{s}$ with $|\tilde{s}| \leq \tilde{t}$. Therefore,

$$\frac{r+s'}{r+s} = d(\hat{p}, \delta_\lambda \hat{q}) \leq d(\hat{p}, \hat{q}) + d(\hat{q}, \delta_\lambda \hat{q}), \tag{3.5}$$

where $\lambda = \frac{\tilde{r}+\tilde{s}}{r+s}$. Note that since $r \geq t\tilde{t}R$, we have $t/r \leq R^{-1}$ and similarly, with t replaced by $\tilde{t}$, it follows that from which it follows that $|s|/r \leq R^{-1}$ and similarly, with s replaced by $\tilde{s}$. From our assumptions $\epsilon \leq \frac{\tilde{r}}{r} \leq 1$. We can now bound

$$\lambda = \frac{\tilde{r}+\tilde{s}}{r+s} = \frac{\tilde{r}/r + \tilde{s}/r}{1 + s/r}$$

above and below by simply using the upper and lower bounds on the sizes of $\tilde{r}/r$,$|s|/r$ and $|\tilde{s}|/r$ term by term. We see that

$$\frac{\epsilon - R^{-1}}{1 + R^{-1}} \leq \lambda \leq \frac{1 + R^{-1}}{1 - R^{-1}}, \tag{3.6}$$

and hence that for R sufficiently large, depending on ϵ, $\lambda \geq \frac{\epsilon}{2}$. Since $\lambda \geq 0$, we have $|1-\lambda|^2 \leq |1-\lambda^2|$, and it follows from the explicit form for the metric (see (3.3)) that

$$d(\hat{q}, \delta_\lambda \hat{q}) \leq \sqrt{|1-\lambda^2|}\, d(0, \hat{q}) = \sqrt{|1-\lambda^2|}.$$

Therefore, from (3.5) we have

$$d(\hat{p}, \hat{q}) \geq 1 + \frac{s'/r - s/r}{1 + s/r} - \sqrt{|1-\lambda^2|} \geq 1 - \frac{2R^{-1}}{1+R^{-1}} - \sqrt{|1-\lambda^2|},$$

and if $\lambda \leq 1$, then by the lower bound in (3.6) for R sufficiently large

$$d(\hat{p}, \hat{q}) \geq 1 - \frac{2R^{-1}}{1+R^{-1}} - \sqrt{1 - \frac{\epsilon^2}{4}} \geq \frac{1}{2}\left(1 - \sqrt{1 - \frac{\epsilon^2}{4}}\right).$$

Otherwise if $\lambda > 1$, then by the upper bound in (3.6) we have

$$|1-\lambda^2| = \lambda^2 - 1 \leq \left(\frac{1+R^{-1}}{1-R^{-1}} - 1\right)\left(\frac{1+R^{-1}}{1-R^{-1}} + 1\right) = \frac{4R^{-1}}{(1-R^{-1})^2},$$

and hence, again for R large enough,

$$d(\hat{p}, \hat{q}) \geq 1 - \frac{2R^{-1}}{1+R^{-1}} - \sqrt{\frac{4R^{-1}}{(1-R^{-1})^2}} \geq \frac{1}{2}\left(1 - \sqrt{1 - \frac{\epsilon^2}{4}}\right). \qquad \square$$

For the purposes of the remainder of this section it is useful to introduce a coordinate system on $\mathbb{H}^n$ which exploits the dilations and the fact that the unit sphere of d is the euclidean unit sphere. It is here that we are directly using properties of $(\mathbb{H}^n, d)$.

Given $p \in \mathbb{H}^n \setminus \{0\}$, let $\lambda_p = d(p, 0) > 0$. Then $\hat{p} = \delta_{1/\lambda_p} p = (z_p, \tau_p)$ for some unique $z_p \in \mathbb{C}^n$ and $\tau_p \in \mathbb{R}$ with $\|z_p\|^2 + \tau_p^2 = 1$. In addition, using complex coordinates, we have $\varrho(p) = (\varrho_i(p))_{j=1}^n$ (with each $\varrho_i(p) \geq 0$) and $\phi(p) = (\phi_j(p))_{j=1}^n \in (-\pi, \pi]^n$ such that $z_p = (\varrho_j(p) \exp[i\phi_j(p)])_{j=1}^n$. Given also $q \in \mathbb{H}^n \setminus \{0\}$ for each $1 \leq j \leq n$, let $\phi_j(p, q) \in [0, \pi)$ denote the magnitude of the angle between $\exp[i\phi_p(j)]$ and $\exp[i\phi_q(j)]$ in $\mathbb{C}$.

By applying Lemma 3.2, we will be able to assume that $\hat{p}_1, \ldots, \hat{p}_m$ are close on the unit sphere, and Lemma 3.4 will then allow us to assume that the radius of each sphere is small compared to the previous one. Then when we will use the small scale separation lemmas, which we prove below, to narrow down the possible positions of $\hat{p}_1, \ldots, \hat{p}_m$ relative to one another.

Lemma 3.5 (Small scale separation 1). *Let $\epsilon > 0$, let $t, \tilde{t} \geq 1$, and let $r, \tilde{r} > 0$ be such that $\tilde{r} \leq \epsilon r$. Suppose that $p, q \in \mathbb{H}^n \setminus \{0\}$ with $0 \in \partial_t B_r(p) \cap \partial_{\tilde{t}} B_{\tilde{r}}(q)$ and $q \in \partial_t B_r(p)$. For all $\bar{\tau} \in (0, 1)$ there exist $\bar{\varrho}, \bar{R}, \bar{\phi}, \bar{\epsilon} > 0$ depending only on $\bar{\tau}$ such that if $R > \max(1, \bar{R})$, $r \geq \tilde{r} \geq t\tilde{t}R$, $\epsilon < \bar{\epsilon}$, $|\tau_p| \leq \bar{\tau}$, and $\max_{1 \leq i \leq n} \phi_i(p, q) < \bar{\phi}$, then $d(\hat{p}, \hat{q}) > \bar{\varrho}$.*

Remark 3.6. The assumption distinguishing the small separation lemmas from the large separation lemma is the relative size of the radii of the thickened spheres. In the large case we have $\tilde{r} \geq \epsilon r$, and in the small case $\tilde{r} \leq \epsilon r$.

The crucial difference between this result and the second small scale separation lemma, which follows, is the condition that τ_p is bounded away from ± 1. This condition ensures that the points 0 and q are not too close to the "poles" of $B_r(p)$, i.e., where $\|z_p\|$ then becomes negligible. This will allow us to control the size of z_q and hence ensure that τ_q is large enough for $\hat{p}$ and $\hat{q}$ to be separated by an appropriate distance $\bar{\varrho}$.

Proof. Using the isometries of d (see (3.4)), we may assume that $\tau_p \geq 0$ and $\phi(p) = 0$ without loss of generality. Setting $\phi = \phi(q)$, we therefore have

$$\max_{1 \leq i \leq n} |\phi_i| = \max_{1 \leq i \leq n} \phi_i(p, q) < \bar{\phi},$$

and of course $z_p = \operatorname{Re} z_p$.

Our assumption on the positions of p and q relative to 0 mean that we can write

$$p = \begin{pmatrix} (r+s)z_p \\ (r+s)^2 \tau_p \end{pmatrix}$$

and

$$q = \begin{pmatrix} (\tilde{r}+\tilde{s})z_q \\ (\tilde{r}+\tilde{s})^2 \tau_q \end{pmatrix}$$

for some $s, \tilde{s}$ with $|s| \leq t$ and $|\tilde{s}| \leq \tilde{t}$. Let $a = \frac{r+s}{r+s'}$, and let $b = \frac{\tilde{r}+\tilde{s}}{r+s'}$. As $d(p, q) = r + s'$, for some $|s'| \leq t$, using equation (3.2) with radius $r + s'$ and p and q as above, we know that

$$\begin{aligned} 1 &= \|az_p - bz_q\|^2 + \left(a^2\tau_p - b^2\tau_q - \frac{1}{2}ab \operatorname{Im}\langle z_p, z_q\rangle\right)^2 \\ &= a^2 \|z_p\|^2 + b^2 \|z_q\|^2 - 2ab \operatorname{Re}\langle z_p, z_q\rangle - 2a^2b^2\tau_p\tau_q \\ &\qquad + a^4\tau_p^2 + b^4\tau_q^2 + \frac{1}{4}(ab \operatorname{Im}\langle z_p, z_q\rangle)^2 - (a^2\tau_p - b^2\tau_q)\, ab \operatorname{Im}\langle z_p, z_q\rangle, \end{aligned} \tag{†}$$

where for the second equality we have used the linearity properties of the inner product to expand the first term and have simply multiplied out the square for the second term. Observe that

$$a-1=\frac{r+s}{r+s'}-1=\frac{\tilde{r}}{r}\frac{s/\tilde{r}-s'/\tilde{r}}{1+s'/r}$$

and

$$b-\frac{\tilde{r}}{r}=\frac{\tilde{r}+\tilde{s}}{r+s'}-\frac{\tilde{r}}{r}=\frac{\tilde{r}}{r}\frac{\tilde{s}/\tilde{r}-s'/r}{1+s'/r},$$

and since $r\geq\tilde{r}\geq t\tilde{t}R$, the rightmost fraction in each of these equalities is $O(R^{-1})$, independent of all other variables, as $R\to\infty$. This means that $a=1+O(\frac{\tilde{r}}{r}R^{-1})$ and $b=\frac{\tilde{r}}{r}+O(\frac{\tilde{r}}{r}R^{-1})$, so by recalling that $\|z_p\|^2+\tau_p^2=1$, and doing similarly with q, we can reduce (†) to

$$1=\|z_p\|^2-2\frac{\tilde{r}}{r}\mathrm{Re}\,\langle z_p,z_q\rangle+\tau_p^2-\tau_p\frac{\tilde{r}}{r}\mathrm{Im}\,\langle z_p,z_q\rangle+O\left(\frac{\tilde{r}^2}{r^2}\right)+\frac{\tilde{r}}{r}E,$$

where E is also an $O(R^{-1})$ error term. We can now subtract $\|z_p\|^2+\tau_p^2=1$ and divide by a factor of $\frac{\tilde{r}}{r}$ to see that

$$2\,\mathrm{Re}\,\langle z_p,z_q\rangle+\tau_p\,\mathrm{Im}\,\langle z_p,z_q\rangle=E+O\left(\frac{\tilde{r}}{r}\right).$$

By using complex coordinates, note that

$$\mathrm{Re}\,\langle z_p,z_q\rangle=\sum_{j=1}^{n}\varrho_j(p)\varrho_j(q)\cos\phi_j$$

and

$$\mathrm{Im}\,\langle z_p,z_q\rangle=\sum_{j=1}^{n}\varrho_j(p)\varrho_j(q)\sin\phi_j.$$

Recall that for each j we have $|\phi_j|<\bar{\phi}$; hence by taking $\bar{\phi}$ small enough, we can ensure that for each j we have

$$\cos\phi_j+\tau_p\sin\phi_j\geq\cos\phi_j-|\sin\phi_j|\geq 0.$$

It follows that $0\leq\mathrm{Re}\,\langle z_p,z_q\rangle\leq 2\,\mathrm{Re}\,\langle z_p,z_q\rangle+\tau_p\,\mathrm{Im}\,\langle z_p,z_q\rangle$, and hence

$$|\mathrm{Re}\,\langle z_p,z_q\rangle|\leq|E|+O\left(\frac{\tilde{r}}{r}\right).$$

Now suppose that $d(\hat{p},\hat{q})\leq 1-\bar{\tau}^2$. Then $\|z_p-z_q\|^2\leq 1-\bar{\tau}^2$ and

$$\begin{aligned}\tau_q^2=1-\|z_q\|^2&=1+\|z_p\|^2-\|z_p-z_q\|^2-2\,\mathrm{Re}\,\langle z_p,z_q\rangle\\&\geq 1+(1-\bar{\tau}^2)-(1-\bar{\tau}^2)-2\,\mathrm{Re}\,\langle z_p,z_q\rangle\\&>\frac{1+\bar{\tau}^2}{2}>\bar{\tau}^2\geq\tau_p^2,\end{aligned}$$

where we have ensured that $\bar{\epsilon}$ and $\bar{R}^{-1}$ small enough for $4|\mathrm{Re}\,\langle z_p,z_q\rangle|<1-\bar{\tau}^2$. It therefore follows that if $d(\hat{p},\hat{q})\leq 1-\bar{\tau}^2$, then

$$d(\hat{p},\hat{q})>\frac{1}{2}d\left(\{h\in S_1(0):\tau_h^2\leq\bar{\tau}^2\},\left\{h\in S_1(0):\tau_h^2\geq\frac{1+\bar{\tau}^2}{2}\right\}\right)>0.$$

Hence, we can take $\bar{\varrho} > 0$ as the minimum of $1 - \bar{\tau}^2$ and this value to complete the proof. □

Lemma 3.7 (Small scale separation 2)**.** *Let $\epsilon > 0$, let $t, \tilde{t} \geq 1$, and let $r, \tilde{r} > 0$ be such that $\tilde{r} \leq \epsilon r$. Suppose that $p, q \in \mathbb{H}^n \setminus \{0\}$ with $0 \in \partial_t B_r(p) \cap \partial_{\tilde{t}} B_{\tilde{r}}(q)$ and $q \in \partial_t B_r(p)$. Let $T \geq t\tilde{t}$, and define $I(p) = \left\{i : \varrho_i(p) < \frac{10T}{r}\right\}$. There exist $\bar{\tau} \in (\frac{1}{2}, 1)$ and $\bar{\varrho}, \bar{R}, \bar{\phi}, \bar{\epsilon} > 0$, all depending only on the metric, such that if $R > \bar{R}$, $r \geq \tilde{r} \geq TR$, $\epsilon < \bar{\epsilon}$, $|\tau_p| \geq \bar{\tau}$, and $\max_{1 \leq i \leq n} \phi_i(p, q) < \bar{\phi}$, then either there exists $i \notin I(p)$ such that $\varrho_i(q) < \frac{10T}{\tilde{r}}$ or $d(\hat{p}, \hat{q}) > \bar{\varrho}$.*

In this lemma we aim for the same conclusion as in the first small scale separation lemma but find an exceptional case. We deal with this later by using a slightly more sophisticated bounding argument.

As remarked above, in the setting of this lemma the first order argument used to prove Lemma 3.5 is not available to us; the argument stalls if z_p can be made arbitrarily small. Instead we must make delicate use of the precise shape of $B_r(p)$ near the poles. This results in a somewhat more technical proof where special care must be paid to the thickenings, which in this case are large enough to easily throw off the estimates.

Proof. As before, we may assume that $\tau_p \geq 0$ and $\phi(p) = 0$, so $z_p = \operatorname{Re} z_p$, without loss of generality. Again we set $\phi = \phi(q)$ so that

$$\max_{1 \leq i \leq n} |\phi_i| = \max_{1 \leq i \leq n} \phi_i(p, q) < \bar{\phi}.$$

To keep track of a large quantity of error terms, we will slightly abuse the big O and little o notations. Throughout we shall write $O(x)$ for any function $f : \mathbb{R} \to \mathbb{R}$ (possibly depending on our variables) for which, by first taking $\bar{\tau}$ sufficiently close to 1, then $\bar{\epsilon}$ sufficiently small and $\bar{R}$ sufficiently large, we can ensure that $|f(x)| \leq K|x|$ for some $K > 0$ independent of all other variables. Similarly we will write $o(x)$ for any function $f(x)$ for which, given any $\delta > 0$, with the same control over $\bar{\epsilon}, \bar{R}$, and $\bar{\tau}$, we can ensure that $|f(x)| \leq \delta|x|$.

Our approach to identify $\bar{\varrho} > 0$ such that $d(\hat{p}, \hat{q}) > \bar{\varrho}$ is to bound τ_q above by some constant $C < 1$. When successful, it will then suffice to take

$$\bar{\varrho} < \frac{1}{2} d\left((0,1), \{h \in \partial B_1 : \tau_h \leq C\}\right)$$

since $\bar{\tau}$ can be increased to ensure $\hat{p}$ is arbitrarily close to $(0, 1)$. However, in our attempt to do this, we will encounter an exceptional case when there exists an $i \notin I(p)$ such that $\varrho_i(q) < \frac{10T}{\tilde{r}}$. This accounts for the "either" option in the statement and so (for the purpose of this proof) we may essentially assume that that is not the case.

Note that if $\|z_q\| \geq \frac{1}{2}$, then $\tau_q^2 \leq \frac{3}{4}$, giving the desired bound on τ_q, so we may assume that $\|z_q\| \leq \frac{1}{2}$.

Step 1 (Relate τ_q to the other variables).

Aim: In this step we are going to express τ_q in terms of other variables in the problem. In particular, we are going to perturb p and q to suppress the thickenings; this will give us more delicate control over the errors induced by the thickening. By re-expressing τ_q in terms of the other variables, we will identify another bounding problem which, when solved, will allow us to bound τ_q.

First we are going to introduce some new points which incorporate the errors due to the thickenings; this enables us to keep the errors under sufficient control to be dealt with later. Let $\eta \in B_t(0)$, which, by recalling that $\eta^{-1} = (-z_\eta, -\tau_\eta)$, we can choose such that $d(\eta^{-1}, p) = r$ and $q' \in B_t(q)$ such that $d(q', p) = r$. Let $P = p\,\eta$, and let $Q = q'\eta$. For notational simplicity we let $P = (z, \tau)$ and let $Q = (w, \sigma)$. We can write these variables more explicitly using the coordinates of p, q, and η: for some s, s_η with $|s|, |s_\eta| \leq t$ we have

$$z = (r+s)z_p + s_\eta z_\eta \quad \text{and} \quad \tau = (r+s)^2\tau_p + s_\eta^2\tau_\eta + \frac{1}{2}\mathrm{Im}\,\langle (r+s)z_p, s_\eta z_\eta\rangle. \tag{3.7}$$

By using $d(q, q') \leq t$, we see that $(y, \zeta) = q'q^{-1} \in B_t(0)$, from which it follows that if $\zeta = (y, \zeta)$, then $\|y\| \leq t$ and $|\zeta| \leq t^2$. Then $Q = \zeta q \eta$, so for some $\tilde{s}$ such that $|\tilde{s}| \leq \tilde{t}$ we have

$$w = (\tilde{r} + \tilde{s})z_q + y + s_\eta z_\eta \tag{3.8}$$

and

$$\sigma = \left[(\tilde{r}+\tilde{s})^2\tau_q + \frac{1}{2}\mathrm{Im}\,\langle y, (\tilde{r}+\tilde{s})z_q\rangle + \zeta\right] + s_\eta^2\tau_\eta + \frac{1}{2}\mathrm{Im}\,\langle (\tilde{r}+\tilde{s})z_q + y, s_\eta z_\eta\rangle.$$

This final expression is somewhat complicated, but by considering the dominant $\tilde{r}^2$ term and noting that $t, \tilde{t} \leq R^{-1}\tilde{r}$, it becomes clear that

$$\sigma = \tau_q \tilde{r}^2 + o(\tilde{r}^2).$$

Therefore, to bound τ_q above by some $C < 1$, it will suffice to do so for $\frac{\sigma}{\tilde{r}^2}$. This is the new bounding problem mentioned in the step aim.

Step 2 (Derive a different expression for σ to help establish the bound). To do this, we will use the fact that, by the right invariance of the metric, $d(P, Q) = r$ and $d(0, P) = r$. The first of these properties ensures that

$$\frac{\|z-w\|^2}{r^2} + \frac{(\tau - \sigma - \frac{1}{2}\mathrm{Im}\,\langle z, w\rangle)^2}{r^4} = 1,$$

and hence that σ is given by one of the roots of the quadratic, i.e.,

$$\sigma = \tau - \frac{1}{2}\mathrm{Im}\,\langle z, w\rangle \pm r^2\sqrt{1 - r^{-2}\|z-w\|^2}.$$

Aim: We need to establish which root gives σ. We will show that the positive root will be too large, and hence that σ is given by the negative root. First, however, it is useful to further simplify the two expressions by applying Taylor's theorem to the square root.

The fact that $d(0, P) = r$ means that

$$\frac{\|z\|^2}{r^2} + \frac{\tau^2}{r^4} = 1,$$

so as long as $\tau > 0$, which we will see just below, we have

$$r^2\sqrt{1 - r^{-2}\|z-w\|^2} = \tau\sqrt{1 - \frac{r^2}{\tau^2}\left(\|w\|^2 - 2\mathrm{Re}\,\langle z, w\rangle\right)}.$$

Let $E = \frac{r^2}{\tau^2}\left(\|w\|^2 - 2\mathrm{Re}\,\langle z, w\rangle\right)$. We are going to show that $E = o(1)$ in order to apply Taylor's theorem.

Using the coordinate expressions in (3.7), we obtain

$$\frac{\tau}{r^2} = \left(1+\frac{s}{r}\right)^2 \tau_p + \frac{s_\eta^2}{r^2}\tau_\eta + \frac{1}{2}\left(1+\frac{s}{r}\right)\frac{s_\eta}{r}\mathrm{Im}\left\langle z_p, z_\eta\right\rangle = \tau_p + o(1).$$

In particular, we are able to ensure that $0 < \frac{1}{2}\bar{\tau} \le r^{-2}\tau \le \frac{3}{2}$. Additionally, we have

$$\left\|\frac{w}{r}\right\| \le \frac{\tilde{r}+|\tilde{s}|}{r} + \frac{t}{r} + \frac{|s_\eta|}{r} \le \epsilon + \frac{3}{R},$$

so $\left\|\frac{w}{r}\right\| = o(1)$. As

$$r^{-2}\left|\|w\|^2 - 2\mathrm{Re}\left\langle z,w\right\rangle\right| \le \left\|\frac{w}{r}\right\|\left(1+2\left\|\frac{w}{r}\right\|\right),$$

we also have $r^{-2}(\|w\|^2 - 2\mathrm{Re}\left\langle z,w\right\rangle) = o(1)$. Combining the above observations shows us that

$$E = \frac{1}{(r^{-2}\tau)^2} r^{-2}\left(\|w\|^2 - 2\mathrm{Re}\left\langle z,w\right\rangle\right) = o(1).$$

With $\bar{\tau}$, $\bar{\epsilon}$, and $\bar{R}$ sufficiently well chosen, we can therefore apply Taylor's theorem to see that

$$\tau\sqrt{1-\frac{r^2}{\tau^2}\left(\|w\|^2 - 2\mathrm{Re}\left\langle z,w\right\rangle\right)} = \tau\sqrt{1-E} = \tau\left(1-\frac{E}{2}+O\left(E^2\right)\right),$$

and hence that

$$\sigma = 2\tau - \frac{1}{2}\mathrm{Im}\left\langle z,w\right\rangle - \frac{\tau E}{2} + O\left(\tau E^2\right)$$

or

$$\sigma = -\frac{1}{2}\mathrm{Im}\left\langle z,w\right\rangle + \frac{\tau E}{2} + O\left(\tau E^2\right).$$

In principle, as $\tau = r^2\tau_p + o(r^2)$ and $\tau_p \ge \bar{\tau} > \frac{1}{2}$, τ can be very large. This will cause problems if σ is given by the positive root. However, we have already seen that $\sigma = O(\tilde{r}^2)$, $\tau = O(r^2)$, and $E = o(1)$, so if σ were given by the positive root, then $\tau = o(r^2)$, contradicting the condition that $\tau_p > \frac{1}{2}$. Therefore,

$$\sigma = -\frac{1}{2}\mathrm{Im}\left\langle z,w\right\rangle + \frac{\tau E}{2} + O\left(\tau E^2\right).$$

This is the expression for σ which we will use to bound $\sigma/\tilde{r}^2$ strictly below 1, and hence to bound τ_q as discussed earlier. We first show that τE^2 is $o(\tilde{r}^2)$, and then it will be enough to show that

$$A = -\frac{1}{2}\mathrm{Im}\left\langle z,w\right\rangle + \frac{\tau E}{2}$$

satisfies $\frac{A}{\tilde{r}^2} < 1$.

Step 3 (Show τE^2 is $o(\tilde{r}^2)$). Observe that

$$\begin{aligned}\tau E^2 &= \frac{r^4}{\tau^3}\left(\|w\|^2 - 2\mathrm{Re}\left\langle z,w\right\rangle\right)^2 \\ &= \frac{\tilde{r}^2}{(r^{-2}\tau)^3}\left(\left\|\frac{w}{r}\right\|\left\|\frac{w}{\tilde{r}}\right\| + 2\left|\mathrm{Re}\left\langle\frac{z}{r},\frac{w}{\tilde{r}}\right\rangle\right|\right)^2 \\ &\le 2^6\left\|\frac{w}{\tilde{r}}\right\|^2\left(\left\|\frac{w}{r}\right\| + 2\left(1+\frac{|s|}{r}\right)\|z_p\| + \frac{2|s_\eta|}{r}\right)^2\tilde{r}^2,\end{aligned}$$

where we have used $r^{-2}\tau \geq \frac{1}{2}\bar{\tau} \geq \frac{1}{4}$ and the coordinate expression for z. Noting that

$$\left\|\frac{w}{\tilde{r}}\right\| \leq 1 + \frac{|\tilde{s}|}{\tilde{r}} + \frac{t}{\tilde{r}} + \frac{|s_\eta|}{\tilde{r}} \leq 2$$

for ϵ and R^{-1} sufficiently small, and that $\|z_p\| = \sqrt{1-\tau_p^2}$, we see that

$$\tau E^2 \leq 2^8 \left(\epsilon + \frac{5}{R} + 2\left(1+R^{-1}\right)\sqrt{1-\bar{\tau}^2}\right)^2 \tilde{r}^2.$$

This means that $\tau E^2 = o(\tilde{r}^2)$, which we will see is sufficient.

Step 4 (Bounding $A/\tilde{r}^2$). This is the most technical step in the proof but is not fundamentally difficult. Recall that

$$\begin{aligned} A = -\frac{1}{2}\mathrm{Im}\,\langle z,w\rangle + \frac{\tau E}{2} &= -\frac{1}{2}\mathrm{Im}\,\langle z,w\rangle + \frac{r^2}{2\tau}\left(\|w\|^2 - 2\mathrm{Re}\,\langle z,w\rangle\right) \\ &= \frac{1}{2}\left(-\mathrm{Im}\,\langle z,w\rangle - 2\frac{r^2}{\tau}\mathrm{Re}\,\langle z,w\rangle\right) + \frac{r^2}{2\tau}\|w\|^2. \end{aligned}$$

Aim: Show that $-\mathrm{Im}\,\langle z,w\rangle - 2\frac{r^2}{\tau}\mathrm{Re}\,\langle z,w\rangle$, the term inside the bracket, is nonpositive modulo an $o(\tilde{r}^2)$ error term.

By recalling that $|s_\eta| \leq t$ and hence is $o(\tilde{r})$, observe that

$$\begin{aligned} \langle z,w\rangle &= \langle (r+s)z_p + s_\eta z_\eta, (\tilde{r}+\tilde{s})z_q + y + s_\eta z_\eta\rangle \\ &= \langle (r+s)z_p, (\tilde{r}+\tilde{s})z_q + y + s_\eta z_\eta\rangle + o(\tilde{r}^2) \\ &= (r+s)\tilde{r}\left\langle z_p, \left(1+\frac{\tilde{s}}{\tilde{r}}\right)z_q + \frac{y+s_\eta z_\eta}{\tilde{r}}\right\rangle + o(\tilde{r}^2). \end{aligned}$$

Let $\upsilon = \tilde{r}^{-1}(y + s_\eta z_\eta)$. Next, by recalling that $I(p) = \left\{i : \varrho_i(p) < \frac{10T}{r}\right\}$ and $r > \tilde{r} > TR$, we see that

$$\begin{aligned} \mathrm{Im}\left\langle z_p, \left(1+\frac{\tilde{s}}{\tilde{r}}\right)z_q + \upsilon\right\rangle &= \sum_{j=1}^n \varrho_j(p)\left(\left(1+\frac{\tilde{s}}{\tilde{r}}\right)\varrho_i(q)\sin\phi_j + \mathrm{Im}\,\upsilon_i\right) \\ &= \sum_{j\notin I(p)} \varrho_j(p)\left(\left(1+\frac{\tilde{s}}{\tilde{r}}\right)\varrho_i(q)\sin\phi_j + \mathrm{Im}\,\upsilon_i\right) + o\left(\frac{\tilde{r}}{r}\right) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Re}\left\langle z_p, \left(1+\frac{\tilde{s}}{\tilde{r}}\right)z_q + \upsilon\right\rangle &= \sum_{j=1}^n \varrho_j(p)\left(\left(1+\frac{\tilde{s}}{\tilde{r}}\right)\varrho_i(q)\cos\phi_j + \mathrm{Re}\,\upsilon_i\right) \\ &= \sum_{j\notin I(p)} \varrho_j(p)\left(\left(1+\frac{\tilde{s}}{\tilde{r}}\right)\varrho_i(q)\cos\phi_j + \mathrm{Re}\,\upsilon_i\right) + o\left(\frac{\tilde{r}}{r}\right). \end{aligned}$$

Therefore, to show that $-\mathrm{Im}\,\langle z,w\rangle - 2\frac{r^2}{\tau}\mathrm{Re}\,\langle z,w\rangle$ is nonpositive up to an error term of the order

$$(r+s)\tilde{r}\,o\left(\frac{\tilde{r}}{r}\right) + o(\tilde{r}^2) = o(\tilde{r}^2),$$

it will be enough to show that

$$-\sum_{j\notin I_p} \varrho_j(p)\left(\left(1+\frac{\tilde{s}}{\tilde{r}}\right)\varrho_i(q)\left(\sin\phi_j+\frac{2r^2}{\tau}\cos\phi_j\right)+\operatorname{Im}\upsilon_i+\frac{2r^2}{\tau}\operatorname{Re}\upsilon_i\right)\le 0.$$

From earlier assumptions $\|\upsilon\| = \tilde{r}^{-1}\|y+s_\eta z_\eta\| \le 2t\tilde{r}^{-1}$, and we can ensure that

$$\sin\phi_j+\frac{2r^2}{\tau}\cos\phi_j \ge 1$$

by further increasing $\bar{\tau}$ and then decreasing $\bar{\epsilon}$, $\bar{R}^{-1}$, and $\bar{\phi}$. So it will be enough for the magnitude of each $\varrho_i(q)$ to be large relative to $2t/\tilde{r}$. This is where we use the assumption from the start of the proof: that for all $j \notin I(p)$ we have $\varrho_j(q) \ge 10T\tilde{r}^{-1}$, so

$$\left(1+\frac{\tilde{s}}{\tilde{r}}\right)\varrho_j(q) \ge \varrho_j(q) \ge 10t\tilde{r}^{-1} \ge 2t\tilde{r}^{-1}\left(1+\frac{2r^2}{\tau}\right)$$

since we ensured that $r^{-2}\tau > \frac{1}{2}$. This means that the terms of the sum above are all nonnegative, giving the required property. It follows that

$$-\operatorname{Im}\langle z,w\rangle - 2\frac{r^2}{\tau}\operatorname{Re}\langle z,w\rangle \le 0 + o(\tilde{r}^2),$$

as outlined in our previous aim.

We can use this to bound A, provided for all $j \notin I(p)$, by considering that

$$\begin{aligned} A &= \frac{1}{2}\left(-\operatorname{Im}\langle z,w\rangle - 2\frac{r^2}{\tau}\operatorname{Re}\langle z,w\rangle\right) + \frac{r^2}{2\tau}\|w\|^2 \\ &\le \frac{r^2}{2\tau}\|w\|^2 + o(\tilde{r}^2) \\ &\le \frac{\tilde{r}^2}{4\bar{\tau}} + o(\tilde{r}^2) \le \frac{\tilde{r}^2}{2} + o(\tilde{r}^2). \end{aligned}$$

Step 5 (Bounding σ and completing the proof). By combining the last two steps, we see that

$$\sigma = A + O\left(\tau E^2\right) \le \frac{\tilde{r}^2}{2} + o(\tilde{r}^2),$$

which allows us to bound $\sigma/\tilde{r}^2$ and hence τ_q by a constant < 1, as we aimed to at the start of the proof. Note that this bound holds only so long as there is no $j \notin I(p)$ for which $\varrho_j(q) < \frac{10T}{\tilde{r}}$, but if there is, then this is the other option allowed by the statement. □

3.3. **Finite intersection dimension.** We can now fit these pieces together to show that property (iv) holds.

Theorem 3.8. *$(\mathbb{H}^n, d)$ has finite intersection dimension.*

Proof. We need to show that there exist $R > 1$ and $\kappa \in \mathbb{N}$ such that if we are given

(1) $t(1),\ldots,t(\kappa) \ge 1$,
(2) $r(1),\ldots,r(\kappa)$ such that each $r(i) \ge t(1),\ldots,t(i)R$,
(3) $p_1,\ldots,p_\kappa \in \mathbb{H}^n$ such that $p_i \in \bigcap_{j<i} \partial_{t(j)}B_{r(j)}(p_j)$ for $j < i$,

then $\bigcap_{i=1}^{\kappa} \partial_{t(i)}B_{r(i)}(p_i) = \emptyset$. Let us assume, by using invariance, that

$$0 \in \bigcap_{i=1}^{\kappa} \partial_{t(i)}B_{r(i)}(p_i),$$

and let us show that κ must be bounded for R sufficiently large.

The logical structure of the proof is to first apply a number of reductions of the form: we start with a sequence of length κ, then we show that given κ', there is an $M(\kappa') \in \mathbb{N}$ such that if $\kappa \geq M$, then there is a subsequence of length κ' with an additional property. We can then relabel and assume that our sequence had the additional property from the beginning. We finish by using all of the gathered properties to show that κ is bounded.

Reduction 1. First we show that we can assume that the $r(i)$ are decreasing, essentially as in [Hoc10]. Let $\kappa'' \leq \kappa$, and assume that $r(i) \geq r(1)$ for all $2 \leq i \leq \kappa''$. By property 3 all of these p_i lie inside $\partial_{t(1)} B_{r(1)}(p_1) \subset B_{2r(1)}(p_1)$, where this containment is due to property 2. Property 3 also ensures that for any pair i, j with $j > i$ there is a point $b \in \partial B_{r(i)}(p_i)$ with $d(b, p_j) \leq t(i)$, and hence by property 2

$$d(p_i, p_j) \geq |d(p_i, b) - d(p_j, b)| \geq r(i) - t(i) \geq r(1)(1 - R^{-1}),$$

so for all $1 \leq i, j \leq \kappa''$ with $i \neq j$

$$d\left(p_1^{-1}\delta_{1/(2r(1))}p_i,\, p_1^{-1}\delta_{1/(2r(1))}p_j\right) \geq \frac{1 - R^{-1}}{2} > 0$$

since each $p_1^{-1}\delta_{1/(2r(1))}p_i \in B_1(0)$ and by Lemma 3.2*(i)* $\kappa'' \leq N_R$, where $N_R = N(\frac{1-R^{-1}}{2})$. Note that N_R decreases as R increases.

Clearly, this argument could be repeated with any chain of κ'' points satisfying the analogous conditions. Therefore, if for some $\kappa' \in \mathbb{N}$ we have $\kappa \geq \kappa'(N_R + 1)$, then there must be $i_1 = 1 < i_2 \leq \cdots < i_{\kappa'} \leq \kappa$ with $r(i_1) \geq r(i_2) \geq \cdots \geq r(i_{\kappa'})$. This means that it suffices for us to prove the claim with the $r(i)$ assumed to be decreasing.

Reduction 2. Let $\epsilon \in (0, 1)$. Next we use Lemma 3.2 and the large scale separation lemma, Lemma 3.4, to ensure that we can assume that $\hat{p}_1, \ldots, \hat{p}_\kappa$ are all within a distance

$$\varrho(\epsilon) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{\epsilon^2}{4}}\right) > 0$$

of one another, and that for all $j > i$ we have $r(j) \leq \epsilon r(i)$. Note that $\varrho(\epsilon)$ decreases as ϵ decreases.

Let $\kappa' \leq \kappa$ again. By Lemma 3.2*(ii)* if $\kappa \geq \kappa' N(\varrho(\epsilon))$, then we have a subcollection $I \subset \{1, \ldots, \kappa\}$ of size at least κ' with $d(\hat{p}_i, \hat{p}_j) < \varrho(\epsilon)$ for all $i, j \in I$. By taking $R > \bar{R}(\epsilon)$ from Lemma 3.4, which is assumed to hold from here onward, the lemma shows that for each pair $i, j \in I$ with $j < i$ we have $r(j) \leq \epsilon r(i)$. The set I therefore gives the desired subsequence.

Reduction 3. Before we do our final reduction, first take $\bar{\tau}$ as given by Lemma 3.7, and we take this as the input for $\bar{\tau}$ in Lemma 3.5. We can then decrease ϵ so that ϵ and $\varrho(\epsilon)$ are small enough to apply Lemmas 3.5 and 3.7 with $\bar{\epsilon} = 2\epsilon$ and $\bar{\varrho} = \varrho(\epsilon)$. Similarly we take R large enough for both lemmas to hold.

It is clear from the pigeonhole principle that given κ' by increasing κ, we can ensure that there is a subcollection $I \subset \{1, \ldots, \kappa\}$ of size κ' such that for all $i, j \in I$ we have $\max_{1 \leq l \leq n} \phi_l(p_i, p_j) < \bar{\phi}$, where $\bar{\phi}$ is small enough for both lemmas to hold. We can therefore assume that the whole sequence also has this property.

κ **is bounded.** With all of this in hand, we can apply Lemmas 3.5 and 3.7 to the sequence at will. Let $T = t_1, \ldots, t_\kappa$, and for each $1 \le i \le \kappa$ set $I(p_i) = \{m : \varrho_m(p_i) < \frac{10T}{r_i}\} \subseteq \{1, \ldots, n\}$, as in Lemma 3.7. By assumption for all $i \neq j$ we have $d(\hat{p}_i, \hat{p}_j) \le \bar{\varrho}$, so by Lemma 3.5 we must have $|\tau_{p_i}| > \bar{\tau}$ for all $i \le \kappa - 1$. By applying this fact along with the same assumption, Lemma 3.7 ensures that for each pair $i < j \le \kappa$ there is some number in $I(p_j)$ which is not in $I(p_i)$. In particular, each of the sets $I(p_1), \ldots, I(p_\kappa) \subseteq \{1, \ldots, n\}$ is pairwise distinct, from which it follows that $\kappa \le 2^n$.

□

Having completed this proof, all that remains is property (i), that the group is well spaced.

3.4. **Well spaced.** We begin with a preliminary lemma.

Lemma 3.9. *There exists an $R > 0$ such that given $p, p' \in \mathbb{H}^n$ and $r > 0$ satisfying $\varrho = d(p, p') > 2Rr$, there is a point q with $d(p', q) \le 2r$ for which $B_r(q) \subseteq B_\varrho(p)$.*

Proof. First using the dilation and isometries of d, we may assume that $r = 1/2$ and $p' = 0$. Moreover, we assume that $\tau_p \ge 0$ and that all coefficients z_p are nonnegative reals.

By right invariance the points in $B_{1/2}(q)$ take the form $(w + q_z, \sigma + \zeta + \frac{1}{2}\mathrm{Im}\,\langle w, q_z\rangle)$, where $\|w\|^2 + 4\sigma^2 \le \frac{1}{4}$. Therefore, by (3.1) it suffices to show that we can choose R large enough such that given (z_p, τ_p) there is $q = (y, \zeta)$ with $\|y\|^2 + \zeta^2 \le 1$ such that

$$\frac{\|w + y - \varrho z_p\|^2}{\varrho^2} + \frac{\left(\sigma + \zeta + \frac{1}{2}\mathrm{Im}\,\langle w, y\rangle - \varrho^2\tau_p - \frac{1}{2}\mathrm{Im}\,\langle w + y, \varrho z_p\rangle\right)^2}{\varrho^4} \le 1,$$

or equivalently (as $d(0, p) = \varrho$) that

$$\begin{aligned} 0 \ge\ & \varrho^3 \left(-2\mathrm{Re}\,\langle w + y, z_p\rangle + \tau_p \mathrm{Im}\,\langle w + y, z_p\rangle\right) \\ & + \varrho^2 \left(\|w + y\|^2 - 2\tau_p \left(\sigma + \zeta + \frac{1}{2}\mathrm{Im}\,\langle w, y\rangle\right) + \frac{1}{4}(\mathrm{Im}\,\langle w + y, z_p\rangle)^2\right) \\ & - \frac{\varrho}{2}\left(\sigma + \zeta + \frac{1}{2}\mathrm{Im}\,\langle w, y\rangle\right)\mathrm{Im}\,\langle w + y, \varrho z_p\rangle + \left(\sigma + \zeta + \frac{1}{2}\mathrm{Im}\,\langle w, y\rangle\right)^2. \end{aligned}$$

Notice that the coefficients of all powers of ϱ have bounds independent of all variables. Let $C > 0$ be strictly greater than the independent bound for the coefficient of ϱ^2, and ensure that $R > C$. Consider the case in which $\|z_p\| \ge \frac{2C}{\varrho} > 0$. Let us take $y = \lambda z_p$, where $\lambda > 0$ is chosen so that $\|y\| = 1$, and hence $\zeta = 0$. Then the coefficient of ϱ^3 above satisfies

$$\begin{aligned} -2\mathrm{Re}\,\langle w + y, z_p\rangle + \tau_p \mathrm{Im}\,\langle w + y, z_p\rangle &= -2\langle y, z_p\rangle - \langle z_p, 2\mathrm{Re}\, w + \tau_p \mathrm{Im}\, w\rangle \\ &\le \|z_p\|(-2 + \frac{3}{2}) \le -\frac{C}{\varrho}. \end{aligned}$$

It follows that the polynomial above is bounded above by a quadratic in ϱ whose coefficients are independent of all variables, and the leading coefficient of which is negative. Hence, we may take R large enough, with the required independence, to ensure that the inequality holds for some appropriate q regardless of the choice of p.

In the case in which $\|z_p\| \leq \frac{2C}{\varrho}$ take $y = 0$ and $\zeta = 1$. Then we have the bounds

$$-2\mathrm{Re}\,\langle w + y, z_p\rangle + \tau_p \mathrm{Im}\,\langle w + y, z_p\rangle \leq \frac{3C}{\varrho}$$

and

$$\|w + y\|^2 - 2\tau_p\left(\sigma + \zeta + \frac{1}{2}\mathrm{Im}\,\langle w, y\rangle\right) + \frac{1}{4}(\mathrm{Im}\,\langle w + y, z_p\rangle)^2 \leq \frac{1}{4} - \frac{3}{2}\tau_p + \frac{C^2}{4\varrho^2}.$$

In particular, we can show that the polynomial is bounded above by a quadratic whose coefficients are independent of all variables, and whose leading coefficient is less than

$$\frac{3C}{\varrho} + \frac{1}{4} - \frac{3}{2}\sqrt{1 - \frac{C^2}{\varrho^2}} + \frac{C^2}{4\varrho^2} \leq -1,$$

where R has been taken sufficiently large relative to C. So, as above, we may increase R to ensure that the required inequality holds. ☐

We call a sequence of balls in a metric space *incremental* if the radii are non-increasing and the center of each ball is not an element of any ball earlier in the sequence. In particular, each center is in only one ball in the sequence.

Proposition 3.10. $(\mathbb{H}^n, d)$ *is well spaced.*

Proof. We mildly adapt a standard technique; see, for example, [Hoc10] or [dG75]. For the purposes of this proof we use ν to denote the right invariant Haar measure on $\mathbb{H}^n$.

Let C be the constant of the BCP, and let D be the constant for the metric doubling property of d. Furthermore, take $m \in \mathbb{N}$ to be large enough that $2^m > R$, with R as in Lemma 3.9. In particular, m depends only on the metric d. Let $\chi = CD^{m+2} + 1$.

Let E be a finite subset of $\mathbb{H}^n$, and let $\mathcal{U}$ be a carpet covering E. By applying the BCP via, for example, [Hoc10, Proposition 2.1], we can find an incremental sequence $U_1, \ldots, U_n$ of elements of $\mathcal{U}$ covering E. We assign colors $1, 2, \ldots, \chi$ to the U_i as follows. Color U_1 as you like, assume that we have colored the U_i for $i \leq k$, and consider U_{k+1}. Taking r to be the radius of U_k, and taking h to be the center of U_{k+1}, by assumption $U_{k+1} \subseteq B_r(h)$ and each U_i with $i \leq k$ has a radius of at least r.

Let $\mathcal{W}$ be the collection of balls $U_1, \ldots, U_k$ which are within distance r of U_{k+1}, and hence of $B_r(h)$, and let $N = |\mathcal{W}|$. Each $U \in \mathcal{W}$ intersects nontrivially with $B_{2r}(h)$. Let us fix such a U. We may take p' from Lemma 3.9 to be a point in its intersection with $B_{2r}(h)$. We may assume that p' is on the boundary of U because the straight line from h to p' is contained by $B_{2r}(h)$ (the balls are euclidean convex), $p' \in U$, and $h \notin U$ (by incrementality), so the intermediate value theorem implies that there is a point on the boundary of U inside $B_{2r}(h)$. Lemma 3.9 then ensures that either the radius of U is at most $2^{m+1}r$ or we can replace U with a ball of radius r centered in $B_{4r}(h)$. Call this new collection of balls $\mathcal{W}'$, and note that it has size N. Each ball in $\mathcal{W}'$ has a radius of at least r and is contained by the ball of radius $2^{m+2}r$ about h. Therefore, by the Besicovitch and metric doubling properties

$$N\nu(B_r(0)) \leq C\nu(B_{2^{m+2}r}(h)) \leq CD^{m+2}\nu(B_r(h)) = CD^{m+2}\nu(B_r(0)),$$

so $N \leq CD^{m+2}$. Since $N \leq \chi - 1$, we assign a color U_k which is different from all those within distance r of U_k.

Once the coloring is complete, the collection $\mathcal{V}_j$ of those balls colored j is evenly spaced precisely because of this property, combined with the fact that the radii are decreasing. □

This proposition completes the proof of Theorem 1.1.

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