TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 372, Number 9, 1 November 2019, Pages 6017–6086
https://doi.org/10.1090/tran/7367
Article electronically published on August 5, 2019

SURJECTIVITY OF EULER TYPE DIFFERENTIAL OPERATORS ON SPACES OF SMOOTH FUNCTIONS

PAWEŁ DOMAŃSKI AND MICHAEL LANGENBRUCH

Dedicated to the memory of Paweł Domański,
a great friend and mathematician who left us far too early

Abstract. We develop a (global) solvability theory for Euler type linear partial differential equations $P(\theta)$ on $C^\infty(\Omega)$, with Ω an open subset of $\mathbb{R}^d$, i.e., for a special type of linear partial differential equation with polynomial coefficients. There is a natural closed upper bound $C^\infty_{I(P)}(\Omega)$ for the range of $P(\theta)$ on $C^\infty(\Omega)$. We characterize by $P(\theta)$-convexity type conditions those Ω such that $P(\theta)$ is surjective on $C^\infty_{I(P)}(\Omega)$. We also clarify when all shifted operators $P(c+\theta)$ are surjective on $C^\infty_{I(P(c+\,\cdot\,))}(\Omega)$. We classify in geometric terms those Ω with $0 \in \Omega$ such that every Euler operator $P(\theta)$ is surjective on $C^\infty_{I(P)}(\Omega)$. Moreover, we determine the operators $P(\theta)$ which are surjective onto $C^\infty_{I(P)}(\Omega)$ for every open set $\Omega \subseteq \mathbb{R}^d$. Under some mild assumptions on Ω, we characterize those Euler operators which are invertible on $C^\infty(\Omega)$. Under the same assumptions we also calculate the spectrum of $P(\theta)$ on $C^\infty(\Omega)$. The results follow from the solvability theory for Hadamard type operators on the space of smooth functions and from a new general Mellin transform, both developed in this paper.

1. Introduction

In the present paper we will discuss the solvability of Euler type differential operators $P(\theta)$ on $C^\infty(\Omega)$ for general open sets $\Omega \subset \mathbb{R}^d$. Recall that Euler type differential operators are special partial differential operators with polynomial coefficients defined as follows: for a polynomial $P(x) := \sum_{|\alpha|\le m} c_\alpha x^\alpha$ let

$$P(\theta) := \sum_{|\alpha|\le m} c_\alpha \theta^\alpha \text{ where } \theta^\alpha := \prod_{j\le d} \theta_j^{\alpha_j} \text{ and } \theta_j := x_j\partial/\partial x_j.$$

Notice that Euler differential operators are singular on

$$Z_1 := \{x \in \mathbb{R}^d \mid x_j = 0 \text{ for some } j \le d\}.$$

So special difficulties are to be expected if $Z_1 \cap \Omega \neq \emptyset$.

The solvability theory for linear partial differential equations with constant coefficients in the class of smooth functions is well established (see [15]). Even though

Received by the editors July 21, 2017.

2010 *Mathematics Subject Classification.* Primary 44A15, 35A01; Secondary 35A09, 35A22, 45E10.

Key words and phrases. Hadamard type operators, linear Euler type partial differential operators, linear partial differential operators with polynomial coefficients, smooth functions, Mellin transform, global solvability, invertibility.

This research was supported by the National Center of Science (Poland), grant no. UMO-2013/10/A/ST1/00091.

there is an extensive literature on linear partial differential equations with polynomial coefficients (see [4], [18]) or on Euler partial differential equations (for instance, see [16]), still a global solvability theory was not known. Let us mention that also the existent solvability results for general partial differential equations with smooth coefficients are not helpful (see [20], [21], [22]).

In case $\Omega \subset]0,\infty[^d$ the Euler operator $P(\theta)$ can be transformed to the constant coefficient linear partial differential operator $P(\partial)$ using the coordinate transform with logarithm (and exponential function, respectively) in each variable. In this way one gets the following.

Theorem 1.1 (Vogt [25, Section 5]). *Let $\Omega \subset]0,\infty[^d$ be open. Then $P(\theta): C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective iff $P(\partial)$ is surjective on $C^\infty(\mathrm{Log}(\Omega))$ iff $\omega := \mathrm{Log}(\Omega)$ is $P(\partial)$-convex (for support, see* [15, 10.6.1]*), that is,*

$$\forall K \Subset \omega\ \exists \widetilde{K} \Subset \omega\ \forall g \in \mathscr{D}(\omega) : \mathrm{supp}(g) \subset \widetilde{K} \text{ if } \mathrm{supp}({}^tP(\partial)g) \subset K. \tag{1.1}$$

Here $\mathrm{Log}(x) := (\log x_1, \ldots, \log x_d)$. In the present paper we treat the more interesting case where

$$Z_1 \cap \Omega \neq \emptyset,$$

so we cannot reduce the problem directly to the constant coefficient case by using Log. Using specific new tools we will prove two substitutes for the above theorem valid in this general case (see the Main Theorem B(a) and (b) below).

When presenting our main results below we will partly use notation and definitions which will be clearly established later in the paper (see the corresponding hints). Firstly, we notice that there is a general a priori bound for the range of $P(\theta)$. In fact, the following inclusion holds (see Lemma 3.3, with obvious notation for the grouping of variables, and see Section 2):

$$P(\theta)C^\infty(\Omega) \subset C^\infty_{I(P)}(\Omega),$$

where

$$C^\infty_{I(P)}(\Omega) := \{f \in C^\infty(\Omega) \mid \forall \emptyset \neq J \subseteq D := \{1,\ldots,d\}\ \forall \alpha \in \mathbb{N}^J\ \forall (0_J, x_{D\setminus J}) \in \Omega : f^{(\alpha)}(0_J, x_{D\setminus J}) = 0 \text{ if } P(\alpha, x_{D\setminus J}) = 0 \text{ for any } x_{D\setminus J} \in \mathbb{R}^{D\setminus J}\}.$$

Hence the best we might prove is that

$$P(\theta)C^\infty(\Omega) = C^\infty_{I(P)}(\Omega). \tag{1.2}$$

We first prove the following theorem (Theorems 9.4 and 9.7; for the definition of m-convex sets see Section 2).

Main Theorem A. *Let $\Omega \subseteq \mathbb{R}^d$ be an open connected set containing $\mathbf{0} = (0,\ldots,0)$. Each Euler operator $0 \neq P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective if and only if Ω is m-convex. In particular, this holds for $\Omega = \mathbb{R}^d$ or $\Omega =]-1,1[^d$.*

This result is not very surprising since it looks like a natural analogue of the well-known fact that linear partial differential operators with constant coefficients are surjective on $C^\infty(\Omega)$ for convex Ω. However, the proof requires essentially new tools as we will explain below. Nevertheless, notice that the Main Theorem A is in sharp contrast to the situation where $P(\theta)$ acts on the space of real analytic functions $\mathscr{A}(\mathbb{R}^d)$. There surjectivity onto $\mathscr{A}_{I(P)}(\mathbb{R}^d)$ is related to the so-called half-plane property of the principal part P_m of the polynomial P, that is,

$$P_m(z) \neq 0 \qquad \text{if Re } z_1, \dots, \text{Re } z_d > 0,$$

which is a rather restrictive condition (see [11]).

Then we look for an analogue of $P(\partial)$-convexity theory [15, Ch. 10] working so nicely in the constant coefficient case: recall that (cf. [15, Section 10.6])

(1.3) $\quad P(\partial)$ is surjective on $C^\infty(\Omega)$ iff Ω is $P(\partial)$-convex (for support).

Notice that $P(\partial)$ satisfies

$$P(\partial)(e^{\langle c,x\rangle} f(x)) = e^{\langle c,x\rangle} P(\partial + c) f(x) \text{ for any } c \in \mathbb{R}^d \text{ and any } f \in C^\infty(\Omega),$$

where $\langle c, x\rangle := \sum_{j=1}^d c_j x_j$.

Since the operator of multiplication by $e^{\langle c,x\rangle}$ is a topological isomorphism on $C^\infty(\Omega)$, Hörmander's theorem in fact also states that

(1.4) $\quad P(\partial + c)$ is surjective on $C^\infty(\Omega)$ for any $c \in \mathbb{R}^d$ iff Ω is $P(\partial)$-convex.

We will prove full analogues for (1.3) and (1.4) for Euler operators in the Main Theorem B(a) and (b) below.

For the proper formulation we need some further notation: let $D := \{1, \dots, d\}$ and set

$$\Omega_{D\setminus J} := \{x_{D\setminus J} \in \mathbb{R}^{D\setminus J} \mid (0_J, x_{D\setminus J}) \in \Omega\} \text{ for any } J \subset D.$$

Here $\Omega_D := \Omega$ and $x_D := x$. Moreover, let

$$\mathcal{E}(\Omega_\sigma) := \{f \in C^\infty(\Omega) \mid \text{supp}(f) \subset \Omega_\sigma\} \text{ for } \sigma \in \{\pm 1\}^d$$

be the space of smooth functions supported in

$$\Omega_\sigma := \{x \in \Omega \mid \sigma_1 x_1, \dots, \sigma_d x_d \geq 0\},$$

that is, in the intersections of Ω with the canonical closed quadrants. Since $\Omega_\sigma \subset \sigma[0, \infty[^d$ we can use $\text{Log}_\sigma(x) := \text{Log}(\sigma x), \sigma x := (\sigma_j x_j)_{j\leq d}$, to transform Ω_σ to a subset of $[-\infty, \infty[^d$ endowed with its canonical topology. We obtain the notion of strict $P(\partial)$-convexity (see (1.5) below) which is a new variant of $P(\partial)$-convexity substituting the spaces $\mathscr{D}(\omega)$ in (1.1) for $\omega \subset [-\infty, \infty[^d$ by the spaces

$$\mathscr{D}^k_{\partial,exp}(\omega) := \{f \in C^\infty(\omega \cap \mathbb{R}^d) \mid \forall \alpha \in \mathbb{N}^d : \\ \exp(\langle x, \mathbf{k}\rangle) f^{(\alpha)}(x) \text{ is bounded and } \text{supp}(f) \Subset \omega\}$$

of a smooth function with exponential growth near $(-\infty, \dots, -\infty)$, here $\mathbf{k} := (k, \dots, k)$. Thus, an open set $\omega \subset [-\infty, \infty[^d$ is called strictly $P(\partial)$-convex (for support) if

$$\forall K \Subset \omega \ \forall k \in \mathbb{N} \ \exists \widetilde{K} \Subset \omega \ \forall g \in \mathscr{D}^k_{\partial,exp}(\omega) : \\ \text{supp}(g) \subset \widetilde{K} \text{ if } \text{supp}(P(-\partial)g) \subset K. \tag{1.5}$$

Notice that the topology of ω is inherited from $[-\infty, \infty[^d$, hence compact subsets of ω need not be bounded in the topology of $\mathbb{R}^d$. The use of the spaces $\mathscr{D}^k_{\partial,exp}(\omega)$ in (1.5) might be unexpected. It is motivated by the fact that we have to regularize elements in $\mathcal{E}([0, \infty[^d)'$ using the $\star$-convolution (see [25] and Sections 9 and 10), which is leading to the space $\mathscr{D}^k_{\partial,exp}([-\infty, \infty[^d)$ via variable transformation with Log.

Now, we can formulate the second main result (see Theorem 9.5 and Theorems 10.13 and 10.14 where further equivalences are given).

Main Theorem B. *Let $\Omega \subset \mathbb{R}^d$ be an open set.*

(a) *The following are equivalent:*
  (i) *The Euler operator $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective.*
  (ii) *Ω is $P(\theta)$-convex, that is,*

$$\forall K \Subset \Omega\ \forall k \in \mathbb{N}\ \exists \widetilde{K} \Subset \Omega\ \forall T \in C^{\mathbf{k}}_{I(P)}(\Omega)' : \quad \operatorname{supp}_{I(P)}(T) \subset \widetilde{K} \text{ if } \operatorname{supp}({}^tP(\theta)T) \subset K. \tag{1.6}$$

(b) *The following are equivalent:*
  (iii) *The Euler operators $P(\theta+c) : C^\infty(\Omega) \to C^\infty_{I(P(\,\cdot\,+c))}(\Omega)$ are surjective for any $c \in \mathbb{R}^d$.*
  (iv) *For any $J \subsetneq D$, any $\sigma \in \{\pm1\}^{D\setminus J}$ with $\Omega_{D\setminus J,\sigma} \neq \emptyset$, and any $\alpha \in \mathbb{N}^J$ such that $P(\alpha,\,\cdot\,) \neq 0$, the sets* $\operatorname{Log}(\sigma\Omega_{D\setminus J,\sigma})$ *are strictly $P(\alpha, \partial_{D\setminus J})$-convex.*

(c) *If Ω is admissible for $P(\theta)$, then the conditions in* (a) *and* (b) *are equivalent.*

In (1.6), $\operatorname{supp}_{I(P)}(T)$ denotes the support of T in the sense of $C^\infty_{I(P)}(\Omega)'$ (cf. (3.8) for the precise definition).

The intuitive geometric notion of admissible sets is introduced and discussed in Section 7. Let us emphasize that statements (a) and (b) of the Main Theorem B are equivalent for *any* open set Ω if $S(P) \subset \{\mathbf{0}\}$, that is, if $I(P)$ contains no "halfline" $\alpha + \mathbb{N}e_j$ (see Corollary 10.15). Many examples of operators $P(\theta)$ with $S(P) \subset \{\mathbf{0}\}$ are discussed in Section 7 including polynomials which are partially hypoelliptic w.r.t. any of the variables, polynomials for which any canonical unit vector e_j is noncharacteristic, and the standard second-order Euler type operators (see Example 7.9(b)).

We emphasize, however, that statements (a) and (b) of the Main Theorem B are not equivalent in general. In fact, already the simplest Euler operators θ_1 and $\theta_1 + 1$ provide a counterexample (see Proposition 4.9).

$P(\theta)$-convexity is used to analyze surjectivity of first-order Euler differential operators in Section 4 showing fundamental differences between constant coefficient partial differential operators and Euler partial differential operators. A surprising consequence is that, contrary to the constant coefficient theory, for some open sets $\Omega \subset \mathbb{R}^d$ and some (first-order) Euler partial differential operators $P(\theta)$ there is no minimal $P(\theta)$-convex open set $\Omega_1 \supsetneq \Omega$ (see Example 4.7). For more examples showing the difference between the constant coefficient case and the Euler case see Example 4.6 and Example 4.8. On the positive side, as a consequence of Main Theorem B, it can be proved that if $P(\theta) : C^\infty(\Omega_j) \to C^\infty_{I(P)}(\Omega_j)$ is surjective for $j = 1, \dots, n$, then $P(\theta) : C^\infty\left(\bigcap_{j=1}^n \Omega_j\right) \to C^\infty_{I(P)}\left(\bigcap_{j=1}^n \Omega_j\right)$ is surjective (Corollary 3.12). Even more surprising: if $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective, then the linear partial differential operator with constant coefficients $P(\partial) : C^\infty(\operatorname{Log}(\sigma\Omega^\circ_\sigma)) \to C^\infty(\operatorname{Log}(\sigma\Omega^\circ_\sigma))$ is surjective for any $\sigma \in \{\pm1\}^d$ (Proposition 10.1).

Our third main result is the following (cf. Theorem 10.7).

Main Theorem C. *A nonzero Euler operator $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective for every open nonempty $\Omega \subseteq \mathbb{R}^d$ if and only if P is elliptic, i.e., the principal part P_m vanishes on $\mathbb{R}^d$ only at $\mathbf{0} = (0, \dots, 0)$.*

This result is similar to the results valid for linear partial differential operators with constant coefficients [15, Corollary 10.8.2, 10.6.6, 10.6.8] although again the proof is quite different.

The fourth main result of the paper is even more intriguing. It is well known that linear partial differential operators with constant coefficients $P(\partial)$ always have an infinite-dimensional kernel in $C^\infty(\Omega)$ for open $\Omega \subset \mathbb{R}^d, d > 1$. So the spectrum of $P(\partial)$ is always equal to $\mathbb{C}$. Surprisingly enough there are plenty of *invertible* Euler partial differential operators $P(\theta)$, and we have the following result (see Corollary 11.8, Theorem 11.5, and Corollary 11.11; for the definition of nearly solid sets see Section 2).

Main Theorem D. *Let $\Omega \subset \mathbb{R}^d$ be an open nearly solid set, i.e., for every $x \in \Omega$ and $t \in [0,1]^d$ also $(x_1t_1, \ldots, x_dt_d) \in \Omega$, and let $\mathbb{C}_{\geq k} := \{0, \ldots, k-1\} \cup \{z \in \mathbb{C} \mid \mathrm{Re}\,(z) \geq k\}$ for $k \in \mathbb{N}$. The spectrum of a nonzero Euler operator $P(\theta)$ on $C^\infty(\Omega)$ is equal to*

$$\sigma(P(\theta)) = \bigcap_{k\in\mathbb{N}} P\left((\mathbb{C}_{\geq k})^d\right).$$

In particular:

(a) *if there is $k \in \mathbb{N}$ such that $P(z) \neq 0$ if $z \in (\mathbb{C}_{\geq k})^d$, then $P(\theta)$ is invertible on $C^\infty(\Omega)$;*

(b) *for a nonzero homogeneous Euler operator $P(\theta)$ we have $\sigma(P(\theta)) \neq \mathbb{C}$ if and only if P has the* half-plane property*, that is, $P(z) \neq 0$ if $\mathrm{Re}\,(z_j) > 0$ for any $j = 1, \ldots, d$.*

There is an extensive literature on polynomials having the half-plane property which is partly motivated by applications to image processing (cf. the survey paper [6] and also [11, Sections 8 and 9]). For example, if P is an elementary symmetric polynomial, then $P(z)$ has the half-plane property, and so $P\big(z+(1,\ldots,1)\big)$ satisfies (a).

The proofs of the main theorems are based on the reduction of surjectivity of $P(\theta)$ on $C^\infty(\Omega)$ to surjectivity of "restricted" operators on $C^\infty(\Omega_{D\setminus J})$ (see Theorem 5.1) and then to surjectivity on the spaces $\mathcal{E}(\Omega_{D\setminus J,\sigma})$ of functions supported in the quadrants $\Omega_{D\setminus J,\sigma}$ (see Theorems 6.4 and 6.5). The sufficiency of the surjectivity on $\mathcal{E}(\Omega_{D\setminus J,\sigma})$ is proved via some inductive procedure for surjectivity of $P(\theta)$ on suitably chosen Whitney jets possibly starting with jets at the point zero. Finally, to get surjectivity on $\mathcal{E}(\Omega_\sigma)$ we introduce a new Mellin transform on $\mathcal{E}([0,\infty[^d)$ (see Theorem 8.2). The latter is one of the technical cores of the paper. In fact we believe that this more general Mellin transform will be of independent interest since it also covers distributions with support in $[0,\infty[^d$ possibly touching the singular set Z_1.

The paper is organized as follows. In Section 2 we collect some basic geometric properties of the set Ω. Then we start to study surjectivity in much more generality, considering Hadamard type operators (containing Euler partial differential operators) in Section 3. These are continuous linear operators $H : C^\infty(\Omega) \to C^\infty(\Omega)$ such that all monomials are eigenvectors, that is, there are $m_\alpha \in \mathbb{C}$ such that

$$H(\xi^\alpha)(x) = m_\alpha x^\alpha \text{ for any } \alpha \in \mathbb{N}^d.$$

The class of Hadamard type operators is the natural framework for the study of Euler operators since the inverse of, the resolvent of, and the semigroups generated by Euler differential operators (if they exist) must be Hadamard type operators.

The basic tool for our consideration is the Representation Theorem 3.1 of Vogt (see [24]) stating that Hadamard operators can be given as multiplicative convolution operators. The following are the main results of Section 3: Lemma 3.3 showing that the image of the Hadamard operator H (in particular, $P(\theta)$) on $C^\infty(\Omega)$ is always contained in the closed space $C^\infty_{I(H)}(\Omega)$ (in particular, $C^\infty_{I(P)}(\Omega)$) and the H-convexity criterion in Theorem 3.8 for surjectivity onto $C^\infty_{I(H)}(\Omega)$. The latter implies important inheritance properties for surjectivity (see Corollary 3.12 and Theorem 3.17) and some useful necessary conditions for the surjectivity of Euler partial differential operators $P(\theta)$; see Corollary 3.14.

In Section 4 we apply $P(\theta)$-convexity theory to first-order Euler operators. In significant cases we prove a complete characterization of surjectivity for such operators by a simple geometric criterion (see Corollary 4.5 and Proposition 4.8). We also present a series of examples showing essential differences between the solvability theory for constant coefficient linear partial differential equations and the corresponding theory for Euler partial differential equations.

The reduction and induction procedure mentioned above is developed for general Hadamard operators in Sections 5 and 6. The main results are Theorems 6.4 and 6.5 which lead to a sufficient condition and a necessary condition for surjectivity of $P(\theta)$ onto $C^\infty_{I(P)}(\Omega)$ in terms of surjectivity of $P(\alpha_J, \theta_{D\setminus J})$ for $\alpha_J \in \mathbb{N}^J$ on the space of functions with the support in the quadrants $Q_\sigma := \{x \mid \sigma x \geq 0\}, \sigma \in \{\pm 1\}^d$.

Section 7 is devoted to the discussion of admissible sets and prepares the proof of part (c) of the Main Theorem B; see Theorems 7.14 and 10.14.

In Section 8 we introduce and study the Mellin transform $\mathscr{M}$ on $\mathcal{E}([0,\infty[^d)'_b$. It turns out that this new version of the classical Mellin transform is a topological isomorphism onto a weighted space of holomorphic germs near $(\infty, \dots, \infty)$ (see Theorem 8.2, which is the main result of this section). The crucial point is that we succeed in defining the Mellin transform for any "distribution" over smooth functions with support contained in the quadrant $[0,\infty[^d$, contrary to the classical theory (comp. [3], [27], [28]).

The proofs of the Main Theorems A, B, and C are given in Sections 9 and 10. The main results in Section 9 are the following: the Local Existence Theorem 9.1, the Approximation Theorem 9.3, the proof of the Main Theorem A (Theorems 9.4 and 9.7) and the proof of part (a) of the Main Theorem B (in Theorem 9.5). In Section 10 we complete the proof of parts (b) and (c) of the Main Theorem B (see Theorems 10.13 and 10.14) and we show the Main Theorem C (see Theorem 10.7).

In Section 11 we prove the Main Theorem D and we study invertibility of $P(\theta)$. We also show cases when the kernel of $P(\theta)$ contains nontrivial smooth functions with support contained in some quadrant (Theorem 11.5). Surprisingly, we also describe the spectrum of some more general Hadamard type operators, namely inverses of invertible $P(\theta)$ (Corollary 11.9).

The solvability theory for general Hadamard operators on $C^\infty(\Omega)$ needs additional tools and will be contained in a subsequent paper (see [13]).

For the basic notions from functional analysis see [19].

2. Geometry of sets

In this section we introduce basic notions and explain some useful geometric facts for open sets $\Omega \subseteq \mathbb{R}^d$.

In the present paper, $\Omega \subset \mathbb{R}^d$ will always denote an open set in $\mathbb{R}^d$. We extend the order on $\mathbb{R}$ to $\mathbb{R}^d$ and set for $x, y \in \mathbb{R}^d$

$$x < y \text{ iff } x_j < y_j \text{ for any } j \le d \quad \text{ and } \quad x \le y \text{ iff } x_j \le y_j \text{ for any } j \le d.$$

Moreover,

$$\mathbf{c} := (c, \ldots, c) \text{ for } c \in \mathbb{R}.$$

Let $\operatorname{Log}(x) := (\log x_1, \ldots, \log x_d)$ for $x \in]0, \infty[^d$ and $\operatorname{Exp}(x) := (\exp x_1, \ldots, \exp x_d)$ for $x \in \mathbb{R}^d$. For $x, y \in \mathbb{C}^d$ the coordinatewise multiplication is defined by $xy := (x_1 y_1, \ldots, x_d y_d)$. The multiplier set of Ω is then defined by

$$V(\Omega) := \{x \in \mathbb{R}^d \mid x\Omega \subset \Omega\}.$$

We say that Ω is *nearly solid* if $V(\Omega) \supset [0,1]^d$, i.e., if for every $x \in \Omega$ and $t \in [0,1]^d$ we have $tx \in \Omega$.

Since we will often argue via restriction of the number of variables, the following notation for grouping of variables will be useful:

for $J \subset D := \{1, \ldots, d\}$ we set $x_{D\setminus J} := (x_i)_{i \in D\setminus J} \in \mathbb{R}^{D\setminus J}$ and $y_J := (y_j)_{j\in J} \in \mathbb{R}^J$.

Correspondingly, $(y_J, x_{D\setminus J}) := (\xi_j)_{j \le d}$ where $\xi_j := y_j$ if $j \in J$ (and $\xi_j := x_j$ if $j \in D \setminus J$, respectively). If $J = \emptyset$ (or $J = D$), then the variable y_J (and $x_{D\setminus J}$, respectively) has to be omitted.

For $J \subsetneq D$ let

$$\Omega_{D\setminus J} := \{x_{D\setminus J} \in \mathbb{R}^{D\setminus J} \mid (0_J, x_{D\setminus J}) \in \Omega\}. \tag{2.1}$$

Here $\Omega_D = \Omega$ and $x_D = x$.

A set $S \subset [0, \infty[^d$ is called multiplicatively convex (or m-convex) if

$$x^\tau y^{\mathbf{1}-\tau} := \begin{pmatrix} x_1^\tau y_1^{1-\tau} \\ \vdots \\ x_d^\tau y_d^{1-\tau} \end{pmatrix} \in S \text{ for any } x, y \in S \text{ and any } 0 < \tau < 1. \tag{2.2}$$

Notice that (2.2) implies that $\operatorname{Log}(S_{>\mathbf{0}})$ is convex in $\mathbb{R}^d$ and that $x_j^\tau y_j^{1-\tau} = 0$ for any $0 < \tau < 1$ if $x_j = 0$.

Let $Q_\sigma := \{x \in \mathbb{R}^d \mid \sigma x \ge \mathbf{0}\}$ denote the canonical closed quadrant for the sign $\sigma \in \{\pm 1\}^d$, and let $\Omega_\sigma := \Omega \cap Q_\sigma$. If $\mathbf{0} \in \Omega$, then Ω is called *m-convex* if

$$\forall \sigma \in \{\pm 1\}^d \ \forall K \Subset \sigma\Omega_\sigma : \overline{\operatorname{mconv}}(K) \Subset \sigma\Omega_\sigma, \tag{2.3}$$

where $\overline{\operatorname{mconv}}(S)$ denotes the closed m-convex hull of $S \subset [0, \infty[^d$. Clearly, just from the definition, $\Omega_{D\setminus J}$ is m-convex for any $J \subset D$ if Ω is m-convex.

For a set $S \subset \mathbb{R}^d$ let S° denote the interior of S. Obviously, $\operatorname{Log}(\sigma\Omega_\sigma^\circ)$ is convex for any $\sigma \in \{\pm 1\}^d$ if Ω is m-convex. The latter property is stronger than expected, as shown below.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be open with $\mathbf{0} \in \Omega$, and let $\operatorname{Log}(\sigma\Omega_\sigma^\circ)$ be convex for any $\sigma \in \{\pm 1\}^d$. Then the cubes $\Gamma_x := \{\xi \mid \mathbf{0} < \sigma\xi \le \sigma x\}, x \in \Omega_\sigma^\circ$, are contained in Ω.*

Proof. Let $B_\delta^\infty(\mathbf{0}) := \{x \in \mathbb{R}^d \mid \|x\|_\infty < \delta\} \subset \Omega$, and let $\sigma = \mathbf{1}$ without loss of generality. Let $x \in \Omega_{\mathbf{1}}^\circ$, i.e., $x > \mathbf{0}$ and $\xi := \operatorname{Log}(x) \in \operatorname{Log}(\Omega_{\mathbf{1}}^\circ)$. Since $\operatorname{Log}(\Omega_{\mathbf{1}}^\circ)$ is convex we get

$$\{y \mid y \le \xi\} \subset \{y \mid y < \xi + \gamma\mathbf{1}\} \subset \operatorname{conv}\{]-\infty, \log(\delta)]^d, \xi + \gamma\mathbf{1}\} \subset \operatorname{Log}(\Omega_{\mathbf{1}}^\circ)$$

for small $\gamma > 0$. $\square$

Let $S_{\{j\}} := \{x \in \mathbb{R}^d \mid x_j = 0\}$, and let π_j be defined by

$$\pi_j(x) := (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_d). \tag{2.4}$$

Then $\Omega \subset \mathbb{R}^d$ has the *projection property* (*pp*) if

$$\pi_j(\Omega) \subset \Omega \text{ for any } j \le d \text{ such that } \Omega \cap S_{\{j\}} \neq \emptyset. \tag{2.5}$$

Obviously,
(2.6)
$\Omega_{D\setminus J}$ has the projection property for any $J \subset D$ if Ω has the projection property.

Using property (*pp*) we now have the following simple description of m-convex open sets.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^d$ be open, and let $\mathbf{0} \in \Omega$. The following are equivalent:*

(a) *Ω is m-convex.*
(b) *$\mathrm{Log}(\sigma\Omega_\sigma^\circ)$ is convex for any $\sigma \in \{\pm 1\}^d$ and Ω satisfies the projection property (pp).*

Proof.
(a) $\Rightarrow$ (b) The first statement is trivial. The second is seen as follows: let $[-\delta, \delta]^d \subset \Omega$, and let $x \in \Omega$. For simplicity we may assume that $x \ge \mathbf{0}$, i.e., $x \in \Omega_{\mathbf{1}}$, and $x_j > 0$ without loss of generality. For $n \in \mathbb{N}$ set $y_n := (y_{n,k})_{k \le d}$ where $y_{n,k} := \delta$ if $k \neq j$ and $y_{n,j} := e^{-n^2}$. Then $y_n \in [0, \delta]^d \subset \Omega_{\mathbf{1}}$ for large n. Hence $y_n^{1/n} x^{1-1/n} \in \overline{\mathrm{mconv}}\{[0,\delta]^d, x\} \subset \Omega_{\mathbf{1}}$ by m-convexity and

$$\pi_j(x) = \lim_{n\to\infty} (y_n^{1/n} x^{1-1/n}) \in \overline{\mathrm{mconv}}\{[0,\delta]^d, x\} \subset \Omega_{\mathbf{1}} \subset \Omega.$$

(b) $\Rightarrow$ (a) i) Let $K \subset \sigma\Omega_\sigma$ be compact. Let $\sigma = \mathbf{1}$ without loss of generality, and let $\gamma_x := \{\xi \mid \mathbf{0} \le \xi < x\}$. Then

$$K \subset \bigcup_{x > \mathbf{0}, x \in \Omega_{\mathbf{1}}} \gamma_x.$$

By compactness we know that $K \subset \bigcup_{k=1}^n \gamma_{x_k} \subset \bigcup_{k=1}^n G_{x_k}$ for some $x_k > \mathbf{0}$ where $G_x := \{\xi \mid \mathbf{0} \le \xi \le x\}$. By Lemma 2.1 we have $\Gamma_x \subset \Omega_{\mathbf{1}}$ if $x \in \Omega_{\mathbf{1}}^\circ$ and hence also $\pi_j(\Gamma_{x_k}) \subset \Omega_{\mathbf{1}}$ for any j by assumption, since $\Omega \cap S_{\{j\}} \neq \emptyset$ for any j, and finally $G_{x_k} \subset \Omega_{\mathbf{1}}$. Hence we can assume that

$$K = \bigcup_{k=1}^n G_{x_k}, \text{ and therefore } K_{>\mathbf{0}} = \bigcup_{k=1}^n (G_{x_k})_{>\mathbf{0}} = \bigcup_{k=1}^n \Gamma_{x_k}.$$

ii) Let $g_x := \{\xi \mid \xi \le \mathrm{Log}(x)\}$. Then

$$\mathrm{Log}(K_{>\mathbf{0}}) = \bigcup_{k=1}^n g_{x_k} = \bigcup_{k=1}^n (\mathrm{Log}(x_k) - [0,\infty[^d) = (\bigcup_{k=1}^n \mathrm{Log}(x_k)) - [0,\infty[^d$$

since $[0,\infty[^d$ is an additive semigroup. The set $S := \mathrm{conv}\{\mathrm{Log}(x_1), \dots, \mathrm{Log}(x_n)\} - [0,\infty[^d$ is contained in $\mathrm{Log}(\Omega_{\mathbf{1}}^\circ)$ by Lemma 2.1 since $\mathrm{conv}\{\mathrm{Log}(x_1), \dots, \mathrm{Log}(x_n)\} \subset \mathrm{Log}(\Omega_{\mathbf{1}}^\circ)$ by convexity. S is closed since $\mathrm{conv}\{\mathrm{Log}(x_1), \dots, \mathrm{Log}(x_n)\}$ is compact and $[0,\infty[^d$ is closed. S is convex since $\mathrm{conv}\{\mathrm{Log}(x_1), \dots, \mathrm{Log}(x_n)\}$ and $[0,\infty[^d$ are convex. We have proved that

$$\overline{\mathrm{mconv}}(K) \cap]0,\infty[^d = \overline{\mathrm{mconv}}(K_{>\mathbf{0}}) \cap]0,\infty[^d \subset \exp(S) \subset \Omega_{\mathbf{1}}^\circ \subset \Omega.$$

iii) Please note that $\overline{\mathrm{Exp}(S)} \setminus \mathrm{Exp}(S) \subset \bigcup_{j\leq d} \pi_j(\Omega) \subset \Omega$. Moreover, $K_{>0}$ is dense in K, so

$$\begin{aligned}\overline{\mathrm{mconv}}(K) &= \overline{\mathrm{mconv}(K_{>0})} = \overline{\mathrm{mconv}(K_{>\mathbf{0}}) \cap]0,\infty[^d} \\ &\subset \overline{\overline{\mathrm{mconv}}(K_{>\mathbf{0}}) \cap]0,\infty[^d} \subset \overline{\mathrm{Exp}(S)} \subset \Omega.\end{aligned}$$

□

Summarizing, by Proposition 2.2 and Lemma 2.1 we have the following.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^d$ be open.*

(a) *If $\mathbf{0} \in \Omega$ and Ω is m-convex, then Ω is nearly solid.*

(b) *If Ω is nearly solid, then $\mathbf{0} \in \Omega$ and Ω has the projection property.*

Proof.

(a) Let $t \in [0,1]^d$ and $x \in \Omega$, and let $x \geq \mathbf{0}$ w.l.o.g. We have to show that $tx \in \Omega$. Let $J := \{j \in D \mid t_j = 0 \text{ or } x_j = 0\}$. Then $tx = \pi_J(t)\pi_J(x)$. Since $\Omega_{D\setminus J}$ is also m-convex we thus can assume that $x > \mathbf{0}$ and $t > \mathbf{0}$; hence $tx \in \Omega$ by Proposition 2.2 and Lemma 2.1.

(b) Let $x \in \Omega$. Then $\mathbf{0} = \mathbf{0}x \in \Omega$ and $\pi_j(x) = tx \in \Omega$ if $t_k := 1$ for $k \neq j$ and $t_j := 0$. □

3. $H-$convexity and surjectivity of Hadamard operators

In this section we introduce and discuss the notion of H-convexity as a criterium for surjectivity of Hadamard operators (and, in particular, of Euler operators). Recall that $V(\Omega) := \{x \in \mathbb{R}^d \mid x\Omega \subset \Omega\}$ and set

$$C^\infty(V(\Omega))' := \{T \in C^\infty(\mathbb{R}^d)' \mid \mathrm{supp}(T) \subset V(\Omega)\}.$$

A continuous linear operator $H : C^\infty(\Omega) \to C^\infty(\Omega)$ is called a *Hadamard operator* if all monomials are eigenvectors, that is, if there are $m_\alpha \in \mathbb{C}$ such that

$$H(\xi^\alpha)(x) = m_\alpha x^\alpha \text{ for any } \alpha \in \mathbb{N}^d.$$

The sequence $(m_\alpha)_{\alpha\in\mathbb{N}^d}$ is then called the multiplier sequence for H. Clearly, any Euler operator $P(\theta)$ is a Hadamard operator with $m_\alpha = P(\alpha)$. The following representation theorem is the basis of the present paper.

Theorem 3.1 ([24, Theorem 3.4])**.** *A continuous linear operator $H : C^\infty(\Omega) \to C^\infty(\Omega)$ is a Hadamard operator if and only if there is $T \in C^\infty(V(\Omega))'$ such that*

$$H(f)(x) := H_T(f)(x) := \langle {}_yT, f(xy)\rangle \text{ for any } f \in C^\infty(\Omega).$$

In this case, the multiplier sequence $(m_\alpha)_{\alpha\in\mathbb{N}^d}$ is the moment sequence of T, that is, $m_\alpha = \langle {}_yT, y^\alpha\rangle$ for any $\alpha \in \mathbb{N}^d$.

In other words, Hadamard operators are multiplicative convolution operators (see Proposition 10.3 for more details). If Ω is contained in the union of the canonical open quadrants, i.e., if

$$\Omega \subset \mathbb{R}^d \setminus Z_1 := \{x \in \mathbb{R}^d \mid \forall j \leq d : x_j \neq 0\},$$

then Hadamard operators on $C^\infty(\Omega)$ can be transformed to usual convolution operators by using log (and exp, respectively) as a transform in each variable (see [25]). This procedure and the solution theory of H_T for $T \in C^\infty(]0,\infty[^d)'$ is explained for this case in [25]. So we will mainly concentrate in the present paper on the case where

$$\Omega \cap Z_1 := \{x \in \Omega \mid x_j = 0 \text{ for some } j \leq d\} \neq \emptyset. \tag{3.1}$$

We start with an a priori bound for the range of Hadamard operators: Let

$$I(H) := \{\alpha \in \mathbb{N}^d \mid m_\alpha = 0\}$$

denote the zero set of the multiplier sequence $(m_\alpha)_{\alpha\in\mathbb{N}^d}$. Set $D := \{1,\dots,d\}$ and

$$\begin{aligned}C^\infty_{I(H)}(\Omega) :=\{f \in C^\infty(\Omega) \mid \forall J \subset D\ \forall\alpha \in \mathbb{N}^J\ \forall(0_J, x_{D\setminus J}) \in \Omega :\\ f^{(\alpha)}(0_J, x_{D\setminus J}) = 0 \text{ if } m_{(\alpha,\beta)} = 0 \text{ for any } \beta \in \mathbb{N}^{D\setminus J}\}.\end{aligned} \tag{3.2}$$

Here we have used the notation for the grouping of variables from Section 2. Notice that for $H = 0$ this means that $C^\infty_{I(H)}(\Omega) = C^\infty_{I(0)}(\Omega) = \{0\}$ is the trivial vector space (this is the only case where we may set $J = \emptyset$ in (3.2)). Also, if $0 \in \Omega$ and $m_\alpha = 0$, then $f^{(\alpha)}(0) = 0$ for any $f \in C^\infty_{I(H)}(\Omega)$ (set $J = D$ in (3.2)).

Recall that if $H = P(\theta)$ is an Euler operator, then $m_\alpha = P(\alpha)$ and $P(\alpha, \beta) = 0$ for any $\beta \in \mathbb{N}^{D\setminus J}$ iff $P(\alpha,\ \cdot\ ) \equiv 0$. Hence, in this case

$$\begin{aligned}C^\infty_{I(P)}(\Omega) := \{f \in C^\infty(\Omega) \mid\ \forall\ J \subset D\ \forall\ \alpha \in \mathbb{N}^J :\\ f^{(\alpha)}(0_J,\ \cdot\ ) \equiv 0 \text{ if } P(\alpha,\ \cdot\ ) \equiv 0\}.\end{aligned} \tag{3.3}$$

The spaces $C^\infty_{I(P)}(\Omega)$ are very sensitive for small changes of the set Ω and of the coefficients of P. We mention a simple but instructive example below.

Example 3.2.

(a) We have $C^\infty_{I(\theta_1-b)}(\Omega) = C^\infty(\Omega)$ if $\omega_1 := \Omega \cap (\{0\} \times \mathbb{R}^{d-1}) = \emptyset$ or if $b \notin \mathbb{N}$.

(b) If $\omega_1 \neq \emptyset$ and $b \in \mathbb{N}$ we get (with $x := (x_1, x')$)

$$\begin{aligned}C^\infty_{I(\theta_1-b)}(\Omega) &= \{f \in C^\infty(\Omega) \mid (\partial_1^b f)(0, x') = 0 \text{ if } (0, x') \in \Omega\}\\ &= \{f \in C^\infty(\Omega) \mid \exists c_j \in C^\infty(\Omega_{D\setminus\{1\}}), g \in C^\infty(\Omega) :\\ &\qquad f(x) = \sum_{j=0}^{b-1} c_j(x')x_1^j + x_1^{b+1}g(x) \text{ near } \omega_1\}.\end{aligned}$$

Proof.

(a) This is evident by (3.3).

(b) The first equation follows from (3.3). To prove "⊂" in the second equation we set

$$F(x) := f(x) - \sum_{j=0}^{b-1} \partial_1^j f(0, x')x_1^j/j! \text{ near } \omega_1$$

and get $\partial_1^j F = 0$ on ω_1 for $j \leq b$. We prove by induction that $F(x) = x_1^{b+1}g(x)$ near ω_1. For $b = 0$ we may set

$$g(x) := \int_0^1 \partial_1 F(tx_1, x')dt$$

near ω_1 (see [15, proof of Theorem 1.1.9]). Let the claim hold for $b - 1 \geq 0$. Then $\partial_1 F = x_1^b g(x)$ near ω_1. By Proposition 4.1 below there is a smooth function h such that $(\theta_1 + b + 1)h = g$ near ω_1. Hence

$$\partial_1(x_1^{b+1}h) = x_1^b(\theta_1 + b + 1)h = x_1^b g = \partial_1 F \text{ near } \omega_1,$$

and therefore $F(x) = x_1^{b+1}h(x)$ near ω_1 since $F = 0$ on ω_1.

The inclusion "⊃" is trivial. ∎

Lemma 3.3. *Let H be a Hadamard operator on $C^\infty(\Omega)$ with multiplier sequence $(m_\alpha)_{\alpha\in\mathbb{N}^d}$. Then $H(C^\infty(\Omega))$ is contained in $C^\infty_{I(H)}(\Omega)$.*

Proof. If Ω does not satisfy (3.1), then $C^\infty_{I(H)}(\Omega) = C^\infty(\Omega)$ and the statement is trivial. Let Ω satisfy (3.1). Notice that we have for any $f \in C^\infty(\Omega)$ and $H := H_T$ by Theorem 3.1

$$\begin{aligned}\partial_x^\alpha H(f)(x) &= \partial_x^\alpha \langle_y T, f(xy)\rangle = \langle_y T, \partial_x^\alpha f(xy)\rangle \\ &= \langle_y T, y^\alpha (\partial^\alpha f)(xy)\rangle \text{ for any } x \text{ and } \alpha.\end{aligned} \tag{3.4}$$

For $\alpha \in \mathbb{N}^J$ the operator

$$G_\alpha : C^\infty(\Omega) \to C^\infty(\Omega_{D\setminus J}), G_\alpha(f) := \partial^\alpha[H(f)]\big|_{\Omega_{D\setminus J}}$$

is linear and continuous, hence $G_\alpha(C^\infty(\Omega))$ is contained in the closure of $G_\alpha(Pol)$ in $C^\infty(\Omega_{D\setminus J})$, where Pol are the polynomials in d variables (recall that Pol is dense in $C^\infty(\Omega)$). If $m_{(\alpha,\beta)} = 0$ for any $\beta \in \mathbb{N}^{D\setminus J}$ we get for any $f(x) := x_J^\gamma x_{D\setminus J}^\eta, \gamma \in \mathbb{N}^J, \eta \in \mathbb{N}^{D\setminus J}$, by (3.4),

$$\begin{aligned}G_\alpha(f)(x_{D\setminus J}) &= \langle_y T, y_J^\alpha \partial^\alpha(x_J^\gamma)(x_{D\setminus J} y_{D\setminus J})^\eta\rangle\big|_{x_J=0} \\ &= \delta_{\alpha,\gamma}\alpha!\langle_y T, y_J^\alpha y_{D\setminus J}^\eta\rangle x_{D\setminus J}^\eta = \delta_{\alpha,\gamma}\alpha! m_{(\alpha,\eta)} x_{D\setminus J}^\eta = 0\end{aligned}$$

by assumption; hence $G_\alpha \equiv 0$. This proves the lemma. □

Remark 3.4. Let Pol denote the vector space of polynomials. Evidently we have

$$H(Pol) = Pol_{I(H)} := \operatorname{span}\{\xi^\alpha \mid m_\alpha \neq 0\}, \text{ and hence } H(C^\infty(\Omega)) \subset \overline{Pol_{I(H)}}^{C^\infty(\Omega)}.$$

The latter space may be essentially smaller than $C^\infty_{I(H)}(\Omega)$. Indeed, if $m_{(\alpha_J,\eta_k)} = 0$ for all odd η_k ($D \setminus J =: \{k\}$ a singleton) and $m_{(\alpha_J,\,\cdot\,)} \neq 0$, then $Pol_{I(H)}$ consists of polynomials which are even w.r.t. x_k; hence by the inclusion above all functions in the range of H also are even w.r.t. x_k.

By Lemma 3.3, the best we can hope for is that

$$H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega) \text{ is surjective.} \tag{3.5}$$

We now start with the description of solvability of Hadamard equations using a variant of $P(\partial)$-convexity (see [15, 10.6.1] and (1.1)) adapted to Hadamard operators. Here and in the rest of the paper the following surjectivity criterion for Fréchet spaces (see [19, Theorem 26.1]) will be helpful: let E, F be Fréchet spaces, and let $H : E \to F$ be continuous and linear. Then H is surjective iff

$$B \subset F' \text{ is bounded in } F_b' \text{ if } {}^tH(B) \text{ is bounded in } E_b'. \tag{3.6}$$

Before using (3.6) for a Hadamard operator $H : E := C^\infty(\Omega) \to F := C^\infty_{I(H)}(\Omega)$ we first have to discuss the meaning of support in $C^\infty_{I(H)}(\Omega)'$. Here we will profit from the fact that $C^\infty_{I(H)}(\Omega)$ is a local space. Notice that this is not true in general for the space $\overline{Pol_{I(H)}}^{C^\infty(\Omega)}$ by Remark 3.4.

Choose $\varphi \in \mathscr{D}(-1,1)$ such that

$$\varphi \equiv 1 \text{ near } 0 \text{ and } \sum_{j\in\mathbb{Z}} \varphi(x-j) \equiv 1 \text{ on } \mathbb{R} \tag{3.7}$$

(see [15, Theorem 1.4.6]). Let $\mathscr{D}_{lc}(\mathbb{R}^d) \subset \mathscr{D}(\mathbb{R}^d)$ be the algebra generated by the functions

$$\psi_{\ell,j,k}(x) := \varphi\big(2^k x_\ell - j\big), \ell \le d, j \in \mathbb{Z}, k \in \mathbb{N},$$

and set

$$\mathscr{D}_{lc}(\Omega) := \mathscr{D}_{lc}(\mathbb{R}^d) \cap \mathscr{D}(\Omega).$$

Proposition 3.5.
(a) *$C^\infty_{I(H)}(\Omega)$ is a $\mathscr{D}_{lc}(\Omega)$-modul by pointwise multiplication.*
(b) *For any covering $\mathfrak{U} := \{U_i \mid i \in I\}$ of Ω and any compact set $K \subset \Omega$ there is a finite resolution of the identity near K in $\mathscr{D}_{lc}(\Omega)$ subordinate to $\mathfrak{U}$.*

Proof.
(a) Since $\mathscr{D}_{lc}(\Omega)$ is generated as an algebra by the functions $\psi_{\ell,j,k}$, the claim has to be proved for these functions only. For any $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}^J$ we obviously have that if $\ell \notin J$,

$$\partial^\alpha_J(\psi_{\ell,j,k}(x)f(x))\big|_{(0_J,x_{D\setminus J})} = \psi_{\ell,j,k}(0_J,x_{D\setminus J})\partial^\alpha_J f(0_J,x_{D\setminus J}).$$

If $\ell \in J$, then $\psi_{\ell,j,k}(x) \equiv 1$ near 0_J by (3.7) (if $j=0$) or $\psi_{\ell,j,k}(x) \equiv 0$ near 0_J (if $j \neq 0$) since $\varphi \in \mathscr{D}(-1,1)$. Therefore

$$\partial^\alpha_J(\psi_{\ell,j,k}(x)f(x))\big|_{(0_J,x_{D\setminus J})} = \psi_{\ell,j,k}(0)\partial^\alpha_J f(0_J,x_{D\setminus J})$$

in this case. Hence $\psi_{\ell,j,k}f \in C^\infty_{I(H)}(\Omega)$ if $f \in C^\infty_{I(H)}(\Omega)$.
(b) This is evident since $\mathscr{D}_{lc}(\mathbb{R}^d)$ for any $k \in \mathbb{N}$ contains the resolution

$$\Psi_{j,k}(x) := \prod_{\ell \le d} \psi_{\ell,j_\ell,k}(x), j \in \mathbb{Z}^d,$$

which is subordinate to the covering of $\mathbb{R}^d$ by $B^\infty_{2^{-k}}(j2^{-k}), j \in \mathbb{Z}^d$. □

Using Proposition 3.5 we could develop a full distribution theory based on $C^\infty_{I(H)}$. We only mention those parts needed in the sequel. Let

$$\mathscr{D}_{I(H)}(\Omega) := C^\infty_{I(H)}(\Omega) \cap \mathscr{D}(\Omega).$$

For $T \in C^\infty_{I(H)}(\Omega)'$ the support $\mathrm{supp}_{I(H)}(T)$ is defined as usual: $x \in \Omega \setminus \mathrm{supp}_{I(H)}(T)$ if there is an open neighborhood U_x of x such that

$$T(f) = 0 \text{ for any } f \in \mathscr{D}_{I(H)}(U_x). \tag{3.8}$$

Proposition 3.6. *Let $T \in C^\infty_{I(H)}(\Omega)'$.*

(a) $\mathrm{supp}_{I(H)}(T)$ *is compact.*
(b) *For any $f \in C^\infty_{I(H)}\big(\Omega \setminus \mathrm{supp}_{I(H)}(T)\big)$ we have $T(f)=0$.*
(c) *If $\psi \in \mathscr{D}_{lc}(\Omega)$ and $\psi \equiv 1$ near $\mathrm{supp}_{I(H)}(T)$, then $T(f) = T(\psi f)$ for any $f \in C^\infty_{I(H)}(\Omega)$.*
(d) *A set $B \subset C^\infty_{I(H)}(\Omega)'_b$ is bounded iff B is bounded in $C^\infty_{I(H)}(W)'_b$ for some open $W \supset \Omega$ (and, equivalently, for any open $W \supset \Omega$) and there is a compact set $K \subset \Omega$ such that $\mathrm{supp}_{I(H)}(T) \subset K$ for any $T \in B$.*

Proof.

(a) $\operatorname{supp}_{I(H)}(T)$ is closed by definition. Since $T \in C^\infty_{I(H)}(\Omega)'$ there is a compact set $K \subset \Omega$ and $k \in \mathbb{N}$ such that

$$|T(f)| \leq C\|f\|_{K,k} \text{ for any } f \in C^\infty_{I(H)}(\Omega), \tag{3.9}$$

where $\|f\|_{K,k} := \sup_{x\in K, \alpha \leq \mathbf{k}} |f^{(\alpha)}(x)|$. Hence $\operatorname{supp}_{I(H)}(T) \subset K$ is compact.

(b) For $f \in \mathscr{D}_{I(H)}\big(\Omega \setminus \operatorname{supp}_{I(H)}(T)\big)$ this follows using a resolution of the identity near $\operatorname{supp}(f)$ from Proposition 3.5(b) subordinate to the covering U_x from above. If $f \in C^\infty_{I(H)}(\Omega)$ we may choose $\psi_n \in \mathscr{D}_{lc}(\Omega)$ such that $\psi_n = 1$ near K_n for an increasing compact exhaustion $(K_n)_n$ of Ω. The existence of such ψ_n follows from Proposition 3.5(b). Then $f\psi_n \in \mathscr{D}_{I(H)}\big(\Omega \setminus \operatorname{supp}_{I(H)}(T)\big)$ by Proposition 3.5(a) and $f\psi_n \to f$ in $C^\infty_{I(H)}(\Omega)$; hence $T(f) = \lim_n T(\psi_n f) = 0$.

(c) This is evident by (b).

(d) "$\Rightarrow$" The first claim clearly holds for any open $W \supset \Omega$. B is equicontinuous on $C^\infty_{I(H)}(\Omega)$ by assumption; hence (3.9) holds with K (and C and k) independent of $T \in B$. Hence $\operatorname{supp}_{I(H)}(T) \subset K$ for any $T \in B$.

"$\Leftarrow$" B is equicontinuous on $C^\infty_{I(H)}(W)$ for some open $W \supset \Omega$ by assumption. Hence there are a compact set $L \subset W$, $k \in \mathbb{N}$, and $C > 0$ such that for any $T \in B$,

$$|T(f)| \leq C\|f\|_{L,k} \text{ for any } f \in C^\infty_{I(H)}(W). \tag{3.10}$$

Let $\psi \in \mathscr{D}_{lc}(\Omega)$ such that $\psi \equiv 1$ near K. Then (c) and (3.10) imply

$$|T(f)| = |T(\psi f)| \leq C\|\psi f\|_{L,k} \leq C_1\|f\|_{\operatorname{supp}(\psi),k} \text{ for any } f \in C^\infty_{I(H)}(\Omega) \tag{3.11}$$

for any $T \in B$ by Leibniz' rule since $\psi f \in D_{I(H)}(\Omega) \subset D_{I(H)}(W)$. This shows the claim since $\operatorname{supp}(\psi) \subset \Omega$ is compact. ☐

Let $C^{\mathbf{k}}_{I(H)}(\Omega)'$ denote the set of all $T \in C^\infty_{I(H)}(\Omega)'$ which are continuous w.r.t. the seminorm $\|f\|_{K,k} := \sup_{x\in K,\alpha\leq\mathbf{k}} |f^{(\alpha)}(x)|$ for some compact set $K \subset \Omega$.

Definition 3.7. Let $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ be a Hadamard operator. An open set $\Omega \subset \mathbb{R}^d$ is called H-convex (for support) if

$$\begin{aligned} &\forall K \Subset \Omega\ \forall k \in \mathbb{N}\ \exists \widetilde{K} \Subset \Omega\ \forall T \in C^{\mathbf{k}}_{I(H)}(\Omega)' : \\ &\qquad\qquad \operatorname{supp}_{I(H)}(T) \subset \widetilde{K} \text{ if } \operatorname{supp}({}^tH(T)) \subset K. \end{aligned} \tag{3.12}$$

Theorem 3.8. *Let $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ be a Hadamard operator.*

(a) *If $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ is surjective, then Ω is H-convex.*
(b) *If Ω is H-convex and $H : C^\infty(W) \to C^\infty_{I(H)}(W)$ is surjective for some open set $W \supset \Omega$, then $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ is surjective.*

Proof.

(a) If Ω is not H-convex there are a compact set $K \subset \Omega$, $k \in \mathbb{N}$, and $T_n \in C^{\mathbf{k}}_{I(H)}(\Omega)'$ such that $\operatorname{supp}({}^tH(T_n)) \subset K$ for any n while $\bigcup_n \operatorname{supp}_{I(H)}(T_n)$ is not contained in any compact set $\widetilde{K} \subset \Omega$. Since $T_n \in C^{\mathbf{k}}_{I(H)}(\Omega)'$ there are $A_n > 0$ and compact sets $K_n \subset \Omega$ such that

$$|T_n(f)| \leq A_n\|f\|_{K_n,k} \text{ for any } f \in C^\infty_{I(H)}(\Omega).$$

Choose $h \in \mathscr{D}(\Omega)$ such that $h \equiv 1$ near K. Since $\operatorname{supp}({}^tH(T_n)) \subset K$ we get

$$\begin{aligned}(3.13)\qquad |{}^tH(T_n)(g)| = |{}^tH(T_n)(gh)| &= |T_n(H(gh))| \leq A_n \|H(gh)\|_{K_n,k} \\ &\leq B_n \|gh\|_{L_n,j} \leq C_n \|g\|_{\operatorname{supp}(h),j} \qquad \text{for any } g \in C^\infty(\Omega)\end{aligned}$$

by Leibniz' rule for suitable compact sets $L_n \subset \Omega$. Notice that j may be chosen independent of n by the formula for H from the Representation Theorem 3.1. Let $B := \{T_n/C_n \mid n \in \mathbb{N}\}$. Then $B \subset C^\infty_{I(H)}(\Omega)'$ and ${}^tH(B)$ is bounded in $C^\infty(\Omega)'_b$ by (3.13). Hence B is bounded in $C^\infty_{I(H)}(\Omega)'_b$ by (3.6). By Proposition 3.6(d) we thus have $\operatorname{supp}_{I(H)}(T_n) = \operatorname{supp}_{I(H)}(T_n/C_n) \subset \widetilde{K}$ for any n for some fixed compact set $\widetilde{K} \subset \Omega$, a contradiction.

(b) We again use (3.6). Let $B \subset C^\infty_{I(H)}(\Omega)'$ such that ${}^tH(B)$ is bounded in $C^\infty(\Omega)'_b$. Then there is a compact set $K \subset \Omega$ such that $\operatorname{supp}({}^tH(T)) \subset K$ for any $T \in B$, and, by H-convexity, there is a compact set $\widetilde{K} \subset \Omega$ such that $\operatorname{supp}_{I(H)}(T) \subset \widetilde{K}$ for any $T \in B$. Also, ${}^tH(B)$ is bounded in $C^\infty(W)'_b$, and hence B is bounded in $C^\infty_{I(H)}(W)'_b$ by (3.6) since $H : C^\infty(W) \to C^\infty_{I(H)}(W)$ is surjective by assumption. Hence B is bounded in $C^\infty_{I(H)}(\Omega)'_b$ by Proposition 3.6(d), proving surjectivity of $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ by (3.6). □

We will see later in Section 9 that Euler differential operators $P(\theta) : C^\infty(\mathbb{R}^d) \to C^\infty_{I(P)}(\mathbb{R}^d)$ are always surjective. So $P(\theta)$-convexity of Ω and surjectivity of $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ are equivalent (see Theorem 9.5).

Remark 3.9. When using H-convexity we must carefully distinguish between $\operatorname{supp}(T)$ and $\operatorname{supp}_{I(H)}(T)$. In fact, if $0 \neq T \in C^\infty(\Omega)'$ such that $T = 0$ on $C^\infty_{I(H)}(\Omega)$, then $\operatorname{supp}(T)$ is nontrivial while $\operatorname{supp}_{I(H)}(T) = \emptyset$. Moreover, we have ${}^tH(T) = 0$, where ${}^tH$ denotes the transpose of $H : C^\infty(\Omega) \to C^\infty(\Omega)$. The latter observation is useful when constructing zero solutions of ${}^tH$ with small support (see Example 3.11 below).

Example 3.10. The transpose of an Euler operator $P(\theta) : C^\infty(\Omega) \to C^\infty(\Omega)$ is given by ${}^tP(\theta) = P(-\theta - 1)$.

Proof. This follows from

$$\langle(-\theta_j - 1)S, f\rangle = \langle S, \partial_j(x_j f) - f\rangle = \langle S, \theta_j f\rangle = \langle {}^t\theta_j S, f\rangle$$

for $S \in C^\infty(\Omega)'$ and $f \in C^\infty(\Omega)$. □

Contrary to constant coefficient operators, Euler operators may have zero solutions with point support. We present a basic example.

Example 3.11. If $P(-\alpha - \mathbf{1}) = 0$ for some $\alpha \in \mathbb{N}^d$, then $P(\theta)\delta_{\mathbf{0}}^{(\alpha)} = 0$.

Proof. Set $Q(\theta) := P(-\theta - \mathbf{1})$. Then ${}^tQ(\theta) = P(\theta)$ by Example 3.10 and $\delta_{\mathbf{0}}^{(\alpha)} = 0$ on $C^\infty_{I(Q)}(\mathbb{R}^d)$ by assumption. The claim follows from Remark 3.9. □

Using Theorem 3.8 we now discuss some inheritance properties for the surjectivity of $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$. Recall that for open sets $\Omega_i \subset \mathbb{R}^d$,

$$\liminf_{i \in I} \Omega_i := \bigcup_{J \subset I \text{finite}} \Big(\bigcap_{i \in I \setminus J} \Omega_i \Big).$$

Corollary 3.12. *Let the Hadamard operator* $H : C^\infty(\Omega_i) \to C^\infty_{I(H)}(\Omega_i), i \in I$, *be surjective.*

(a) *If* $\emptyset \neq \Omega := \bigcap_{i\in I} \Omega_i$ *is open, then* $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ *is surjective.*
(b) *If* $\bigcap_{i\in I\setminus J} \Omega_i$ *is open for all finite* $J \subset I$, *then* $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ *is surjective if* $\emptyset \neq \Omega := \liminf_{i\in I} \Omega_i$.

Proof.
(a) Let $H = H_T$ where $\operatorname{supp}(T) \subset \bigcap_i V(\Omega_i)$ by the Representation Theorem 3.1. Since $\bigcap_i V(\Omega_i) \subset V(\cap_i \Omega_i) = V(\Omega)$ and Ω is open, $H = H_T$ operates on $C^\infty(\Omega)$ by Theorem 3.1 again. Since $H : C^\infty(\Omega_i) \to C^\infty_{I(H)}(\Omega_i)$ is surjective and $\Omega_i \supset \Omega$ we need to show by Theorem 3.8(b) that Ω is H-convex. Fix $k \in \mathbb{N}$ and let $K \subset \Omega$ be compact. Let $T \in C^{\mathbf{k}}_{I(H)}(\Omega)'$, and let $\operatorname{supp}({}^tH(T)) \subset K$. Since Ω_i is H-convex for any i by Theorem 3.8(a) there are compact sets $\widetilde{K}_i \subset \Omega_i$ (depending only on $\mathbf{k}$ and K) such that $\operatorname{supp}_{I(H)}(T) \subset \widetilde{K}_i$. Hence $\operatorname{supp}_{I(H)}(T) \subset \widetilde{K} := \bigcap_{i\in I} \widetilde{K}_i$. Since $\widetilde{K} \subset \Omega$ is compact, Ω is H-convex.

(b) As proven in (a), H operates on $C^\infty(\bigcap_{i\in I\setminus J} \Omega_i)$ for any finite $J \subset I$, hence H operates on $C^\infty(\Omega)$. Let $B \subset C^\infty_{I(H)}(\Omega)'$ such that ${}^tH(B)$ is bounded in $C^\infty(\Omega)'_b$. Then there is a compact set $K \subset \Omega$ such that $\operatorname{supp}({}^tH(T)) \subset K$ for any $T \in B$. Since $\bigcap_{i\in I\setminus J} \Omega_i$ is open for any finite $J \subset I$, by compactness there is $J \subset I$ finite such that $K \subset \omega := \bigcap_{i\in I\setminus J} \Omega_i \subset \Omega$. Hence ${}^tH(B)$ is bounded in $C^\infty(\omega)'_b$ and B is bounded in $C^\infty_{I(H)}(\omega)'_b$ by (3.6) since $H : C^\infty(\omega) \to C^\infty_{I(H)}(\omega)$ is surjective by (a). Hence $B \subset C^\infty_{I(H)}(\Omega)'_b$ is bounded, and $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ is surjective by (3.6). □

Notice that Ω in Corollary 3.12(a) and (b) need not be open. For constant coefficient partial differential operators the corresponding result holds for Ω° instead of Ω (without our extra assumption). However, in our case these assumptions cannot be omitted (see Example 4.6).

Corollary 3.12(a) especially applies to finite intersections. We mention a significant case.

Corollary 3.13. *Let* $H = H_T$ *be a Hadamard operator with* $\operatorname{supp}(T) \subset [0,\infty[^d$, *and let* $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ *and* $H : C^\infty(]0,\infty[^d) \to C^\infty(]0,\infty[^d)$ *be surjective. Then* $H : C^\infty(\Omega^\circ_\sigma) \to C^\infty(\Omega^\circ_\sigma)$ *is surjective for any* $\sigma \in \{\pm 1\}^d$ *with* $\Omega_\sigma \neq \emptyset$.

Proof. Recall that $\Omega_\sigma := \{x \in \Omega \mid \sigma x \geq 0\}$. Since the mapping $M_\sigma(f)(x) := f(\sigma x)$ is a Hadamard operator, M_σ commutes with H and hence $H : C^\infty(\sigma]0,\infty[^d) \to C^\infty_{I(H)}(\sigma]0,\infty[^d)$ is surjective. The claim now follows from Corollary 3.12(a) since $\Omega \cap \sigma]0,\infty[^d = \Omega^\circ_\sigma$. □

Corollary 3.14. *Let* $0 \neq P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ *be a surjective Euler operator. Then* $P(\theta) : C^\infty(\Omega^\circ_\sigma) \to C^\infty(\Omega^\circ_\sigma)$ *is surjective for any* $\sigma \in \{\pm 1\}^d$ *with* $\Omega_\sigma \neq \emptyset$.

Proof. This follows from Corollary 3.13 since any $0 \neq P(\theta) : C^\infty(]0,\infty[^d) \to C^\infty(]0,\infty[^d)$ can be transformed to the surjective operator $P(\partial) : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ using $\operatorname{Log}(x) = (\log(x_1),\ldots,\log(x_d))$ as a transformation of variables; cf. [25]. □

Further significant applications of Corollary 3.12 are gathered in the following theorem.

Theorem 3.15. *Let the Hadamard operator $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ be surjective.*

(a) *If $0 \in \Omega$, then $H : C^\infty(\mathbb{R}^d) \to C^\infty_{I(H)}(\mathbb{R}^d)$ is surjective.*

(b) *Let $\Omega := \mathbb{R}^d \setminus S$ where $S \subset \mathbb{R}^d$ is compact and $\mathbf{0} \in S$. Then $H : C^\infty(\mathbb{R}^d \setminus \{\mathbf{0}\}) \to C^\infty_{I(H)}(\mathbb{R}^d \setminus \{\mathbf{0}\})$ is surjective. Also, $\mathbb{R}^d$ is H-convex.*

Proof.

(a) We use Corollary 3.12(b) for $\Omega_n := n\Omega, n \in \mathbb{N}$. Since H commutes with dilations, $H : C^\infty(n\Omega) \to C^\infty_{I(H)}(n\Omega)$ is surjective by assumption. Notice that

$$\mathbb{R}^d \supset \liminf_{n\in\mathbb{N}} n\Omega \supset \bigcup_{k\in\mathbb{N}} (\cap_{n\geq k} n\Omega) = \mathbb{R}^d$$

since $B^\infty_\varepsilon(0) \subset \Omega$ for some $\varepsilon > 0$. Let $J \subset \mathbb{N}$ be finite. Then the set $\bigcap_{n\notin J} n\Omega$ is open. Indeed, if $x \in \bigcap_{n\notin J} n\Omega$, then $x/n \in \Omega$ for any $n \notin J$. Since $x/n \in B^\infty_{\varepsilon/2}(0)$ for $n \geq k_0$ we get

$$B^\infty_\delta(x)/n = x/n + B^\infty_\delta(0)/n \subset B^\infty_\varepsilon(0) \subset \Omega \text{ if } \delta \leq \varepsilon/2 \text{ and } n \geq k_0.$$

For $k_0 \geq n$ and $n \notin J$ we similarly get for $2\delta \leq \delta_0 := \min_{n\notin J, k_0\geq n} \operatorname{dist}(x/n, \partial\Omega)$

$$\operatorname{dist}(B^\infty_\delta(x)/n, \partial\Omega) \geq \delta_0 - \delta/n \geq \delta_0/2.$$

(b) (i) We argue as above for $\Omega_n := \Omega/n, n \in \mathbb{N}$. Notice that

$$\mathbb{R}^d \setminus \{0\} \supset \liminf_{n\in\mathbb{N}} \Omega/n \supset \bigcup_{k\in\mathbb{N}} (\bigcap_{n\geq k} \Omega/n) = \mathbb{R}^d \setminus \{0\}$$

by the assumptions on Ω. The set $\bigcap_{n\notin J} \Omega/n$ is open if $J \subset \mathbb{N}$ is finite. Indeed, if $x \in \bigcap_{n\notin J} \Omega/n$, then $x \neq 0$ and $nx \in \Omega$ for any $n \notin J$. Let $S \subset B^\infty_C(0)$. Let $\xi \in B^\infty_{|x|/2}(0)$. Then

$$|n(x + \xi)| \geq n|x|/2 > C \text{ if } n > k_0 := 2C/|x|.$$

Hence $nB^\infty_{|x|/2}(x) \subset \Omega$ for $n > k_0$. Obviously, there is $\delta > 0$ such that $nB^\infty_\delta(x) \subset \Omega$ if $n \leq k_0$ and $n \notin J$. Thus $H : C^\infty(\mathbb{R}^d \setminus \{\mathbf{0}\}) \to C^\infty_{I(H)}(\mathbb{R}^d \setminus \{\mathbf{0}\})$ is surjective by Corollary 3.12(b).

(ii) By using the Representation Theorem 3.1 and (i), $H = H_T$ for some $T \in C^\infty(V(\mathbb{R}^d \setminus \{0\}))' = C^\infty((\mathbb{R} \setminus \{0\})^d)'$, since $V(\mathbb{R}^d \setminus \{0\}) = (\mathbb{R} \setminus \{0\})^d$. Specifically, there is $\varepsilon > 0$ such that $|x_j| \geq \varepsilon$ for any $j \leq d$ if $x \in \operatorname{supp}(T)$. Fix a compact set $K \subset \mathbb{R}^d$ and $k \in \mathbb{N}$, and let $S \in C^{\mathbf{k}}_{I(H)}(\mathbb{R}^d)'$ with $\operatorname{supp}({}^tH(S)) \subset K$. Choose $\psi \in \mathscr{D}_{lc}(B^\infty_2(0))$ such that $\psi(x) = 1$ for $\|x\|_\infty \leq 1$. By Proposition 10.3 we get

$$\operatorname{supp}({}^tH_T((1-\psi)S)) = \operatorname{supp}((1-\psi)S \star T) \subset \operatorname{supp}(T) \cdot \operatorname{supp}((1-\psi)S) \subset \mathbb{R}^d \setminus B^\infty_\varepsilon(0)$$

and

$$\begin{aligned}\operatorname{supp}({}^tH_T((1-\psi)S)) &\subset \operatorname{supp}({}^tH_T(S)) \cup \operatorname{supp}({}^tH_T(\psi S))\\ &\subset K \cup (\operatorname{supp}(T) \cdot \overline{B^\infty_2(0)}) =: \widehat{K}.\end{aligned}$$

Hence, $\operatorname{supp}({}^tH_T((1-\psi)S)) \subset \widehat{K} \setminus B^\infty_\varepsilon(0)$. By Theorem 3.8(a) and (i) there is a compact set $\widetilde{K} \subset \mathbb{R}^d \setminus \{0\}$ (only depending on K and k) such that $\operatorname{supp}_{I(H)}((1-\psi)S) \subset \widetilde{K}$. Since $\operatorname{supp}_{I(H)}(\psi S) \subset B^\infty_2(0)$ we have $\operatorname{supp}_{I(H)}(S) \subset \widetilde{K} \cup \overline{B^\infty_2(0)}$. This proves the H-convexity of $\mathbb{R}^d$. □

Recall that

$$\Omega_{D\setminus J} := \{x_{D\setminus J} \in \mathbb{R}^{D\setminus J} \mid (0_J, x_{D\setminus J}) \in \Omega\}.$$

If $J = \emptyset$ this means that $\Omega_D = \Omega$. The set $\Omega_{D\setminus J}$ may be void while $\Omega_{D\setminus J} \neq \emptyset$ for some $\emptyset \neq J \subset D$ if (3.1) holds.

Notice that if $\Omega_{D\setminus J} \neq \emptyset$, then

$$V(\Omega) \subset \mathbb{R}^J \times V(\Omega_{D\setminus J}) = V(\mathbb{R}^J \times \Omega_{D\setminus J}). \tag{3.14}$$

Indeed, let $x \in V(\Omega)$ and $y_{D\setminus J} \in \Omega_{D\setminus J}$. Then $(0_J, y_{D\setminus J}) \in \Omega$ and $(0_J, x_{D\setminus J} y_{D\setminus J}) = x(0_J, y_{D\setminus J}) \in \Omega$ by the definition of $V(\Omega)$, that is, $x_{D\setminus J} y_{D\setminus J} \in \Omega_{D\setminus J}$, and therefore $x_{D\setminus J} \in V(\Omega_{D\setminus J})$, proving the inclusion in (3.14). The equality in (3.14) is easy.

By (3.14) and the Representation Theorem 3.1, any Hadamard operator H on $C^\infty(\Omega)$ also defines a Hadamard operator on $C^\infty(\mathbb{R}^J \times \Omega_{D\setminus J})$.

Theorem 3.16. *Let $\Omega \subset \mathbb{R}^J \times \Omega_{D\setminus J}$ for some $\emptyset \neq J \subsetneqq D$. If the Hadamard operator $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ is surjective, then also $H : C^\infty(\mathbb{R}^J \times \Omega_{D\setminus J}) \to C^\infty_{I(H)}(\mathbb{R}^J \times \Omega_{D\setminus J})$ is surjective.*

Proof. We modify the proof of Theorem 3.15(a) using Corollary 3.12(b) for $\Omega_n := \widetilde{n}\Omega, n \in \mathbb{N}$, where $\widetilde{n} := (n\mathbf{1}_J, \mathbf{1}_{D\setminus J})$. As before, $H : C^\infty(\widetilde{n}\Omega) \to C^\infty_{I(H)}(\widetilde{n}\Omega)$ is surjective. Let $W := \liminf_{n\in\mathbb{N}} \widetilde{n}\Omega = \bigcup_{S\subset\mathbb{N}\text{finite}}(\bigcap_{n\notin S} \widetilde{n}\Omega)$. If $x \in \bigcap_{n\notin S} \widetilde{n}\Omega$, then $x/\widetilde{n} = (x_J/n, x_{D\setminus J}) \in \Omega \subset \mathbb{R}^J \times \Omega_{D\setminus J}$ for any $n \notin S$. Hence $W \subset \mathbb{R}^J \times \Omega_{D\setminus J}$. If $x \in \mathbb{R}^J \times \Omega_{D\setminus J}$, then $x/\widetilde{n} \to (0_J, x_{D\setminus J}) \in \{0_J\} \times \Omega_{D\setminus J} \subset \Omega$; hence $x/\widetilde{n} \in \Omega$ for large n and therefore $W \supset \mathbb{R}^J \times \Omega_{D\setminus J}$. Also, by a slight modification of the argument in the proof of Theorem 3.15(a) we see that $\bigcap_{n\notin S} \widetilde{n}\Omega$ is open. The claim follows from Corollary 3.12(b). □

The following result will be a central tool in the reduction procedure for the surjectivity of Hadamard operators discussed in Section 5 (see the proof of Theorem 5.1).

Theorem 3.17. *Let the Hadamard operator $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ be surjective. Then $H : C^\infty(\mathbb{R}^J \times \Omega_{D\setminus J}) \to C^\infty_{I(H)}(\mathbb{R}^J \times \Omega_{D\setminus J})$ is surjective for any $\emptyset \neq J \subsetneqq D$ with $\Omega_{D\setminus J} \neq \emptyset$.*

Proof. Let $W := \Omega \cap (\mathbb{R}^J \times \Omega_{D\setminus J})$. Then H operates on $C^\infty(W)$ by (3.14) and $\Omega_{D\setminus J} = W_{D\setminus J}$ and $W \subset \mathbb{R}^J \times W_{D\setminus J}$. By Theorem 3.16 it is sufficient to prove that $H : C^\infty(W) \to C^\infty_{I(H)}(W)$ is surjective. Since $W \subset \Omega$ and since $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ is surjective by assumption we only need to prove that W is H-convex by Theorem 3.8(b).

(i) Let $\Sigma \subset W$ be compact. Since $W \subset \mathbb{R}^J \times \Omega_{D\setminus J}$ there are compact sets $L \subset \mathbb{R}^J$ and $K \subset \Omega_{D\setminus J}$ such that $\Sigma \subset L \times K$. Since $K \subset \Omega_{D\setminus J}$ there is $\varepsilon > 0$ such that $\overline{B^\infty_\varepsilon(0_J)} \times K \subset \Omega$. Substituting Σ by $\Sigma \cup (\overline{B^\infty_\varepsilon(0_J)} \times K)$ we may assume that

$$\overline{B^\infty_\varepsilon(0_J)} \times K \subset \Sigma. \tag{3.15}$$

Let $T \in C^\infty_{I(H)}(W)'$ such that $\operatorname{supp}({}^tH(T)) \subset \Sigma$. Since $W \subset \Omega$ and since Ω is H-convex by Theorem 3.8(a), there is a compact set $\widetilde{\Sigma} \subset \Omega$ such that

$$\operatorname{supp}_{I(H)}(T) \subset \widetilde{\Sigma}. \tag{3.16}$$

For $\varepsilon > 0$ chosen small enough there is a compact set $\widehat{K} \subset \Omega_{D\setminus J}$ such that

$$\widetilde{\Sigma} \cap \left(\overline{B^\infty_\varepsilon(0_J)} \times \mathbb{R}^{D\setminus J}\right) \subset \overline{B^\infty_\varepsilon(0_J)} \times \widehat{K}. \tag{3.17}$$

Indeed, let $\{K_n \mid n \in \mathbb{N}\}$ be an increasing compact exhaustion of $\Omega_{D\setminus J}$. If (3.17) is not true, then there are $x^n \in \widetilde{\Sigma}$ such that $|x^n_J|_\infty \leq 1/n$ and $x^n_{D\setminus J} \notin K_n$. We can assume that $x^n \to x = (0_J, x_{D\setminus J}) \in \widetilde{\Sigma} \subset \Omega$. Hence $x_{D\setminus J} \in \Omega_{D\setminus J}$, a contradiction. If we show that

$$\operatorname{supp}_{I(H)}(T) \subset \mathbb{R}^J \times \widehat{K}; \tag{3.18}$$

then W is H-convex since then $\operatorname{supp}_{I(H)}(T) \subset (\mathbb{R}^J \times \widehat{K}) \cap \widetilde{\Sigma} =: X$ by (3.16). Clearly, X is a compact set in W.

(ii) *Proof of* (3.18). For $a > 0$ let D_a be the partial dilation operator defined by $D_a(f)(x) := f(\widetilde{a}x)$ where $\widetilde{a}_j = a$ if $j \in J$ (and $\widetilde{a}_j = 1$ if $j \in D \setminus J$, respectively). Notice that

$$\operatorname{supp}({}^tD_a(R)) = \widetilde{a} \operatorname{supp}(R) \text{ for } R \in C^\infty(\mathbb{R}^J \times \Omega_{D\setminus J})' \tag{3.19}$$

and that the same equation holds for $\operatorname{supp}_{I(H)}({}^tD_a(R))$ if $R \in C^\infty_{I(H)}(\mathbb{R}^J \times \Omega_{D\setminus J})'$.

Since $W \subset \mathbb{R}^J \times \Omega_{D\setminus J}$ there are compact sets $L_T \subset \mathbb{R}^J$ and $K_T \subset \Omega_{D\setminus J}$ such that $\operatorname{supp}_{I(H)}(T) \subset L_T \times K_T$. We first choose $0 < \varepsilon_T \leq \varepsilon$ and then $0 < b_T \leq 1$ such that

$$\widetilde{b_T}\,(L_T \times K_T) = (b_T\, L_T) \times K_T \subset \overline{B^\infty_{\varepsilon_T}(0_J)} \times K_T \subset \Omega.$$

Then

$$\operatorname{supp}_{I(H)}({}^tD_{b_T}(T)) \subset \overline{B^\infty_{\varepsilon_T}(0_J)} \times K_T \subset \Omega. \tag{3.20}$$

By (3.19) and (3.15) we get, decreasing b_T if necessary,

$$\begin{aligned} \operatorname{supp}({}^tH({}^tD_{b_T}(T))) &= \operatorname{supp}({}^tD_{b_T}({}^tH(T))) \\ &\subset (b_T\, L) \times K \subset \overline{B^\infty_\varepsilon(0_J)} \times K \subset \Sigma \end{aligned} \tag{3.21}$$

since D_{b_T} and H are Hadamard operators, and so they commute (see [24, 25]). Hence by (3.20), (3.21), (3.16), and (3.17)

$$\operatorname{supp}_{I(H)}({}^tD_{b_T}(T)) \subset \widetilde{\Sigma} \cap \left(\overline{B^\infty_\varepsilon(0_J)} \times K_T\right) \subset \overline{B^\infty_\varepsilon(0_J)} \times \widehat{K}$$

since $\varepsilon_T \leq \varepsilon$. Thus (3.18) is proved by again using (3.19). □

4. First order Euler operators

Using $P(\theta)$-convexity from the previous section and additionally some elementary analytical tools, in this section we discuss solvability of first-order Euler equations. In this way, we will anticipate some of the questions which will be discussed for general Hadamard and Euler operators in the subsequent sections. Also, we will present several interesting examples showing that Euler operators behave quite differently than linear partial differential operators with constant coefficients. For $0 \neq a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ let

$$\langle a, \theta\rangle + b := a_1\theta_1 + \cdots + a_d\theta_d + b$$

denote a first-order Euler differential operator with real coefficients. We start with an invertibility criterion.

Proposition 4.1. *Let $0 \neq a \geq \mathbf{0}$, and let $b > 0$. Let $\Omega \subset \mathbb{R}^d$ be an open set such that*
$$A := \{(t^{a_1}, \ldots, t^{a_d}) \mid t \in [0,1]\} \subseteq V(\Omega).$$
Then $\langle a, \theta\rangle + b$ is invertible on $C^\infty(\Omega)$.

Proof. We define a distribution $T \in C^\infty(\Omega)'$ by
$$T(f) = \int_0^1 f(t^{a_1}, \ldots, t^{a_d}) t^{b-1} dt.$$
Then $\mathrm{supp}(T) = A \subseteq V(\Omega)$, so by Theorem 3.1, the Hadamard operator $H_T : C^\infty(\Omega) \to C^\infty(\Omega)$ is defined, and H_T is the inverse of $\langle a, \theta\rangle + b$ because
$$H_T(\xi^\alpha)(x) = \int_0^1 (x_1 t^{a_1})^{\alpha_1}, \ldots, (x_d t^{a_d})^{\alpha_d} t^{b-1} dt = \frac{1}{a_1\alpha_1 + \cdots + a_d\alpha_d + b} x^\alpha$$
while
$$(\langle a, \theta\rangle + b)(x^\alpha) = (a_1\alpha_1 + \cdots + a_d\alpha_d + b)x^\alpha$$
(also use the density of the polynomials in $C^\infty(\Omega)$). □

Notice that, in particular, the Euler operators in Proposition 4.1 are injective on $C^\infty(\Omega)$ while constant coefficient partial differential operators in at least 2 variables always have an infinite-dimensional kernel in $C^\infty(\Omega)$. We will resume the discussion of zero solutions for Euler equations in Section 11.

We next consider solvability on open sets $\Omega := \mathbb{R}^d \setminus S$ where $S \subset \mathbb{R}^d$ is compact.

Proposition 4.2. *Let $a > \mathbf{0}$, and let $b > 0$. Let $\Omega := \mathbb{R}^d \setminus S$ for some compact set S with $\mathbf{0} \in S$ such that*
$$B := \{(t^{a_1}, \ldots, t^{a_d}) \mid t \geq 1\} \subseteq V(\Omega). \tag{4.1}$$
Then $\langle a, \theta\rangle + b : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective.

Proof. We decompose $f \in C^\infty(\Omega)$ as $f = f_1 + f_2$ with $f_1 \in C^\infty(\mathbb{R}^d)$ and $f_2 \in C^\infty(\Omega)$ with $f_2(x) = 0$ if $|x|$ is large. By Proposition 4.1 there is $g_1 \in C^\infty(\mathbb{R}^d)$ such that $(\langle a, \theta\rangle + b)g_1 = f_1$ on $\mathbb{R}^d$. We define $g_2 \in C^\infty(\Omega)$ by
$$g_2(x) := -\int_1^\infty f_2(x_1 t^{a_1}, \ldots, x_d t^{a_d}) t^{b-1} dt \text{ for } x \in \Omega.$$
Then g_2 is defined and smooth on Ω by (4.1) since $f_2 = 0$ near ∞ and since $x \neq 0$ for $x \in \Omega$ and $a > \mathbf{0}$,
$$\begin{aligned}(\langle a, \theta_x\rangle + b)g_2(x) &= -\int_1^\infty \left[\left(\sum_{j\leq d} a_j x_j t^{a_j}\partial_j + b\right) f_2\right](x_1 t^{a_1}, \ldots, x_d t^{a_d}) t^{b-1} dt \\ &= -\int_1^\infty \frac{d}{dt}\left[t^b f_2(x_1 t^{a_1}, \ldots, x_d t^{a_d})\right] dt \\ &= -\left[t^b f_2(x_1 t^{a_1}, \ldots, x_d t^{a_d})\right]_{t=1}^{t=\infty} = f_2(x),\end{aligned}$$
since $f_2 = 0$ near ∞. Hence, $(\langle a, \theta\rangle + b)(g_1 + g_2) = f_1 + f_2 = f$ on $\Omega = \mathbb{R}^d \setminus S$. □

Notice that $\langle a, \theta\rangle + b$ in general is not injective in the situation of Proposition 4.2. Indeed, $(\sum_{j\leq d} \theta_j + 2)f = 0$ for $f(x) := 1/\|x\|_2^2 \in C^\infty(\mathbb{R}^d \setminus \{0\})$.

It is well known that constant coefficient linear differential operators $P(\partial)$ are elliptic if $P(\partial)$ is surjective on $C^\infty(\mathbb{R}^d \setminus \{x\})$ for some $x \in \mathbb{R}^d$. This no longer holds for Euler operators and $x := 0$ by the above proposition since $P(x) = \langle a, x\rangle + b$ is not elliptic. Notice, however, that we will prove the corresponding result for Euler operators in Theorem 10.7 for $x \in (\mathbb{R}_*)^d$.

Next we show that first-order Euler partial differential equations often have a distributional zero solution supported in a path.

Proposition 4.3. *Let $a, u \in \mathbb{R}^d$ and $0 \neq b \in \mathbb{R}$ such that $a_j/b > 0$ and $u_j \neq 0$ for some j. Then the distribution T defined by*

$$\langle T, \varphi\rangle = \int_0^\infty \varphi\left(u_1 t^{\frac{a_1}{b}}, \dots, u_d t^{\frac{a_d}{b}}\right) dt$$

is a zero solution of ${}^t(\langle a, \theta\rangle + b)$ in $\mathscr{D}'(\mathbb{R}^d)$.

Proof. The formula defines a distribution on $\mathbb{R}^d$ since $\lim_{t\to\infty} |u_j t^{\frac{a_j}{b}}| = \infty$ for some j by assumption. Let us observe that

$$\begin{aligned}
&\int_0^\infty (\theta_t + 1)\left[\varphi\left(u_1 t^{\frac{a_1}{b}}, \dots, u_d t^{\frac{a_d}{b}}\right)\right] dt \\
&\quad = \int_0^\infty t \sum_{j=1}^d \left[(\partial_j \varphi)\left(u_1 t^{\frac{a_1}{b}}, \dots, u_d t^{\frac{a_d}{b}}\right) u_j \frac{a_j}{b} t^{\frac{a_j}{b}-1}\right] + \varphi\left(u_1 t^{\frac{a_1}{b}}, \dots, u_d t^{\frac{a_d}{b}}\right) dt \\
&\quad = \int_0^\infty \left(\sum_{j=1}^d \frac{a_j}{b}\theta_j \varphi + \varphi\right)\left(u_1 t^{\frac{a_1}{b}}, \dots, u_d t^{\frac{a_d}{b}}\right) dt \\
&\quad = \frac{1}{b}\langle T, (\langle a, \theta\rangle + b)\,\varphi\rangle = \frac{1}{b}\langle {}^t(\langle a, \theta\rangle + b) T, \varphi\rangle.
\end{aligned}$$

On the other hand,

$$\int_0^\infty (\theta_t + 1)\left[\varphi\left(u_1 t^{\frac{a_1}{b}}, \dots, u_d t^{\frac{a_d}{b}}\right)\right]\, dt = \int_0^\infty \frac{d}{dt}\left[t\varphi\left(u_1 t^{\frac{a_1}{b}}, \dots, u_d t^{\frac{a_d}{b}}\right)\right] dt = 0,$$

since $\lim_{t\to\infty} |u_j t^{\frac{a_j}{b}}| = \infty$ for some j. ☐

Notice that the above zero solution may start in Z_1. Specifically, if $\langle a, \theta\rangle + b := \theta_1 + 1$, then the construction in Proposition 4.3 leads to a zero solution with support equal to the half line $\{(tu_1, u_2, \dots, u_d) \mid t \geq 0\}$ for $u_1 \neq 0$. Recall that the support of zero solutions of constant coefficient partial differential operators cannot be a half line by a corollary to Holmgren's uniqueness theorem (cf. [15, Theorem 8.6.8]).

Finally, we get a simple geometric necessary condition for $P(\theta)$-convexity.

Proposition 4.4. *Let $0 \neq a \geq \mathbf{0}$, let $b > 0$, and let $J := \{j \mid a_j \neq 0\}$. If $\Omega \subseteq \mathbb{R}^d$ is open connected, $\Omega_{D\setminus J} \neq \emptyset$, and Ω is $(\langle a, \theta\rangle + b)$-convex, then*

$$A = \{(t^{a_1}, \dots, t^{a_d}) \mid t \in [0, 1]\} \subseteq V(\Omega).$$

Proof. We have to show that $\Omega = \Omega_0$, where

$$\Omega_0 := \{x \in \Omega \mid \ \forall\, t \in [0, 1] : (x_1 t^{a_1}, \dots, x_d t^{a_d}) \in \Omega\}.$$

Observe that $J \neq \emptyset$ and that $\Omega_{D\setminus J} \subset \Omega_0$, hence $\Omega_0 \neq \emptyset$ by assumption. Secondly, we observe that Ω_0 is open. Indeed, if $x \in \Omega_0$, then

$$K_x := \{(x_1 t^{a_1}, \dots, x_d t^{a_d}) \mid t \in [0, 1]\}$$

is a compact subset of Ω. Clearly, there is $\varepsilon > 0$ such that if $y \in B_\varepsilon(x)$, then $K_y \subset \Omega$ as well, so $B_\varepsilon(x) \subset \Omega_0$.

Finally, we will prove that Ω_0 is closed in Ω. Indeed, if $(x^{(n)}) \subset \Omega_0$, $\lim x^{(n)} = x \in \Omega$, then without loss of generality we may assume that $B_\varepsilon(x) \subset \Omega$ and $(x^{(n)}) \subseteq \overline{B_{\varepsilon/4}(x)}$. Since $K_{x^{(n)}} \subset \Omega$, we define $\varphi_n \in D(\Omega)$ so that $\varphi_n \equiv 1$ near $K_{x^{(n)}}$ but

$$\{(x_1 t^{a_1}, \ldots, x_d t^{a_d}) \mid t \in [0,\infty)\} \cap \{y \mid 0 < |\varphi_n(y)| < 1\} \subset \overline{B_{\varepsilon/2}(x)}.$$

We define $T_n \in D'(\Omega)$ by

$$\langle T_n, \varphi\rangle := \int_0^\infty (\varphi\varphi_n)\left(x_1^{(n)} t^{\frac{a_1}{b}}, \ldots, x_d^{(n)} t^{\frac{a_d}{b}}\right) dt \text{ for } \varphi \in D(\Omega).$$

Then

$$K_{x^{(n)}} \subset \operatorname{supp}(T_n) \subset K_{x^{(n)}} \cup \overline{B_{\varepsilon/2}(x)}.$$

On the other hand, by Proposition 4.3, $\operatorname{supp}{}^t(\langle a,\theta\rangle + b)T_n \subset \overline{B_{\varepsilon/2}(x)}$. Now, by $(\langle a,\theta\rangle + b)$-convexity of Ω it follows that there is a compact set $\widetilde{K} \subset \Omega$ such that

$$K_{x^{(n)}} \subset \operatorname{supp} T_n \subset \widetilde{K} \text{ for any } n \in \mathbb{N}.$$

Clearly, then, $K_x \subset \widetilde{K} \subset \Omega$, and so $x \in \Omega_0$.

Since Ω_0 is nonempty and both closed and open in the connected set Ω, we get $\Omega_0 = \Omega$. □

Summarizing we get a simple geometric characterization of $P(\theta)$-convexity (invertibility, surjectivity) for the Euler operators considered in Propositions 4.1 and 4.4.

Corollary 4.5. *Let $0 \neq a \geq \mathbf{0}$, let $b > 0$, and let $J := \{j \mid a_j \neq 0\}$. Let $\Omega \subseteq \mathbb{R}^d$ be an open connected set with $\Omega_{D\setminus J} \neq \emptyset$, and let $P(\theta) := \langle a,\theta\rangle + b$. The following assertions are equivalent:*

(a) *Ω is $P(\theta)$-convex;*
(b) *$P(\theta) : C^\infty(\Omega) \to C^\infty(\Omega) = C^\infty_{I(P)}(\Omega)$ is surjective;*
(c) *$P(\theta) : C^\infty(\Omega) \to C^\infty(\Omega)$ is invertible;*
(d) *$A := \{(t^{a_1}, \ldots, t^{a_d}) \mid t \in [0,1]\} \subseteq V(\Omega)$.*

Proof. (a)⇒(d): Proposition 4.4. (d)⇒(c): Proposition 4.1. (c)⇒(b): Obvious. (b)⇒(a): Theorem 3.8(a). □

Please note that by the above result, for instance, the connected set $\Omega \subseteq \mathbb{R}^d$, $\mathbf{0} \in \Omega$, is $(\theta_1 + \cdots + \theta_d + b)$-convex, $b > 0$, if and only if Ω is star-like around $\mathbf{0}$, i.e., if $x \in \Omega$, then the whole interval $[0,x] \subset \Omega$.

We now collect further examples showing the difference between linear partial differential operators with constant coefficients and linear Euler partial differential operators.

Example 4.6. Set

$$\Omega_n := \{(x,y) \mid |x|, |y| < 1\} \setminus \{(x,y) \mid y \leq -1/2, x \leq -1/n\}.$$

Then, by Corollary 4.5, Ω_n is $(\theta_1 + 1)$-convex and $(\theta_1 + 1)$ is invertible on $C^\infty(\Omega_n)$ since $A = \{(t,1) \mid t \in [0,1]\} = [0,1] \times \{1\} \subset V(\Omega_n)$. On the other hand,

$$\Omega := \left(\bigcap_n \Omega_n\right)^\circ = \{(x,y) \mid |x|, |y| < 1\} \setminus \{(x,y) \mid y \leq -1/2, x \leq 0\},$$

and then, by Corollary 4.5, $\theta_1 + 1$ is not surjective on $C^\infty(\Omega)$ since $A = [0,1] \times \{1\} \not\subseteq V(\Omega)$. Indeed, $(0,1) \in A$ but $(0,1)\Omega = \{0\} \times [-1,1] \not\subseteq \Omega$.

Please note that also $\Omega = (\liminf_n \Omega_n)^\circ$. This example is in strong contrast to facts known for linear partial differential operators with constant coefficients. There it is known that if $P(\partial)$ is surjective on $C^\infty(\Omega_i)$ for all $i \in I$, then $P(\partial)$ is surjective both on $C^\infty\left(\left(\bigcap_{i\in I}\Omega_i\right)^\circ\right)$ and on $C^\infty\left((\liminf_{i\in I}\Omega_i)^\circ\right)$; see [15, Theorem 10.6.4]. On the other hand, by Corollary 3.12, surjectivity of $P(\theta)$ inherits to $C^\infty\left(\bigcap_{i\in I}\Omega_i\right)$ (and to $C^\infty(\liminf_{i\in I}\Omega_i)$) if $\bigcap_{i\in I}\Omega_i$ (and any $\bigcap_{i\in I\setminus J}\Omega_i$ for $J \subset I$ finite, respectively) are open.

Example 4.7. Take Ω_n and Ω as in Example 4.6. (a) There is no minimal open $W \supset \Omega$ such that $\theta_1 + 1$ is surjective on $C^\infty(W)$. Indeed, W would be contained in any Ω_n, hence $\bigcap_n \Omega_n \supset W \supset \Omega$, and $W = \Omega$ since W is open, a contradiction, since $\theta_1 + 1$ is not surjective on $C^\infty(\Omega)$ by Example 4.6. This again is in strong contrast to the situation for constant coefficient partial differential operators. There it is known that for every open set $\Omega \subseteq \mathbb{R}^d$ and every $P(\partial)$ there is a minimal open $W \supset \Omega$ such that $P(\partial)$ is surjective on $C^\infty(W)$; see [15, Corollary 10.6.5].

(b) It is well known that a constant coefficient operator $P(\partial)$ in two variables is surjective on $C^\infty(\Omega)$ iff

$$\text{the intersection of } \Omega \text{ with any characteristic line is an interval.} \tag{4.2}$$

This is no longer true for Euler differential operators since the characteristic lines for $\theta_1 + 1$ are $\mathbb{R} \times \{x_2\}, x_2 \in \mathbb{R}$, and hence Ω satisfies (4.2) while $\theta_1 + 1$ is not surjective on $C^\infty(\Omega)$.

Proposition 4.8. *Let $b \in \mathbb{N}$, and let Ω be connected. The following are equivalent:*

(a) *The operator $\theta_1 - b : C^\infty(\Omega) \to C^\infty_{I(\theta_1 - b)}(\Omega)$ is surjective.*
(b) *The operator $\partial_1 : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective.*
(c) *The intersection of Ω with each characteristic line $L := \mathbb{R}\times\{x'\}, x' \in \mathbb{R}^{d-1}$, is an interval.*

Proof.
"$(a) \Rightarrow (b)$" (i) We first show that

$$\theta_1 : C^\infty(\Omega) \to C^\infty_{I(\theta_1)}(\Omega) \text{ is surjective.} \tag{4.3}$$

Indeed, let $b \neq 0$. Let $f \in C^\infty(\Omega)$, and let $h(x) := x_1^{b+1} f(x)$. Then $h \in C^\infty_{I(\theta_1 - b)}(\Omega)$ by Example 3.2 and there is $g \in C^\infty(\Omega)$ by assumption such that $(\theta_1 - b)g = h$. Then

$$x_1\partial_1 g(x) - bg(x) = (\theta_1 - b)g(x) = x_1^{b+1} f(x),$$

and hence

$$g_1(x) := g(x)/x_1 = \frac{1}{b}(\partial_1 g(x) - x_1^b f(x))$$

is smooth on Ω. Moreover, by a simple calculation,

$$\begin{aligned}(\theta_1 - (b-1))g_1(x) &= (x_1\partial_1 - b + 1)[g(x)/x_1] \\ &= \frac{1}{x_1}[\theta_1 - b]g(x) = x_1^b f(x) \text{ if } x \in \Omega, x_1 \neq 0.\end{aligned}$$

By continuity this equation holds on Ω, and (4.3) follows by Example 3.2 and induction.

(ii) Since $C^\infty_{I(\theta_1)}(\Omega) = \{h \in C^\infty(\Omega) \mid \exists g \in C^\infty(\Omega) : h(x) = x_1 g(x) \text{ on } \Omega\}$ by Example 3.2, we have for $h \in C^\infty_{I(\theta_1)}(\Omega)$

$$x_1\partial_1 f(x) = \theta_1 f(x) = h(x) = x_1 g(x) \text{ on } \Omega \text{ iff } \partial_1 f(x) = g(x) \text{ on } \Omega. \tag{4.4}$$

Hence, (b) follows.

"$(b) \Rightarrow (a)$" Let $b \neq 0$, and let $f \in C^\infty_{I(\theta_1 - b)}(\Omega)$. By Example 3.2, we have

$$f(x) = \sum_{j \leq b-1} f_j(x')x_1^j + x_1^{b+1}F(x) \text{ near } \omega_1 := \{x \in \Omega \mid x_1 = 0\}$$

for some $f_j \in C^\infty(\omega_1)$ and $F \in C^\infty(\Omega)$. We may choose $\psi \in C^\infty(\mathbb{R} \times \omega_1)$ such that $\psi(x) = 1$ near ω_1 and such that $g(x) := \psi(x) \sum_{j \leq b-1} f_j(x')x_1^j/(j - b)$ is smooth on Ω. Notice that $(\theta_1 - b)g(x) = \sum_{j \leq b-1} f_j(x')x_1^j$ near ω_1, hence

$$F_1(x) := f(x) - (\theta_1 - b)g(x) = x_1^{b+1}F(x) \text{ near } \omega_1.$$

Since $F_1 \in C^\infty(\Omega)$ there is $F_2 \in C^\infty(\Omega)$ such that

$$F_1(x) = x_1^{b+1}F_2(x) \text{ on } \Omega.$$

By (4.4) and the assumption we know that $\theta_1 : C^\infty(\Omega) \to C^\infty_{I(\theta_1)}(\Omega)$ is surjective. Hence there is $G_1 \in C^\infty(\Omega)$ such that $\theta_1 G_1(x) = x_1 F_2(x)$ on Ω and we get for $G_2(x) := x_1^b G_1(x)$

$$(\theta_1 - b)G_2(x) = (x_1\partial_1 - b)[x_1^b G_1(x)] = x_1^b \theta_1 G_1(x) = x_1^{b+1}F_2(x) = F_1(x) \text{ on } \Omega,$$

and therefore $(\theta_1 - b)[G_2 + g] = F_1 + (\theta_1 - b)g = f$ on Ω. This shows (a).

"$(b) \Rightarrow (c)$" This follows similarly to Proposition 4.4 using the zero solutions $T \in D'(\mathbb{R}^d)$ of ∂_1 defined by $T(\varphi) := \int \varphi(t, x')dt$ (cf. also [15, Theorems 10.8.5 and 10.8.3]).

"$(c) \Rightarrow (b)$" By (c) we can choose $y^{(n)} \in \Omega$ and $\varepsilon_n > 0$ such that $B^\infty_{\varepsilon_n}(y^{(n)}) \subset \Omega$ and such that $\big(B^\infty_{\varepsilon_n}(0, y'^{(n)})\big)_{n\in\mathbb{N}}$ is a locally finite open covering of $\pi_1(\Omega)$. We choose a C^∞-resolution of the identity $(\varphi_n)_{n\in\mathbb{N}}$ of $\pi_1(\Omega)$ subordinate to $(B^\infty_{\varepsilon_n}(0, y'^{(n)}))_{n\in\mathbb{N}}$. For an interval I and $h \in C(I)$ and $c \in I$, let $I_c(h)(t) := \int_c^t h(\tau)\, d\tau$ for $t \in I$. For $f \in C^\infty(\Omega)$ set

$$R(f)(x_1, x') := \sum_{n\in\mathbb{N}} {}_\tau I_{y_1^{(n)}}\big(f(\tau, x')\big)(x_1)\varphi_n(x') \text{ for } (x_1, x') \in \Omega.$$

$R(f)$ is defined and smooth on Ω by (c). Clearly, $\partial_1(R(f)) = f$ on Ω. □

A more sophisticated variant of Proposition 4.8 will be useful in the discussion of admissible sets (see Proposition 7.11).

Corollary 4.5 and Proposition 4.8 provide a basic example that the statements (a) and (b) in the Main Theorem B are not equivalent.

Proposition 4.9. *If $\theta_1 + 1 : C^\infty(\Omega) \to C^\infty_{I(\theta_1+1)}(\Omega) = C^\infty(\Omega)$ is surjective, then $\theta_1 : C^\infty(\Omega) \to C^\infty_{I(\theta_1)}(\Omega)$ is surjective. The converse implication is not true.*

Proof.

(a) Let $\theta_1 + 1 : C^\infty(\Omega) \to C^\infty(\Omega)$ be surjective. Then

$$(tx_1, x') \in \Omega \text{ for any } t \in [0, 1] \text{ if } x \in \Omega \tag{4.5}$$

by Corollary 4.5. Let $(x_1, x'), (\xi_1, x') \in \Omega$. Then $(t, x') \in \Omega$ for any t between 0 and x_1 and between 0 and ξ_1, respectively, by (4.5), and hence for any t between x_1 and ξ_1. Therefore, $\theta_1 : C^\infty(\Omega) \to C^\infty_{I(\theta_1)}(\Omega)$ is surjective by Proposition 4.8.

(b) The open set $\Omega := \mathbb{R}^2 \setminus (]-\infty, 0] \times \{0\})$ satisfies the condition in Proposition 4.8(c) for $a := e_1$ but not the one in Corollary 4.5(d). Indeed, $x := (x_1, 0) \in \Omega$

for $x_1 > 0$ while $x(0,1) = (0, x_2) = 0 \notin \Omega$. Hence $(0,1) \notin V(\Omega)$. The claim follows by those propositions. □

The different behavior of $P(\theta) := \theta_1 + 1$ and $Q(\theta) := \theta_1$ in Proposition 4.9 holds in spite of the fact that both $P(\theta)$ and $Q(\theta)$ have the same principal parts. This is in strong contrast to [15, Theorem 10.8.3] which states that for constant coefficient linear partial differential operators in two variables surjectivity is fully determined by the principal part.

5. Reducing the surjectivity problem for Hadamard operators

In this section we develop necessary and sufficient conditions for surjectivity of all shifted Hadamard operators for H in terms of surjectivity of the restricted Hadamard operators H_α on spaces of functions which are flat on the singular set Z_1. In this way we take some principal steps towards the proof of part (b) of the Main Theorem B from the introduction.

Our first aim is to show that surjectivity is inherited to restricted operators defined as follows: for a Hadamard operator $H = H_T$ on $C^\infty(\Omega)$ the restrictions H_α are Hadamard operators on $C^\infty(\Omega_{D\setminus J})$ for $\emptyset \neq J \subsetneq D := \{1, \dots, d\}$ and $\alpha = \alpha_J \in \mathbb{N}^J$, and they are defined as follows: as we noticed already before Theorem 3.16 any Hadamard operator on $C^\infty(\Omega)$ defines a Hadamard operator on $C^\infty(\mathbb{R}^J \times \Omega_{D\setminus J})$ if $\Omega_{D\setminus J} \neq \emptyset$. Hence, for $\alpha \in \mathbb{N}^J$ the following operator $H_\alpha : C^\infty(\Omega_{D\setminus J}) \to C^\infty(\Omega_{D\setminus J})$,

$$\begin{aligned} H_\alpha(f)(x_{D\setminus J}) &:= \left[\delta_{\mathbf{1}_J} \circ H_T(\xi_J^\alpha f(\xi_{D\setminus J}))\right](x_{D\setminus J}) \\ &= \langle {}_yT, y_J^\alpha f(x_{D\setminus J} y_{D\setminus J})\rangle =: \langle {}_{y_{D\setminus J}}T_\alpha, f(x_{D\setminus J} y_{D\setminus J})\rangle, \end{aligned} \tag{5.1}$$

is a well-defined continuous linear mapping. Moreover, H_α is a Hadamard operator since

$$H_\alpha(\xi^{\beta_{D\setminus J}})(x_{D\setminus J}) = \langle {}_yT, y_J^\alpha y_{D\setminus J}^{\beta_{D\setminus J}}\rangle x_{D\setminus J}^{\beta_{D\setminus J}} = m_{(\alpha,\beta_{D\setminus J})} x_{D\setminus J}^{\beta_{D\setminus J}}. \tag{5.2}$$

Hence by Theorem 3.1 we have $T_\alpha \in C^\infty(V(\Omega_{D\setminus J}))'$, $H_\alpha = H_{T_\alpha}$, and

$$m_{\beta_{D\setminus J}}(T_\alpha) = m_{(\alpha,\beta_{D\setminus J})}(T) \text{ for any } \alpha \in \mathbb{N}^J \text{ and any } \beta_{D\setminus J} \in \mathbb{N}^{D\setminus J}. \tag{5.3}$$

Please note that for an Euler operator $H = P(\theta)$ we have

$$H_\alpha = P(\alpha_J, \theta_{D\setminus J}) \tag{5.4}$$

since both operators have the same multiplier sequences.

We now prove that surjectivity of Hadamard operators H is inherited to the restricted Hadamard operators H_α.

Theorem 5.1. *Let $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ be a surjective Hadamard operator. Then for any $\emptyset \neq J \subsetneq \{1, \dots, d\}$ with $\Omega_{D\setminus J} \neq \emptyset$ and any $\alpha = \alpha_J \in \mathbb{N}^J$ the restricted Hadamard operators*

$$H_\alpha : C^\infty(\Omega_{D\setminus J}) \to C^\infty_{I(H_\alpha)}(\Omega_{D\setminus J})$$

are surjective.

Proof. By Theorem 3.17 we can assume that $\Omega = \mathbb{R}^J \times \Omega_{D\setminus J}$. For fixed $\alpha \in \mathbb{N}^J$ take $f_\alpha \in C^\infty_{I(H_\alpha)}(\Omega_{D\setminus J})$. This means by (5.3) that

$$\begin{aligned} &\forall \emptyset \neq M \subset D \setminus J\ \forall \gamma \in \mathbb{N}^M\ \forall x_{D\setminus(J\cup M)} \in \Omega_{D\setminus(J\cup M)} : \\ &\qquad f_\alpha^{(\gamma)}(0_M, x_{D\setminus(J\cup M)}) = 0 \text{ if } m_{(\alpha,\gamma,\beta)} = 0 \text{ for any } \beta \in \mathbb{N}^{D\setminus(J\cup M)}. \end{aligned} \tag{5.5}$$

Set $f(x) := f_\alpha(x_{D\setminus J}) x_J^\alpha/\alpha!$. Then $f \in C^\infty(\Omega)$ since $\Omega = \mathbb{R}^J \times \Omega_{D\setminus J}$; more precisely, $f \in C^\infty_{I(H)}(\Omega)$. This is seen as follows: For $\emptyset \neq L \subset D$ let $\gamma \in \mathbb{N}^L$ such that

$$m_{(\gamma,\beta)} = 0 \text{ for any } \beta \in \mathbb{N}^{D\setminus L}. \tag{5.6}$$

Then

$$\partial^\gamma f(0_L, x_{D\setminus L}) = \partial^{\gamma_{L\setminus J}} f_\alpha(0_{L\setminus J}, x_{D\setminus(J\cup L)}) \partial^{\gamma_{L\cap J}} x_{L\cap J}^{\alpha_{L\cap J}}/\alpha_{L\cap J}!\big|_{0_{L\cap J}} x_{J\setminus L}^{\alpha_{J\setminus L}}/\alpha_{J\setminus L}! = 0$$

if $\gamma_{L\cap J} \neq \alpha_{L\cap J}$. If $\gamma_{L\cap J} = \alpha_{L\cap J}$, then

$$\partial^\gamma f(0_L, x_{D\setminus L}) = \partial^{\gamma_{L\setminus J}} f_\alpha(0_{L\setminus J}, x_{D\setminus(J\cup L)}) x_{\alpha_{J\setminus L}}^{\alpha_{J\setminus L}}/\alpha_{J\setminus L}! = 0$$

by (5.5) since $f_\alpha \in C^\infty_{I(H_\alpha)}(\Omega_{D\setminus J})$ and since for any $\delta \in \mathbb{N}^{D\setminus(J\cup L)}$

$$m_{(\alpha,\gamma_{L\setminus J},\delta)} = m_{(\alpha_{L\cap J},\gamma_{L\setminus J},\alpha_{J\setminus L},\delta)} = m_{(\gamma,\alpha_{J\setminus L},\delta)} = 0$$

by (5.6) since $(\alpha_{J\setminus L}, \delta) \in \mathbb{N}^{D\setminus L}$.

Since $f \in C^\infty_{I(H)}(\Omega)$ there is $g \in C^\infty(\Omega)$ by assumption such that $H(g) = f$. Then we get for any $\alpha \in \mathbb{N}^J$ by the definition of H_α

$$\begin{aligned} H_\alpha\left(\partial^\alpha g(0_J, x_{D\setminus J})\right)(x_{D\setminus J}) &= \langle {}_yT, y_J^\alpha (\partial^\alpha g)(0_J, x_{D\setminus J} y_{D\setminus J})\rangle \\ &= \partial_{x_J}^\alpha \langle {}_yT, g(x_J y_J, x_{D\setminus J} y_{D\setminus J})\rangle\Big|_{x_J=0} \\ &= (\partial_{x_J}^\alpha H(g))(0_J, x_{D\setminus J}) = (\partial_{x_J}^\alpha f)(0_J, x_{D\setminus J}) = f_\alpha(x_{D\setminus J}) \end{aligned}$$

by the definition of f. This proves the claim. □

Recall that $C^\infty_{I(H_\alpha)}(\Omega_{D\setminus J}) = \{0\}$, the trivial vector space, in Theorem 5.1 if $H_\alpha = 0$.

Our next necessary condition for surjectivity of Hadamard operators concerns surjectivity within smooth functions which are flat on $\Omega \cap Z_1$. For this we introduce "shifted" Hadamard operators for $H = H_T$ defined by

$$H_{T,k} := H_{T_k} \text{ where } T_k := y^k T \text{ for } k \in \mathbb{N}^d. \tag{5.7}$$

Notice that the operators $H_{T,k}$ satisfy

$$x^k H_{T,k}(f)(x) = \langle {}_yT y^k, x^k f(xy)\rangle = \langle {}_yT, (xy)^k f(xy)\rangle = H_T(\xi^k f)(x) \text{ on } \Omega \tag{5.8}$$

for $f \in C^\infty(\Omega)$, and therefore

$$H_{T,k}(\xi^\alpha)(x) = \langle {}_yT, (xy)^k (xy)^\alpha\rangle x^{-k} = \langle {}_yT, y^{\alpha+k}\rangle x^\alpha; \tag{5.9}$$

the multiplier sequence for H_{T_k} is thus the multiplier sequence of H_T shifted by k.

Definition 5.2. Let

$$\begin{aligned} C^\infty_{\mathbf{k}\text{-}flat}(\Omega) &:= \{f \in C^\infty(\Omega) \mid \forall \alpha \leq \mathbf{k} : \partial^\alpha(f) = 0 \text{ on } Z_1 \cap \Omega\} \text{ and} \\ C^\infty_{flat}(\Omega) &:= C^\infty_{\infty\text{-}flat}(\Omega). \end{aligned}$$

Proposition 5.3. *If the shifted Hadamard operators* $H_{T,k} : C^\infty(\Omega) \to C^\infty_{I(H_{T,k})}(\Omega)$ *are surjective for any* $k \in \mathbb{N}^d$*, then*

$$H\Big(C^\infty_{\mathbf{k}\text{-}flat}(\Omega)\Big) \supset C^\infty_{flat}(\Omega) \text{ for any } k \in \mathbb{N}.$$

Proof. For $f \in C^\infty_{flat}(\Omega)$ let $F(x) := f(x)x^{-\mathbf{k}}, x \in \Omega$. Then $F \in C^\infty_{flat}(\Omega)$, and by assumption there is $G \in C^\infty(\Omega)$ such that $H_{T,k}(G) = F$. Let $g(x) := x^{\mathbf{k}}G(x), x \in \Omega$. Then $g \in C^\infty_{\mathbf{k}-flat}(\Omega)$ and (5.8) implies that

$$H_T(g)(x) = H_T(\xi^{\mathbf{k}}G)(x) = x^{\mathbf{k}}H_{T,k}(G)(x) = x^{\mathbf{k}}F(x) = f(x)$$

on Ω. ☐

To shorten the subsequent statements we set $H_\emptyset := H$ and $\Omega_D := \Omega$ (corresponding to $J = \emptyset$ in the statements below). Combining Theorem 5.1 and Proposition 5.3 we get the following theorem.

Theorem 5.4 (Necessary condition for surjectivity)**.** *Let the shifted Hadamard operators*

$$H_{T,k} : C^\infty(\Omega) \to C^\infty_{I(H_{T,k})}(\Omega)$$

be surjective for any $k \in \mathbb{N}^d$*. Then*

$$H_\alpha\Big(C^\infty_{\mathbf{k}\text{-}flat}(\Omega_{D\setminus J})\Big) \supset C^\infty_{flat}(\Omega_{D\setminus J})$$

for any $k \in \mathbb{N}$*, any* $J \subsetneq D$ *with* $\Omega_{D\setminus J} \neq \emptyset$ *and any* $\alpha = \alpha_J \in \mathbb{N}^J$ *with* $H_\alpha \neq 0$*.*

Proof. For $J = \emptyset$ this is Proposition 5.3. Fix $\alpha \in \mathbb{N}^J$. Then we have for $k \in \mathbb{N}^d$ by (5.1) and (5.8)

$$x_{D\setminus J}^{k_{D\setminus J}} \left(H_{T,k}\right)_\alpha (f)(x_{D\setminus J}) = \langle {}_yT_k, y_J^\alpha x_{D\setminus J}^{k_{D\setminus J}} f(x_{D\setminus J}y_{D\setminus J})\rangle$$

$$(5.10) \qquad = \langle {}_yT, y_J^{k_J+\alpha}[x_{D\setminus J}y_{D\setminus J}]^{k_{D\setminus J}} f(x_{D\setminus J}y_{D\setminus J})\rangle = H_{k_J+\alpha}(\xi_{D\setminus J}^{k_{D\setminus J}} f)(x_{D\setminus J}).$$

We therefore get for $k = (0_J, k_{D\setminus J})$

$$(5.11) \qquad \left(H_{T,k}\right)_\alpha = H_{T_\alpha, k_{D\setminus J}},$$

and hence the shifted restricted operators

$$H_{T_\alpha,k_{D\setminus J}} : C^\infty(\Omega_{D\setminus J}) \to C^\infty_{I(H_{T_\alpha,k})}(\Omega_{D\setminus J}), k_{D\setminus J} \in \mathbb{N}^{D\setminus J},$$

are surjective by Theorem 5.1. Now apply Proposition 5.3 to H_{T_α}. ☐

We will next show that a condition slightly stronger than the one in Theorem 5.4 is also sufficient for the surjectivity of the shifted Hadamard operators. For this we notice that Hadamard operators canonically operate on certain Whitney jets on Ω. Recall that Whitney jets are defined as follows: for $\Omega \subset \mathbb{R}^d$ open and $S \subset \Omega$ let

$$(5.12) \qquad W(S) := \{(f_\alpha)_{\alpha\in\mathbb{N}^d} \in C(S)^{\mathbb{N}^d} \mid \exists f \in C^\infty(\Omega) : f^{(\alpha)}\Big|_S = f_\alpha \text{ for any } \alpha\}.$$

Especially we will use Whitney jets on the sets

$$Z_k := Z_k(\Omega) := \{x \in \Omega \mid x_j = 0 \text{ for at least } k \text{ indices } j\}$$

for $0 \leq k \leq d$. Notice that

$$Z_0 = \Omega, \qquad Z_1 = \{x \in \Omega \mid \exists j \leq d : x_j = 0\}, \qquad Z_d = \{0\} \text{ if } 0 \in \Omega.$$

Hadamard operators canonically operate on $W(Z_k)$ as follows: For $(f_\alpha)_{\alpha\in\mathbb{N}^d} \in W(Z_k)$ choose f as in (5.12) and set

$$(H((f_\alpha)_{\alpha\in\mathbb{N}^d}))_\beta := \partial^\beta H(f)\Big|_{Z_k} \text{ for } \beta \in \mathbb{N}^d.$$

This is a well-defined mapping $H : W(Z_k) \to W_{I(H)}(Z_k)$ for

$$W_{I(H)}(Z_k) := \{(f_\alpha)_{\alpha\in\mathbb{N}^d} \in W(Z_k) \mid \forall\emptyset \neq J \subset D\ \forall\alpha \in \mathbb{N}^J\ \forall(0_J, x_{D\setminus J}) \in \Omega : \\ f_\alpha(0_J, x_{D\setminus J}) = 0 \text{ if } m_{(\alpha,\beta)} = 0 \text{ for any } \beta \in \mathbb{N}^{D\setminus J}\}.$$

Indeed, if f and $\widetilde{f}$ are chosen as in (5.12), then $f^{(\alpha)}|_{Z_k} = \widetilde{f}^{(\alpha)}|_{Z_k}$ for any α and $f - \widetilde{f}$ is flat on Z_k, and therefore we get for any $x \in Z_k$ by (3.4) for $H = H_T$

$$\partial_x^\beta H(f)(x) = \langle {}_yT, y^\beta f^{(\beta)}(xy)\rangle = \langle {}_yT, y^\beta \widetilde{f}^{(\beta)}(xy)\rangle = \partial_x^\beta H(\widetilde{f})(x) \text{ for any } \beta$$

since $xy \in Z_k$ for any $y \in \mathbb{R}^d$ if $x \in Z_k$. It is evident from Lemma 3.3 that $H((f_\alpha)_\alpha) \in W_{I(H)}(Z_k)$.

For $0 \le k \le d-1$ let

$$W_{flat}(Z_k) := \{(f_\alpha)_{\alpha\in\mathbb{N}^d} \in W(Z_k) \mid f_\alpha\big|_{Z_{k+1}} = 0 \text{ for any } \alpha\},$$

especially,

$$W_{flat}(Z_0) = \{f \in C^\infty(\Omega) \mid f \text{ is flat on } Z_1\} = C^\infty_{flat}(\Omega).$$

Notice that $x \in Z_k \setminus Z_{k+1}$ iff $x_j = 0$ exactly for $j \in J$ where $|J| = k$. Thus $Z_k \setminus Z_{k+1}$ is the disjoint union of the sets

$$S_{D\setminus J} := \{0_J\} \times \big(\Omega_{D\setminus J} \setminus Z_1(\Omega_{D\setminus J})\big) = \big(\{0_J\} \times \Omega_{D\setminus J}\big) \setminus Z_{k+1}, |J| = k.$$

Hence $h \in W_{flat}(Z_k)$ if and only if for any J with $|J| = k$ there are Whitney jets $h_{D\setminus J}$ on $\{0_J\} \times \Omega_{D\setminus J}$ defined by $(h_{D\setminus J,\alpha})_{\alpha\in\mathbb{N}^J} \in C^\infty_{flat}(\Omega_{D\setminus J})^{\mathbb{N}^J}$, i.e.,

$$(h_{D\setminus J})_{(\alpha,\beta)}(0_J, x_{D\setminus J}) = \partial^\beta_{x_{D\setminus J}} h_{D\setminus J,\alpha}(x_{D\setminus J})$$

for any $(\alpha, \beta) \in \mathbb{N}^J \times \mathbb{N}^{D\setminus J}, (0_J, x_{D\setminus J}) \in \Omega$ such that

$$h = \sum_{J\subset D, |J|=k} h_{D\setminus J}, \tag{5.13}$$

where $h_{D\setminus J}$ is trivially extended to a Whitney jet on Z_k, that is, for any J with $|J| = k$ we have

$$h_{(\alpha,\beta)}(0_J, x_{D\setminus J}) = \partial^\beta_{x_{D\setminus J}} h_{D\setminus J,\alpha}(x_{D\setminus J}) \text{ for any } (\alpha,\beta) \in \mathbb{N}^J \times \mathbb{N}^{D\setminus J} \text{ if } (0_J, x_{D\setminus J}) \in \Omega.$$

Notice that any $h_{D\setminus J}$ is flat on $Z_1(\Omega_{D\setminus J})$ (and on $Z_{k+1}(\Omega)$ by trivial extension).

Our sufficient condition now reads as follows.

Theorem 5.5 (Sufficient condition for surjectivity)**.** *Let $H = H_T$ be a Hadamard operator on $C^\infty(\Omega)$. Then the shifted operators*

$$H_{T,k} : C^\infty(\Omega) \to C^\infty_{I(H_{T,k})}(\Omega) \text{ are surjective for any } k \in \mathbb{N}^d$$

if the operators

$$H_\alpha : C^\infty_{flat}(\Omega_{D\setminus J}) \to C^\infty_{flat}(\Omega_{D\setminus J}) \text{ are surjective} \tag{5.14}$$

for any $J \subsetneqq D$ such that $\Omega_{D\setminus J} \neq \emptyset$ and any $\alpha = \alpha_J \in \mathbb{N}^J$ such that $H_\alpha \neq 0$.

Proof. Notice that $C^\infty_{flat}(\Omega)$ is an invariant subspace for any Hadamard operator on $C^\infty(\Omega)$ by (3.4) since $xy \in Z_1$ for any $y \in \mathbb{R}^d$ if $x \in Z_1$. Moreover, the operator of multiplication by $x_{D\setminus J}^{k_{D\setminus J}}$ is an isomorphism on $C^\infty_{flat}(\Omega_{D\setminus J})$ for $k \in \mathbb{N}^d$. Hence, the shifted operator $H_{T,k} = H_{T_k}$ also satisfies (5.14) by (5.11). We may thus assume that $k = \mathbf{0}$, i.e., $H_{T,k} = H_{T,\mathbf{0}} = H_T$.

If $Z_1 \cap \Omega = \emptyset$ we set $J = \emptyset$ in (5.14) and

$$H_T : C^\infty(\Omega) = C^\infty_{flat}(\Omega_D) \to C^\infty_{flat}(\Omega_D) = C^\infty_{I(H_T)}(\Omega)$$

is surjective by assumption.

Let $Z_1 \cap \Omega \neq \emptyset$. Then the number $d_0 := \max\{d \geq k \geq 1 \mid Z_k \cap \Omega \neq \emptyset\}$ exists. We will show by induction that the mappings

$$H = H_T : W(Z_k) \to W_{I(H)}(Z_k)$$

are surjective for $k = d_0, \dots, 0$. Since $W(Z_0) = C^\infty(\Omega)$ this will prove the claim.

(i)(a) Let $d_0 = d$ (and hence $Z_d = \{0\} \subset \Omega$). Let

$$f = (f_\alpha)_\alpha \in W_{I(H)}(Z_d) = \{(c_\alpha)_\alpha \in \mathbb{C}^{\mathbb{N}^d} \mid c_\alpha = 0 \text{ if } m_\alpha := \langle T, \xi^\alpha \rangle = 0\}.$$

By E. Borel's theorem we find $h \in C^\infty(\mathbb{R}^d)$ such that $h^{(\alpha)}(0) = f^{(\alpha)}(0)/m_\alpha$ if $\alpha \notin I(H)$. Hence $\widetilde{h} := (h^{(\alpha)}(0))_\alpha \in W(Z_d)$ satisfies $H(\widetilde{h}) = f$ since by the definition of $H(\widetilde{h})$ and (3.4)

$$\begin{aligned} H(\widetilde{h})_\beta = \partial^\beta H(h)\Big|_{Z_d} &= \langle {}_yT, (\partial^\beta h)(xy) y^\beta \rangle\Big|_{x=0} \\ &= m_\beta h^{(\beta)}(0) = f^{(\beta)}(0) \text{ for any } \beta \in \mathbb{N}^d \end{aligned}$$

(notice that the last equation also holds if $m_\beta = 0$ since $f \in W_{I(H)}(Z_d)$).

(i)(b) Let $d_0 < d$. Then $Z_{d_0+1} = \emptyset$ and we have for any $J \subset D$ with $|J| = d_0$

$$\{0_J\} \times Z_1(\Omega_{D\setminus J}) \subset Z_{d_0+1} \cap (\{0_J\} \times \Omega_{D\setminus J}) \subset Z_{d_0+1} \cap \Omega = \emptyset \tag{5.15}$$

by the definition of d_0. Moreover,

$$W_{I(H)}(Z_{d_0}) \subset W(Z_{d_0}) = W_{flat}(Z_{d_0}), \tag{5.16}$$

and by (5.13) any $f \in W_{I(H)}(Z_{d_0})$ can be written as

$$f = \sum_{J \subset D, |J| = d_0} f_{D\setminus J},$$

where $f_{D\setminus J} = (f_{D\setminus J,\alpha})_{\alpha \in \mathbb{N}^J} \in C^\infty_{flat}(\Omega_{D\setminus J})^{\mathbb{N}^J}$ is trivially extended to Z_{d_0}. Since $Z_1(\Omega_{D\setminus J}) = \emptyset$ by (5.15), we get $C^\infty_{flat}(\Omega_{D\setminus J}) = C^\infty(\Omega_{D\setminus J})$, and by (5.14)

$$H_\alpha = H_{T_\alpha} : C^\infty(\Omega_{D\setminus J}) \to C^\infty(\Omega_{D\setminus J}) \tag{5.17}$$

is surjective for any $\alpha = \alpha_J \in \mathbb{N}^J$ such that $m_{(\alpha,\beta)}(T) \neq 0$ for some $\beta \in \mathbb{N}^{D\setminus J}$. By (5.17) for any $f_{D\setminus J,\alpha}, \alpha \in \mathbb{N}^J$, we thus find $h_{D\setminus J,\alpha} \in C^\infty(\Omega_{D\setminus J})$ such that $H_\alpha(h_{D\setminus J,\alpha}) = f_{D\setminus J,\alpha}$. If $m_{(\alpha,\beta)}(T) = 0$ for any $\beta \in \mathbb{N}^{D\setminus J}$, then $H_\alpha = 0$ and $f_{D\setminus J,\alpha} = 0$ since $f \in W_{I(H)}(Z_{d_0})$, and we take $h_{D\setminus J,\alpha} := 0$. The Whitney jets $h_{D\setminus J} := (h_{D\setminus J,\alpha})_{\alpha \in \mathbb{N}^J}, |J| = d_0$, define a Whitney jet $h \in W(Z_{d_0})$ as in (5.13) (also use (5.15)) and, clearly, $H_T(h) = f$.

(ii) Let the claim hold for some $1 \leq k \leq d_0$. For $f := (f_\alpha)_\alpha \in W_{I(H)}(Z_{k-1})$ we have $\widetilde{f} := (f_\alpha\Big|_{Z_k})_\alpha \in W_{I(H)}(Z_k)$, and by the induction hypothesis there is $g \in W(Z_k)$ such that $H_T(g) = \widetilde{f}$. Let $\widetilde{g} \in W(Z_{k-1})$ be an extension of g (first, to

Ω by Whitney's Extension Theorem (see [26]) and then restricted to Z_{k-1}). Then $h := f - H_T(\widetilde{g}) \in W_{flat}(Z_{k-1})$ since

$$H_T(\widetilde{g})\Big|_{Z_k} = H_T(\widetilde{g}\big|_{Z_k}) = H_T(g) = \widetilde{f} = f\Big|_{Z_k}$$

by the action of H_T on Whitney jets. Hence we have by (5.13)

$$h = \sum_{J\subset D, |J|=k-1} h_{D\setminus J},$$

where $h_{D\setminus J} = (h_{D\setminus J,\alpha})_{\alpha\in\mathbb{N}^J} \in C^\infty_{flat}(\Omega_{D\setminus J})^{\mathbb{N}^J}$. By the assumption there are $\widetilde{h}_{J,\alpha} \in C^\infty_{flat}(\Omega_{D\setminus J})$ such that $H_\alpha(\widetilde{h}_{D\setminus J,\alpha}) = h_{D\setminus J,\alpha}$ for $\alpha \in \mathbb{N}^J, |J| = k-1$. These again define Whitney jets $\widetilde{h}_{D\setminus J} \in W(\Omega_{D\setminus J}), |J| = k-1$, and then $\widetilde{h} \in W(Z_{k-1})$ as in (5.13) satisfying $H(\widetilde{h}) = h$. This proves the claim for $k-1$. □

Let us emphasize that the basis of the induction in the preceding proof is either automatically given (if $0 \in \Omega$) or it is concerned with surjectivity of Hadamard operators on spaces $C^\infty_{flat}(\omega)$ where $\omega \subset \mathbb{R}^\rho$ is open and contained in $\mathbb{R}^\rho \setminus Z_1$; hence $C^\infty_{flat}(\omega) = C^\infty(\omega)$ in that case, and surjectivity holds by assumption (5.14). Moreover, checking (5.14) in that case can be transferred to a surjectivity problem for convolution operators in the classical sense using the transformation of variables by Log and Exp, respectively, as mentioned in the introduction (see [25]).

6. Hadamard operators H_T with $\operatorname{supp}(T) \subset [0,\infty[^d$

In the preceding section we reduced the question of surjectivity of the shifted Hadamard operators to a corresponding problem for functions which are flat on Z_1. In the present section we will take a step further and reduce the surjectivity problem to solvability within functions supported in any of the canonical quadrants. Since Hadamard operators $H = H_T$ in general do not even operate on such functions we have to assume here that $\operatorname{supp}(T) \subset [0,\infty[^d$. Significant examples are Euler partial differential operators (see Sections 9 and 10) where $P(\theta) = H_T$ for some $T \in C^\infty(V(\Omega))'$ with $\operatorname{supp} T = \{\mathbf{1}\}$ (cf. (9.1)).

We recall some useful notation. For $\sigma \in \{\pm1\}^d$ and $\Omega \subset \mathbb{R}^d$ open we set

$$Q_\sigma := \{x \in \mathbb{R}^d \mid \sigma x \geq \mathbf{0}\} \text{ and } \Omega_\sigma := \Omega \cap Q_\sigma.$$

Notice that Ω_σ is neither open nor closed in general; in fact, Ω_σ is open iff $\partial Q_\sigma \cap \Omega = \emptyset$, and Ω_σ is closed if either $Q_\sigma \cap \Omega = \emptyset$ or $Q_\sigma \subset \Omega$. Let $C^\infty(\Omega_\sigma)$ denote the C^∞-functions on Ω_σ in the sense of Whitney, and let $C^\infty_{I(H)}(\Omega_\sigma)$ be defined accordingly.

We now extend the notion of Hadamard operators as follows: a continuous linear mapping $H : C^\infty(\Omega_\sigma) \to C^\infty(\Omega_\sigma)$ is a Hadamard operator if

$$H(\xi^\alpha\big|_{\Omega_\sigma})(x) = m_\alpha x^\alpha, x \in \Omega_\sigma, \text{ for any } \alpha \in \mathbb{N}^d.$$

Remark 6.1. Let $H = H_T$ be a Hadamard operator on $C^\infty(\Omega)$ with $\operatorname{supp}(T) \subset [0,\infty[^d$. Then H can be uniquely extended to a Hadamard operator

$$H_\sigma : C^\infty(\Omega_\sigma) \to C^\infty_{I(H)}(\Omega_\sigma)$$

for any $\sigma \in \{\pm1\}^d$.

Proof. By Whitney's Extension Theorem, any $f \in C^\infty(\Omega_\sigma)$ can be extended to $F \in C^\infty(\Omega)$. Notice that $C^\infty(\Omega_\sigma)$ is endowed with the corresponding quotient topology. Set

$$H_\sigma(f)(x) := H(F)(x), x \in \Omega_\sigma.$$

Clearly, $H_\sigma(f) \in C^\infty(\Omega_\sigma)$ and

$$H_\sigma(f)(x) = H(F)(x) = \langle {}_yT, F(xy)\rangle = \langle {}_yT, f(xy)\rangle, \qquad x \in \Omega_\sigma^\circ,$$

since $\operatorname{supp}(T) \subset [0,\infty[^d$. Hence H_σ is well defined and extends H. The extension is unique since it is fixed on the polynomials which are dense in $C^\infty(\Omega_\sigma)$. □

Proposition 6.2. *For any $f \in C^\infty_{I(H)}(\Omega_\sigma)$ there is $F \in C^\infty_{I(H)}(\Omega)$ such that $F\big|_{\Omega_\sigma} = f$.*

Proof. (a) We first solve the extension problem locally near $S := \partial\Omega_\sigma \cap \Omega$. We can assume that $\sigma = \mathbf{1}$. For $\xi \in S$ let $\overline{B^\infty_\delta(\xi)} \subset \Omega$. Then there is $\emptyset \neq J \subset D$ such that $\xi_j = 0$ exactly for $j \in J$ and

$$\overline{B^\infty_\delta(\xi)} \cap \Omega_{\mathbf{1}} = [0,\delta]^J \times \overline{B^\infty_\delta(\xi_{D\setminus J})} =: \Sigma \Subset \Omega.$$

We may assume that $J = \{1,\dots,n\}$ for $1 \le n \le d$. Let $E : C^\infty([0,\delta]) \to C^\infty([-\delta,\delta])$ be a continuous linear extension operator (see, e.g., [23]) and define

$$L(f)(x) := E\big(f(\,\cdot\,,x')\big)(x_1), f \in C^\infty_{I(H)}(\Omega_{\mathbf{1}}),$$

for $x \in [-\delta,\delta] \times [0,\delta]^{n-1} \times \overline{B^\infty_\delta(\xi)} =: \Sigma_1$ where the middle term (and the last term) in Σ_1 is to be omitted if $n = 1$ (and if $n = d$, respectively). Then $L(f) \in C^\infty(\Sigma_1)$ since for $j \ge 2$,

$$(6.1) \qquad \partial_j(L(f))(x) = \lim_{t\to 0} E\big([f(\,\cdot\,,x'+he_j) - f(\,\cdot\,,x')]/h\big)(x_1) = L(\partial_j f)(x)$$

(where $t > 0$ (and $t < 0$) if $x_j = 0$ (and $x_j = \delta$, respectively) and $j \le n$). Clearly, $L(f)$ is an extension of $f\big|_\Sigma$. We now prove that $L(f) \in C^\infty_{I(H)}(\Sigma_1)$. Let $m_{(\alpha,\,\cdot\,)} = 0$ for $\alpha \in \mathbb{N}^K$. If $1 \in K$, then $(0_K, x_{D\setminus K}) \in \Sigma$ if $(0_K, x_{D\setminus K}) \in \Sigma_1$, and hence

$$\partial^\alpha(L(f))(0_K, x_{D\setminus K}) = \partial^\alpha f(0_K, x_{D\setminus K}) = 0,$$

since L is an extension operator and $f \in C^\infty_{I(H)}(\Omega_{\mathbf{1}})$. If $1 \notin K$, then

$$\partial^\alpha(L(f))(0_K, x_{D\setminus K}) = E\big(\partial^\alpha f(\,\cdot\,, 0_K, x_{D\setminus(K\cup\{1\})})\big)(x_1) = 0 \text{ for } (0_K, x_{D\setminus K}) \in \Sigma_1$$

by (6.1) since E is linear and $\partial^\alpha f(x_1, 0_K, x_{D\setminus(K\cup\{1\})}) = 0$ for $(x_1, 0_K, x_{D\setminus(K\cup\{1\})}) \in \Sigma$ since $f \in C^\infty_{I(H)}(\Omega_{\mathbf{1}})$. The proof is finished (directly, if $n = 1$, and by induction on n, if $n > 1$).

(b) Since S is closed in Ω we can find a covering $B^\infty_{\delta_k}(x_k), x_k \in S$, of S which is locally finite in Ω and such that the extension problem on $\overline{B^\infty_{\delta_k}(x_k)}$ is solved by means of the extension operators L_k from (a). Choose a corresponding resolution of the identity $\phi_n \in \mathscr{D}_{lc}(B^\infty_{\delta_k}(x_k))$ for S by Proposition 3.5(b). For $f \in C^\infty_{I(H)}(\Omega_\sigma)$ let

$$L(f)(x) := \sum_k L_k(f)(x)\phi_k(x) \text{ for } x \in \Omega \setminus \Omega_\sigma \text{ and } L(f)(x) := f(x) \text{ for } x \in \Omega_\sigma.$$

Then $L(f) \in C^\infty(\Omega)$ by (a) and since $\sum_k L_k(f)\phi_k = f$ near $\partial\Omega_\sigma \cap \Omega$. Clearly, $L(f)$ extends f. By (a) and Proposition 3.5(a) we finally have $L(f) \in C^\infty_{I(H)}(\Omega)$, and $L : C^\infty_{I(H)}(\Omega_\sigma) \to C^\infty_{I(H)}(\Omega)$ is an extension operator. □

Theorem 6.3. *Let* $H = H_T$ *with* $\text{supp}(T) \subset [0,\infty[^d$. *If* $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ *is surjective, then* $H_\sigma : C^\infty(\Omega_\sigma) \to C^\infty_{I(H)}(\Omega_\sigma)$ *is surjective for any* $\sigma \in \{\pm 1\}^d$ *with* $\Omega_\sigma \neq \emptyset$.

Proof. For $f \in C^\infty_{I(H)}(\Omega_\sigma)$ we choose an extension $F \in C^\infty_{I(H)}(\Omega)$ by Proposition 6.2. By assumption there is $G \in C^\infty(\Omega)$ such that $H(G) = F$; hence $g := G\big|_{\Omega_\sigma}$ satisfies

$$H_\sigma(g) = H(G)\big|_{\Omega_\sigma} = F\big|_{\Omega_\sigma} = f$$

since $\text{supp}(T) \subset [0,\infty[^d$. □

Let

$$\mathcal{E}(\Omega_\sigma) := \{f \in C^\infty(\Omega) \mid \text{supp}(f) \subset \Omega_\sigma\} \text{ and}$$
$$\mathcal{E}^{\mathbf{k}}(\Omega_\sigma) := \{f \in C^{\mathbf{k}}(\Omega) \mid \text{supp}(f) \subset \Omega_\sigma\}$$

for $\sigma \in \{\pm 1\}^d$ with the topologies induced by

$$C^\infty(\Omega) \text{ (and } C^{\mathbf{k}}(\Omega) := \{f \in C(\Omega) \mid \partial^\alpha f \in C(\Omega) \text{ if } \alpha \leq \mathbf{k}\}, \text{ respectively).}$$

We now get the following necessary condition for surjectivity of shifted Hadamard operators.

Theorem 6.4 (Necessary condition for surjectivity)**.** *Let* $H = H_T$ *be a Hadamard operator on* $C^\infty(\Omega)$ *with* $\text{supp}(T) \subset [0,\infty[^d$. *If the shifted operators*

$$H_{T,k} : C^\infty(\Omega) \to C^\infty_{I(H_{T,k})}(\Omega) \textit{ are surjective for any } k \in \mathbb{N}^d,$$

then there is $l \in \mathbb{N}$ *such that for any* $k \in \mathbb{N}, k \geq l$,

$$H_\alpha(\mathcal{E}^{\mathbf{k}}(\Omega_{D\setminus J,\sigma})) \supset \mathcal{E}(\Omega_{D\setminus J,\sigma})$$

for any $J \subsetneq D$ *and* $\sigma \in \{\pm 1\}^{D\setminus J}$ *with* $\Omega_{D\setminus J,\sigma} \neq \emptyset$ *and any* $\alpha = \alpha_J \in \mathbb{N}^J$ *with* $H_\alpha \neq 0$.

Proof. Since surjectivity is inherited to restrictions by Theorem 5.1 and since the restrictions of the shifted operators are the shifts of the restricted operators by (5.11), we only need to prove the claim $J = \emptyset$, i.e., for H. Since $(H_\sigma)_{T,k} = (H_{T,k})_\sigma : C^\infty(\Omega_\sigma) \to C^\infty_{I(H)}(\Omega_\sigma)$ is surjective for any σ with $\Omega_\sigma \neq \emptyset$ and any $k \in \mathbb{N}^d$ by Theorem 6.3, we can now argue as in Proposition 5.3 with Ω substituted by Ω_σ. □

If $\text{supp}(T) \subset]0,\infty[^d$, then $T_c := y^c T$ is defined for any $c \in \mathbb{R}^d$. If H_T operates on $C^\infty(\Omega)$, then also H_{T_c} operates on $C^\infty(\Omega)$ and, moreover, on $C^\infty(\Omega \setminus Z_1)$ by the Representation Theorem 3.1, and

(6.2) $x^c H_{T,c}(f)(x) = \langle_y T y^c, x^c f(xy)\rangle = \langle_y T, (xy)^c f(xy)\rangle = H_T(\xi^c f)(x)$ on $\Omega \setminus Z_1$

for any $f \in C^\infty(\Omega \setminus Z_1)$. Hence for any $c \in \mathbb{R}^d$

$$H_{T,c}(\xi^\alpha)(x) = \langle_y T, y^{\alpha+c}\rangle x^\alpha \text{ on } \Omega. \tag{6.3}$$

For Euler differential operators the shifted operator is therefore given by

$$P(\theta)_c = P(\theta + c) \text{ for any } c \in \mathbb{R}^d. \tag{6.4}$$

Clearly, by the Representation Theorem 3.1, the Hadamard operator

$$H = H_T : \mathcal{E}(\Omega_\sigma) \to \mathcal{E}(\Omega_\sigma), \sigma \in \{\pm 1\}^d, \text{ is linear and continuous}$$

if $\text{supp}(T) \subset [0,\infty[^d \cap V(\Omega)$. Theorem 5.5 implies the following sufficient condition for surjectivity.

Theorem 6.5 (Sufficient condition for surjectivity)**.** *Let $H = H_T$ be a Hadamard operator on $C^\infty(\Omega)$ with $\operatorname{supp}(T) \subset [0,\infty[^d$ (respectively, with $\operatorname{supp}(T) \subset]0,\infty[^d$). Then the shifted operators*

$$H_{T,k} : C^\infty(\Omega) \to C^\infty_{I(H_{T,k})}(\Omega) \text{ are surjective}$$

for any $k \in \mathbb{N}^d$ (and for any $k \in \mathbb{R}^d$, respectively) if the operators

$$H_\alpha : \mathcal{E}(\Omega_{D\setminus J,\sigma}) \to \mathcal{E}(\Omega_{D\setminus J,\sigma}) \text{ are surjective}$$

for any $J \subsetneq D$ and $\sigma \in \{\pm 1\}^{D\setminus J}$ such that $\Omega_{D\setminus J,\sigma} \neq \emptyset$ and any $\alpha = \alpha_J \in \mathbb{N}^J$ such that $H_\alpha \neq 0$.

Proof. For $f \in C^\infty_{flat}(\Omega)$ there are $f_\sigma \in \mathcal{E}(\Omega_\sigma)$ such that $f = \sum_{\sigma\in\{\pm1\}^d} f_\sigma$. By assumption there are $g_\sigma \in \mathcal{E}(\Omega_\sigma)$ such that $H(g_\sigma) = f_\sigma$. Then $g := \sum_\sigma g_\sigma \in C^\infty_{flat}(\Omega)$ and $H(g) = f$. Since this argument also applies to any of the restricted operators H_α, the claim now follows for $k \in \mathbb{N}^d$ from Theorem 5.5. For $\operatorname{supp}(T) \subset]0,\infty[^d$ we may use the remarks above to reduce the statement to be proved to the case $k = \mathbf{0}$ (cf. the beginning of the proof of Theorem 5.5). □

The reader might suspect that substituting the spaces $\mathcal{E}(\Omega_{D\setminus J,\sigma})$ in the sufficient condition from Theorem 6.5 by the spaces $C^\infty(\Omega_{D\setminus J,\sigma})$ will produce a condition sufficient at least for the surjectivity of H. This however is not the case. An elementary example is provided by $H := \theta_1$ and $\Omega := \mathbb{R}^2 \setminus (\{0\}\times]-\infty, 0])$. Indeed, $\theta_1 : C^\infty(\Omega) \to C^\infty_{I(\theta_1)}(\Omega)$ is not surjective by Proposition 4.8 while $\theta_1 : C^\infty(\Omega_\sigma) \to C^\infty_{I(\theta_1)}(\Omega_\sigma) = \{f \in C^\infty(\Omega_\sigma) \mid f(0,x_2) = 0 \text{ if } (0,x_2) \in \Omega_\sigma\}$ (and, similarly, any of the nonzero restricted operators) are surjective for any σ since, e.g., $\theta_1 R(f) = f$ for

$$R(f)(x) := \int_1^{x_1} \frac{f(t,x_2)}{t} dt, x \in \Omega_{\mathbf{1}}, f \in C^\infty_{I(\theta_1)}(\Omega_{\mathbf{1}}),$$

and similarly in the other cases (notice that $R(f) \in C^\infty(\Omega_{\mathbf{1}}), \Omega_{\mathbf{1}} = [0,\infty[^2\setminus\{0\})$.

7. Admissible sets

In the present section we will discuss statement (c) of the Main Theorem B, i.e., for which sets Ω the Euler operator $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective iff all shifted operators $P(k+\theta) : C^\infty(\Omega) \to C^\infty_{I(P(k+\,\cdot\,))}(\Omega), k \in \mathbb{R}^d$, are surjective. Clearly, some extra assumption on Ω depending on P is needed since we have already seen in Proposition 4.9 that both statements are not equivalent in general. Nevertheless we will show that the equivalence holds for a large class of sets which we call *admissible* sets. To start with, the above implication clearly holds if $\Omega \cap Z_1 = \emptyset$ (use the Theorem of Vogt, (1.3) and (1.4) from the introduction). Another important class is the sets Ω having the projection property (see (2.5)). We state the result for Hadamard operators obtaining the necessary conditions from Theorem 5.4.

Theorem 7.1. *Let Ω have the projection property* (2.5)*. If the Hadamard operator $H : C^\infty(\Omega) \to C^\infty_{I(H)}(\Omega)$ is surjective, then*

$$H_\alpha\Big(C^\infty_{\mathbf{k}-flat}(\Omega_{D\setminus J})\Big) \supset C^\infty_{flat}(\Omega_{D\setminus J})$$

for any $k \in \mathbb{N}$, any $J \subsetneq D$ with $\Omega_{D\setminus J} \neq \emptyset$, and any $\alpha = \alpha_J \in \mathbb{N}^J$ with $H_\alpha \neq 0$.

Proof. Since the projection property is inherited to $\Omega_{D\setminus J} \neq \emptyset$, it is sufficient to prove the claim for H (i.e., for $J = \emptyset$) by Theorem 5.1 .

Let $S_{\{j\}} := \{x \in \mathbb{R}^d \mid x_j = 0\}$, and let $K := \{j \mid S_{\{j\}} \cap \Omega \neq \emptyset\}$. Fix $k \in \mathbb{N}$. The canonical projections $\Pi_{j,k} : C^\infty(\Omega) \to C^\infty(\Omega)$ can be defined by (2.5) for $j \in J$ and $k \in \mathbb{N}$ by setting

$$\Pi_{j,k}(f)(x) := f(x) - \sum_{n=0}^{k} \partial_j^n f(\pi_j(x)) x_j^n / n!. \tag{7.1}$$

Set $L_k := \prod_{j \in K} \Pi_{j,k}$. For $g \in C^\infty_{flat}(\Omega)$ there is $f \in C^\infty(\Omega)$ by assumption such that $H(f) = g$. Then $L_k(f) \in C^\infty_{\mathbf{k}-flat}(\Omega)$ since $S_{\{j\}} \cap \Omega = \emptyset$ for $j \notin K$. Notice that L_k is a multiplier projection, and therefore L_k commutes with H. Moreover, $L_k(g) = g$ since $g \in C^\infty_{flat}(\Omega)$. Hence

$$H(L_k(f)) = L_k(H(f)) = L_k(g) = g.$$

The claim is proved. □

For the rest of this section we will consider Euler operators $P(\theta)$. We first single out those subspaces where some conditions are imposed on the functions in $C^\infty_{I(P)}(\Omega)$. Let

$$S_K := \{x \in \mathbb{R}^d \mid x_K = \mathbf{0}_K\} \text{ for } \emptyset \neq K \subset D.$$

Definition 7.2. For an Euler operator $P(\theta)$ we define

$$S(P) := \bigcup_{\emptyset \neq K \subset D} \left\{S_K \mid \exists \alpha = \alpha_K \in \mathbb{N}^K : P(\alpha_K, \,\cdot\,) = 0\right\} \text{ and}$$

$$S_J(P) := \bigcup \{S_K \mid S_K \subset S(P), J \subset K \subset D\} \subset S_J \text{ for } \emptyset \neq J \subset D.$$

We will consider admissible sets defined by the following intuitive geometric condition connected to $S(P)$. Let

$$\omega_K := \Omega \cap S_K \text{ for } \emptyset \neq K \subset D \text{ and } \omega_\emptyset := \Omega.$$

Definition 7.3. An open set $\Omega \subset \mathbb{R}^d$ is called admissible for $P(\theta)$ if for any $\emptyset \neq J \subsetneq D$

$$\Omega \cap L_{x,j} = \emptyset \text{ for any } j \in J \text{ and any } x \in \partial_J \left(S_J(P) \cap \omega_J\right). \tag{7.2}$$

Here $L_{x,j} := x + \mathbb{R}e_j$ is the line passing x in direction e_j and ∂_J denotes the boundary in S_J.

Admissibility is most restrictive if $S(P)$ contains hyperplanes. Let

$$J_{max} := \{j \in D \mid S_{\{j\}} \subset S(P)\}.$$

Remark 7.4. If Ω is connected and admissible for $P(\theta)$, then

$$\pi_j(\Omega) \subset \Omega \text{ for any } j \in J_{max} \text{ with } \omega_{\{j\}} \neq \emptyset.$$

Proof. Let $j \in J_{max}$ and $\omega_{\{j\}} \neq \emptyset$. Let $j = 1$ w.l.o.g. Then $S_{\{1\}}(P) = S_{\{1\}}$ since $1 \in J_{max}$. Thus

$$\partial_{\{1\}} \left(S_{\{1\}}(P) \cap \omega_{\{1\}}\right) = \partial_{\{1\}} \omega_{\{1\}} = \{0\} \times \partial\Omega_{D\setminus\{1\}},$$

and hence $\Omega \cap (\mathbb{R} \times \partial\Omega_{D\setminus\{1\}}) = \emptyset$ by admissibility. Since Ω is connected and $\omega_{\{1\}} \neq \emptyset$, this implies that $\Omega \subset \mathbb{R} \times \Omega_{D\setminus\{1\}}$. The claim is proved. □

Corollary 7.5. *Let Ω be connected. Then Ω is admissible for all $P(\theta)$ iff Ω has the projection property.*

Proof. The necessity follows from Remark 7.4 while the sufficiency is seen as follows: if $\omega_{\{j\}} \neq \emptyset$ and $\Omega \cap L_{x,j} \neq \emptyset$ for some $x \in S_{\{j\}}$, then $x \in \Omega$ by the projection property; hence $x \in \omega_{\{j\}}$ and $x \notin \partial_{\{j\}} \left(S_{\{j\}}(P) \cap \omega_{\{j\}}\right)$. So (7.2) holds for $J := \{j\}$. Since the projection property is inherited to ω_K for any $\emptyset \neq K \subset D$, this implies that Ω is admissible. □

Notice, however, that we mostly can assume that $J_{max} = \emptyset$ by the following proposition.

Proposition 7.6. *Let $P(\theta)$ be an Euler operator with P irreducible. Then $J_{max} = \emptyset$ or $P(x) = c(x_j - \alpha_j)$ for some $\alpha_j \in \mathbb{N}, c \in \mathbb{C}_*$, and $j \in D$. If $d = 2$, then $S(P) \subset \{\mathbf{0}\}$ in the former case.*

Proof. Let $j \in J_{max}$, and let $j = 1$ w.l.o.g. Then $P(\alpha_1, \,\cdot\,) = 0$ for some $\alpha_1 \in \mathbb{N}$, and therefore

$$P(x_1, y') = \sum_{j=1}^{m} \partial_1^j P(\alpha_1, y')(x_1 - \alpha_1)^j / j! = (x_1 - \alpha_1) \sum_{j=1}^{m} \partial_1^j P(\alpha_1, y')(x - \alpha_1)^{j-1} / j!\ .$$

Hence $P(x_1, y') = c(x_1 - \alpha_1)$ since P is irreducible. If $J_{max} = \emptyset$, then $|K| \geq 2$ if $S_K \subset S(P)$. Thus $S_K = S_{\{1,2\}} = \{\mathbf{0}\}$ (or $S(P) = \emptyset$) if $d = 2$. □

Recall that surjectivity of the operators $\theta_j - \alpha_j, \alpha_j \in \mathbb{N}$, was discussed in Proposition 4.8. Clearly, the assumption (7.2) becomes weaker the smaller $S(P)$ is. This is striking in the extreme case that $S(P) \subset \{\mathbf{0}\}$.

Proposition 7.7.
(a) *If $S(P) \subset \Omega$ or $S(P) \cap \Omega = \emptyset$, then Ω is admissible for $P(\theta)$.*
(b) *If $S(P) \subset \{\mathbf{0}\}$, then all open sets Ω are admissible for $P(\theta)$.*

Proof.
(a) If $S(P) \cap \Omega = \emptyset$, then $S_J(P) \cap \omega_J = \emptyset$ for any $\emptyset \neq J$, hence (7.2) is void. If $S(P) \subset \Omega$, then

$$\partial_J \left(S_J(P) \cap \omega_J\right) = \partial_J \left(S_J(P)\right) = \emptyset$$

for any $\emptyset \neq J$ and (7.2) is void again.
(b) This follows from (a) since $S(P) \subset \{\mathbf{0}\} \subset \Omega$ if $\mathbf{0} \in \Omega$, and $S(P) \cap \Omega = \emptyset$ if $\mathbf{0} \notin \Omega$. □

We now prove a useful estimate for $S(P)$. For $j \in D$ we expand $P(x)$ as

$$P(x) = \sum_{k=0}^{m_j} Q_k(x_{D \setminus \{j\}}) x_j^k, \tag{7.3}$$

where $m_j \in \mathbb{N}$ is the degree of P w.r.t. x_j.

Proposition 7.8. *Let P be written as in* (7.3).
(a) *$S(P) \subset S_{\{j\}}$ iff there is no $\alpha \in \mathbb{N}^{D \setminus \{j\}}$ such that $Q_k(\alpha) = 0$ for any $k = 0, \ldots, m_j$.*
(b) *Specifically, $S(P) \subset S_{\{j\}}$ if $Q_k \equiv c \neq 0$ for some $0 \leq k \leq m_j$.*

Proof.

(a) *Necessity.* If there is $\alpha \in \mathbb{N}^{D\setminus\{j\}}$ such that $Q_k(\alpha) = 0$ for any $k = 0, \dots, m_j$, then $P(\alpha, \cdot) = 0$ and hence $S_{D\setminus\{j\}} \subset S(P)$, that is, $S(P) \not\subseteq S_{\{j\}}$.

Sufficiency. If $S(P) \not\subseteq S_{\{j\}}$, then there is $S_K \in S(P)$ with $j \notin K$, that is, $K \subset D \setminus \{j\}$, and $P(\alpha, \cdot) = 0$ for some $\alpha \in \mathbb{N}^K$. Therefore

$$0 = P(\alpha, \mathbf{0}_{D\setminus(K\cup\{j\})}, x_j) = \sum_{k=0}^{m_j} Q_k(\alpha, \mathbf{0}_{D\setminus(K\cup\{j\})})x_j^k \text{ for any } x_j \in \mathbb{R},$$

and hence $Q_k(\alpha, \mathbf{0}_{D\setminus(K\cup\{j\})}) = 0$ for any k.

(b) This directly follows from (a). □

Using the bounds for $S(P)$ from Proposition 7.8 we see that the class of polynomials with $S(P) \subset \{\mathbf{0}\}$ is much larger than might be expected.

Example 7.9.

(a) $S(P) \subset \{\mathbf{0}\}$ if P contains some power of any variable with coefficient independent of the other variables, that is, if for any $j \in D$ there is $1 \leq \beta_j \in \mathbb{N}$ and $c_j \neq 0$ such that

$$P(x) = \sum_{j=1}^{d} c_j x_j^{\beta_j} + \sum_{\alpha_j \neq \beta_j \text{ for all } j} c_\alpha x^\alpha.$$

(b) Specifically, $S(P) \subset \{\mathbf{0}\}$ in each of the following cases.

(i) P is partially hypoelliptic w.r.t. any variable (cf. [15, Example 11.2.8]), i.e.,

$$P(x) = \sum_{j=1}^{d} c_j x^{m_j} + \sum_{\alpha<(m_1,\dots,m_d)} c_\alpha x^\alpha, \tag{7.4}$$

where $c_j \neq 0$ for any j and $m_j \geq 1$ denotes the degree of P w.r.t. x_j.

(ii) P is hypoelliptic.

(iii) Any e_j is noncharacteristic for P.

(iv) P is of second order, that is, $P(x) = P_2(x) + \langle a, x\rangle + b$, and $a_j \neq 0$ if e_j is characteristic.

Proof.

(a) This follows from Proposition 7.8(b) since $Q_{\beta_j} \equiv c_j \neq 0$ in the expansion (7.3) w.r.t. x_j. Notice that the term $\sum_{k\neq j}^{d} c_k x_k^{\beta_k}$ is a part of Q_0 in this expansion and thus does not disturb Q_{β_j} since $\beta_j \geq 1$.

(b)(i) By [15, Example 11.2.8], P is partially hypoelliptic w.r.t. x_1 iff P satisfies (7.3) for $m_1 := \deg_{x_1}(P) \geq 1$, that is,

$$P(x) = c_1 x_1^{m_1} + \sum_{1\leq k<m_1} Q_k(x')x_1^k + Q_0(x'),$$

where $c_1 \neq 0$. Since P is also partially hypoelliptic w.r.t. x_2, the highest power of x_2 is contained in Q_0, and we may argue by induction to prove that P is partially hypoelliptic w.r.t. any variable iff (7.4) holds. The claim now follows from (7.4) and (a).

(ii) This is evident by (i) (cf. [15, Chapter XI]).

(iii) This follows from (i) with $m_j = m := \deg(P)$ for any j.

(iv) This follows from (i) with $m_j = 2$ if e_j is noncharacteristic and $m_j = 1$ otherwise. □

Notice that the assumption $\beta_j \geq 1$ cannot be omitted in Example 7.9(a) (take, e.g., $P(x) := x_2$; then $S(P) = S_{\{2\}}$). However β_j may be far from being the degree of P w.r.t. x_j (take, e.g., $P(x) := \sum_{j\leq d} x_j + \prod_{j\leq d} x_j^m$ for $m \geq 2$).

By Example 7.9(b)(iv) we have $S(P) \subset \{\mathbf{0}\}$ for any of the Euler analogon of a classical second order operator like $P(\theta) := \sum_{j\leq d} \theta_j^2$ ("Euler–Lapace operator"), $P(\theta) := \sum_{j\leq d-1} \theta_j^2 + i\theta_d$ ("Euler heat equation"), $P(\theta) := \sum_{j\leq d-1} \theta_j^2 + \theta_d$ ("Euler Schrödinger operator"), and $P(\theta) := \sum_{j\leq d-1} \theta_j^2 - \theta_d^2$ ("Euler wave equation").

After so many examples with $S(P) \subset \{\mathbf{0}\}$ the reader might wonder if there are natural examples with large $S(P)$ (apart from Remark 7.4). These should contain only mixed powers of x_j by the above discussion. An example with a minimal number of variables is the following.

Example 7.10. Let $P(x) := x_1x_2 + x_2x_3 + x_3x_1$. Then $S(P) = \mathbb{R}e_1 \cup \mathbb{R}e_2 \cup \mathbb{R}e_3$.

Proof. "$\supset$" This is evident since $P(e_jt) \equiv 0$ for any j.

"$\subset$" Clearly, $P(\alpha_j, x_{D\setminus\{j\}}) \neq 0$ for any j and any $\alpha_j \in \mathbb{N}$. □

We now want to obtain the necessary conditions from Theorem 6.4 assuming only that Ω is admissible for $P(\theta)$ and that $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective. Since the former conditions are concerned with functions on $\Omega_\sigma, \sigma \in \{\pm 1\}^d$, we will argue on these sets from now on. The following variant of Proposition 4.8 shows that we can solve the equation $\partial_j^k g = f$ on Ω_σ for admissible Ω taking care of $C^\infty_{I(P)}(\Omega_\sigma)$. For simplicity of notation we assume that $j = 1$ and then write $x := (x_1, x')$ for $x \in \mathbb{R}^d$. Similarly as before, let

$$\omega_{\sigma,K} := \Omega_\sigma \cap S_K \text{ for } \emptyset \neq K \subset D \text{ and } \omega_{\sigma,\emptyset} := \Omega_\sigma.$$

Proposition 7.11. *Let $\Omega \subset \mathbb{R}^d$ be open. Let $\omega_{\sigma,\{1\}} \neq \emptyset$, and let*

$$\Omega \cap L_{x,1} = \emptyset \text{ for any } x \in \partial_{\{1\}} \left(S_{\{1\}}(P) \cap \omega_{\{1\}}\right). \tag{7.5}$$

Let U be a neighbourhood of $\omega_{\sigma,\{1\}}$ in $\Omega_\sigma \cap ([0,\infty[\times \Omega_{D',\sigma'})$, $D' := \{2,\ldots,d\}$, such that $|x_1| < 1$ on U and such that

$$U \cap L_{x',1} \text{ is a (possibly void) interval for any } x' \in \mathbb{R}^{D'}. \tag{7.6}$$

Then for any $k \in \mathbb{N}$ and any $f \in C^\infty_{I(P(ke_1+\ \cdot\ ))}(\Omega_\sigma)$ with $\operatorname{supp}(f) \subset U$ there is $g \in C^\infty_{I(P)}(\Omega_\sigma)$ such that $\partial_1^k g = f$ on Ω_σ.

Proof. We modify the proof of "(c) ⇒ (b)" in Proposition 4.8. We may assume that $\sigma = \mathbf{1}$, that is, $\Omega_\sigma = \Omega_{\mathbf{1}} = \{x \in \Omega \mid x \geq 0\}$. Let $w := \Omega_{D'} = \{x' \in \mathbb{R}^{d-1} \mid (0,x') \in \Omega\}$, $w_{\mathbf{1}'} := \Omega_{D',\mathbf{1}'} = \{x' \in w \mid x' \geq 0\}$, and $S := \{x' \in w_{\mathbf{1}'} \mid (0,x') \in S_{\{1\}}(P)\}$.

(a) We construct a cutoff function Ψ as follows: Since S is closed in $w_{\mathbf{1}'}$ we can choose $y'_n \in S$ and

$$0 < \gamma_n < \min\{1, \operatorname{dist}(y'_n, \partial w_{\mathbf{1}'})/2\}$$

such that $\left(B^\infty_{\gamma_n}(y'_n)\right)_{n\in\mathbb{N}}$ is an open covering of S which is locally finite in $w_{\mathbf{1}'}$. Set $V := \bigcup_n B^\infty_{\gamma_n}(y'_n)$. Choose a C^∞-resolution of the identity on S consisting of $\varphi_n \in \mathscr{D}_{lc}(B^\infty_{\gamma_n}(y'_n))$ and set $\Phi := \sum \varphi_n$. For an interval I, $f \in C(I)$, and $c \in I$ let $I^k_c(f)(t), t \in I$, be the k-fold integral of f from c to t. For $f \in C^\infty(\Omega_{\mathbf{1}})$ with $\operatorname{supp}(f) \subset U$ we define

$$g(x) := {}_\tau I^k_0\big(f(\tau,x')\big)(x_1)\Phi(x') + {}_\tau I^k_1\big(f(\tau,x')\big)(x_1)(1-\Phi(x')) =: g_1(x) + g_2(x).$$

g is defined and smooth on $[0,\infty[\times w_{\mathbf{1}'} = [0,\infty[\times\Omega_{D',\mathbf{1}'}$ by (7.6). To define g_2 we integrate starting at 1, and hence

$$\mathrm{supp}(g_2) \subset \mathrm{supp}(f) \subset U \subset \Omega_{\mathbf{1}} \cap ([0,\infty[\times\Omega_{D',\mathbf{1}'})$$

since $|x_1| < 1$ on U. Thus g_2 can be trivially extended to $\Omega_{\mathbf{1}}$. We show that this also holds for g_1. If not, there are

$$(t_n, x_n') \in \mathrm{supp}(g_1) \subset [0,\infty[\times \mathrm{supp}(\Phi) \subset [0,\infty[\times\Omega_{D',\mathbf{1}'}$$

such that

$$(t_n, x_n') \to (t, x') \in \Omega_{\mathbf{1}} \setminus ([0,\infty[\times\Omega_{D',\mathbf{1}'}) \subset [0,\infty[\times\partial w.$$

Since $\mathrm{supp}(\Phi) \subset V$ we may choose y_n' such that $x_n' \in B^\infty_{\gamma_n}(y_n')$. Since $x_n' \to x' \in \partial w$ we have $\gamma_n \to 0$ by the choice of γ_n, and hence $y_n' \to x'$ and therefore $x' \in S$ since $y_n' \in S$. Since $x' \in \partial w$, this implies by (7.5) that $(t, x') \notin \Omega$, a contradiction. Clearly, $\partial_1^k g = f$ on $\Omega_{\mathbf{1}}$.

(b) Let $f \in C^\infty_{I(P(ke_1+\,\cdot\,))}(\Omega_{\mathbf{1}})$. We show that $g \in C^\infty_{I(P)}(\Omega_{\mathbf{1}})$. Let $x = (0_J, x_{D\setminus J}) \in \Omega_{\mathbf{1}} \cap S_J$ for $S_J \subset S(P)$, and let $\alpha \in \mathbb{N}^J$ such that $P(\alpha,\,\cdot\,) = 0$. We consider two cases.

(i) Let $1 \in J$. Then $S_J \subset S_{\{1\}}(P) \subset S$ and hence $x_1 = 0$ and $\Phi = 1$ near $\Omega_{\mathbf{1}} \cap S_J$, and therefore $g = g_1$ near x. If $\alpha_1 < k$, then

$$\begin{aligned}\partial^\alpha g(0_J, x_{D\setminus J}) &= \partial_1^{\alpha_1}\, {}_\tau I_0^k\big(\partial^{\alpha'}[f(\tau, 0_{J'}, x_{D\setminus J})\Phi(0_{J'}, x_{D\setminus J})]\big)(0)\\ &= \partial_1^{\alpha_1}\, {}_\tau I_0^k\big(\partial^{\alpha'} f(\tau, 0_{J'}, x_{D\setminus J})\big)(0) = 0\end{aligned}$$

since the k-fold integral starts from 0. If $\alpha_1 \geq k$, then

$$\partial^\alpha g(0_J, x_{D\setminus J}) = \partial_1^{\alpha_1-k}\partial^{\alpha'}[f(0_J, x_{D\setminus J})\Phi(0_{J'}, x_{D\setminus J})] = \partial^{\alpha-ke_1} f(0_J, x_{D\setminus J}) = 0$$

since $P(ke_1 + (\alpha - ke_1),\,\cdot\,) = 0$, and hence $\partial_1^{\alpha-ke_1} f(0_J, x_{D\setminus J}) = 0$ since $f \in C^\infty_{I(P(ke_1+\,\cdot\,))}(\Omega_{\mathbf{1}})$. Thus, $g \in C^\infty_{I(P)}(\Omega_{\mathbf{1}})$ in any case.

(ii) Let $1 \notin J$. For $x \in \mathrm{supp}(\Phi) \cap S_J$ we thus get

$$\begin{aligned}\partial^\alpha g_1(0_J, x_{D\setminus J}) &= {}_\tau I_0^k\left(\partial^\alpha[f(\tau, x_J, x_{D\setminus(J\cup\{1\})})\Phi(x_J, x_{D\setminus(J\cup\{1\})})]\right)(x_1)\big|_{x_J=0}\\ &= {}_\tau I_0^k\left(\partial^\alpha[f\Phi](0_J, (\tau, x_{D\setminus(J\cup\{1\})}))\right)(x_1) = 0\end{aligned}$$

since $f\Phi \in C^\infty_{I(P(ke_1+\,\cdot\,))}(\Omega_\sigma)$ by Proposition 3.5 since $\varphi_n \in \mathscr{D}_{lc}(B^\infty_{\gamma_n}(y_n'))$, and hence

$$\partial^\alpha[f\Phi](0_J, (\tau, x_{D\setminus(J\cup\{1\})})) = 0 \text{ for } \tau \geq 0$$

since $P(ke_1 + (\alpha,\,\cdot\,)) = P(\alpha, ke_1 +\,\cdot\,) = 0$ by assumption. This shows the claim for g_1. The claim for g_2 follows in the same way since $f(1-\Phi) \in C^\infty_{I(P(ke_1+\,\cdot\,))}(\Omega_{\mathbf{1}})$ also. □

Theorem 7.12. *Let $P(\theta)$ be an Euler operator, and let Ω be admissible for $P(\theta)$. If $P(\theta): C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective, then $P(k+\theta): C^\infty(\Omega_\sigma) \to C^\infty_{I(P(k+\,\cdot\,))}(\Omega_\sigma)$ is surjective for any $k \in \mathbb{N}^d$ and any $\sigma \in \{\pm 1\}^d$ such that $\Omega_\sigma \neq \emptyset$.*

Proof. Let $\sigma \in \{\pm 1\}^d$ such that $\Omega_\sigma \neq \emptyset$. Then $P(\theta) : C^\infty(\Omega_\sigma) \to C^\infty_{I(P)}(\Omega_\sigma)$ is surjective by Theorem 6.3.

(a) The operator $P(ke_1 + \theta) : C^\infty(\Omega_\sigma) \to C^\infty_{I(P(ke_1 + \,\cdot\,))}(\Omega_\sigma)$ is surjective for any $k \in \mathbb{N}$.

(i) For $f \in C^\infty_{I(P(ke_1+\,\cdot\,))}(\Omega_\sigma)$ let $f_1(x) := x_1^k f(x)$. Then $f_1 \in C^\infty_{I(P)}(\Omega_\sigma)$. Indeed, let $S_J \subset S(P)$ for $\emptyset \neq J \subset D$, and let $\alpha \in \mathbb{N}^J$ such that $P(\alpha, \,\cdot\,) = 0$. Let $1 \in J$. Then $\partial^\alpha f_1(0_J, x_{D\setminus J}) = 0$ if $\alpha_1 < k$. If $\alpha_1 \geq k$, then

$$\partial^\alpha f_1(0_J, x_{D\setminus J}) = k!\partial^{\alpha - ke_1} f_1(0_J, x_{D\setminus J}) = 0$$

since $f \in C^\infty_{I(P(ke_1+\,\cdot\,))}(\Omega_\sigma)$ and since $P(ke_1 + (\alpha - ke_1), \,\cdot\,) = 0$ by assumption. Let $1 \notin J$. Then

$$\partial^\alpha f_1(0_J, x_{D\setminus J}) = x_1^k \partial^\alpha f(0_J, x_{D\setminus J}) = 0$$

since $P(ke_1 + (\alpha, \,\cdot\,)) = P(\alpha, ke_1 + \,\cdot\,) = 0$ by assumption.

(ii) Since $f_1 \in C^\infty_{I(P)}(\Omega_\sigma)$ there is $g \in C^\infty(\Omega_\sigma)$ such that $P(\theta)g = f_1$. Let U be an open neighbourhood of $\omega_{\sigma,\{1\}}$ in $\Omega_\sigma \cap (\sigma_1[0,\infty[\times \Omega_{D',\sigma})$ satisfying (7.6), and choose $\phi \in C^\infty(\Omega_\sigma)$ such that $\phi = 1$ near $\Omega_\sigma \setminus U$ and $\phi = 0$ near $\omega_{\sigma,\{1\}}$. Set $g_1(x) := \phi(x)g(x)x_1^{-k}$. Then $g_1 \in C^\infty(\Omega_\sigma)$ and $f_2 := f - P(ke_1 + \theta)g_1 \in C^\infty_{I(P(ke_1+\,\cdot\,))}(\Omega_\sigma)$ by Lemma 3.3. If $\phi = 1$ near x we get by (5.8)

$$P(ke_1 + \theta)g_1(x) = x_1^{-k}P(\theta)(x_1^k g_1(x)) = x_1^{-k}P(\theta)g(x) = x_1^{-k} f_1(x) = f(x).$$

Therefore, $\operatorname{supp}(f_2) \subset U$.

(iii) By (ii) and admissibility (for $J = \{1\}$) we can use Proposition 7.11 and obtain $f_3 \in C^\infty_{I(P)}(\Omega_\sigma)$ such that $\partial_1^k f_3 = f_2$ on Ω_σ. Hence there is $g_3 \in C^\infty(\Omega_\sigma)$ such that $P(\theta)g_3 = f_3$. Since $L := x_1^k \partial_1^k$ is an Euler operator, L commutes with $P(\theta)$ and

$$\begin{aligned} x_1^k P(ke_1 + \theta)(\partial_1^k g_3)(x) &= P(\theta)(\xi_1^k \partial_1^k g_3)(x) \\ &= x_1^k \partial_1^k P(\theta) g_3(x) = x_1^k \partial_1^k f_3(x) = x_1^k f_2(x) \end{aligned}$$

on Ω_σ, and hence $P(ke_1 + \theta)(\partial_1^k g_3) = f_2$ on Ω_σ by continuity. This finishes the proof of (a).

(b) Ω is admissible for $P(n + \theta)$ for any $n \in \mathbb{N}^d$ since $S_J \subset S(P)$ if $S_J \subset S(P(n + \,\cdot\,))$. Therefore we can use (a) for $P(ke_1 + \theta)$ instead of $P(\theta)$ (and Proposition 7.11 for $j = 2$ instead of $j = 1$) and conclude that $P(k_1e_1 + k_2e_2 + \theta) : C^\infty(\Omega_\sigma) \to C^\infty_{I(P(k_1e_1+k_2e_2+\,\cdot\,))}(\Omega_\sigma)$ is surjective. Inductively this shows the theorem. □

So far we have used (7.2) only for $J = \{j\}, j \in D$. The general form is needed to guarantee inheritance of admissibility w.r.t. restrictions.

Lemma 7.13. *Let $\Omega \subset \mathbb{R}^d$ be admissible for the Euler operator $P(\theta)$. Then for any $\emptyset \neq K \subsetneq D$ and any $\alpha \in \mathbb{N}^K$ the set $\Omega_{D\setminus K}$ is admissible for the restricted operators $P(\alpha, \theta_{D\setminus K})$ if $\Omega_{D\setminus K} \neq \emptyset$ and $P(\alpha, \,\cdot\,) \neq 0$.*

Proof. We may assume that $K := \{1\}$ and then apply this case inductively. Set $W := \Omega_{D'} = \{x' \in \mathbb{R}^{D'} \mid (0, x') \in \Omega\}$ where $D' := D \setminus \{1\}$:

$$\begin{aligned} &W = \Omega_{D'} \text{ corresponds to } \Omega \cap S_{\{1\}} = \omega_{\{1\}} \text{ and} \\ &w_{J'} \text{ corresponds to } \omega_{J' \cup \{1\}} \text{ for } J' \subset D'. \end{aligned} \tag{7.7}$$

Fix $\alpha \in \mathbb{N}$ and $J' \subset D'$. If $S_{L'} \subset S_{J'}(P(\alpha, \cdot\,))$ for some $L' \subset D'$, then there is $\gamma \in \mathbb{N}^{L'}$ such that $P(\alpha, \gamma, \cdot\,) = 0$, and therefore $S_{\{1\}\cup L'} \subset S_{\{1\}\cup J'}(P)$. If $x' \in \partial_{L'}(S_{L'} \cap w_{L'})$, then $(0, x') \in \partial_{\{1\}\cup L'}\left(S_{\{1\}\cup L'} \cap \omega_{\{1\}\cup L'}\right)$, and by assumption $L_{(0,x'),j} \cap \Omega = \emptyset$ for $j \in J' \cup \{1\}$. Hence $L_{x',j} \cap w = \emptyset$ for $j \in J'$. □

We finally obtain the main result of this section.

Theorem 7.14 (Necessary condition for surjectivity)**.** *Let $P(\theta)$ be an Euler operator, and let Ω be admissible for $P(\theta)$. If $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective, then for any $k \in \mathbb{N}, k \geq l$,*

$$P(\alpha, \theta_{D\setminus J})(\mathcal{E}^{\mathbf{k}}(\Omega_{D\setminus J,\sigma})) \supset \mathcal{E}(\Omega_{D\setminus J,\sigma})$$

for any $J \subsetneq D$ and $\sigma \in \{\pm 1\}^{D\setminus J}$ with $\Omega_{D\setminus J,\sigma} \neq \emptyset$ and any $\alpha \in \mathbb{N}^J$ with $P(\alpha, \cdot\,) \neq 0$.

Proof. (a) By Theorem 7.12, $P(k+\theta) : C^\infty(\Omega_\sigma) \to C^\infty_{I(P(k+\,\cdot\,))}(\Omega_\sigma)$ is surjective for any $\sigma \in \{\pm 1\}^d$ such that $\Omega_\sigma \neq \emptyset$. By the proof of Theorem 6.4 we get $P(\theta)(\mathcal{E}^{\mathbf{k}}(\Omega_\sigma)) \supset \mathcal{E}(\Omega_\sigma)$.

(b) Let $\emptyset \neq J \subsetneq D$, and let $\alpha \in \mathbb{N}^J$ with $P(\alpha, \cdot\,) \neq 0$. By Theorem 5.1 we know that $P(\alpha, \theta_{D\setminus J}) : C^\infty(\Omega_{D\setminus J}) \to C^\infty_{I(P(\alpha,\,\cdot\,))}(\Omega_{D\setminus J})$ is surjective. Since $\Omega_{D\setminus J}$ is admissible for $P(\alpha, \theta_{D\setminus J})$ by Lemma 7.13, we get as in (a) that $P(\alpha, k+\theta_{D\setminus J}) : C^\infty(\Omega_{D\setminus J,\sigma}) \to C^\infty_{I(P(\alpha,k+\,\cdot\,))}(\Omega_{D\setminus J,\sigma})$ is surjective for any $k \in \mathbb{N}^{D\setminus J}$ and any $\sigma \in \{\pm 1\}^{D\setminus J}$ with $\Omega_{D\setminus J,\sigma} \neq \emptyset$. The proof is completed as in (a). □

8. The Mellin transform

So far we have reduced the surjectivity problem for all shifted Hadamard operators

$$H_{T,k} : C^\infty(\Omega) \to C^\infty_{I(H_{T,k})}(\Omega), k \in \mathbb{N}^d,$$

for $H = H_T$ with $\mathrm{supp}(T) \subset [0,\infty[^d$ to the surjectivity problem for H (and all its restrictions H_α) on the spaces $\mathcal{E}(\Omega_{D\setminus J,\sigma})$ of smooth functions on $\Omega_{D\setminus J}$ supported in the closed canonical quadrants (see Theorems 6.4 and 6.5). For Hadamard operators H on $C^\infty(\mathbb{R}^d)$ this means that we have to check if H and the restrictions H_α are surjective on $\mathcal{E}(\sigma[0,\infty[^{D\setminus J})$ for $J \subset D$ and $\sigma \in \{\pm 1\}^{D\setminus J}$. To solve the latter question in Section 9 for Euler differential operators we will use a suitably defined Mellin transform $\mathscr{M}$ on $\mathcal{E}([0,\infty[^d)'$ which is developed in the present section. Recall that according to our previous notation

$$\mathcal{E}([0,\infty[^d) := \{f \in C^\infty(\mathbb{R}^d) \mid \mathrm{supp}(f) \subset [0,\infty[^d\}.$$

We start, however, with the Mellin transform $\mathscr{M}$ defined on

$$\mathscr{D}([0,\infty[^d) := \{f \in C^\infty(\mathbb{R}^d) \mid \mathrm{supp}(f) \text{ is compact in } [0,\infty[^d\}$$

by

$$\mathscr{M}(g)(z) := \int_{[0,\infty[^d} g(t) t^{z-\mathbf{1}}\, dt \text{ for } z \in \mathbb{C}^d. \tag{8.1}$$

For a precise estimate of $\mathscr{M}(g)(z)$ we need the following logarithmic version of the classical support functional used in Paley–Wiener type theorems for the Fourier transformation: For a compact set $K \subset [0,\infty[^d$ let

$$h_K(x) := H_{\mathrm{Log}(K_{>0})}(x) := \sup_{\mathbf{0}<t\in K} \langle \mathrm{Log}(t), x\rangle \text{ for } x \in \mathbb{R}^d,$$

where $\mathrm{Log}(K_{>0}) := \{\mathrm{Log}(t) := (\log(t_1),\dots,\log(t_d)) \mid \mathbf{0} < t \in K\}$.

Recall that $\theta_j := x_j\partial_j$ and $\theta^\alpha := \theta_1^{\alpha_1}\dots\theta_d^{\alpha_d}, \alpha \in \mathbb{N}^d$, are the standard Euler differentials.

Proposition 8.1. *Let $g \in \mathscr{D}([0,\infty[^d)$, and let $\mathrm{supp}(g) \subset K$ where $K \subset [0,\infty[^d$ is compact.*

(a) *$\mathscr{M}(g)$ is an entire function.*

(b) *For any $z \in \mathbb{C}^d$ and any $\alpha \in \mathbb{N}^d$ we have*

$$\mathscr{M}(\theta^\alpha g)(z) = (-z)^\alpha \mathscr{M}(g)(z). \tag{8.2}$$

(c) *For any $k \in \mathbb{N}$ we have*

$$|\mathscr{M}(g)(z)| \le C_k(1+|z|)^{-k} \max_{|\beta|_\infty \le k} \|g^{(\beta)}\|_\infty e^{h_K(\mathrm{Re}_+(z))} \text{ if } \mathrm{Re}\,(z) \ge -\mathbf{k}+\mathbf{1},$$

where $\mathrm{Re}_+(z) := (\max\{\mathrm{Re}\,(z_j),0\})_{j\le d}$.

Proof.

(a) $\mathscr{M}(g)(z)$ is defined for any $z \in \mathbb{C}^d$ since for any $k \in \mathbb{N}$ such that $\mathrm{Re}\,(z)+\mathbf{k} \ge \mathbf{1}$,

$$\begin{aligned} \Big|\int_{[0,\infty[^d} g(t)t^{z-\mathbf{1}}\,dt\Big| &\le \int_K |g(t)t^{-\mathbf{k}}|t^{\mathrm{Re}\,(z)+\mathbf{k}-\mathbf{1}}\,dt \\ &\le C_k\|g(t)t^{-\mathbf{k}}\|_\infty \le C_k\|g^{(\mathbf{k})}\|_\infty/\mathbf{k}!. \end{aligned} \tag{8.3}$$

The last inequality follows from Taylor's theorem applied in each variable separately since we have for any $\beta \in \mathbb{N}^d$

$$|g(t)| = |t^\beta g^{(\beta)}(\tau t)|/\beta! \le |t^\beta|\|g^{(\beta)}\|_\infty/\beta! \text{ for any } t > \mathbf{0} \tag{8.4}$$

for $g \in \mathscr{D}([0,\infty[^d)$, where $\mathbf{0} \le \tau \le \mathbf{1}$ is chosen suitably.

$\mathscr{M}(g)$ is holomorphic with $\partial_{z_j}\mathscr{M}(g)(z) = \mathscr{M}\big(\ln(t_j)g(t)\big)(z)$ by the theorem of dominated convergence and (8.4) since for any $\zeta \in \mathbb{C}^d$ and any $t > \mathbf{0}$,

$$\begin{aligned} |(t^{\zeta+he_j}-t^\zeta)/h - \ln(t_j)t^\zeta| &= |t^\zeta\ln(t_j)|\Big|\int_0^1 (t_j^{\tau h}-1)\,d\tau\Big| \\ &= t^{\mathrm{Re}\,(\zeta)}(\ln(t_j))^2\left|\int_0^1 \tau\int_0^1 t_j^{\tau\lambda h}\,d\lambda\,d\tau\right||h| \\ &\le t^{\mathrm{Re}\,(\zeta)}(\ln(t_j))^2\Big(t_j^{-|\mathrm{Re}\,(h)|}+t_j^{|\mathrm{Re}\,(h)|}\Big)|h|. \end{aligned} \tag{8.5}$$

(b) By partial integration we have

$$\mathscr{M}(\theta_j g)(z) = \int_{[0,\infty[^d} \partial_j g(t)t_jt^{z-\mathbf{1}}\,dt = -z_j\int_{[0,\infty[^d} g(t)t^{z-\mathbf{1}}\,dt = -z_j\mathscr{M}(g)(z)$$

since $g \in \mathscr{D}([0,\infty[^d)$.

(c) Reasoning by induction on $|\beta|$ it is easily seen that for any $\beta \in \mathbb{N}^d$

$$\frac{\theta^\beta g(x)}{\beta!} = \sum_{\alpha\le\beta} \frac{c_{\beta,\alpha}}{\alpha!}x^\alpha g^{(\alpha)}(x) \tag{8.6}$$

for certain $c_{\beta,\alpha}$ (independent of g). Using (8.4) we get for $x \geq \mathbf{0}$ with $C_\beta := \sup_{\alpha\leq\beta} |c_{\beta,\alpha}|$ that

$$\begin{aligned}|\theta^\beta g(x)| &\leq C_\beta \sum_{\alpha\leq\beta} \frac{\beta!}{\alpha!} |x^\alpha g^{(\alpha)}(x)| \\ &\leq C_\beta \sum_{\alpha\leq\beta} \frac{\beta!}{\alpha!(\beta-\alpha)!} |x^\beta| \sup_{\mathbf{0}\leq y\leq x} |g^{(\beta)}(y)| \leq 2^{|\beta|} C_\beta |x^\beta| \|g^{(\beta)}\|_\infty\end{aligned} \tag{8.7}$$

since $g \in \mathscr{D}([0,\infty[^d)$. We now group the variables such that $\widetilde{z}$ satisfies $|\widetilde{z}_j| \geq 1$ and z' satisfies $|z'_j| \leq 1$. Using (8.7) for variables $\widetilde{z}$ and (8.4) for the variables z' we get

$$\left| \frac{(\theta^{\widetilde{\mathbf{k}}} g)(t)}{t^{\mathbf{k}}} \right| \leq C_k \max_{|\beta|_\infty \leq k} \|g^{(\beta)}\|_\infty.$$

Thus using (b) we get for $\mathrm{Re}\,(z) \geq -\mathbf{k}+\mathbf{1}$

$$\begin{aligned}\prod_{j\leq d} (1+|z_j|)^k |\mathscr{M}(g)(z)| &\leq 2^{kd} |\widetilde{z}^{\widetilde{\mathbf{k}}} \mathscr{M}(g)(z)| \\ &= 2^{kd} \left| \int_{[0,\infty[^d} \frac{(\theta^{\widetilde{\mathbf{k}}} g)(t)}{t^{\mathbf{k}}} t^{z+\mathbf{k}-\mathbf{1}}\, dt \right| \\ &\leq C_k \max_{|\beta|_\infty\leq k} \|g^{(\beta)}\|_\infty \int_K t^{\mathrm{Re}\,(z)+\mathbf{k}-\mathbf{1}}\, dt \\ &\leq \widetilde{C}_k \max_{|\beta|_\infty\leq k} \|g^{(\beta)}\|_\infty e^{h_K(\mathrm{Re}\,(z)+\mathbf{k}-\mathbf{1})} \\ &\leq \widehat{C}_k \max_{|\beta|_\infty\leq k} \|g^{(\beta)}\|_\infty e^{h_K(\mathrm{Re}_+(z))}\end{aligned} \tag{8.8}$$

since

$$\begin{aligned}h_K(\mathrm{Re}\,(z)+\mathbf{k}-\mathbf{1}) &= \sup_{\mathbf{0}\leq t\in \ln(K)} \langle t, \mathrm{Re}\,(z)+\mathbf{k}-\mathbf{1}\rangle \\ &\leq \sup_{\mathbf{0}\leq t\in \ln(K)} \langle t, \mathrm{Re}_+(z)\rangle + C = h_K(\mathrm{Re}_+(z)) + C.\end{aligned} \tag{8.9}$$

The inequality (8.9) holds since $\mathrm{Re}\,(z)+\mathbf{k}-\mathbf{1} \geq \mathbf{0}$. □

For a compact set $K \subset [0,\infty[^d$ let

$$\mathcal{E}(K)'_b := \{T \in \mathcal{E}([0,\infty[^d)' \mid \mathrm{supp}_+(T) \subset K\}$$

with the topology induced by $\mathcal{E}([0,\infty[^d)'_b$. Here $\mathrm{supp}_+(T)$ denotes the support in the sense of $\mathcal{E}([0,\infty[)'$. Notice that for $T \in \mathcal{E}([0,\infty[^d)'$ and $K := \mathrm{supp}_+(T)$ we have

$$K = \overline{K_{>0}}. \tag{8.10}$$

Indeed, if $x \notin \overline{K_{>0}}$, then there is a neighborhood U_x of x such that $U_x \cap K_{>0} = \emptyset$, i.e., $U_x \cap K \subset Z_1$. Hence, if $f \in \mathcal{E}([0,\infty[^d)$ and $\mathrm{supp}(f) \subset U_x$, then f is flat on K by definition and thus $T(f) = 0$; hence $x \notin K$.

Let $T \in \mathcal{E}(K)'$ where K is regular in the sense that there is $C > 0$ such that any two points x, y in the same component of K can be joined by a rectifiable curve of length $\leq C|x-y|$. Let $F := \{f \in C^\infty(\mathbb{R}^d) \mid \mathrm{supp}(f) \subset \mathbb{R}^d \setminus [0,\infty[^d\}$. Then T is extended to the topological direct sum $E := \mathcal{E}([0,\infty[^d) \bigoplus F \subset C^\infty(\mathbb{R}^d)$ by setting $T(f) := 0$ if $f \in F$. T is continuous, and hence T can be continuously extended to

$T' \in C^\infty(\mathbb{R}^d)'$ and $\operatorname{supp}(T') \subset K \cup S$, where $S \subset Z_1$ is a regular connected compact set. Hence $K \cup S$ is regular and there is $k \in \mathbb{N}$ such that (see [15, 2.3.10])

$$|T'(g)| \leq C \sup_{t \in K \cup S, \beta \leq \mathbf{k}} |g^{(\beta)}(t)| \text{ for } g \in C^\infty(\mathbb{R}^d),$$

and therefore

$$(8.11) \quad \begin{aligned} |T(g)| = |T'(g)| &\leq C \sup_{t \in K \cup S, \beta \leq \mathbf{k}} |g^{(\beta)}(t)| \\ &= C \sup_{t \in K, \beta \leq \mathbf{k}} |g^{(\beta)}(t)| \text{ for } g \in \mathcal{E}([0,\infty[^d). \end{aligned}$$

Obviously, for $f \in \mathcal{E}^{\mathbf{k}}([0,\infty[^d)$ we have $f_n := f(\,\cdot - \mathbf{1}/n) \to f$ in $\mathcal{E}^{\mathbf{k}}([0,\infty[^d)$, and $\mathcal{E}([0,\infty[^d)$ is therefore dense in $\mathcal{E}^{\mathbf{k}}([0,\infty[^d)$ by convolution. Thus T can be uniquely extended to $\widetilde{T} \in \mathcal{E}^{\mathbf{k}}([0,\infty[^d)'$ and

$$(8.12) \quad |\widetilde{T}(g)| \leq C \sup_{t \in K, \beta \leq \mathbf{k}} |g^{(\beta)}(t)| \text{ for } g \in \mathcal{E}^{\mathbf{k}}([0,\infty[^d).$$

Since the functions

$$g_z(t) := t^z, t \geq \mathbf{0},$$

are naturally in $\mathcal{E}^{\mathbf{k}}([0,\infty[^d)$ for $\operatorname{Re}(z) > \mathbf{k}$, the Mellin transform

$$\mathscr{M}(T)(z) := \langle \widetilde{T}, g_{z-\mathbf{1}} \rangle = \langle {}_t\widetilde{T}, t^{z-\mathbf{1}} \rangle, \operatorname{Re}(z) > \mathbf{k} + \mathbf{1},$$

is uniquely defined, and using (8.9)

$$(8.13) \quad \begin{aligned} |\mathscr{M}(T)(z)| &\leq C \sup_{t \in K, \beta \leq \mathbf{k}} |g^{(\beta)}_{z-\mathbf{1}}(t)| \leq C_1 \sup_{t \in K, \beta \leq \mathbf{k}} |c_\beta(z) t^{z-\beta-\mathbf{1}}| \\ &\leq C_2 |z^{\mathbf{k}}| e^{h_K(\operatorname{Re}(z)-\mathbf{1})} \leq C_3 |z^{\mathbf{k}}| e^{h_K(\operatorname{Re}(z))} \text{ if } \operatorname{Re}(z) > \mathbf{k} + \mathbf{1}, \end{aligned}$$

where $c_\beta(z) := \prod_{j=1}^d \prod_{\ell=1}^{\beta_j} (z_j - \ell) = z^\beta \prod_{j=1}^d \prod_{\ell=1}^{\beta_j} (1 - \ell/z_j)$.

We will show that for regular compact sets K the transform $\mathscr{M}$ maps $\mathcal{E}(K)'$ into the weighted space $\mathcal{H}_{\mathcal{M}}(K)$ of holomorphic germs near $(\infty, \dots, \infty)$ defined as follows: for $k \in \mathbb{N}$ let $\Omega_k := \{z \in \mathbb{C}^d \mid \operatorname{Re}(z) > \mathbf{k}\}$ and

$$\mathcal{H}_{K,k}(\Omega_k) := \{f \in \mathcal{H}(\Omega_k) \mid \|f\|_k := \sup_{z \in \Omega_k} |f(z)| |z^{-\mathbf{k}}| e^{-h_K(\operatorname{Re}(z))} < \infty\}$$

and

$$\mathcal{H}_{\mathcal{M}}(K) := \operatorname{ind}_{k \to \infty} \mathcal{H}_{K,k}(\Omega_k).$$

For $\mathcal{E}(K)'_b$ the following theorem is the Mellin analogue of the Paley–Wiener theorem as well as the Fourier inversion theorem for distributions with compact support.

Theorem 8.2. *Let $K \subset [0,\infty[^d$ be compact and m-convex with $K = \overline{K_{>0}}$, and let $[0,\delta]^d \subset K$ for some $\delta > 0$. Then the mapping*

$$\mathscr{M} : \mathcal{E}(K)'_b \to \mathcal{H}_{\mathcal{M}}(K)$$

is a topological isomorphism. For $f \in \mathcal{H}_{K,k}(\Omega_k)$ the inverse mapping $\mathscr{M}^{-1}(f)$ is given by

$$(8.14) \quad \langle \mathscr{M}^{-1}(f), g \rangle := \frac{1}{(2\pi i)^d} \int_{\mathbf{c}+i\mathbb{R}^d} f(\tau + \mathbf{1}) \mathscr{M}(g)(-\tau)\, d\tau \text{ for } g \in \mathscr{D}([0,\infty[^d),$$

where $c > k + 2$.

Proof. (a) If $T \in \mathcal{E}(K)'$ is of order $\mathbf{k}$, then $\mathscr{M}(T)(\zeta)$ is holomorphic for $\mathrm{Re}\,(\zeta) > \mathbf{k}+\mathbf{1}$ since we have for $\beta \le \mathbf{k}$ using (8.13) and (8.5)

$$(g^{(\beta)}_{\zeta-\mathbf{1}+he_j}(t) - g^{(\beta)}_{\zeta-\mathbf{1}}(t))/h - \partial_{\zeta_j} c_\beta(\zeta) t^{\zeta-\mathbf{1}-\beta} - \ln(t_j) g^{(\beta)}_{\zeta-\mathbf{1}}(t)$$
$$= (c_\beta(\zeta+he_j)t^{\zeta+he_j-\mathbf{1}-\beta} - c_\beta(\zeta)t^{\zeta-\mathbf{1}-\beta})/h - \partial_{\zeta_j} c_\beta(\zeta)t^{\zeta-\mathbf{1}-\beta} - c_\beta(\zeta)\ln(t_j)t^{\zeta-\mathbf{1}-\beta}$$
$$= t^{\zeta-\mathbf{1}-\beta}\Big[\frac{1}{h}(c_\beta(\zeta+he_j)-c_\beta(\zeta))t_j^h - \partial_{\zeta_j}c_\beta(\zeta) + c_\beta(\zeta)\big(\frac{1}{h}(t_j^h-1)-\ln(t_j)\big)\Big] \to 0$$

as $h \to 0$ uniformly on compact sets $J \subset [0,\infty[^d$ since $\mathrm{Re}\,(\zeta) > \beta + \mathbf{1}$.

(b) We show that K is regular. Indeed, let $x, y \in K_{>0}$. Then

$$\gamma(\tau) := (\gamma_1(\tau), \dots, \gamma_d(\tau)) := x^\tau y^{1-\tau} \in K_{>0} \text{ for } \tau \in [0,1]$$

by (2.2) and the arclength $L_0^1(\gamma)$ of γ can be estimated as follows:

$$L_0^1(\gamma) = \int_0^1 \|\gamma'(t)\|_2 dt \le \sqrt{d}\int_0^1 \|\gamma'(t)\|_\infty dt \le \sqrt{d}\int_0^1 \|\gamma'(t)\|_1 dt$$
$$\le d\sqrt{d}\max_{j\le d}\int_0^1 |\gamma_j'(t)|dt = d\sqrt{d}\|x-y\|_\infty \le d\sqrt{d}\|x-y\|_2$$

since γ_j is a parametrization of the interval $\mathrm{conv}\{x_j, y_j\}$. Since $K = \overline{K_{>0}}$ this shows that K is regular. Hence, $\mathscr{M} : \mathcal{E}(K)'_b \to \mathcal{H}_\mathcal{M}(K)$ is defined and linear by (8.13) and (a).

The rest of the proof of Theorem 8.2 will be obtained in a sequence of propositions. The injectivity of $\mathscr{M}$ on $\mathcal{E}([0,\infty[^d)'$ is a direct consequence of the following easy lemma.

Lemma 8.3. *The following are equivalent for $T \in C^\infty(\mathbb{R}^d)'$:*

(a) *There is $k \in \mathbb{N}$ such that $\langle_y T, y^\beta\rangle = 0$ if $\beta > \mathbf{k}$.*

(b) $\mathrm{supp}(T) \subset Z_1 := \{x \in \mathbb{R}^d \mid \exists j \le d : x_j = 0\}$.

(c) *There is $k \in \mathbb{N}$ such that*

$$T = \sum_{j\le d}\sum_{\ell \le k} T_{j,\ell}(x_{\widehat{j}}) \otimes \delta^{(\ell)}(x_j) \text{ where } x_{\widehat{j}} := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$$

for some $T_{j,\ell} \in C^\infty(\mathbb{R}^{d-1})'$.

Proof. *(c)* $\Rightarrow$ *(b)*: This is trivial.

(b) $\Rightarrow$ *(a)*: By assumption there is k such that $\mathrm{supp}(T) \subset K := \{x \in \mathbb{R}^d \mid x \in Z_1, |x|_\infty \le k\}$ and such that T is of order $\mathbf{k}$. By [15, 2.3.11] we thus get

$$|T(g)| \le C \sup_{x\in K, \beta \le \mathbf{k}} |g^{(\beta)}(x)| \text{ for any } g \in C^\infty(\mathbb{R}^d).$$

This implies that

$$|\langle_y T, y^\alpha\rangle| \le C_1 \sup_{y \in K, \beta\le \mathbf{k}} |y^{\alpha-\beta}| = 0$$

if $\alpha > \mathbf{k}$.

(a) $\Rightarrow$ *(c)*: Let $\mathscr{N} := \{\alpha \in \mathbb{N}^d \mid \alpha_j \le k \text{ for some } j \le d\}$, $\mathscr{N}_1 := \{\alpha \in \mathbb{N}^d \mid \alpha_1 \le k\}$, and

$$\mathscr{N}_j := \{\alpha \in \mathbb{N}^d \mid \alpha_j \le k\} \setminus \bigcup_{\ell<j} \mathscr{N}_\ell.$$

Obviously, this defines a disjoint decomposition of $\mathscr{N}$ and hence we get by Taylor expansion of the Fourier transform $\widehat{T}$ using the assumption

$$\widehat{T}(z) = \sum_{\alpha\in\mathscr{N}} \langle {}_yT, y^\alpha\rangle \frac{(-iz)^\alpha}{\alpha!} = \sum_{j\le d}\sum_{\alpha\in\mathscr{N}_j} \langle {}_yT, y^\alpha\rangle \frac{(-iz)^\alpha}{\alpha!} = \sum_{j\le d}\sum_{\ell\le k} f_{j,\ell}(z_{\widehat{j}})z_j^\ell,$$

where $z_{\widehat{j}} := (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_d)$. Since for $j < n \le d$ we have $\alpha_j > k$ if $\alpha \in \mathscr{N}_n$ we get for $\ell \le k$

$$f_{1,\ell}(z_{\widehat{1}}) = (\partial_{z_1}^\ell \widehat{T})(0, z_2, \dots, z_d)\ell!,$$

and hence by the Paley–Wiener theorem $f_{1,\ell} = \widehat{T}_{1,\ell}$ for some distribution $T_{1,\ell}$ with compact support since the necessary estimates are satisfied by $(\partial_{z_1}^\ell \widehat{T})(0, z_2, \dots, z_d)$. For $j \ge 2$ we can write

$$f_{j,\ell}(z_{\widehat{j}}) = \partial_{z_j}^\ell \Big(\widehat{T}(z_1, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_d) - \sum_{n<j}\sum_{\ell\le k} \widehat{T}_{j,\ell}(z_{\widehat{n}})z_n^\ell\Big)\ell!$$

and reason by induction. □

Proposition 8.4. *The mapping $\mathscr{M} : \mathcal{E}(K)' \to \mathcal{H}_{\mathcal{M}}(K)$ is injective.*

Proof. If $\mathscr{M}(T) = 0$ in $\mathcal{H}_{\mathcal{M}}(K)$, then there is k such that $\mathscr{M}(T)(z) = 0$ if $\mathrm{Re}\,(z) > \mathbf{k}$ and hence $\langle {}_t\widetilde{T}, t^\beta\rangle = 0$ if $\beta > \mathbf{k} + \mathbf{1}$ by the definition of $\mathscr{M}(T)$. Hence $\widetilde{T}|_{C^\infty(\mathbb{R}^d)}$ can be written as in Lemma 8.3(c), and thus $T = \widetilde{T}\Big|_{\mathcal{E}([0,\infty[^d)} = 0$. □

To show (8.14) we first notice that the functions in $\mathcal{H}_{\mathcal{M}}(K)$ are uniquely determined by their values on $\mathbf{j} + \mathbb{N}^d$.

Lemma 8.5. *If $f \in \mathcal{H}_{K,k}(\Omega_k)$ vanishes on $\mathbf{j} + \mathbb{N}^d$ for some $k \le j \in \mathbb{N}$, then $f = 0$ on Ω_k.*

Proof. By the identity theorem we can assume that $j = k$ and that

$$|f(z)| \le C_1|z^{\mathbf{k}}|e^{\langle \mathbf{k}, \mathrm{Re}\,(z)\rangle} \text{ on } \Omega_k \tag{8.15}$$

since $h_K(\mathrm{Re}\,(z)) \le \langle \mathbf{m}, \mathrm{Re}\,(z)\rangle$ for some $m \in \mathbb{N}$. We also assume that $d = 1$ (otherwise the argument below is applied to each variable separately). By assumption,

$$g_n(z) := \frac{f(z)e^{nz}}{z^k \sin(\pi z)}$$

is a holomorphic function on Ω_k. To estimate g_n on the rays $z := \gamma_n(t) := k + 1/2 + (1 \pm in)t, t \ge 0$, we notice that

$$|\sin(x + iy)|^2 = \sinh^2(y) + \sin^2(x) \text{ for } x, y \in \mathbb{R}, \tag{8.16}$$

and hence there is $C_0 > 0$ independent of n such that for $z = \gamma_n(t)$

$$|\sin(\pi z)| \ge e^{n\pi t}/C_0. \tag{8.17}$$

Indeed, for $t \ge 1/\pi$ we get by (8.16) for $n \ge 1$

$$|\sin(\pi z)| \ge |\sinh(n\pi t)| \ge e^{n\pi t}/4,$$

while for $0 \le t \le 1/\pi$ we have

$$|\sin(\pi z)| \ge |\sin((k + 1/2 + t)\pi)| = |\cos(\pi t)| \ge \cos(1).$$

By (8.17) and (8.15) we get for $n \geq k/(\pi - 1)$

$$|g_n(z)| \leq C_0 C_1 e^{(k+n)(k+1/2)} e^{(k+n-\pi n)t} \leq C_0 C_1 e^{(k+n)(k+1/2)} \tag{8.18}$$

for $z := \gamma_n(t), t \geq 0$. These rays define a cone Γ_n. Notice that C_1 is also independent of n. Since g_n is of finite exponential type inside Γ_n we get the estimate (8.18) on $\overline{\Gamma}_n$ by the Phragmen–Lindelöf Theorem [17, Chap. I, Theorem 21]. This implies that for $n \geq k$ and $x \geq 4k + 2$

$$\begin{aligned} |f(x)/x^k| = |g_n(x)\sin(\pi x)e^{-nx}| &\leq C_0 C_1 e^{(k+n)(k+1/2)-nx} \\ &\leq C_0 C_1 e^{n(2k+1)-nx} \leq C_0 C_1 e^{-nx/2}. \end{aligned}$$

Hence $f(x) = 0$ for these x, and $f = 0$ on Ω_k by the identity theorem. □

The proof of Theorem 8.2 is now completed by the following proposition.

Proposition 8.6. *Let $K \subset [0, \infty[^d$ be m-convex, and let $[0, \delta]^d \subset K$ for some $\delta > 0$. For $f \in \mathcal{H}_{K,k}(\Omega_k)$ let*

$$\langle R(f), g\rangle := \frac{1}{(2\pi i)^d} \int_{\mathbf{c}+i\mathbb{R}^d} f(\tau + \mathbf{1}) \mathscr{M}(g)(-\tau)\, d\tau \text{ for } g \in \mathscr{D}([0, \infty[^d), \tag{8.19}$$

where $c \geq k + 2$. Then $R(f) \in \mathcal{E}(K)'$ and the mapping $R : \mathcal{H}_{\mathscr{M}}(K) \to \mathcal{E}(K)'_b$ is a topological isomorphism. Moreover, $R = \mathscr{M}^{-1}$, and thus $\mathscr{M} : \mathcal{E}(K)'_b \to \mathcal{H}_{\mathcal{M}}(K)$ is a topological isomorphism.

Proof. Let $f \in \mathcal{H}_{K,k}(\Omega_k)$ and $g \in \mathscr{D}(J)$ for some $J \Subset [0, \infty[^d$, and let $c \geq k + 2$ be fixed. First notice that

$$\langle R(f), g\rangle = \frac{1}{(2\pi i)^d} \int_{\mathbf{c}+i\mathbb{R}^d} \frac{f(\tau + \mathbf{1})}{\tau^{\mathbf{k+2}}} \mathscr{M}(\theta^{\mathbf{k+2}} g)(-\tau)\, d\tau \tag{8.20}$$

by Proposition 8.1(b).

(i) By (8.6) and (8.4) (used twice) we have

$$\begin{aligned} |\theta^{\mathbf{k+2}} g(t)| &\leq C_1 \sup_{\alpha \leq \mathbf{k+2}} |t^\alpha g^{(\alpha)}(t)| \\ &\leq C_1 |t^{\mathbf{a}}| \sup_{\alpha \leq \mathbf{k+2}} \|g^{(\alpha+\mathbf{a})}\|_\infty \leq C_2 |t^{\mathbf{a}}| \|g^{(\mathbf{k+2+a})}\|_\infty \quad \text{ if } t \in J, \end{aligned}$$

and hence for $a := \min\{j \in \mathbb{N} \mid j \geq c + 1\}$ and $\mathrm{Re}\,(\tau) = c$

$$\begin{aligned} |\mathscr{M}(\theta^{\mathbf{k+2}} g)(-\tau)| &= \left| \int_{[0,\infty[^d} \theta^{\mathbf{k+2}} g(t) t^{-\tau-\mathbf{1}}\, dt \right| \\ &\leq C_2 \|g^{(\mathbf{k+2+a})}\|_\infty \int_J t^{-c-\mathbf{1}+\mathbf{a}}\, dt \leq C_3 \|g^{(\mathbf{k+2+a})}\|_\infty. \end{aligned}$$

Since $f \in \mathcal{H}_{K,k}(\Omega_k)$ we thus get by (8.20)

$$|\langle R(f), g\rangle| \leq C_4 \|f\|_k \|g^{(\mathbf{k+2+a})}\|_\infty \int_{\mathbf{c}+i\mathbb{R}^d} \left| \frac{(\tau + \mathbf{1})^{\mathbf{k}}}{\tau^{\mathbf{k+2}}} \right| d\tau\; e^{h_K(\mathbf{c+1})} \tag{8.21}$$

and $R(f) \in \mathscr{D}(J)'$, since $c > 0$ and hence $h_K(\mathbf{c} + \mathbf{1}) < \infty$.

(ii) To show that $\mathrm{supp}_+(R(f)) \subset K$ let $\mathbf{0} < x_0 \notin K$ and set $y_0 := \mathrm{Log}(x_0)$. Since K is m-convex (and hence $\mathrm{Log}(K_{>0})$ is convex) we can choose $\varepsilon > 0$ and $b \in \mathbb{R}^d$ (by the geometric version of the theorem of Hahn–Banach) such that

$$h_K(b) =: a_0 < a_1 := \inf_{\tau \in B_\varepsilon(y_0)} \langle b, \tau\rangle = - \sup_{\tau \in B_\varepsilon(y_0)} \langle -b, \tau\rangle =: -H_{B_\varepsilon(y_0)}(-b). \tag{8.22}$$

Notice that $b > \mathbf{0}$ since $]-\infty, \log\delta]^d \subset \operatorname{Log}(K_{>0})$ by assumption. Take $g \in \mathscr{D}\big(\operatorname{Exp}(B_\varepsilon(y_0))\big)$. Then $G := (\theta^{\mathbf{k+2}} g)\circ \operatorname{Exp} \in \mathscr{D}(B_\varepsilon(y_0))$ and $\mathscr{M}(\theta^{\mathbf{k+2}}g)(-\tau) = \widehat{G}(-i\tau)$. Since $b > \mathbf{0}$ we may thus use the Paley–Wiener theorem for G and Cauchy's theorem to improve (8.21) for any $\eta > 0$ as follows:

$$|\langle R(f), g\rangle| \le C_2 \int_{\mathbf{c}+\eta\mathbf{b}+i\mathbb{R}^d} \left|\frac{(\tau+\mathbf{1})^{\mathbf{k}}}{\tau^{\mathbf{k+2}}}\right| d\tau\; e^{h_K(\mathbf{c}+\eta\mathbf{b}+\mathbf{1}) + H_{B_\varepsilon(y_0)}(-\mathbf{c}-\eta\mathbf{b})}. \tag{8.23}$$

Notice that

$$\begin{aligned} h_K(\mathbf{c}+\eta\mathbf{b}+\mathbf{1}) + H_{B_\varepsilon(y_0)}(-\mathbf{c}-\eta\mathbf{b}) &\le C_3 + h_K(\eta\mathbf{b}) + H_{B_\varepsilon(y_0)}(-\eta\mathbf{b}) \\ &= C_3 + \eta(h_K(\mathbf{b}) + H_{B_\varepsilon(y_0)}(-\mathbf{b})) \\ &= C_3\eta(a_0 - a_1). \end{aligned}$$

Hence (8.23) implies that

$$|\langle R(f), g\rangle| \le C_4 \int_{\mathbf{c}+\eta\mathbf{b}+i\mathbb{R}^d} \frac{1}{|\tau^{\mathbf{2}}|}\, d\tau\; e^{C_3(a_0-a_1)\eta} \to 0 \text{ if } \eta\to\infty \tag{8.24}$$

since $a_1 > a_0$. Since $\operatorname{Exp}(B_\varepsilon(y_0))$ is a neighbourhood of x_0 this shows that

$$\operatorname{supp}_+(R(f)) \cap]0,\infty[^d \subset K,$$

and hence $\operatorname{supp}_+(R(f)) \subset K$ since $\mathscr{D}(]0,\infty[^d)$ is dense in $\mathcal{E}([0,\infty[^d)$.

(iii) Notice that (8.20) is independent of c by Cauchy's theorem. This shows that R is well defined on the inductive limit $\mathcal{H}_{\mathcal{M}}(K)$. Summarizing, we have shown so far that

$$R : \mathcal{H}_{\mathcal{M}}(K) \to \mathcal{E}(K)'_b$$

is defined, linear, and continuous by (8.21).

(iv) Since $\operatorname{supp}_+(R(f)) \subset K$ we may fix $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that $\varphi = 1$ near K and define

$$\langle R(f), g\rangle := \langle R(f), \varphi g\rangle \text{ for any } g \in \mathcal{E}([0,\infty[^d).$$

By (8.21) we know that (8.20) (with $\mathscr{M}$ defined by (8.1)) defines a continuous extension $\widetilde{R(f)}$ of $R(f)$ to $\mathcal{E}^{\mathbf{k+2+a}}([0,\infty[^d)$. To show that $\mathscr{M}(R(f)) = f$ in $\mathcal{H}_{\mathcal{M}}(K)$, it is thus sufficient to show that

$$\langle \widetilde{R(f)}, x^{\alpha-\mathbf{1}}\rangle = f(\alpha) \text{ for } \alpha \ge \mathbf{k}+\mathbf{a}+\mathbf{2} \tag{8.25}$$

since then $\mathscr{M}(R(f))(\alpha) = f(\alpha)$ for these α and hence $\mathscr{M}(R(f)) = f$ by Lemma 8.5. The formula (8.25) is seen as follows: by (8.20) we have for $g_\alpha(t) := t^\alpha, \alpha \ge \mathbf{k}+\mathbf{a}+\mathbf{2}$, and any $c \ge k+2$,

$$\begin{aligned} (2\pi i)^d \langle \widetilde{R(f)}, g_{\alpha-\mathbf{1}}\rangle &= \int_{\mathbf{c}+i\mathbb{R}^d} \frac{f(\tau+\mathbf{1})}{\tau^{\mathbf{k+2}}} \mathscr{M}(\theta^{\mathbf{k+2}}(\varphi g_{\alpha-\mathbf{1}}))(-\tau)\, d\tau \\ &= \int_{\mathbf{c}+i\mathbb{R}^d} \frac{f(\tau+\mathbf{1})}{\tau^{\mathbf{k+2}}} \mathscr{M}(\varphi\theta^{\mathbf{k+2}} g_{\alpha-\mathbf{1}})(-\tau)\, d\tau \\ &= (\alpha-\mathbf{1})^{\mathbf{k+2}} \int_{\mathbf{c}+i\mathbb{R}^d} \frac{f(\tau+\mathbf{1})}{\tau^{\mathbf{k+2}}} \mathscr{M}(\varphi g_{\alpha-\mathbf{1}})(-\tau)\, d\tau, \end{aligned}$$

where the second equation holds since $\operatorname{supp}_+(R(f)) \subset K$ and $\varphi = 1$ near K; hence

$$\operatorname{supp}\big(\theta^{\mathbf{k+2}}(\varphi g_{\alpha-\mathbf{1}}) - \varphi\theta^{\mathbf{k+2}} g_{\alpha-\mathbf{1}}\big) \cap K = \emptyset.$$

By partial integration we get for $j \in \mathbb{N}$ since $\operatorname{Re}(\tau) = \mathbf{c} \le \mathbf{a} \le \alpha - \mathbf{2}$,

$$\begin{aligned}\mathscr{M}(\varphi g_{\alpha-\mathbf{1}})(-\tau) &= \int_{[0,\infty[^d} \varphi(t) t^{\alpha-\tau-\mathbf{2}}\, dt \\ &= \int_{[0,\infty[^d} (\partial_1 \cdots \partial_d)\varphi(t) t^{\alpha-\tau-\mathbf{1}}\, dt / \prod_{n\le d} (\tau_n - \alpha_n + 1) \\ &= \int_{[0,\infty[^d} (\partial_1 \cdots \partial_d \varphi)(e^j x) x^{\alpha-\tau-\mathbf{1}}\, dx \frac{e^{-\langle \mathbf{j}, \tau-\alpha\rangle}}{\prod_{n\le d}(\tau_n - \alpha_n + 1)},\end{aligned}$$

and hence

$$\begin{aligned}&\langle \widetilde{R(f)}, g_{\alpha-\mathbf{1}}\rangle \\ &= \frac{(\alpha-\mathbf{1})^{\mathbf{k}+\mathbf{2}}}{(2\pi i)^d} \int_{\mathbf{c}+i\mathbb{R}^d} \frac{f(\tau+\mathbf{1}) e^{-\langle \mathbf{j},\tau-\alpha\rangle}}{\tau^{\mathbf{k}+\mathbf{2}} \prod_{n\le d}(\tau_n-\alpha_n+1)} \int_{[0,\infty[^d} (\partial_1 \ldots \partial_d \varphi)(e^j x) x^{\alpha-\tau-\mathbf{1}} dx d\tau \\ &= \int_{[0,\infty[^d} (\partial_1 \ldots \partial_d\varphi)(e^j x) \Big(\frac{(\alpha-\mathbf{1})^{\mathbf{k}+\mathbf{2}}}{(2\pi i)^d} \int_{\mathbf{c}+i\mathbb{R}^d} \frac{f(\tau+\mathbf{1}) e^{-\langle \mathbf{j},\tau-\alpha\rangle}}{\tau^{\mathbf{k}+\mathbf{2}} \prod_{n\le d}(\tau_n-\alpha_n+1)}\, d\tau \Big)\, dx.\end{aligned}$$

For large j we can now close the path of integration for τ near ∞ by Cauchy's theorem and get for $\alpha \ge \mathbf{k} + \mathbf{a} + \mathbf{2}$ by Cauchy's integral formula,

$$\langle \widetilde{R(f)}, g_{\alpha-\mathbf{1}}\rangle = (-1)^d f(\alpha) \int_{[0,\infty[^d} (\partial_1 \ldots \partial_d \varphi)(e^j x) e^{jd}\, dx = f(\alpha)\varphi(0) = f(\alpha),$$

since $\varphi(0) = 1$ since $0 \in K$ by assumption (the factor $(-1)^d$ comes from the orientation of the path of integration).

(d) We have shown so far that $R = \mathscr{M}^{-1} : \mathcal{H}_{\mathcal{M}}(K) \to \mathcal{E}(K)'_b$ is bijective and continuous. Clearly, $\mathcal{E}(K)'_b$ is a (DFS)-space being a closed subspace of the (DFS)-space $C^\infty(\mathbb{R}^d)'_b$. Notice also that $\mathcal{H}_{\mathcal{M}}(K)$ is a (DFS)-space since the restrictions $R : \mathcal{H}_{K,k}(\Omega_k) \to \mathcal{H}_{K,k+1}(\Omega_{k+1})$ are compact. Indeed, if (f_n) is a sequence in $\mathcal{H}_{K,k}(\Omega_k)$ such that $\|f_n\|_k \le 1$ for any n, then (f_n) is a bounded sequence in $\mathcal{H}(\Omega_{k+1/2})$, hence we can assume by Montel's theorem that $f_n \to f \in \mathcal{H}(\Omega_{k+1/2})$ uniformly on compact sets in $\Omega_{k+1/2}$. Then

$$\begin{aligned}|f_n(z) - f(z)| &= \lim_m |f_n(z) - f_m(z)| \\ &\le \lim_m (\|f_n\|_k + \|f_m\|_k)/(|z_1 \ldots z_d|) \le 2\varepsilon \text{ if } z \in \Omega_{k+1} \text{ and } \|z\|_\infty \ge 1/\varepsilon\end{aligned}$$

since $|z_j| \ge k+1$ for $j \le d$ if $z \in \Omega_{k+1}$, and hence

$$|z_1 \ldots z_d| \ge (k+1)^{d-1} \|z\|_\infty \ge 1/\varepsilon.$$

Since (f_n) converges in $\mathcal{H}(\Omega_{k+1/2})$ this shows that $f_n \to f \in \mathcal{H}_{K,k+1}(\Omega_{k+1})$ with respect to $\|\cdot\|_{k+1}$.

Thus, $\mathscr{M}^{-1}$ is a topological isomorphism by the open mapping theorem. Hence $\mathscr{M}$ is a topological isomorphism. □

The preceding Mellin isomorphism theorem can be formulated for other locations of m-convex compact sets $K \subset [0,\infty[^d$ (including the case $K \subset]0,\infty[^d$) with larger holomorphy region for the Mellin transforms (depending on the location of K).

Corollary 8.7. *For $g \in \mathscr{D}([0,\infty[^d)$ we have*

$$\frac{1}{(2\pi i)^d}\int_{\xi+i\mathbb{R}^d} \mathscr{M}(g)(-\tau)\, d\tau = g(\mathbf{1})$$

for any $\xi > \mathbf{0}$.

Proof. Set $T := \delta_{\mathbf{1}}$ (and hence $f := \mathscr{M}(T) = 1$) in Theorem 8.2 and use Cauchy's theorem. □

Remark 8.8. Let $S, T \in \mathcal{E}([0,\infty[^d)'$.

(a) $\mathscr{M}({}^tH_T(S))(z) = \mathscr{M}(T)(z)\mathscr{M}(S)(z)$ for $\mathrm{Re}\,(z) > \mathbf{k_0}$, where $k_0 \in \mathbb{N}$ is sufficiently large.

(b) For $J \subset \{1,\dots,d\}$ and $\alpha \in \mathbb{N}^J$ let T_α be the restriction of T defined in (5.1). For any $z_{D\setminus J} \in \mathbb{C}^{D\setminus J}$ we then have

$$\mathscr{M}(T_\alpha)(z_{D\setminus J}) = \mathscr{M}(T)(\alpha+\mathbf{1}, z_{D\setminus J}) \text{ if } \mathrm{Re}\,(\alpha+\mathbf{1}, z_{D\setminus J}) > \mathbf{k}.$$

Proof.

(a) Since $T \in \mathcal{E}([0,\infty[^d)'$ we have $H_T : \mathcal{E}([0,\infty[^d) \to \mathcal{E}([0,\infty[^d)$ and ${}^tH_T(S) \in \mathcal{E}([0,\infty[^d)'$. So $\mathscr{M}({}^tH_T(S))$ is defined. By Lemma 8.5 we need to show the equation only for $z := \alpha$, where $\mathbf{k_0} < \alpha \in \mathbb{N}^d$. This is easy:

$$\begin{aligned}\mathscr{M}({}^tH_T(S))(\alpha) &= \langle {}^tH_T(S), \xi^{\alpha-\mathbf{1}}\rangle = \langle S, H_T(\xi^{\alpha-\mathbf{1}})\rangle = \langle_x T, x^{\alpha-\mathbf{1}}\rangle\langle_y S, y^{\alpha-\mathbf{1}}\rangle \\ &= \mathscr{M}(T)(\alpha)\mathscr{M}(S)(\alpha).\end{aligned}$$

(b) By the definition of T_α and the Mellin transform we have for $z := z_{D\setminus J}$

$$\mathscr{M}(T_\alpha)(z) = \langle_t \widetilde{T}_\alpha, t^{z-\mathbf{1}}\rangle = \langle_y \widetilde{T}, y_J^\alpha y_{D\setminus J}^{z-\mathbf{1}}\rangle = \langle_y \widetilde{T}, y^{(\alpha, z-\mathbf{1})}\rangle = \mathscr{M}(T)(\alpha+\mathbf{1}, z).$$

□

Corollary 8.9. *Let $0 \neq R$ be a polynomial, and let $T \in \mathcal{E}([0,\infty[^d)'$. Let $K \subset [0,\infty[^d$ be m-convex and compact, and let $[0,\delta]^d \subset K$ for some $\delta > 0$. Then*

$$\mathrm{supp}_+(T) \subset K \textit{ if } \mathrm{supp}_+({}^tR(\theta)T) \subset K.$$

Proof. By Theorem 8.2 $\mathscr{M}(T)$ is holomorphic on $\Omega_k = \{z \in \mathbb{C}^d \mid \mathrm{Re}\,(z) > \mathbf{k}\}$ for some $k \in \mathbb{N}$. By the Malgrange Lemma (see [15, 7.3.3]) we get

$$\begin{aligned}|\mathscr{M}(T)(z)| &\le C \sup_{|\zeta|\le 1} |R(z+\zeta-\mathbf{1})\mathscr{M}(T)(z+\zeta)| \\ &= C \sup_{|\zeta|\le 1} |\langle_x T, R(\theta_x)x^{z+\zeta-\mathbf{1}}\rangle| \qquad (8.26)\\ &= C \sup_{|\zeta|\le 1} |\mathscr{M}({}^tR(\theta)T)(z+\zeta)| \text{ if } \mathrm{Re}\,(z) > \mathbf{k}+\mathbf{2}.\end{aligned}$$

Using Theorem 8.2 again we get $\mathrm{supp}_+(T) \subset K$ since $\mathrm{supp}_+({}^tR(\theta)T) \subset K$. □

9. Surjectivity of Euler operators on $C^\infty_{I(P)}(\Omega)$ for m-convex Ω

This section is devoted to the proof of the Main Theorem A from the introduction, stating that if $0 \in \Omega$, then all Euler type operators $P(\theta)$ are surjective on $C^\infty_{I(P)}(\Omega)$ if and only if Ω is m-convex in the sense explained in Section 2. As a consequence we will obtain part (a) of the Main Theorem B.

Euler type differential operators $H := P(\theta)$ are Hadamard operators with $H = H_T$, where $\operatorname{supp}(T) \subset \{\mathbf{1}\}$. In fact, there are $c_\alpha \in \mathbb{C}$ such that

$$P(\theta) = H_T \text{ for } T = \sum_{|\alpha|\le m} c_\alpha \delta_{\mathbf{1}}^{(\alpha)}. \tag{9.1}$$

Indeed, by (8.6) there are $c_\alpha \in \mathbb{C}$ such that $P(\theta) = \sum_{|\alpha|\le m}(-1)^{|\alpha|}c_\alpha x^\alpha \partial^\alpha$ and then (9.1) follows. Thus the theory developed in Sections 3 and 6 can be applied. Recall that the restricted Euler operators are $P_\alpha(\theta_{D\setminus J}) = P(\alpha, \theta_{D\setminus J})$ by (5.4), where $\alpha \in \mathbb{N}^J$, $J \subset D := \{1,\dots,d\}$, and that

$$C^\infty_{I(P)}(\Omega) := \{f \in C^\infty(\Omega) \mid \forall \emptyset \ne J \subset D\ \forall \alpha \in \mathbb{N}^J : f^{(\alpha)}(0_J,\ \cdot\ ) = 0 \text{ if } P(\alpha,\ \cdot\ ) = 0\}$$

by (3.3).

We start with a Local Existence Theorem.

Theorem 9.1. *Let $P \ne 0$ be a polynomial. Then*

$$P(\theta)\left(\mathscr{E}^{\mathbf{k}}([0,\infty[^d)\right) \supset \mathscr{D}([0,\infty[^d) \text{ for any } k \in \mathbb{N}.$$

Proof. Fix $g \in \mathscr{D}([0,\infty[^d)$ and $k \in \mathbb{N}$.

(i) We will use a variant of the formula [15, (7.3.22)] for elementary solutions of partial differential operators with constant coefficients. Let $Pol(m)$ denote the polynomials of order $\le m$, and let $Pol^\circ(m) := Pol(m) \setminus \{0\}$. Also, let $\Phi \in C^\infty(Pol^\circ(m) \times \mathbb{C}^d)$ be a function as in [15, Lemma 7.3.12]. For $\zeta \in \mathbb{C}^d$ let $P_\zeta(\eta) := P(\zeta+\eta)$. Let

$$G(x) := \left(\frac{1}{2\pi i}\right)^d \int_{\mathbf{k+2}+i\mathbb{R}^d} \int \frac{\mathscr{M}(g(x\ \cdot\ ))(-\zeta-\eta)}{P(\zeta+\eta)} \Phi(P_\zeta,\eta)\, d\eta\, d\zeta \text{ for } x > \mathbf{0}. \tag{9.2}$$

To study existence and properties of G we use the following formula valid for $z \in \mathbb{C}^d$:

$$\begin{aligned}\partial_x^\alpha \mathscr{M}(g(x\ \cdot\ ))(-z) &= \int \partial_x^\alpha[g(xt)]t^{-z-\mathbf{1}}dt \\ &= \int t^\alpha(\partial^\alpha g)(xt)t^{-z-\mathbf{1}}dt = x^{z-\alpha}\int(\partial^\alpha g)(\tau)\tau^{-z+\alpha-\mathbf{1}}d\tau \\ &= x^{z-\alpha}\mathscr{M}(\partial^\alpha g)(-z+\alpha), x > \mathbf{0}.\end{aligned} \tag{9.3}$$

Since g is flat on Z_1 and has compact support, we thus get for $\operatorname{Re}(z) \le \mathbf{k+3}$ (also using Proposition 8.1(c) for $\widetilde{k} := k+4 \ge 2d$)

$$|\partial_x^\alpha \mathscr{M}(g(x\ \cdot\ ))(-z)| \le C_\alpha x^{\operatorname{Re}(z)-\alpha}(1+|z|)^{-2d}.$$

We apply this estimate in (9.2) for $z = \zeta+\eta$ with $\operatorname{Re}(\zeta) = k+\mathbf{2}$ and $\|\eta\|_\infty \le 1$ (notice that $\operatorname{supp}(\Phi(P,\ \cdot\ )) \subset B_1(0)$). We thus get exactly as in the proof of [15, Theorem 7.3.10] that $G \in C^\infty(]0,\infty[^d)$ and

$$|\partial^\alpha G(x)| \le B_\alpha \sup_{|t|_\infty\le 1} x^{\mathbf{k+2}+t-\alpha} \text{ for } x > \mathbf{0}.$$

For $\alpha \le \mathbf{k}$ the right-hand side vanishes on Z_1, hence $G \in \mathscr{E}^{\mathbf{k}}([0,\infty[^d)$.

(ii) By (9.3) and (8.2) we have

$$\begin{aligned}\theta_{x_j}\mathscr{M}(g(x\ \cdot))(z) &= \int x_j t_j(\partial_j g)(xt)t^{z-\mathbf{1}}dt = \int t_j \partial_{t_j}[g(xt)]t^{z-\mathbf{1}}dt \\ &= \mathscr{M}(\theta_j[g(x\ \cdot)])(z) = -z_j\mathscr{M}(g(x\ \cdot))(z),\end{aligned}$$

and hence for $k > \deg(P)$ and $x > \mathbf{0}$,

$$\begin{aligned} P(\theta_x)G(x) &= \Big(\frac{1}{2\pi i}\Big)^d \int_{\mathbf{k+2}+i\mathbb{R}^d} \int \frac{\mathscr{M}(P(\theta)[g(x\,\cdot\,)])(-\zeta-\eta)}{P(\zeta+\eta)} \Phi(P_\zeta,\eta)\; d\eta\; d\zeta \\ &= \Big(\frac{1}{2\pi i}\Big)^d \int_{\mathbf{k+2}+i\mathbb{R}^d} \int \mathscr{M}(g(x\,\cdot\,))(-\zeta-\eta)\Phi(P_\zeta,\eta)\; d\eta\; d\zeta \\ &= \Big(\frac{1}{2\pi i}\Big)^d \int_{\mathbf{k+2}+i\mathbb{R}^d} \mathscr{M}(g(x\,\cdot\,))(-\zeta)\; d\zeta = g(x\xi)\big|_{\xi=1} = g(x) \end{aligned}$$

(the last line follows from [15, (7.3.19)] and Corollary 8.7). Since $G \in \mathcal{E}^{\mathbf{k}}([0,\infty[^d)$ and $k > \deg(P)$ we get $P(\theta)G = g$. The theorem is proved. ☐

Next we need an approximation theorem for the zero solutions of $P(\theta)$. This is based on the following result on "extension of divisibility" in $\mathcal{H}_{K,k}(\Omega_k)$ (recall the corresponding notation from the preceding section).

Lemma 9.2. *Let P be an irreducible polynomial. For any $k \in \mathbb{N}$ there is $k < k_1 \in \mathbb{N}$ such that the following holds for any m-convex compact set $K \subset [0,\infty[^d$ with $[0,\delta]^d \subset K$ for some $\delta > 0$: if for $F \in \mathcal{H}_{K,k}(\Omega_k)$ there is $g \in \mathcal{H}(\Omega_n)$ such that $F = Pg$ on Ω_n, then there is $G \in \mathcal{H}_{K,k_1}(\Omega_{k_1})$ such that $F = PG$ on Ω_{k_1}.*

Proof. (a) For any $k \in \mathbb{N}$ there is $k < k_0 \in \mathbb{N}$ such that the following holds: if for $F \in \mathcal{H}(\Omega_k)$ there is $g \in \mathcal{H}(\Omega_n)$ such that $F = Pg$ on Ω_n, then there is $G \in \mathcal{H}(\Omega_{k_0})$ such that $F = PG$ on Ω_{k_0}.

Proof. For $k \in \mathbb{N}$ let

$$V_k := \{z \in \Omega_k \mid P(z) = 0\}.$$

Since P is irreducible, it is sufficient to prove that $F = 0$ on V_{k_0}. Hence we need to show that for any $k \in \mathbb{N}$ there is $k_0 \in \mathbb{N}$ such that for all $F \in \mathcal{H}(\Omega_k)$ the following holds: $F = 0$ on V_{k_0} if $F = 0$ on V_n for some n.

Proof. We order the variables such that $z = (w,t)$ with $w \in \mathbb{C}^{d-1}, t \in \mathbb{C}$, and

$$P(w,t) = \sum_{j\le m} a_j(w)t^j \text{ with } a_m \neq 0 \text{ and } m \geq 1.$$

Let $\Delta = \Delta(w)$ be the discriminant of P. The set $W := \{w \in \mathbb{C}^{d-1} \mid (a_m\Delta)(w) \neq 0\}$ is semialgebraic and open. By [15, Appendix, Lemma A.1.2], for any $w \in W$ there are a neighbourhood $V \subset W$ and m holomorphic functions $t_j(w)$ such that $t_j(w) \neq t_\ell(w)$ if $j \neq \ell$ and $w \in V$ and such that

$$\{(w,t_1(w)),\dots,(w,t_m(w)) \mid w \in V\} = \{(w,t) \in V\times\mathbb{C} \mid P(w,t) = 0\}. \tag{9.4}$$

Since $W_k := V_k \cap (W\times\mathbb{C})$ is semialgebraic, W_k has finitely many connected components by [5, Theorem 2.4.5]. Each component S of W_k is a complex manifold by (9.4). Notice that W_k is dense in V_{k_0} for $k_0 \geq k$, hence $F = 0$ on V_{k_0} if $F = 0$ on any component S of W_k such that $S \cap V_{k_0} \neq \emptyset$. Since F is holomorphic on W_k we therefore only need to show that there is $k_0 \in \mathbb{N}$ such that for any component S of W_k with $S \cap V_{k_0} \neq \emptyset$ we have $S \cap \Omega_n \neq \emptyset$ for any n. Since the sets Ω_n are decreasing, for any component S of W_k either there is $\ell \in \mathbb{N}$ such that $S \cap \Omega_n = \emptyset$ for any $n \geq \ell$ or $S \cap \Omega_n \neq \emptyset$ for any n. Since there are only finitely many components S, there is k_0 such that either $V_{k_0} = \emptyset$ or $S \cap \Omega_n \neq \emptyset$ for any n if S is a connected component of W_k with $S \cap V_{k_0} \neq \emptyset$. The claim is proved.

(b) Let $F \in \mathcal{H}_{K,k}(\Omega_k)$ such that $F = Pg$ on Ω_n for some $g \in \mathcal{H}(\Omega_n)$. Choose k_0 and $G \in \mathcal{H}(\Omega_{k_0})$ by (a) such that $F = PG$ on Ω_{k_0}. By the Malgrange Lemma [15, 7.3.3] we get as in (8.26) for $z \in \Omega_{k_0+2}$

$$\begin{aligned}|G(z)| &\leq C \sup_{|\zeta|\leq 1} |P(z+\zeta)G(z+\zeta)| = \sup_{|\zeta|\leq 1} |F(z+\zeta)| \\ &\leq C_1 \sup_{|\zeta|\leq 1} |(z+\zeta)^{\mathbf{k}}| e^{h_K(\mathrm{Re}\,(z+\zeta))} \leq C_2 |z^{\mathbf{k}}| e^{h_K(\mathrm{Re}\,(z))},\end{aligned}$$

hence $G \in \mathcal{H}_{K,k_1}(\Omega_{k_1})$ for $k_1 := k_0 + 2$. □

For U open (in $[0,\infty[^d$) let

$$\mathscr{N}_P^{\mathbf{k}}(U) := \{f \in \mathcal{E}^{\mathbf{k}}(U) \mid P(\theta)f = 0\}.$$

Theorem 9.3. *Let P be an irreducible polynomial. For any $k \in \mathbb{N}$ there is $k < \kappa = \kappa(k) \in \mathbb{N}$ such that for any m-convex open set $U \subset \mathbb{R}^d$ with $\mathbf{0} \in U$, any $n \geq \kappa$ and any $\sigma \in \{\pm 1\}^d$: $\mathscr{N}_P^{\mathbf{n}}([0,\infty[^d)$ is dense in $\mathscr{N}_P^{\kappa}(\sigma U_\sigma)$ for the topology of $\mathscr{N}_P^{\mathbf{k}}(\sigma U_\sigma)$.*

Proof. Let $T \in \mathcal{E}^{\mathbf{k}}(\sigma U_\sigma)'$. Notice that $\sigma U_\sigma \subset [0,\infty[^d$. Since $0 \in U$ we may choose an m-convex compact set $[0,\delta]^d \subset K \subset \sigma U_\sigma$ such that $\mathrm{supp}(T) \subset K$. Hence $F := \mathscr{M}(T) \in \mathcal{H}_{K,k+1}(\Omega_{k+1})$ by (8.13) and part (a) of the proof of Theorem 8.2. If $T\big|_{\mathscr{N}_P^{\mathbf{n}}([0,\infty[^d)} = 0$, then

$$0 = \langle T, \xi^{z-\mathbf{1}} \rangle = \mathscr{M}(T)(z) = F(z) \text{ if } z \in \Omega_{n+1} \text{ and } P(z-\mathbf{1}) = 0.$$

Since P is irreducible, this implies that there is $g \in \mathcal{H}(\Omega_{n+1})$ such that $F = P(\,\cdot\, - \mathbf{1})g$ on Ω_{n+1}. By Lemma 9.2 (applied to $k+1$ and $P(\,\cdot\, - \mathbf{1})$) there is $k_1 > k+1$ such that $F = P(\,\cdot\, - \mathbf{1})G$ on Ω_{k_1} for some $G \in \mathcal{H}_{K,k_1}(\Omega_{k_1})$. By Proposition 8.6 there is $k_2 \in \mathbb{N}$ such that $\mathscr{M}^{-1} : \mathcal{H}_{K,k_1}(\Omega_{k_1}) \to \mathcal{E}^{\mathbf{k_2}}(K)'_b$ is continuous, hence $S := \mathscr{M}^{-1}(G) \in \mathcal{E}^{\mathbf{k_2}}(K)'$ and ${}^tP(\theta)S = T$ in $\mathcal{E}^{\kappa}(\sigma U_\sigma)'$ for $\kappa := k_2 + \deg(P)$. Indeed, this equation holds in $\mathcal{E}([0,\infty[^d)'$ by Theorem 8.2 since $\mathscr{M}({}^tP(\theta)S) = P(\,\cdot\, - \mathbf{1})G = F = \mathscr{M}(T)$ in $\mathcal{H}_{\mathscr{M}}$ by Remark 8.8(a), and therefore it holds in $\mathcal{E}^{\kappa}(\sigma U_\sigma)'$ since $\mathcal{E}([0,\infty[^d)'$ is dense in $\mathcal{E}^{\kappa}(\sigma U_\sigma)'$. Hence we get for every $f \in \mathscr{N}_P^{\kappa}(\sigma U_\sigma)$

$$\langle T, f \rangle = \langle {}^tP(\theta)S, f \rangle = \langle S, P(\theta)f \rangle = 0.$$

The theorem is proved. □

We can now prove one implication of the Main Theorem A.

Theorem 9.4. *All Euler differential operators $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ are surjective if Ω is m-convex and $\mathbf{0} \in \Omega$. In particular, $P(\theta) : C^\infty(\mathbb{R}^d) \to C^\infty_{I(P)}(\mathbb{R}^d)$ is surjective for each polynomial P.*

Combining Theorems 9.4 and 3.8 we obtain part (a) of the Main Theorem B.

Theorem 9.5. *An Euler differential operator $0 \neq P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective if and only if Ω is $P(\theta)$-convex.*

Proof. The necessity follows from Theorem 3.8(a), while sufficiency comes from Theorem 9.4 and Theorem 3.8(b). □

Proof of Theorem 9.4. The statement is trivial for $P = 0$ since then $C^\infty_{I(P)}(\Omega) = \{0\}$. Let $P \neq 0$. Since m-convexity is inherited from Ω to $\Omega_{D\setminus J}$ for $J \subsetneqq D$, the theorem is proved by Theorem 6.5 if we show that for any polynomial $0 \neq R$,

$$R(\theta) : \mathcal{E}(\Omega_\sigma) \to \mathcal{E}(\Omega_\sigma) \text{ is surjective if } \Omega \text{ is m-convex and } \Omega_\sigma \neq \emptyset. \tag{9.5}$$

Proof. Since $R(\theta)[f(\sigma x)] = [R(\theta)f](\sigma x)$ we have to show that $R(\theta) : \mathcal{E}(\sigma\Omega_\sigma) \to \mathcal{E}(\sigma\Omega_\sigma)$ is surjective. We may assume that R is irreducible. Since local existence is guaranteed by Theorem 9.1 we can use the Approximation Theorem 9.3 and the classical Mittag–Leffler procedure to complete the proof of Theorem 9.4: define κ_n by $\kappa_1 := \kappa(1)$ and $\kappa_{n+1} := \kappa(\kappa_n), 0 \neq n \in \mathbb{N}$, for κ from Theorem 9.3. Choose m-convex open sets $U_n \subset \Omega$ such that $\mathbf{0} \in U_n, \overline{U_n} \subset U_{n+1}$, and $\bigcup_n U_n = \Omega$. Choose $\varphi_n \in \mathscr{D}(\Omega)$ such that $\varphi_n = 1$ near $\overline{U_n}$. For $g \in \mathcal{E}(\sigma\Omega_\sigma)$ there are $g_n \in \mathcal{E}^{\kappa_\mathbf{n}}([0,\infty[^d)$ by Theorem 9.1 such that $R(\theta)g_n = g\varphi_n$. Set $G_1 := g_1$. If $G_n \in \mathcal{E}^{\kappa_\mathbf{n}}([0,\infty[^d)$ is defined for some $n \geq 1$ such that $R(\theta)G_n = g\varphi_n$, then $(g_{n+1} - G_n) \in \mathscr{N}_R^{\kappa_\mathbf{n}}(\sigma(U_n)_\sigma)$. Hence there is $F_{n+1} \in \mathscr{N}_R^{\kappa_{n+1}}([0,\infty[^d)$ by Theorem 9.3 such that $G_{n+1} := (g_{n+1} + F_{n+1}) \in \mathcal{E}^{\kappa_{\mathbf{n+1}}}([0,\infty[^d)$ and $R(\theta)G_{n+1} = g\varphi_{n+1}$ and

$$\sup_{x \in \sigma(U_{n-1})_\sigma, \alpha \leq \kappa_{\mathbf{n-1}}} |\partial^\alpha (G_{n+1}(x) - G_n(x))| \leq 2^{-n}.$$

Clearly, G_n converges to $G \in \mathcal{E}(\sigma\Omega_\sigma)$ and $R(\theta)G = g$. □

Let us emphasize that the question if $P(\theta) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective may lead to difficult problems for diophantine equations. We present a striking example.

Theorem 9.6. *Let Ω be m-convex and $\mathbf{0} \in \Omega$. Then*

$$P(\theta) := (\theta_1 + 1)^m + (\theta_2 + 1)^m - (\theta_3 + 1)^m : C^\infty(\Omega) \to C^\infty(\Omega)$$

is surjective for $m \geq 3$.

Proof. By Theorem 9.4 we have to prove that $C^\infty_{I(P)}(\Omega) = C^\infty(\Omega)$, i.e., that $I(P) = \emptyset$, which means that $P(\alpha) \neq 0$ for any $\alpha \in \mathbb{N}^3$. This is equivalent to Fermat's Last Theorem. □

The same argument was used in [10, Example 10.3] to show that $P(\theta) := (\theta_1 + 1)^m + (\theta_2 + 1)^m - (\theta_3 + 1)^m$ is injective on $\mathscr{A}(\mathbb{R}^3)$. Notice that $P(\theta)$ is not injective on $C^\infty(\mathbb{R}^3)$ by Theorem 11.5.

We now prove the converse implication of the Main Theorem A.

Theorem 9.7. *Let $\Omega \subseteq \mathbb{R}^d$ be an open connected set containing $\mathbf{0}$. If Ω is $P(\theta)$-convex for all nonzero polynomials P, then Ω is m-convex.*

Proof. By Corollary 4.5, if Ω is $(\theta_j + 1)$-convex, then for any $x = (x_n) \in \Omega$

$$\{(x_1, \ldots, x_{j-1}, tx_j, x_{j+1}, \ldots, x_d) \mid t \in [0,1]\} \subset \Omega.$$

Using this inductively for $j = 1, \ldots, d$ we get

$$\{(t_1x_1, \ldots, t_dx_d) \mid (t_1, \ldots, t_d) \in [0,1]^d\} \subset \Omega.$$

This means by definition that Ω is nearly solid, and so Ω_σ° is connected for any $\sigma \in \{\pm 1\}^d$ and Ω has the projection property (see Proposition 2.3).

If Ω is $P(\theta)$-convex for every nonzero P, then, by Theorem 9.5 and Corollary 3.14, $P(\theta)$ is surjective on $C^\infty(\Omega_\sigma^\circ)$ for every nonzero P and every $\sigma \in \{\pm 1\}^d$. Without loss of generality we may assume that $\sigma = \mathbf{1}$. Let us observe that $P(\partial) = C_{\mathrm{Exp}} \circ P(\theta) \circ C_{\mathrm{Log}}$, where

$$C_{\mathrm{Exp}}(f)(x) = f(\exp x_1, \ldots, \exp x_d), \quad C_{\mathrm{Log}}(f)(x) = f(\log x_1, \ldots, \log x_d).$$

Thus every nonzero constant coefficient linear partial differential operator $P(\partial)$ is surjective on $C^\infty(\mathrm{Log}(\Omega_\mathbf{1}^\circ))$ and, by [15, Cor. 10.8.4], $\mathrm{Log}(\Omega_\mathbf{1}^\circ)$ is convex (since $\mathrm{Log}\,\Omega_\mathbf{1}^\circ$ is connected as we have seen above). Hence Ω is m-convex by Proposition 2.2. □

10. Strict $P(\theta)$-convexity and surjectivity of Euler operators

In this section we will continue the discussion from Sections 3 and 5–7 for Euler type differential operators $P(\theta)$ relating the solvability theory for $P(\theta)$ (and the shifted operators $P(k+\theta), k \in \mathbb{N}^d$) on $C^\infty(\Omega)$ to the solvability of $P(\theta)$ in $C^\infty(\Omega_\sigma^\circ)$ and in $\mathcal{E}(\Omega_\sigma)$, respectively, for $\sigma \in \{\pm 1\}^d$, and, moreover, to the solvability of the constant coefficient operator $P(\partial)$ on $\mathrm{Log}_\sigma(\Omega_\sigma^\circ)$ and corresponding P-convexity conditions. In this way we will also prove the Main Theorems B and C from the introduction in a sequence of steps.

Recall that $Q_\sigma := \{x \in \mathbb{R}^d \mid \sigma x \geq 0\}$ for $\sigma \in \{\pm 1\}^d$ and that

$$\begin{aligned} &\mathrm{Log}_\sigma : Q_\sigma^\circ \to \mathbb{R}^d, \qquad \mathrm{Log}_\sigma(x) := (\log(\sigma_j x_j))_{j \leq d}, \\ &\mathrm{Exp}_\sigma : \mathbb{R}^d \to Q_\sigma^\circ, \quad \mathrm{Exp}_\sigma(x) := (\sigma_j \exp(x_j))_{j \leq d}. \end{aligned} \tag{10.1}$$

Log_σ and Exp_σ are diffeomorphisms, and the composition operators

$$C_{\mathrm{Log}_\sigma} : C^\infty(\mathbb{R}^d) \to C^\infty(Q_\sigma^\circ), \quad C_{\mathrm{Log}_\sigma}(f) := f \circ \mathrm{Log}_\sigma$$

and

$$C_{\mathrm{Exp}_\sigma} : C^\infty(Q_\sigma^\circ) \to C^\infty(\mathbb{R}^d), \quad C_{\mathrm{Exp}_\sigma}(f) := f \circ \mathrm{Exp}_\sigma$$

clearly satisfy

$$P(\partial) \circ C_{\mathrm{Exp}_\sigma} = C_{\mathrm{Exp}_\sigma} \circ P(\theta) \quad \text{and} \quad P(\theta) \circ C_{\mathrm{Log}_\sigma} = C_{\mathrm{Log}_\sigma} \circ P(\partial). \tag{10.2}$$

Recall that by (5.4) the restricted Euler operators on $C^\infty(\Omega_{D\setminus J})$ are given by $P(\alpha, x_{D\setminus J})$ for $J \subsetneq D$ and $\alpha \in \mathbb{N}^J$. For $J = \emptyset$, α is to be omitted and $x_D = x$ as well as $\Omega_D := \Omega$.

Proposition 10.1. *Let $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ be surjective. Then*

$$\mathrm{Log}_\sigma(\Omega^\circ_{D\setminus J,\sigma}) \text{ is } P(\alpha, \partial_{D\setminus J})\text{-convex for supports,} \tag{10.3}$$

that is,

$$P(\alpha, \partial_{D\setminus J}) \text{ is surjective on } C^\infty(\mathrm{Log}_\sigma(\Omega^\circ_{D\setminus J,\sigma})) \tag{10.4}$$

for any $J \subsetneq D$, any $\alpha \in \mathbb{N}^J$ with $P(\alpha, \cdot\,) \neq 0$, and any $\sigma \in \{\pm 1\}^{D\setminus J}$ with $\Omega_{D\setminus J,\sigma} \neq \emptyset$.

Proof. Theorem 5.1 and Corollary 3.14 imply that $P(\alpha, \theta_{D\setminus J}) : C^\infty(\Omega^\circ_{D\setminus J,\sigma}) \to C^\infty(\Omega^\circ_{D\setminus J,\sigma})$ is surjective if $P(\alpha, \cdot\,) \neq 0$. By (10.2) this implies (10.4) which is equivalent to (10.3) by [15, Section 10.6]. □

Notice, however, that the surjectivity of $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ cannot be characterized by (10.4). We present an elementary example.

Example 10.2. Let $\Omega := \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Then

$$P(\theta) := \theta_1 - \theta_2 : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega) = C^\infty(\Omega)$$

is not surjective although (10.4) is satisfied.

Proof. (a) For any choices of σ the sets $\mathrm{Log}_\sigma(\Omega_\sigma^\circ) = \mathbb{R}^2$ (and $\mathrm{Log}_\sigma(\Omega^\circ_{\{1\},\sigma}) = \mathrm{Log}_\sigma(\Omega^\circ_{\{2\},\sigma}) = \mathbb{R}$, respectively) are clearly $R(\partial)$-convex for any polynomial $0 \neq R$. We have $C^\infty_{I(P)}(\Omega) = C^\infty(\Omega)$ since $0 \notin \Omega$ and $S(P) \subset \{0\}$ by Example 7.9(b)(iii).

(b) We show that Ω is not $P(\theta)$-convex (cf. (3.12)). Choose $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(x) \neq 0$ if $x \in]1/2, 1[$ and $\varphi(x) = 0$ otherwise. Let $f_n(x) := \varphi(n x_1 x_2)$. Then

$P(\theta)f_n = 0$ and $\text{supp}(f_n) = \{x \in \mathbb{R}^2 \mid 1/(2n) \leq x_1x_2 \leq 1/n\}$. Choose $h \in C^\infty(\mathbb{R}^2)$ such that $h(x) = 1$ if $\|x\|_\infty \leq 1$ and $h(x) = 0$ if $\|x\|_\infty \geq 2$. Then $g_n := f_nh \in \mathscr{D}(\Omega) \subset C^{\mathbf{0}}(\Omega)'$ and $\text{supp}(P(\theta)g_n) \subset K := \{x \in \mathbb{R}^2 \mid 1 \leq \|x\|_\infty \leq 2\} \Subset \Omega$, while

$$\text{supp}(g_n) \supset \{(x_1, 1/(2nx_1)) \in \mathbb{R}^2 \mid x_1 \in [1/(2n), 1]\}$$
$$\ni (1/\sqrt{n}, 1/(2\sqrt{n})) \to 0 \text{ if } n \to \infty.$$

Hence $P(\theta)$ is not surjective on $C^\infty(\Omega)$ by Theorem 3.8. □

For the remaining part of this section we resume the discussion of $P(\theta)$-convexity (see Definition 3.7). It would be tempting to get rid of the order k in the definition of $P(\theta)$-convexity using regularization by means of the $\star$-convolution which was first studied for distributions in [25]. We recall some of the basic results (see [25, Proposition 1.1]).

Proposition 10.3. *For $S, T \in C^\infty(\mathbb{R}^d)'$ let*

$$(S \star T)(h) := \langle_x S, \langle_y T, h(xy)\rangle\rangle \text{ for } h \in C^\infty(\mathbb{R}^d).$$

(a) *$(C^\infty(\mathbb{R}^d), \star)$ is a commutative algebra with unit $\delta_{\mathbf{1}}$, and*

$$\star : C^\infty(\mathbb{R}^d)'_b \times C^\infty(\mathbb{R}^d)'_b \to C^\infty(\mathbb{R}^d)'_b$$

is hypocontinuous.

(b) *We have*

$${}^tP(\theta)(S \star T) = ({}^tP(\theta)S) \star T = S \star ({}^tP(\theta)T)$$

and

$$\text{supp}(S \star T) \subset \text{supp}(S) \cdot \text{supp}(T).$$

We emphasize that regularization by multiplicative convolution does not work for $T \in C^\infty(\mathbb{R}^d)'$ as the following surprisingly simple example shows.

Example 10.4. For $S \in C^\infty(\mathbb{R}^d)'$ and $\alpha \in \mathbb{N}^d$ we have

$$S \star \partial^\alpha \delta_{\mathbf{0}} = \langle S, x^\alpha\rangle \partial^\alpha \delta_{\mathbf{0}},$$

where $\delta_{\mathbf{0}}$ is Dirac's distribution. Thus, the order of $\partial^\alpha \delta_{\mathbf{0}} \star S$ is $|\alpha|$ if $\langle S, x^\alpha\rangle \neq 0$. So convolving with even a "very smooth" distribution might not improve smoothness.

Proof. Let $f \in C^\infty(\mathbb{R}^d)$. Then

$$\begin{aligned}\langle S \star \partial^\alpha \delta_{\mathbf{0}}, f\rangle &= \langle_x S, \langle_y \partial^\alpha \delta_{\mathbf{0}}, f(xy)\rangle\rangle \\ &= (-1)^\alpha \langle_x S, \langle_y \delta_{\mathbf{0}}, x^\alpha (\partial^\alpha f)(xy)\rangle\rangle = \langle S, x^\alpha\rangle \langle \partial^\alpha \delta_{\mathbf{0}}, f\rangle.\end{aligned}$$

□

Notice however that we only need a characterization of surjectivity of Euler operators on $\mathcal{E}(\Omega_\sigma)$ if the small gap between the conclusion of Theorem 6.4 and the assumption of Theorem 6.5 can be closed, and we will see in Lemma 10.8 that on $\mathcal{E}(\Omega_\sigma)'$ the $\star$-convolution indeed can be used for regularization in spite of the preceding example. The following is thus the appropriate convexity condition for surjectivity of $P(\theta)$ on $\mathcal{E}(\Omega_\sigma)$.

Definition 10.5. An open set $\Omega \subset \mathbb{R}^d$ is called strictly $P(\theta)$-convex if

$$\forall \sigma \in \{\pm 1\}^d \ \forall K \Subset \Omega_\sigma \ \forall k \in \mathbb{N} \ \exists \widetilde{K} \Subset \Omega_\sigma \ \forall T \in \mathcal{E}^{\mathbf{k}}(\Omega_\sigma)' :$$

$$\operatorname{supp}_+(T) \subset \widetilde{K} \text{ if } \operatorname{supp}_+(P(-\theta - \mathbf{1})T) \subset K, \tag{10.5}$$

where $\operatorname{supp}_+(T)$ is the support in the sense of $\mathcal{E}(\Omega_\sigma)'$.

Notice that ${}^tP(\theta) = P(-\theta - \mathbf{1})$ by Example 3.10.

Theorem 10.6.
(a) *Let Ω be strictly $R(\theta)$-convex. Then $R(\theta) : \mathcal{E}(\Omega_\sigma) \to \mathcal{E}(\Omega_\sigma)$ is surjective for any $\sigma \in \{\pm 1\}^d$ with $\Omega_\sigma \neq \emptyset$.*
(b) *The shifted Euler operators $P(c+\theta) : C^\infty(\Omega) \to C^\infty_{I(P(c+\,\cdot\,))}(\Omega)$ are surjective for any $c \in \mathbb{R}^d$ if $\Omega_{D\setminus J}$ is strictly $P(\alpha, \theta_{D\setminus J})$-convex for any $J \subsetneq D$ and any $\alpha \in \mathbb{N}^J$ such that $P(\alpha, \,\cdot\,) \neq 0$.*

Proof.
(a) We use the surjectivity criterion (3.6). Take $B \subset \mathcal{E}(\Omega_\sigma)'$ such that $R(-\theta - \mathbf{1})B$ is bounded in $\mathcal{E}(\Omega_\sigma)'_b$. Then $R(-\theta - \mathbf{1})B$ is bounded in $\mathcal{E}(Q_\sigma)'_b, Q_\sigma := \{x \in \mathbb{R}^d \mid \sigma x \geq 0\}$, and hence B is bounded in $\mathcal{E}(Q_\sigma)'_b$ by (9.5) and (3.6). On the other hand, there is a compact set $K \subset \Omega_\sigma$ such that $\operatorname{supp}_+(R(-\theta - \mathbf{1})S) \subset K$ for any $S \in B$. By $R(\theta)$-convexity there is a compact set $\widetilde{K} \subset \Omega_\sigma$ such that $\operatorname{supp}_+(S) \subset \widetilde{K}$ for any $S \in B$. This implies that $B \subset \mathcal{E}(\Omega_\sigma)'_b$ is bounded since B is bounded in $\mathcal{E}(Q_\sigma)'_b$.
(b) By (a) we know that for any $J \subsetneq D$, any $\sigma \in \{\pm 1\}^{D\setminus J}$, and any $\alpha \in \mathbb{N}^J$ the operator

$$R(\theta_{D\setminus J}) := P(\alpha, \theta_{D\setminus J}) : \mathcal{E}(\omega_\sigma) \to \mathcal{E}(\omega_\sigma), \omega := \Omega_{D\setminus J}, \text{ is surjective}$$

if $R \neq 0$ and $\omega_\sigma \neq \emptyset$. The theorem now follows from Theorem 6.5. ☐

Using Theorem 10.6 we obtain the following interesting result including the Main Theorem C from the introduction.

Theorem 10.7. *The following are equivalent for a polynomial $P \neq 0$:*
- (a) *The Euler operator $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective for any open $\Omega \subset \mathbb{R}^d$.*
- (b) *The Euler operator $P(\theta) : C^\infty(\mathbb{R}^d \setminus \{x\}) \to C^\infty_{I(P)}(\mathbb{R}^d \setminus \{x\})$ is surjective for some $x \in (\mathbb{R}_*)^d$.*
- (c) *The polynomial P is elliptic.*

In that case $\operatorname{codim}[P(\theta)C^\infty(\Omega)] < \infty$, *and $P(\theta) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective if and only if $0 \notin \Omega$ or $P(\alpha) \neq 0$ for any $\alpha \in \mathbb{N}^d$.*

Proof.
(a)⇒(b): This is trivial.
(b)⇒(c): Since $P(\theta)$ commutes with the dilation operators $D_\sigma, \sigma \in \{\pm 1\}^d$, we may assume that $x > \mathbf{0}$. By the assumption and Proposition 10.1 (for $J = \emptyset$ and $\sigma = \mathbf{1}$), the open set $\omega := \mathbb{R}^d \setminus \{\operatorname{Log}(x)\}$ is $P(\partial)$-convex. If P is not elliptic there is a characteristic vector $0 \neq N$. Hence on the characteristic hyperplane $S := \xi + N^\perp, \operatorname{Log}(x) \notin S$, the distance function $\operatorname{dist}_{\partial\omega}(\,\cdot\,) = \operatorname{dist}(\operatorname{Log}(x), \,\cdot\,)$ does not satisfy the minimum principle, a contradiction to [15, Theorem 10.8.1].

(c)⇒(a): We apply Theorem 10.6(b). Since $P(\alpha,\ \cdot\ )$ is elliptic for any $\alpha \in \mathbb{N}^J, J \subsetneq D$ we only need to show (10.5) for P. Let $K \Subset \Omega_\sigma$, and let $T \in \mathcal{E}(\Omega_\sigma)'$ such that $\mathrm{supp}_+(P(-\theta-\mathbf{1})T) \subset K$. Then $\mathrm{supp}_+(T) \subset \overline{\mathrm{mconv}}(K)$ by Corollary 8.9. Let $\sigma = \mathbf{1}$ without loss of generality, and let $\delta := \mathrm{dist}(K, \partial\Omega)$ (measured by the max-norm). Let $\mathbf{0} \le x \in \partial\Omega$. Then $P(-\theta-1)T\big|_{B_\delta^\infty(x)_{>\mathbf{0}}} = 0$, that is, ${}^tP(\partial)^tC_{\mathrm{Log}}(T)\big|_V = 0$ where $V := \mathrm{Log}(B_\delta^\infty(x)_{>\mathbf{0}})$, since T has compact support in Ω and $x \in \partial\Omega$ so ${}^tC_{\mathrm{Log}}(T)$ vanishes on some open nonvoid subset of V. Since P and ${}^tP$ are elliptic, by [15, Corollary 4.4.4 and the preceding remarks], ${}^tP(\partial)^tC_{\mathrm{Log}}(T)|_V = 0$ implies that ${}^tC_{\mathrm{Log}}(T)$ must vanish on V, so $\mathrm{supp}_+(T)$ is disjoint from $B_\delta^\infty(x)_{>\mathbf{0}}$. By (8.10), it follows that $\mathrm{supp}_+(T)$ is disjoint from $B_\delta^\infty(x)$. This shows (10.5) for $\widetilde{K} := \{x \in \Omega_{\mathbf{1}} \mid \mathrm{dist}(x, \partial\Omega) \ge \delta)\} \cap \overline{\mathrm{mconv}}(K)$.

The remaining statements easily follow from the fact that P is elliptic, hence $I(P)$ is finite and $S(P) \subset \{\mathbf{0}\}$ (see Example 7.9(b)(ii)), and $C^\infty_{I(P)}(\Omega) = \{f \in C^\infty(\Omega) \mid f^{(\alpha)}(0) = 0 \text{ if } P(\alpha) = 0, \alpha \in \mathbb{N}^d\}$ if $0 \in \Omega$ and $C^\infty_{I(P)}(\Omega) = C^\infty(\Omega)$ if $0 \notin \Omega$. □

Notice that Theorem 10.7 does not hold for $x = \mathbf{0}$ by Proposition 4.2.

We want to reformulate strict $P(\theta)$-convexity using regularized elements of $\mathcal{E}(\Omega_\sigma)'$. Somehow unexpectedly the resulting space is the following:

$$\mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma) := \{f \in C^\infty(\Omega_\sigma^\circ) \mid \forall \alpha \in \mathbb{N}^d : x^{\mathbf{k}}\theta^\alpha(f)(x) \text{ is bounded and } \overline{\mathrm{supp}(f)} \Subset \Omega_\sigma\}$$

(with the canonical inductive topology with respect to supports). Let us emphasize that the elements of $\mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma)$ should be viewed as objects on Ω_σ (rather than on Ω_σ°). We denote

$$\mathscr{D}_{\theta,pol}(\Omega_\sigma) = \bigcup_k \mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma)$$

again with the inductive topology.

Lemma 10.8. *Let $S \in C^\infty(\mathbb{R}^d)'$ with $\mathrm{supp}(S) \subset [0,\infty[^d$. Then there is $k \in \mathbb{N}$ such that $g \star S \in \mathscr{D}^{\mathbf{k}}_{\theta,pol}(Q_\sigma)$ for any $g \in \mathscr{D}(Q_\sigma)$, specifically,*

$$(g \star S)(x) = \langle {}_yS, y^{-\mathbf{1}}g(x/y)\rangle \text{ for } x \in Q_\sigma^\circ.$$

Proof. (a) Let $\mathrm{supp}(S) \subset B^\infty_{C_1}(0) := \{x \in \mathbb{R}^d \mid \|x\|_\infty < C_1\}$ and $\mathrm{supp}(g) \subset B^\infty_C(0)$. Set

$$R(S)(x) := \langle {}_yS, y^{-\mathbf{1}}g(x/y)\rangle \text{ for } x \in Q_\sigma^\circ.$$

$R(S)(x)$ is defined for $x \in Q_\sigma^\circ$ since $g_x := g(x/\ \cdot\ ) \in C^\infty(\mathbb{R}^d)$. Indeed, g_x is clearly smooth on $(\mathbb{R}_*)^d$ and

$$\text{(10.6)} \qquad \mathrm{supp}(g_x) \subset \{y \in \mathbb{R}^d \mid y \ge x/C\} \text{ and } \mathrm{supp}(R(S)) \subset B^\infty_{CC_1}(0)_{>\mathbf{0}}.$$

Clearly, $R(S) \in C^\infty(Q_\sigma^\circ)$ and $\theta_jR(S)(x) = \langle {}_yS, y^{-\mathbf{1}}(\theta_jg)(x/y)\rangle$. Thus $R(S) \in \mathscr{D}_{\theta,pol}(Q_\sigma)$ by (10.6) if $R(S)(x)x^{\mathbf{k}}$ is bounded for some $k \in \mathbb{N}$ (independent of g). We start the proof with a useful formula: for $\alpha \in \mathbb{N}^d$ and $g \in \mathscr{D}(Q_\sigma)$ we have

$$\text{(10.7)} \qquad y^\alpha(-\partial_y)^\alpha\big[g(x/y)y^{-\mathbf{1}}\big] = \big[(\theta+\alpha)!g\big](x/y)y^{-\mathbf{1}},$$

where $(\theta+\alpha)! := \prod_{j=1}^d(\theta_j+1)\dots(\theta_j+\alpha_j)$.

Proof. Since the polynomials are dense in $C^\infty(Q_\sigma^\circ)$ we need to treat monomials g only. For these the formula follows by an easy direct calculation.

There are $k \in \mathbb{N}$ and Radon measures μ_α with $\mathrm{supp}(\mu_\alpha) \subset \overline{B^\infty_{C_1}(0)_{\geq 0}}$ such that $S = \sum_{\alpha<\mathbf{k}} \partial^\alpha \mu_\alpha$. To show that $R(S) \in \mathscr{D}_{\theta,pol}(Q_\sigma)$ we notice that by (10.7),

$$x^{\alpha+\mathbf{1}}\big\langle {}_y\mu_\alpha, (-\partial_y)^\alpha \big[g(x/y)y^{-\mathbf{1}}\big]\big\rangle = \big\langle {}_y\mu_\alpha, (x/y)^{\alpha+\mathbf{1}}\big[(\theta+\alpha)!g\big](x/y)\big\rangle. \tag{10.8}$$

By (10.6) this shows the claim since $\sup_{\xi\in Q_\sigma} |\xi^{\alpha-\mathbf{1}}\phi(\xi)| < \infty$ for any $\phi \in \mathscr{D}(Q_\sigma)$. Since the mappings $R_x := \delta_x \circ R, x \in Q_\sigma^\circ$, are continuous on $C^\infty([0,\infty[^d)'_b$, by the closed graph theorem also

$$R : C^\infty([0,\infty[^d)'_b \to \mathscr{D}_{\theta,pol}(Q_\sigma) \text{ is continuous.} \tag{10.9}$$

(b) By (10.9) and Proposition 10.3(a), to show that $g \star S = R(S)$ we only need to prove this equation on the set of point evaluations $\{\delta_\eta \mid \eta > \mathbf{0}\}$ which is a total subset of $C^\infty([0,\infty[^d)'_b$. This is easy:

$$\langle g\star\delta_\eta, f\rangle = \langle {}_xg, f(x\eta)\rangle = \int g(x)f(x\eta)\,dx = \int g(\xi/\eta)f(\xi)\eta^{-\mathbf{1}}d\xi = \langle R(\delta_\eta), f\rangle$$

for $f \in \mathscr{D}(Q_\sigma^\circ)$. The lemma is proved. $\square$

Using the spaces $\mathscr{D}_{\theta,pol}(\Omega_\sigma)$ we can now reformulate strict $P(\theta)$-convexity.

Proposition 10.9. *Ω is strictly $P(\theta)$-convex if and only if*

$$\forall\sigma\in\{\pm1\}^d\ \forall K\Subset\Omega_\sigma\ \forall k\in\mathbb{N}\ \exists\widetilde{K}\Subset\Omega_\sigma\ \forall g\in\mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma): \tag{10.10}$$

$$\mathrm{supp}(g)\subset\widetilde{K} \textit{ if } \mathrm{supp}(P(-\theta-\mathbf{1})g)\subset K.$$

Proof. Necessity. This is clear since $\mathscr{D}_{\theta,pol}(\Omega_\sigma) \subset \mathcal{E}(\Omega_\sigma)'$ and $\mathrm{supp}_+(g) = \overline{\mathrm{supp}(g)}$.

Sufficiency. Fix a compact set $K \subset \Omega_\sigma$ and $k \in \mathbb{N}$. Let $S \in \mathcal{E}^{\mathbf{k}}(\Omega_\sigma)'$ with $\mathrm{supp}_+\big(P(-\theta-\mathbf{1})S\big) \subset K$. Take $g_n \in \mathscr{D}(B^\infty_{1/n}(\mathbf{1}))$ such that $\int g_n(x)dx = 1$. Notice that by Proposition 10.3(b)

$$\mathrm{supp}\big(P(-\theta-\mathbf{1})(S\star g_n)\big) = \mathrm{supp}\big((P(-\theta-\mathbf{1})S)\star g_n\big)$$

$$\subset \mathrm{supp}\big((P(-\theta-\mathbf{1})S)\big)B^\infty_{1/n}(\mathbf{1}) \subset KB^\infty_{1/n}(\mathbf{1}) \subset (K+B^\infty_{C/n}(0))\cap Q_\sigma =: K_n$$

since

$$\|xy-x\|_\infty = \|x(\mathbf{1}-y)\|_\infty \leq \|x\|_\infty\|\mathbf{1}-y\|_\infty \leq C/n$$

if $y\in B^\infty_{1/n}(\mathbf{1})$ and $x \in K \subset B^\infty_C(0)$. Notice that K_n is decreasing and a compact set in Ω_σ for large n. Also, $S\star g_n \in \mathscr{D}^{\mathbf{k+1}}_{\theta,pol}(Q_\sigma^\circ)$ by Lemma 10.8. As above we see that $S\star g_n \in \mathscr{D}^{\mathbf{k+1}}_{\theta,pol}(\Omega_\sigma)$ for large n. Hence there is a compact set $\widetilde{K}\subset\Omega_\sigma$ by (10.10) such that $\mathrm{supp}(S\star g_n)\subset\widetilde{K}$ for large n. This implies that $\mathrm{supp}_+(S)\subset\widetilde{K}$. Indeed, if $x\notin\widetilde{K}$, then there is $\varepsilon>0$ such that $B_\varepsilon(x)\cap\widetilde{K}=\emptyset$. For $\phi\in\mathscr{D}(B_\varepsilon(x)\cap Q_\sigma)$ we then have $S(\phi)\leftarrow(S\star g_n)(\phi)=0$ by Proposition 10.3 since $g_n\to\delta_{\mathbf{1}}$ in $C^\infty(\mathbb{R}^d)'_b$. Hence $x\notin\mathrm{supp}_+(S)$, and we have proved that Ω is strictly $P(\theta)$-convex. $\square$

The following result is the essential tool for closing the gap between Theorems 6.4 and 6.5 for Euler operators.

Theorem 10.10. *The following are equivalent for any polynomial* $0 \neq R$*:*

(a) *The Euler operator* $R(\theta) : \mathcal{E}(\Omega_\sigma) \to \mathcal{E}(\Omega_\sigma)$ *is surjective for any* $\sigma \in \{\pm 1\}^d$ *with* $\Omega_\sigma \neq \emptyset$.

(b) *For any* $k \in \mathbb{N}$ *we have* $R(\theta)\big(\mathcal{E}^{\mathbf{k}}(\Omega_\sigma)\big) \supset \mathcal{E}(\Omega_\sigma)$ *for any* $\sigma \in \{\pm 1\}^d$ *with* $\Omega_\sigma \neq \emptyset$.

(c) Ω *is strictly* $R(\theta)$*-convex.*

(d) $R(\theta)$ *satisfies* (10.10).

Proof.

"$(a) \Rightarrow (b)$" This is obvious.

"$(d) \Rightarrow (c) \Rightarrow (a)$" This holds by Proposition 10.9 and Theorem 10.6.

"$(b) \Rightarrow (d)$" This follows essentially as in [15, Theorem 10.6.6]. The crucial points are the following: fix $k \in \mathbb{N}$ and a compact set $K \subset \Omega_\sigma$. Consider the bilinear form

$$b(\phi, f) := \int_{x > \mathbf{0}} \phi(x) f(x)\, dx \text{ for } f \in \mathcal{E}(\Omega_\sigma) \text{ and } \phi \in \Phi,$$

where $\Phi := \{\phi \in \mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma^\circ) \mid \operatorname{supp}(R(-\theta - \mathbf{1})\phi) \subset K\}$. The topology is given by the seminorms $p_\alpha(\phi) := \sup_{x \in \Omega_\sigma^\circ} |x^{\mathbf{k}} \theta^\alpha R(-\theta - \mathbf{1})\phi(x)|, \phi \in \Phi$, for $\alpha \in \mathbb{N}^d$. The seminorms p_α are finite on Φ since $R(-\theta - \mathbf{1}) : \mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma) \to \mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma)$.

Notice that $R(-\theta - 1)$ is injective on $\mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma)$. To prove this we may assume that $\sigma = \mathbf{1}$. Let $R(-\theta - \mathbf{1})\phi = 0$ for some $\phi \in \mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_{\mathbf{1}}^\circ)$. Since $\mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_{\mathbf{1}}) \subset \mathcal{E}(Q_{\mathbf{1}})'$ canonically, we may apply the Mellin transform $\mathscr{M}$ and get (cf. (8.26))

$$0 = \mathscr{M}(R(-\theta - \mathbf{1})\phi) = R(\,\cdot\, - \mathbf{1})\mathscr{M}(\phi) \text{ in } \mathcal{H}_{\mathscr{M}}.$$

So $\mathscr{M}(\phi) = 0$ in $\mathcal{H}_{\mathscr{M}}$, and $\phi = 0$ in $\mathcal{E}(Q_\sigma)'$ by Theorem 8.2, i.e., $\phi = 0$ in $\mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma)$.

Since $R(-\theta - 1)$ is injective on $\mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma)$ the topology of Φ is separated. The bilinear form b is continuous w.r.t. f if ϕ is fixed. On the other hand, if $f \in \mathcal{E}(\Omega_\sigma)$ is fixed there is $g \in \mathcal{E}^{\mathbf{k}+\mathbf{m}+\mathbf{1}}(\Omega_\sigma), m := \deg(P)$, by assumption such that $R(\theta)g = f$. By partial integration we thus get

$$b(\phi, f) = b(\phi, R(\theta)g) = b\left(x^{\mathbf{k}} R(-\theta - \mathbf{1})\phi, g x^{-\mathbf{k}}\right). \tag{10.11}$$

We will prove this for $R(\theta) := \theta_1$ (the general case then follows easily):

$$\begin{aligned}
b(\phi, \theta_1 g) &= \int_{x > \mathbf{0}} (x_1 \phi(x)) \partial_1 g(x)\, dx \\
&= \lim_{\varepsilon \searrow 0} \int_{x' > \varepsilon} \int_\varepsilon^\infty (x_1 \phi(x_1, x')) \partial_1 g(x_1, x')\, dx_1\, dx' \\
&= \lim_{\varepsilon \searrow 0} \Big[-\varepsilon \int_{x' > \varepsilon} \phi(\varepsilon, x') g(\varepsilon, x')\, dx' + \int_{x' > \varepsilon} \int_\varepsilon^\infty -\partial_1 (x_1 \phi(x_1, x')) g(x_1, x')\, dx_1\, dx' \Big] \\
&= b\left((-\theta_1 - 1)\phi, g\right) = b\left(x^{\mathbf{k}} (-\theta_1 - 1)\phi, g x^{-\mathbf{k}}\right).
\end{aligned}$$

By (10.11), b is clearly continuous w.r.t. ϕ for the topology of Φ since $g x^{-\mathbf{k}}$ is locally bounded on Ω_σ since $g \in \mathcal{E}^{\mathbf{k}}(\Omega_\sigma)$.

The proof is now completed as in [15, Theorem 10.6.6]: since $\mathcal{E}(\Omega_\sigma)$ is a Fréchet space and Φ is metrizable, b is jointly continuous. Thus there are constants C, N_1, and N_2 and a compact set $\widetilde{K} \subset \Omega_\sigma$ such that for any $f \in \mathcal{E}(\Omega_\sigma)$ and any $\phi \in \Phi$,

$$\left| \int_{x > \mathbf{0}} \phi(x) f(x)\, dx \right| = |b(\phi, f)| \leq C \left(\sum_{|\alpha| \leq N_1} p_\alpha(\phi) \right) \left(\sum_{\beta \leq N_2} \sup_{x \in \widetilde{K}} |\partial^\alpha f(x)| \right).$$

Therefore $\operatorname{supp}(\phi) \subseteq \widetilde{K}$. This proves (10.10). $\square$

We finally translate strict $P(\theta)$-convexity (or rather (10.10)) into a new variant of the classical $P(\partial)$-convexity for supports. For this we have to extend (10.1) as follows: for $\sigma \in \{\pm 1\}^d$ we have the following topological isomorphisms:

$$\operatorname{Log}_\sigma : Q_\sigma \to \widetilde{\mathbb{R}}^d := [-\infty, \infty[^d \text{ and } \operatorname{Exp}_\sigma : \widetilde{\mathbb{R}}^d \to Q_\sigma, \tag{10.12}$$

where $\log(0) := -\infty$ and $\exp(-\infty) := 0$ and $\widetilde{\mathbb{R}}^d$ is endowed with its natural topology (like $[0,1[^d$). In this way we can transform spaces of (generalized) functions on Ω_σ into functions on $\operatorname{Log}_\sigma(\Omega_\sigma)$. Specifically we define for open sets $\omega \subset \widetilde{\mathbb{R}}^d$

$$\mathscr{D}^{\mathbf{k}}_{\partial,exp}(\omega) := \{f \in C^\infty(\omega \cap \mathbb{R}^d) \mid \forall \alpha \in \mathbb{N}^d : e^{\langle x, \mathbf{k}\rangle} f^{(\alpha)}(x) \text{ is bounded and } \operatorname{supp}(f) \Subset \omega\}$$

$$\text{and } \mathscr{D}_{\partial,exp}(\omega) := \bigcup_{k \in \mathbb{N}} \mathscr{D}^{\mathbf{k}}_{\partial,exp}(\omega)$$

endowed with the inductive topology. We emphasize that the topology of ω is induced from $\widetilde{\mathbb{R}}^d$, hence compact subsets of ω need not be bounded. Strict $P(\partial)$-convexity now is defined as follows.

Definition 10.11. An open set $\omega \subset \widetilde{\mathbb{R}}^d$ is strictly $P(\partial)$-convex if

$$\begin{aligned} &\forall K \Subset \omega \ \forall k \in \mathbb{N} \ \exists \widetilde{K} \Subset \omega \ \forall g \in \mathscr{D}^{\mathbf{k}}_{\partial,exp}(\omega) : \\ &\qquad \operatorname{supp}(g) \subset \widetilde{K} \text{ if } \operatorname{supp}(P(-\partial)g) \subset K. \end{aligned} \tag{10.13}$$

Since strict $P(\theta)$-convexity is formulated in the dual spaces we now consider the transpose of (10.1) (in the sense of (10.12)).

Proposition 10.12.

(a) *Let* ${}^tC_{\operatorname{Log}_\sigma}(f)(x) := f(\operatorname{Exp}_\sigma(x)) \operatorname{Exp}_\sigma(x)^{\mathbf{1}}$, *let* $f \in \mathscr{D}_{\theta,pol}(\Omega_\sigma)$, *and let* ${}^tC_{\operatorname{Exp}_\sigma}(g)(x) := g(Log_\sigma(x))x^{-\mathbf{1}}, g \in \mathscr{D}_{\partial,exp}(\operatorname{Log}_\sigma \Omega_\sigma)$. *Then*

$${}^tC_{\operatorname{Log}_\sigma} : \mathscr{D}_{\theta,pol}(\Omega_\sigma) \to \mathscr{D}_{\partial,exp}(\operatorname{Log}_\sigma \Omega_\sigma)$$

is a topological isomorphism with inverse ${}^tC_{\operatorname{Exp}_\sigma}$.

(b) Ω *satisfies* (10.10) *if and only if* $\operatorname{Log}_\sigma(\Omega_\sigma)$ *is strictly* $P(\partial)$-*convex for any* $\sigma \in \{\pm 1\}^d$.

Proof.

(a) Since $e^{\langle x,\mathbf{k}\rangle} = \sigma^{\mathbf{k}} \operatorname{Exp}_\sigma(x)^{\mathbf{k}}$ this follows from

$$\begin{aligned} \operatorname{Exp}_\sigma(x)^{\mathbf{k}} \partial_j \left[f(\operatorname{Exp}_\sigma(x)) \operatorname{Exp}_\sigma(x)^{\mathbf{1}} \right] &= [(\theta_j + 1)f](\operatorname{Exp}_\sigma(x)) \operatorname{Exp}_\sigma(x)^{\mathbf{k}+\mathbf{1}} \\ &= [x^{\mathbf{k}+\mathbf{1}}(\theta_j + 1)f](\operatorname{Exp}_\sigma(x)) \end{aligned}$$

for $f \in \mathscr{D}^{\mathbf{k}}_{\theta,pol}(\Omega_\sigma)$ and a similar line for ${}^tC_{\operatorname{Exp}_\sigma}$.

(b) This follows from (a) since the above formula for $k = 0$ implies that $P(-\partial)\left[{}^tC_{\operatorname{Log}_\sigma}(f)\right] = {}^tC_{\operatorname{Log}_\sigma}[P(-\theta - \mathbf{1})f]$. $\square$

Summarizing we have proved that each of the sufficient or necessary conditions obtained so far are in fact equivalent characterizations of surjectivity of all shifted Euler operators. Specifically, this includes part (b) of the Main Theorem B from the introduction.

Theorem 10.13. *The following are equivalent for a polynomial $P \neq 0$ and $\Omega \subset \mathbb{R}^d$ open:*

(a) *The Euler operators $P(\theta + c) : C^\infty(\Omega) \to C^\infty_{I(P(\,\cdot\,+c))}(\Omega)$ are surjective for any $c \in \mathbb{N}^d$ (and, equivalently, for any $c \in \mathbb{R}^d$).*

(b) *For any $J \subsetneq D$, any $\sigma \in \{\pm 1\}^{D\setminus J}$ with $\Omega_{D\setminus J,\sigma} \neq \emptyset$, and any $\alpha \in \mathbb{N}^J$ such that $P(\alpha,\,\cdot\,) \neq 0$, we have*

$$P(\alpha, \theta_{D\setminus J})\left(\mathcal{E}^{\mathbf{k}}(\Omega_{D\setminus J,\sigma})\right) \supset \mathcal{E}(\Omega_{D\setminus J,\sigma}).$$

(c) *For any $J \subsetneq D$, any $\sigma \in \{\pm 1\}^{D\setminus J}$ with $\Omega_{D\setminus J,\sigma} \neq \emptyset$, and any $\alpha \in \mathbb{N}^J$ such that $P(\alpha,\,\cdot\,) \neq 0$, the Euler operator*

$$P(\alpha, \theta_{D\setminus J}) : \mathcal{E}(\Omega_{D\setminus J,\sigma}) \to \mathcal{E}(\Omega_{D\setminus J,\sigma}) \text{ is surjective.}$$

(d) *For any $J \subsetneq D$ with $\Omega_{D\setminus J} \neq \emptyset$ and any $\alpha \in \mathbb{N}^J$ such that $P(\alpha,\,\cdot\,) \neq 0$, the sets $\Omega_{D\setminus J}$ are strictly $P(\alpha, \theta_{D\setminus J})$-convex.*

(e) *For any $J \subsetneq D$ with $\Omega_{D\setminus J} \neq \emptyset$ and any $\alpha \in \mathbb{N}^J$ such that $P(\alpha,\,\cdot\,) \neq 0$, the sets $\Omega_{D\setminus J}$ satisfy* (10.10).

(f) *For any $J \subsetneq D$, any $\sigma \in \{\pm 1\}^{D\setminus J}$ with $\Omega_{D\setminus J,\sigma} \neq \emptyset$, and any $\alpha \in \mathbb{N}^J$ such that $P(\alpha,\,\cdot\,) \neq 0$, the sets $\operatorname{Log}_\sigma(\Omega_{D\setminus J,\sigma})$ are strictly $P(\alpha, \partial_{D\setminus J})$-convex.*

Proof.
"$(a) \Rightarrow (b)$" By (6.4), the Hadamard operators $P(\theta + c)$ are the shifted operators for $H := P(\theta)$. Thus, if $P(\theta + c) : C^\infty(\Omega) \to C^\infty_{I(P(\,\cdot\,+c))}(\Omega)$ are surjective for any $c \in \mathbb{N}^d$, then (b) follows (for large k) by Theorem 6.4.

"$(b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$" This follows from Theorem 10.10 applied to $P(\alpha, \theta_{D\setminus J})$.

"$(e) \Leftrightarrow (f)$" This follows from Proposition 10.12 applied to $P(\alpha, \theta_{D\setminus J})$.

"$(c) \Rightarrow (a)$" By Theorem 6.5 and (c), the operators $P(\theta + c) : C^\infty(\Omega) \to C^\infty_{I(P(\,\cdot\,+c))}(\Omega)$ are surjective for any $c \in \mathbb{R}^d$ since $P(\theta) = H_T$ with $\operatorname{supp}(T) = \{\mathbf{1}\}$ by (9.1). ☐

We finally obtain part (c) of the Main Theorem B.

Theorem 10.14. *Let $P \neq 0$ be a polynomial, and let $\Omega \subset \mathbb{R}^d$ be admissible for $P(\theta)$ (cf. Definition* 7.3*). The following are equivalent to the statements of Theorem* 10.13*:*

(a) *The Euler operator $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective.*

(b) *Ω is $P(\theta)$-convex.*

Proof. The equivalence "$(a) \Leftrightarrow (b)$" was proved in Theorem 9.5. (a) implies the condition in Theorem 10.13(b) by Theorem 7.14, since Ω is admissible for $P(\theta)$. ☐

Recall that the solvability theory for Euler differential operators on $\mathscr{A}(\mathbb{R}^d)$ is very different (see the introduction and [11]).

Corollary 10.15. *Let $P \neq 0$ be a polynomial with $S(P) \subset \{\mathbf{0}\}$, and let $\Omega \subset \mathbb{R}^d$ be open. Then all statements in Theorems* 10.13 *and* 10.14 *are equivalent.*

Proof. By Proposition 7.7(b), Ω is admissible for $P(\theta)$. The claim now follows from Theorem 10.14. ☐

The reader should recall the examples of operators $P(\theta)$ with $S(P) \subset \{\mathbf{0}\}$ discussed in Example 7.9 including polynomials which are partially hypoelliptic w.r.t.

any of the variables, polynomials for which any canonical unit vector e_j is noncharacteristic, and the standard second-order Euler type operators.

Finally, we discuss the solvability of Euler operators $P(\theta)$ in two variables. Since $P_1(\theta)P_2(\theta) : C^\infty(\Omega) \to C^\infty_{I(P_1P_2)}(\Omega)$ is surjective iff $P_j(\theta) : C^\infty(\Omega) \to C^\infty_{I(P_j)}(\Omega)$ is surjective for $j = 1, 2$, we may assume that P is irreducible. By Proposition 7.6 we know that then either $P(\theta) = \theta_j - b$ for some $b \in \mathbb{N}$ and $j = 1, 2$ or $S(P) \subset \{\mathbf{0}\}$. The surjectivity of the operators $\theta_j - b, b \in \mathbb{N}, j = 1, 2$, was completely characterized in Proposition 4.8. The remaining case is solved in the next corollary.

Corollary 10.16. *Let $0 \neq P$ be a polynomial in two variables with $S(P) \subset \{\mathbf{0}\}$. For each open set $\Omega \subset \mathbb{R}^2$ the following are equivalent:*

(a) *The Euler operator $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective.*
(b) *For any $\sigma \in \{\pm 1\}^2$ with $\Omega_\sigma \neq \emptyset$ the sets $\mathrm{Log}_\sigma(\Omega_\sigma)$ are strictly $P(\partial)$-convex.*
(c) *The shifted Euler operators $P(\theta + c) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ are surjective for any $c \in \mathbb{R}^2$.*

Proof. "$(a) \Rightarrow (b)$" This follows from Corollary 10.15 and Theorem 10.13(f) (for $J := \emptyset$).

"$(b) \Rightarrow (c)$" The components of $\mathrm{Log}_\sigma(\Omega_{D\setminus J,\sigma})$ are intervals for $\emptyset \neq J \neq D$ and the restricted operators are ordinary differential operators since $d = 2$. Hence all conditions in Theorem 10.13(f) are satisfied by (b), and (c) follows. □

11. Zero solutions of Euler type operators

In this section we will study zero solutions of Euler equations (i.e., kernels of Euler operators), invertibility of Euler operators, and their spectrum. Please note that the last two topics are not interesting for linear partial differential operators $P(\partial)$ with constant coefficients since $P(\partial)$ is never invertible on $C^\infty(\Omega)$ and the spectrum therefore is always the whole complex plane $\mathbb{C}$ if $d > 1$. This also holds for Euler operators $P(\theta)$ on $C^\infty(\Omega)$ if $\Omega \cap Z_1 = \emptyset$ (use Log and Exp to transform $P(\theta)$ to $P(\partial)$). So we will mostly assume that $\mathbf{0} \in \Omega$. Notice also that $P(\theta)$ is never injective on $C^\infty(\Omega)$ if $\mathbf{0} \in P(\mathbb{N}^d)$, since then $P(\theta)p = 0$ for some monomial $p \neq 0$.

The questions we are going to discuss are the following: is it possible that the kernel of $P(\theta)$ is trivial? What happens if $P(\mathbb{N}^d)$ does not contain zero? In fact, if $P(\mathbb{N}^d)$ does not contain zero, then, by Theorem 9.4, for any m-convex set Ω (in particular, for $\Omega = \mathbb{R}^d$), the operators $P(\theta) : C^\infty(\Omega) \to C^\infty(\Omega)$ are surjective. In this case, $P(\theta)$ has a nontrivial kernel if and only if it is not invertible.

As in the preceding sections, solutions supported in the canonical quadrants are of special interest.

Theorem 11.1. *Let $\mathbf{0} \in \Omega$.*

(a) *The implications "i) $\Rightarrow$ ii) $\Rightarrow$ iii)" hold where*
  (i) *The operator $P(\theta)$ is injective on $C^\infty(\Omega)$.*
  (ii) *The operator $P(\theta)$ is injective on $C^\infty(\mathbb{R}^d)$.*
  (iii) *For any $J \subsetneq D$ and any $\alpha \in \mathbb{N}^J$ the operators $P(\alpha, \theta_{D\setminus J})$ are injective on $\mathcal{E}([0,\infty[^{D\setminus J})$ and $I(P) = \emptyset$, that is,*

$$P(\beta) \neq 0 \text{ for any } \beta \in \mathbb{N}^d. \tag{11.1}$$

(b) *Conversely,* $P(\theta)$ *is injective on* $C^\infty(\Omega)$ *if* P *satisfies* (11.1) *and also if* $P(\alpha, \theta_{D\setminus J})$ *is injective on* $\mathcal{E}(\Omega_{D\setminus J,\sigma})$ *for any* $\alpha \in \mathbb{N}^J$*, any* $J \subsetneq D$*, and any* $\sigma \in \mathbb{N}^{D\setminus J}$*.*

Proof.

(a) "i) $\Rightarrow$ ii)" For $\xi \in \mathbb{R}^d$ there is $a > \mathbf{0}$ such that $x_0 := \xi/a \in \Omega$ since $\mathbf{0} \in \Omega$. For $f \in C^\infty(\mathbb{R}^d)$ set $f_a(x) := f(ax)$. If $P(\theta)f = 0$, then $P(\theta)f_a = 0$ on Ω since dilations commute with $P(\theta)$, and therefore $f_a = 0$ on Ω by assumption, and hence $f(\xi) = f_a(x_0) = 0$.

"ii) $\Rightarrow$ iii)" The second claim is trivial since otherwise there were nontrivial monomial zero solutions for $P(\theta)$. We next show that $P(\alpha, \theta_{D\setminus J})$ is injective on $C^\infty(\mathbb{R}^{D\setminus J})$ for any $\alpha \in \mathbb{N}^J$ and any $J \subsetneq D$. Notice that we only need to show this for $J := \{j\}, j \in D$, and then use induction. Let $j = 1$ w.l.o.g. and assume that there is $\alpha \in \mathbb{N}$ such that $P(\alpha, \theta_{D'}), D' := D\setminus\{1\}$, is not injective on $C^\infty(\mathbb{R}^{D'})$. Then there is $0 \neq f \in C^\infty(\mathbb{R}^{D'})$ such that $P(\alpha, \theta')f = 0$ on $\mathbb{R}^{D'}$. Set $F(x) := f(x')x_1^\alpha$ for $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{D'}$. Then $0 \neq F \in C^\infty(\mathbb{R}^d)$ and

$$P(\theta)F(x) = x_1^\alpha P(\alpha, \theta')f(x') = 0 \tag{11.2}$$

by a simple calculation. Hence $P(\theta)$ is not injective on $C^\infty(\mathbb{R}^d)$.

Since $\mathcal{E}([0,\infty[^{D\setminus J}) \subset C^\infty(\mathbb{R}^{D\setminus J})$, (iii) follows.

(b) (I) $P(\theta)$ is injective on $C^\infty(\Omega)$ if $P(\theta)$ is injective on $\mathcal{E}(\Omega_\sigma)$ for any $\sigma \in \{\pm1\}^d$ and if $P(\alpha_j, \theta_{D'}), D' := D \setminus \{j\}$, is injective on $C^\infty(\Omega_{D'})$ for any $\alpha_j \in \mathbb{N}$ and any $j \in D$.

Proof. Let $j = 1$. Notice that for $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}$ we have

$$P(\alpha, \theta')\left[(\partial_1^\alpha f)(0, x')\right] = [\partial_1^\alpha P(\theta)f](0, x') \text{ on } \Omega_{D'}. \tag{11.3}$$

Indeed, (11.3) is to be checked only for monomials f, and this case is easy.

If $P(\theta)F = 0$ for $F \in C^\infty(\Omega)$, then $P(\alpha, \theta')\left[(\partial_1^\alpha F)(0, x')\right] = 0$ on $\Omega_{D'}$ for any $\alpha \in \mathbb{N}$ by (11.3), hence $\partial_1^\alpha F = 0$ on $\Omega_{D'}$ for any $\alpha \in \mathbb{N}$ since $P(\alpha, \theta')$ is injective on $C^\infty(\Omega_{D'})$, that is, F is flat on $\Omega \cap \{x_1 = 0\}$. By the same argument used for any $j \in D$ we see that F is flat on $\Omega \cap Z_1$, hence $F = \sum_\sigma F_\sigma$ for suitable $F_\sigma \in \mathcal{E}(\Omega_\sigma)$ satisfying $P(\theta)F_\sigma = 0$. Hence, $F_\sigma = 0$ for any σ since $P(\theta)$ is injective on $\mathcal{E}(\Omega_\sigma)$, and in conclusion $F = 0$.

(II) $P(\alpha, \theta_j)$ is injective on $C^\infty(\Omega_{\{j\}})$ for any $j \in D$ and any $\alpha \in \mathbb{N}^{D'}, D' := D \setminus \{j\}$.

Proof. Let $j = 1$, and let $P(\alpha', \theta_1)f = 0$ for some $f \in C^\infty(\Omega_{\{1\}})$. Then $\partial_1^k f(0) = 0$ for any $k \in \mathbb{N}$ since $P(\alpha', k) \neq 0$ by assumption and since

$$[\partial_1^k P(\alpha', \theta_1)f](0) = P(\alpha', k)\partial_1^k f(0)$$

(as (11.3) this is proved by checking on monomials). Hence f decomposes into two zero solutions of $P(\alpha', \theta_1)$ in $\mathcal{E}(\Omega_{\{1\},\pm1})$ which vanish by assumption. This proves the claim for $j = 1$.

(III) Using (I) (and (II)) inductively (for $\widetilde{D} := D \setminus J$ instead of D) we show that $P(\alpha, \theta_{D\setminus J})$ is injective on $C^\infty(\Omega_{D\setminus J})$ for any $\alpha \in \mathbb{N}^J$ and any $J \subset D$ with $|J| = d-2, \ldots, 0$. This proves (b). $\square$

If we additionally assume that Ω satisfies the projection property (2.5), the implications from Theorem 11.1 provide the following characterization of injectivity of $P(\theta)$. Specifically, injectivity is then inherited from $P(\theta)$ to the shifted operators $P(c + \theta)$ for $c \in \mathbb{N}^d$.

Corollary 11.2. *Let* $\mathbf{0} \in \Omega$*, and let* Ω *satisfy the projection property* (2.5)*. The following are equivalent:*

(a) *The operator* $P(\theta)$ *is injective on* $C^\infty(\Omega)$*.*
(b) *The shifted operators* $P(c+\theta)$ *are injective on* $C^\infty(\Omega)$ *for any* $c \in \mathbb{N}^d$*.*
(c) P *satisfies* (11.1) *and* $P(\alpha, \theta_{D\setminus J})$ *is injective on* $\mathcal{E}(\Omega_{D\setminus J,\sigma})$ *for any* $\alpha \in \mathbb{N}^J$*, any* $J \subsetneq D$*, and any* $\sigma \in \mathbb{N}^{D\setminus J}$*.*

Proof.
"(a) $\Rightarrow$ (c)" This is proved by a slight modification of the proof of "ii) $\Rightarrow$ iii)" in Theorem 11.1 since $F \in C^\infty(\Omega)$ if Ω has the projection property.

"(c) $\Rightarrow$ (b)" Clearly, (11.1) is inherited to the shifted polynomial $P(c+x)$ for $c \in \mathbb{N}^d$. This also holds for the second condition in (c), since the multiplication by x^c is an isomorphism in $\mathcal{E}(\Omega_\sigma)$ and since for $f \in \mathcal{E}(\Omega_\sigma)$

$$P(\theta)(x^c f(x)) = x^c P(c+\theta) f(x) \text{ on } \Omega_\sigma \tag{11.4}$$

by (6.2) and (6.4). Hence we may assume that $c = 0$, i.e., we have to show that (a) holds. Notice that (a) follows from Theorem 11.1(b). □

Clearly, injectivity (more precisely (11.1)) in general is not inherited from $P(\theta)$ to $P(c+\theta)$ for $c \in \mathbb{R}^d$, e.g., $\theta_1 + 1/2$ is injective on $C^\infty(\mathbb{R})$ while θ_1 is not.

To characterize bijectivity of Euler operators on $C^\infty(\Omega)$ we now prove a characterization of bijective Euler operators on $\mathcal{E}(\Omega_\sigma)$. We start however with a counterexample, i.e., an explicit example of a nontrivial zero solution of $P(\theta)$ with support contained in $[0,\infty[^d$.

Example 11.3. Let P be a homogeneous polynomial such that $P(y) = 0$ for some $y = (y_1, \ldots, y_d) > \mathbf{0}$. For $f \in C^\infty(\mathbb{R})$ flat at zero let g be defined by

$$g(x_1, \ldots, x_d) := \begin{cases} f(x_1^{y_1} \cdots x_d^{y_d}) & \text{if } x_1, \ldots, x_d > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then g belongs to $\ker_{C^\infty(\mathbb{R}^d)} P(\theta)$ and $\operatorname{supp} g \subseteq [0,\infty[^d$.

Proof. If $x = (x_1, \ldots, x_d)$ tends to the boundary of $[0,\infty[^d$, then $|x_1^{y_1} \cdots x_d^{y_d}|$ tends to zero since $y > 0$. Therefore g is flat at the boundary of $[0,\infty[^d$, and so $g \in C^\infty(\mathbb{R}^d)$.

Observe that

$$\theta_j g(x_1, \ldots, x_d) = y_j f'(x_1^{y_1} \cdots x_d^{y_d}) x_1^{y_1} \cdots x_d^{y_d} = y_j (\theta f)(x_1^{y_1} \cdots x_d^{y_d}).$$

Hence we have for $|\alpha| = m := \deg(P)$

$$\theta^\alpha g(x_1, \ldots, x_d) = y^\alpha (\theta^m f)(x_1^{y_1} \cdots x_d^{y_d}),$$

and therefore

$$P(\theta) g(x_1, \ldots, x_d) = P(y)(\theta^m f)(x_1^{y_1} \cdots x_d^{y_d}) = 0.$$

□

More generally, we have the following characterization if Ω_σ is connected. Slightly extending our previous definition we say here that Ω_σ is strictly $P(\theta)$-convex if (10.5) holds for Ω_σ.

Theorem 11.4. *Let $\mathbf{0} \in \Omega$, and let Ω_σ be connected. The following are equivalent:*

(a) *The operator $P(\theta) : \mathcal{E}(\Omega_\sigma) \to \mathcal{E}(\Omega_\sigma)$ is bijective.*

(b) *There is $E \in \mathcal{E}([0,\infty[^d)'$ such that $P(-\theta - \mathbf{1})E = \delta_{\mathbf{1}}$ and*

$$\mathrm{supp}_+(E) \subset V(\Omega_\sigma) := \{a \in [0,\infty[^d | \, a\Omega_\sigma \subset \Omega_\sigma\}.$$

(c) *Ω_σ is strictly $P(\theta)$-convex and there is $k \in \mathbb{N}$ such that*

$$P(z) \neq 0 \text{ for any } z \in \mathbb{C}^d \text{ such that } \mathrm{Re}\,(z) \geq \mathbf{k}. \tag{11.5}$$

(d) *Ω_σ is strictly $P(\theta)$-convex and there is $k \in \mathbb{N}$ such that*

$$|P(z)| \geq C \min_{j \leq d} |\,\mathrm{Re}\,(z_j) - k|^m \text{ for any } z \in \mathbb{C}^d \text{ such that } \mathrm{Re}\,(z) \geq \mathbf{k} \tag{11.6}$$

(here $m := \deg(P)$).

Proof.

"(a) ⇒ (c)" The first claim holds by (the proof of) Theorem 10.10. Since $P(\theta)$ is injective on $\mathcal{E}(\Omega_\sigma)$ the operator $P(\theta)$ is injective on $\mathcal{E}([0,\infty[^d)$ (use dilations as in the proof of Theorem 11.1(a)"i) ⇒ ii)"). Hence $P(\theta) : \mathcal{E}([0,\infty[^d) \to \mathcal{E}([0,\infty[^d)$ is a bijection between Fréchet spaces since $P(\theta) : \mathcal{E}([0,\infty[^d) \to \mathcal{E}([0,\infty[^d)$ is surjective by (9.5). Thus there is a continuous linear inverse

$$H : \mathcal{E}([0,\infty[^d) \to \mathcal{E}([0,\infty[^d).$$

Clearly, $\delta_{\mathbf{1}} \circ H$ is a continuous linear functional on $\mathcal{E}([0,\infty[^d)$ which extends to a distribution T with compact support contained in $[0,\infty[^d$. There are a compact set $L \subset [0,\infty[^d$, $C > 0$, and $q \in \mathbb{N}$ such that

$$|T(f)| \leq C\|f\|_{L,q} \qquad \text{with } \|f\| = \sup_{x \in L, |\alpha| \leq q} |\partial^\alpha f(x)|.$$

Therefore $T \in \mathcal{E}^{\mathbf{q}}([0,\infty[^d)'$ and $P(\theta) : \mathcal{E}^{\mathbf{q}+\mathbf{m}}([0,\infty[^d) \to \mathcal{E}^{\mathbf{q}}([0,\infty[^d)$ for $m := \deg(P)$. Hence

$$T \circ P(\theta) \in \mathcal{E}^{\mathbf{q}+\mathbf{m}}([0,\infty[^d)'.$$

Since

$$T \circ P(\theta) = \delta_{\mathbf{1}} \circ H \circ P(\theta) = \delta_{\mathbf{1}} \circ \mathrm{id} \, = \delta_{\mathbf{1}} \quad \text{on}\, \mathcal{E}([0,\infty[^d),$$

by density

$$T \circ P(\theta) = \delta_{\mathbf{1}} \text{ on } \mathcal{E}^{\mathbf{q}+\mathbf{m}}([0,\infty[^d).$$

On the other hand, if $\mathrm{Re}\,(z) > \mathbf{q} + \mathbf{m} + \mathbf{1}$ and $P(z - \mathbf{1}) \neq 0$ we get by the definition of the Mellin transform

$$\begin{aligned}\mathscr{M}(T)(z) = T(g_{z-\mathbf{1}}) &= T\left(\frac{1}{P(z-\mathbf{1})} \cdot P(z-\mathbf{1}) \cdot g_{z-\mathbf{1}}\right) \\ &= \frac{1}{P(z-\mathbf{1})}\left(T \circ P(\theta)\right)(g_{z-\mathbf{1}}) = \frac{1}{P(z-\mathbf{1})} \cdot \delta_{\mathbf{1}}(g_{z-\mathbf{1}}) = \frac{1}{P(z-\mathbf{1})}.\end{aligned}$$

Since the left hand side is holomorphic for Re $z > \mathbf{q}+\mathbf{1}$, thus $P(z) \neq 0$ if Re $z > \mathbf{q}$.

"(c) ⇒ (d)" This follows by the proof of [11, Theorem 8.1"(c) ⇒ (d)"] applied to $P(\mathbf{k} + \;\cdot\;)$.

"(d) ⇒ (b)" (i) By (11.6) we know that $\frac{1}{|P(z)|} \le C$ if $\mathrm{Re}\,(z) \ge \mathbf{k}+\mathbf{1}$. By Theorem 8.2, it follows that there is $E \in \mathcal{E}([0,\infty[^d)'$, $\mathrm{supp}_+(E) \subset [0,1]^d$, with Mellin transform equal to $\frac{1}{P(z-\mathbf{1})}$. For $g \in \mathscr{D}([0,\infty[^d)$ we get for $C > k+2$ fixed,

$$\langle P(-\theta-1)E, g\rangle = \langle E, P(\theta)g\rangle = \frac{1}{(2\pi i)^d}\int_{\mathbf{C}+i\mathbb{R}^d} \frac{\mathscr{M}(P(\theta)g)(-\tau)}{P(\tau)}\,d\tau$$
$$= \frac{1}{(2\pi i)^d}\int_{\mathbf{C}+i\mathbb{R}^d} \mathscr{M}(g)(-\tau)\,d\tau = g(\mathbf{1}) = \langle \delta_{\mathbf{1}}, g\rangle$$

by (8.14), (8.2) and Corollary 8.7.

(ii) Let $W := \{a \in \Omega_\sigma \mid a\,\mathrm{supp}_+(E) \subset \Omega_\sigma\}$. If we show that $W = \Omega_\sigma$, then $\mathrm{supp}_+(E) \subset V(\Omega_\sigma)$. Obviously, $W \ne \emptyset$ since $[0,\delta]^d \subset W$ for some $\delta > 0$ since $\mathbf{0} \in \Omega$. Clearly, W is open in Ω_σ since $\mathrm{supp}_+(E)$ is compact. Let $a_n \in W$, and let $a_n \to a \in \Omega_\sigma$. Then $K := \{a, a_n \mid n \in \mathbb{N}\}$ is compact in Ω_σ. Choose a compact set $\widetilde{K} \subset \Omega_\sigma$ for K by (10.5) (for the order k of E). Then

$$a_n\,\mathrm{supp}_+(E) = \mathrm{supp}_+({}^tD_{a_n}(E)) \subset \widetilde{K}$$

since $\mathrm{supp}_+(P(-\theta-\mathbf{1})\,{}^tD_{a_n}(E)) = \mathrm{supp}_+(\delta_{a_n}) = a_n \subset K$ (notice that ${}^tD_{a_n}(E) \in \mathcal{E}(\Omega_\sigma)'$ since $\mathrm{supp}_+({}^tD_{a_n}(E)) \subset \Omega_\sigma$ since $a_n \in W$). Hence $a\,\mathrm{supp}_+(E) \subset \widetilde{K} \subset \Omega_\sigma$, that is, $a \in W$. Since Ω_σ is connected we have shown that $W = \Omega_\sigma$.

"(b) ⇒ (a)" Similarly as in the Representation Theorem 3.1 we set for $f \in \mathcal{E}(\Omega_\sigma)$

$$H(f)(x) := \langle {}_yE, f(xy)\rangle \text{ for } x \in \Omega_\sigma.$$

$H(f)$ is defined on Ω_σ since $\mathrm{supp}_+(E) \subset V(\Omega_\sigma)$. Also, $H(f) \in \mathcal{E}(\Omega_\sigma)$ (use (3.4) first on Ω_σ°). H is an inverse of $P(\theta)$ on $\mathcal{E}(\Omega_\sigma)$. Indeed, we have for $f \in \mathcal{E}(\Omega_\sigma)$

$$\begin{aligned} P(\theta_x)H(f)(x) &= \langle {}_yE, P(\theta_x)f(xy)\rangle = \langle {}_yE, (P(\theta)f)(xy)\rangle = H(P(\theta)f)(x)\\ &= \langle {}_yE, P(\theta_y)f(xy)\rangle = \langle P(-\theta_y-\mathbf{1})_yE, f(xy)\rangle\\ &= \langle {}_y\delta_{\mathbf{1}}, f(xy)\rangle = f(x) \text{ on } \Omega_\sigma. \end{aligned}$$

□

Now the central result of this section is the following theorem. Here Ω is called *sufficiently connected* if $\mathbf{0} \in \Omega$ and if $\Omega_{J,\sigma}$ is connected for any $\emptyset \ne J \subset D$ and any $\sigma \in \{\pm 1\}^J$.

Theorem 11.5. *Let Ω be sufficiently connected. The following are equivalent:*

(a) *The operator $P(\theta) : C^\infty(\Omega) \to C^\infty(\Omega)$ is bijective.*
(b) *The shifted operators $P(c+\theta) : C^\infty(\Omega) \to C^\infty(\Omega)$ are bijective for any $c \in \mathbb{N}^d$.*
(c) *The operator $P(\theta) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective and $P(\theta)$ is injective on $C^\infty(\mathbb{R}^d)$.*
(d) *$\Omega_{D\setminus J}$ is strictly $P(\alpha, \theta_{D\setminus J})$-convex for any $J \subsetneq D$ and any $\alpha \in \mathbb{N}^J$, and there is $k \in \mathbb{N}$ such that*

$$P(z) \ne 0 \text{ if } z \in (\mathbb{C}_{\ge k})^d, \tag{11.7}$$

where $\mathbb{C}_{\ge k} := \{0, \dots, k-1\} \cup \{z \in \mathbb{C} \mid \mathrm{Re}\,(z) \ge k\} \supset \mathbb{N}$.

Proof. "(a) ⇒ (c)" This holds by Theorem 11.1(a).

"(c) ⇒ (d)" (i) Notice that $\Omega_{D\setminus J} \ne \emptyset$ since $0 \in \Omega$ by assumption. Hence $\Omega_{D\setminus J}$ is strictly $P(\alpha, \theta_{D\setminus J})$-convex for any $J \subsetneq D$ and any $\alpha \in \mathbb{N}^J$ by Corollary 10.15

since $I(P)=\emptyset$ by Theorem 11.1(a). By Theorem 11.1(a) and (9.5), $P(\alpha,\theta_{D\setminus J}) : \mathcal{E}([0,\infty[^{D\setminus J}) \to \mathcal{E}([0,\infty[^{D\setminus J})$ is bijective for any α and J as above, hence Theorem 11.4 implies that

$$\forall J\subset D\ \forall \alpha_J\in\mathbb{N}^J\ \exists k_{\alpha_J}\in\mathbb{N}\ \forall z_{D\setminus J}\in\mathbb{C}^{D\setminus J}: \quad P(\alpha_J,z_{D\setminus J})\neq 0 \text{ if } \operatorname{Re}(z_{D\setminus J})\geq \mathbf{k}_{\alpha_J}. \tag{11.8}$$

Notice that (11.8) for $J=D$ coincides with (11.1), while (11.8) for $J=\emptyset$ means that there is $k_\emptyset\in\mathbb{N}$ such that $P(z)\neq 0$ if $\operatorname{Re}(z)\geq \mathbf{k}_\emptyset$.

(ii) The conditions (11.7) and (11.8) are equivalent.

Proof. "⇒" Clearly, (11.8) is satisfied with $k_{\alpha_J}:=k$ for k from (11.7) since $\mathbb{N}\subset\mathbb{C}_{\geq k}$.

"⇐" Let $k_0:=k_\emptyset$ from (11.8). Using k_{α_J} from (11.8) we define k_j inductively for $j=1,\dots,d-1$ by

$$k_j:=\max\{k_{j-1},k_{\alpha_J}\mid J\subset D,\alpha_J\in\mathbb{N}^J:|J|=j \text{ and } \alpha_J\leq \mathbf{k_{j-1}}\}.$$

Then (11.7) is satisfied for k_{d-1}.

"(d) ⇒ (b)" Notice that the conditions in (d) are inherited to $P(\theta+c)$ for $c\in\mathbb{N}^d$ (cf. (11.4)). Hence we can assume that $c=0$, that is, we have to show that (a) holds. $P(\theta):C^\infty(\Omega)\to C^\infty(\Omega)$ is surjective by Theorem 10.13 since $I(P)=\emptyset$ by (11.7) since $\mathbb{N}\subset\mathbb{C}_{\geq k}$. By (11.8) and strict $P(\alpha,\theta_{D\setminus J})$-convexity of $\Omega_{D\setminus J}$ we get from Theorem 11.4 that the operators $P(\alpha_J,\theta_{D\setminus J}):\mathcal{E}(\Omega_{D\setminus J,\sigma})\to\mathcal{E}(\Omega_{D\setminus J,\sigma})$ are bijective (hence injective) for all σ,α, and J as above. By Theorem 11.1(b) we conclude that $P(\theta):C^\infty(\Omega)\to C^\infty(\Omega)$ is injective.

"(b) ⇒ (a)" This is trivial. □

Remark 11.6. Ω is sufficiently connected in each of the following cases:

(i) $\mathbf{0}\in\Omega$, Ω satisfies the projection property, and Ω_σ is connected for any $\sigma\in\{\pm1\}^d$.

(ii) Ω is star shaped with center $\mathbf{0}$.

(iii) Ω is nearly solid.

(iv) Ω is m-convex and $\mathbf{0}\in\Omega$.

Proof.

(i) By (i), the canonical projection $\pi_j:\Omega\to\{0\}\times\Omega_{D\setminus\{j\}}$ is surjective for any $j\in D$, and therefore $\pi_J:=\prod_{j\in J}\pi_j:\Omega_\sigma\to\{0_J\}\times\Omega_{D\setminus J,\sigma_{D\setminus J}}$ is surjective for any $\emptyset\neq J\subsetneq D$ and any $\sigma\in\{\pm1\}^d$. Hence connectedness is inherited to $\Omega_{D\setminus J,\sigma_{D\setminus J}}$.

(ii) This is evident, since Ω_σ is star shaped for any σ, and since this property is inherited by the canonical projections.

(iii) This follows from ii), since nearly solid sets are star shaped since $tx=(t,\dots,t)x\in\Omega$ for any $t\in[0,1]$ and any $x\in\Omega$.

(iv) This follows from (iii) by Proposition 2.3. □

Using Theorem 9.4 as well we get the following.

Corollary 11.7. *Let Ω be nearly solid or let Ω be m-convex and $\mathbf{0}\in\Omega$. The following are equivalent:*

(a) *The operator $P(\theta):C^\infty(\Omega)\to C^\infty(\Omega)$ is bijective.*
(b) *The shifted operators $P(c+\theta):C^\infty(\Omega)\to C^\infty(\Omega)$ are bijective for any $c\in\mathbb{N}^d$.*

(c) *The operator* $P(\theta)$ *is injective on* $C^\infty(\mathbb{R}^d)$.

(d) P *satisfies* (11.7).

Proof. "(a) $\Leftrightarrow$ (b) $\Rightarrow$ (c)" This holds by Remark 11.6 and Theorem 11.5.

"(c) $\Rightarrow$ (d)" This follows by Theorem 11.5 and Theorem 9.4 (applied to $\Omega := \mathbb{R}^d$) since $I(P) = \emptyset$ by Theorem 11.1(a).

"(d) $\Rightarrow$ (a)" If $\mathbf{0} \in \Omega$ and Ω is m-convex, then Ω is nearly solid by Proposition 2.3. Hence Ω is nearly solid in any case and

$$[0,1]^d \subset V(\Omega_\sigma) \text{ for any } \sigma \in \{\pm 1\}^d. \tag{11.9}$$

By the proof of Theorem 11.4 "(d) $\Rightarrow$ (b)" (i), (11.7) implies that there is $E \in \mathcal{E}([0,\infty[^d)'$ such that $P(-\theta - \mathbf{1})E = \delta_{\mathbf{1}}$ and $\operatorname{supp}_+(E) \subset [0,1]^d \subset V(\Omega_\sigma)$ by (11.9). Hence $P(\theta) : \mathcal{E}(\Omega_\sigma) \to \mathcal{E}(\Omega_\sigma)$ is bijective for any σ by Theorem 11.4. Since the assumptions are inherited to restrictions of variables, the same argument shows that $P(\alpha, \theta_{D\setminus J}) : \mathcal{E}(\Omega_{D\setminus J,\sigma}) \to \mathcal{E}(\Omega_{D\setminus J,\sigma})$ is bijective for any $J \subsetneq D$, any $\alpha \in \mathbb{N}^J$, and σ. Hence $\Omega_{D\setminus J}$ is strictly $P(\alpha, \theta_{D\setminus J})$-convex for these J and α, and (a) follows by Theorem 11.5 and (11.7). $\square$

We emphasize that the above result implies that there are plenty of polynomials P such that $P(\theta)$ has a trivial kernel in $C^\infty(\mathbb{R}^d)$ (compare also Proposition 4.1), a phenomenon which cannot occur for partial differential operators with constant coefficients if $d > 1$. Standard examples are connected to polynomials having some half-plane property. These properties also turned out to be essential when characterizing surjectivity of $P(\theta)$ on the spaces $\mathscr{A}(\Omega)$ of real analytic functions; see [11]. Recall that a polynomial P has the (closed) half-plane property if

$$P(z) \neq 0 \text{ if } \operatorname{Re}(z) > \mathbf{0} \text{ (and if } \operatorname{Re}(z) \geq \mathbf{0}\text{, respectively).} \tag{11.10}$$

A collection of results on these properties is presented in the survey paper [6]. Specifically, the following classes of polynomials have the half-plane property (cf. [11, Theorem 9.1]):

(i) Elementary symmetric polynomials.

(ii) Each polynomial $P(x) := \det(A\operatorname{diag}(x)A^*)$, where A is a complex $r \times d$ matrix and $\operatorname{diag}(x)$ denotes the $d \times d$ diagonal matrix with coefficients $x_1, \dots, x_d$.

(iii) Each quadratic form P proportional to a form with real nonnegative coefficients such that the matrix has exactly one strictly positive eigenvector. P has the closed half-plane property if additionally all diagonal coefficients of the matrix are strictly positive.

Obviously, each polynomial having the closed half-plane property satisfies (11.7), specifically, each shifted polynomial $P(c + \,\cdot\,), c > \mathbf{0}$, satisfies (11.7) if P has the half-plane property. To present a corresponding operator of order two: if $Q(x) = x_1x_2 + x_1x_3 + x_2x_3 + 2(x_1 + x_2 + x_3) + 3$, then $Q(\theta)$ is invertible on $C^\infty(\Omega)$ for nearly solid Ω. Indeed, $P(x) := x_1x_2 + x_1x_3 + x_2x_3$ has the half-plane property by (i) so $Q(x) = P(x + \mathbf{1})$ has the closed half-plane property and Corollary 11.7 above applies (cf. also Corollary 11.11).

Corollary 11.8. *For every open nearly solid set* $\Omega \subseteq \mathbb{R}^d$ *the spectrum of* $0 \neq P(\theta)$ *on* $C^\infty(\Omega)$ *is equal to*

$$\sigma(P(\theta)) = \bigcap_{k\in\mathbb{N}} P\left((\mathbb{C}_{\geq k})^d\right) \supset P(\mathbb{N}^d).$$

In particular, for $d = 1$ *we have* $\sigma(P(\theta)) = P(\mathbb{N})$.

Proof. This follows from Corollary 11.7 (recall that $\mathbb{N} \subset \mathbb{C}_{\geq k}$ for any $k \in \mathbb{N}$). In case $d = 1$,

$$\bigcap_{k\in\mathbb{N}} P\left(\{z \in \mathbb{C} \mid \operatorname{Re} z \geq k\}\right) = \emptyset$$

since $|P(z)| \to \infty$ if $|z| \to \infty$. So the claim follows. □

Using the spectral mapping theorem for resolvents on general locally convex spaces ([1, Theorem 1.1]; see also [2]) we obtain the following.

Corollary 11.9. *For every open nearly solid set $\Omega \subseteq \mathbb{R}^d$ the spectrum of the inverse $P(\theta)^{-1}$ of an invertible operator $P(\theta)$ on $C^\infty(\Omega)$ is equal to*

$$\sigma(P(\theta)^{-1}) = \bigcap_{k\in\mathbb{N}} \left\{ \frac{1}{P(z)} \mid z \in (\mathbb{C}_{\geq k})^d \right\}.$$

We finally come back to the half-plane property and its connection to the present theory. For a polynomial P let P_{m_J} be the principal part of $P(0_J, \,\cdot\,)$ for $J \subsetneq D$. Specifically, we have for the principal part $P_m = P_{m_\emptyset}$ in this notation. Notice that P_{m_J} in general is not equal to $P_m(0_J, \,\cdot\,)$.

Proposition 11.10. *Let P satisfy* (11.7)*. Then P_{m_J} has the half-plane property for any $J \subsetneq D$.*

Proof. Since (11.7) is inherited by setting some variables equal to 0, we need to show this only for $J = \emptyset$, i.e., for the principal part P_m of P. For $z \in \mathbb{C}^d$ with $\operatorname{Re}(z) > \mathbf{0}$ fixed we set $N \in \mathbb{R}^d$ such that $Q(\xi) := P_m(z + \xi N), \xi \in \mathbb{C}$ is not identically 0. Set $Q_t(\xi) := P(t(z + \xi N))t^{-m}$ for $\xi \in \mathbb{C}$. Then $Q_t(\xi) \to Q(\xi)$ for $t \to \infty$ uniformly on $D_\varepsilon := \{\xi \in \mathbb{C} \mid |\xi| \leq \varepsilon\}, \varepsilon > 0$. If ε is small, then $\operatorname{Re}(t(z + \xi N)) > \mathbf{k}$ for large t and $\xi \in D_\varepsilon$, and hence $Q_t(\xi) \neq 0$ for $\xi \in D_\varepsilon$ if t is large. Hence the theorem of Hurwitz implies that $Q(\xi) \neq 0$ if $\xi \in D_\varepsilon$; hence $0 \neq Q(0) = P_m(z)$. □

Corollary 11.11. *Let $\Omega \subset \mathbb{R}^d$ be an open nearly solid set, and let P be a nonzero homogeneous polynomial. Let $\sigma(P(\theta))$ denote the spectrum of $P(\theta)$ on $C^\infty(\Omega)$. The following are equivalent:*

(a) *P has the half-plane property.*
(b) *We have $\sigma(P(\theta)) \neq \mathbb{C}$.*
(c) *There is $r > 0$ such that*

$$\{0 \neq z \in \mathbb{C} \mid |z| < r\} \cap \sigma(P(\theta)) = \emptyset.$$

(d) *The operator $P(\theta)$ is injective on $\mathcal{E}([0, \infty[^d)$.*
(e) *There is a lower-order term perturbation Q of P such that $Q(\theta)$ is invertible (or, equivalently, injective) on $C^\infty(\Omega)$.*
(f) *The operators $P(c+\theta)$ are invertible (or, equivalently, injective) on $C^\infty(\Omega)$ for $\mathbf{0} < c \in \mathbb{R}^d$.*

Proof. "(a) ⇒ (c)" By [6, Prop.2.1], for every $J \subset D$, the polynomial $P(0_J, z_{D\setminus J})$ is either $\equiv 0$ or it has the half-plane property. In the first case, for $\lambda \neq 0$ the operator $P(0_J, \theta_{D\setminus J}) - \lambda = -\lambda$ is always invertible. In the second case, by [11, Th. 8.1 (d)], there is a constant $c_j > 0$ such that

$$|P(0_J, z_{D\setminus J})| \geq c_J \min_{j\in D\setminus J} |\operatorname{Re}\, z_j|^m \text{ for } m := \deg(P).$$

Thus if $r = \min_J c_J$, $\lambda \neq 0$, and $|\lambda| < r$ the polynomial $P(0_J, z_{D\setminus J}) - \lambda$ has no zeros for Re $z_{D\setminus J} \geq \mathbf{1}$. Repeating this for all $J \subset D$, by Corollary 11.7, we get that $P(\theta) - \lambda$ is invertible on $C^\infty(\Omega)$.

"(c) $\Rightarrow$ (b)" This is trivial.

"(b) $\Rightarrow$ (a)" Since $P(\theta)$ vanishes on constant functions we have $0 \in \sigma(P(\theta))$. Fix $0 \neq \lambda \in \mathbb{C}$. If P does not have the half-plane property there is z with $P(z) = 0$ and Re $z > \mathbf{2}$. Since $P \neq 0$ there is $w \in \mathbb{C}^d$, $|w| < 1$, such that $P(z + w) \neq 0$. Let $Q(t) := P(z + tw), t \in \mathbb{C}$. Then $Q \neq 0$ and $Q(0) = 0$. Since Q is an open map, there are $t_n \to 0$ such that $0 \neq Q(t_n)$ has the same argument as λ. Thus

$$P\left(|\lambda/Q(t_n)|^{1/m}(z + t_n w)\right) = |\lambda/Q(t_n)|\, P(z + t_n w) = \lambda.$$

Since $|Q(t_n)| \to 0$ as $n \to \infty$, for any $k \in \mathbb{N}$ we have $\mathrm{Re}\,(|\lambda/Q(t_n)|^{1/m}(z+t_n w)) \geq \mathbf{k}$ for large n. By Corollary 11.7, $P(\theta) - \lambda$ is not invertible on $C^\infty(\Omega)$.

"(a) $\Leftrightarrow$ (d)" This follows by Theorem 11.4 since $P(\theta)$ is surjective on $\mathcal{E}([0, \infty[^d)$ by (9.5).

"(a) $\Rightarrow$ (f)" Clearly $P(c + z) \neq 0$ if $\mathrm{Re}\,(z) \geq \mathbf{0}$ and $c > \mathbf{0}$. Hence (f) holds by Corollary 11.7.

"(f) $\Rightarrow$ (e)" This is trivial.

"(e) $\Rightarrow$ (a)" If $Q(\theta)$ is injective on $C^\infty(\Omega)$, then $Q(\theta)$ is injective on $\mathcal{E}([0, \infty[^d)$ by Theorem 11.1(a). Hence Q satisfies (11.5) by Theorem 11.4, and (a) follows by the proof of Proposition 11.10. □

References

[1] Angela A. Albanese, José Bonet, and Werner J. Ricker, *Dynamics and spectrum of the Cesàro operator on $C^\infty(\mathbb{R}_+)$*, Monatsh. Math. **181** (2016), no. 2, 267–283. MR3539935

[2] Angela A. Albanese, José Bonet, and Werner J. Ricker, *Erratum to: Dynamics and spectrum of the Cesàro operator on $C^\infty(\mathbb{R}_+)$ [MR3539935]*, Monatsh. Math. **181** (2016), no. 4, 991–993. MR3563310

[3] Alexander D. Poularikas (ed.), *The transforms and applications handbook*, The Electrical Engineering Handbook Series, CRC Press, Boca Raton, FL; IEEE Press, New York, 1996. MR1407750

[4] Jan-Erik Björk, *Analytic $\mathcal{D}$-modules and applications*, Mathematics and its Applications, vol. 247, Kluwer Academic Publishers Group, Dordrecht, 1993. MR1232191

[5] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy, *Real algebraic geometry*, Translated from the 1987 French original; Revised by the authors, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 36, Springer-Verlag, Berlin, 1998. MR1659509

[6] Young-Bin Choe, James G. Oxley, Alan D. Sokal, and David G. Wagner, *Homogeneous multivariate polynomials with the half-plane property*, Special issue on the Tutte polynomial, Adv. in Appl. Math. **32** (2004), no. 1-2, 88–187. MR2037144

[7] Paweł Domański and Michael Langenbruch, *Representation of multipliers on spaces of real analytic functions*, Analysis (Munich) **32** (2012), no. 2, 137–162. MR3043719

[8] Paweł Domański and Michael Langenbruch, *Algebra of multipliers on the space of real analytic functions of one variable*, Studia Math. **212** (2012), no. 2, 155–171. MR3008439

[9] Paweł Domański and Michael Langenbruch, *Hadamard multipliers on spaces of real analytic functions*, Adv. Math. **240** (2013), 575–612. MR3046319

[10] Paweł Domański, Michael Langenbruch, and Dietmar Vogt, *Hadamard type operators on spaces of real analytic functions in several variables*, J. Funct. Anal. **269** (2015), no. 12, 3868–3913. MR3418073

[11] Paweł Domański and Michael Langenbruch, *Interpolation of holomorphic functions and surjectivity of Taylor coefficient multipliers*, Adv. Math. **293** (2016), 782–855. MR3474335

[12] Paweł Domański and Michael Langenbruch, *Multiplier projections on spaces of real analytic functions in several variables*, Complex Var. Elliptic Equ. **62** (2017), no. 2, 241–268. MR3575864

[13] P. Domański and M. Langenbruch, *Surjectivity of Hadamard type operators on spaces of smooth functions*, Rev. R. Acad. Cienc. Exactas Ffs. Nat. Ser. A Mat. RACSAM 113(2) (2018), 1625–1676.

[14] L. Ehrenpreis, *Solution of some problems of division. IV. Invertible and elliptic operators*, Amer. J. Math. **82** (1960), 522–588. MR0119082

[15] Lars Hörmander, *The analysis of linear partial differential operators. II*, Differential operators with constant coefficients, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 257, Springer-Verlag, Berlin, 1983. MR705278

[16] Ryuichi Ishimura, *Sur les équations différentielles d'ordre infini d'Euler* (French), Mem. Fac. Sci. Kyushu Univ. Ser. A **44** (1990), no. 1, 1–10. MR1052859

[17] B. Ja. Levin, *Distribution of zeros of entire functions*, Revised edition, Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields, and H. P. Thielman, Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980. MR589888

[18] B. Malgrange, *Équations différentielles à coefficients polynomiaux* (French), Progress in Mathematics, vol. 96, Birkhäuser Boston, Inc., Boston, MA, 1991. MR1117227

[19] Reinhold Meise and Dietmar Vogt, *Introduction to functional analysis*, Translated from the German by M. S. Ramanujan and revised by the authors, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press, Oxford University Press, New York, 1997. MR1483073

[20] Reinhard Mennicken, *Perturbations of semi-Fredholm operators in locally convex spaces*, Functional analysis, holomorphy, and approximation theory (Rio de Janeiro, 1979), Lecture Notes in Pure and Appl. Math., vol. 83, Dekker, New York, 1983, pp. 233–304. MR687995

[21] Boris Sagraloff, *Normale Auflösbarkeit bei linearen partiellen Differentialoperatoren in lokalen gewichteten Sobolevräumen* (German), J. Reine Angew. Math. **310** (1979), 131–150. MR546668

[22] Boris Sagraloff, *Normal solvability of linear partial differential operators in $C^\infty(\Omega)$*, Geometrical approaches to differential equations (Proc. Fourth Scheveningen Conf., Scheveningen, 1979), Lecture Notes in Math., vol. 810, Springer, Berlin, 1980, pp. 290–305. MR589214

[23] R. T. Seeley, *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. **15** (1964), 625–626. MR0165392

[24] Dietmar Vogt, *Operators of Hadamard type on spaces of smooth functions*, Math. Nachr. **288** (2015), no. 2-3, 353–361. MR3310518

[25] D. Vogt, *$\mathcal{E}'$ as an algebra by multiplicative convolution*, Funct. Approx. Comment. Math. 59.1 (2018), 117–128.

[26] Hassler Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63–89. MR1501735

[27] A. H. Zemanian, *Distribution theory and transform analysis*, An introduction to generalized functions, with applications, 2nd ed., Dover Publications, Inc., New York, 1987. MR918977

[28] A. H. Zemanian, *Generalized integral transformations*, 2nd ed., Dover Publications, Inc., New York, 1987. MR896486

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland

Department of Mathematics, University of Oldenburg, D–26111 Oldenburg, Germany
Email address: michael.langenbruch@uni-oldenburg.de