TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 372, Number 9, 1 November 2019, Pages 6183–6211
https://doi.org/10.1090/tran/7745
Article electronically published on January 16, 2019

MIXED MULTIPLICITIES OF FILTRATIONS

STEVEN DALE CUTKOSKY, PARANGAMA SARKAR, AND HEMA SRINIVASAN

Abstract. In this paper we define and explore properties of mixed multiplicities of (not necessarily Noetherian) filtrations of m_R-primary ideals in a Noetherian local ring R, generalizing the classical theory for m_R-primary ideals. We construct a real polynomial whose coefficients give the mixed multiplicities. This polynomial exists if and only if the dimension of the nilradical of the completion of R is less than the dimension of R, which holds, for instance, if R is excellent and reduced. We show that many of the classical theorems for mixed multiplicities of m_R-primary ideals hold for filtrations, including the famous Minkowski inequalities of Teissier, and of Rees and Sharp.

1. Introduction

The theory of mixed multiplicities of m_R-primary ideals in a Noetherian local ring R with maximal ideal m_R was initiated by Bhattacharya [2], Rees [28], and Teissier and Risler in the paper [33]. In this paper we extend mixed multiplicities to arbitrary—that is, not necessarily Noetherian—filtrations of R by m_R-primary ideals and explore their properties.

An account of the history of the Minkowski inequalities of mixed multiplicities is given in [12]. This article explains the origins of this subject in Teissier's work on equisingularity [33] and gives many important references. A survey of the theory of mixed multiplicities of ideals, with proofs, can be found in [32, Chapter 17]. We refer to this book for references to many important results in this area. We particularly mention [32, Sections 17.1–17.3], which develops the theory of joint reductions, including discussion of the results of the papers of Rees [29] and of Swanson [31]. A further development is by Katz and Verma [19] , who generalized mixed multiplicities to ideals which are not all m_R-primary. Trung and Verma [36] computed mixed multiplicities of monomial ideals from mixed volumes of suitable polytopes. Mixed multiplicities are used by Huh in the analysis of the coefficients of the chromatic polynomial of graph theory in [14].

The starting point of our investigation is the following theorem, which allows one to define the multiplicity of a filtration of R by m_R-primary ideals. As the theorem shows, one must impose the condition that the dimension of the nilradical of the completion $\hat{R}$ of R is less than the dimension of R. Let $\lambda(M)$ denote the length of an R-module M.

Received by the editors May 3, 2018, and, in revised form, October 25, 2018.
2010 *Mathematics Subject Classification.* Primary 13H15; Secondary 14C17.
The first author was partially supported by NSF grant DMS-1700046.
The second author was supported by IUSSTF, SERB Indo-U.S. Postdoctoral Fellowship 2017/145, and DST-INSPIRE India.

Theorem 1.1 ([7, Theorem 1.1], [8, Theorem 4.2]). *Suppose that R is a Noetherian local ring of dimension d, and that $N(\hat{R})$ is the nilradical of the m_R-adic completion $\hat{R}$ of R. Then the limit*

$$\lim_{n\to\infty} \frac{\lambda(R/I_n)}{n^d} \tag{1}$$

exists for any filtration $\mathcal{I}=\{I_n\}$ of R by m_R-primary ideals if and only if $\dim N(\hat{R}) < d$.

The nilradical $N(R)$ of a d-dimensional ring R is

$$N(R) = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}.$$

We have that the dimension of the R-module $N(R)$ is $\dim N(R) = d$ if and only if there exists a minimal prime P of R such that $\dim R/P = d$ and R_P is not reduced.

The problem of existence of such limits (1) has also been considered by Ein, Lazarsfeld, and Smith [11] and Mustaţă [25]. In the case in which the ring R is a domain and is essentially of finite type over an algebraically closed field k with $R/m_R = k$, Lazarsfeld and Mustaţă [22] showed that the limit exists for all filtrations of R by m_R-primary ideals. All of these assumptions are necessary in their proof.

The following is a very simple example of a filtration of m_R-primary ideals such that the above limit is not rational. Let k be a field, and let $R = k[[x]]$ be a power series ring over k. Let $I_n = (x^{\lceil n\sqrt{2}\rceil})$, where $\lceil \alpha \rceil$ is the round up of a real number α (the smallest integer which is greater than or equal to α). Then $\{I_n\}$ is a graded family of m_R-primary ideals such that

$$\lim_{n\to\infty} \frac{\lambda(R/I_n)}{n} = \sqrt{2}$$

is an irrational number.

There are also irrational examples determined by the valuation ideals of a discrete valuation. In [10, Example 6] an example is given of a normal three-dimensional local ring R which is essentially of finite type over a field of arbitrary characteristic and a divisorial valuation ν on the quotient field of R which dominates R such that the filtration of m_R-primary ideals $\{I_n\}$ defined by

$$I_n = \{f \in R \mid \nu(f) \geq n\}$$

satisfies the condition that the limit

$$\lim_{n\to\infty} \frac{\lambda(R/I_n)}{n^3}$$

is irrational.

Non-Noetherian filtrations ($\oplus_{n\geq 0} I_n$ not Noetherian) occur naturally in commutative algebra. The filtration of ideals determined by a divisorial valuation which dominates a normal local ring is generally not Noetherian. For instance, the condition that a two-dimensional normal local ring R satisfies the condition that this filtration is Noetherian for all divisorial valuations dominating R is the condition (N) of Muhly and Sakuma [24]. It is proven in [5] that a complete normal local ring of dimension 2 satisfies condition (N) if and only if its divisor class group is a torsion group.

The existence of mixed multiplicities of (not necessarily Noetherian) filtrations $\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$ of m_R-primary ideals is established in Theorem 6.1. Let M be a finitely generated R-module. In Theorems 6.1 and 6.6 it is shown that the function

$$(2) \qquad P(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\lambda(M/I(1)_{mn_1} \cdots I(r)_{mn_r} M)}{m^d}$$

is equal to a homogeneous polynomial $G(n_1, \ldots, n_r)$ of total degree d with real coefficients for all $n_1, \ldots, n_r \in \mathbb{N}$. This limit always exists if and only if the dimension of the nilradial $N(\hat{R})$ of the m_R-adic completion of R is less than $d = \dim R$, as follows from Theorem 1.1. We must thus impose the condition that $\dim N(\hat{R}) < d$. This condition holds if R is analytically unramified; that is, $\hat{R}$ is reduced. We may then define the mixed multiplicities of M from the coefficients of G, generalizing the definition of mixed multiplicities for m_R-primary ideals. Specifically, we write

$$G(n_1, \ldots, n_r) = \sum_{d_1+\cdots+d_r=d} \frac{1}{d_1! \cdots d_r!} e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M) n_1^{d_1} \cdots n_r^{d_r}.$$

We say that $e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M)$ is the mixed multiplicity of M of type $(d_1, \ldots, d_r)$ with respect to the filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$. Here we are using the notation

$$e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M)$$

for the coefficients of $G(n_1, \ldots, n_r)$ to be consistent with the classical notation for mixed multiplicities of M for m_R-primary ideals from [33]. The mixed multiplicity of M of type $(d_1, \ldots, d_r)$ with respect to m_R-primary ideals $I_1, \ldots, I_r$, denoted by $e_R(I_1^{[d_1]}, \ldots, I_r^{[d_r]}; M)$ [33], [32, Definition 17.4.3] is equal to the mixed multiplicity $e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M)$, where the filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$ are defined by $\mathcal{I}(1) = \{I_1^i\}_{i\in\mathbb{N}}, \ldots, \mathcal{I}(r) = \{I_r^i\}_{i\in\mathbb{N}}$.

We write the multiplicity $e_R(\mathcal{I}; M) = e_R(\mathcal{I}^{[d]}; M)$ if $r = 1$, and $\mathcal{I} = \{I_i\}$ is a filtration of R by m_R-primary ideals. We have

$$e_R(\mathcal{I}; M) = \lim_{m\to\infty} d! \frac{\lambda(M/I_m M)}{m^d}.$$

We have by Proposition 6.5 that for $1 \le i \le r$

$$e_R(\mathcal{I}(i); M) = e_R(\mathcal{I}(1)^{[0]}, \ldots, \mathcal{I}(i-1)^{[0]}, \mathcal{I}(i)^{[d]}, \mathcal{I}(i+1)^{[0]}, \ldots, \mathcal{I}(r)^{[0]}; M),$$

generalizing the equality for m_R-primary ideals by Rees in [28, Lemma 2.4].

We show that many of the classical properties of mixed multiplicities for m_R-primary ideals continue to hold for filtrations, including the famous "Minkowski inequalities", proven in Theorem 6.3 and stated below. The Minkowski inequalities were formulated and proven for m_R-primary ideals by Teissier [33], [34], and proven in full generality, for Noetherian local rings, by Rees and Sharp [30]. We prove the strong inequality (1), from which inequalities (2)–(4) follow. The fourth inequality, (4), was proven for filtrations of R by m_R-primary ideals in a regular local ring with algebraically closed residue field by Mustaţă [25, Corollary 1.9], and more recently by Kaveh and Khovanskii [18, Corollary 7.14]. Inequality (4) was proven with our assumption that $\dim N(\hat{R}) < d$ in [8, Theorem 3.1]. Inequalities (2)–(4) can be deduced directly from inequality (1), as explained in [33], [34], [30], [32, Corollary 17.7.3].

Theorem 1.2 (Minkowski inequalities)**.** *Suppose that R is a Noetherian d-dimensional local ring with $\dim N(\hat{R}) < d$, that M is a finitely generated R-module, and that $\mathcal{I}(1) = \{I(1)_j\}$ and $\mathcal{I}(2) = \{I(2)_j\}$ are filtrations of R by m_R-primary ideals. Then*

(1) $e_R(\mathcal{I}(1)^{[i]}, \ \mathcal{I}(2)^{[d-i]}; M)^2 \leq e_R(\mathcal{I}(1)^{[i+1]}, \ \mathcal{I}(2)^{[d-i-1]}; M) e_R(\mathcal{I}(1)^{[i-1]}, \mathcal{I}(2)^{[d-i+1]}; M)$ *for* $1 \leq i \leq d-1$*;*

(2) *for* $0 \leq i \leq d$,

$$e_R(\mathcal{I}(1)^{[i]}, \mathcal{I}(2)^{[d-i]}; M) e_R(\mathcal{I}(1)^{[d-i]}, \mathcal{I}(2)^{[i]}; M) \leq e_R(\mathcal{I}(1); M) e_R(\mathcal{I}(2); M);$$

(3) *for* $0 \leq i \leq d$, $e_R(\mathcal{I}(1)^{[d-i]}, \mathcal{I}(2)^{[i]}; M)^d \leq e_R(\mathcal{I}(1); M)^{d-i} e_R(\mathcal{I}(2); M)^i$*; and*

(4) $e_R(\mathcal{I}(1)\mathcal{I}(2)); M)^{\frac{1}{d}} \leq e_R(\mathcal{I}(1); M)^{\frac{1}{d}} + e_R(\mathcal{I}(2); M)^{\frac{1}{d}}$,
where $\mathcal{I}(1)\mathcal{I}(2) = \{I(1)_j I(2)_j\}$.

In Section 7 we give an example showing that Theorem 6.1 does not have a good extension to arbitrary multigraded non-Noetherian filtrations $\mathcal{I} = \{I_{n_1,\ldots,n_r}\}$ of m_R-primary ideals, even in a power series ring in one variable over a field. In our example ($d = 1$)

$$P(n_1, n_2) = \lim_{m\to\infty} \frac{\lambda(R/I_{mn_1,mn_2})}{m} = \lceil \sqrt{n_1^2 + n_2^2} \rceil$$

for $n_1, n_2 \in \mathbb{N}$, where $\lceil x \rceil$ is the round up of a real number x. The function $P(n_1, n_2)$ is far from polynomial like.

We will show however that the function $P(n_1, \ldots, n_r)$ is polynomial like in an important situation. We show that the multigraded filtration of m_R-primary ideals measuring vanishing along the exceptional divisors of a resolution of singularities of an excellent, normal, two-dimensional local ring is such that the function

$$P(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\lambda(R/I_{mn_1,\ldots,mn_r})}{m^2}$$

is a piecewise polynomial function (a polynomial with rational coefficients when restricted to a member of an abstract complex of polyhedral sets whose union is $\mathbb{Q}_{\geq 0}$). The function $P(n_1, \ldots, n_r)$ is in fact an intersection product on the resolution of singularities. These formulas hold, even though the filtration $\{I_{n_1,\ldots,n_r}\}$ is generally not Noetherian.

The first step in the construction of mixed multiplicities for m_R-primary filtrations is to construct them for Noetherian filtrations. In this case the associated multigraded Hilbert function is a quasi polynomial whose highest degree terms are constant, rational numbers, as we show in Proposition 3.5. Next, in Section 4, we restrict to the case $M = R$ and assume that R is analytically irreducible. Using methods of volumes of Newton–Okounkov bodies adapted to our situation, we show in Proposition 4.3 and Corollary 4.4 that the coefficients of the polynomials $P_a(n_1, \ldots, n_r)$ of (2) for successive Noetherian approximations $\mathcal{I}_a(1), \ldots, \mathcal{I}_a(r)$ of $\mathcal{I}(1), \ldots, \mathcal{I}(r)$, all have a limit as $a \to \infty$. We then define $G(x_1, \ldots, x_n)$ to be the real polynomial with these limit coefficients, and we show in Theorem 4.5 that for $n_1, \ldots, n_r \in \mathbb{Z}_+$, $G(n_1, \ldots, n_r)$ is the function $P(n_1, \ldots, n_r)$ of (2) for the filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$. In Section 5 we obtain the reductions necessary to prove Theorem 6.1, allowing us to define mixed multiplicities for filtrations of m_R-primary ideals in Section 6.

We will denote the nonnegative integers by $\mathbb{N}$ and the positive integers by $\mathbb{Z}_+$. We will denote the set of nonnegative rational numbers by $\mathbb{Q}_{\geq 0}$, and the positive rational numbers by $\mathbb{Q}_+$. We will denote the set of nonnegative real numbers by $\mathbb{R}_{\geq 0}$.

The maximal ideal of a local ring R will be denoted by m_R. The quotient field of a domain R will be denoted by $\mathrm{Q}(R)$. We will denote the length of an R-module M by $\lambda_R(M)$ or $\lambda(M)$ if the ring R is clear from the context.

A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of ideals on a ring R is a descending chain

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

of ideals such that $I_iI_j \subset I_{i+j}$ for all $i, j \in \mathbb{N}$. A filtration $\mathcal{I} = \{I_n\}$ of ideals on a local ring R is a filtration of R by m_R-primary ideals if I_j is m_R-primary for $j \geq 1$. A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of ideals on a ring R is said to be Noetherian if $\bigoplus_{n\geq 0} I_n$ is a finitely generated R-algebra.

2. Polynomials, quasi polynomials, and multiplicities I

A map $\sigma : \mathbb{N}^r \to \mathbb{Q}$ is said to be periodic if there exists an $\alpha \in \mathbb{N}$ such that

$$\sigma(n_1, n_2, \ldots, n_i + \alpha, \ldots, n_r) = \sigma(n_1, n_2, \ldots, n_i, \ldots, n_r)$$

for all $(n_1, \ldots, n_r) \in \mathbb{N}^r$ and $1 \leq i \leq r$. If this condition holds, then α is said to be a period of σ.

In this section we suppose that (R, m_R) is a Noetherian local ring, that M is a finitely generated R-module, and that $\{I(j)_i\}$ are Noetherian filtrations of R by m_R-primary ideals for all $1 \leq j \leq r$. Then for all $1 \leq j \leq r$ there exists an integer $\alpha \geq 1$ such that $R_j^{(\alpha)} = \bigoplus_{n\geq 0} I(j)_{\alpha n}$ are Noetherian standard $\mathbb{N}$-graded rings (by [3, Proposition 3, Section 1.3, Chapter III]). Therefore

$$S = \bigoplus_{n_1,\ldots,n_r\geq 0} I(1)_{\alpha n_1} \cdots I(r)_{\alpha n_r}$$

is a Noetherian standard $\mathbb{N}^r$-graded ring where $S_{(n_1,\ldots,n_r)} = I(1)_{\alpha n_1} \cdots I(r)_{\alpha n_r}$. For all $1 \leq j \leq r$ consider the ideals

$$K_j = \bigoplus_{n_1,\ldots,n_r\geq 0} I(1)_{\alpha n_1} \cdots I(j-1)_{\alpha n_{j-1}} I(j)_{\alpha n_j+1} I(j+1)_{\alpha n_{j+1}} \cdots I(r)_{\alpha n_r}$$

of S where $(K_j)_{(n_1,\ldots,n_r)} = I(1)_{\alpha n_1} \cdots I(j-1)_{\alpha n_{j-1}} I(j)_{\alpha n_j+1} I(j+1)_{\alpha n_{j+1}} \cdots I(r)_{\alpha n_r}$. Then for all $1 \leq j \leq r$

$$\begin{aligned} G_j^{(\alpha)} &:= S/K_j \\ &= \bigoplus_{n_1,\ldots,n_r\geq 0} \frac{I(1)_{\alpha n_1} \cdots I(j)_{\alpha n_j} \cdots I(r)_{\alpha n_r}}{I(1)_{\alpha n_1} \cdots I(j-1)_{\alpha n_{j-1}} I(j)_{\alpha n_j+1} I(j+1)_{\alpha n_{j+1}} \cdots I(r)_{\alpha n_r}} \end{aligned}$$

are standard graded algebras over $R/I(j)_1$.

For all $1 \leq j \leq r$ and integers $0 \leq b_j \leq \alpha - 1$ we have finitely generated $G_j^{(\alpha)}$-modules

$$G_j^{(b_1,\ldots,b_r)}(M) := \bigoplus_{n_1,\ldots,n_r\geq 0} \frac{I(1)_{\alpha n_1+b_1}\cdots I(j)_{\alpha n_j+b_j}\cdots I(r)_{\alpha n_r+b_r}M}{I(1)_{\alpha n_1+b_1}\cdots I(j-1)_{\alpha n_{j-1}+b_{j-1}}I(j)_{\alpha n_j+b_j+1}I(j+1)_{\alpha n_{j+1}+b_{j+1}}\cdots I(r)_{\alpha n_r+b_r}M}.$$

By [13, Theorem 4.1] for all $1 \leq j \leq r$ and integers $0 \leq b_j \leq \alpha - 1$, there exist polynomials $P^{(j)}_{(b_1,\ldots,b_r)}(X_1,\ldots,X_r) \in \mathbb{Q}[X_1,\ldots,X_r]$ and an integer $m \in \mathbb{Z}_+$ such that for all $n_1,\ldots,n_r \geq m$ we have

$$\begin{aligned} &H^{(j)}_{(b_1,\ldots,b_r)}(n_1,\ldots,n_r)\\ &\quad := \lambda\Big(\frac{I(1)_{\alpha n_1+b_1}\cdots I(j)_{\alpha n_j+b_j}\cdots I(r)_{\alpha n_r+b_r}M}{I(1)_{\alpha n_1+b_1}\cdots I(j-1)_{\alpha n_{j-1}+b_{j-1}}I(j)_{\alpha n_j+b_j+1}I(j+1)_{\alpha n_{j+1}+b_{j+1}}\cdots I(r)_{\alpha n_r+b_r}M}\Big)\\ &\quad = P^{(j)}_{(b_1,\ldots,b_r)}(n_1,\ldots,n_r).\end{aligned}$$

Proposition 2.1. *Let $Q_1(X_1,\ldots,X_r),\ldots,Q_k(X_1,\ldots,X_r) \in \mathbb{Q}[X_1,\ldots,X_r]$ be numerical polynomials, and let $1 \leq l$ be a fixed integer. Then for any integer $t \geq 1$ and $j \in \{1,\ldots,r\}$*

$$\sum_{n=0}^{t}\sum_{m=1}^{k} Q_m(n_1,\ldots,n_{j-1},l+n,n_{j+1},\ldots,n_r)$$

is a polynomial $Q(n_1,\ldots,n_{j-1},t,n_{j+1},\ldots,n_r)$ in $n_1,\ldots,n_{j-1},t,n_{j+1},\ldots,n_r$ with coefficients in $\mathbb{Q}$.

Proof. Fix j. For all $m \in \{1,\ldots,k\}$, we have

$$Q_m(n_1,\ldots,n_r) = \sum_{\substack{\beta=(\beta_1,\ldots\beta_r)\in\mathbb{N}^r\\ |\beta|\leq d_m}} e^m_\beta \binom{n_1+\beta_1}{\beta_1}\cdots\binom{n_r+\beta_r}{\beta_r},$$

where d_m is the total degree of Q_m. Then

$$\tilde{Q}(n_1,\ldots,n_r) = \sum_{m=1}^{k} Q_m(n_1,\ldots,n_r) = \sum_{\substack{\beta=(\beta_1,\ldots\beta_r)\in\mathbb{N}^r\\ |\beta|\leq d}} u_\beta \binom{n_1+\beta_1}{\beta_1}\cdots\binom{n_r+\beta_r}{\beta_r}$$

is a numerical polynomial of total degree less than or equal to d with $d = \max\{d_1,\ldots,d_k\}$ and $u_\beta = \sum\limits_{m=1}^{k} e^m_\beta$ for all $\beta = (\beta_1,\ldots\beta_r) \in \mathbb{N}^r$ with $|\beta| \leq d$ (note that $e^m_\beta = 0$ if $|\beta| > d_m$).

Let $A_\beta := u_\beta\binom{n_1+\beta_1}{\beta_1}\cdots\binom{n_{j-1}+\beta_{j-1}}{\beta_{j-1}}\binom{n_{j+1}+\beta_{j+1}}{\beta_{j+1}}\cdots\binom{n_r+\beta_r}{\beta_r}$. Then for any integer $t\geq 1$

$$\begin{aligned}
&\sum_{n=0}^{t}\sum_{m=1}^{k} Q_m(n_1,\ldots,n_{j-1},l+n,n_{j+1},\ldots,n_r)\\
&\quad=\sum_{n=0}^{t}\tilde{Q}(n_1,\ldots,n_{j-1},l+n,n_{j+1},\ldots,n_r)\\
&\quad=\sum_{n=0}^{t}\sum_{\substack{\beta=(\beta_1,\ldots\beta_r)\in\mathbb{N}^r\\|\beta|\leq d}} A_\beta\binom{l+n+\beta_j}{\beta_j}\\
&\quad=\sum_{\substack{\beta=(\beta_1,\ldots\beta_r)\in\mathbb{N}^r\\|\beta|\leq d}} A_\beta\Big[\sum_{n=0}^{t}\binom{l+n+\beta_j}{\beta_j}\Big]\\
&\quad=\sum_{\substack{\beta=(\beta_1,\ldots\beta_r)\in\mathbb{N}^r\\|\beta|\leq d}} A_\beta\Big[\binom{t+l+\beta_j+1}{\beta_j+1}-\binom{l+\beta_j}{\beta_j+1}\Big]\\
&\quad=Q(n_1,\ldots,n_{j-1},t,n_{j+1},\ldots,n_r).
\end{aligned}$$

□

For all $1\leq j\leq r$ and integers $0\leq b_j\leq\alpha-1$ we define

$$(\alpha;b_1,\ldots,b_r)=\{(n_1,\ldots,n_r)\in\mathbb{N}^r : n_j\equiv b_j(\mathrm{mod}\ \alpha)\text{ for all }1\leq j\leq r\}.$$

Proposition 2.2. *Suppose that R is a Noetherian local ring, that M is a finitely generated R-module, and that $\mathcal{I}(1)=\{I(1)_i\},\ldots,\mathcal{I}(r)=\{I(r)_i\}$ are Noetherian filtrations of R by m_R-primary ideals. Consider the function $\lambda(M/I(1)_{n_1}\cdots I(r)_{n_r}M)$ of $n_1,\ldots,n_r\in\mathbb{N}$, where λ is the length as an R-module. Then there exist $c\in\mathbb{Z}_+$, $s\in\mathbb{N}$, and periodic functions $\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)$ such that whenever $n_1,\ldots,n_r\geq c$, we have*

$$\lambda(M/I(1)_{n_1}\cdots I(r)_{n_r}M)=\sum_{i_1+\cdots+i_r\leq s}\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)n_1^{i_1}n_2^{i_2}\cdots n_r^{i_r}.$$

Proof. (We use the integer α and the polynomials $P^{(j)}_{(b_1,\ldots,b_r)}$ mentioned in the above discussion.)
For all $1\leq j\leq r$ and integers $0\leq b_j\leq\alpha-1$ we define the polynomials

$$Q^{(j)}_{(b_1,\ldots,b_j,0,\ldots,0)}(n_1,\ldots,n_r)=\begin{cases}0 & \text{if } b_j=0,\\ \sum\limits_{i(j)=1}^{b_j}P^{(j)}_{(b_1,\ldots,b_{j-1},i(j)-1,0,\ldots,0)}(n_1,\ldots,n_r) & \text{if } 1\leq b_j\leq\alpha-1.\end{cases}$$

Let $\alpha\leq t=\alpha l\in\mathbb{N}$ be such that $H^{(j)}_{(b_1,\ldots,b_r)}(m_1,\ldots,m_r)=P^{(j)}_{(b_1,\ldots,b_r)}(m_1,\ldots,m_r)$ for all $m_1,\ldots,m_r\geq l$ with $0\leq b_1,\ldots,b_r\leq\alpha-1$. Let $(n_1,\ldots,n_r)\in(\alpha;b_1,\ldots,b_r)$

with $n_j \geq c = t + \alpha$ for all $1 \leq j \leq r$. Then

$$\begin{aligned}
&\lambda(M/I(1)_{n_1}\cdots I(r)_{n_r}M)\\
&= \lambda(M/I(1)_t\cdots I(r)_tM) + \sum_{i=0}^{n_1-t-1}\lambda\Big(\tfrac{I(1)_{t+i}I(2)_t\cdots I(r)_tM}{I(1)_{t+i+1}I(2)_t\cdots I(r)_tM}\Big)\\
&\quad + \sum_{i=0}^{n_2-t-1}\lambda\Big(\tfrac{I(1)_{n_1}I(2)_{t+i}I(3)_t\cdots I(r)_tM}{I(1)_{n_1}I(2)_{t+i+1}I(3)_t\cdots I(r)_tM}\Big)\\
&\quad + \cdots + \sum_{i=0}^{n_r-t-1}\lambda\Big(\tfrac{I(1)_{n_1}I(2)_{n_2}\cdots I(r-1)_{n_{r-1}}I(r)_{t+i}M}{I(1)_{n_1}I(2)_{n_2}\cdots I(r-1)_{n_{r-1}}I(r)_{t+i+1}M}\Big)\\
&= \lambda(M/I(1)_t\cdots I(r)_tM)\\
&\quad + \sum_{p(1)=0}^{\lfloor\frac{n_1-t}{\alpha}\rfloor-1}\sum_{i(1)=1}^{\alpha} P^{(1)}_{(i(1)-1,0,\ldots,0)}(l+p(1),l,\ldots,l) + Q^{(1)}_{(b_1,0,\ldots,0)}(l+\lfloor\tfrac{n_1-t}{\alpha}\rfloor,l,\ldots,l)\\
&\quad + \sum_{p(2)=0}^{\lfloor\frac{n_2-t}{\alpha}\rfloor-1}\sum_{i(2)=1}^{\alpha} P^{(2)}_{(b_1,i(2)-1,0,\ldots,0)}(l+\lfloor\tfrac{n_1-t}{\alpha}\rfloor,l+p(2),l,\ldots,l)\\
&\quad + Q^{(2)}_{(b_1,b_2,0,\ldots,0)}(l+\lfloor\tfrac{n_1-t}{\alpha}\rfloor,l+\lfloor\tfrac{n_2-t}{\alpha}\rfloor,l,\ldots,l)\\
&\quad\vdots\\
&\quad + \sum_{p(r)=0}^{\lfloor\frac{n_r-t}{\alpha}\rfloor-1}\sum_{i(r)=1}^{\alpha} P^{(r)}_{(b_1,\ldots,b_{r-1},i(r)-1)}(l+\lfloor\tfrac{n_1-t}{\alpha}\rfloor,\ldots,l+\lfloor\tfrac{n_{r-1}-t}{\alpha}\rfloor,l+p(r))\\
&\quad + Q^{(r)}_{(b_1,\ldots,b_r)}(l+\lfloor\tfrac{n_1-t}{\alpha}\rfloor,\ldots,l+\lfloor\tfrac{n_r-t}{\alpha}\rfloor)\\
&= \lambda(M/I(1)_t\cdots I(r)_tM)\\
&\quad + \sum_{j=1}^{r}\Big[\sum_{p(j)=0}^{\lfloor\frac{n_j-t}{\alpha}\rfloor-1}\sum_{i(j)=1}^{\alpha} P^{(j)}_{(b_1,\ldots,b_{j-1},i(j)-1,0,\ldots,0)}(l+\lfloor\tfrac{n_1-t}{\alpha}\rfloor,\ldots,l+\lfloor\tfrac{n_{j-1}-t}{\alpha}\rfloor,l\\
&\quad + p(j),l,\ldots,l) + Q^{(j)}_{(b_1,\ldots,b_j,0,\ldots,0)}(l+\lfloor\tfrac{n_1-t}{\alpha}\rfloor,\ldots,l+\lfloor\tfrac{n_j-t}{\alpha}\rfloor,l,\ldots,l)\Big]
\end{aligned}$$

Using Proposition 2.1, we have a multigraded polynomial

$$T_{(b_1,\ldots,b_r)}(X_1,\ldots,X_r) := \sum_{i_1+\cdots+i_r\leq u(b_1,\ldots,b_r)} e^{(b_1,\ldots,b_r)}_{(i_1,\ldots,i_r)}X_1^{i_1}\cdots X_r^{i_r} \in \mathbb{Q}[X_1,\ldots,X_r]$$

such that

$$\begin{aligned}
&T_{(b_1,\ldots,b_r)}(m_1,\ldots,m_r)\\
&\quad = \lambda(M/I(1)_t\cdots I(r)_tM)\\
&\qquad + \sum_{j=1}^{r}\Big[\sum_{p(j)=0}^{m_j-1}\Big(\sum_{i(j)=1}^{\alpha} P^{(j)}_{(b_1,\ldots,b_{j-1},i(j)-1,0,\ldots,0)}(l+m_1,\ldots,l+m_{j-1},l\\
&\qquad + p(j),l,\ldots,l)\Big) + Q^{(j)}_{(b_1,\ldots,b_j,0,\ldots,0)}(l+m_1,\ldots,l+m_j,l,\ldots,l)\Big]
\end{aligned}$$

and for all $(n_1,\ldots,n_r) \in (\alpha; b_1,\ldots,b_r)$ with $n_j \geq c$ for $1 \leq j \leq r$, we get

$$\lambda(M/I(1)_{n_1}\cdots I(r)_{n_r}M) = T_{(b_1,\ldots,b_r)}(a(n_1),\ldots,a(n_r)),$$

where $a(n_j) := \lfloor\frac{n_j-t}{\alpha}\rfloor = \frac{n_j-t-b_j}{\alpha}$ for all $1 \leq j \leq r$, and we let $u(b_1,\ldots,b_r)$ be the total degree of $T_{(b_1,\ldots,b_r)}$.

Now for all $(n_1,\ldots,n_r)\in(\alpha;b_1,\ldots,b_r)$, we have

$$\begin{aligned}
&T_{(b_1,\ldots,b_r)}(a(n_1),\ldots,a(n_r))\\
&\quad=\sum_{i_1+\cdots+i_r\le u(b_1,\ldots,b_r)} e^{(b_1,\ldots,b_r)}_{(i_1,\ldots,i_r)}a(n_1)^{i_1}\cdots a(n_r)^{i_r}\\
&\quad=\sum_{i_1+\cdots+i_r\le u(b_1,\ldots,b_r)} e^{(b_1,\ldots,b_r)}_{(i_1,\ldots,i_r)}\frac{(n_1-t-b_1)^{i_1}\cdots(n_r-t-b_r)^{i_r}}{\alpha^{i_1+\cdots+i_r}}.
\end{aligned}$$

Let $\sigma^{(b_1,\ldots,b_r)}_{(i_1,\ldots,i_r)}(n_1,\ldots,n_r)$ denote the coefficient of $n_1^{i_1}\cdots n_r^{i_r}$ in the above equation. Let $e_j=(0,\ldots,0,1,0,\ldots,0)\in\mathbb{N}^r$ with 1 at the jth position, where $j\in\{1,\ldots,r\}$. Note that

$$(n_1,\ldots,n_r)+\alpha e_j\in(\alpha;b_1,\ldots,b_r)$$

and

$$a(n_j+\alpha)=\lfloor\frac{n_j+\alpha-t}{\alpha}\rfloor=\frac{n_j+\alpha-t-b_j}{\alpha}$$

Thus for all $1\le j\le r$, we have

$$\sigma^{(b_1,\ldots,b_r)}_{(i_1,\ldots,i_r)}(n_1,\ldots,n_r)=\sigma^{(b_1,\ldots,b_r)}_{(i_1,\ldots,i_r)}(n_1,\ldots,n_{j-1},n_j+\alpha,n_{j+1}\ldots,n_r).$$

For all $(m_1,\ldots,m_r)\in(\alpha;b_1,\ldots,b_r)$ we define

$$\sigma_{i_1,\ldots,i_r}(m_1,\ldots,m_r)=\begin{cases}\sigma^{(b_1,\ldots,b_r)}_{(i_1,\ldots,i_r)}(m_1,,\ldots,m_r) & \text{if } i_1+\cdots+i_r\le u(b_1,\ldots,b_r),\\ 0 & \text{if } i_1+\cdots+i_r> u(b_1,\ldots,b_r).\end{cases}$$

Therefore for all $n_1,\ldots,n_r\ge c$ and $s=\max\{u(b_1,\ldots,b_r):0\le b_1,\ldots,b_r\le\alpha-1\}$ we get

$$\lambda(M/I(1)_{n_1}\cdots I(r)_{n_r}M)=\sum_{i_1+\cdots+i_r\le s}\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)n_1^{i_1}\cdots n_r^{i_r}.\qquad\square$$

3. Polynomials, quasi polynomials, and multiplicities II

Lemma 3.1. *Suppose that $r,d\ge 1$, and that $a=\binom{r-1+d}{r-1}$. Then there exist $n_1(i),\ldots n_r(i)\in\mathbb{Z}_+$ for $1\le i\le a$ such that the set of vectors consisting of all monomials of degree d in $n_1(i),\ldots,n_r(i)$ for $1\le i\le a$,*

$$\{(n_1(1)^d,n_1(1)^{d-1}n_2(1),\ldots,n_r(1)^d),\ldots,(n_1(a)^d,n_1(a)^{d-1}n_2(a),\ldots,n_r(a)^d)\},$$

is a $\mathbb{Q}$-basis of $\mathbb{Q}^a$.

Proof. Let $\Lambda:(\mathbb{Q}_+)^r\to\mathbb{Q}^a$ be defined by $\Lambda(s_1,\ldots,s_r)=(s_1^d,s_1^{d-1}s_2,\ldots,s_r^d)$. We will first show that the image of Λ is not contained in a proper $\mathbb{Q}$-linear subspace of $\mathbb{Q}^a$. Suppose otherwise. Then there exists a nonzero linear form

$$L(y_{d,0,\ldots,0},y_{d-1,1,0,\ldots,0},\ldots,y_{0,\ldots,0,d})=\sum_{i_1+\cdots+i_r=d}\alpha_{i_1,\ldots,i_r}y_{i_1,\ldots,i_r}$$

on $\mathbb{Q}^a$ such that $L(s_1^d,s_1^{d-1}s_2,\ldots,s_r^d)=0$ for all $(s_1,\ldots,s_r)\in(\mathbb{Q}_+)^r$. The degree d form $G(x_1,\ldots,x_r):=L(x_1^d,x_1^{d-1}x_2,\ldots,x_r^d)$ vanishes on $(\mathbb{Q}_+)^r$. Since $\mathbb{Q}$ is an infinite field, this implies that $G(x_1,\ldots,x_r)$ is the zero polynomial (as follows from [15, proof of Theorem 2.19]). But $G(x_1,\ldots,x_r)$ is a nontrivial linear combination of the monomials in $x_1,\ldots,x_r$ of degree d, so it cannot be 0. So Image(Λ) is not contained in a proper linear subspace of $\mathbb{Q}^a$. Thus there exist $(s_1(i),\ldots,s_r(i))\in(\mathbb{Q}_+)^r$ for $1\le i\le a$ such that

$$\{(s_1(1)^d,s_1(1)^{d-1}s_2(1),\ldots,s_r(1)^d),\ldots,(s_1(a)^d,s_1(a)^{d-1}s_2(a),\ldots,s_r(a)^d)\}$$

is a $\mathbb{Q}$-basis of $\mathbb{Q}^a$. There exists a positive integer u such that $n_i(j) = us_i(j) \in \mathbb{Z}_+$ for all i, j, and since

$$(n_1(j)^d, n_1(j)^{d-1}n_2(j), \ldots, n_r(j)^d) = u^d(s_1(j)^d, s_1(j)^{d-1}s_2(j), \ldots, s_r(j)^d)$$

for $1 \le j \le a$, we have

$$\{(n_1(1)^d, n_1(1)^{d-1}n_2(1), \ldots, n_r(1)^d), \ldots, (n_1(a)^d, n_1(a)^{d-1}n_2(a), \ldots, n_r(a)^d)\}$$

as a $\mathbb{Q}$-basis of $\mathbb{Q}^d$. □

Lemma 3.2. *Let $g = \binom{r-1+d}{r-1}$. There exist $n_1(i), \ldots, n_r(i) \in \mathbb{Z}_+$ for $1 \le i \le g$ and $c_j(i_1, \ldots, i_r) \in \mathbb{Q}$ for $1 \le j \le g$ and $i_1, \ldots, i_r \in \mathbb{N}$ with $i_1 + \cdots + i_r = d$ such that if $F(x_1, \ldots, x_r) \in \mathbb{Q}[x_1, \ldots, x_r]$ is a polynomial of total degree d, with an expansion*

$$F(x_1, \ldots, x_r) = \sum_{i_1+\cdots+i_r \le d} a_{i_1,\ldots,i_r} x_1^{i_1} \cdots x_r^{i_r} \in \mathbb{Q}[x_1, \ldots, x_r] \tag{3}$$

with $a_{i_1,\ldots,i_r} \in \mathbb{Q}$, then for $i_1, \ldots, i_r \in \mathbb{N}$ with $i_1 + \cdots + i_r = d$,

$$a_{i_1,\ldots,i_r} = \sum_{j=1}^{g} c_j(i_1, \ldots, i_r) b_j, \tag{4}$$

where

$$b_j = \lim_{m\to\infty} \frac{F(mn_1(j), \ldots, mn_r(j))}{m^d}. \tag{5}$$

Proof. By Lemma 3.1 we can choose $n_i(j) \in \mathbb{Z}_+$ for $1 \le i \le r$ and $1 \le j \le g$ so that

$$B = \begin{pmatrix} n_1(1)^d & n_1(1)^{d-1}n_2(1) & \cdots & n_r(1)^d \\ \vdots & & & \vdots \\ n_1(g)^d & n_1(g)^{d-1}n_2(g) & \cdots & n_r(g)^d \end{pmatrix}$$

has rank g. Write

$$B^{-1} = \begin{pmatrix} c_1(d, 0, \ldots, 0) & \cdots & c_g(d, 0, \ldots, 0) \\ c_1(d-1, 1, 0, \ldots, 0) & \cdots & c_g(d-1, 1, 0, \ldots, 0) \\ \vdots & \vdots & \\ c_1(0, \ldots, 0, d) & \cdots & c_g(0, \ldots, 0, d) \end{pmatrix}. \tag{6}$$

Suppose that $F(x_1, \ldots, x_r) \in \mathbb{Q}[x_1, \ldots, x_r]$ has the expression (3). By (3) and (5)

$$b_j = \lim_{t\to\infty} \frac{F(tn_1(j), \ldots, tn_r(j))}{t^d} = \sum_{i_1+\cdots+i_r=d} a_{i_1,\ldots,i_r} n_1(j)^{i_1} \cdots n_r(j)^{i_r}$$

for $1 \le j \le g$. We thus have that

$$B \begin{pmatrix} a_{d,0,\ldots,0} \\ a_{d-1,1,0,\ldots,0} \\ \vdots \\ a_{0,\ldots,0,d} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_g \end{pmatrix},$$

so (4) holds by (6). □

Suppose that R is a Noetherian local ring of dimension d, that M is a finitely generated R-module, and that J is an m_R-primary ideal in R. Recall that the multiplicity $e_R(J;M)$ is defined by the expansion of the Hilbert polynomial of M, which is equal to $\lambda(M/J^mM)$ for $m \gg 0$,

$$\frac{e_R(J;M)}{d!}m^d + \text{ lower order terms in } m,$$

so

$$e_R(J;M) = \lim_{m\to\infty} d!\frac{\lambda(M/J^mM)}{m^d}.$$

Lemma 3.3. *Suppose that R is a Noetherian local ring of dimension d, that M is a finitely generated R-module, and that*

$$\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$$

are Noetherian filtrations of R by m_R-primary ideals. Let $a \in \mathbb{Z}_+$ be such that $I(j)_{ia} = I(j)_a^i$ for $1 \le j \le r$ and $i \ge 0$. Suppose that $n_1, \ldots, n_r \in \mathbb{N}$. Then

$$\lim_{m\to\infty} \frac{\lambda(M/I(1)_{mn_1}\cdots I(r)_{mn_r}M)}{m^d} = \frac{1}{d!a^d}e_R(I(1)_{an_1}\cdots I(r)_{an_r};M) \in \mathbb{Q}_+.$$

Proof. For $m \in \mathbb{Z}_+$, write $m = ua+v$ with $0 \le v < a$. Then we have a short exact sequence of R-modules

$$\begin{aligned}0 &\to I(1)_{uan_1}\cdots I(r)_{uan_r}M/I(1)_{mn_1}\cdots I(r)_{mn_r}M \to M/I(1)_{mn_1}\cdots I(r)_{mn_r}M\\ &\to M/I(1)_{uan_1}\cdots I(r)_{uan_r}M \to 0.\end{aligned}$$

We have for $m \gg 0$

$$\begin{aligned}&\lambda(I(1)_{uan_1}\cdots I(r)_{uan_r}M/I(1)_{mn_1}\cdots I(r)_{mn_r}M)\\ &\quad\le \lambda(M/I(1)_{(u+1)an_1}\cdots I(r)_{(u+1)an_r}M) - \lambda(M/I(1)_{uan_1}\cdots I(r)_{uan_r}M)\\ &\quad= \tfrac{e_R(I(1)_{an_1}\cdots I(r)_{an_r};M)}{(d-1)!}u^{d-1} + \text{ lower order terms in } u.\end{aligned}$$

So

$$\begin{aligned}&\lim_{m\to\infty} \tfrac{\lambda(I(1)_{uan_1}\cdots I(r)_{uan_r}M/I(1)_{mn_1}\cdots I(r)_{mn_r}M)}{m^d}\\ &\quad\le \lim_{u\to\infty} \frac{\frac{e_R(I(1)_{an_1}\cdots I(r)_{an_r};M)}{(d-1)!}u^{d-1} + \text{ lower order terms in } u}{(ua+v)^d} = 0.\end{aligned}$$

Thus

$$\begin{aligned}\lim_{m\to\infty} \frac{\lambda(M/I(1)_{mn_1}\cdots I(r)_{mn_r}M)}{m^d} &= \lim_{u\to\infty} \frac{\lambda(M/I(1)_{uan_1}\cdots I(r)_{uan_r}M)}{(ua+v)^d}\\ &= \frac{1}{d!a^d}e_R(I(1)_{an_1}\cdots I(r)_{an_r};M). \qquad \square\end{aligned}$$

Define the total degree of a quasi polynomial $\sum \sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)n_1^{i_1}\cdots n_r^{i_r}$ to be the largest t such that there exist $i_1,\ldots,i_r \in \mathbb{N}$ with $i_1+\cdots+i_r = t$ such that $\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)$ is not (identically) 0.

Proposition 3.4. *Let*

$$P(n_1,\ldots,n_r) = \sum \sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)n_1^{i_1}\cdots n_r^{i_r}$$

be the quasi polynomial of the conclusions of Proposition 2.2. Then the total degree of $P(n_1,\ldots,n_r)$ is $\dim M$, and $\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)$ is a constant function if $i_1 + \cdots + i_r = \dim M$.

Proof. Let t be the total degree of $P(n_1,\ldots,n_r)$, and let $a \in \mathbb{Z}_+$ be such that $I(j)_{ai} = I(j)_a^i$ for all $i \geq 0$ and $1 \leq j \leq r$, so a is a common period of the coefficients $\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)$ of $P(n_1,\ldots,n_r)$ (by the proof of Proposition 2.2). Suppose that $b_1,\ldots,b_r \in \mathbb{N}$ with $0 \leq b_i < a$ for all i. Suppose that $n_1,\ldots,n_r \in \mathbb{Z}_+$. Then for $n_1,\ldots,n_r \gg 0$

$$\lambda(M/I(1)_{an_1+b_1}\cdots I(r)_{an_r+b_r}M) = P(an_1+b_1,\ldots,an_r+b_r).$$

Define

$$\begin{aligned}
P_{(b_1,\ldots,b_r)}(n_1,\ldots,n_r) &:= P(an_1+b_1,\ldots,an_r+b_r)\\
&= \textstyle\sum_{i_1+\cdots+i_r\leq t}\sigma_{i_1,\ldots,i_r}(an_1+b_1,\ldots,an_r+b_r)(an_1+b_1)^{i_1}\cdots(an_r+b_r)^{i_r}\\
&= \textstyle\sum_{i_1+\cdots+i_r\leq t}\sigma_{i_1,\ldots,i_r}(b_1,\ldots,b_r)(an_1+b_1)^{i_1}\cdots(an_r+b_r)^{i_r}\\
&= \textstyle\sum_{i_1+\cdots+i_r= t}\sigma_{i_1,\ldots,i_r}(b_1,\ldots,b_r)a^t n_1^{i_1}\cdots n_r^{i_r}\\
&\quad + \text{lower total order terms in } n_1,\ldots,n_r.
\end{aligned}$$

We have that $P_{(b_1,\ldots,b_r)}(n_1,\ldots,n_r) \in \mathbb{Q}[n_1,\ldots,n_r]$ is a polynomial. For fixed $n_1,\ldots,n_r \in \mathbb{Z}_+$ and $m >> 0$ we have

$$\begin{aligned}
P_{(0,\ldots,0)}(mn_1,\ldots,mn_r) &= \lambda(M/I(1)_{amn_1}\cdots I(r)_{amn_r}M)\\
&= \lambda(M/(I(1)_{an_1}\cdots I(r)_{an_r})^m M).
\end{aligned}$$

Thus by [32, Lemma 11.1.3]

$$\lim_{m\to\infty}\frac{P_{(0,\ldots,0)}(mn_1,\ldots,mn_r)}{m^{\dim M}} \in \mathbb{Q}_+.$$

Therefore the total degree of $P_{(0,\ldots,0)}(n_1,\ldots,n_r)$ is $\dim M$.

Fix $n_1,\ldots,n_r \in \mathbb{Z}_+$ and $b_i \in \mathbb{N}$ with $0 \leq b_i < a$ for $1 \leq i \leq r$. For $m \in \mathbb{Z}_+$ we have short exact sequences of R-modules,

$$\begin{aligned}
&0 \to I(1)_{man_1}\cdots I(r)_{man_r}M/I(1)_{man_1+b_1}\cdots I(r)_{man_r+b_r}M\\
&\quad \to M/I(1)_{man_1+b_1}\cdots I(r)_{man_r+b_r}M \to M/I(1)_{man_1}\cdots I(r)_{man_r}M \to 0.
\end{aligned}$$

Now for $m \gg 0$

$$\begin{aligned}
&\lambda(I(1)_{man_1}\cdots I(r)_{man_r}M/I(1)_{man_1+b_1}\cdots I(r)_{man_r+b_r}M)\\
&\quad \leq \lambda(I(1)_{man_1}\cdots I(r)_{man_r}M/I(1)_{(m+1)an_1}\cdots I(r)_{(m+1)an_r}M)\\
&\quad = P_{(0,\ldots,0)}((m+1)n_1,\ldots,(m+1)n_r) - P_{(0,\ldots,0)}(mn_1,\ldots,mn_r)
\end{aligned}$$

is a polynomial of degree less than $\dim M$ in m. Thus

$$\begin{aligned}
\lim_{m\to\infty}\frac{P_{(b_1,\ldots,b_r)}(mn_1,\ldots,mn_r)}{m^{\dim M}} &= \lim_{m\to\infty}\frac{\lambda(M/I(1)_{man_1}\cdots I(r)_{man_r}M)}{m^{\dim M}}\\
&= \lim_{m\to\infty}\frac{P_{(0,\ldots,0)}(mn_1,\ldots,mn_r)}{m^{\dim M}}
\end{aligned}$$

and

$$\sigma_{i_1,\ldots,i_r}(b_1,\ldots,b_r) = \sigma_{i_1,\ldots,i_r}(0,\ldots,0)$$

if $i_1+\cdots+i_r = \dim M$, by Lemma 3.2. □

Proposition 3.5. *Suppose that R is a Noetherian local ring, that M is a finitely generated R-module, and that $\mathcal{I}(1) = \{I(1)_i\},\ldots,\mathcal{I}(r) = \{I(r)_i\}$ are Noetherian filtrations of R by m_R-primary ideals. Then there exist a positive integer c and periodic functions $\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)$ such that whenever $n_1,\ldots,n_r \geq c$, we have that*

$$\lambda(M/I(1)_{n_1}\cdots I(r)_{n_r}M) = \sum_{i_1+\cdots+i_r\leq \dim M}\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)n_1^{i_1}n_2^{i_2}\cdots n_r^{i_r}$$

is a quasi polynomial of total degree equal to $\dim M$*, and the coefficients* $\sigma_{i_1,\ldots,i_r}(n_1,\ldots,n_r)$ *are constants whenever* $i_1+\cdots+i_r=\dim M$*.*

Proof. This follows from Propositions 2.2 and 3.4. □

4. Volumes on analytically irreducible local domains

Definition 4.1. Suppose that $\mathcal{I}=\{I_i\}$ is a filtration of ideals on a local ring R. For $a\in\mathbb{Z}_+$ the ath truncated filtration $\mathcal{I}_a=\{I_{a,i}\}$ of $\mathcal{I}$ is defined by $I_{a,n}=I_n$ if $n\le a$ and if $n>a$, then $I_{a,n}=\sum I_{a,i}I_{a,j}$, where the sum is over $i,j>0$ such that $i+j=n$.

We give an algebraic proof of the following lemma. A geometric proof is given in [7, page 9].

Lemma 4.2. *Suppose that* R *is an excellent* d*-dimensional local domain. Then there exists an excellent regular local ring* S *of dimension* d *which birationally dominates* R*.*

Proof. Let $d=\dim R$. Let $z_1,\ldots,z_d$ be a system of parameters in R, and let $Q=(z_1,\ldots,z_d)$, which is an m_R-primary ideal in R. Let T be the integral closure of $B=R[\frac{z_2}{z_1},\ldots,\frac{z_d}{z_1}]$ in $\mathrm{Q}(R)$. The ring T is an excellent ring and is a finitely generated R-algebra by [23, Theorem 78, page 257].

We will now show that z_1 is not a unit in B, using [1, an argument from (1.3.1), page 15]. Suppose that z_1 is a unit in B. Then there exists $y\in B$ such that $z_1y=1$, so there exists a nonzero polynomial $f(X_2,\ldots,X_d)$ of some degree n with coefficients in R such that $y=f\left(\frac{z_2}{z_1},\ldots,\frac{z_d}{z_1}\right)$. Then $z_1^n=z_1^{n+1}y=z_1g(z_1,\ldots,z_d)$, where $g(X_1,\ldots,X_d)$ is a nonzero homogeneous polynomial of degree n with coefficients in R. Thus $z_1^n\in m_RQ^n$, which is a contradiction by [37, Theorem 21, page 292]. We further have that z_1 is not a unit in T since T is finite over B. Now $QT=z_1T$ and z_1 is not a unit in T, so $\mathrm{ht}(P)=1$ if P is a minimal prime of m_RT by Krull's principal ideal theorem.

We next show that T has dimension d. The ring R is universally catenary since R is excellent, so the dimension formula holds between R and T (the inequality (*) in [23, page 85] is an equality). Let n be a maximal ideal of T which contains z_1. Then $n\cap R=m_R$. We have that T/n is a finitely generated algebra over the field R/m_R and T/n is a field, so T/n is a finite R/m_R-module by [20, Corollary 1.2, page 379]. By the dimension formula, we have

$$\mathrm{ht}(n)=\mathrm{ht}(m_R)+\mathrm{trdeg}_{\mathrm{Q}(R)}\mathrm{Q}(T)-\mathrm{trdeg}_{R/m_R}T/nT=\mathrm{ht}(m_R)=d.$$

Since the dimension formula gives us $\mathrm{ht}(m)\le d$ for all maximal ideals m in T, we have $\dim T=d$. Let

$$\mathrm{NR}(T)=\{P\in\mathrm{Spec}(T)\mid T_P \text{ is not a regular local ring}\}.$$

The set $\mathrm{NR}(T)$ is a closed set since T is excellent. Let I be an ideal of T such that $\mathrm{NR}(T)=\mathrm{Spec}(T/I)$. If P is a minimal prime of I, then $\mathrm{ht}(P)>1$ since T is normal (Serre's criterion for normality). The Jacobson radical of T/m_RT (the intersection of all maximal ideals of T/m_RT) is the nilradical of T/m_RT by [23, Theorem 25, page 93] since T/m_RT is a finitely generated algebra over the field R/m_R. Let $\overline{I}=I(T/m_RT)$. There exists a maximal ideal $\overline{n}$ of T/m_RT such that $\overline{I}\not\subset\overline{n}$ since otherwise $I\subset\sqrt{m_RT}$, which is impossible since all minimal primes of I have height larger than 1 and all minimal primes of m_RT have height equal to 1. Let n be the lift of $\overline{n}$ to a maximal ideal of T. Then $S := T_n$ is a regular local ring of dimension d which birationally dominates R. $\square$

In this section we suppose that R is a Noetherian local ring of dimension d which is analytically irreducible. Suppose that $\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$ are (not necessarily Noetherian) filtrations of R by m_R-primary ideals. Define a function $F : \mathbb{N}^r \to \mathbb{R}$ by

$$F(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\lambda(R/I(1)_{mn_1} \cdots I(r)_{mn_r})}{m^d} \tag{7}$$

for $n_1, \ldots, n_r \in \mathbb{N}$ where the limit is over $m \in \mathbb{Z}_+$. This limit exists by Theorem 1.1.

For $a \in \mathbb{Z}_+$ let $\{I_a(j)_i\}$ be the ath truncated filtration of $\{I(j)_i\}$ for $1 \le j \le r$ (defined in Definition 4.1). By Proposition 3.5, for $a \in \mathbb{Z}_+$, there is a homogeneous polynomial $F_a(x_1, \ldots, x_r)$ of total degree d in $\mathbb{Q}[x_1, \ldots, x_r]$ such that

$$\lim_{m\to\infty} \frac{\lambda(R/I_a(1)_{mn_1} \cdots I_a(r)_{mn_r})}{m^d} = F_a(n_1, \ldots, n_r)$$

if $n_1, \ldots, n_r \in \mathbb{Z}_+$. Expand

$$F_a(x_1, \ldots, x_r) = \sum_{i_1+\cdots+i_r=d} b_{i_1,\ldots,i_r}(a) x_1^{i_1} \cdots x_r^{i_r}$$

with $b_{i_1,\ldots,i_r}(a) \in \mathbb{Q}$.

Proposition 4.3. *For fixed $n_1, \ldots, n_r \in \mathbb{Z}_+$*

$$\lim_{a\to\infty} F_a(n_1, \ldots, n_r) = F(n_1, \ldots, n_r).$$

Proof. Define filtrations of ideals $\{J_i\}$ and $\{J(a)_i\}$ by $J_i = I(1)_{in_1} \cdots I(r)_{in_r}$ and $J(a)_i = I_a(1)_{in_1} \cdots I_a(r)_{in_r}$.

We use a construction and method from [7, proof of Theorem 4.2]. We begin by reviewing the construction in the context of this proposition. Since $\lambda(R/J_i) = \lambda_{\hat{R}}(\hat{R}/\hat{J}_i)$ and $\lambda(R/J(a)_i) = \lambda_{\hat{R}}(\hat{R}/J(a)_i\hat{R})$ for all i and a and since $\hat{R}$ is a domain, we may assume that R is complete and thus is excellent. By Lemma 4.2 there exists a regular local ring S of dimension d which birationally dominates R. Choosing a regular system of parameters $y_1, \ldots, y_d$ in S and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ which are rationally independent real numbers such that $\lambda_i \ge 1$ for all i, we define a valuation ν on the quotient field of R such that ν dominates S by prescribing $\nu(y_1^{a_1} \cdots y_d^{a_d}) = a_1\lambda_1 + \cdots + a_d\lambda_d$ for $a_1, \ldots, a_d \in \mathbb{N}$ and $\nu(\gamma) = 0$ if $\gamma \in S$ is a unit. Let $k = R/m_R$ and $k' = S/m_S$.

We will show that the residue field $V_\nu/m_\nu = k'$. Given an element $h \in V_\nu$, let $[h]$ denote the class of f in the residue field V_ν/m_ν. Suppose that $h \in V_\nu$ and that $\nu(h) = 0$. Write $h = \frac{f}{g}$ with $f, g \in S$. There exist a unit $\alpha \in S$, $i_1, \ldots, i_d \in \mathbb{N}$, and $a \in S$ such that $f = \alpha y_1^{i_1} \cdots y_d^{i_d} + a$ and $\nu(a) > \nu(f) = i_1\lambda_1 + \cdots + i_d\lambda_d$. Similarly, there exist a unit $\beta \in S$, $j_1, \ldots, j_d \in \mathbb{N}$, and $b \in S$ such that $g = \beta y_1^{j_1} \cdots y_d^{j_d} + b$ with $\nu(b) > \nu(g) = j_1\lambda_1 + \cdots + j_d\lambda_d$. We have $y_1^{i_1} \cdots y_d^{i_d} = y_1^{j_1} \cdots y_d^{j_d}$ since $\nu(f) = \nu(g)$. Now $[\frac{f}{y_1^{i_1}\cdots y_d^{i_d}}] = [\alpha]$ and $[\frac{g}{y_1^{i_1}\cdots y_d^{i_d}}] = [\beta]$, so $[h] = \frac{[\alpha]}{[\beta]} \in S/m_S = k'$.

For $\lambda \in \mathbb{R}_{\ge 0}$ define ideals K_λ and K_λ^+ in the valuation ring V_ν of ν by

$$K_\lambda = \{f \in \mathrm{Q}(R) \mid \nu(f) \ge \lambda\}$$

and

$$K^+_\lambda = \{f \in \mathrm{Q}(R) \mid \nu(f) > \lambda\}.$$

For $t \geq 1$ define semigroups

$$\begin{array}{rcl}\Gamma^{(t)} &=& \{(m_1,\ldots,m_d,i) \in \mathbb{N}^{d+1} \mid \dim_k J_i \cap K_{m_1\lambda_1+\cdots+m_d\lambda_d}/ \\ && J_i \cap K^+_{m_1\lambda_1+\cdots+m_d\lambda_d} \geq t \text{ and } m_1+\cdots+m_d \leq \beta i\},\\ \Gamma(a)^{(t)} &=& \{(m_1,\ldots,m_d,i) \in \mathbb{N}^{d+1} \mid \dim_k J(a)_i \cap K_{m_1\lambda_1+\cdots+m_d\lambda_d}/ \\ && J(a)_i \cap K^+_{m_1\lambda_1+\cdots+m_d\lambda_d} \geq t \text{ and } m_1+\cdots+m_d \leq \beta i\}\end{array}$$

and

$$\begin{array}{rcl}\hat{\Gamma}^{(t)} &=& \{(m_1,\ldots,m_d,i) \in \mathbb{N}^{d+1} \mid \dim_k R \cap K_{m_1\lambda_1+\cdots+m_d\lambda_d}/ \\ && R \cap K^+_{m_1\lambda_1+\cdots+m_d\lambda_d} \geq t \text{ and } m_1+\cdots+m_d \leq \beta i\}.\end{array}$$

Here $\beta = \alpha c$, where $c \in \mathbb{Z}_+$, is chosen so that $m_R^c \subset J_1 = J(a)_1 = I(1)_{n_1}\cdots I(r)_{n_r}$, and $\alpha \in \mathbb{Z}_+$ is such that $K_{\alpha n} \cap R \subset m_R^n$ for all $n \in \mathbb{N}$. Such an α exists by [6, Lemma 4.3]. Define $\Gamma^{(t)}_m = \Gamma^{(t)} \cap (\mathbb{N}^d \times \{m\})$, $\Gamma(a)^{(t)}_m = \Gamma(a)^{(t)} \cap (\mathbb{N}^d \times \{m\})$, and $\hat{\Gamma}^{(t)}_m = \hat{\Gamma}^{(t)} \cap (\mathbb{N}^d \times \{m\})$ for $m \in \mathbb{N}$.

The Newton–Okounkov body of a (strongly nonnegative) subsemigroup S of $\mathbb{Z}^d \times \mathbb{N}$ is defined as

$$\Delta(S) = \mathrm{con}(S) \cap (\mathbb{R}^d \times \{1\}),$$

where $\mathrm{con}(S)$ is the closed convex cone which is the closure of the set of all linear combinations $\sum \lambda_i s_i$ with $s_i \in S$ and λ_i being a nonnegative real number. This theory is developed in [27], [22], [17] and is summarized in [7, Section 3].

By [7, Lemmas 4.5 and 4.6], [7, Theorem 3.2]

$$\lim_{m\to\infty} \frac{\#\Gamma^{(t)}_m}{m^d} = \mathrm{Vol}(\Delta(\Gamma^{(t)})), \tag{8}$$

$$\lim_{m\to\infty} \frac{\#\Gamma(a)^{(t)}_m}{m^d} = \mathrm{Vol}(\Delta(\Gamma(a)^{(t)})), \tag{9}$$

and

$$\lim_{m\to\infty} \frac{\#\hat{\Gamma}^{(t)}_m}{m^d} = \mathrm{Vol}(\Delta(\hat{\Gamma}^{(t)})) \tag{10}$$

all exist (where $\#T$ is the number of elements in a finite set T).

By [7, (19), page 11]

$$\begin{array}{rcl}F_a(n_1,\ldots,n_r) &=& \lim_{m\to\infty} \frac{\lambda(R/J(a)_m)}{m^d} \\ &=& \sum_{t=1}^{[k':k]} \lim_{m\to\infty} \frac{\#\hat{\Gamma}^{(t)}_m}{m^d} - \sum_{t=1}^{[k':k]} \lim_{m\to\infty} \frac{\#\Gamma(a)^{(t)}_m}{m^d},\end{array} \tag{11}$$

with a similar formula

$$\begin{array}{rcl}F(n_1,\ldots,n_r) &=& \lim_{m\to\infty} \frac{\lambda(R/J_m)}{m^d} \\ &=& \sum_{t=1}^{[k':k]} \lim_{m\to\infty} \frac{\#\hat{\Gamma}^{(t)}_m}{m^d} - \sum_{t=1}^{[k':k]} \lim_{m\to\infty} \frac{\#\Gamma^{(t)}_m}{m^d}.\end{array} \tag{12}$$

Let

$$\overline{a} = \lfloor a/\max\{n_1,\ldots,n_r\}\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer in a real number x. We have

$$\Gamma^{(t)}_i = \Gamma(a)^{(t)}_i \qquad \text{for } i \leq \overline{a}, \tag{13}$$

so

$$n * \Gamma^{(t)}_{\overline{a}} := \{x_1+\cdots+x_n \mid x_1,\ldots,x_n \in \Gamma^{(t)}_{\overline{a}}\} \subset \Gamma(a)^{(t)}_{n\overline{a}} \text{ for all } n \geq 1.$$

By [22, Proposition 3.1] (recalled in [7, Theorem 3.3]) and since $\overline{a} \mapsto \infty$ as $a \mapsto \infty$, given $\varepsilon > 0$, there exists an $a_0 > 0$ such that for all $a \geq a_0$ we have

$$\begin{aligned} \mathrm{Vol}(\Delta(\Gamma^{(t)})) &\geq \mathrm{Vol}(\Delta(\Gamma(a)^{(t)}) = \lim_{n\to\infty} \frac{\#\Gamma(a)^{(t)}_n}{n^d} = \lim_{n\to\infty} \frac{\#\Gamma(a)^{(t)}_{n\overline{a}}}{(n\overline{a})^d} \\ &\geq \lim_{n\to\infty} \frac{\#(n*\Gamma^{(t)}_{\overline{a}})}{(n\overline{a})^d} \geq \mathrm{Vol}(\Delta(\Gamma^{(t)})) - \varepsilon. \end{aligned} \tag{14}$$

By (11)–(14), the proposition holds. □

The following corollary now follows from Lemma 3.2 and Proposition 4.3.

Corollary 4.4. *For all* $i_1, \ldots, i_r \in \mathbb{N}$ *with* $i_1 + \cdots + i_r = d$

$$b_{i_1,\ldots,i_r} := \lim_{a\to\infty} b_{i_1,\ldots,i_r}(a) \tag{15}$$

exists (in $\mathbb{R}$*).*

Now define a homogeneous polynomial

$$G(x_1, \ldots, x_r) = \sum_{i_1+\cdots+i_r=d} b_{i_1,\ldots,i_r} x_1^{i_1} \cdots x_r^{i_r} \in \mathbb{R}[x_1, \ldots, x_r],$$

where the $b_{i_1,\ldots,i_r}$ are defined by (15).

Theorem 4.5. *For all* $n_1, \ldots, n_r \in \mathbb{Z}_+$

$$F(n_1, \ldots, n_r) = G(n_1, \ldots, n_r).$$

Proof. For fixed $n_1, \ldots, n_r \in \mathbb{Z}_+^r$ and $a \in \mathbb{Z}_+$

$$\begin{aligned} |F(n_1, \ldots, n_r) - G(n_1, \ldots, n_r)| &\leq |F(n_1, \ldots, n_r) - F_a(n_1, \ldots, n_r)| \\ &\quad + |F_a(n_1, \ldots, n_r) - G(n_1, \ldots, n_r)|, \end{aligned}$$

which is arbitrarily small for $a \gg 0$ by Proposition 4.3 and Corollary 4.4. □

5. Reduction to local domains

Lemma 5.1. *Suppose that* R *is a Noetherian domain and that* M *is a torsion free finitely generated* R*-module. Then there exists a short exact sequence of* R*-modules*

$$0 \to R^s \to M \to F \to 0,$$

where $s = \mathrm{rank}(M)$ *and* $\dim F < \dim R$*.*

Proof. Let K be the quotient field of R, and let $\{e_1, \ldots, e_s\}$ be a K-basis of $M \otimes_R K$. Since M is torsion free, we have a natural inclusion $M \subset M \otimes K$. For all i, there exists $0 \neq x_i \in R$ such that $x_i e_i \in M$, so after replacing e_i with $x_i e_i$, we may assume that $e_i \in M$. Let $\varphi : R^s \to M$ be the R-module homomorphism $\varphi = (e_1, \ldots, e_s)$. Let L be the kernel of φ, and let F be the cokernel. We have a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & L & \to & R^s & \overset{\varphi}{\to} & M & \to & F & \to & 0 \\ & & & & \downarrow & & \downarrow & & & & \\ & & & & K^s & \overset{\varphi}{\to} & M \otimes_R K & \to & F \otimes_R K & \to & 0, \end{array}$$

where the vertical arrows are injective and the rows are exact. By our construction of φ $K^s \overset{\varphi}{\to} M \otimes_R K$ is an isomorphism. Thus $L = 0$ and $\dim F < \dim R$. □

Lemma 5.2. *Suppose that R is a Noetherian local ring of dimension d and that M is a finitely generated R-module. Let T be a submodule of M such that* $\dim T < d$, *so there is a short exact sequence of R-modules*

$$0 \to T \to M \to M/T := \overline{M} \to 0.$$

Suppose that $\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$ are filtrations of R by m_R-primary ideals. Then for fixed $n_1, \ldots, n_r \in \mathbb{N}$,

$$\lim_{m\to\infty} \frac{\lambda(M/I(1)_{mn_1} \cdots I(r)_{mn_r} M)}{m^d} = \lim_{m\to\infty} \frac{\lambda(\overline{M}/I(1)_{mn_1} \cdots I(r)_{mn_r} \overline{M})}{m^d}.$$

Proof. Define a filtration of R by m_R-primary ideals by $J_m = I(1)_{mn_1} \cdots I(r)_{mn_r}$. We have short exact sequences of R-modules

$$0 \to T/T \cap (J_m M) \to M/J_m M \to \overline{M}/J_m \overline{M} \to 0.$$

There exists a positive integer c such that $m_R^c \subset J_1$. Thus $m_R^{cm} T \subset T \cap (J_m M)$ for all m and

$$\lambda(T/T \cap J_m M) \le \lambda(T/m_R^{cm} T).$$

Since $\dim T < d$,

$$\lim_{m\to\infty} \frac{\lambda(T/m_R^{cm} T)}{m^d} = 0,$$

and the lemma follows. □

Lemma 5.3. *Suppose that R is a Noetherian local domain of dimension d and that M is a finitely generated R-module. Suppose that $\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$ are filtrations of R by m_R-primary ideals. Let $s = \mathrm{rank}(M)$. Suppose that $n_1, \ldots, n_r \in \mathbb{N}$. Then*

$$\lim_{m\to\infty} \frac{\lambda(M/I(1)_{mn_1} \cdots I(r)_{mn_r} M)}{m^d} = s \left(\lim_{m\to\infty} \frac{\lambda(R/I(1)_{mn_1} \cdots I(r)_{mn_r})}{m^d} \right).$$

Proof. Define a filtration of m_R-primary ideals by $J_m = I(1)_{mn_1} \cdots I(r)_{mn_r}$. By Lemma 5.2 we may assume that M is torsion free, so there exists by Lemma 5.1 a short exact sequence of R-modules

$$0 \to R^s \to M \to F \to 0$$

where $\dim F < d$. There exists $c > 0$ such that $m_R^c \subset J_1$. There exists a $0 \ne x \in R$ such that $xM \subset R^s$. We have exact sequences for all $m \in \mathbb{Z}_+$,

$$\begin{aligned} 0 \to R^s \cap (J_m M)/J_m R^s \to R^s/J_m R^s \\ \to M/J_m M \to N_m \to 0, \end{aligned}$$

where N_m is defined to be the cokernel of the last map, and we have an exact sequence

$$0 \to A_m \to R^s/J_m R^s \xrightarrow{x} R^s/J_m R^s \to W_m \to 0, \tag{16}$$

where A_m is the kernel of the first map and W_m is the cokernel of the last map. We have

$$A_m = \left[(J_m : x)/J_m\right]^s.$$

We have

$$x(R^s \cap J_m M) \subset J_m R^s,$$

so

$$\lambda(R^s \cap (J_m M)/J_m R^s) \le \lambda(A_m).$$

We have

$$W_m \cong [(R/(x))/J_m(R/(x))]^s,$$

so

$$\lambda(W_m) \le \lambda((R/(x))/m_R^{cm}(R/(x)))^s$$

for all m. Thus

$$\lim_{m\to\infty} \frac{\lambda(A_m)}{m^d} = \lim_{m\to\infty} \frac{\lambda(W_m)}{m^d} = 0$$

by (16), so

$$\lim_{m\to\infty} \frac{\lambda(R^s \cap J_m M/J_m R^s)}{m^d} = 0.$$

Now $xM \subset R^s$ implies that

$$N_m \cong M/R^s + J_m M = M/(R^s + J_m M + xM).$$

Thus

$$\lambda(N_m) \le \lambda((M/xM)/m_R^{cm}(M/xM),$$

so

$$\lim_{m\to\infty} \frac{\lambda(N_m)}{m^d} = 0$$

since $\dim M/xM < d$, and the lemma follows. ☐

Lemma 5.4. *Suppose that R is a d-dimensional reduced Noetherian local ring and that M is a finitely generated R-module. Let $\{P_1, \ldots, P_s\}$ be the minimal primes of R and $S = \bigoplus_{i=1}^s R/P_i$. Suppose that $\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$ are filtrations of R by m_R-primary ideals. Suppose that $n_1, \ldots, n_r \in \mathbb{Z}_+$ are fixed. Then*

$$\lim_{m\to\infty} \frac{\lambda(M/I(1)_{mn_1} \cdots I(r)_{mn_r})}{m^d} = \lim_{m\to\infty} \frac{\lambda(M \otimes_R S/I(1)_{mn_1} \cdots I(r)_{mn_r} M \otimes_R S)}{m^d}.$$

Proof. Define a filtration of R by m_R-primary ideals by $J_m = I(1)_{mn_1} \cdots I(r)_{mn_r}$. There exists a $c \in \mathbb{Z}_+$ such that $m_R^c \subset J_1$. Since S is a finitely generated R submodule of the total ring of fractions $T = \bigoplus_{i=1}^s Q(R/P_i)$ of R, there exists a nonzero divisor $x \in R$ such that $xS \subset R$. Tensoring the short exact sequence

$$0 \to R \to S \to S/R \to 0$$

of R-modules with M, we have a natural short exact sequence of R-modules,

$$M \stackrel{\gamma}{\to} M \otimes_R S \to M \otimes_R (S/R) \to 0.$$

Let $K = \text{kernel } \gamma$ and $U = \text{Image } \gamma$. We have that $(\text{Kernel } \gamma)_{P_i} = 0$ for $1 \le i \le s$ since $R_{P_i} \cong S_{P_i}$ for all i. Thus $\dim \text{Kernel } \gamma < d$, and by Lemma 5.2

$$\lim_{m\to\infty} \frac{\lambda(U/J_m U)}{m^d} = \lim_{m\to\infty} \frac{\lambda(M/J_m M)}{m^d}.$$

Let $V = M \otimes_R S$. We have short exact sequences of R-modules

$$0 \to U \cap J_m V/J_m U \to U/J_m U \to V/J_m V \to N_m \to 0,$$

where $N_m = V/U + J_m V$. We also have short exact sequences

$$0 \to A_m \to U/J_m U \stackrel{x}{\to} U/J_m U \to W_m \to 0, \tag{17}$$

where A_m is the kernel of multiplication by x and W_m is the cokernel. Now $x(U \cap J_mV) \subset J_mU$, so $U \cap J_mV/J_mU \subset A_m$ for all m. Now $W_m \cong (U/xU)/J_m(U/xU)$ and $\dim U/xU < d$. We have

$$\lambda(W_m) \le \lambda((U/xU)/m_R^{mc}(U/xU)),$$

and thus

$$\lim_{m\to\infty} \frac{\lambda(W_m)}{m^d} = 0.$$

From (17) we have

$$\lim_{m\to\infty} \frac{\lambda(U \cap J_mV/J_mU)}{m^d} \le \lim_{m\to\infty} \frac{\lambda(A_m)}{m^d} = \lim_{m\to\infty} \frac{\lambda(W_m)}{m^d} = 0.$$

Since $xV \subset U$, we have

$$N_m \cong V/U + J_mV = V/(U + J_mV + xV).$$

Thus

$$\lambda(N_m) \le \lambda((V/xV)/m_R^{mc}(V/xV))$$

for all m, so

$$\lim_{m\to\infty} \frac{\lambda(N_m)}{m^d} = 0$$

since $\dim V/xV < d$. □

6. Mixed multiplicities of filtrations

The following theorem allows us to define mixed multiplicities for arbitrary (not necessarily Noetherian) filtrations of m_R-ideals in a Noetherian local ring with $\dim N(\hat{R}) < \dim R$. By Theorem 1.1, if the assumption $\dim N(\hat{R}) < d$ is removed from the hypotheses of Theorem 6.1, then the conclusions of Theorem 6.1 will no longer be true. Theorem 6.1 generalizes a theorem of Bhattacharya [2] and Teissier and Risler in the paper [33] (also proven in [32, Theorem 17.4.2]) for m_R-primary ideals to filtrations of m_R-primary ideals.

Theorem 6.1. *Suppose that R is a Noetherian local ring of dimension d such that*

$$\dim N(\hat{R}) < d,$$

and that $\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$ are (not necessarily Noetherian) filtrations of R by m_R-primary ideals. Suppose that M is a finitely generated R-module. Then there exists a homogeneous polynomial $G(x_1, \ldots, x_r) \in \mathbb{R}[x_1, \ldots, x_r]$ which is of total degree d if G is nonzero such that for all $n_1, \ldots, n_r \in \mathbb{Z}_+$

$$\lim_{m\to\infty} \frac{\lambda(M/I(1)_{mn_1} \cdots I(r)_{nm_r}M)}{m^d} = G(n_1, \ldots, n_r).$$

We will see in Theorem 6.6 that the conclusions of the theorem hold for all $n_1, \ldots, n_r \in \mathbb{N}$.

Proof. Replacing R with $\hat{R}$, $I(j)_i$ with $I(j)_i\hat{R}$ and M with $M \otimes_R \hat{R}$, we may assume that R is complete. By Lemma 5.2 (taking $T = N(R)M$) we reduce to the case where R is analytically unramified. By Lemma 5.4 we reduce to the case in which R is analytically irreducible. By Lemma 5.3 we reduce to the case in which R is analytically irreducible and $M = R$. Theorem 6.1 now follows from Theorem 4.5. □

Let assumptions be as in the statement of Theorem 6.1. Generalizing the classical definition of mixed multiplicities for m_R-primary ideals [2], [28], [33], [32, Definition 17.4.3], we define the mixed multiplicities of M of type $(d_1,\ldots,d_r)$ with respect to the filtrations $\mathcal{I}(1),\ldots,\mathcal{I}(r)$ of R by m_R-primary ideals

$$e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M)$$

from the coefficients of the homogeneous polynomial $G(n_1,\ldots,n_r)$. Specifically, we write

$$G(n_1,\ldots,n_r)=\sum_{d_1+\cdots d_r=d}\frac{1}{d_1!\cdots d_r!}e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M)n_1^{d_1}\cdots n_r^{d_r}.$$

We write the multiplicity $e_R(\mathcal{I};M)=e_R(\mathcal{I}^{[d]};M)$ if $r=1$, and $\mathcal{I}=\{I_i\}$ is a filtration of R by m_R-primary ideals. We have

$$e_R(\mathcal{I};M)=\lim_{m\rightarrow\infty}d!\frac{\lambda(M/I_mM)}{m^d}.$$

Proposition 6.2. *Suppose that R is a d-dimensional Noetherian local ring with $\dim N(\hat{R})<d$. Suppose that $\mathcal{I}(j)=\{I(j)_i\}$ for $1\le j\le r$ are filtrations of R by m_R-primary ideals, and that M is a finitely generated R-module. Then for all $d_1,\ldots,d_r$ with $d_1+\cdots+d_r=d$, we have*

$$\lim_{a\rightarrow\infty}e_R(\mathcal{I}_a(1)^{[d_1]},\ldots,\mathcal{I}_a(r)^{[d_r]};M)=e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M).$$

Proof. The proof of Theorem 6.1 gives a reduction to the case in which R is analytically irreducible and $M=R$. The proposition now follows from Corollary 4.4. ☐

The following theorem extends to filtrations of R by m_R-primary ideals the Minkowski inequalities of m_R-primary ideals of Teissier [33], [34] and Rees and Sharp [30]. Inequality (4) of Theorem 6.3 was proven for graded families of m_R-primary ideals in a regular local ring with algebraically closed residue field by Mustaţă [25, Corollary 1.9] and more recently by Kaveh and Khovanskii [18, Corollary 7.14]. Inequality (4) was proven with our assumption that $\dim N(\hat{R})<d$ in [8, Theorem 3.1]. Inequalities (2)–(4) can be deduced directly from inequality (1), as in the proof of [32, Corollary 17.7.3], as explained in [34], [30], [32].

Theorem 6.3 (Minkowski inequalities)**.** *Suppose that R is a Noetherian d-dimensional local ring with $\dim N(\hat{R})<d$, that M is a finitely generated R-module, and that $\mathcal{I}(1)=\{I(1)_j\}$ and $\mathcal{I}(2)=\{I(2)_j\}$ are filtrations of R by m_R-primary ideals. Then*

(1) $e_R(\mathcal{I}(1)^{[i]},\ \mathcal{I}(2)^{[d-i]};\ M)^2\le e_R(\mathcal{I}(1)^{[i+1]},\ \mathcal{I}(2)^{[d-i-1]};\ M)e_R(\mathcal{I}(1)^{[i-1]},\ \mathcal{I}(2)^{[d-i+1]};M)$
for $1\le i\le d-1$;

(2) *for* $0\le i\le d$,
$e_R(\mathcal{I}(1)^{[i]},\mathcal{I}(2)^{[d-i]};M)e_R(\mathcal{I}(1)^{[d-i]},\mathcal{I}(2)^{[i]};M)\le e_R(\mathcal{I}(1);M)e_R(\mathcal{I}(2);M)$;

(3) *for* $0\le i\le d$, $e_R(\mathcal{I}(1)^{[d-i]},\mathcal{I}(2)^{[i]};M)^d\le e_R(\mathcal{I}(1);M)^{d-i}e_R(\mathcal{I}(2);M)^i$; *and*

(4) $e_R(\mathcal{I}(1)\mathcal{I}(2));M)^{\frac{1}{d}}\le e_R(\mathcal{I}(1);M)^{\frac{1}{d}}+e_R(\mathcal{I}(2);M)^{\frac{1}{d}}$,
where $\mathcal{I}(1)\mathcal{I}(2)=\{I(1)_jI(2)_j\}$.

Proof. By the reduction of the proof of Theorem 6.1, it suffices to prove the theorem for R an analytically irreducible domain and $M=R$. We first will show that for all $a\in\mathbb{Z}_+$ the Minkowski inequalities hold for the ath truncated filtrations $\mathcal{I}_a(1)=\{I_a(1)_m\}$ and $\mathcal{I}_a(2)=\{I_a(2)_m\}$ (defined in Definition 4.1).

Given $a\in\mathbb{Z}_+$, there exists an $f_a\in\mathbb{Z}_+$ such that $I_a(i)_{f_am}=(I_a(i)_{f_a})^m$ for all $m\geq 0$ and $i=1,2$. Define filtrations of R by m_R-primary ideals by $J_a(i)_m=I_a(i)_{f_am}$. Then for $n_1,n_2\in\mathbb{Z}_+$,

$$\lim_{m\to\infty}\frac{\lambda(R/J_a(1)_{mn_1}J_a(2)_{mn_2})}{m^d}=\sum_{d_1+d_2=d}\frac{1}{d_1!d_2!}e_R(J_a(1)_1^{[d_1]},J_a(2)_1^{[d_2]};R)n_1^{d_1}n_2^{d_2},$$

$$\lim_{m\to\infty}\frac{\lambda(R/J_a(k)_1^m)}{m^d}=\frac{1}{d!}e_R(J_a(k)_1;R)$$

for $k=1$ and 2 and

$$\lim_{m\to\infty}\frac{\lambda(R/(J_a(1)_1J_a(2)_1)^m)}{m^d}=\frac{1}{d!}e_R(J_a(1)_1J_a(2)_1;R),$$

where $e_R(J_a(1)^{[d_1]},\ J_a(2)^{[d_2]};R)$, $\mathrm{e}_R(J_a(1)_1;R)$, $e_R(J_a(2)_1;R)$, $e_R(J_a(1)J_a(2)_1;R)$ are the usual mixed multiplicities of ideals [32, Theorem 17.4.2 and Definition 17.4.3].

Now the Minkowski inequalities hold for the mixed multiplicities of ideals

$$e_R(J_a(1)_1^{[d_1]},J_a(2)_1^{[d_2]};R),\quad e_R(J_a(1)_1;R),\quad e_R(J_a(2)_1;R),$$

and

$$e_R(J_a(1)_1J_a(2)_1;R)$$

by [30] or [32, Theorem 17.7.2 and Corollary 17.7.3]. By Lemma 3.3

$$\lim_{m\to\infty}\frac{\lambda(R/I_a(1)_{mn_1},I_a(2)_{mn_2})}{m^d}=\frac{1}{f_a^d}\left(\lim_{m\to\infty}\frac{\lambda(R/J_a(1)_1^{mn_1}J_a(2)_1^{mn_2})}{m^d}\right)$$

for all $n_1,n_2\in\mathbb{N}$,

$$\lim_{m\to\infty}\frac{\lambda(R/I_a(k)_m)}{m^d}=\frac{1}{f_a^d}\lim_{m\to\infty}\frac{\lambda(R/J_a(k)_1^m)}{m^d}$$

for $k=1$ and 2, and

$$\lim_{m\to\infty}\frac{\lambda(R/I_a(1)_mI_a(2)_m)}{m^d}=\frac{1}{f_a^d}\lim_{m\to\infty}\frac{\lambda(R/(J_a(1)_1J_a(2)_1)^m)}{m^d}.$$

By Lemma 3.2

$$e_R(\mathcal{I}_a(1)^{[d_1]},\mathcal{I}_a(2)^{[d_2]};R)=\frac{1}{f_a^d}e_R(J_a(1)_1^{[d_1]},J_a(2)_1^{[d_2]};R)$$

for all d_1,d_2,

$$e_R(\mathcal{I}_a(1);R)=\frac{1}{f_a^d}e_R(J_a(1)_1;R),e_R(\mathcal{I}_a(2)_1;R)=\frac{1}{f_a^d}e_R(J_a(2)_1;R),$$

and

$$e_R(\mathcal{I}_a(1)\mathcal{I}_a(2);R)=\frac{1}{f_a^d}e_R(J_a(1)_1J_a(2)_1;R).$$

Thus the Minkowski inequalities hold for the $e_R(\mathcal{I}_a(1)^{[d_1]},\mathcal{I}_a(2)^{[d_2]};R)$, $e_R(\mathcal{I}_a(1);R)$, $e_R(\mathcal{I}_a(2);R)$, and $e_R(\mathcal{I}_a(1)\mathcal{I}_a(2);R)$. Now the Minkowski inequalities hold for

$$e_R(\mathcal{I}(1)^{[d_1]},\mathcal{I}(2)^{[d_2]};R),\quad e_R(\mathcal{I}(1);R),\quad e_R(\mathcal{I}(2);R),$$

and

$$e_R(\mathcal{I}(1)\mathcal{I}(2);R)$$

by Proposition 6.2. ∎

Remark 6.4 (Minkowski equality). Teissier [35] (for Cohen–Macaulay normal complex analytic R), Rees and Sharp [30] (in dimension 2), and Katz [16] (in complete generality) have proven that if R is a d-dimensional formally equidimensional Noetherian local ring and $I(1)$, $I(2)$ are m_R-primary ideals such that the Minkowski equality

$$e_R((I(1)I(2));R)^{\frac{1}{d}} = e_R(I(1);R)^{\frac{1}{d}} + e_R(I(2);R)^{\frac{1}{d}}$$

holds, then there exist positive integers r and s such that the complete ideals $\overline{I(1)^r}$ and $\overline{I(2)^s}$ are equal, which is equivalent to the statement that the R-algebras $\bigoplus_{n\geq 0} I(1)^n$ and $\bigoplus_{n\geq 0} I(2)^n$ have the same integral closure.

This statement is not true for filtrations, even in a regular local ring, as is shown by the following simple example. Let k be a field and R be the power series ring $R = k[[x_1,\ldots,x_d]]$. Let $\mathcal{I}(1) = \{I(1)_i\}$, where $I(1)_i = m_R^i$, and $\mathcal{I}(2) = \{I(2)_i\}$, where $I(2)_i = m_R^{i+1}$. Then the Minkowski equality

$$e_R((\mathcal{I}(1)\mathcal{I}(2));R)^{\frac{1}{d}} = e_R(\mathcal{I}(1);R)^{\frac{1}{d}} + e_R(\mathcal{I}(2);R)^{\frac{1}{d}}$$

is satisfied, but $\bigoplus_{i\geq 0} I(1)_i$ and $\bigoplus_{i\geq 0} I(2)_i$ do not have the same integral closure.

The following proposition generalizes an identity of Rees [28, Lemma 2.4].

Proposition 6.5. *Suppose that R is a Noetherian local ring of dimension d such that $\dim N(\hat{R}) < d$ and $\mathcal{I}(1) = \{I(1)_i\},\ldots,\mathcal{I}(r) = \{I(r)_i\}$ are filtrations of R by m_R-primary ideals. Suppose that M is a finitely generated R-module. Then for $1 \leq i \leq r$*

$$\begin{aligned} &e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(i-1)^{[d_{i-1}]},\mathcal{I}(i)^{[0]},\mathcal{I}(i+1)^{[d_{i+1}]},\ldots,\mathcal{I}(r)^{[d_r]};M)\\ &\quad = e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(i-1)^{[d_{i-1}]},\mathcal{I}(i+1)^{[d_{i+1}]},\ldots,\mathcal{I}(r)^{[d_r]};M)\end{aligned}$$

whenever $d_1+\cdots+d_{i-1}+d_{i+1}\cdots+d_r = d$.

In particular,

$$e_R(\mathcal{I}(i);M) = e_R(\mathcal{I}(1)^{[0]},\ldots,\mathcal{I}(i-1)^{[0]},\mathcal{I}(i)^{[d]},\mathcal{I}(i+1)^{[0]},\ldots,\mathcal{I}(r)^{[0]};M).$$

Proof. By the proof of Theorem 6.3, we need only show that the identities hold for m_R-primary ideals $I(1),\ldots,I(r)$. We may assume that $i = r$. Let $G(x_1,\ldots,x_r) \in \mathbb{Q}[x_1,\ldots,x_r]$ be the homogeneous polynomial of degree d such that

$$\lim_{m\to\infty} \frac{\lambda(M/I(1)^{mn_1}\cdots I(r)^{mn_r}M)}{m^d} = G(n_1,\ldots,n_r)$$

whenever $n_1,\ldots,n_r \in \mathbb{Z}_+$, and let $Q(x_1,\ldots,x_{r-1}) \in \mathbb{Q}[x_1,\ldots,x_{r-1}]$ be the homogeneous polynomial of degree d such that

$$\lim_{m\to\infty} \frac{\lambda(M/I(1)^{mn_1}\cdots I(r-1)^{mn_{r-1}}M)}{m^d} = Q(n_1,\ldots,n_{r-1})$$

whenever $n_1,\ldots,n_{r-1} \in \mathbb{Z}_+$. Then for all $n_1,\ldots,n_{r-1} \in \mathbb{Z}_+$

$$\lim_{m\to\infty} \frac{G(mn_1,\ldots,mn_{r-1},1)}{m^d} = \lim_{m\to\infty} \frac{G(mn_1,\ldots,mn_{r-1},0)}{m^d},$$

and for $\alpha \in \mathbb{Z}_+$

$$\lim_{m\to\infty} \frac{Q(mn_1,\ldots,mn_{r-1}+\alpha)}{m^d} = \lim_{m\to\infty} \frac{Q(mn_1,\ldots,mn_{r-1})}{m^d}.$$

There exists an $\alpha \in \mathbb{Z}_+$ such that $I(r-1)^\alpha \subset I(r)$. Thus for $n_1,\ldots,n_{r-1} \in \mathbb{Z}_+$

$$Q(n_1,\ldots,n_{r-1}) \leq G(n_1,\ldots,n_{r-1},1) \leq Q(n_1,\ldots,n_{r-1}+\alpha),$$

and thus we have equality of polynomials,

$$\begin{array}{rcl} Q(n_1,\ldots,n_{r-1}) &=& \lim_{m\to\infty}\frac{Q(mn_1,\ldots,mn_{r-1})}{m^d}\\ &=& \lim_{m\to\infty}\frac{G(mn_1,\ldots,mn_{r-1},0)}{m^d}\\ &=& G(n_1,\ldots,n_{r-1},0),\end{array}$$

and the theorem holds (for m_R-primary ideals). □

As a consequence of the above proposition, we extend the conclusions of Theorem 6.1 to all $n_1,\ldots,n_r\in\mathbb{N}$.

Theorem 6.6. *Suppose that R is a Noetherian local ring of dimension d such that*

$$\dim N(\hat{R})<d$$

and $\mathcal{I}(1)=\{I(1)_i\},\ldots,\mathcal{I}(r)=\{I(r)_i\}$ are (not necessarily Noetherian) filtrations of R by m_R-primary ideals. Suppose that M is a finitely generated R-module. Then there exists a homogeneous polynomial $G(x_1,\ldots,x_r)\in\mathbb{R}[x_1,\ldots,x_r]$ which is of total degree d if G is nonzero such that for all $n_1,\ldots,n_r\in\mathbb{N}$

$$\lim_{m\to\infty}\frac{\lambda(M/I(1)_{mn_1}\cdots I(r)_{nm_r}M)}{m^d}=G(n_1,\ldots,n_r).$$

The proof of the following proposition is by the same method as the proof of Theorem 6.3, starting with the fact that the identities of Proposition 6.7 hold for m_R-primary ideals by [32, Lemma 17.4.4].

Proposition 6.7. *Suppose that R is a Noetherian local ring of dimension d such that $\dim N(\hat{R})<d$, and that $\mathcal{I}(1)=\{I(1)_i\},\ldots,\mathcal{I}(r)=\{I(r)_i\}$ are filtrations of R by m_R-primary ideals. Suppose that*

$$0\to M_1\to M_2\to M_3\to 0$$

is a short exact sequence of finitely generated R-modules. Then for any $d_1,\ldots,d_r\in\mathbb{N}$ with $d_1+\cdots+d_r=d$, we have

$$\begin{aligned}&e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M_2)\\ &\quad=e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M_1)+e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M_3).\end{aligned}$$

The following associativity formula is proven for m_R-primary ideals in [32, Theorem 17.4.8].

Theorem 6.8 (Associativity formula)**.** *Suppose that R is a Noetherian local ring of dimension d with $\dim N(\hat{R})<d$. Suppose that $\mathcal{I}(j)=\{I(j)_i\}$ for $1\le j\le r$ are filtrations of R by m_R-primary ideals, that M is a finitely generated R-module, and that $\mathcal{I}(1)=\{I(1)_i\},\ldots\mathcal{I}(r)=\{I(r)_i\}$ are filtrations of R by m_R-primary ideals. Let P be a minimal prime of R. Then $\dim N(\widehat{R/P})<d$. For any $d_1,\ldots,d_r\in\mathbb{N}$ with $d_1+\cdots+d_r=d$,*

$$\begin{aligned}&e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M)\\ &\quad=\sum\lambda_{R_P}(M_P)e_{R/P}((\mathcal{I}(1)R/P)^{[d_1]},\ldots,(\mathcal{I}(r)R/P)^{[d_r]};R/P),\end{aligned}$$

where the sum is over the minimal primes of R such that $\dim R/P=d$ and $\mathcal{I}(j)R/P=\{I(j)_iR/P\}$.

Proof. Let $\overline{R} = R/N(R)$. We have $N(\hat{\overline{R}}) = N(\hat{R})\hat{\overline{R}}$, so $\dim N(\hat{\overline{R}}) < d = \dim \hat{\overline{R}}$.

Let $P_1, \ldots, P_s$ be the minimal primes of R, and let $S = \bigoplus_{i=1}^s R/P_i$. As in the proof of Lemma 5.4, we have a natural inclusion $\overline{R} \to S$, and there exists a non zerodivisor $x \in \overline{R}$ such that $xS \subset \overline{R}$. Further, x is a non zerodivisor on S since S is a subring of the total quotient ring of $\overline{R}$. Since completion is flat, we have an induced inclusion

$$\hat{\overline{R}} \to \hat{S} = \bigoplus_{i=1}^{s} \widehat{R/P_i}.$$

We have $xN(\hat{S}) \subset N(\hat{\overline{R}})$. Now x is a non zerodivisor on $\hat{S}$ since it is on S and the completion is flat. Thus $\dim N(\hat{S}) \le \dim N(\hat{\overline{R}}) < d$, so $\dim N(\widehat{R/P_i}) < d$ for all i.

By Theorem 6.1 and Lemmas 5.2 and 3.2, we have

$$e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M) = e_{\overline{R}}(\overline{\mathcal{I}}(1)^{[d_1]}, \ldots, \overline{\mathcal{I}}(r)^{[d_r]}; \overline{M}),$$

where $\overline{\mathcal{I}}(j) = \{I(j)_i\overline{R}\}$ for $1 \le j \le r$ and $\overline{M} = M/N(R)M$.

By Theorem 6.1 and Lemmas 5.4 and 3.2

$$e_{\overline{R}}(\overline{\mathcal{I}}(1)^{[d_1]}, \ldots, \overline{\mathcal{I}}(r)^{[d_r]}; \overline{M}) = \sum_{i=1}^{s} e_{R/P_i}((\mathcal{I}(1)R/P_i)^{[d_1]}, \ldots, (\mathcal{I}(r)R/P_i)^{[d_r]}; M/P_iM).$$

Now for $1 \le i \le r$

$$\begin{aligned} &e_{R/P_i}((\mathcal{I}(1)R/P_i)^{[d_1]}, \ldots, (\mathcal{I}(r)R/P_i)^{[d_r]}; M/P_iM) \\ &\quad = \lambda_{R_{P_i}}(M_{P_i})e_{R/P_i}((\mathcal{I}(1)R/P_i)^{[d_1]}, \ldots, (\mathcal{I}(r)R/P_i)^{[d_r]}; R/P_i) \end{aligned}$$

by Lemma 5.3 since $R_{P_i} = Q(R/P_i)$. ☐

The following theorem generalizes [32, Proposition 11.2.1] for m_R-primary ideals to filtrations of R by m_R-primary ideals.

Theorem 6.9. *Suppose that R is a Noetherian d-dimensional local ring such that*

$$\dim N(\hat{R}) < d,$$

and such that M is a finitely generated R-module. Suppose that $\mathcal{I}' = \{I'_i\}$ and $\mathcal{I} = \{I_i\}$ are filtrations of R by m_R-primary ideals. Suppose that $\mathcal{I}' \subset \mathcal{I}$ ($I'_i \subset I_i$ for all i) and that the ring $\bigoplus_{n\ge0} I_n$ is integral over $\bigoplus I'_n$. Then

$$e_R(\mathcal{I}; M) = e_R(\mathcal{I}'; M).$$

The converse of Theorem 6.9 is false. Taking R to be a power series ring $R = k[[x_1, \ldots, x_d]]$ over a field k, let $I_i = m_R^i$ and $I'_i = m_R^{i+1}$. Then $e_R(\mathcal{I}; R) = e_R(\mathcal{I}'; R)$, but $\bigoplus_{n\ge0} I_n$ is not integral $\bigoplus_{n\ge0} I'_n$. This is in contrast to a theorem of Rees, in [28], [32, Theorem 11.3.1], showing that if R is a formally equidimensional Noetherian local ring and $I' \subset I$ are m_R-primary ideals, then $\bigoplus_{n\ge0} I^n$ is integral over $\bigoplus_{n\ge0}(I')^n$ if and only if $e_R(I; R) = e_R(I'; R)$.

Proof of Theorem 6.9.

(1) We first observe that if $I' \subset I$ are m_R-primary ideals and that $\bigoplus_{n\ge0} I^n$ is integral over $\bigoplus_{n\ge0}(I')^n$, then, by [32, Theorem 8.2.1, Corollary 1.2.5, and Proposition 11.2.1], $e_R(I; R) = e_R(I'; R)$.

(2) Suppose that $\mathcal{I} = \{I_i\}$ and $\mathcal{I}' = \{I'_i\}$ are Noetherian filtrations of R by m_R-primary ideals and $\mathcal{I}' \subset \mathcal{I}$. Suppose that $b \in \mathbb{Z}_+$. Define $\mathcal{I}^{(b)} = \{I_i^{(b)}\}$,

where $I_i^{(b)} = I_{bi}$, and define $(\mathcal{I}')^{(b)} = \{(I')_i^{(b)}\}$, where $(I')_i^{(b)} = (I')_{bi}$. Then from Lemma 3.3 we deduce that

$$e_R(\mathcal{I};R) = e_R(\mathcal{I}';R) \qquad \text{if and only if } e_R(\mathcal{I}^{(b)};R) = e_R((\mathcal{I}')^{(b)};R).$$

(3) Suppose $\mathcal{I}' \subset \mathcal{I}$ are filtrations of R by m_R-primary ideals. Suppose that $a \in \mathbb{Z}_+$. Let $\mathcal{I}_a = \{I_{a,n}\}$ be the ath truncated filtration of $\mathcal{I}$ defined in Definition 4.1. Then there exists $\overline{a} \in \mathbb{Z}$ such that every element of $\bigoplus_{n\ge 0} I_{a,n}$ (considered as a subring of $\bigoplus_{n\ge 0} I_n$) is integral over $\bigoplus_{n\ge 0} I'_{\overline{a},n}$, where $\mathcal{I}'_{\overline{a}} = \{I'_{\overline{a},i}\}$ is the $\overline{a}$th truncated filtration of $\mathcal{I}'$ defined in Definition 4.1.

Define a Noetherian filtration $\mathcal{A}_a = \{A_{a,i}\}$ of R by m_R-primary ideals by $A_{a,i} = I_{a,i} + I'_{\overline{a},i}$. Thus we have inclusions of graded rings $\bigoplus_{n\ge 0} I'_{\overline{a},n} \subset \bigoplus_{n\ge 0} A_{a,n}$, and $\bigoplus_{n\ge 0} A_{a,n}$ is finite over $\bigoplus_{n\ge 0} I'_{\overline{a},n}$. By steps (2) and (1)

$$e_R(\mathcal{I}'_{\overline{a}};R) = e_R(\mathcal{A}_a;R).$$

By Proposition 4.3

$$\lim_{a\to\infty} e_R(\mathcal{I}'_{\overline{a}};R) = e_R(\mathcal{I}';R),$$

and thus

$$\lim_{a\to\infty} e_R(\mathcal{A}_a;R) = e_R(\mathcal{I}';R).$$

(4) Let the notation be as in the proof of Proposition 4.3, but taking $J_i = I_i$ and $J(a)_i = I'_{ai}$. Define

$$\Gamma(\mathcal{A}_a)^{(t)} = \{(m_1,\ldots,m_d,i) \in \mathbb{N}^{d+1} \mid \dim_k A_{a,i} \cap K_{m_1\lambda_1+\cdots+m_d\lambda_d}/A_{a,i} \cap K^+_{m_1\lambda_1+\cdots+m_d\lambda_d} \ge t \text{ and } m_1+\cdots+m_d \le \beta i\}.$$

Now $\Gamma(a)^{(t)} \subset \Gamma(\mathcal{A}_a)^{(t)} \subset \Gamma^{(t)}$ for all t, so

$$\Delta(\Gamma(a)^{(t)}) \subset \Delta(\Gamma(\mathcal{A}_a)^{(t)}) \subset \Delta(\Gamma^{(t)})$$

for all a. By (14)

$$\lim_{a\to\infty} \mathrm{Vol}(\Delta(\Gamma(a)^{(t)})) = \mathrm{Vol}(\Delta(\Gamma^{(t)})),$$

so

$$\lim_{a\to\infty} \mathrm{Vol}(\Delta(\Gamma(A_a)^{(t)})) = \mathrm{Vol}(\Delta(\Gamma^{(t)})).$$

Thus

$$\lim_{a\to\infty} e_R(\mathcal{A}_a;R) = e_R(\mathcal{I};R)$$

by (12) of the proof of Proposition 4.3 applied to $\mathcal{A}_a$.

(5) We have $e_R(\mathcal{I};R) = e_R(\mathcal{I}';R)$ by steps (3) and (4). Now $e_R(\mathcal{I};M) = e_R(\mathcal{I}';M)$ by Theorem 6.8 (with $r=1$). ☐

Corollary 6.10. *Suppose that R is a Noetherian d-dimensional local ring such that*

$$\dim N(\hat{R}) < d,$$

and that M is a finitely generated R-module. Suppose that $\mathcal{I}(j)' = \{I(j)'_i\}$ and $\mathcal{I}(j) = \{I(j)_i\}$ are filtrations of R by m_R-primary ideals for $1 \le j \le r$. Suppose that $\mathcal{I}(j)' \subset \mathcal{I}(j)$ for $1 \le j \le r$ and that the ring

$$\bigoplus_{n_1,\ldots,n_r\ge 0} I(1)_{n_1} I(2)_{n_2} \cdots I(r)_{n_r}$$

is integral over

$$\bigoplus_{n_1,\ldots,n_r\ge 0} I(1)'_{n_1} I(2)'_{n_2} \cdots I(r)'_{n_r}.$$

Then

(18)
$$e_R(\mathcal{I}(q)^{[d_1]},\mathcal{I}(2)^{[d_2]},\ldots,\mathcal{I}(r)^{[d_r]};M)=e_R((\mathcal{I}(1)')^{[d_1]},(\mathcal{I}(2)')^{[d_2]},\ldots,(\mathcal{I}(r)')^{[d_r]};M)$$

for all $d_1,\ldots,d_r\in\mathbb{N}$ *with* $d_1+\cdots+d_r=d$.

Proof. For $n_1,\ldots,n_r\in\mathbb{Z}_+$ the ring $\bigoplus_{m\ge 0}I(1)_{mn_1}I(2)_{mn_2}\cdots I(r)_{mn_r}$ is integral over $\bigoplus_{m\ge 0}I(1)'_{mn_1}I(2)'_{mn_2}\cdots I(r)'_{mn_r}$, so

$$\lim_{m\to\infty}\frac{\lambda(M/I(1)_{mn_1}I(2)_{mn_2}\cdots I(r)_{mn_r}M)}{m^d}=\lim_{m\to\infty}\frac{\lambda(M/I(1)'_{mn_1}I(2)'_{mn_2}\cdots I(r)'_{mn_r}M)}{m^d}$$

by Theorem 6.9. Thus we have the equalities (18) by Lemma 3.2 and Theorem 6.1. □

7. Multigraded filtrations

We define a multigraded filtration $\mathcal{I}=\{I_{n_1,\ldots,n_r}\}_{n_1,\ldots,n_r\in\mathbb{N}}$ of ideals on a ring R to be a collection of ideals of R such that $R=I_{0,\ldots,0}$,

$$I_{n_1,\ldots,n_{n_{j-1}},n_j+1,n_{j+1},\ldots,n_r}\subset I_{n_1,\ldots,n_{j-1},n_j,n_{j+1},\ldots,n_r}$$

for all $n_1,\ldots,n_r\in\mathbb{N}$ and $I_{a_1,\ldots,a_r}I_{b_1,\ldots,b_r}\subset I_{a_1+b_1,\ldots,a_r+b_r}$ whenever $a_1,\ldots,a_r$, $b_1,\ldots,b_r\in\mathbb{N}$.

A multigraded filtration $\mathcal{I}=\{I_{n_1,\ldots,n_r}\}$ of ideals on a local ring R is a multigraded filtration of R by m_R-primary ideals if $I_{n_1,\ldots,n_r}$ is m_R-primary whenever $n_1+\cdots+n_r>0$.

If R is a Noetherian local ring of dimension d with $\dim N(\hat{R})<d$ and $\mathcal{I}=\{I_{n_1,\ldots,n_r}\}$ is a multigraded filtration of R by m_R-primary ideals, then we can define (by Theorem 1.1) the function

$$P(n_1,\ldots,n_r)=\lim_{m\to\infty}\frac{\lambda(R/I_{mn_1,\ldots,mn_r})}{m^d} \tag{19}$$

and ask if it has polynomial like behavior. The following example shows that it can be far from polynomial like, so Theorem 6.1 does not have a good generalization to arbitrary multigraded filtrations of m_R-primary ideals.

Let $R=k[[t]]$ be a power series ring over a field k. For $(n_1,n_2)\in\mathbb{N}^2$ define $\alpha:\mathbb{N}^2\to\mathbb{N}$ by

$$\alpha(n_1,n_2)=\lceil\sqrt{n_1^2+n_2^2}\rceil,$$

where for a real number x $\lceil x\rceil$ is the smallest integer a such that $x\le a$.

Define $I_{n_1,n_2}=(t^{\alpha(n_1,n_2)})$ and $\mathcal{I}=\{I_{n_1,n_2}\}$. Then $\mathcal{I}$ is a multigraded filtration of R by m_R-primary ideals. For $(n_1,n_2)\in\mathbb{N}^2$ we have

$$P(n_1,n_2)=\lim_{m\to\infty}\frac{\lambda(R/I_{mn_1,mn_2})}{m}=\lim_{m\to\infty}\frac{\lceil m\sqrt{n_1^2+n_2^2}\rceil}{m}=\lceil\sqrt{n_1^2+n_2^2}\rceil.$$

We now show that the function (19) is polynomial like in an important situation. Let R be an excellent, normal local ring of dimension 2, and let $f:X\to\operatorname{Spec}(R)$ be a resolution of singularities, with integral exceptional divisors $E_1,\ldots,E_r$. A resolution of singularities of a two-dimensional, excellent local domain always exists by [21] or [4]. If $n_1,\ldots,n_r\in\mathbb{N}$, let $D_{n_1,\ldots,n_r}=\sum_{i=1}^r n_iE_i$, and define

$$I_{n_1,\ldots,n_r}=\Gamma(X,\mathcal{O}_X(-D_{n_1,\ldots,n_r})),$$

which is an m_R-primary ideal in R. Then $\{I_{n_1,\ldots,n_r}\}$ is a multigraded filtration of R by m_R-primary ideals. By [5, Theorem 4] if the divisor class group $\mathrm{Cl}(R)$ is not a torsion group, then there exists a resolution of singularities $f : X \to \mathrm{Spec}(R)$ and an exceptional divisor F on X such that $\bigoplus_{n\geq 0}\Gamma(X,\mathcal{O}_X(-nF))$ is not a finitely generated R-algebra, so $\bigoplus_{n_1,\ldots,n_r\geq 0} I_{n_1,\ldots,n_r}$ is not a finitely generated R-algebra, and thus the multigraded filtration $\{I_{n_1,\ldots,n_r}\}$ is not Noetherian. In Proposition 6.3 [9] it is shown that there exists an abstract complex of polyhedral sets $\mathcal{P}$ whose union is $\mathbb{Q}_{\geq 0}$ [9, Definition 4.4] such that for $P\in\mathcal{P}$ and $(n_1,\ldots,n_r)\in P\cap\mathbb{N}^r$

$$\lambda(R/I_{n_1,\ldots,n_r}) = Q_P(n_1,\ldots,n_r) + L_P(n_1,\ldots,n_r) + \Phi_P(n_1,\ldots,n_r),$$

where $Q_P(n_1,\ldots,n_r)$ is a quadratic polynomial with rational coefficients, $L_P(n_1,\ldots,n_r)$ is a linear function with periodic coefficients (a linear quasi polynomial), and $\Phi_P(n_1,\ldots,n_r)$ is a bounded function ($|\Phi_P(n_1,\ldots,n_r)|$ is bounded). Thus the function defined in (19) is piecewise polynomial, with

$$P(n_1,\ldots,n_r) = \lim_{m\to\infty}\frac{\lambda(R/I_{mn_1,\ldots,mn_r})}{m^2} = Q_P(n_1,\ldots,n_r)$$

if $(n_1,\ldots,n_r)\in P$. We have the further interpretation of $P(n_1,\ldots,n_r)$ as the intersection product

$$P(n_1,\ldots,n_r) = -\frac{1}{2}(\Delta^2_{n_1,\ldots,n_r}),$$

where $\Delta_{n_1,\ldots,n_r}$ is the Zariski $\mathbb{Q}$-divisor associated to $-n_1E_1-\cdots-n_rE_r$ [9, Formula (7)].

References

[1] S. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces*, Pure and Applied Mathematics, vol. 24, Academic Press, New York–London, 1966. MR0217069

[2] P. B. Bhattacharya, *The Hilbert function of two ideals*, Proc. Cambridge Philos. Soc. **53** (1957), 568–575. MR0089835

[3] N. Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989. Translated from the French; Reprint of the 1972 edition. MR979760

[4] V. Cossart, U. Jannsen, and S. Saito, *Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes*, arXiv:0905.2191 (2009).

[5] S. D. Cutkosky, *On unique and almost unique factorization of complete ideals. II*, Invent. Math. **98** (1989), no. 1, 59–74, DOI 10.1007/BF01388844. MR1010155

[6] S. D. Cutkosky, *Multiplicities associated to graded families of ideals*, Algebra Number Theory **7** (2013), no. 9, 2059–2083, DOI 10.2140/ant.2013.7.2059. MR3152008

[7] S. D. Cutkosky, *Asymptotic multiplicities of graded families of ideals and linear series*, Adv. Math. **264** (2014), 55–113, DOI 10.1016/j.aim.2014.07.004. MR3250280

[8] S. D. Cutkosky, *Asymptotic multiplicities*, J. Algebra **442** (2015), 260–298, DOI 10.1016/j.jalgebra.2015.03.017. MR3395062

[9] S. D. Cutkosky, J. Herzog, and A. Reguera, *Poincaré series of resolutions of surface singularities*, Trans. Amer. Math. Soc. **356** (2004), no. 5, 1833–1874, DOI 10.1090/S0002-9947-03-03346-4. MR2031043

[10] S. D. Cutkosky and V. Srinivas, *On a problem of Zariski on dimensions of linear systems*, Ann. of Math. (2) **137** (1993), no. 3, 531–559, DOI 10.2307/2946531. MR1217347

[11] L. Ein, R. Lazarsfeld, and K. E. Smith, *Uniform approximation of Abhyankar valuation ideals in smooth function fields*, Amer. J. Math. **125** (2003), no. 2, 409–440. MR1963690

[12] K. Goel, R. V. Gurjar, and J. K. Verma, *The Minkowski (in)equality for multiplicity of ideals*, arXiv:1807.06309 (2018).

[13] M. Herrmann, E. Hyry, J. Ribbe, and Z. Tang, *Reduction numbers and multiplicities of multigraded structures*, J. Algebra **197** (1997), no. 2, 311–341, DOI 10.1006/jabr.1997.7128. MR1483767

[14] J. Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, J. Amer. Math. Soc. **25** (2012), no. 3, 907–927, DOI 10.1090/S0894-0347-2012-00731-0. MR2904577

[15] N. Jacobson, *Basic algebra. I*, 2nd ed., W. H. Freeman and Company, New York, 1985. MR780184

[16] D. Katz, *Note on multiplicity*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1021–1026, DOI 10.2307/2047584. MR929434

[17] K. Kaveh and A. G. Khovanskii, *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), no. 2, 925–978, DOI 10.4007/annals.2012.176.2.5. MR2950767

[18] K. Kaveh and A. Khovanskii, *Convex bodies and multiplicities of ideals*, Proc. Steklov Inst. Math. **286** (2014), no. 1, 268–284, DOI 10.1134/S0081543814060169. Reprint of Tr. Mat. Inst. Steklova **286** (2014), 291–307. MR3482603

[19] D. Katz and J. K. Verma, *Extended Rees algebras and mixed multiplicities*, Math. Z. **202** (1989), no. 1, 111–128, DOI 10.1007/BF01180686. MR1007742

[20] S. Lang, *Algebra*, Springer-Verlag, New York–Berlin–Heidelberg, 2002. Revised 3rd ed.

[21] J. Lipman, *Desingularization of two-dimensional schemes*, Ann. of Math. (2) **107** (1978), no. 1, 151–207. MR0491722

[22] R. Lazarsfeld and M. Mustaţă, *Convex bodies associated to linear series* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 5, 783–835, DOI 10.24033/asens.2109. MR2571958

[23] H. Matsumura, *Commutative algebra*, 2nd ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980. MR575344

[24] H. T. Muhly and M. Sakuma, *Asymptotic factorization of ideals*, J. London Math. Soc. **38** (1963), 341–350, DOI 10.1112/jlms/s1-38.1.341. MR0159837

[25] M. Mustaţă, *On multiplicities of graded sequences of ideals*, J. Algebra **256** (2002), no. 1, 229–249, DOI 10.1016/S0021-8693(02)00112-6. MR1936888

[26] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers, a division of John Wiley & Sons, New York–London, 1962. MR0155856

[27] A. Okounkov, *Why would multiplicities be log-concave?*, The orbit method in geometry and physics (Marseille, 2000), Progr. Math., vol. 213, Birkhäuser Boston, Boston, MA, 2003, pp. 329–347. MR1995384

[28] D. Rees, *$\mathfrak{a}$-transforms of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. **57** (1961), 8–17. MR0118750

[29] D. Rees, *Multiplicities, Hilbert functions and degree functions*, Commutative algebra: Durham 1981 (Durham, 1981), London Math. Soc. Lecture Note Ser., vol. 72, Cambridge University Press, Cambridge, England–New York, 1982, pp. 170–178. MR693635

[30] D. Rees and R. Y. Sharp, *On a theorem of B. Teissier on multiplicities of ideals in local rings*, J. London Math. Soc. (2) **18** (1978), no. 3, 449–463, DOI 10.1112/jlms/s2-18.3.449. MR518229

[31] I. Swanson, *Mixed multiplicities, joint reductions and quasi-unmixed local rings*, J. London Math. Soc. (2) **48** (1993), no. 1, 1–14, DOI 10.1112/jlms/s2-48.1.1. MR1223888

[32] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, England, 2006. MR2266432

[33] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney* (French), Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), Soc. Math. France, Paris, 1973, pp. 285–362. Astérisque, Nos. 7 et 8. MR0374482

[34] B. Teissier, *Sur une inégalité pour les multiplicitiés*, Ann. of Math. **106** (1977), 38–44. Appendix to a paper by D. Eisenbud and H. Levine.

[35] B. Teissier, *On a Minkowski-type inequality for multiplicities. II*, C. P. Ramanujam-A tribute, Tata Inst. Fund. Res. Studies in Math., vol. 8, Springer, Berlin–New York, 1978, pp. 347–361. MR541030

[36] Ngo Viet Trung and J. Verma, *Mixed multiplicities of ideals versus mixed volumes of polytopes*, Trans. Amer. Math. Soc. **359** (2007), no. 10, 4711–4727, DOI 10.1090/S0002-9947-07-04054-8. MR2320648

[37] O. Zariski and P. Samuel, *Commutative algebra, Volume I*, The University Series in Higher Mathematics, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1958. With the cooperation of I. S. Cohen. MR0090581

Department of Mathematics, University of Missouri, Columbia, Missouri 65211
Email address: cutkoskys@missouri.edu

Department of Mathematics, University of Missouri, Columbia, Missouri 65211
Email address: parangamasarkar@gmail.com

Department of Mathematics, University of Missouri, Columbia, Missouri 65211
Email address: srinivasanh@missouri.edu