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THE SKOROKHOD REPRESENTATION THEOREM FOR YOUNG MEASURES

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ABSTRACT. In this paper, we extend the well-known Skorokhod representation theorem for Young measures and show that the Skorokhod representation property is transmitted between spaces. The open mapping theorem for Young measures stated by Tateishi in the case of compact metric spaces is also generalized to the case of metrizable Souslin spaces.

1. INTRODUCTION

Let $\mathbb{S}$ be a Polish space (that is, a completely metrizable, separable topological space), and let $\nu^n (n \in \mathbb{N} \cup \{0\})$ be probability measures on $\mathbb{S}$ such that $\nu^n \to \nu^0$ weakly*. The Skorokhod representation theorem states that there exists a sequence $\xi_n (n \in \mathbb{N} \cup \{0\})$ of random variables on some probability space such that the law of ξ_n coincides with ν^n and $\xi_n \to \xi_0$ almost surely. This theorem was first proved by Skorokhod [12], and later Dudley [6] extends it to the case of a general separable metric space. In the case of a Polish space, this theorem is further strengthened by Blackwell and Dubins [2] and Fernique [7]. They demonstrate the existence of a function $\xi : \mathcal{M}_1^+(\mathbb{S}) \times [0,1] \to \mathbb{S}$ such that the law of $\xi(\nu, \cdot)$ coincides with ν and that the map $\nu \mapsto \xi(\nu, x)$ is continuous for almost all $x \in [0,1]$. An extension to nonmetrizable topological spaces was also studied by Banakh, Bogachev, and Kolesnikov [3], Bogachev and Kolesnikov [4], and Jakubowski [8].

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. A probability measure on $(\Omega \times \mathbb{S}, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}))$ whose projection on Ω is equal to μ is called a Young measure. The notion of Young measures was introduced by Young [15] to represent the "generalized curves" in the relaxation problem of optimal control and calculus of variation. Young measures are also called "transition probabilities", "Markov kernels", and "relaxed control functions", depending on the context, and the theory of Young measures has been extensively researched by several authors (see, e.g., [1, 5, 10, 14]).

The purpose of this paper is to extend the Skorokhod representation theorem to Young measures. The Skorokhod representation theorem is hitherto stated for probability measures defined on the purely topological space. It is of some interest whether some version of the Skorokhod representation theorem holds for the probability measures defined on the product of the nontopological measurable set and the topological set. This paper is the first attempt to consider the problem. First we show that if $\mathbb{S}$ is a Polish space, the space of Young measures $\mathcal{Y}(\Omega; \mathbb{S})$ has the Skorokhod representation property (for the definition, see Section 2). Secondly, we

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show that the Skorokhod representation property is transmitted between spaces. More specifically, let $\mathbb{S}, \mathbb{T}$ be metrizable Souslin spaces, and assume that there exists a continuous, open, and surjective map $\varphi : \mathbb{S} \to \mathbb{T}$. We show that if the space $\mathcal{Y}(\Omega; \mathbb{S})$ has the Skorokhod representation property, then the space $\mathcal{Y}(\Omega; \mathbb{T})$ also has the Skorokhod representation property. This transmission property of the Skorokhod representation is used to establish the Skorokhod representation property for metrizable Souslin spaces.

We also consider an open mapping theorem for Young measures to prove the transmission property of the Skorokhod representation. An open mapping theorem for Young measures was previously considered as the subject of independent interest in a paper by Tateishi [13]. However, it is also crucial to consider the problem to demonstrate the transmission property of the Skorokhod representation. In this paper, we generalize the open mapping theorem for Young measures stated in Tateishi [13] in the case of compact metric spaces to the case of metrizable Souslin spaces.

The idea that this paper follows is not new. The transmission property of the Skorokhod representation for measures between spaces is considered in Banakh, Bogachev, and Kolesnikov [3] and Bogachev and Kolesnikov [4]. The author is inspired by their papers in studying the problem.

Following Section 2, we first consider, in Section 3, the Skorokhod representation theorem for Young measures in the case in which $\mathbb{S}$ is the unit interval. We extend it, in Section 4, to the case in which $\mathbb{S}$ is a Polish space. Section 5 is devoted to the transmission property of the Skorokhod representation for Young measures. The Skorokhod representation propety will be established for metrizable Souslin spaces, thanks to the transmission property and the Skorokhod representation theorem for Young measures in the case in which $\mathbb{S}$ is the unit interval.

2. Preliminaries

Let $\mathbb{S}$ be a metrizable Souslin space, and let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space. The Borel σ-algebra on $\mathbb{S}$ is denoted by $\mathcal{B}(\mathbb{S})$, the set of probability measures on $\mathcal{B}(\mathbb{S})$ by $\mathcal{M}_1^+(\mathbb{S})$, the set of nonnegative measures on $\mathcal{B}(\mathbb{S})$ by $\mathcal{M}^+(\mathbb{S})$, and the set of bounded measures on $\mathcal{B}(\mathbb{S})$ by $\mathcal{M}^b(\mathbb{S})$. These sets are endowed with the weak* topology, that is, the topology generated by the seminorms:

$$\nu \mapsto |\nu(\psi)| = \left| \int_{\mathbb{S}} \psi(\omega)\nu(d\omega) \right|,$$

where ψ is a continuous and bounded function on $\mathbb{S}$. A probability measure on $(\Omega \times \mathbb{S}, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}))$ whose projection on Ω is equal to μ is called a Young measure. By the theorem of disintegration of measures, there corresponds, to each Young measure ν, a function $\nu^* : \Omega \times \mathcal{B}(\mathbb{S}) \to [0, 1]$ called a disintegration of measure ν which satisfies the following conditions:

(i) $\nu^*(\omega, \cdot)$ is a probability measure on $\mathbb{S}$ for each $\omega \in \Omega$,
(ii) $\nu^*(\cdot, A)$ is measurable for each $A \in \mathcal{B}(\mathbb{S})$,
(iii) $\nu(A) = \int_\Omega \left[\int_{\mathbb{S}} \chi_A(\omega, s)\nu^*_\omega(ds)\right] \mu(d\omega)$ for each $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{S})$,

where χ_A stands for the characteristic function of A. Note that condition (ii) is equivalent to the condition that the map $\omega \mapsto \nu^*(\omega, \cdot) : \Omega \to \mathcal{M}_1^+(\mathbb{S})$ is measurable with respect to the Borel σ-algebra generated by the weak* topology on $\mathcal{M}_1^+(\mathbb{S})$.

Note also that to each Young measure there correspond many disintegrations. However, there is exactly one one-to-one correspondence between the space of Young measures and the equivalence classes of μ-almost everywhere equal disintegrations. We shall identify hereafter each Young measure ν with its disintegration ν^*. We denote by $\mathcal{Y}(\Omega;\mathbb{S})$ the space of all Young measures on $\Omega \times \mathbb{S}$.

A real-valued function ψ defined on $\Omega\times\mathbb{S}$ is said to be a Carathéodory integrand if the following conditions hold:

(i) ψ is $\mathcal{F}\otimes\mathcal{B}(\mathbb{S})$-measurable,

(ii) for each $\omega\in\Omega$, the map $s\mapsto\psi(\omega,s)$ is continuous and bounded,

(iii) the map $\omega\mapsto\|\psi(\omega,\cdot)\|_\infty$ is μ-integrable,

where $\|\cdot\|_\infty$ stands for the sup-norm. The set of all Carathéodory integrands on $\Omega\times\mathbb{S}$ is denoted by $\mathcal{G}_C(\Omega;\mathbb{S})$. The space $\mathcal{Y}(\Omega;\mathbb{S})$ of Young measures is endowed with the topology generated by the seminorms:

$$\nu\mapsto|\nu(\psi)| = \left|\int_{\Omega\times\mathbb{S}}\psi d\nu\right| = \left|\int_\Omega\left[\int_{\mathbb{S}}\psi(\omega,s)\nu_\omega(ds)\right]\mu(d\omega)\right|,$$

where $\psi\in\mathcal{G}_C(\Omega;\mathbb{S})$. The topology on $\mathcal{Y}(\Omega;\mathbb{S})$ is called the stable topology. We also consider, on $\mathcal{Y}(\Omega;\mathbb{S})$, the topology generated by the seminorms:

$$\nu\mapsto|\nu(\chi_A\otimes f)|,$$

where $A\in\mathcal{F}$ and where f is a bounded continuous function on $\mathbb{S}$. This topology is said to be the W-stable topology. We remark that the stable topology is finer than the W-stable topology, and a net $\{\nu^\alpha\}$ converges W-stably to ν^0 if and only if, for each $A\in\mathcal{F}$, the net $\{\nu^\alpha(A\times\cdot)\}$ of elements of $\mathcal{M}^+(\mathbb{S})$ converges weakly* to $\nu^0(A\times\cdot)$ (see, e.g., Castaing, Raynaud de Fitte, and Valadier [5, p. 21]). We remark also that if $\mathbb{S}$ is a metrizable Souslin space, the stable topology coincides with the W-stable topology (see Castaing, Raynaud de Fitte, and Valadier [5, p. 23, Theorem 2.1.3]). If $\mathbb{S}$ is a compact metric space, the space $\mathcal{G}_C(\Omega;\mathbb{S})$ of Carathéodory integrands is isometrically isomorphic to the space $L_1(\Omega, C(\mathbb{S}))$ of Bochner integrable functions defined on Ω in the space $C(\mathbb{S})$ of continuous real-valued functions defined on $\mathbb{S}$. The dual of $L_1(\Omega, C(\mathbb{S}))$ is the linear space $L_\infty(\Omega,\mathcal{M}^b(\mathbb{S}))$ of equivalence classes of scalarly measurable functions defined on Ω in the space $\mathcal{M}^b(\mathbb{S})$ of bounded measures. We can consider the space $\mathcal{Y}(\Omega;\mathbb{S})$ of Young measures as a subset of $L_\infty(\Omega,\mathcal{M}^b(\mathbb{S}))$. Then the stable topology on $\mathcal{Y}(\Omega;\mathbb{S})$ is equal to the relative topology of the topology on $L_\infty(\Omega,\mathcal{M}^b(\mathbb{S}))$ defined by the duality $\sigma(L_\infty, L_1)$. We remark also that each metrizable Souslin space $\mathbb{S}$ can be embedded in a (nonunique) compact metric space $\hat{\mathbb{S}}$ as a subspace, and the stable topology on $\mathcal{Y}(\Omega;\mathbb{S})$ is the relative topology of $\mathcal{Y}(\Omega;\hat{\mathbb{S}})$ considered as a subspace.

The Lebesgue measure on the unit interval $[0,1]$ is denoted by λ.

We say that the space $\mathbb{S}$ has the Skorokhod representation property for Young measures if there exists a map $\xi:\mathcal{Y}(\Omega;\mathbb{S})\times\mathcal{F}\times[0,1]\to\mathbb{S}$ such that

(i) for each $\nu\in\mathcal{Y}(\Omega;\mathbb{S})$ and $A\in\mathcal{F}$, the law of $\xi(\nu,A,\cdot)$ coincides with $\mathbb{E}(\nu\mid A)$,

(ii) for each $A\in\mathcal{F}$, the map $\nu\mapsto\xi(\nu,A,x)$ is continuous for almost all x in $[0,1]$,

where

$$\mathbb{E}(\nu\mid A) = \frac{1}{\mu(A)}\int_A\nu_\omega\mu(d\omega)$$

stands for the conditional expectation of ν given A.

Remark 1. Since the W-stable convergence of Young measures is equivalent to the weak* convergence of its conditional expectations, conditions (i) and (ii) fully characterize the stable topology of Young measures.

3. The Skorokhod representation theorem for Young measures: $\mathbb{S} = [0,1]$

First we state the Skorokhod representation theorem for Young measures in the case in which $\mathbb{S}$ is the unit interval. In the next section, we establish the Skorokhod representation theorem on Polish spaces. The following theorem is used to establish the Skorokhod representation property for metrizable Souslin spaces by making use of the transmission property of the Skorokhod representation, which will also be established later.

Theorem 1. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space, and let $\mathcal{Y}(\Omega; [0,1])$ be endowed with the stable topology. Then there exists a map $\xi : \mathcal{Y}(\Omega; [0,1]) \times \mathcal{F} \times [0,1] \to [0,1]$ such that*

(i) *for each $\nu \in \mathcal{Y}(\Omega; [0,1])$ and $A \in \mathcal{F}$, the map $x \mapsto \xi(\nu, A, x)$ is nondecreasing;*
(ii) *for each $\nu \in \mathcal{Y}(\Omega; [0,1])$ and $A \in \mathcal{F}$, the law of $\xi(\nu, A, \cdot)$ coincides with $\mathbb{E}(\nu \mid A)$; and*
(iii) *for each $A \in \mathcal{F}$, the map $\nu \mapsto \xi(\nu, A, x)$ is continuous for almost all x in $[0,1]$.*

Lemma 1. [1] *Let ν^n, ν^0 be in $\mathcal{Y}(\Omega; [0,1])$. Assume that $\nu^n \to \nu^0$ stably. Then, for almost all $x \in [0,1]$ and $A \in \mathcal{F}$,*

$$\int_A \nu^n_\omega([0,x])\mu(d\omega) \to \int_A \nu^0_\omega([0,x])\mu(d\omega).$$

Proof. Since $[0,1]$ is closed and the sequence $\{\nu^n(A \times \cdot)\}$ converges weakly* to $\{\nu^0(A \times \cdot)\}$, we have

$$\limsup_{n\to\infty} \int_A \nu^n_\omega([0,x])\mu(d\omega) \le \int_A \nu^0_\omega([0,x])\mu(d\omega).$$

Also, we have the inequality

$$\liminf_{n\to\infty} \int_A \nu^n_\omega([0,x[)\mu(d\omega) \ge \int_A \nu^0_\omega([0,x[)\mu(d\omega).$$

Thus, the equality

$$\lim_{n\to\infty} \int_A \nu^n_\omega([0,x])\mu(d\omega) = \int_A \nu^0_\omega([0,x])\mu(d\omega)$$

holds for the continuity point x of the function $\theta : t \mapsto \int_A \nu^0_\omega([0,t])\mu(d\omega)$. Since the function θ is nondecreasing, the complement of the set of continuity points of θ is countable and, thus, Lebesgue negligible. □

Now we prove the Skorokhod representation theorem for Young measures in the case in which $\mathbb{S} = [0,1]$.

[1] The assertion of the lemma of the early version of my manuscript had some mistakes. The anonymous referee kindly told me about the mistake and gave me the complete proof of the modified lemma. The proof of the lemma is owed to the anonymous referee.

Proof of Theorem 1. Define $\xi : \mathcal{Y}(\Omega; \mathbb{S}) \times \mathcal{F} \times [0,1] \to [0,1]$ by

$$\xi(\nu, A, x) = \inf \left\{ y : \mu(A)x \le \int_A \nu_\omega([0,y])\mu(d\omega) \right\}.$$

Property (i) is a direct consequence of the definition of ξ. Furthermore, since, for each $A \in \mathcal{F}$, $\mu(A)x \le \int_A \nu_\omega([0,y])\mu(d\omega)$ is equivalent to $y \ge \xi(\nu, A, x)$, by definition, we have

$$\begin{aligned}\lambda[x : \xi(\nu, A, x) \le y] &= \lambda[x : \mu(A)x \le \int_A \nu_\omega([0,y])\mu(d\omega)] \\ &= \frac{1}{\mu(A)} \int_A \nu_\omega([0,y])\mu(d\omega) \\ &= \mathbb{E}(\nu \mid A)([0,y]).\end{aligned}$$

Hence, we obtain property (ii).

Finally, we have to check property (iii). For this purpose, let $\nu^n (n \in \mathbb{N} \cup \{0\})$ be a sequence in $\mathcal{Y}(\Omega; [0,1])$ such that $\nu^n \to \nu^0$ stably, and let $x \in [0,1]$, $A \in \mathcal{F}$ with $\mu(A) > 0$ and $\epsilon > 0$ be given. Let y be such that $\xi(\nu^0, A, x) - \epsilon < y < \xi(\nu^0, A, x)$, and the convergence of Lemma 1 holds. Then, by the second inequality, we have

$$\int_A \nu^0_\omega([0,y])\mu(d\omega) < \nu(A)x.$$

Thus, in view of Lemma 1, there exists $n_0 \in \mathbb{N}$ such that

$$\int_A \nu^n_\omega([0,y])\mu(d\omega) < \mu(A)x$$

for all $n > n_0$, and we obtain $\xi(\nu^0, A, x) - \epsilon < y \le \xi(\nu^n, A, x)$ for all $n > n_0$. By taking limit inferior with respect to n of the inequality and letting $\epsilon \downarrow 0$, we get $\liminf_{n\to\infty} \xi(\nu^n, A, x) \ge \xi(\nu^0, A, x)$.

Let us now fix $x, x' \in [0,1]$ with $x < x'$, $A \in \mathcal{F}$ with $\mu(A) > 0$ and $\epsilon > 0$. Let y be such that $\xi(\nu^0, A, x') < y < \xi(\nu^0, A, x') + \epsilon$, and the convergence of Lemma 1 holds. Then

$$\mu(A)x < \mu(A)x' \le \int_A \nu^0_\omega([0, \xi(\nu^0, A, x')])\mu(d\omega) \le \int_A \nu^0_\omega([0,y])\mu(d\omega),$$

where the second inequality follows from the definition of ξ, and the third is the consequence of the monotonicity of the map $t \mapsto \int_A \nu^0_\omega([0,t])\mu(d\omega)$. Hence, again by Lemma 1, $\mu(A)x < \int_A \nu^n_\omega([0,y])\mu(d\omega)$ for sufficiently large n, and we obtain the following inequality, $\xi(\nu^n, A, x) \le y < \xi(\nu^0, A, x') + \epsilon$, and hence, by taking the limit superior with respect to n and letting $\epsilon \downarrow 0$, we get $\limsup_{n\to\infty} \xi(\nu^n, A, x) \le \xi(\nu^0, A, x')$. It follows that $\xi(\nu^n, A, x) \to \xi(\nu^0, A, x)$ if the map $t \mapsto \xi(\nu^0, A, t)$ is continuous at x. Since $t \mapsto \xi(\nu^0, A, t)$ is a nondecreasing map by (i), the set of discontinuous points of $\xi(\nu^0, A, \cdot)$ has measure 0, and thus $\xi(\nu^n, A, x) \to \xi(\nu^0, A, x)$ as $n \to +\infty$ for almost all $x \in [0,1]$. □

4. The Skorokhod representation theorem for Young measures on Polish spaces

In this section, we state and prove the Skorokhod representation theorem for Young measures on Polish spaces. Let us begin by stating the Skorokhod representation theorem for measures due to Blackwell and Dubins [2]. This theorem is used to prove the Skorokhod representation theorem for Young measures.

Theorem 2 (Blackwell and Dubins [2, Theorem])**.** *Let $\mathbb{S}$ be a Polish space. Then there exists a function $\xi : \mathcal{M}_1^+(\mathbb{S}) \times [0,1] \to \mathbb{S}$ such that*

(i) *the law of $\xi(\nu, \cdot)$ coincides with ν, and*
(ii) *the map $\nu \mapsto \xi(\nu, x)$ is continuous for almost all $x \in [0,1]$.*

Now we state the Skorokhod representation theorem for Young measures on Polish spaces.

Theorem 3. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space, and let $\mathbb{S}$ be a Polish space. Then $\mathbb{S}$ has the Skorokhod representation property for Young measures.*

Proof. Recall that for a net $\nu^\alpha \in \mathcal{Y}(\Omega; \mathbb{S})$, the stable convergence of ν^α to ν^0 implies the W-stable convergence of ν^α to ν^0, and the W-stable convergence of ν^α to ν^0 is equivalent to $\mathbb{E}(\nu^\alpha \mid A) \to \mathbb{E}(\nu^0 \mid A)$ weakly* for all $A \in \mathcal{F}$. Let $\eta : \mathcal{M}_1^+(\mathbb{S}) \times [0,1] \to \mathbb{S}$ be the Skorokhod map for measures on $\mathbb{S}$ obtained in Theorem 2 and define $\xi : \mathcal{Y}(\Omega; \mathbb{S}) \times \mathcal{F} \times [0,1] \to \mathbb{S}$ by

$$\xi(\nu, A, x) = \eta(\mathbb{E}(\nu \mid A), x).$$

By property (i) of Theorem 2, the law of $\xi(\nu, A, \cdot)$ equals $\mathbb{E}(\nu \mid A)$. Hence, the Skorokhod representation property (i) for Young measures follows. Moreover, since $\nu^\alpha \to \nu^0$ stably implies that $\mathbb{E}(\nu^\alpha \mid A) \to \mathbb{E}(\nu^0 \mid A)$ weakly* for all $A \in \mathcal{F}$, and $\eta(\cdot, x)$ is weak* continuous on $\mathcal{M}_1^+(\mathbb{S})$ for almost all $x \in [0,1]$ by property (ii) of Theorem 2, the Skorokhod representation property (ii) for Young measures can be obtained, and this completes the proof of the theorem. □

5. The transmission of the Skorokhod representation property

In this section, we consider the transmission of the Skorokhod representation property across spaces. The Skorokhod representation property will be established for all metrizable Souslin spaces, thanks to the transmission property. The Skorokhod representation property is transmitted across spaces which are connected by a continuous, open, and surjective map. In fact, the following result holds.

Theorem 4. *Let $(\Omega, \mathcal{F}, \mu)$ be a nonatomic complete finite positive measure space, and let $\mathbb{S}, \mathbb{T}$ be compact metric spaces such that $\mathbb{S}$ has the Skorokhod representation property for Young measures. Assume that a continuous, open, and surjective mapping $\varphi : \mathbb{S} \to \mathbb{T}$ is given. Then $\mathbb{T}$ also has the Skorokhod representation property for Young measures.*

We consider an open mapping theorem for Young measures to prove Theorem 4. Let $\mathbb{S}, \mathbb{T}$ be metrizable Souslin spaces, and let $\varphi : \mathbb{S} \to \mathbb{T}$ be a continuous, open, and surjective map. This map induces a map $\pi : \mathcal{M}_1^+(\mathbb{S}) \to \mathcal{M}_1^+(\mathbb{T})$ by the relation $\pi(\nu) = \nu \circ \varphi^{-1}$. We define a map $\Pi : \mathcal{Y}(\Omega; \mathbb{S}) \to \mathcal{Y}(\Omega; \mathbb{T})$ by the relation: $\Pi(\nu)_\omega = \pi(\nu_\omega)$. We remark that, since φ is continuous, $\nu \mapsto \pi(\nu)$ is continuous with respect to the weak* topology, and hence $\nu \mapsto \pi(\nu)$ is measurable. The map $\omega \mapsto \Pi(\nu)_\omega$ is measurable as the composition of two measurable maps: $\omega \mapsto \nu_\omega$ and $\nu \mapsto \pi(\nu)$. Thus, $\Pi(\nu) \in \mathcal{Y}(\Omega; \mathbb{T})$. The following is immediate.

Lemma 2. $\Pi : \mathcal{Y}(\Omega; \mathbb{S}) \to \mathcal{Y}(\Omega; \mathbb{T})$ *satisfies the following relation:*

(1)
$$\int_\Omega \left[\int_\mathbb{T} \psi(\omega, t) \Pi(\nu)_\omega(dt) \right] \mu(d\omega) = \int_\Omega \left[\int_\mathbb{S} \psi(\omega, \varphi(s)) \nu_\omega(ds) \right] \mu(d\omega); \psi \in \mathcal{G}_C(\Omega, \mathbb{T}).$$

The following open mapping theorem for Young measures appeared as a subject of independent interest in Tateishi [13]. However, the theorem is also crucial to proving the transmission property of the Skorokhod representation between spaces.

Theorem 5 (Tateishi [13, Theorem 1]). *Let $(\Omega, \mathcal{F}, \mu)$ be a nonatomic complete finite positive measure space, and let $\mathbb{S}, \mathbb{T}$ be compact metric spaces. Assume that $\varphi : \mathbb{S} \to \mathbb{T}$ is a continuous, open, and surjective mapping. Then $\Pi : \mathcal{Y}(\Omega; \mathbb{S}) \to \mathcal{Y}(\Omega; \mathbb{T})$ is also a continuous, open, and surjective mapping.*

The following lemma is an open mapping theorem for measures due to Schief [11].

Lemma 3 (Schief [11, Theorem 2.3]). *Let $\mathbb{S}, \mathbb{T}$ be Souslin spaces, and let $\varphi : \mathbb{S} \to \mathbb{T}$ be a continuous, open, and surjective mapping. Then the map $\pi : \mathcal{M}_1^+(\mathbb{S}) \to \mathcal{M}_1^+(\mathbb{T})$ defined by $\pi(\nu) = \nu \circ \varphi^{-1}$ is also a continuous, open, and surjective mapping, where $\mathcal{M}_1^+(\mathbb{S})$ and $\mathcal{M}_1^+(\mathbb{T})$ are each endowed with the weak* topology.*

Also, we use Michael's continuous selection theorem.

Lemma 4 (Michael [9, Theorem 1.2]). *Let $\mathbb{S}$ be a paracompact space, and let $\mathbb{T}$ be a metrizable set in a complete locally convex linear topological space $\mathfrak{X}$. Suppose also that the correspondence $\Gamma : \mathbb{S} \to \mathbb{T}$ is lower hemicontinuous and nonempty and closed valued such that $\Gamma(s)$ is complete for each $s \in \mathbb{S}$. Then there exists a continuous map $f : \mathbb{S} \to \mathbb{T}$ such that $f(s) \in \overline{\text{co}}\, \Gamma(s)$ for all $s \in \mathbb{S}$, where $\overline{\text{co}}\, A$ stands for the closed convex hull of A.*

Remark 2. The completeness of the locally convex topological vector space $\mathfrak{X}$ assumed in Lemma 4 can be replaced by the following condition: for every compact subset K of $\mathfrak{X}$, the closed convex hull of K in $\mathfrak{X}$ is compact.

Now we prove Theorem 4.

Proof of Theorem 4. Let $\Pi : \mathcal{Y}(\Omega; \mathbb{S}) \to \mathcal{Y}(\Omega; \mathbb{T})$ be defined by $\Pi(\nu)_\omega = \pi(\nu_\omega)$. Then by Theorem 5, Π is continuous and open with respect to the stable topology and is surjective. We remark that, by virtue of the assumption that the underlying space which defines the space of Young measures is a compact metric space, the space of Young measures is also compact and metrizable. Therefore, we can apply Michael's continuous selection theorem (Lemma 4) to the (multivalued) inverse map $\Pi^{-1} : \mathcal{Y}(\Omega; \mathbb{T}) \twoheadrightarrow \mathcal{Y}(\Omega; \mathbb{S})$ and deduce the existence of a continuous map $\Psi : \mathcal{Y}(\Omega; \mathbb{T}) \to \mathcal{Y}(\Omega; \mathbb{S})$ such that $\Pi \circ \Psi$ is the identity map on $\mathcal{Y}(\Omega; \mathbb{T})$.

We define a map $\Xi : \mathcal{Y}(\Omega; \mathbb{T}) \times \mathcal{F} \times [0, 1] \to \mathbb{T}$ by

$$\Xi(\tau, A, x) = \varphi(\xi(\Psi(\tau), A, x)),$$

where ξ is the Skorokhod representation for the space of Young measures $\mathcal{Y}(\Omega; \mathbb{S})$ obtained in Theorem 3. Then the map $\tau \mapsto \Xi(\tau, A, x)$ is easily seen to be continuous for almost all $x \in [0, 1]$. Hence, the Skorokhod representation property (ii) is immediate. Furthermore, the law of $\lambda \circ \Xi^{-1}(\tau, A, \cdot)$ equals $\mathbb{E}(\tau \mid A)$, as the following estimates indicate: for each $\psi \in C(\mathbb{T})$,

$$\begin{aligned}
&\int_{\mathbb{T}} \psi(t)(\lambda \circ \Xi^{-1}(\tau, A, \cdot))(dt) \\
&\quad = \int_{\mathbb{T}} \psi(t)(\lambda \circ \xi^{-1}(\Psi(\tau), A, \cdot) \circ \varphi^{-1})(dt) \\
&\quad = \int_{\mathbb{S}} \psi(\varphi(s))(\lambda \circ \xi^{-1}(\Psi(\tau), A, \cdot))(ds) \\
&\quad = \int_{\mathbb{S}} \psi(\varphi(s))\mathbb{E}(\Psi(\tau) \mid A)(ds) \\
&\quad = \frac{1}{\mu(A)} \int_A \left[\int_{\mathbb{S}} \psi(\varphi(s))\Psi(\tau)_\omega(ds)\right] \mu(d\omega) \\
&\quad = \frac{1}{\mu(A)} \int_A \left[\int_{\mathbb{T}} \psi(t)\tau_\omega(dt)\right] \mu(d\omega) \\
&\quad = \int_{\mathbb{T}} \psi(t)\mathbb{E}(\tau \mid A)\mu(d\omega),
\end{aligned}$$

where the fifth equality follows from the fact that Ψ is the right-inverse of Π and relation (1). □

In order to generalize Theorem 4 to the case in which $\mathbb{S}, \mathbb{T}$ are metrizable Souslin spaces, we have to generalize the open mapping theorem for Young measures to the case of metrizable Souslin spaces. The following theorem is the generalized version of the open mapping theorem for Young measures.

Theorem 6. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space, and let $\mathbb{S}, \mathbb{T}$ be metrizable Souslin spaces. Assume that $\varphi : \mathbb{S} \to \mathbb{T}$ is a continuous, open, and surjective mapping. Then $\Pi : \mathcal{Y}(\Omega; \mathbb{S}) \to \mathcal{Y}(\Omega; \mathbb{T})$ is also a continuous, open, and surjective mapping.*

To prove this theorem, we use the following lemmata.

Lemma 5 (Bogachev and Kolesnikov [4, Lemma 2.2])**.** *Let ν be a Radon probability measure on a complete regular space $\mathfrak{X}$, and let*

$$G_0 = \{\tau \in \mathcal{M}_1^+(\mathfrak{X}) : \tau(V_j) - \nu(V_j) > -\epsilon_0, j = 1, \dots, m\},$$

where $V_1, \dots, V_m$ are open sets in $\mathfrak{X}$ and where $\epsilon_0 > 0$. Then for every $\epsilon > 0$, there exist pairwise disjoint open sets $W_1, \dots, W_p$ and a number $\delta > 0$ such that

(i) $\nu(V_j) - \sum_{\{k: W_k \subset V_j\}} \nu(W_k) < \epsilon$,
(ii) $G = \{\tau \in \mathcal{M}_1^+(\mathfrak{X}) : \tau(W_k) - \nu(W_k) > -\delta, k = 1, \dots, p\} \subset G_0$.

The following lemma is a characterization of the stable topology on the space of Young measures.

Lemma 6. *The stable topology on $\mathcal{Y}(\Omega; \mathbb{S})$ can be defined by means of the base generated by the sets of the form: for a finite number of measurable sets $A_i \in \mathcal{F}, i = 1, \dots, m$ and a finite number of open sets $V_j^i, i = 1, \dots, m, j = 1, \dots, n$ in $\mathbb{S}$ and positive real numbers $\epsilon_i, i = 1, \dots, m$,*

$$\{\tau \in \mathcal{Y}(\Omega; \mathbb{S}) : \mathbb{E}(\tau \mid A_i)(V_j^i) - \mathbb{E}(\nu \mid A_i)(V_j^i) > -\epsilon_i, i = 1, \dots, m, j = 1, \dots, n\}.$$

Proof. Since the stable topology on $\mathcal{Y}(\Omega;\mathbb{S})$ is generated by the seminorms

$$\nu \mapsto \int_A \left[\int_{\mathbb{S}} \psi(s)\nu_\omega(ds)\right]\mu(d\omega), \qquad A \in \mathcal{F},\ \psi \in C^b(\mathbb{S}),$$

it can be defined by means of the base generated by sets of the form: for a finite number of measurable sets $A_1, \dots, A_m \in \mathcal{F}$, a finite number of continuous and bounded functions $\psi_1, \dots, \psi_q \in C^b(\mathbb{S})$ and $\epsilon > 0$,

$$\left\{\tau \in \mathcal{Y}(\Omega;\mathbb{S}) : \int_{A_i} \left[\int_{\mathbb{S}} \psi_j(s)[\tau_\omega - \nu_\omega](ds)\right]\mu(d\omega) < \epsilon \text{ for all } i, j\right\}.$$

However, this set can be rewritten by setting $\epsilon_i = \epsilon/\mu(A_i)$ as

$$\left\{\tau \in \mathcal{Y}(\Omega;\mathbb{S}) : \int_{\mathbb{S}} \psi_j(s)\mathbb{E}(\tau - \nu \mid A_i)(ds) < \epsilon_i \text{ for all } i, j\right\}.$$

The weak* topology on $\mathcal{M}_1^+(\mathbb{S})$ can be defined by means of the base generated by the sets of the form: $\{\tau \in \mathcal{M}_1^+(\mathbb{S}) : \tau(V) > \nu(V) - \epsilon\}$, where V is an open set and $\epsilon > 0$. Since the condition $\int_{\mathbb{S}} \psi_j(s)\mathbb{E}(\tau - \nu \mid A_i)(ds) < \epsilon_i$ for all j is equivalent to the condition that $\mathbb{E}(\tau - \nu \mid A_i)$ belongs to some neighborhood of 0 in $\mathcal{M}^b(\mathbb{S})$, this condition can be rewritten for some open sets $V_j^i, i = 1, \dots, m, j = 1, \dots, n$ in $\mathbb{S}$ and positive numbers $\epsilon_i, i = 1, \dots, m$ as follows:

$$\mathbb{E}(\tau \mid A_i)(V_j^i) - \mathbb{E}(\nu \mid A_i)(V_j^i) > -\epsilon_i, \qquad i = 1, \dots, m,\ j = 1, \dots, n.$$

Thus, we conclude that the assertion of the lemma holds true. □

We now show the open mapping theorem for Young measures. The proof makes use of the reasoning stated in Bogachev and Kolesnikov [4, proof of Theorem 2.3]. In fact, the proof follows closely the proof of Bogachev and Kolesnikov with some minor modifications. We run the complete proof for the reader's convenience.

Proof of Theorem 6. The continuity of Π is immediate by relation (1). By Lemmas 3 and 4, $\pi(\nu)$ defined by $\nu \circ \varphi^{-1}$ admits a continuous right-inverse $\hat{\pi}$. Define $\hat{\Pi}$ as follows: $\hat{\Pi}(\tau)_\omega = \hat{\pi}(\tau_\omega)$. Then, since $\hat{\pi}$ is continuous, the map $\omega \mapsto \hat{\Pi}(\tau)_\omega$ is measurable, and hence $\hat{\Pi}(\tau) \in \mathcal{Y}(\Omega;\mathbb{S})$ and $\Pi \circ \hat{\Pi}(\tau) = \tau$. Thus, Π is surjective. Hence, we have to show only that Π is an open mapping.

Let $\nu^0 \in \mathcal{Y}(\Omega;\mathbb{S})$ be given, and let $\tau^0 = \Pi(\nu^0)$. By Lemma 6, we may assume, without loss of generality, that a neighborhood of ν^0 is of the form

$$U = \{\nu \in \mathcal{Y}(\Omega;\mathbb{S}) : \mathbb{E}(\nu \mid A_i)(V_j^i) - \mathbb{E}(\nu^0 \mid A_i)(V_j^i) > -\epsilon_i, i = 1, \dots, m, j = 1, \dots, n\},$$

where $A_i, i = 1, \dots, m$ are pairwise disjoint measurable subsets of Ω, and $V_j^i, i = 1, \dots, m, j = 1, \dots, n$ are open sets on $\mathbb{S}$, which can be taken to be pairwise disjoint with respect to the index j thanks to Lemma 5.

We have to show that, for some neighborhood O of $\tau^0 = \Pi(\nu^0)$, each measure $\tau \in O$ can be associated with a measure $\nu \in U$ such that $\tau = \Pi(\nu)$. We first consider the case in which A_i is independent of i. Then the set U can be reformulated as follows:

$$U = \{\nu \in \mathcal{Y}(\Omega;\mathbb{T}) : \mathbb{E}(\nu \mid A)(V_j) - \mathbb{E}(\nu^0 \mid A)(V_j) > -\epsilon, j = 1, \dots, n\}.$$

Since the map φ is open, $\varphi(V_j)$ is open for any open set $V_j \subset \mathbb{S}$. By Lemma 5, there exist pairwise disjoint open sets $W_1, \dots, W_p \subset \mathbb{T}$ such that

$$\mathbb{E}(\tau^0 \mid A)(\varphi(V_j)) - \sum_{W_k \subset \varphi(V_j)} \mathbb{E}(\tau^0 \mid A)(W_k) < \epsilon/2.$$

Let $\epsilon_1 < \frac{\epsilon}{2p}$ be positive, and let us fix a neighborhood of τ^0 of the form

$$O = \{\tau \in \mathcal{Y}(\Omega; \mathbb{T}) : \mathbb{E}(\tau \mid A)(W_k) - \mathbb{E}(\tau^0 \mid A)(W_k) > -\epsilon_1, k = 1, \dots, p\}.$$

We set $I_k = \{j : W_k \subset \varphi(V_j)\}$. Since $V_j, j = 1, \dots, n$ is pairwise disjoint, we may find nonnegative numbers α_{jk} such that $\sum_{j \in I_k} \alpha_{jk} = 1$ and

$$\alpha_{jk}\mathbb{E}(\tau^0 \mid A)(W_k) \geq \mathbb{E}(\nu^0 \mid A)(\varphi^{-1}(W_k) \cap V_j).$$

For every $j \in I_k$, the map $\varphi : \varphi^{-1}(W_k) \cap V_j \to W_k$ is continuous, open, and surjective. Thus, the map $\pi_{jk} : \mathcal{M}_1^+(\varphi^{-1}(W_k) \cap V_j) \to \mathcal{M}_1^+(W_k)$ defined, for each $\nu \in \mathcal{M}_1^+(\varphi^{-1}(W_k) \cap V_j)$, by $\pi_{jk}(\nu) = \nu \circ \varphi^{-1}$ has a continuous right-inverse $\hat{\pi}_{jk}$ thanks to Lemma 4. We define ν_ω^{jk} as $\nu_\omega^{jk} = \hat{\pi}_{jk}(\tau_\omega/\tau_\omega(W_k)) \cdot \tau_\omega(W_k)$. Then, since both of the maps $\omega \mapsto \hat{\pi}_{jk}(\tau_\omega/\tau_\omega(W_k))$ and $\omega \mapsto \tau_\omega(W_k)$ are measurable, the map $\omega \mapsto \nu_\omega^{jk}$ is measurable. We also define $\tilde{\nu}$ as follows: the map $\varphi : \mathbb{S} \setminus \bigcup_k \varphi^{-1}(W_k) \to \mathbb{T} \setminus \bigcup_k W_k$ is continuous, open, and surjective, and hence the map $\pi_{-k} : \mathcal{M}_1^+(\mathbb{S} \setminus \bigcup_k \varphi^{-1}(W_k)) \to \mathcal{M}_1^+(\mathbb{T} \setminus \bigcup_k W_k)$ defined, for each $\nu \in \mathcal{M}_1^+(\mathbb{S} \setminus \bigcup_k \varphi^{-1}(W_k))$, by $\pi_{-k}(\nu) = \nu \circ \varphi^{-1}$ has a continuous right-inverse $\hat{\pi}_{-k}$ thanks to Lemma 4. We define $\tilde{\nu}_\omega = \hat{\pi}_{-k}(\tau_\omega/\tau_\omega(\mathbb{T} \setminus \bigcup_k W_k)) \cdot \tau_\omega(\mathbb{T} \setminus \bigcup_k W_k)$. It follows that $\omega \mapsto \tilde{\nu}_\omega$ is also measurable.

Let $\nu = \tilde{\nu} + \sum_k \sum_{j \in I_k} \alpha_{jk}\nu^{jk}$. Then $\tau = \Pi(\nu)$. Let us show that $\nu \in U$.

$$\begin{aligned}
\mathbb{E}(\nu \mid A)(V_j) &\geq \sum_{k:j \in I_k} \alpha_{jk}\mathbb{E}(\nu^{jk} \mid A)(V_j) \\
&= \sum_{k:j \in I_k} \alpha_{jk}\mathbb{E}(\tau \mid A)(W_k) \\
&> \sum_{k:j \in I_k} \alpha_{jk}(\mathbb{E}(\tau^0 \mid A)(W_k) - \epsilon_1) \\
&\geq \sum_{k:j \in I_k} \alpha_{jk}\mathbb{E}(\tau^0 \mid A)(W_k) - p\epsilon_1 \\
&\geq \sum_{k:j \in I_k} \mathbb{E}(\nu^0 \mid A)(\varphi^{-1}(W_k) \cap V_j) - p\epsilon_1 \\
&= \mathbb{E}(\nu^0 \mid A)(V_j) - \mathbb{E}(\nu^0 \mid A)(V_j \setminus \bigcup_{k:j \in I_k} \varphi^{-1}(W_k)) - p\epsilon_1 \\
&\geq \mathbb{E}(\nu^0 \mid A)(V_j) - \mathbb{E}(\tau^0 \mid A)(\varphi(V_j) \setminus \bigcup_{k:W_k \in \varphi(V_j)} W_k) - p\epsilon_1 \\
&\geq \mathbb{E}(\nu^0 \mid A)(V_j) - \frac{\epsilon}{2} - p\epsilon_1 \\
&> \mathbb{E}(\nu^0 \mid A)(V_j) - \epsilon.
\end{aligned}$$

Now we consider the general case in which A_i depends on i. Then the U, O are of the form

$$\begin{aligned} U = \{\nu \in \mathcal{Y}(\Omega; \mathbb{S}) : \mathbb{E}(\nu \mid A_i)(V_j^i) - \mathbb{E}(\nu^0 \mid A_i)(V_j^i) \\ > -\epsilon_i, i = 1, \dots, m, j = 1, \dots, n\}, \\ O = \{\tau \in \mathcal{Y}(\Omega; \mathbb{T}) : \mathbb{E}(\tau \mid A_i)(W_k^i) - \mathbb{E}(\tau^0 \mid A_i)(W_k^i) \\ > -\epsilon_{i1}, i = 1, \dots, m, k = 1, \dots, p\}. \end{aligned}$$

For each $\tau \in O$, the condition

$$\mathbb{E}(\tau \mid A_i)(W_k^i) - \mathbb{E}(\tau^0 \mid A_i)(W_k^i) > -\epsilon_{i1}, k = 1, \dots, p$$

implies that we have a Young measure $\nu^i \in \mathcal{Y}(\Omega; \mathbb{S})$ such that

$$\mathbb{E}(\nu^i \mid A_i)(V_j^i) - \mathbb{E}(\nu^0 \mid A_i)(V_j^i) > -\epsilon_i, j = 1, \dots, n.$$

We consider the Young measure $\nu \in \mathcal{Y}(\Omega; \mathbb{S})$ defined by

$$\nu_\omega = \begin{cases} \nu_\omega^i & \text{for } \omega \in A_i, \\ \hat{\pi}(\tau_\omega) & \text{for } \omega \in \Omega \setminus \bigcup_i A_i. \end{cases}$$

Then the Young measure ν satisfies $\tau = \Pi(\nu)$ and $\nu \in U$. □

As an application of the open mapping theorem for Young measures in the case of metrizable Souslin spaces, we obtain the following theorem. We should mention that the theorem also will be followed by a more general theorem (Theorem 9).

Theorem 7. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space such that $(\Omega, \mathcal{F})$ is countably generated. Let $\mathbb{S}, \mathbb{T}$ be metrizable Souslin spaces such that $\mathbb{S}$ has the Skorokhod representation property for Young measures. Assume that a continuous, open, and surjective mapping $\varphi : \mathbb{S} \to \mathbb{T}$ is given. Then $\mathbb{T}$ also has the Skorokhod representation property for Young measures.*

Proof. Since $(\Omega, \mathcal{F})$ is countably generated and $\mathbb{S}, \mathbb{T}$ are metrizable Souslin spaces, both of the spaces $\mathcal{Y}(\Omega; \mathbb{S})$ and $\mathcal{Y}(\Omega; \mathbb{T})$ are metrizable (see Castaing, Raynaud de Fitte, and Valadier [5, Proposition 2.3.1]). In particular, $\mathcal{Y}(\Omega; \mathbb{T})$ is a paracompact space. Let $\hat{\mathbb{S}}$ be a compact metric space such that $\mathbb{S}$ is a subspace of $\hat{\mathbb{S}}$. Then $\mathcal{Y}(\Omega; \mathbb{S})$ is a metrizable subset of the locally convex Hausdorff topological vector space $L_\infty(\Omega, \mathcal{M}^b(\hat{\mathbb{S}}))$, and the condition stated in Remark 2 also holds. Furthermore, the map Π^{-1} is compact valued, and hence $\Pi^{-1}(\nu); \nu \in \mathcal{Y}(\Omega; \mathbb{T})$ is complete for any metric compatible with the topology on $\mathcal{Y}(\Omega; \hat{\mathbb{S}})$. Therefore, we can apply Michael's continuous selection theorem (Lemma 4) to the map Π^{-1}, and by following the reasoning used in the proof of Theorem 4, we can confirm that the assertion of the theorem is true. □

We establish that the map Π is open, and we make clear how the open mapping property of Π induces the transition property of Skorokhod representation across spaces. We remark that the open mapping property of Π is necessary to apply Michael's continuous selection theorem and deduce that the inverse map Π^{-1} admits a continuous selection. In order to circumvent this issue, we have to shed light directly on the conditions which guarantee that Π admits a continuous right-inverse. The following theorem is motivated by this purpose.

Theorem 8. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a complete finite positive measure space, and let* $\mathbb{S}, \mathbb{T}$ *be metrizable Souslin spaces. Assume that* $\varphi : \mathbb{S} \to \mathbb{T}$ *is a continuous surjection. We assume also that the following condition is satisfied:* $\pi(\nu)$ *defined by* $\pi(\nu) = \nu \circ \varphi^{-1}$ *admits a continuous right-inverse. Then* $\Pi : \mathcal{Y}(\Omega; \mathbb{S}) \to \mathcal{Y}(\Omega; \mathbb{T})$ *is a continuous surjection which admits a continuous right-inverse.*

Remark 3. A sufficient condition for the existence of a continuous right-inverse of π is that φ is an open map in addition to the continuity and the surjection of π thanks to Michael's continuous selection theorem (Lemma 4).

Proof. Since $\mathbb{S}$ is a metrizable Souslin space, there exists a compact metric space $\hat{\mathbb{S}}$ such that $\mathbb{S}$ is a subspace of $\hat{\mathbb{S}}$. Let π be defined by $\pi(\nu) = \nu \circ \varphi^{-1}$. Then, since $\mathbb{S} \subset \hat{\mathbb{S}}$ is universally measurable, we can consider the range of the (multivalued) inverse map π^{-1} as $\mathcal{M}_1^+(\hat{\mathbb{S}})$. The set $\mathcal{M}_1^+(\mathbb{T})$ is paracompact since it is metrizable. $\mathcal{M}_1^+(\hat{\mathbb{S}})$ is a metrizable set in a complete locally convex linear topological space $\mathcal{M}^b(\hat{\mathbb{S}})$. Furthermore, the map π^{-1} is lower hemicontinuous (see Lemma 3) with nonempty and complete values; we can apply the continuous selection theorem (Lemma 4) to the map π^{-1} and deduce the existence of a continuous right-inverse of π. Define $\hat{\Pi}$ as follows: $\hat{\Pi}(\tau)_\omega = \hat{\pi}(\tau_\omega)$. Then, since $\hat{\pi}$ is continuous, the map $\omega \mapsto \hat{\Pi}(\tau)_\omega$ is measurable, and hence $\hat{\Pi}(\tau) \in \mathcal{Y}(\Omega; \mathbb{S})$. Since $\hat{\pi}$ is the right-inverse of π, we have $\Pi \circ \hat{\Pi}_\omega(\tau) = \pi \circ \hat{\pi}(\tau_\omega) = \tau_\omega$. Hence, $\hat{\Pi}$ is the right-inverse of Π. The continuity of $\hat{\Pi}$ is also immediate by the relation

$$\int_\Omega \left[\int_{\mathbb{S}} \psi(\omega, \varphi(s)) \hat{\Pi}(\tau)(ds) \right] \mu(d\omega) = \int_\Omega \left[\int_{\mathbb{T}} \psi(\omega, t) \tau_\omega(dt) \right] \mu(d\omega). \qquad \square$$

It follows that the following theorem holds true.

Theorem 9. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a complete finite positive measure space, and let* $\mathbb{S}, \mathbb{T}$ *be metrizable Souslin spaces such that* $\mathbb{S}$ *has the Skorokhod representation property for Young measures. Assume that there exists a continuous, surjective map* $\varphi : \mathbb{S} \to \mathbb{T}$ *such that* $\pi(\nu)$ *defined by* $\pi(\nu) = \nu \circ \varphi^{-1}$ *admits a continuous right-inverse. Then* $\mathbb{T}$ *also has the Skorokhod representation property for Young measures.*

Proof. Since, by Theorem 8, Π admits a continuous right-inverse, Theorem 9 can be proved in the same manner as in the proof of Theorem 4. $\square$

The following lemma offers the proof that a metrizable Souslin space has the Skorokhod representation property for Young measures.

Lemma 7 (Banakh, Bogachev, and Kolesnikov [3, Lemma 2.3])**.** *Let* $\mathbb{S}$ *be a nonempty metrizable compact space. Then there exists a continuous surjection* $\varphi : C \to \mathbb{S}$ *such that the map* $\pi : \mathcal{M}_1^+(C) \to \mathcal{M}_1^+(\mathbb{S})$ *defined by* $\pi(\nu) = \nu \circ \varphi^{-1}$ *has a linear continuous right-inverse, where* C *is the classical Cantor set on* $[0, 1]$.

Since the classical Cantor set $C \subset [0, 1]$ is a universally measurable set, each measure $\nu \in \mathcal{M}_1^+(C)$ can be extended to a Borel measure on $[0, 1]$ by $\nu([0, 1] \setminus C) = 0$. Therefore, if $[0, 1]$ has the Skorokhod representation property for Young measures, then C also has the Skorokhod representation property for Young measures.

Theorem 10. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a complete finite positive measure space, and let* $\mathbb{S}$ *be a metrizable Souslin space. Then* $\mathbb{S}$ *has the Skorokhod representation property for Young measures.*

Proof. By Theorem 1, $[0, 1]$ has the Skorokhod representation property for Young measures. It follows that the universally measurable subset C of $[0, 1]$ also has the Skorokhod representation property for Young measures. By Lemma 7, for any nonempty metrizable compact space $\mathbb{T}$, there exists a continuous surjection $\varphi : C \to \mathbb{T}$ such that the map $\pi : \mathcal{M}_1^+(C) \to \mathcal{M}_1^+(\mathbb{T})$ has a linear continuous right-inverse. Hence, by Theorem 8, $\mathbb{T}$ has the Skorokhod representation property for Young measures. For any metrizable Souslin space $\mathbb{S}$, there exists a compact metric space $\hat{\mathbb{S}}$ such that $\mathbb{S}$ is a subspace of $\hat{\mathbb{S}}$ and a metrizable Souslin space is a universally measurable set. Then, by following the path of reasoning $[0, 1] \Rightarrow C \Rightarrow \hat{\mathbb{S}} \Rightarrow \mathbb{S}$, we conclude that the assertion of the theorem is true. □

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