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COMPLEX OSCILLATION AND NONOSCILLATION RESULTS

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Abstract. For an entire coefficient $A(z)$, classifying the oscillation of solutions f of the linear differential equation $f'' + A(z)f = 0$ has been a long-standing problem since the early 1980s. New results on the following three typical questions are proved: Under which conditions on $A(z)$ does there exist a solution f such that

$$\text{(Q1) } f \text{ has no zeros}, \quad \text{(Q2) } \lambda(f) \geq \sigma(A), \quad \text{(Q3) } \lambda(f) = \infty?$$

Here $\lambda(g)$ and $\sigma(g)$ denote the exponent of convergence of the zeros of g and the order of growth of g, respectively. Several nontrivial examples are given.

1. Introduction

Let $A(z)$ be an entire function. Then all solutions f of

$$f'' + A(z)f = 0 \tag{1.1}$$

are entire functions. Moreover, it is known [14], [17] that f is of finite order of growth $\sigma(f)$ if and only if $A(z)$ is a polynomial, and that zero is the only possible finite deficient value of admissible solutions of (1.1).

Example 1.1. We observe that the equation

$$f'' - \left(\gamma^2 e^z - \frac{\gamma}{2} e^{z/2} + \frac{1}{4}\right) f = 0$$

possesses linearly independent solutions $f_1(z) = \exp\left(2\gamma e^{z/2} - z/2\right)$ and

$$f_2(z) = (4\gamma e^{z/2} + 1) \exp\left(-2\gamma e^{z/2} - z/2\right)$$

for any choice of the constant $\gamma \in \mathbb{C} \setminus \{0\}$. Clearly, we see that f_1 has no zeros, that $\lambda(f_2) = 1$, and that $\lambda(f_1 + f_2) = \infty$. Here $\lambda(f)$ denotes the exponent of convergence of the zeros of f.

The situation in the previous example is not uncommon. In fact, three typical questions regarding the oscillation of nontrivial solutions f of (1.1) can be phrased as follows: Under which conditions on $A(z)$ does there exist a solution f of (1.1) such that

$$\text{(Q1) } f \text{ has no zeros}, \quad \text{(Q2) } \lambda(f) \geq \sigma(A), \quad \text{(Q3) } \lambda(f) = \infty?$$

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The cases

$$0 < \lambda(f) < \sigma(A), \qquad \sigma(A) < \lambda(f) < \infty,$$

and

$$\lambda(f) = \sigma(A) \notin \mathbb{N}$$

are also possible, as will be seen in Section 7, but they seem to be quite exceptional. Regarding (Q1), we recall a result on zero-free solution bases for (1.1) due to Bank and Laine.

Theorem A ([2, p. 356]). *There exist two linearly independent solutions f_1, f_2 of (1.1) which have no zeros in the complex plane if and only if $A(z)$ may be represented as*

$$-4A(z) = h'(z)^2 + \varphi'(z)^2 - 2\varphi''(z), \tag{1.2}$$

where φ is a nonconstant entire function and h is a primitive of e^{φ}. Furthermore, the solution base $\{f_1, f_2\}$ is given by

$$\left\{\exp\left(-\frac{1}{2}(\varphi \pm h)\right)\right\}.$$

If (1.1) possesses one zero-free solution f, then it must be of the form $f(z) = e^{h(s)}$, where $h(z)$ is some entire function. In this case,

$$A(z) = -h''(z) - h'(z)^2. \tag{1.3}$$

In typical situations, however, $A(z)$ is an exponential polynomial, and then it is not at all clear whether an entire $h(z)$ satisfying (1.3) can be found. In fact, the discussions that will follow illustrate that in many cases it is more natural to expect the opposite. Denoting $g = h'$, we see that the problem is equivalent to finding an entire solution to the Riccati equation $g' = -A - g^2$.

An example of a condition guaranteeing (Q2) is $\overline{N}(r, 1/A) = S(r, A)$; see [1, Satz 1]. An intriguing special case of this is

$$\lambda(A) < \rho(A); \tag{1.4}$$

that is, zero is a Borel exceptional value for $A(z)$. Note that the second fundamental theorem forbids $A(z)$ from having another finite Borel exceptional value. Regarding (Q3), we recall the following special case of the main result in [3].

Theorem B. *Suppose that the following hold:*

(a) *$P(z), Q(z)$ are polynomials such that $P(z) = a_n z^n + \cdots + a_0$ and $\deg(Q) + 2 < 2n$.*
(b) *$\Pi(z)$ is an entire function with $\sigma(\Pi) < n$.*
(c) *There exist a $\theta_0 \in \mathbb{R}$ such that $\Re\left(a_n e^{in\theta_0}\right) = 0$ and an $\varepsilon > 0$ such that $\Pi(z)$ has at most finitely many zeros in the sector $|\arg(z) - \theta_0| < \varepsilon$.*

Then every solution $f \not\equiv 0$ of

$$f'' + \left(\Pi(z)e^{P(z)} + Q(z)\right) f = 0$$

satisfies $\lambda(f) = \infty$.

The discussions in [3] are continued in [5]. For example, if $P(z), Q(z)$ are polynomials and if $A(z) = e^{P(z)} + Q(z)$, then either $\lambda(f) \geq \sigma(A)$ holds or $Q(z)$ reduces to the specific form $Q(z) = -P'(z)^2/16 + P''(z)/4$.

We move on to considering (1.1) from the point of view of (Q2) and (Q3) in the case in which $A(z)$ has two exponential terms. Namely, we consider

$$f'' + \left(e^{P_1(z)} + e^{P_2(z)} + Q(z)\right) f = 0, \tag{1.5}$$

where $P_1(z) = \zeta_1 z^n + \cdots$ and $P_2(z) = \zeta_2 z^m + \cdots$ are nonconstant polynomials such that $e^{P_1(z)}$ and $e^{P_2(z)}$ are linearly independent, and such that $Q(z)$ is an entire function of order $< \max\{n, m\}$. We summarize [12, Theorem 1], [13, Theorem 3.2] regarding (1.5) as follows.

Theorem C ([12], [13]). *Let $f \not\equiv 0$ be a solution of* (1.5). *If $n \neq m$, then $\lambda(f) = \infty$, while if $n = m$, we have the following four assertions.*

(a) *If $\zeta_1 = \zeta_2$, then $\lambda(f) \geq n$.*
(b) *If $\zeta_1 \neq \zeta_2$ and ζ_1/ζ_2 is nonreal, then $\lambda(f) = \infty$.*
(c) *If $0 < \zeta_1/\zeta_2 < 1/2$, then $\lambda(f) \geq n$.*
(d) *If $3/4 < \zeta_1/\zeta_2 < 1$ and $Q(z) \equiv 0$, then $\lambda(f) \geq n$.*

The condition $0 < \zeta_1/\zeta_2 < 1$ means geometrically that the points $0, \zeta_1, \zeta_2$ are collinear, and that ζ_1, ζ_2 are on the same side of the origin. Suppose then that

$$A(z) = P_1(z)e^{\zeta_1 z^n} + P_2(z)e^{\zeta_2 z^n} + Q(z),$$

where $P_1(z)$, $P_2(z) \not\equiv 0$, and $Q(z)$ are polynomials such that the points 0, ζ_1, ζ_2 are collinear but ζ_1, ζ_2 are on the opposite sides of the origin. Then as an immediate consequence of [5, Theorem 4.3], it was observed in [9] that every solution $f \not\equiv 0$ of (1.1) satisfies $\lambda(f) = \infty$.

There are examples of zero-free solutions in the following two cases [12]: (1) $\zeta_1/\zeta_2 = 1/2$ and $Q(z) \equiv 0$, and (2) $\zeta_1/\zeta_2 = 3/4$ and $Q(z) \not\equiv 0$. These examples are related to parts (c) and (d) above. This opens up a question, to be discussed in Section 2, as to whether zero-free solutions exist in the cases $\zeta_1/\zeta_2 \in (1/2, 3/4)$ and $\zeta_1/\zeta_2 \in (3/4, 1)$ under the assumption $Q(z) \not\equiv 0$.

2. New results and discussions

In this section, we state and discuss new results related to questions (Q1), (Q2), and (Q3), stated in Section 1.

2.1. **New results on (Q2) and (Q3).** We proceed to consider the case in which $\rho = \zeta_1/\zeta_2 \in (0, 1)$ in (1.5). For future reference, we write

$$f'' + \left(e^{P(z)} + e^{\rho P(z)} + Q(z)\right) f = 0, \tag{2.1}$$

and ask whether it is possible to obtain $\lambda(f) = \infty$ in place of $\lambda(f) \geq n$ in parts (c) and (d) of Theorem C, and whether the condition $Q(z) \equiv 0$ in part (d) can be ignored. While doing this, we will also discuss the gap $1/2 < \zeta_1/\zeta_2 < 3/4$, which corresponds to the case $m = 1$ in the following result.

Theorem 2.1. *Suppose that $P(z)$ and $Q(z)$ are polynomials satisfying*

$$\begin{aligned} \deg(Q) + 2 < 2\deg(P) \quad & \textit{if } \deg(P) \geq 2, \\ Q(z) \equiv 0 \quad & \textit{if } \deg(P) = 1, \end{aligned} \tag{2.2}$$

and that ρ is a constant satisfying

$$0 < \rho < \frac{1}{2} \quad \text{or} \quad \frac{2m-1}{2m} < \rho < \frac{2m+1}{2(m+1)} \tag{2.3}$$

for some $m \in \mathbb{N}$. Then every solution $f \not\equiv 0$ of (2.1) *satisfies $\lambda(f) = \infty$.*

If the assumptions (2.2) and (2.3) are violated, then zero-free solutions of (2.1) may occur.

Example 2.2.

(a) Let $\rho = 1/2$, and let $\varphi(z)$ be a polynomial with $\deg(\varphi) \geq 2$. Then

$$f(z) = \exp\left(i \int^z e^{-2\varphi(\zeta)+i\zeta}\, d\zeta + \varphi(z)\right)$$

is a zero-free solution of (2.1), where the polynomials

$$P(z) = -4\varphi(z) + 2iz$$

and

$$Q(z) = -\varphi'(z)^2 - \varphi''(z)$$

satisfy $\deg(P) = \deg(\varphi) \geq 2$ and $\deg(Q) + 2 = 2\deg(P)$. Note also that if $\deg(P) = \deg(\varphi) = 1$, then $Q(z)$ is a nonzero constant.

(b) Let $\rho = 3/4$. Then

$$f(z) = \exp\left(-2e^{-\frac{i}{2}z} - 2e^{-\frac{i}{4}z} + \frac{i}{8}z\right)$$

is a zero-free solution of (2.1), where $P(z) = -iz$ and $Q(z) = -1/64$.

Let $A(z) = e^{P(z)} + e^{\rho P(z)} + Q(z)$ be the coefficient function in (2.1), where $\rho \in (0,1)$. We make two observations based on [15, Satz 1–2]: If $Q(z) \not\equiv 0$, then $A(z)$ has no finite deficient values, while if $Q(z) \equiv 0$, then zero is the only finite deficient value of $A(z)$ with $\delta(0, A) = \rho$. In other words, even if zero would be a Nevanlinna exceptional (deficient) value for $A(z)$, it is not always guaranteed that the situation in (Q2)—let alone in (Q3)—would hold. This is in contrast to the situation in (1.4) regarding Borel exceptional values.

The proof of Theorem 2.1 is postponed to Section 3, where we also pinpoint where and how the proof collapses if $\rho = (2m-1)/(2m)$ for some $m \in \mathbb{N}$.

2.2. **New results on (Q1).** If $Q(z)$ is transcendental in (2.1), we are forced to give up the assumption (2.2). Keeping Theorem 2.1 in mind, we now ask whether (2.1) could have a zero-free solution if ρ satisfies $\frac{2m-1}{2m} < \rho < \frac{2m+1}{2(m+1)}$ for some $m \in \mathbb{N}$? The reasoning in this direction becomes overly technical, starting from the case $\deg(P) = 1$. For an exact future reference, we rewrite (2.1) in this case as

$$f'' + (H_1 e^z + H_2 e^{\rho z} + q(z))\, f = 0, \tag{2.4}$$

where $q(z)$ is an entire function of order $\sigma(q) < 1$ and $H_1, H_2 \in \mathbb{C}\setminus\{0\}$. We consider in detail the cases

$$1/2 < \rho < 3/4 \qquad (m = 1), \tag{2.5}$$

$$3/4 < \rho < 5/6 \qquad (m = 2) \tag{2.6}$$

and give some suggestive discussions on the case $\frac{5}{6} < \rho < 1$. Observe that, for the case (2.6) with $q(z) \equiv 0$, there are no zero-free solutions to (2.4) by part (d) of Theorem C. Hence we may assume that $q(z) \not\equiv 0$ in (2.6).

Theorem 2.3. *Equation* (2.4) *under either of the conditions* (2.5) *or* (2.6) *possesses no zero-free solutions.*

The lengthy proof of Theorem 2.3 is postponed to Sections 4 and 5. It seems that in the general case $\frac{2m-1}{2m} < \rho < \frac{2m+1}{2(m+1)}$ there are no zero-free solutions either, but the method used in this paper can give only one specific interval at the time, and the amount of considerations increases exponentially along with m. We will illustrate the complexity of the case $\frac{5}{6} < \rho < 1$ in more detail in Section 6. The proof of Theorem 2.3 yields general forms of zero-free solutions in the cases $\rho = 1/2$ and $\rho = 3/4$, including two examples given in [12], [13]. The details are given in Section 7, where we will also discuss the somewhat unusual situations $0 < \lambda(f) < \sigma(A)$, $\sigma(A) < \lambda(f) < \infty$, and $\lambda(f) = \sigma(A) \notin \mathbb{N}$ mentioned in Section 1.

3. Proof of Theorem 2.1

We follow the idea given in the proof of [5, Theorem 4.1]. Write

$$A(z) = e^{P(z)} + e^{\rho P(z)} + Q(z).$$

Let $f = \Pi e^{\kappa}$ be a solution of (1.5), where we suppose on the contrary the assertion that $\sigma(\Pi) < \infty$. Set $p = \deg P \geq 1$, and set

$$H(z) = A(z) - Q(z) = e^{P(z)} + e^{\rho P(z)} = e^{\rho P(z)}\left(e^{(1-\rho)P(z)} + 1\right).$$

The Phragmén-Lindelöf indicator $h_{e^P}(\theta)$ of $e^{P(z)}$ changes sign at $2p$ evenly spaced points $\theta_1, \ldots, \theta_{2p} \in [0, 2\pi)$. Without loss of generality, we may suppose that $\theta_1 = 0$, in which case

$$\theta_j = \pi(j-1)/p, \qquad j = 1, \ldots, 2p.$$

For a small $\varepsilon > 0$, we know by [15, Satz 2] that the majority of the zeros of $H(z)$ are in the ε-sectors $|\arg(z) - \theta_j| < \varepsilon$, $j = 1, \ldots, 2p$. In fact, since $H(z)$ is an exponential polynomial with zero-free coefficients, [15, proof of Satz 2] shows that $H(z)$ has at most finitely many zeros outside of the aforementioned sectors. Thus there exists a constant $r_1 > 0$ such that $H(z)$ has no zeros in the domains

$$S(r_1, \theta_j + \varepsilon, \theta_{j+1} - \varepsilon) = \{z : |z| > r_1,\ \theta_j + \varepsilon < \arg(z) < \theta_{j+1} - \varepsilon\}$$

for $j = 1, 2, \ldots, 2p-1$ and in $S(r_1, \theta_{2p} + \varepsilon, 2\pi - \varepsilon)$. In particular, the square root $H^{1/2}$ is well-defined in these domains. Without loss of generality, we may assume that $h_{e^P}(\theta) > 0$ on $(0, \pi/p)$; thus $h_{e^P}(\theta) < 0$ on $(-\pi/p, 0)$.

Define $u(z) = e^{P(z)/2}$, and write

$$H = u^2 + u^{2\rho} = u^2(1 + u^{\gamma}), \qquad \gamma = 2(\rho - 1) < 0.$$

By the standard binomial theorem, we have the formal expansion

$$H^{1/2} = u(1 + u^{\gamma})^{1/2} = \sum_{n=0}^{\infty} c_n u^{\gamma n + 1}, \tag{3.1}$$

where

$$c_0 = 1$$

and

$$c_n = \frac{\Gamma(3/2)}{n!\Gamma(3/2 - n)}, \qquad n \geq 1.$$

The convergence of the formal series is guaranteed if $|u^{\gamma}| < 1$. Since $\gamma < 0$, this happens when $|u(z)|$ grows exponentially, for example, in $S(r_2, \varepsilon, \pi/p - \varepsilon)$ for some

$r_2 > r_1$. Using the well-known half-integer formulas for the gamma function, we get

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and

$$\Gamma\left(\frac{3}{2} - n\right) = \frac{(-4)^{n-1}(n-1)!}{(2(n-1))!}\sqrt{\pi}.$$

Using Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ or its familiar consequence

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} \qquad \text{as } n \to \infty,$$

it follows that

$$|c_n| = \frac{(2(n-1))!}{2 \cdot 4^{n-1} n!(n-1)!} = \frac{\sqrt{\pi}}{4^n(2n-1)}\binom{2n}{n} \sim \frac{1}{(2n-1)\sqrt{n}}.$$

Next, we will find a unique integer N such that

$$\gamma N + 1 > 0 \quad \text{and} \quad \gamma(N+1) + 1 < 0. \tag{3.2}$$

Such N is the largest integer for which the powers in the formal expansion (3.1) are still positive. Using $\gamma = 2(\rho - 1)$, we may rewrite (3.2) in the equivalent form

$$\frac{1}{2(1-\rho)} - 1 < N < \frac{1}{2(1-\rho)}.$$

If $0 < \rho < 1/2$, we may choose $N = 0$. If $\frac{2m-1}{2m} < \rho < \frac{2m+1}{2(m+1)}$ for some $m \in \mathbb{N}$, then we may choose $N = m$. Since there are no other possibilities for ρ, we have found an N such that (3.2) holds.

Define now $G_1(z)$ up to a constant term by

$$\begin{aligned} G_1'(z) &= u, \qquad N = 0, \\ G_1'(z) &= u + c_1 u^{\gamma+1} + \cdots + c_N u^{\gamma N+1}, \qquad N \geq 1. \end{aligned} \tag{3.3}$$

Clearly, $G_1(z)$ is entire. For the same branch of $H^{1/2}$ as above, we deduce that

$$|H(z)^{1/2} - G_1'(z)| \leq \sum_{n=N+1}^{\infty} |c_n||u(z)|^{\gamma n+1} \leq C|u(z)|^{\gamma(N+1)+1} \leq |z|^{-\mu} \tag{3.4}$$

in $S(r_3, \varepsilon, \pi/p - \varepsilon)$, where $r_3 > r_2$ depends on ε and $\mu > 0$, while μ can be chosen arbitrarily large.

Recall that $h_{e^P}(\theta) > 0$ on $(0, \pi/p)$ and $h_{e^P}(\theta) < 0$ on $(-\pi/p, 0)$. Thus, for $r_4 > r_3$ large enough, [5, Lemma 5.1] yields

$$\kappa'(z) = O(r^q), \qquad z \in S(r_4, -\pi/p + \varepsilon, -\varepsilon), \tag{3.5}$$

$$\kappa'(z) = iH(z)^{1/2} + O(r^q), \qquad z \in S(r_4, \varepsilon, \pi/p - \varepsilon), \tag{3.6}$$

where $q = \deg(Q)$. Defining $G(z) = iG_1(z)$, we have by (3.3)–(3.6)

$$\kappa'(z) - G'(z) = O(r^q) \tag{3.7}$$

in $S(r_4, -\pi/p + \varepsilon, -\varepsilon) \cup S(r_4, \varepsilon, \pi/p - \varepsilon)$. By the Phragmén-Lindelöf principle it follows that (3.7) holds uniformly in $S(r_4, -\pi/p + \varepsilon, \pi/p - \varepsilon)$. Since κ and G are entire, this reasoning can be repeated on any two consecutive intervals where the indicator $h_{e^P}(\theta)$ changes sign. Thus (3.7) holds for $|z| > r_4$. This means that $\kappa(z) = G(z) + R(z)$, where $R(z)$ is a polynomial of degree $\leq q + 1$.

Defining $W(z) = \Pi(z)e^{R(z)}$, we may write $f = We^G$. Since $W(z)$ is an entire function and of finite order, we have

$$\frac{W^{(j)}(z)}{W(z)} = O\left(r^M\right), \qquad j = 1, 2, \tag{3.8}$$

for some $M > 0$ and almost every ray $\arg(z) = \theta$ [8].

Let $\delta > \varepsilon > 0$ be a small constant such that the estimates (3.8) hold on the ray $L : \arg(z) = \delta$. We show that there is a constant $J \neq 0$ such that

$$W(z)^2 G'(z) \to J \tag{3.9}$$

as $z \to \infty$ along L. This leads to the desired contradiction as follows. On the ray $-L : \arg(z) = -\delta$, we have $|A(z)| = |Q(z)| + o(1)$ so that

$$\log^+ |f(z)| = O\left(|z|^\eta\right)$$

by [10, Theorem 5.1], where $\eta = 0$ if $p = 1$, and $\eta = \frac{q+2}{2}$ if $p \geq 2$. In both cases, $\eta < p$; see (2.2). Therefore $f(z)^2 G'(z) \to 0$ as $z \to \infty$ along $-L$ by (3.3) and the fact that $h_{e^P}(-\delta) < 0$. Since $G(z) = iG_1(z)$, we have by (3.3) that $\exp(G(z)) = \exp(O(1))$ on $-L$ so that $W(z)^2 G'(z) \to 0$. But now, according to the Phragmén-Lindelöf principle, we must have $J = 0$, which is a contradiction. Thus it remains to prove (3.9).

Note that $1 > \gamma + 1 > \cdots > \gamma N + 1 > 0$ in (3.3). Thus, in a small sector around the ray L, we have

$$G'(z) = iG_1'(z) = iu(z)(1 + o(1))$$

so that

$$\frac{G''(z)}{G'(z)} = \frac{u'(z)}{u(z)}(1 + o(1)) = \frac{P'(z)}{2}(1 + o(1)) = O\left(r^{p-1}\right).$$

Substituting $f = We^G$ into (1.5), we obtain

$$\frac{W''}{W} + 2G'\frac{W'}{W} + G'' + (G')^2 + A = 0.$$

Since $A(z) = H(z) + O\left(r^q\right)$, we have by (3.8)

$$O\left(r^M\right) + 2G'\frac{W'}{W} + G'' + (G')^2 + H = 0 \tag{3.10}$$

for some $M > 0$ along L. Using (3.4), we get

$$|H - (G_1')^2| = |H^{1/2} - G_1'| \cdot |H^{1/2} + G_1'| = O\left(|z|^{-\mu}\right)|G_1'|,$$

that is,

$$H = (G_1')^2 + O\left(|z|^{-\mu}\right)G_1' = -(G')^2 + O\left(|z|^{-\mu}\right)G'$$

along L. Substituting this into (3.10) yields

$$G'\left(2\frac{W'}{W} + \frac{G''}{G'} + O\left(|z|^{-\mu}\right)\right) = O\left(r^M\right)$$

along L. Thus $2\frac{W'}{W} + \frac{G''}{G'} \to 0$ exponentially along L, and, a fortiori, for some $z_0 \in L$, the line integral

$$\int_{z_0}^{z} \left(2\frac{W'(\zeta)}{W(\zeta)} + \frac{G''(\zeta)}{G'(\zeta)}\right) d\zeta = \log W(z)^2 G'(z) - \log W(z_0)^2 G'(z_0)$$

converges to a finite constant (depending on z_0) as $z \to \infty$ along L. This yields (3.9) and completes the proof. $\square$

Remark 3.1. Instead of (2.3), suppose that $\rho = (2m-1)/(2m)$ for some $m \in \mathbb{N}$. Then an integer N satisfying (3.2) cannot be found, so the square root expansion in (3.1) has a constant term. This leads to two possibilities, depending on whether this constant term is included or not included in the definition of G_1' in (3.3): The proof collapses in (3.9) or in (3.4), respectively.

4. Proof of Theorem 2.3 when $1/2 < \rho < 3/4$

For convenience, set

$$A(z) = H_1 e^z + H_2 e^{\rho z} + q(z). \tag{4.1}$$

Suppose that (2.4) possesses a zero-free solution $f(z) = e^{h(z)}$, where $h(z)$ is an entire function satisfying (1.3). Writing $g = h'$, we have

$$g' + g^2 + A(z) = 0. \tag{4.2}$$

The rest of the proof is organized as follows. In Subsection 4.1, we introduce the notation used in the proof. In Subsection 4.2, we construct differential equations satisfied by g, $U = g' - g/2$, and $V = \left(\frac{1}{2} - \rho\right) U + U' + \frac{\rho}{8} - \frac{1}{16}$ by eliminating first e^z and then $e^{\rho z}$ from (2.4). The asymptotic properties of the functions g, U, and V are investigated, respectively, in Subsections 4.3–4.5 by relying on Gundersen's estimates for logarithmic derivatives [8]. Using these asymptotic properties, we then estimate the Nevanlinna characteristic functions of V and $C(U) = 8Ug$ in Subsection 4.6. The proof is completed in Subsections 4.7 and 4.8 by dividing the reasoning into the cases $V(z) \not\equiv 0$ and $V(z) \equiv 0$.

4.1. **Notation.**

For two real nonnegative functions $\phi(r)$, $\psi(r)$, where $r \in [r_0, \infty)$, we write $\phi(r) \asymp \psi(r)$ as $r \to \infty$ if $\phi(r) = O(\psi(r))$ as $r \to \infty$ and, if $\psi(r) = O(\phi(r))$, as $r \to \infty$. For θ on a fixed interval, set

$$L_\theta = \{z = re^{i\theta} \,:\, 0 \le r < \infty\}.$$

Using this notation, we write, for instance, $|e^z| \asymp e^{r\cos\theta}$ uniformly as $z \in L_\theta$, $r \to \infty$. For brevity, we drop the word "uniformly" from now on. Any estimate associated with $r \ge R$ means that the estimate in question holds for all "sufficiently large r". We use the symbol E for a set of linear measure zero in $[0, 2\pi)$ of arbitrary appearance. A finite number of steps is needed in the proof, and we finally regard R as the maximum of the constants in question and E as a finite union of sets of linear measure zero.

Since we assume that $\sigma(q) < 1$, we have $|q(z)| < e^{r^{\delta_1}}$ for any $\delta_1 \in (\sigma(q), 1)$ and for $r = |z| \ge R$. Along the proof, the constant δ_1 will be replaced by other constants $\delta_j \in (0,1)$, where δ_{n+1} may depend on the earlier constants $\delta_1, \dots, \delta_n$. We use constants $k_j > 0$ to express polynomial growth r^{k_j} such that k_{n+1} may depend on $k_1, \dots, k_n$. Multiplicative constants are denoted by $K_j > 0$, and again K_{n+1} may depend on $K_1, \dots, K_n$.

4.2. **Differential equations satisfied by g.**

By the Clunie lemma and (4.2), we have $\sigma(g) \le 1$. We set

$$U = g' - \frac{1}{2} g. \tag{4.3}$$

From (4.2) and (4.3),

$$g^2 + \frac{g}{2} + A(z) + U = 0. \tag{4.4}$$

Differentiating (4.4) and combining it with (4.2) and (4.3), then eliminating e^z, we obtain

$$g - 8Ug + 2U - 4U' + B(z) = 0, \tag{4.5}$$

where

$$B(z) = 4H_2(1-\rho)e^{\rho z} + 4(q - q'). \tag{4.6}$$

Further, we differentiate (4.5), and combining this with (4.5), we eliminate $e^{\rho z}$. Combining this result with (4.3), we obtain

$$\begin{aligned}\left((4-8\rho)U + 8U' + \rho - \frac{1}{2}\right) g + 8U^2 + (2\rho - 1)U - 2(1+2\rho)U' \\ + 4U'' + 4(\rho q - q' - \rho q' + q'') = 0.\end{aligned} \tag{4.7}$$

For convenience, we set the coefficient of g in (4.7) as $8V$—namely,

$$V = \left(\frac{1}{2} - \rho\right) U + U' + \frac{\rho}{8} - \frac{1}{16}. \tag{4.8}$$

4.3. **Asymptotic growth of g.** We consider the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ on which $\cos\theta > 0$. From (4.1),

$$|A(z)| \geq |e^z| \left(|H_1| - |H_2||e^{(\rho-1)z}| - \left|\frac{q(z)}{e^z}\right| \right) \geq e^{r\cos\theta}(|H_1| - o(1))$$

as $z \in L_\theta$, $r \to \infty$. Similarly, $|A(z)| \leq e^{r\cos\theta}(|H_1| + o(1))$ as $z \in L_\theta$, $r \to \infty$. Combining these inequalities, we obtain

$$|A(z)| \asymp e^{r\cos\theta} \qquad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.9}$$

We apply [8, Corollary 1, p. 89] to g in (4.2), which is of order at most 1, as mentioned above. Then, for $z \in L_\theta$, $\theta \notin E$, and some $k_1 > 0$, the estimate

$$|g'(z) + g(z)^2| \leq \left|\frac{g'(z)}{g(z)}\right| |g(z)| + |g(z)|^2 \leq r^{k_1}|g(z)| + |g(z)|^2 \tag{4.10}$$

holds for $r = |z| \geq R$. Assume that there exists a sequence $\{r_n\}$ with $r_n \to \infty$ such that $|g(r_n e^{i\theta})| \leq r_n^{k_2}$. Then, by (4.2), we have $e^{r_n \cos\theta} \asymp |A(r_n e^{i\theta})| \leq K_1 r_n^{2k_2}$, which is a contradiction. Hence $|g(re^{i\theta})| \geq r^{k_2}$ for $r \geq R$. This gives, for $\theta \notin E$,

$$|g'(z) + g(z)^2| \leq |g(z)|^2(1 + o(1)) \qquad \text{as } z \in L_\theta,\ r \to \infty.$$

By the inequality above and (4.2), for $\theta \notin E$, we obtain $|A(z)| \leq |g(z)|^2(1+o(1))$ as $z \in L_\theta$, $r \to \infty$. Similarly, we obtain $|A(z)| \geq |g(z)|^2(1 - o(1))$ as $z \in L_\theta$, $r \to \infty$. Thus, by (4.9), we have for $\theta \notin E$

$$|g(z)| \asymp e^{\frac{1}{2}r\cos\theta} \qquad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.11}$$

We next consider the interval $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ on which $\cos\theta < 0$. In this case, we have $|H_1 e^z + H_2 e^{\rho z}| \to 0$ as $z \in L_\theta$, $r \to \infty$. This implies that, for $r = |z| \geq R$, we have

$$|A(z)| \leq e^{r^{\delta_1}}, \qquad z \in L_\theta. \tag{4.12}$$

We show that the following inequality holds for $r = |z| \geq R$:

$$|g(z)| \leq e^{r^{\delta_1}}, \qquad z \in L_\theta. \tag{4.13}$$

By [8, Corollary 1] and (4.2), we see that, for $\theta \notin E$ and some $k_3 > 0$,

$$|g'(z) + g(z)^2| \geq \left| |g(z)|^2 - \left|\frac{g'(z)}{g(z)}\right| |g(z)| \right| \geq |g(z)|^2 - r^{k_3}|g(z)| \tag{4.14}$$

holds for $r \geq R$. If $|g(z)| \leq 2r^{k_3}$ for $z \in L_\theta$ and $r = |z| \geq R$, then (4.13) clearly holds. Assume that there exists a sequence $\{r'_n\}$ with $r'_n \to \infty$ such that $|g(r'_n e^{i\theta})| \geq 2(r'_n)^{k_3}$. By (4.14),

$$\left|g'\left(r'_n e^{i\theta}\right) + g\left(r'_n e^{i\theta}\right)^2\right| \geq \frac{1}{2}\left|g\left(r'_n e^{i\theta}\right)\right|^2,$$

which gives $|g(r'_n e^{i\theta})| \leq \sqrt{2}e^{\frac{1}{2}(r'_n)^{\delta_1}} < e^{(r'_n)^{\delta_1}}$ for n sufficiently large by (4.2) and (4.12). This shows that (4.13) also holds for such $\{r'_n\}$. We have thus confirmed that (4.13) holds for $r = |z| \geq R$.

4.4. **Asymptotic growth of U.** We consider the asymptotic growth of $U(z)$ defined in (4.3) having the property $\sigma(U) \leq \sigma(g)$. Consider first the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. By the assumption (2.5), we see that $(\rho - \frac{1}{2})\cos\theta > 0$ in this case. By means of [8, Corollary 1] and (4.11), we have

$$|U(z)| \leq |g(z)| \left(\left|\frac{g'(z)}{g(z)}\right| + \frac{1}{2}\right) \leq |g(z)| \left(r^{k_1} + \frac{1}{2}\right) \leq K_2 e^{\frac{1}{2}r\cos\theta + r^{\delta_2}} \tag{4.15}$$

for $\theta \notin E$, $z \in L_\theta$, and $r \geq R$. Similarly, we see by [8, Corollary 1] that

$$|U'(z)| \leq \left|\frac{U'(z)}{U(z)}\right| |U(z)| \leq K_3 e^{\frac{1}{2}r\cos\theta + r^{\delta_3}} \tag{4.16}$$

for $\theta \notin E$, $z \in L_\theta$, and $r \geq R$. By means of (4.11), (4.15), and (4.16), we obtain for $\theta \notin E$

$$|U(z)| = o(|e^{\rho z}|), \quad |U'(z)| = o(|e^{\rho z}|) \quad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.17}$$

As we mentioned above, we have $|q(z)| < e^{r^{\delta_1}}$ with $\sigma(q) < \delta_1 < 1$ for $r = |z| \geq R$, and hence, by [8, Corollary 1],

$$|q'(z)| < \left|\frac{q'(z)}{q(z)}\right| |q(z)| \leq K_4 e^{r^{\delta_4}} \qquad \text{as } z \in L_\theta,\ \theta \notin E,\ r \to \infty.$$

Hence we obtain

$$|q(z)| = o(|e^{\rho z}|), \quad |q'(z)| = o(|e^{\rho z}|) \quad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.18}$$

It follows from (4.5), (4.6), (4.11), (4.17), and (4.18) that

$$\begin{aligned} |U(z)g(z)| &= \left|\frac{1}{2}H_2(1-\rho)e^{\rho z} + \frac{1}{2}(q(z) - q'(z)) + \frac{1}{8}(g(z) + 2U(z) - 4U'(z))\right| \\ &= \frac{1}{2}|H_2|(1-\rho)e^{\rho r\cos\theta}(1 + o(1)) \qquad \text{as } z \in L_\theta,\ r \to \infty. \end{aligned} \tag{4.19}$$

Combining (4.11) and (4.19), we obtain

$$|U(z)| \asymp e^{(\rho - \frac{1}{2})r\cos\theta} \qquad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.20}$$

Using (4.20), we have

$$|U'(z)| \leq \left|\frac{U'(z)}{U(z)}\right| |U(z)| \leq K_5 e^{(\rho - \frac{1}{2})r\cos\theta + r^{\delta_5}} \tag{4.21}$$

instead of (4.16), for $\theta \notin E$, $z \in L_\theta$, and $r \geq R$.

We next consider the interval $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ for $U(z)$. By [8, Corollary 1], (4.3), and (4.13), we see that, for $\theta \notin E$,

$$|U(z)| \le \left|\frac{g'(z)}{g(z)} - \frac{1}{2}\right| |g(z)| \le K_6 e^{r^{\delta_6}} \qquad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.22}$$

4.5. **Asymptotic growth of V.** Next, we consider the asymptotic growth of $V(z)$ defined in (4.8) having the property $\sigma(V) \le \sigma(U) \le \sigma(g)$. On the interval $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, we see by (4.8) and (4.22) that

$$|V(z)| \le |U(z)| \left(\left|\frac{1}{2} - \rho\right| + \left|\frac{U'(z)}{U(z)}\right| \right) + K_7 \le K_8 e^{r^{\delta_7}} \tag{4.23}$$

for $\theta \notin E$, $z \in L_\theta$, and $r \ge R$.

We next estimate the growth of $V(z)$ on the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. To do this, we write (4.7) as

$$8Vg = C(U), \tag{4.24}$$

where

$$C(U) = -8U^2 - (2\rho - 1)U + 2(1 + 2\rho)U' - 4U'' - 4(\rho q - q' - \rho q' + q''). \tag{4.25}$$

It follows from (4.20) that, for $\theta \notin E$,

$$|U(z)|^2 \asymp e^{(2\rho-1)r\cos\theta} \qquad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.26}$$

By means of [8, Corollary 1] and (4.26),

$$|U''(z)| \le \left|\frac{U''(z)}{U(z)}\right| |U(z)| \le K_9 e^{(\rho - \frac{1}{2})r\cos\theta + r^{\delta_8}} \tag{4.27}$$

and

$$|q''(z)| < \left|\frac{q''(z)}{q(z)}\right| |q(z)| \le K_{10} e^{r^{\delta_9}} \tag{4.28}$$

for $\theta \notin E$, $z \in L_\theta$, and $r \ge R$. Hence by (4.20), (4.21), (4.26), (4.27), and (4.28), we obtain

$$|C(U(z))| = |U(z)|^2(1 + o(1)) \qquad \text{as } r \to \infty$$

for $\theta \notin E$, $z \in L_\theta$, and hence by (4.26) we have

$$|C(U(z))| \asymp e^{(2\rho-1)r\cos\theta} \qquad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.29}$$

We note that $2\rho - 1 < 1/2$ by (2.5). Hence by (4.11), (4.24), and (4.29), we conclude that on the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$V(z) = o(1) \qquad \text{as } z \in L_\theta,\ r \to \infty. \tag{4.30}$$

By means of (4.23), (4.30), and the Phragmén-Lindelöf theorem, for any $0 \le \theta < 2\pi$, we obtain

$$|V(z)| \le K_{11} e^{r^{\delta_{10}}}, \qquad r \ge R. \tag{4.31}$$

4.6. **Nevanlinna characteristic functions of V and $C(U)$.** We begin by observing that (4.31) yields an upper estimate for the characteristic of V:

$$\begin{aligned} T(r,V) = m(r,V) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |V(re^{i\theta})|d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ |K_{11}e^{r^{\delta_{10}}}|d\theta \leq r^{\delta_{10}}(1+o(1)). \end{aligned} \tag{4.32}$$

To estimate the characteristic of $C(U)$, we need some preparations.

First, we derive a lower estimate for the characteristic of $A(z)$. Fix $0<\eta<\frac{\pi}{2}$, and let $-\frac{\pi}{2}+\eta<\theta<\frac{\pi}{2}-\eta$ be arbitrary. We consider

$$|H_2||e^{(\rho-1)z}| < |H_1|/4, \tag{4.33}$$

$$|q(z)/e^z| < |H_1|/4. \tag{4.34}$$

Inequality (4.33) is equivalent to $\log(4|H_2/H_1|) < (1-\rho)r\cos\theta$. But this is clearly valid for all $r > r_1 = \max\left\{0, \frac{\log|4H_2/H_1|}{(1-\rho)\cos\left(\frac{\pi}{2}-\eta\right)}\right\}$. Recall from Subsection 4.1 that there exists a constant $r_2>0$ such that $|q(z)| \leq \exp\left(r^{\delta_1}\right)$ for all $r=|z|\geq r_2$. Thus (4.34) follows from $r^{\delta_1} - r\cos\left(\frac{\pi}{2}-\eta\right) < \log(|H_1|/4)$, where the left-hand side tends to $-\infty$ as $r\to\infty$. Hence there exists an $r_3>0$ such that this inequality is valid for all $r>r_3$. We conclude that there exists a constant $J_1>0$ such that

$$|A(z)| \geq |e^z|\left(|H_1| - |H_2||e^{(\rho-1)z}| - \left|\frac{q(z)}{e^z}\right|\right) \geq J_1 e^{r\cos\theta} \tag{4.35}$$

for all $-\frac{\pi}{2}+\eta<\theta<\frac{\pi}{2}-\eta$ and $|z|\geq\max\{r_1,r_2,r_3\}$. Hence we obtain

$$m(r,A) \geq \frac{1}{2\pi}\int_{-\frac{\pi}{2}+\eta}^{\frac{\pi}{2}-\eta} \log^+|A(re^{i\theta})|d\theta \geq \frac{1-\kappa(\eta)}{\pi}\, r(1+o(1)) \qquad \text{as } r\to\infty,$$

where $\kappa(\eta) = 2\left(1-\sin\left(\frac{\pi}{2}-\eta\right)\right) > 0$ tends to zero as $\eta\to 0$.

Second, we derive a lower estimate for the characteristic of g. Fix $0<K_{12}<J_1$. The auxiliary function

$$h_*(x) = (2k_1\log x + \log 2 - \log K_{12})/x$$

is differentiable on $(0,\infty)$ and has precisely one critical point, $x_c\in(0,\infty)$, at which it reaches its maximal value $h_*(x_c)>0$. It follows that $h_*(x)$ is strictly increasing on $(0,x_c)$ and strictly decreasing on (x_c,∞) tending to zero as $x\to\infty$. Let $-\frac{\pi}{2}+\eta<\theta<\frac{\pi}{2}-\eta$. If $\cos\theta > h_*(x_c)$, we choose $r_\theta=0$, while if $\cos\theta\leq h_*(x_c)$, we choose r_θ to be the unique point satisfying $r_\theta\geq x_c$ and $h_*(r_\theta)=\cos\theta$. Then, for $r\geq r_\theta$, we have

$$K_{12}e^{r\cos\theta} \geq 2r^{2k_1}. \tag{4.36}$$

Clearly, if $0<\theta_1<\theta_2<\frac{\pi}{2}$, then $r_{\theta_1}\leq r_{\theta_2}$. Hence we see that (4.36) holds for all $-\frac{\pi}{2}+\eta<\theta<\frac{\pi}{2}-\eta$ and $r\geq r_{\frac{\pi}{2}-\eta}$. Assume that, for some $r_*\geq r_{\frac{\pi}{2}-\eta}$ and $\theta\notin E$, we have $r_*^{k_1}\geq|g(r_*e^{i\theta})|$. Then, by (4.2), (4.10), and (4.35), we obtain $J_1e^{r_*\cos\theta} < 2r_*^{2k_1}$, which contradicts (4.36). Thus there exists a constant $J_2>0$ such that, for any $r\geq r_{\frac{\pi}{2}-\eta}$ and $-\frac{\pi}{2}+\eta<\theta<\frac{\pi}{2}-\eta$, $\theta\notin E$, we get

$$|g(re^{i\theta})| \geq J_2 e^{\frac{1}{2}r\cos\theta},$$

which yields

$$m(r,g) \geq \frac{1}{2\pi}\int_{-\frac{\pi}{2}+\eta}^{\frac{\pi}{2}-\eta} \log^+ |g(re^{i\theta})|d\theta \geq \frac{1-\kappa(\eta)}{2\pi}\, r(1+o(1)) \tag{4.37}$$

as $r \to \infty$.

We are now ready to derive an upper estimate for $T(r, C(U)) = m(r, C(U))$. For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $r \notin E$, we have (4.29). We consider the asymptotic behavior of $C(U)$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. By means of [8, Corollary 1] and (4.22), we see that

$$|C(U(z))| \leq K_{13} e^{r^{\delta_{11}}} \tag{4.38}$$

for $\theta \notin E$, $z \in L_\theta$, and $r \geq R$. By (4.18), (4.22), (4.27), (4.29), and (4.38), we have

$$\begin{aligned} T(r, C(U)) &\leq \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^+ |C(U(re^{i\theta}))|d\theta + \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log^+ |C(U(re^{i\theta}))|d\theta \\ &\leq K_{13} r^{\delta_{11}} + \frac{(2\rho-1)}{\pi}\, r(1+o(1)) \qquad \text{as } r \to \infty. \end{aligned} \tag{4.39}$$

4.7. **Completion of proof in the case $V \not\equiv 0$.** Assume that $V(z) \not\equiv 0$. By (4.24) and the first main theorem due to Nevanlinna,

$$T(r, g) \leq T(r, V) + T(r, C(U)) + O(1). \tag{4.40}$$

Using (4.40), (4.37), (4.31), and (4.39), we have

$$\frac{1-\kappa(\eta)}{2\pi}\, r(1+o(1)) \leq r^{\delta_{10}} + K_{13} r^{\delta_{11}} + \frac{2\rho-1}{\pi}\, r(1+o(1))$$

as $r \to \infty$. This yields $(1-\kappa(\eta))/2 \leq 2\rho - 1$. Since $\eta > 0$ can be chosen arbitrarily, we obtain $\rho \geq 3/4$, which contradicts the assumption (2.5).

4.8. **Completion of proof in the case $V \equiv 0$.** Assume that $V(z) \equiv 0$. By (4.3) and (4.8), we see that g satisfies the nonhomogeneous linear differential equation

$$g'' - \rho g' + \left(\frac{\rho}{2} - \frac{1}{4}\right) g = \frac{1}{16} - \frac{\rho}{8}$$

with constant coefficients. Thus

$$g(z) = \gamma_1 e^{\frac{1}{2}z} + \gamma_2 e^{(\rho - \frac{1}{2})z} - \frac{1}{4},$$

where γ_1, γ_2 are constants. Substituting this into (4.2), we have

$$A(z) = -\gamma_1^2 e^z - 2\gamma_1\gamma_2 e^{\rho z} - (\rho-1)\gamma_2 e^{(\rho-\frac{1}{2})z} - \gamma_2^2 e^{(2\rho-1)z} - \frac{1}{16}.$$

By (4.1) this is possible only when $\rho = 1/2$, which contradicts the assumption (2.5). This completes the proof of Theorem 2.3 under the assumption (2.5).

5. Proof of Theorem 2.3 when $3/4 < \rho < 5/6$

We suppose, contrary to the assertion, that under the assumption (2.6), equation (2.4) has a zero-free solution $f(z) = e^{h(z)}$, where $h(z)$ is an entire function. We adopt the notation from Subsection 4.1 and make use of the identities (4.1)–(4.8), which hold for any fixed $\rho \in (1/2, 1)$.

5.1. **New algebraic equation for U.** We differentiate (4.7) and combine it with (4.3) to obtain

$$\begin{aligned}&\left((2-4\rho)U-8(\rho-1)U'+8U''+\frac{\rho}{2}-\frac{1}{4}\right)g\\ &\quad-4(2\rho-1)U^2+24UU'+\left(\rho-\frac{1}{2}\right)U+(2\rho-1)U'\\ &\quad-2(2\rho+1)U''+4U'''+4(\rho q'-q''-\rho q''+q''')=0.\end{aligned}\tag{5.1}$$

By appealing to Subsection 4.7, we can assume that $V(z)$ does not vanish identically. Then we eliminate g using (4.7) and (5.1). Further, eliminating U', U'', and U''' by (4.8), we obtain a new algebraic equation for U,

$$C_2(V)U^2+C_1(V)U+C_0(V)=0,\tag{5.2}$$

where

$$C_2(V)=32\left((4\rho-3)V-2V'\right),\tag{5.3}$$

$$C_1(V)=8\left(24V^2-\rho(2\rho-1)V+(2\rho-1)V'\right),\tag{5.4}$$

$$\begin{aligned}C_0(V)&=-8(2\rho-3)V^2-16VV'-32(V')^2+32VV''\\ &\quad+\left(\frac{3}{2}-4\rho+2\rho^2-16(\rho q-q'-3\rho q'+3q''+2\rho q''-2q''')\right)V\\ &\quad+\left(2-4\rho-32(\rho q-q'-\rho q'+q'')\right)V'.\end{aligned}\tag{5.5}$$

5.2. **Asymptotic growth of g, U, and V.** Concerning the asymptotic growth of g, U, V, the results (4.9)–(4.29) are still valid under the assumption (2.6). In particular, the estimate (4.23) for V holds on the interval $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$. However, the estimate (4.30) for V does not hold on the interval $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ under the assumption (2.6). Instead, by (4.24), (4.11), and (4.29), we have

$$|V(z)|\asymp e^{(2\rho-\frac{3}{2})r\cos\theta}\qquad \text{as } z\in L_\theta,\ r\to\infty.\tag{5.6}$$

It follows from (2.6) that $2\rho-\frac{3}{2}=2(\rho-\frac{3}{4})>0$, so $|V(z)|\neq o(1)$ as $r\to\infty$ in this case. For the sake of brevity, we set $W(z)=C_2(V(z))$ and $Y(z)=-C_1(V(z))U(z)-C_0(V(z))$ in (5.2). Then we may write

$$WU^2=Y.\tag{5.7}$$

5.3. **Nevanlinna characteristic functions of W, Y, and U.** We first consider the interval $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Applying [8, Corollary 1] to (5.4) and (5.5) and using similar arguments as in (4.19) and (4.29), we obtain by (4.20) and (5.6) that

$$|Y(z)|\le 192|U(z)||V(z)|^2(1+o(1))\asymp e^{(5\rho-\frac{7}{2})r\cos\theta}\tag{5.8}$$

as $z\in L_\theta$, $r\to\infty$. Combining (5.7) with (5.8), we have

$$|W(z)|\asymp e^{(3\rho-\frac{5}{2})r\cos\theta}\qquad \text{as } z\in L_\theta,\ r\to\infty.\tag{5.9}$$

Since we assume (2.6), we have $3\rho-\frac{5}{2}=3(\rho-\frac{5}{6})<0$, so $|W(z)|=o(1)$ in this case. We next consider the interval $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$. By the definition of Y and using (4.22) and (4.23), we have

$$|Y(z)|\le K_{14}e^{r^{\delta_{12}}}\tag{5.10}$$

for $\theta \notin E$, $z \in L_\theta$, and for $r \geq R$. By the definition of W and by (4.23),

$$|W(z)| \leq K_{15} e^{r^{\delta_{13}}} \tag{5.11}$$

for $\theta \notin E$, $z \in L_\theta$, and $r \geq R$. As in (4.32) and (4.39), we have

$$T(r, W) \leq r^{\delta_{13}}(1 + o(1)) \tag{5.12}$$

and

$$T(r, Y) \leq r^{\delta_{14}} + \frac{5\rho - \frac{7}{2}}{\pi} r(1 + o(1)) \qquad \text{as } r \to \infty. \tag{5.13}$$

Arguing similarly as for the lower estimate (4.37) of $m(r, g)$, we see by (4.20) that, for a small $\eta > 0$, we get

$$T(r, U) = m(r, U) \geq \frac{\rho - \frac{1}{2} - \kappa(\eta)}{\pi} r(1 + o(1)) \qquad \text{as } r \to \infty. \tag{5.14}$$

5.4. **Completion of proof in the case $W \not\equiv 0$.** Assume that $W(z) \not\equiv 0$. Then, by means of the first fundamental theorem due to Nevanlinna, we have

$$T(r, U^2) \leq T(r, W) + T(r, Y).$$

Using $T(r, U^2) = 2T(r, U)$ together with (5.7), (5.12), (5.13), and (5.14), we obtain

$$\frac{2\rho - 1 - 2\kappa(\eta)}{\pi} r(1 + o(1)) \leq r^{\delta_{13}} + r^{\delta_{14}} + \frac{5\rho - \frac{7}{2}}{\pi} r(1 + o(1)) \tag{5.15}$$

as $r \to \infty$. This yields $2\rho - 1 - 2\kappa(\eta) \leq 5\rho - 7/2$. Since $\eta > 0$ can be chosen arbitrarily, we obtain $\rho \geq 5/6$, which contradicts the assumption (2.6).

5.5. **Completion of proof in the case $W \equiv 0$.** Assume that $W(z) \equiv 0$. By (5.6), we see that $V(z)$ is a transcendental function. Using (5.3), (5.4), and (5.5), we have

$$U = -\frac{C_0(V)}{C_1(V)} = \frac{96(\rho - 1)V + P(z)}{48V + 4\rho^2 - 8\rho + 3}, \tag{5.16}$$

where

$$\begin{aligned} P(z) = \ & 64\rho(2\rho - 1)q + 64q' - 128\rho q' - 128\rho^2 q' \\ & + 192\rho q'' - 64q''' + 12\rho^2 - 12\rho + 3. \end{aligned}$$

By (5.14), we see that $U(z)$ is a transcendental function, which shows that the right-hand side of (5.16) does not reduce to a constant. Thus, by (5.16), we obtain

$$T(r, U) \leq T(r, V) + T(r, q) + O(\log r). \tag{5.17}$$

It follows from (5.6) that

$$T(r, V) \leq \frac{2\rho - \frac{3}{2}}{\pi} r(1 + o(1)) \qquad \text{as } r \to \infty. \tag{5.18}$$

We note that we have $T(r, q) \leq r^{\delta_1}$ for $r \geq R$; see Subsection 4.1. Hence, by combining (5.17) with (5.14) and (5.18), it follows that

$$\frac{\rho - \frac{1}{2} - \kappa(\eta)}{\pi} r(1 + o(1)) \leq r^{\delta_1} + \frac{2\rho - \frac{3}{2}}{\pi} r(1 + o(1)) \qquad \text{as } r \to \infty.$$

This yields $\rho - 1/2 - \kappa(\eta) \leq 2\rho - 3/2$. Since $\eta > 0$ can be chosen arbitrarily, we obtain $\rho \geq 1$, which contradicts the assumption (2.6).

6. Discussions on the case $5/6 < \rho < 1$

This section is devoted to discussing zero-free solutions of (2.4) in the case $5/6 < \rho < 1$, focusing on $5/6 < \rho < 7/8$. The amount of calculations becomes so substantial that the use of suitable computer software is beneficial.

As earlier, we suppose that (2.4) has a zero-free solution $f(z) = e^{h(z)}$. Differentiating both sides of (5.2) and eliminating U' by (4.8), we obtain a quadratic polynomial in U, call it $Z(U)$, whose coefficients are differential polynomials in V. If $Z(U)$ does not coincide with the left-hand side of (5.2), we may eliminate U using $Z(U) = 0$ and (5.2). Setting $W = C_0(V)$, we obtain

$$S(W)V^7 + F_6(W)V^6 + \cdots + F_1(W)V + F_0(W) = 0, \tag{6.1}$$

where $S(W) = \tilde{S}_0((6\rho - 5)W - 2W')$, $\tilde{S}_0 \in \mathbb{C}$, and the functions $F_j(W)$ are differential polynomials in W for $j = 0, 1, \ldots, 6$. In a rigorous proof, we should check whether the coefficients in (6.1) vanish identically or not. We omit these considerations and simply admit (6.1). We now divide our discussion into two parts; see Subsections 6.1 and 6.2. Our discussion leads to $A(z)$ having multiple exponential terms, which is not the case in (2.4), but may be interesting on its own; see Subsection 6.3.

6.1. **The case $S(W) \not\equiv 0$.**

Assume that $S(W(z)) \not\equiv 0$. In this case, the growth of $S(W(z))$ is governed by that of $(6\rho - 5)W$. Using suitable computer software, we find that $\deg F_6 = 2$, $\deg F_5 = 2$, $\deg F_4 = 3$, $\deg F_3 = 4$, $\deg F_2 = 5$, $\deg F_1 = 5$, and $\deg F_0 = 5$. Our assumption $\rho > 5/6$ then implies, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, that

$$|V(z)| \asymp e^{(2\rho - \frac{3}{2})r\cos\theta} \quad \text{and} \quad |W(z)| \asymp e^{(3\rho - \frac{5}{2})r\cos\theta} \quad \text{as } z \in L_\theta,\ r \to \infty.$$

Combining these estimates with (6.1), we obtain

$$|S(W(z))| \asymp e^{(4\rho - \frac{7}{2})r\cos\theta} \qquad \text{as } z \in L_\theta,\ r \to \infty, \tag{6.2}$$

which is of the same growth as $F_6(W)V^6/V^7$. Indeed, the other terms $F_j(W)V^j/V^7$, $j = 0, 1, \ldots, 5$, reduce to functions of smaller order due to $\rho < 1$. Using arguments similar to those given in Sections 4 and 5 together with (6.2), we have $T(r, S(W))$ reducing to a smaller order function when $5/6 < \rho < 7/8$. Further, we obtain $2\rho - \frac{3}{2} \leq 6\rho - 5$ in this case, which is obviously a contradiction.

The discussion above is a template for the proof that equation (2.4) has no zero-free solutions in the case $5/6 < \rho < 7/8$. It appears that, in the general case $\frac{2m-1}{2m} < \rho < \frac{2m+1}{2(m+1)}$, there are no zero-free solutions either, but the amount of considerations increases exponentially along with m.

6.2. **The case $S(W) \equiv 0$.**

Assume that $S(W(z)) \equiv 0$, and aim for a contradiction. Using the relations $W = 32\left((4\rho - 3)V - 2V'\right)$, (4.8), and (4.3), we see that $g(z)$ satisfies a nonhomogeneous linear differential equation

$$\begin{aligned} 32g^{(4)} &+ (128 - 192\rho)g''' + (112 - 416\rho + 352\rho^2)g'' \\ &- 16(3\rho - 2)(4\rho^2 - 2\rho - 1)g' + 2(6\rho - 5)(4\rho - 3)g \\ &= -\frac{1}{2}(6\rho - 7)(4\rho - 5)(2\rho - 1) \end{aligned}$$

with constant coefficients. Thus

$$g(z) = \gamma_1 e^{\frac{z}{2}} + \gamma_2 e^{\frac{1}{2}(6\rho - 5)z} + \gamma_3 e^{\frac{1}{2}(4\rho - 3)z} + \gamma_4 e^{\frac{1}{2}(2\rho - 1)z} + G, \tag{6.3}$$

where

$$G = -\frac{(6\rho-7)(4\rho-5)}{4(6\rho-5)(4\rho-3)},$$

and where $\gamma_1, \ldots, \gamma_4$ are arbitrary constants. From (4.2), we infer

$$\begin{aligned} A(z) &= -g'(z) - g(z)^2 \\ &= -G^2 - \gamma_1^2 e^z - \frac{1}{2}(4G+1)\gamma_1 e^{\frac{z}{2}} - \frac{1}{2}(6\rho-5+4G)\gamma_2 e^{(3\rho-\frac{5}{2})z} \\ &\quad - \gamma_2^2 e^{(6\rho-5)z} - \frac{1}{2}(4\rho-3+4G)\gamma_3 e^{(2\rho-\frac{3}{2})z} - 2\gamma_2\gamma_3 e^{(5\rho-4)z} \\ &\quad - \frac{1}{2}(2\rho-1+4G)\gamma_4 e^{(\rho-\frac{1}{2})z} - 2\gamma_1\gamma_4 e^{\rho z} - (\gamma_3^2 + 2\gamma_2\gamma_4)e^{(4\rho-3)z} \\ &\quad - 2(\gamma_1\gamma_2 + \gamma_3\gamma_4)e^{(3\rho-2)z} - (\gamma_4^2 + 2\gamma_1\gamma_3)e^{(2\rho-1)z}. \end{aligned} \tag{6.4}$$

The coefficient $A(z)$ has at most 11 exponential terms here, whereas in (2.4) there are just two. In order to see how much cancellation in (6.4) could possibly happen, we make two observations in the case $5/6 < \rho < 7/8$:

(1) None of the constants $4G+1$, $6\rho-5+4G$, $4\rho-3+4G$, $2\rho-1+4G$ in (6.4) vanish.
(2) The leading coefficients $1, 1/2, 3\rho-5/2, 6\rho-5, 2\rho-3/2, 5\rho-4, \rho-1/2, \rho, 4\rho-3, 3\rho-2, 2\rho-1$ of $A(z)$ are pairwise distinct.

Thus the question of possible cancellation of exponential terms reduces down to checking whether or not the constants $\gamma_1, \ldots, \gamma_4$ should vanish. For example, in order to keep the $e^{\rho z}$ term as in (2.4), we need to assume that $\gamma_1\gamma_4 \neq 0$. But then the coefficient $A(z)$ in (6.4) has at least five exponential terms, which is a contradiction. In fact, assuming that precisely j of the constants $\gamma_1, \ldots, \gamma_4$ are nonzero for $j \in \{1, 2, 3, 4\}$ leads us to $4+6+4+1 = 15$ different cases, in each of which either $A(z)$ has at least three exponential terms or we have a contradiction with our assumption $5/6 < \rho < 7/8$. Hence the case $S(W(z)) \equiv 0$ is not possible.

6.3. **Multiple exponential terms.** Although the reasoning in the case $S(W(z)) \equiv 0$ led to a failure from the equation (2.4) point of view, it can be applied to create new examples of zero-free solutions in the case in which $A(z)$ has multiple exponential terms. This case is discussed from the oscillation theory point of view in [16], for example.

Instead of (6.3), the starting point could be any function of the form

$$g(z) = \gamma_1 e^{\tau_1 z} + \cdots + \gamma_m e^{\tau_m z} + G, \tag{6.5}$$

independent of the condition $\frac{5}{6} < \rho < 1$. Setting $A(z) = -g'(z) - g(z)^2$, we see that $f(z) = \exp(\int^z g(\zeta)\,d\zeta)$ satisfies (1.1). Then the number of exponential terms appearing in $A(z)$ can be estimated to be at most

$$m + \sum_{j=1}^{m+1} j = m + \frac{m(m+1)}{2} = \frac{m(m+3)}{2}.$$

Cancellation of terms may happen, for example, when $\tau_i + \tau_j = \tau_k$ holds.

Example 6.1. An example of the three exponential term case can be obtained by choosing $g(z) = \gamma e^{\frac{z}{6}} - 6\gamma^2 e^{\frac{z}{3}} - 18\gamma^3 e^{\frac{z}{2}} - \frac{1}{12}$, where $\gamma \in \mathbb{C} \setminus \{0\}$. Then

$$f(z) = \exp\left(6\gamma e^{\frac{z}{6}} - 18\gamma^2 e^{\frac{z}{3}} - 36\gamma^3 e^{\frac{z}{2}} - \frac{z}{12}\right)$$

is a zero-free solution of

$$f'' + \left(-324\gamma^6 e^z - 216\gamma^5 e^{\frac{5z}{6}} + 18\gamma^3 e^{\frac{z}{2}} - \frac{1}{144}\right) f = 0.$$

7. Examples

We begin by unfolding how the proof of Theorem 2.3 yields general forms of zero-free solutions in the cases $\rho = 1/2$ and $\rho = 3/4$. We then proceed to construct/recall examples showing that the cases

$$0 < \lambda(f) < \sigma(A), \qquad \sigma(A) < \lambda(f) < \infty,$$

and

$$\lambda(f) = \sigma(A) \notin \mathbb{N}_0$$

are all possible, although somewhat exceptional. Finally, we note that an example of a zero-free solution for (2.1) in the case in which $\rho = 3/4$ and $Q(z) \equiv 0$ is still not known.

7.1. **Examples related to Theorem 2.3.** We begin by giving explicit forms of zero-free solutions in the cases $\rho = 1/2$ and $\rho = 3/4$, which illustrate the sharpness of Theorem 2.3 in some sense. Moreover, by using a suitable change of a variable, we show that [13, Example 1], [12, Example 2] reduce to special cases of these two general examples, respectively.

Example 7.1. The computations in Subsection 4.8 show that the differential equation

$$f'' + \left(H_1 e^z + H_2 e^{\frac{1}{2}z} + q\right) f = 0$$

has a zero-free solution $f(z) = \exp\left(2\gamma_1 e^{\frac{1}{2}z} + (\gamma_2 - \frac{1}{4})z\right)$ precisely when

$$H_1 = -\gamma_1^2, \quad H_2 = -2\gamma_1\gamma_2, \quad q = -\left(\gamma_2 - \frac{1}{4}\right)^2. \tag{7.1}$$

Recall [13, Example 1], according to which $f(z) = \exp\left(e^{2iz}\right)$ satisfies

$$f'' + \left(4e^{4iz} + 4e^{2iz}\right) f = 0.$$

However, this reduces to (7.1) with $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = \frac{1}{4}$ after the change of variable $t = 4iz$ and $\tilde{f}(t) = f(z)$. Indeed, the equation

$$\tilde{f}'' + \left(-\frac{1}{4}e^t - \frac{1}{4}e^{\frac{1}{2}t}\right) \tilde{f} = 0$$

has a zero-free solution $\tilde{f}(t) = \exp\left(e^{\frac{1}{2}t}\right)$.

Example 7.2. In Subsection 5.5, we obtained $W(z) \not\equiv 0$ under the assumption that $\rho > 3/4$. We now carry forward the computations in the case with $\rho = 3/4$ and $W(z) \equiv 0$. It follows from (5.3) that $V(z)$ is a constant. Hence by (5.4), we have $V(z) = 1/64$ in this case. By (5.5), we see that $q(z)$ satisfies a nonhomogeneous linear differential equation of third order,

$$q''' - \frac{9}{4}q'' + \frac{13}{8}q' - \frac{3}{8}q = \frac{3}{512},$$

with constant coefficients. Thus

$$q(z) = \tilde{\gamma}_1 e^{\frac{3}{4}z} + \tilde{\gamma}_2 e^{\frac{1}{2}z} + \tilde{\gamma}_3 e^{z} - \frac{1}{64},$$

where $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ are constants. If at least one of the $\tilde{\gamma}_j$ is not zero, then $\sigma(q) = 1$, which contradicts our assumption that $\sigma(q) < 1$. Hence $\tilde{\gamma}_j = 0$ for all j, which shows that $q(z) = -1/64$.

Setting $V(z) = 1/64$ and $\rho = 3/4$ and using (4.3) and (4.8), we see that g satisfies a nonhomogeneous linear differential equation

$$g'' - \frac{3}{4}g' + \frac{1}{8}g = -\frac{1}{64}$$

with constant coefficients. Thus

$$g(z) = \gamma_1 e^{\frac{1}{2}z} + \gamma_2 e^{\frac{1}{4}z} - \frac{1}{8},$$

where γ_1, γ_2 are constants. Substituting this into (4.2), we have

$$A(z) = -\gamma_1^2 e^{z} - 2\gamma_1\gamma_2 e^{\frac{3}{4}z} - \frac{1}{4}(\gamma_1 + 4\gamma_2^2)e^{\frac{1}{2}z} - \frac{1}{64}. \tag{7.2}$$

By setting $\gamma_1 + 4\gamma_2^2 = 0$, we see that the differential equation

$$f'' + (H_1 e^{z} + H_2 e^{\frac{3}{4}z} + q)f = 0$$

has a zero-free solution $f(z) = \exp\left(2\gamma_1 e^{\frac{1}{2}z} + 4\gamma_2 e^{\frac{1}{4}z} - \frac{1}{8}z\right)$ precisely when

$$H_1 = -\gamma_1^2, \quad H_2 = -2\gamma_1\gamma_2, \quad q(z) \equiv -\frac{1}{64}, \quad \gamma_1 + 4\gamma_2^2 = 0. \tag{7.3}$$

Recall [12, Example 2], according to which $f(z) = \exp\left(\frac{1}{2}e^{2z} + ie^{z} - \frac{1}{2}z\right)$ satisfies

$$f'' + \left(e^{4z+\log(-1)} + e^{3z+\log(-2i)} - \frac{1}{4}\right) f = 0.$$

However, this reduces to (7.3) with $\gamma_1 = \frac{1}{4}$ and $\gamma_2 = \frac{i}{4}$ after the change of variable $t = 4z$ and $\tilde{f}(t) = f(z)$. Indeed, the equation

$$\tilde{f}'' + \left(-\frac{1}{16}e^{t} - \frac{i}{8}e^{\frac{3}{4}t} - \frac{1}{64}\right)\tilde{f} = 0$$

has a zero-free solution $\tilde{f}(t) = \exp\left(ie^{\frac{1}{4}t} + \frac{1}{2}e^{\frac{1}{2}t} - \frac{t}{8}\right)$.

7.2. **The cases $0 < \lambda(f) < \sigma(A)$ and $\sigma(A) < \lambda(f) < \infty$.** The discussions in the previous sections lead us to asking whether the cases $0 < \lambda(f) < \sigma(A)$ and $\sigma(A) < \lambda(f) < \infty$ are possible. The answer is yes, there are examples of both cases, but these seem to be quite exceptional.

The only examples on the case $\sigma(A) < \lambda(f) < \infty$ known to the authors are due to Bergweiler and Eremenko disproving the long-standing Bank–Laine conjecture [2]. We state one of their results as follows.

Theorem D ([6, Theorem 1.1])**.** *For every $\sigma \in (1/2, 1)$, there exists an entire function $A(z)$ of order σ such that the differential equation* (1.1) *has two linearly independent solutions whose product E satisfies*

$$\frac{1}{\sigma(A)} + \frac{1}{\lambda(E)} = 2.$$

Moreover, $\lambda(E) = \sigma(E)$, and one of these solutions is zero free.

The following example on the case $0 < \lambda(f) < \sigma(A)$ seems to be new.

Example 7.3. Recalling that $\sin z$ and $\cos z$ are both exponential polynomials of order 1, we see that the function

$$A(z) = -\left(\frac{3}{2}\cos z + z\sin z\right)e^{z^2} - \frac{1}{4}\left(\sin^2 z\right)e^{2z^2} + 1$$

is an exponential polynomial of order 2. For this particular coefficient $A(z)$, equation (1.1) has a solution

$$f(z) = (\sin z)\exp\left(\frac{1}{2}\int^z e^{\zeta^2}\sin\zeta\, d\zeta\right).$$

In particular, $0 < \lambda(f) = 1 < 2 = \sigma(A)$.

More generally, given an entire function $\Phi(z)$ of order $\sigma(\Phi) > 1$, we see that $h(z) = \sin z$ is a solution of the second order linear equation

$$h'' + (\Phi(z)\sin z)h' + (1 - \Phi(z)\cos z)h = 0.$$

Then standard reasoning as in [14, p. 74] shows that

$$f(z) = (\sin z)\exp\left(\frac{1}{2}\int^z \Phi(\zeta)\sin\zeta\, d\zeta\right)$$

is a solution of (1.1), where

$$A(z) = -\frac{1}{4}\left(\Phi(z)\sin z\right)^2 - \frac{1}{2}\left(\Phi(z)\sin z\right)' + 1 - \Phi(z)\cos z.$$

Note that $A(z)$ is not in general an exponential polynomial. We prove that $\sigma(A) = \sigma(\Phi)$. The inequality $\sigma(A) \le \sigma(\Phi)$ being clear, we define $g(z) = \Phi(z)\sin z$ and write

$$g^2 = 4 - 4A(z) - g\left(2\frac{g'}{g} + 4\frac{\cos z}{\sin z}\right).$$

Then

$$\begin{aligned} m(r,g) &\le m(r, 4-4A) + m\left(r, 2\frac{g'}{g} + 4\frac{\cos z}{\sin z}\right) + O(1) \\ &\le m(r,A) + O(r), \qquad r \ge R, \end{aligned}$$

which shows that $\sigma(\Phi) = \sigma(g) \le \sigma(A)$. Thus $0 < \lambda(f) = 1 < \sigma(\Phi) = \sigma(A)$.

7.3. **The case $\lambda(f) = \sigma(A) \notin \mathbb{N}_0$.** So far all examples for the case $\lambda(f) = \sigma(A) > 0$ have appeared for an exponential polynomial coefficient $A(z)$, in which case $\sigma(A) \in \mathbb{N}_0$. Here $\sigma(A) = 0$ means that $A(z)$ is a polynomial. We next show that the case $\lambda(f) = \sigma(A) \notin \mathbb{N}_0$ is also possible.

Example 7.4. Choose

$$f(z) = (\cos\sqrt{z})e^{h(z)}$$

and

$$h'(z) = \left(\frac{\sin\sqrt{z}}{2\sqrt{z}}\right)^2,$$

where we require $h(0) = 0$ in order to fix h uniquely. Clearly, h is entire and of order $\frac{1}{2}$. Observe that the ratio

$$\frac{(\cos\sqrt{z})'' + 2h'(z)(\cos\sqrt{z})'}{\cos\sqrt{z}} = \frac{\frac{\sin 2\sqrt{z}}{2\sqrt{z}} - 1}{4z}$$

is also an entire function of order $\frac{1}{2}$. This guarantees that

$$A(z) = -\frac{f''(z)}{f(z)} = -\frac{(\cos\sqrt{z})''}{\cos\sqrt{z}} - 2h'(z)\frac{(\cos\sqrt{z})'}{\cos\sqrt{z}} - h''(z) - h'(z)^2$$
$$= \frac{1 - \frac{\sin 2\sqrt{z}}{2\sqrt{z}}}{4z} - \left(\left(\frac{\sin\sqrt{z}}{2\sqrt{z}}\right)^2\right)' - \left(\frac{\sin\sqrt{z}}{2\sqrt{z}}\right)^4$$

is an entire function and of order $\sigma(A) \leq \frac{1}{2}$. Suppose that $\sigma(A) < \frac{1}{2}$, and aim for a contradiction. Define $g(z) = \frac{\sin\sqrt{z}}{2\sqrt{z}}$, which is clearly an entire function of order $\frac{1}{2}$ and of lower order $\frac{1}{2}$; see [7, Subsection 2.2]. Thus g is of regular growth so that $m(r, A) = S(r, g)$. Using Clunie's lemma to

$$g^3 \cdot g = \frac{1-g}{4z} - 2gg' - A(z),$$

we obtain $m(r, g) = S(r, g)$. But this is a contradiction, and hence $\sigma(A) = \frac{1}{2} = \lambda(f)$.

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