TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 372, Number 10, 15 November 2019, Pages 6771–6803
https://doi.org/10.1090/tran/7949
Article electronically published on August 28, 2019

LOCALLY CONSTRAINED INVERSE CURVATURE FLOWS

JULIAN SCHEUER AND CHAO XIA

ABSTRACT. We consider inverse curvature flows in warped product manifolds, which are constrained subject to local terms of lower order—namely, the radial coordinate and the generalized support function. Under various assumptions we prove longtime existence and smooth convergence to a coordinate slice. We apply this result to deduce a new Minkowski-type inequality in the anti–de Sitter Schwarzschild manifolds and a weighted isoperimetric-type inequality in hyperbolic space.

CONTENTS

1. INTRODUCTION

In this paper we deduce convergence results for hypersurface flows in $(n+1)$-dimensional warped product spaces

$$N^{n+1} = (a,b) \times \mathbb{S}^n. \tag{1.1}$$

The metric on N is supposed to have the form

$$\bar{g} = dr^2 + \lambda^2(r)\sigma, \tag{1.2}$$

Received by the editors August 20, 2017.
2010 *Mathematics Subject Classification.* Primary 53C21, 53C24, 53C44.
Key words and phrases. Inverse curvature flow, constrained curvature flow, geometric inequality, warped product space.
The first author was being supported by the "Deutsche Forschungsgemeinschaft" (DFG, German research foundation), research grant "Quermassintegral preserving local curvature flows", number SCHE 1879/3-1. The research of the second author was supported in part by NSFC (Grant No. 11501480, 11871406), the Natural Science Foundation of Fujian Province of China (Grant No. 2017J06003) and the Fundamental Research Funds for the Central Universities (Grant No. 20720180009). Part of this work was done while he was visiting the Institute of Mathematics at Albert-Ludwigs-Universität Freiburg. He would like to thank the Institute for its hospitality.

where λ is a positive warping factor and σ is the round metric on $\mathbb{S}^n$. Precisely, let $M = M^n$ be a closed, connected, and orientable smooth manifold. Then for a family of embeddings

$$x\colon [0, T^*) \times M \to N \tag{1.3}$$

which satisfy the flow equation

$$\begin{aligned} \dot{x} &= \left(\frac{n}{F} - \frac{u}{\lambda'(r)}\right)\nu \\ x(0,\cdot) &= x_0, \end{aligned} \tag{1.4}$$

we will prove longtime existence and smooth convergence to a slice $\{r = \mathrm{const}\}$. F is a function of the principal curvatures satisfying several natural properties to be specified later, u is the *support function*

$$u = \bar{g}(\lambda(r)\partial_r, \nu), \tag{1.5}$$

and x_0 is an initial embedding of M, the image of which is a graph over $\mathbb{S}^n$,

$$M_0 = x_0(M) = \{(r_0(y), y)\colon y \in \mathbb{S}^n\}. \tag{1.6}$$

Before we state the main results in detail—see Theorems 1.1, 1.3, and 1.5—let us give a brief overview over recent related work and our motivation to consider this flow.

Curvature driven hypersurface flows have attracted a lot of attention for about the last four decades, starting with the mean curvature flow of convex hypersurfaces [5, 29, 30] and several fully nonlinear (1-homogeneous) analogues involving the scalar curvature, the Gaussian curvature, and more general functions of the principal curvatures [2, 3, 10, 11]. Besides these contracting flows, expanding flows for star-shaped hypersurfaces have also been considered [17, 21–23, 39, 40, 42]. The most prominent example of an expanding flow is the inverse mean curvature flow, a weak notion of which was used by Huisken and Ilmanen to prove the Riemannian Penrose inequality [32]. Various other applications of contracting and expanding flows include a classification of 2-convex n-dimensional hypersurfaces using the mean curvature flow with surgery, due to Huisken and Sinestrari for $n \geq 3$ [34], various extensions of geometric inequalities of Alexandrov–Fenchel-type to nonconvex hypersurfaces [8, 25], and new Alexandrov–Fenchel-type inequalities in hyperbolic space [12, 15, 44, 45] and in the sphere [24, 36, 45].

These contracting and expanding flows all have the property of some sort of singularity formation, where, however, in the optimal case, the singularities in the expanding case are quite easy to deal with and manifest themselves only in a uniform convergence to infinity or to a minimal hypersurface, if present. Still it seems tempting to directly define a flow which prevents this singularity formation—for example, by adding a constraining term. The first example of such a flow is the volume preserving mean curvature flow which has the form

$$\dot{x} = \left(\frac{1}{|M_t|}\int_{M_t} H - H\right)\nu. \tag{1.7}$$

It has the nice property that, in additional to keeping the enclosed volume fixed, it also decreases the surface area, making it a natural candidate to prove the isoperimetric inequality once one can show that it drives hypersurfaces to round spheres. In [31] this was accomplished for strictly convex hypersurfaces of Euclidean space. Similar flows, which preserve higher order curvature integrals, were considered, for example, in [37, 38] and in [9] for flows in hyperbolic space. Note, however, that the global term involved in this equation adds such heavy complications that these nonlocal flows until now have only allowed a quite restricted class of hypersurfaces—namely, convex ones in Euclidean space and horo-convex[1] ones in hyperbolic space. Besides some perturbation results, in the sphere there are no results at all [1].

However, using the Minkowski identity in $\mathbb{R}^{n+1}$,

$$\int_M H \langle x, \nu \rangle = n|M|, \tag{1.8}$$

it is possible to define a constrained flow, which involves no global term and still preserves enclosed volume while decreasing the surface area. In Euclidean space it reads

$$\dot{x} = (n - H \langle x, \nu \rangle)\nu, \tag{1.9}$$

and in warped products as above with warping factor $\lambda(r)$, it has to be

$$\dot{x} = (n\lambda'(r) - Hu)\nu, \tag{1.10}$$

where u is defined as in (1.5). This beautiful flow was invented by Guan and Li in [26], where they proved longtime existence and smooth convergence to a round sphere when the ambient space is a space form. Together with Wang, they generalized this result to a broader class of ambient warped products with mild assumptions on λ in [28]. The major advantage compared to the classical volume preserving mean curvature flow (1.7) is that the C^0-estimates, a.k.a. barriers, are for free due to the maximum principle. Hence only the star-shapedness of the initial hypersurface is required—namely, that it is a graph in the warped product $(a, b) \times \mathbb{S}^n$ over the base $\mathbb{S}^n$. This result allows one to deduce an isoperimetric inequality for such graphs in quite general warped products. See also [27] for a fully nonlinear extension of this flow.

On the other hand, Brendle, Guan, and Li [7] designed an inverse-type constrained curvature flow in space forms,

$$\dot{x} = \left(\frac{n\lambda'}{F} - u \right) \nu. \tag{1.11}$$

Compared to the mean curvature–type constrained flow (1.10), this flow seems more appropriate for higher order isoperimetric-type inequalities—the Alexandrov–Fenchel-type inequalities for quermassintegrals—in space forms, for the reason that the higher order Minkowski identities imply that, for

$$F = n\frac{H_k}{H_{k-1}}, \tag{1.12}$$

the kth quermassintegral is preserved, while the $(k + 1)$th quermassintegral is decreasing. However, the study of (1.11) is quite subtle from the partial differential equation point of view, and until today no satisfactory complete result has been achieved. Some convergence results are proved in [7] when the initial hypersurface is already close to a sphere. A full convergence result for closed, star-shaped, and k-convex initial hypersurfaces would prove the quermass Alexandrov–Fenchel

[1] A hypersurface in hyperbolic space is called horo-convex if all of its principal curvatures are greater that or equal to 1.

inequalities for such hypersurfaces. For horo-convex domains these have been established by Wang and the second author [44] using a global quermassintegral preserving curvature flow.

Guan-Li's considerations motivate us to study another kind of constrained flow, the constrained inverse curvature flow (1.4) in general warped product spaces. Compared to (1.11), we are able to prove the longtime existence and smooth convergence of (1.4) to a coordinate slice under milder assumptions on the curvature function F, the warping factor λ, and the initial hypersurface. We use this result to deduce a new geometric inequality in the anti–de Sitter Schwarzschild manifolds, see Theorem 1.5, upon which we will make more comment later.

Let us first state the main results of this paper. Since our assumptions on the curvature function and the initial embedding depend on the structure of the warping factor λ, we split our flow results into two theorems. We start with the ambient space $N = \mathbb{S}^{n+1}_+$, in which case $\lambda(r) = \sin r, r \in [0, \frac{\pi}{2})$.

Theorem 1.1. *Let $x_0(M)$ be the embedding of a closed n-dimensional manifold M in $\mathbb{S}^{n+1}$ such that $x_0(M)$ is strictly convex. Let*

$$F = n\frac{H_k}{H_{k-1}}, \tag{1.13}$$

where H_k is the kth normalized elementary symmetric polynomial of the principal curvatures. Then any solution x of (1.4) *exists for all positive times and converges to a geodesic slice in the C^∞-topology.*

Now we come to ambient spaces satisfying $\lambda'' \geq 0$. We obtain convergence results for a large class of speeds and therefore make the following assumption.

Assumption 1.2. Let $\Gamma \subset \mathbb{R}^n$ be a symmetric, convex, open cone containing

$$\Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n \colon \kappa_i > 0\}, \tag{1.14}$$

and suppose that F is positive in Γ, strictly monotone, homogeneous of degree 1, and concave with

$$F_{|\partial\Gamma} = 0, \qquad F(1, \ldots, 1) = n. \tag{1.15}$$

Theorem 1.3. *Let $a, b \in \mathbb{R}$, and let $(N, \bar{g})$ be the warped space $((a, b) \times \mathbb{S}^n, dr^2 + \lambda^2(r)\sigma)$ with $\lambda > 0$, $\lambda' > 0$, and $\lambda'' \geq 0$. Let $F \in C^\infty(\Gamma)$ satisfy Assumption* 1.2, *and let $x_0(M)$ be the embedding of a closed n-dimensional manifold M in N such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$, and such that $\kappa \in \Gamma$ for all n-tuples of principal curvatures along $x_0(M)$. Then any solution x of* (1.4) *exists for all positive times and converges to a geodesic slice in the C^∞-topology.*

Remark 1.4. The assumption $\lambda'' \geq 0$ is used only for deriving the uniform lower bound for F. This assumption resembles the nonpositivity of the ambient sectional curvature in the radial direction, a property which was also crucial in the deduction of longtime existence of the inverse mean curvature flow in warped product spaces; see [40].

Note that, compared to the purely expanding inverse mean curvature flow

$$\dot{x} = \frac{1}{H}\nu, \tag{1.16}$$

which was treated in general warped products in [40], the set of assumptions on the warping factor in Theorem 1.3 is quite small. In order to obtain convergence results of a purely expanding flow, ones needs a lot of more global information about the ambient space. From the viewpoint of geometric inequalities for hypersurfaces, only local information is required, and hence a constrained flow seems to be more promising than a flow of the form (1.16). Indeed, in this paper we use Theorem 1.3 to obtain the following geometric inequalities, one weighted Minkowski-type inequality and one weighted isoperimetric-type inequality.

Theorem 1.5. *Let $N = (a,b) \times \mathbb{S}^n$ be equipped with one of the anti–de Sitter Schwarzschild metrics or the hyperbolic metric, i.e.,*

$$\lambda' = \sqrt{1+\lambda^2 - m\lambda^{1-n}}, \qquad m \geq 0. \tag{1.17}$$

Let $\Sigma \subset N$ be a closed, star-shaped, and mean-convex hypersurface given by the function $r\colon \mathbb{S}^n \to (a,b)$, and let

$$\Omega = \{(s,y) \in N\colon a \leq s \leq r(y),\ y \in \mathbb{S}^n\}. \tag{1.18}$$

Then it holds that

$$\int_\Sigma H\lambda' d\mu - 2n\int_\Omega \frac{\lambda'\lambda''}{\lambda} dN \geq \xi_1(|\Sigma|) \tag{1.19}$$

and

$$\int_\Sigma H\lambda' d\mu - 2n\int_\Omega \frac{\lambda'\lambda''}{\lambda} dN \geq \xi_0\left(\int_\Omega \lambda' dN\right), \tag{1.20}$$

where ξ_0, ξ_1 are the associated monotonically increasing functions for radial coordinate slices. Equality holds if and only if Σ is a radial coordinate slice.

In particular, in hyperbolic space, due to $\lambda'' = \lambda$, inequality (1.20) reduces to

$$\int_\Sigma H\lambda' d\mu - (n+1)n\int_\Omega \lambda' dN \geq n|\mathbb{S}^n|^{\frac{2}{n+1}}\left((n+1)\int_\Omega \lambda' dN\right)^{\frac{n-1}{n+1}}, \tag{1.21}$$

where $\lambda'(r) = \cosh r$. Equality in (1.21) holds if and only if Σ is a geodesic sphere centered at the origin. The second author proved a Minkowski-type inequality in [46] stating that, for a closed horo-convex hypersurface $\Sigma \subset \mathbb{H}^{n+1}$, it holds that

$$\left(\int_\Sigma \lambda' d\mu\right)^2 \geq \frac{n+1}{n}\int_\Sigma H\lambda' d\mu \int_\Omega \lambda' dN. \tag{1.22}$$

Combining this with (1.21), we get the following theorem.

Theorem 1.6. *Let Σ be a closed horo-convex hypersurface in $\mathbb{H}^{n+1}$ with the origin lying inside Ω. Then*

$$\int_\Sigma \lambda' d\mu \geq \left[\left((n+1)\int_\Omega \lambda' dN\right)^2 + |\mathbb{S}^n|^{\frac{2}{n+1}}\left((n+1)\int_\Omega \lambda' dN\right)^{\frac{2n}{n+1}}\right]^{\frac{1}{2}}.$$

Equality holds if and only if Σ is a geodesic sphere centered at the origin.

Remark 1.7. Theorem 1.6 already appeared in the paper [16], where it is the case that $k=0$ in Theorem 9.2. However, their proof relies on an *invalid* inequality—namely, [16, eq. (9.8)]—which states that

$$|\Sigma|^{\frac{n+1}{n}} \geq |\mathbb{S}^n|^{\frac{1}{n}}\int_\Sigma u d\mu \quad = (n+1)|\mathbb{S}^n|^{\frac{1}{n}}\int_\Omega \lambda' dN. \tag{1.23}$$

This inequality is already incorrect on geodesic spheres not centered at the origin. Theorem 1.6 fixes this gap in the proof of [16, Thm. 9.2].

Remark 1.8. By using the classical inverse mean curvature flow, Brendle, Hung, and Wang proved in [8] for a closed, star-shaped, and mean-convex hypersurface Σ in anti–de Sitter Schwarzschild space that

$$\int_\Sigma H\lambda' d\mu - (n+1)n\int_\Omega \lambda' dN \geq n|\mathbb{S}^n|^{\frac{1}{n}}\left(|\Sigma|^{\frac{n-1}{n}} - |\partial N|^{\frac{n-1}{n}}\right). \tag{1.24}$$

In particular, in hyperbolic space they get

$$\int_\Sigma H\lambda' d\mu - (n+1)n\int_\Omega \lambda' dN \geq n|\mathbb{S}^n|^{\frac{1}{n}}|\Sigma|^{\frac{n-1}{n}}. \tag{1.25}$$

Equation (1.24) is different from (1.19) in the sense that the right-hand side of (1.19) does not depend on the horizon $\{a\} \times \mathbb{S}^n$.

Another nice corollary is given by the following area bound for star-shaped and mean-convex hypersurfaces in ambient spaces of nonpositive radial curvature. It is clear to the authors neither whether this bound is evident by other means nor if it has been recorded before. The monotonicity of the area in these spaces follows from Theorem 1.3; see (8.6) and Remark 4.2.

Corollary 1.9. *Let $a, b \in \mathbb{R}$, and let $(N, \bar{g})$ be the warped space $((a,b)\times\mathbb{S}^n, dr^2 + \lambda^2(r)\sigma)$ with $\lambda > 0$, $\lambda' > 0$, and $\lambda'' \geq 0$. Let $\Sigma \subset N$ be a closed, star-shaped, and mean-convex hypersurface,*

$$\Sigma = \{(r(y), y) \in N : y \in \mathbb{S}^n\}. \tag{1.26}$$

Then the area of Σ satisfies

$$|\Sigma| \leq |\mathbb{S}^n|\lambda\left(r_{max}\right)^n, \tag{1.27}$$

where $r_{max} = \max_{\mathbb{S}^n} r$.

It would be very interesting to find further monotone quantities along these flows, in particular in a spherical ambient space.

The paper is organized as follows. In sections 2 and 3 we collect the notation and derive the fundamental evolution equations for several geometric quantities. In sections 4–7 we derive a priori estimates under various conditions on F and λ, and in section 8 we complete the proof of Theorems 1.1 and 1.3. Section 9 is devoted to prove monotonicity for various geometric quantities and, in turn, the geometric inequalities in Theorem 1.5.

2. Notation and conventions

2.1. Conventions on Riemannian geometry.

2.1.1. *Intrinsic curvature.* Let (M^n, g) be a Riemannian manifold. With respect to a local frame $(e_i)_{1\leq i\leq n}$ of the tangent bundle, let (g_{ij}) denote the coordinate functions of g with respect to the basis $(\epsilon^i \otimes \epsilon^j)_{1\leq i,j\leq n}$, where ϵ^i denote the basis elements dual to e_i. Let (g^{ij}) denote the inverse matrix of (g_{ij}). For a (k,l) tensor field T, the coordinates of which with respect to this frame are given by

$$T = (T^{i_1\dots i_k}_{j_1\dots j_l}), \tag{2.1}$$

we can define $(k+1, l-1)$ tensor fields by using the tangent-cotangent isomorphism induced by g, e.g.,

$$T^{i_1\dots i_{k+1}}_{j_1\dots j_{l-1}} = T^{i_1\dots i_k}_{j_1\dots j_l} g^{j_l i_{k+1}}. \tag{2.2}$$

Of course we can also raise other indices to different slots, but it will always be apparent, or explicitly stated, which one is meant.

The Lie bracket of two vector fields X, Y on M is given by

$$[X,Y]\varphi = X(Y\varphi) - Y(X\varphi) \qquad \forall \varphi \in C^\infty(M). \tag{2.3}$$

Let ∇ be the Levi-Civita connection of g. Then, for a (k,l) tensor field T, its covariant derivative ∇T is a $(k, l+1)$ tensor field given by

$$\begin{aligned}(\nabla T)(Y^1,\dots,Y^k,X_1,\dots,X_l,X) &= (\nabla_X T)(Y^1,\dots,Y^k,X_1,\dots,X_l)\\ &= X(T(Y^1,\dots,Y^k,X_1,\dots,X_l))\\ &\quad - T(\nabla_X Y^1, Y^2,\dots,Y^k,X_1,\dots,X_l) - \cdots\\ &\quad - T(Y^1,\dots,Y^k,X_1,\dots,X_{l-1}\nabla_X X_l).\end{aligned} \tag{2.4}$$

We denote by $\nabla^m T$ the mth covariant derivative of T, and its coordinates with respect to a basis $(e_i)_{1\leq i\leq n}$ are denoted by

$$\nabla^m T = \left(T^{i_1\dots i_k}_{j_1\dots j_l;j_{l+1}\dots j_{l+m}}\right), \tag{2.5}$$

where all indices appearing after the semicolon indicate covariant derivatives. The $(1,3)$ Riemannian curvature tensor is defined by

$$\mathrm{Rm}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{2.6}$$

or with respect to the basis (e_i),

$$\mathrm{Rm}(e_i,e_j)e_k = {R_{ijk}}^l e_l, \tag{2.7}$$

where we use the summation convention (and will henceforth do so). The coordinate expression of (2.6), the so-called Ricci identities, read

$$X^k_{;ij} - X^k_{;ji} = -{R_{ijm}}^k X^m \tag{2.8}$$

for all vector fields $X = (X^k)$. We also denote the $(0,4)$ version of the curvature tensor by Rm,

$$\mathrm{Rm}(W,X,Y,Z) = g(\mathrm{Rm}(W,X)Y,Z). \tag{2.9}$$

The Ricci curvature can unambiguously be defined in coordinates by

$$\mathrm{Rc}(e_i,e_j) = R_{ij} = {R_{kij}}^k. \tag{2.10}$$

The scalar curvature is

$$R = {R_i}^i = g^{ki}R_{ki}. \tag{2.11}$$

2.1.2. *Extrinsic curvature.* When dealing with immersed hypersurfaces

$$x\colon M \hookrightarrow N \tag{2.12}$$

of a Riemannian manifold M^n into an ambient Riemannian manifold N^{n+1}, we furnish all of the previous geometric quantities of N with an overbar, e.g., $\bar g = (\bar g_{\alpha\beta})$, where greek indices run from 0 to n, $\bar\nabla$, etc. We keep using latin indices, running from 1 to n, for geometric quantities of M, e.g., the induced metric $g = (g_{ij})$. The induced geometry of M is governed by the following relations. The (local) second fundamental form $h = (h_{ij})$ is given by the Gaussian formula

$$\bar\nabla_X Y = \nabla_X Y - h(X,Y)\nu, \tag{2.13}$$

where ν is a local normal field. Note that here (and in the rest of the paper) we will abuse notation by disregarding the necessity to distinguish between a vector $X \in T_pM$ and its push-forward $x_*X \in T_pN$. The Weingarten endomorphism $A = (h^i_j)$ is given by $h^i_j = g^{ki}h_{kj}$, and the Weingarten equation

$$\bar{\nabla}_X \nu = A(X), \tag{2.14}$$

holds there, or in coordinates

$$\nu^\alpha_{\ ;i} = h^k_i x^\alpha_{\ ;k}. \tag{2.15}$$

We also have the Codazzi equation

$$\nabla_Z h(X,Y) - \nabla_Y h(X,Z) = -\overline{\mathrm{Rm}}(\nu, X, Y, Z) \tag{2.16}$$

or

$$h_{ij;k} - h_{ik;j} = -\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i} x^\gamma_{\ ;j} x^\delta_{\ ;k}, \tag{2.17}$$

and the Gauss equation

$$\mathrm{Rm}(W,X,Y,Z) = \overline{\mathrm{Rm}}(W,X,Y,Z) + h(W,Z)h(X,Y) - h(W,Y)h(X,Z) \tag{2.18}$$

or

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{\ ;i} x^\beta_{\ ;j} x^\gamma_{\ ;k} x^\delta_{\ ;l} + h_{il}h_{jk} - h_{ik}h_{jl}. \tag{2.19}$$

2.1.3. *Graphs in warped products.* In this paper we deal with warped products

$$N = (a,b) \times \mathbb{S}^n \tag{2.20}$$

with metric

$$\bar{g} = dr^2 + \lambda^2(r)\sigma, \tag{2.21}$$

where σ is the round metric of $\mathbb{S}^n$. We need the specific structure of the Ricci curvature tensor in such a warped product. It holds that

$$\overline{\mathrm{Rc}} = -\left(\frac{\lambda''}{\lambda} - (n-1)\frac{1-\lambda'^2}{\lambda^2}\right)\bar{g} - (n-1)\left(\frac{\lambda''}{\lambda} + \frac{1-\lambda'^2}{\lambda^2}\right) dr \otimes dr; \tag{2.22}$$

cf. [6, Prop. 2.1].

Our hypersurfaces

$$x \colon M \hookrightarrow N \tag{2.23}$$

will all be graphs over $\mathbb{S}^n$,

$$x(M) = \{(r(y), y) \colon y \in \mathbb{S}^n\} = \{(r(y(\xi)), y(\xi)) \colon \xi \in M\}, \tag{2.24}$$

where we do not make a notational difference between the radial coordinate r of N and the function $r_{|M}$. Along M, we will always pick the *outward* pointing normal

$$\nu = v^{-1}(1, -\lambda^{-2}\sigma^{ik}\partial_k r), \tag{2.25}$$

where

$$v^2 = 1 + \lambda^{-2}\sigma^{ij}\partial_i r \partial_j r, \tag{2.26}$$

and we use this normal in the Gaussian formula (2.13). The support function of M is defined by

$$u = \bar{g}(\lambda\partial_r, \nu) = \frac{\lambda}{v}. \tag{2.27}$$

There is also a relation between the second fundamental form and the radial function on the hypersurface. Let

$$\bar{h} = \lambda' \lambda \sigma. \tag{2.28}$$

Then

$$v^{-1} h = -\nabla^2 r + \bar{h} \tag{2.29}$$

holds; cf. [20, eq. (1.5.10)]. Since the induced metric is given by

$$g_{ij} = r_{;i} r_{;j} + \lambda^2 \sigma_{ij}, \tag{2.30}$$

we obtain

$$v^{-1} h_{ij} = -r_{;ij} + \frac{\lambda'}{\lambda} g_{ij} - \frac{\lambda'}{\lambda} r_{;i} r_{;j}. \tag{2.31}$$

Define

$$\varphi(r) = \int_a^r \frac{1}{\lambda}. \tag{2.32}$$

Regarding r as a function on $\mathbb{S}^n$, we have

$$h_i^j = \frac{\lambda'}{\lambda v} \delta_i^j - \frac{1}{\lambda v} \tilde{g}^{jk} \varphi_{,ki}, \tag{2.33}$$

where

$$\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi_,{}^i \varphi_,{}^j}{v^2}, \tag{2.34}$$

and the covariant derivative and index raising is performed with respect to the spherical metric σ_{ij}; cf. [21, eq. (3.26)]. We will use $\hat{\nabla}$ to denote the covariant derivative on $\mathbb{S}^n$ throughout this paper.

2.1.4. *Anti–de Sitter Schwarzschild space.* The anti–de Sitter Schwarzschild manifolds are asymptotically hyperbolic Riemannian warped products of the form

$$N = (r_0, \infty) \times \mathbb{S}^n \tag{2.35}$$

equipped with the warped product metric

$$\bar{g} = dr^2 + \lambda^2(r) \sigma, \tag{2.36}$$

where λ satisfies

$$\lambda' = \sqrt{1 + \lambda^2 - m\lambda^{1-n}} \tag{2.37}$$

with $m > 0$ and horizon $\partial N = \{r_0\} \times \mathbb{S}^n$. The limiting case $m = 0$ is the hyperbolic metric. These Riemannian manifolds carry the property to be *static*, i.e.,

$$\bar{\Delta} \lambda' \bar{g} - \bar{\nabla}^2 \lambda' + \lambda' \overline{\mathrm{Rc}} = 0, \tag{2.38}$$

which ensures that the Lorentzian warped product $-\lambda'^2 dt^2 + \bar{g}$ is a solution to Einstein's equation.

2.2. **Curvature functions.** In Assumption 1.2 the part of our normal variation that depends on the curvature of the hypersurface was stipulated to depend on the principal curvatures

$$F = F(\kappa_i). \tag{2.39}$$

However, in the calculation of the evolution equations, it is often useful to consider F as a function of the diagonalizable Weingarten operator A,

$$F = F(A) := F(\mathrm{EV}(A)), \tag{2.40}$$

where $\mathrm{EV}(A)$ is the unordered n-tuple of eigenvalues of A. This is well-defined due to the symmetry of F. However, when using this definition, F is defined not on the whole endomorphism bundle but on the diagonalizable operators only. It is thus most convenient to consider the function defined by

$$\hat{F}(g,h) := F\left(\frac{1}{2}g^{ik}(h_{kj}+h_{jk})\right) \tag{2.41}$$

for all positive definite g and all bilinear forms $h \in T^{0,2}_p M$. Then

$$\hat{F}^{ij} = \frac{\partial F}{\partial h_{ij}} \tag{2.42}$$

is a $(2,0)$ tensor, and we also write

$$\hat{F}^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}. \tag{2.43}$$

Furthermore, if $F = F(\kappa_i)$ is strictly monotone, then $\hat{F}^{ij}$ is strictly elliptic. If F is concave, then

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \leq 0 \tag{2.44}$$

for all symmetric (η_{ij}). We refer to [4], [20, Ch. 2], [41] for more details on curvature functions.

Furthermore, we will abuse notation and also write F for $\hat{F}$ since no confusion will be possible. For example, when writing F^{ij}, we can only mean $\hat{F}^{ij}$ since there are two contravariant indices.

Let us denote by σ_k the kth elementary symmetric polynomial, and define the kth normalized elementary symmetric polynomial by

$$H_k = \frac{1}{\binom{n}{k}}\sigma_k. \tag{2.45}$$

Denote by Γ_k the connected component of $\{\sigma_k > 0\}$, which contains the point $(1,\dots,1)$.

3. Evolution equations

In this section we deduce the evolution equations relevant to studying the flow

$$\dot{x} = \left(\frac{n}{F} - \frac{u}{\lambda'}\right)\nu \equiv \mathcal{F}\nu. \tag{3.1}$$

The following basic evolution equations are well known and can be found in many places. We use the reference [20, Ch. 2.3], where we note that we use the other sign on the curvature tensor.

Lemma 3.1. *Along* (3.1), *the following evolution equations hold:*

$$\dot{g} = 2\mathcal{F}h, \tag{3.2}$$

$$\frac{\bar{\nabla}}{dt}\nu = -\operatorname{grad}\mathcal{F}, \tag{3.3}$$

$$\dot{h}_i^j = -\mathcal{F}_{;i}{}^{j} - \mathcal{F}h_k^j h_i^k - \mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^{\alpha}_{;i}\nu^{\beta}\nu^{\gamma}x^{\delta}_{;k}g^{kj}, \tag{3.4}$$

and

$$\dot{h}_{ij} = -\mathcal{F}_{;ij} + \mathcal{F}h_{ik}h_j^k - \mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^{\alpha}_{;i}\nu^{\beta}\nu^{\gamma}x^{\delta}_{;j}. \tag{3.5}$$

We need some further special evolution equations.

Lemma 3.2. *Define the operator* $\mathcal{L}$ *by*

$$\mathcal{L} = \partial_t - \frac{n}{F^2}F^{ij}\nabla^2_{ij} - \frac{\lambda}{\lambda'}r_;{}^{k}\nabla_k. \tag{3.6}$$

Along the flow (3.1) *of graphs*

$$M_t = \{(r(t,y),y)\colon y \in \mathbb{S}^n\}, \tag{3.7}$$

we have the following evolution equations for the radial function r*, the support function* u*, and the curvature function* F*:*

$$\mathcal{L}r = \frac{2n}{vF} - \frac{\lambda}{\lambda'} - \frac{n\lambda'}{\lambda F^2}F^{ij}g_{ij} + \frac{n\lambda'}{\lambda F^2}F^{ij}r_{;i}r_{;j}, \tag{3.8}$$

$$\begin{aligned}\mathcal{L}u = {}& \frac{n}{F^2}\left(F^{ij}h_{ik}h_j^k - \frac{1}{n}F^2\right)u - \frac{\lambda''\lambda}{\lambda'^2}\|\nabla r\|^2 u \\ &+ \frac{n\lambda}{F^2}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^{\alpha}x^{\beta}_{;i}x^{\gamma}_{;m}x^{\delta}_{;j}r_;{}^{m},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathcal{L}F = {}& -\frac{2n}{F^3}F^{ij}F_{;i}F_{;j} - \frac{n}{F}\left(F^{ij}h_{jk}h_i^k - \frac{1}{n}F^2\right) + \frac{u^2\lambda''}{\lambda\lambda'^2}F - \frac{u\lambda''}{\lambda\lambda'}F^{ij}g_{ij} \\ &+ \frac{u}{\lambda'^2}\left(\frac{\lambda'\lambda''}{\lambda} - \lambda''' + \frac{2\lambda''^2}{\lambda'}\right)F^{ij}r_{;i}r_{;j} - \frac{2\lambda''}{\lambda'^2}F^{ij}u_{;i}r_{;j} \\ &- \frac{\lambda}{\lambda'}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^{\alpha}x^{\beta}_{;i}x^{\gamma}_{;m}x^{\delta}_{;j}r_;{}^{m} - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x^{\alpha}_{;i}\nu^{\beta}\nu^{\gamma}x^{\delta}_{;j}.\end{aligned} \tag{3.10}$$

Proof. The 0-component of (3.1) gives

$$\dot{r} = \mathcal{F}v^{-1} = \left(\frac{n}{F} - \frac{u}{\lambda'}\right)v^{-1}, \tag{3.11}$$

while from (2.31) we see, using the 1-homogeneity of F,

$$-\frac{n}{F^2}F^{ij}r_{;ij} = \frac{n}{vF} - \frac{n\lambda'}{\lambda F^2}F^{ij}g_{ij} + \frac{n\lambda'}{\lambda F^2}F^{ij}r_{;i}r_{;j}. \tag{3.12}$$

Adding up gives (3.8).

To prove (3.9), note that $\lambda\partial_r$ is a conformal vector field, i.e., for all ambient vector fields $\bar{X}$ it holds that

$$\bar{\nabla}_{\bar{X}}(\lambda\partial_r) = \lambda'\bar{X}. \tag{3.13}$$

Hence

$$\dot{u} = \bar{g}(\lambda'\dot{x},\nu) + \bar{g}(\lambda\partial_r, \bar{\nabla}_{\dot{x}}\nu) = \lambda'\mathcal{F} - \bar{g}(\lambda\partial_r, \operatorname{grad}\mathcal{F}). \tag{3.14}$$

Furthermore, it holds that

$$Xu = \bar{g}(\lambda\partial_r, A(X)) \tag{3.15}$$

and

$$\begin{aligned}\nabla^2 u(X,Y) &= Y(Xu) - (\nabla_Y X)u \\ &= \lambda' h(X,Y) + \bar{g}(\lambda\partial_r, \nabla_Y A(X)) - h(Y, A(X))u \qquad \forall X,Y \in TM.\end{aligned} \tag{3.16}$$

We use the Codazzi equation (2.16) to deduce

$$\begin{aligned}\bar{g}(\lambda\partial_r, \nabla_Y A(X)) &= \lambda\bar{g}_{\alpha\beta}r^\alpha x^\beta_{;k} h^k_{i;j} X^i Y^j \\ &= \lambda\bar{g}_{\alpha\beta}r^\alpha x^\beta_{;k} h_{ij;}{}^k X^i Y^j - \lambda\bar{g}_{\alpha\beta}r^\alpha x^\beta_{;k}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i} g^{km} x^\gamma_{;m} x^\delta_{;j}.\end{aligned} \tag{3.17}$$

Note that

$$\bar{g}_{\alpha\beta}r^\alpha x^\beta_{;k} = r_{;k}; \tag{3.18}$$

we thus get

$$u_{;ij} = \lambda' h_{ij} + \lambda r_{;k} h_{ij;}{}^k - h^k_i h_{kj} u - \lambda\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i} x^\gamma_{;m} x^\delta_{;j} r_;{}^m. \tag{3.19}$$

Since

$$\mathcal{F}_{;k} = -\frac{n}{F^2}F^{ij}h_{ij;k} - \frac{u_{;k}}{\lambda'} + \frac{\lambda'' u}{\lambda'^2} r_{;k}, \tag{3.20}$$

we obtain (3.9).

From (3.4), we have

$$\begin{aligned}\dot{F} &= -F^{ij}\mathcal{F}_{;ij} - F^{ij}\mathcal{F}h_{jk}h^k_i - F^{ij}\mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;i}\nu^\beta\nu^\gamma x^\delta_{;j} \\ &= \frac{n}{F^2}F^{ij}F_{;ij} - \frac{2n}{F^3}F^{ij}F_{;i}F_{;j} + \frac{1}{\lambda'}F^{ij}u_{;ij} - \frac{u}{\lambda'^2}F^{ij}\lambda'_{;ij} + \frac{2u}{\lambda'^3}F^{ij}\lambda'_{;i}\lambda'_{;j} \\ &\quad - \frac{2\lambda''}{\lambda'^2}F^{ij}u_{;i}r_{;j} - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)F^{ij}h_{jk}h^k_i - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;i}\nu^\beta\nu^\gamma x^\delta_{;j}.\end{aligned} \tag{3.21}$$

Using (3.19) and (2.31), we get (3.10). □

We also need the parabolic equation satisfied by the Weingarten operator. A similar calculation was performed in [20, Lemma 2.4.1], but since our flow speed is not directly covered by this reference, we deduce it for convenience.

Lemma 3.3. *Along* (3.1) *the following evolution equation holds.*

$$
\begin{aligned}
\mathcal{L}h_i^j &= -\frac{2n}{F^3}F_{;i}F_;{}^j + \frac{n}{F^2}F^{kl,rs}h_{kl;i}h_{rs;}{}^j - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_;{}^j + r_{;i}u_;{}^j\right) \\
&\quad - \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_;{}^j + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h_i^j \\
&\quad - \frac{u\lambda''}{\lambda'\lambda}\delta_i^j - \frac{\lambda}{\lambda'}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}x^\gamma_{;m}x^\delta_{;l}r_;{}^m g^{lj} \\
&\quad - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;i}\nu^\beta\nu^\gamma x^\delta_{;m}g^{mj} + \frac{n}{F^2}F^{kl}h_{rk}h_l^r h_i^j - \frac{2n}{F}h_k^j h_i^k \\
&\quad + \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h_i^m + x^\alpha_{;l}x^\beta_{;i}x^\gamma_{;k}x^\delta_{;m}h_r^m\right)g^{rj} \\
&\quad + \frac{2n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;i}x^\delta_{;m}h_k^m g^{rj} + \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;k}x^\gamma_{;l}\nu^\delta h_i^j \\
&\quad + \frac{n}{F}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}\nu^\gamma x^\delta_{;m}g^{mj} - \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{;k}x^\gamma_{;l}x^\delta_{;i}x^\epsilon_{;m}g^{mj} \\
&\quad - \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{;i}x^\gamma_{;k}x^\delta_{;m}x^\epsilon_{;l}g^{mj}.
\end{aligned}
\tag{3.22}
$$

Proof. We use (3.5) and calculate $-\mathcal{F}_{;ij}$ step by step. We use

$$
\mathcal{F} = \frac{n}{F} - \frac{u}{\lambda'}, \qquad -\mathcal{F}_{;i} = \frac{n}{F^2}F_{;i} + \frac{u_{;i}}{\lambda'} - \frac{u\lambda'' r_{;i}}{\lambda'^2},
\tag{3.23}
$$

and (3.19), as well as (2.31), to deduce

$$
\begin{aligned}
-\mathcal{F}_{;ij} &= -\frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F_{;ij} + \frac{u_{;ij}}{\lambda'} - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) \\
&\quad - u\left(\frac{\lambda'''}{\lambda'^2} - \frac{2\lambda''^2}{\lambda'^3}\right)r_{;i}r_{;j} - \frac{u\lambda''}{\lambda'^2}r_{;ij} \\
&= -\frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F_{;ij} - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) \\
&\quad - u\left(\frac{\lambda'''}{\lambda'^2} - \frac{2\lambda''^2}{\lambda'^3}\right)r_{;i}r_{;j} + h_{ij} + \frac{\lambda}{\lambda'}r_{;k}h_{ij;}{}^k - \frac{u}{\lambda'}h_{ik}h_j^k \\
&\quad - \frac{\lambda}{\lambda'}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}x^\gamma_{;m}x^\delta_{;j}r_;{}^m + \frac{u\lambda''}{\lambda'^2}\left(v^{-1}h_{ij} - \frac{\lambda'}{\lambda}g_{ij} + \frac{\lambda'}{\lambda}r_{;i}r_{;j}\right) \\
&= -\frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F_{;ij} - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) + \frac{\lambda}{\lambda'}r_{;k}h_{ij;}{}^k \\
&\quad - \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_{;j} + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h_{ij} - \frac{u}{\lambda'}h_{ik}h_j^k \\
&\quad - \frac{u\lambda''}{\lambda'\lambda}g_{ij} - \frac{\lambda}{\lambda'}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}x^\gamma_{;m}x^\delta_{;j}r_;{}^m.
\end{aligned}
\tag{3.24}
$$

We have to transform $F_{;ij}$. Using the Codazzi equation (2.16) and the Ricci identities (2.8), we obtain

$$
\begin{aligned}
F_{;ij} &= F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{kl;ij} \\
&= F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{ki;lj} - F^{kl}\left(\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}\right)_{;j} \\
&= F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{ki;jl} + F^{kl}R_{ljk}{}^a h_{ai} + F^{kl}R_{lji}{}^a h_{ka} \\
&\quad - F^{kl}\left(\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}\right)_{;j} \\
&= F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}R_{ljk}{}^a h_{ai} + F^{kl}R_{lji}{}^a h_{ka} + F^{kl}h_{ij;kl} \\
&\quad - F^{kl}\left(\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}\right)_{;j} - F^{kl}\left(\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}\right)_{;l}.
\end{aligned}
\tag{3.25}
$$

Differentiating the big brackets by the product rule gives, using the Weingarten equation (2.14) and the Gauss equation (2.18),

$$
\begin{aligned}
F_{;ij} &= F^{kl}h_{ij;kl} + F^{kl,rs}h_{kl;i}h_{rs;j} \\
&\quad + F^{kl}(h_{la}h_{jk} - h_{lk}h_{ja} + \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;k}x^\delta_{\ ;a})h^a_i \\
&\quad + F^{kl}(h_{la}h_{ji} - h_{li}h_{ja} + \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;i}x^\delta_{\ ;a})h^a_k \\
&\quad - F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}x^\epsilon_{\ ;j} - F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;m}x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}h^m_j \\
&\quad + F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}\nu^\gamma x^\delta_{\ ;i}h_{lj} + F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}\nu^\delta h_{ij} \\
&\quad - F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}x^\epsilon_{\ ;l} - F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;m}x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}h^m_l \\
&\quad + F^{kl}h_{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}\nu^\gamma x^\delta_{\ ;j} + F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}\nu^\delta h_{jl},
\end{aligned}
\tag{3.26}
$$

and after some rearranging, using the homogeneity of F,

$$
\begin{aligned}
F_{;ij} &= F^{kl}h_{ij;kl} + F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{rk}h^r_l h_{ij} - Fh_{ik}h^k_j \\
&\quad + F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;k}x^\delta_{\ ;m}h^m_i + x^\alpha_{\ ;l}x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;m}h^m_j\right) \\
&\quad + 2F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;i}x^\delta_{\ ;m}h^m_k + F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}\nu^\delta h_{ij} \\
&\quad + F\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}\nu^\gamma x^\delta_{\ ;j} - F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}x^\epsilon_{\ ;j} \\
&\quad - F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}x^\epsilon_{\ ;l}.
\end{aligned}
\tag{3.27}
$$

From (3.5), inserting (3.24), we get

(3.28)

$$
\begin{aligned}
\dot{h}_{ij} &= -\mathcal{F}_{;ij} + \mathcal{F}h_{ik}h^k_j - \mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j} \\
&= -\frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F_{;ij} - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) + \frac{\lambda}{\lambda'}r_{;k}h_{ij;}{}^k \\
&\quad - \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_{;j} + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h_{ij} - \frac{u}{\lambda'}h_{ik}h^k_j - \frac{u\lambda''}{\lambda'\lambda}g_{ij} \\
&\quad - \frac{\lambda}{\lambda'}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;m}x^\delta_{\ ;j}r_;{}^m + \left(\frac{n}{F} - \frac{u}{\lambda'}\right)h_{ik}h^k_j \\
&\quad - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j}.
\end{aligned}
$$

Inserting (3.27) into this equation gives
(3.29)

$$\begin{aligned}\dot h_{ij} =& \frac{n}{F^2}F^{kl}h_{ij;kl} + \frac{\lambda}{\lambda'}r_{;k}h_{ij;}{}^k - \frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F^{kl,rs}h_{kl;i}h_{rs;j}\\ &- \frac{\lambda''}{\lambda'^2}(u_{;i}r_{;j} + r_{;i}u_{;j}) - \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_{;j} + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h_{ij}\\ &- \frac{2u}{\lambda'}h_{ik}h^k_j - \frac{u\lambda''}{\lambda'\lambda}g_{ij} - \frac{\lambda}{\lambda'}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;i}x^\gamma{}_{;m}x^\delta{}_{;j}r_;{}^m\\ &- \left(\frac{n}{F} - \frac{u}{\lambda'}\right)\bar R_{\alpha\beta\gamma\delta}x^\alpha{}_{;i}\nu^\beta\nu^\gamma x^\delta{}_{;j} + \frac{n}{F^2}F^{kl}h_{rk}h^r_l h_{ij}\\ &+ \frac{n}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta}\left(x^\alpha{}_{;l}x^\beta{}_{;j}x^\gamma{}_{;k}x^\delta{}_{;m}h^m_i + x^\alpha{}_{;l}x^\beta{}_{;i}x^\gamma{}_{;k}x^\delta{}_{;m}h^m_j\right)\\ &+ \frac{2n}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta}x^\alpha{}_{;l}x^\beta{}_{;j}x^\gamma{}_{;i}x^\delta{}_{;m}h^m_k + \frac{n}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;k}x^\gamma{}_{;l}\nu^\delta h_{ij}\\ &+ \frac{n}{F}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;i}\nu^\gamma x^\delta{}_{;j} - \frac{n}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta{}_{;k}x^\gamma{}_{;l}x^\delta{}_{;i}x^\epsilon{}_{;j}\\ &- \frac{n}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta{}_{;i}x^\gamma{}_{;k}x^\delta{}_{;j}x^\epsilon{}_{;l}.\end{aligned}$$

Using
(3.30)

$$\dot h^i_j = \dot g^{ik}h_{kj} + g^{ik}\dot h_{kj} = -g^{il}\dot g_{lm}g^{mk}h_{kj} + g^{ik}\dot h_{kj} = 2\left(\frac{u}{\lambda'} - \frac{n}{F}\right)h^i_k h^k_j + g^{ik}\dot h_{kj}$$

gives the result. □

In particular, when the ambient space N is a space form of sectional curvature K_N, then

$$\bar R_{\alpha\beta\gamma\delta} = K_N(\bar g_{\alpha\delta}\bar g_{\beta\gamma} - \bar g_{\alpha\gamma}\bar g_{\beta\delta}) \tag{3.31}$$

and

$$\lambda'' = -K_N\lambda, \qquad \lambda''' = -K_N\lambda' = \frac{\lambda''\lambda'}{\lambda}, \tag{3.32}$$

and (3.22) reduces to

$$\begin{aligned}\mathcal{L}h^j_i =& -\frac{2n}{F^3}F_{;i}F_;{}^j + \frac{n}{F^2}F^{kl,rs}h_{kl;i}h_{rs;}{}^j - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_;{}^j + r_{;i}u_;{}^j\right)\\ &- \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_;{}^j + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h^j_i\\ &- \frac{u\lambda''}{\lambda'\lambda}\delta^j_i - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)K_N\delta^j_i + \frac{n}{F^2}F^{kl}h_{rk}h^r_l h^j_i - \frac{2n}{F}h^j_k h^k_i\\ &+ \frac{n}{F^2}F^{kl}K_N(h_{il}\delta^j_k + h^j_l g_{ik} - 2g_{kl}h^j_i)\\ &+ \frac{2n}{F^2}F^{kl}K_N(h_{kl}\delta^j_i - g_{li}h^j_k) + K_N\frac{n}{F^2}F^{kl}g_{kl}h^j_i - K_N\frac{n}{F}\delta^j_i\\ =& -\frac{2n}{F^3}F_{;i}F_;{}^j + \frac{n}{F^2}F^{kl,rs}h_{kl;i}h_{rs;}{}^j + K_N\frac{\lambda}{\lambda'^2}\left(u_{;i}r_;{}^j + r_{;i}u_;{}^j\right)\\ &+ K_N^2\frac{2u\lambda^2}{\lambda'^3}r_{;i}r_;{}^j + \left(1 - K_N\frac{u^2}{\lambda'^2} + \frac{n}{F^2}F^{kl}h_{rk}h^r_l - \frac{n}{F^2}K_N F^{kl}g_{kl}\right)h^j_i\\ &+ 2\frac{u}{\lambda'}K_N\delta^j_i - \frac{2n}{F}h^j_k h^k_i.\end{aligned} \tag{3.33}$$

4. Upper bounds for the curvature function

In this section we show that the curvature function F is bounded from above along the flow (3.1) in the case $F = n\frac{H_k}{H_{k-1}}$ for very general λ. For this paper we apply it only in the case $\lambda = \sin$, but due to its generality it might be of use in further situations.

Proposition 4.1. *Let $a, b \in \mathbb{R}$, and let $(N, \bar{g})$ be the warped space $((a,b)\times\mathbb{S}^n, dr^2 + \lambda^2(r)\sigma)$ with $\lambda, \lambda' > 0$. Let*

$$F = n\frac{H_k}{H_{k-1}}, \tag{4.1}$$

and let $x_0(M)$be the embedding of a closed n-dimensional manifold M in N such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$, and such that $\kappa \in \Gamma_k$ for all n-tupels of principal curvatures along $x_0(M)$. Then along with any solution x of (3.1) with initial embedding x_0 there exists a constant $c = c(n, k, \sup r_0, \inf r_0, \lambda)$ such that

$$F \leq c. \tag{4.2}$$

Remark 4.2. Note that under the hypothesis of Proposition 4.1 we have

$$r \leq \sup r_0, \qquad r \geq \inf r_0 \tag{4.3}$$

along the flow, due to the maximum principle. This assertion also holds for arbitrary monotone curvature functions F.

Now we prove this proposition.

Proof. Consider the test function

$$\Phi = \log F + \frac{u}{\lambda} + \alpha r \tag{4.4}$$

with a large constant α to be determined. Assume that Φ attains its maximum at p. By a suitable choice of coordinate, we can assume that $g_{ij}|_p = \delta_{ij}$, $h_{ij}|_p$ is diagonal and that, in turn, F^{ij} is diagonal at p. Assume that $F|_p \geq C$ for some sufficient large constant C. In the following we compute at p.

From Lemma 3.2 we deduce

$$\begin{aligned}
\mathcal{L}\left(\frac{u}{\lambda}\right) &= \frac{1}{\lambda}\mathcal{L}u - \frac{u\lambda'}{\lambda^2}\mathcal{L}r + \frac{n}{F^2}\frac{2\lambda'}{\lambda^2}F^{ij}u_{;i}r_{;j} - \frac{n}{F^2}\frac{u}{\lambda^3}\left(2\lambda'^2 - \lambda\lambda''\right)F^{ij}r_{;i}r_{;j} \\
&= \frac{n}{F^2}\left(F^{ij}h_{ik}h^k_j - \frac{1}{n}F^2\right)\bar{g}(\partial_r, \nu) - \frac{\lambda''}{\lambda'^2}\|\nabla r\|^2 u \\
&\quad - \frac{u\lambda'}{\lambda^2}\left(\frac{2n}{vF} - \frac{\lambda}{\lambda'} - \frac{n\lambda'}{\lambda F^2}F^{ij}g_{ij} + \frac{n\lambda'}{\lambda F^2}F^{ij}r_{;i}r_{;j}\right) \\
&\quad + \frac{n}{F^2}\frac{2\lambda'}{\lambda^2}F^{ij}u_{;i}r_{;j} - \frac{n}{F^2}\frac{u}{\lambda^3}\left(2\lambda'^2 - \lambda\lambda''\right)F^{ij}r_{;i}r_{;j} \\
&\quad + \frac{n}{F^2}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}x^\gamma_{;m}x^\delta_{;j}r_;{}^m \\
&\leq \frac{n}{F^2}\left(F^{ii}h^2_{ii} - \frac{1}{n}F^2\right)\bar{g}(\partial_r, \nu) + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C + \sum_i CF^{ii}.
\end{aligned} \tag{4.5}$$

We also have

(4.6)
$$\begin{aligned}\mathcal{L}\log F &= -\frac{n}{F^2}F^{ij}(\log F)_{;i}(\log F)_{;j} - \frac{n}{F^2}\left(F^{ij}h_{jk}h_i^k - \frac{1}{n}F^2\right) + \frac{u^2\lambda''}{\lambda\lambda'^2}\\ &\quad - \frac{u\lambda''}{\lambda\lambda' F}F^{ij}g_{ij} + \frac{u}{\lambda'^2 F}\left(\frac{\lambda'\lambda''}{\lambda} - \lambda''' + \frac{2\lambda''^2}{\lambda'}\right)F^{ij}r_{;i}r_{;j} - \frac{2\lambda''}{\lambda'^2 F}F^{ij}u_{;i}r_{;j}\\ &\quad - \frac{\lambda}{\lambda' F}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;m}x^\delta_{\ ;j}r_;{}^m - \frac{1}{F}\left(\frac{n}{F} - \frac{u}{\lambda'}\right)F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j}\\ &\le -\frac{n}{F^2}F^{ii}(\log F)_{;i}(\log F)_{;i} - \frac{n}{F^2}\left(F^{ii}h_{ii}^2 - \frac{1}{n}F^2\right)\\ &\quad + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C + \sum_i CF^{ii}.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{L}\Phi &= \mathcal{L}\log F + \mathcal{L}\left(\frac{u}{\lambda}\right) + \alpha\mathcal{L}r\\ &\le -\frac{n}{F^2}F^{ii}(\log F)_{;i}(\log F)_{;i} - \frac{n}{F^2}\left(F^{ii}h_{ii}^2 - \frac{1}{n}F^2\right)(1 - \bar{g}(\partial_r,\nu))\\ &\quad - \alpha\frac{\lambda}{\lambda'} + \alpha\frac{C}{F} + \alpha\frac{C}{F}\sum_i F^{ii} + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C + C\sum_i F^{ii}.\end{aligned}\tag{4.7}$$

A calculation using the Newton–Maclaurin inequalities gives, for our special F,

$$F^{ii}h_{ii}^2 - \frac{1}{n}F^2 \ge 0\tag{4.8}$$

and

$$F_i^i \le C(n,k);\tag{4.9}$$

see [33, Prop. 2.2] for useful formulas for this calculation.

Thus

$$\mathcal{L}\Phi \le -\frac{n}{F^2}F^{ii}(\log F)_{;i}(\log F)_{;i} - \alpha\frac{\lambda}{\lambda'} + \alpha\frac{C}{F} + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C.\tag{4.10}$$

From the maximal property of Φ at p, we have

$$\nabla\log F = -\frac{1}{\lambda}\nabla u + \frac{u\lambda'}{\lambda^2}\nabla r - \alpha\nabla r.\tag{4.11}$$

Therefore

(4.12)

$$
\begin{aligned}
0 \le \mathcal{L}\Phi &\le -\frac{n}{F^2}F^{ii}\left(-\frac{1}{\lambda}u_{;i}+\frac{u\lambda'}{\lambda^2}r_{;i}-\alpha r_{;i}\right)^2-\alpha\frac{\lambda}{\lambda'}+\alpha\frac{C}{F}+\frac{C}{F}F^{ii}|u_{;i}||r_{;i}|+C\\
&\le -\frac{n}{2F^2\lambda^2}F^{ii}{u_{;i}}^2+\frac{n}{F^2}F^{ii}\left(\frac{u\lambda'}{\lambda^2}r_{;i}-\alpha r_{;i}\right)^2+\frac{C}{F}F^{ii}|u_{;i}||r_{;i}|\\
&\quad -\alpha\frac{\lambda}{\lambda'}+\alpha\frac{C}{F}+C\\
&\le -\frac{n}{2F^2\lambda^2}F^{ii}\left(|u_{;i}|-\frac{CF\lambda^2}{n}|r_{;i}|\right)^2-\alpha\frac{\lambda}{\lambda'}+\alpha\frac{C}{F}+C+C\frac{\alpha^2}{F^2}\\
&\le -\alpha\frac{\lambda}{\lambda'}+\alpha\frac{C}{F}+C+C\frac{\alpha^2}{F^2}.
\end{aligned}
$$

Assume that $F|_p \ge \alpha$. Then by choosing α large enough, we get the right-hand side (RHS) of the above inequality to be negative, which is a contradiction. Therefore $F|_p \le \alpha$ for our choice of α and, in turn, $\Phi|_p$ is bounded. Since Φ attains its maximum at p, we conclude that F is bounded from above. □

5. Gradient estimates

In this section we show that the graph function has a uniform C^1 bound along the flow (3.1) for very general F and λ.

Proposition 5.1. *Let $a, b \in \mathbb{R}$, and let $(N, \bar{g})$ be the warped space $((a,b)\times\mathbb{S}^n, dr^2+\lambda^2(r)\sigma)$ with $\lambda, \lambda' > 0$. Let $F \in C^\infty(\Gamma)$ be a positive, 1-homogeneous and strictly monotone curvature function, and let $x_0(M)$ be the embedding of a closed n-dimensional manifold M in N such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$ with graph function r, and such that $\kappa \in \Gamma$ for all n-tuples of principal curvatures along $x_0(M)$. Then along any solution x of (3.1) with initial embedding x_0 there exists a constant $c = c(n, \sup r_0, \inf r_0, \lambda)$ such that*

$$|\hat{\nabla} r| \le c. \tag{5.1}$$

Proof. Recall that

$$\varphi = \int_a^r \frac{1}{\lambda(s)}\, ds. \tag{5.2}$$

To simplify the notation, we use just $\varphi_i = \varphi_{,i}$, etc.; i.e., we omit the comma when taking the covariant derivative on $\mathbb{S}^n$.

We rewrite the flow equation as a scalar equation for φ,

$$
\begin{aligned}
\partial_t\varphi &= \frac{1}{\lambda}\left(\frac{n}{F\left(\frac{\lambda'}{\lambda v}\delta_i^j-\frac{1}{\lambda v}\tilde{g}^{jk}\varphi_{ki}\right)}-\frac{u}{\lambda'}\right)v\\
&= \frac{nv^2}{F(\lambda'\delta_i^j-\tilde{g}^{jk}\varphi_{ki})}-\frac{1}{\lambda'} =: G(\varphi,\hat{\nabla}\varphi,\hat{\nabla}^2\varphi),
\end{aligned}
\tag{5.3}
$$

where $\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$. For simplicity, we denote by $F = F(\lambda'\delta_i^j - \tilde{g}^{jk}\varphi_{ki})$ and F_j^i the derivative of F with respect to its argument.

We compute

$$
G^{ij} := \frac{\partial G}{\partial \varphi_{ij}} = \frac{nv^2}{F^2} F^i_k \tilde{g}^{kj},
$$
$$
G^{\varphi_p} := \frac{\partial G}{\partial \varphi_p} = \frac{2n\varphi^p}{F} + \frac{nv^2}{F^2} F^i_j \left(-\frac{\sigma^{jp}\varphi^k + \sigma^{kp}\varphi^j}{v^2} + \frac{2\varphi^j\varphi^k\varphi^p}{v^4} \right) \varphi_{ki}, \tag{5.4}
$$
$$
G^{\varphi} := \frac{\partial G}{\partial \varphi} = -\frac{nv^2\lambda''\lambda}{F^2} F^i_i + \frac{\lambda''\lambda}{\lambda'^2}.
$$

Using the 1-homogeneity of F, we have

$$
\begin{aligned}
G^{ij}\varphi_{ij} &= \frac{nv^2}{F^2} F^i_k \tilde{g}^{kj}\varphi_{ij} \\
&= -\frac{nv^2}{F^2} F^i_k (\lambda'\delta^k_i - \tilde{g}^{kj}\varphi_{ij}) + \frac{nv^2\lambda'}{F^2} F^i_i = -\frac{nv^2}{F} + \frac{nv^2\lambda'}{F^2} F^i_i
\end{aligned} \tag{5.5}
$$

and

$$
G^{ij}\varphi_i\varphi_j = \frac{nv^2}{F^2} F^i_k \tilde{g}^{kj}\varphi_i\varphi_j = \frac{n}{F^2} F^i_k \varphi^k \varphi_i. \tag{5.6}
$$

Let $\mathcal{L} = \partial_t - G^{ij}\nabla^2_{ij}$ be the parabolic operator. Using the Ricci identities on $\mathbb{S}^n$, we get

$$
\begin{aligned}
\mathcal{L}|\hat{\nabla}\varphi|^2 = &-2G^{ij}\varphi_{ik}\varphi_j{}^k - 2G^{ij}\sigma_{ij}|\hat{\nabla}\varphi|^2 + 2G^{ij}\varphi_i\varphi_j \\
&+ G^{\varphi_p}(|\hat{\nabla}\varphi|^2)_p + 2G^{\varphi}|\hat{\nabla}\varphi|^2.
\end{aligned} \tag{5.7}
$$

Let $f : [0, \infty) \to (0, \infty)$ be an auxiliary function to be determined. Consider a test function

$$
\Phi = \log \frac{|\hat{\nabla}\varphi|^2}{f(\varphi)}.
$$

In the following we compute at a maximal point of Φ. Due to the maximal property,

$$
\hat{\nabla}|\hat{\nabla}\varphi|^2 = \frac{f'}{f}|\hat{\nabla}\varphi|^2\hat{\nabla}\varphi. \tag{5.8}
$$

By a suitable choice of the coordinates, we may assume that $\sigma_{ij} = \delta_{ij}$ and $|\hat{\nabla}\varphi| = \varphi_1$. Then

$$
\varphi_{11} = \frac{1}{2}\frac{f'}{f}|\hat{\nabla}\varphi|^2, \qquad \varphi_{1j} = 0 \text{ for } j = 2, \dots, n. \tag{5.9}
$$

Then $\tilde{g}^{ij}$ is diagonal with

$$
\tilde{g}^{11} = \frac{1}{v^2}, \qquad \tilde{g}^{ii} = 1 \text{ for } i \neq 1. \tag{5.10}
$$

We may further assume that φ_{ij} is diagonal and, in turn, that F^k_i is diagonal. Thus we have

$$
\begin{aligned}
-2G^{ij}\varphi_{ik}\varphi_j{}^k &= -\frac{2nv^2}{F^2}F^i_l\tilde{g}^{lj}\varphi_{ik}\varphi_j{}^k \\
&= -\frac{2n}{F^2}F^{11}\frac{1}{4}\left(\frac{f'}{f}\right)^2|\hat{\nabla}\varphi|^4 - \frac{2nv^2}{F^2}\sum_{k\geq 2}F^{kk}\varphi^2_{kk},
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
-2G^{ij}\sigma_{ij}|\nabla\varphi|^2 + 2G^{ij}\varphi_i\varphi_j &= -\frac{2nv^2}{F^2}F^i_k\tilde{g}^{kj}(\sigma_{ij}|\hat{\nabla}\varphi|^2 - \varphi_i\varphi_j) \\
&= -\frac{2nv^2}{F^2}\sum_{k\geq 2}F^{kk}|\hat{\nabla}\varphi|^2,
\end{aligned} \tag{5.12}
$$

(5.13)

$$
\begin{aligned}
G^{\varphi_p}(|\hat{\nabla}\varphi|^2)_p &= \left(\frac{2n\varphi^p}{F} + \frac{nv^2}{F^2}F^i_j\left(-\frac{\sigma^{jp}\varphi^k + \sigma^{kp}\varphi^j}{v^2} + \frac{2\varphi^j\varphi^k\varphi^p}{v^4}\right)\varphi_{ki}\right)\frac{f'}{f}|\hat{\nabla}\varphi|^2\varphi_p \\
&= \frac{f'}{f}\frac{2n}{F}|\hat{\nabla}\varphi|^4 - \left(\frac{f'}{f}\right)^2\frac{n}{v^2F^2}F^{11}|\hat{\nabla}\varphi|^6,
\end{aligned}
$$

and

$$
2G^{\varphi}|\hat{\nabla}\varphi|^2 = \left(-\frac{2nv^2\lambda''\lambda}{F^2}F^i_i + \frac{2\lambda''\lambda}{\lambda'^2}\right)|\hat{\nabla}\varphi|^2. \tag{5.14}
$$

On the other hand, using (5.3), (5.5), and (5.6), we get

$$
\begin{aligned}
\mathcal{L}(f(\varphi)) &= f'\left(\frac{nv^2}{F} - \frac{1}{\lambda'}\right) - f'G^{ij}\varphi_{ij} - f''G^{ij}\varphi_i\varphi_j \\
&= f'\left(\frac{2nv^2}{F} - \frac{1}{\lambda'}\right) - f'\frac{nv^2\lambda'}{F^2}F^i_i - f''\frac{n}{F^2}F^{11}|\hat{\nabla}\varphi|^2.
\end{aligned} \tag{5.15}
$$

Using (5.7)–(5.15) and the maximal property of Φ at p, we have

$$
\begin{aligned}
0 &\leq \frac{\mathcal{L}(|\hat{\nabla}\varphi|^2)}{|\nabla\varphi|^2} - \frac{\mathcal{L}f}{f} \\
&= -\frac{2n}{F^2}F^{11}\frac{1}{4}\left(\frac{f'}{f}\right)^2|\hat{\nabla}\varphi|^2 - \frac{2nv^2}{F^2|\hat{\nabla}\varphi|^2}\sum_{k\geq 2}F^{kk}\varphi^2_{kk} - \frac{2nv^2}{F^2}\sum_{k\geq 2}F^{kk} \\
&\quad + \frac{f'}{f}\frac{2n}{F}|\hat{\nabla}\varphi|^2 - \left(\frac{f'}{f}\right)^2\frac{n}{v^2F^2}F^{11}|\hat{\nabla}\varphi|^4 - \frac{2nv^2\lambda''\lambda}{F^2}F^i_i + 2\frac{\lambda''\lambda}{\lambda'^2} \\
&\quad - \frac{f'}{f}\left(\frac{2nv^2}{F} - \frac{1}{\lambda'}\right) + \frac{f'}{f}\frac{nv^2\lambda'}{F^2}F^i_i + \frac{f''}{f}\frac{n}{F^2}F^{11}|\hat{\nabla}\varphi|^2 \\
&= -\frac{2n}{F^2}F^{11}\frac{1}{4}\left(\frac{f'}{f}\right)^2|\hat{\nabla}\varphi|^2 - \frac{2nv^2}{F^2|\hat{\nabla}\varphi|^2}\sum_{k\geq 2}F^{kk}\varphi^2_{kk} - \frac{2nv^2}{F^2}\sum_{k\geq 2}F^{kk} \\
&\quad - \frac{f'}{f}\frac{2n}{F} - \left[\left(\frac{f'}{f}\right)^2\frac{|\hat{\nabla}\varphi|^2}{v^2} - \frac{f''}{f}\right]\frac{n}{F^2}F^{11}|\hat{\nabla}\varphi|^2 \\
&\quad - \frac{nv^2}{F^2}\left(2\lambda''\lambda - \lambda'\frac{f'}{f}\right)F^i_i + \frac{f'}{f}\frac{1}{\lambda'} + 2\frac{\lambda''\lambda}{\lambda'^2}.
\end{aligned} \tag{5.16}
$$

Note that

$$
\begin{aligned}
-\frac{f'}{f}\frac{2n}{F} &= -\frac{f'}{f}\frac{2n}{F^2}F_k^i(\lambda'\delta_i^k - \tilde{g}^{kj}\varphi_{ij}) \\
&= -\frac{f'}{f}\frac{2n\lambda'}{F^2}F_i^i + \left(\frac{f'}{f}\right)^2\frac{n}{F^2}\frac{1}{v^2}F^{11}|\hat{\nabla}\varphi|^2 + \frac{f'}{f}\frac{2n}{F^2}\sum_{k\geq 2}F^{kk}\varphi_{kk}.
\end{aligned} \tag{5.17}
$$

Therefore

$$
\begin{aligned}
0 \leq &-\frac{2n}{F^2}F^{11}\frac{1}{4}\left(\frac{f'}{f}\right)^2|\hat{\nabla}\varphi|^2 - \frac{2nv^2}{F^2|\hat{\nabla}\varphi|^2}\sum_{k\geq 2}F^{kk}\varphi_{kk}^2 - \frac{2nv^2}{F^2}\sum_{k\geq 2}F^{kk} \\
&-\frac{f'}{f}\frac{2n\lambda'}{F^2}F_i^i + \left(\frac{f'}{f}\right)^2\frac{n}{F^2}\frac{1}{v^2}F^{11}|\hat{\nabla}\varphi|^2 + \frac{f'}{f}\frac{2n}{F^2}\sum_{k\geq 2}F^{kk}\varphi_{kk} \\
&-\left[\left(\frac{f'}{f}\right)^2\frac{|\hat{\nabla}\varphi|^2}{v^2} - \frac{f''}{f}\right]\frac{n}{F^2}F^{11}|\hat{\nabla}\varphi|^2 \\
&-\frac{nv^2}{F^2}\left(2\lambda''\lambda - \lambda'\frac{f'}{f}\right)F_i^i + \frac{f'}{f}\frac{1}{\lambda'} + 2\frac{\lambda''\lambda}{\lambda'^2}
\end{aligned} \tag{5.18}
$$

and, completing the square,

$$
\begin{aligned}
0 \leq &-\frac{n}{F^2}F^{11}\left[\frac{1}{2}\left(\frac{f'}{f}\right)^2 + \left(\frac{f'}{f}\right)^2\frac{|\hat{\nabla}\varphi|^2}{v^2} - \frac{f''}{f} + 2\lambda''\lambda - \lambda'\frac{f'}{f}\right]|\hat{\nabla}\varphi|^2 \\
&+\frac{n}{F^2}F^{11}\left[-2\lambda'\frac{f'}{f} + \left(\frac{f'}{f}\right)^2\frac{|\hat{\nabla}\varphi|^2}{v^2} - 2\lambda''\lambda + \lambda'\frac{f'}{f}\right] \\
&-\frac{2n}{F^2}\sum_{k\geq 2}F^{kk}\left(\varphi_{kk} - \frac{1}{2}\frac{f'}{f}\right)^2 - \frac{2n}{F^2|\hat{\nabla}\varphi|^2}\sum_{k\geq 2}F^{kk}\varphi_{kk}^2 \\
&+\frac{2n}{F^2}\left(\frac{1}{4}\left(\frac{f'}{f}\right)^2 - \lambda'\frac{f'}{f} - v^2\left(1 + \lambda''\lambda - \frac{1}{2}\lambda'\frac{f'}{f}\right)\right)\sum_{k\geq 2}F^{kk} \\
&+\frac{f'}{f}\frac{1}{\lambda'} + 2\frac{\lambda''\lambda}{\lambda'^2}.
\end{aligned} \tag{5.19}
$$

Choose $f(\varphi) = e^{-a\varphi}$ with $a > 0$ large enough so that the first term on the RHS of (5.19) has a negative sign, and when $|\hat{\nabla}\varphi|^2$ is large enough, this term dominates the second term. Also, by choosing $a > 0$ large enough and when $|\hat{\nabla}\varphi|^2$ is large enough, the fourth line and the fifth line are both negative. We get a contradiction. Thus $|\hat{\nabla}\varphi|^2 \leq C$. □

6. Preserved convexity in the sphere

In ambient spaces where λ'' can be negative, it is very difficult to control F from below if the flow hypersurfaces are not convex. Hence we assume strict convexity in these cases, and we have to restrict to space forms to show that this property is preserved.

Proposition 6.1. *Let $x_0(M)$ be the embedding of a closed n-dimensional manifold M in $\mathbb{S}^{n+1}$ such that $x_0(M)$ is strictly convex. Let F be a positive, 1-homogeneous, strictly monotone, and inverse concave[2] curvature function. Then along any solution x of (3.1) with initial embedding x_0 all flow hypersurfaces are strictly convex.*

Proof. Let b be the inverse of the Weingarten map, which exists at least for a short time. We show that, for a smooth solution

$$x\colon [0,T^*)\times M\to\mathbb{S}^{n+1}, \tag{6.1}$$

all M_t, $t<T^*$ are strictly convex. From

$$\dot{b}^k_m=-b^k_j\dot{h}^j_i b^i_m,\qquad F^{qs}b^k_{m;qs}=2F^{qs}b^k_jh^j_{p;q}b^p_rh^r_{i;s}b^i_m-F^{qs}b^k_ph^p_{l;qs}b^l_m, \tag{6.2}$$

$$u_{;i}=\lambda h^k_i r_{;k}, \tag{6.3}$$

and (3.33) we deduce

$$\begin{aligned}\mathcal{L}b^k_m=&\frac{n}{F^2}\left(\frac{2}{F}F^{rs}F^{pq}-2F^{qs}b^{pr}-F^{pq,rs}\right)b^k_jb^i_mh_{rs;i}h_{pq;}{}^j\\&-\frac{\lambda^2}{\lambda'^2}\left(b^k_lr_;{}^lr_{;m}+r_;{}^kb^l_mr_{;l}\right)-\frac{2u\lambda^2}{\lambda'^3}b^k_jr_;{}^jb^i_mr_{;i}\\&+\psi_1b^k_m-\frac{2u}{\lambda'}b^k_lb^l_m+\psi_2\delta^k_m,\end{aligned} \tag{6.4}$$

where ψ_i, $i=1,2$ are some functions which are bounded on every compact interval $[0,T_0]\subset[0,T^*)$. If the convexity is lost at some time $T_0<T^*$, then the largest eigenvalue of b blows up at T_0. Although the largest eigenvalue is not a smooth function, we can still apply (6.4) to estimate it by using the following well-known trick; compare this to, e.g., the proof of [18, Lemma 6.1]:

Define

$$\phi=\sup\{b_{ij}\eta^i\eta^j : g_{ij}\eta^i\eta^j=1\}, \tag{6.5}$$

and suppose that this function attains a maximum at (t_0,ξ_0), $t_0<T_0$. Use normal coordinates around (t_0,ξ_0), such that

$$g_{ij}=\delta_{ij},\quad b_{ij}=\kappa_i^{-1}\delta_{ij},\quad \kappa_1^{-1}\leq\cdots\leq\kappa_n^{-1} \tag{6.6}$$

at (t_0,ξ_0).
Around (t_0,ξ_0) let η be the vector field

$$\eta=(0,\dots,0,1), \tag{6.7}$$

and define

$$\tilde{\phi}=\frac{b_{ij}\eta^i\eta^j}{g_{ij}\eta^i\eta^j}. \tag{6.8}$$

Then locally around (t_0,ξ_0) we have $\tilde{\phi}\leq\phi$, and at this point it holds that

$$\dot{\tilde{\phi}}=\dot{b}_{nn}-2\mathcal{F}=\dot{b}^n_n, \tag{6.9}$$

and the spatial derivatives also coincide. Thus at (t_0,ξ_0) the functions $\tilde{\phi}$ and b^n_n satisfy the same evolution equation, whence it suffices to show that the RHS of (6.4) is negative at the point (t_0,ξ_0).

[2] $F(\kappa_1,\dots,\kappa_n)$ is called inverse concave if $\tilde{F}(\kappa_1,\dots,\kappa_n)=F^{-1}(\kappa_1^{-1},\dots,\kappa_n^{-1})$ is concave.

The first line is negative due to the inverse concavity of F—compare this to the proof in [43, p. 112]—while for the rest the good terms involving $b^k_l b^l_m$ surely dominate. This completes the proof. $\square$

7. Bounds on the speed and the curvature

In this section we deduce the remaining ingredients which are necessary to obtain longtime existence. Namely, we need a full bound on the second fundamental form and, in turn, to apply the Krylov–Safonov theory, we need a lower bound on the curvature function to show that the operator $\mathcal{L}$ is uniformly parabolic along the flow. We start with the spherical case.

7.1. **The spherical case.**

Lemma 7.1. *Let $x_0(M)$ be the embedding of a closed n-dimensional manifold M in $\mathbb{S}^{n+1}$ such that $x_0(M)$ is strictly convex. Let*

$$F = n\frac{H_k}{H_{k-1}}. \tag{7.1}$$

Then along any solution x of (3.1) *with initial embedding x_0, there exists a constant $c = c(n, k, \sup r_0, \inf r_0, \lambda)$ such that*

$$\|A\|^2 \leq c. \tag{7.2}$$

Proof. Due to the convexity preservation, Proposition 6.1, it suffices to bound the mean curvature H from above. Note that $u \geq c_0 > 0$ by Proposition 5.1 (we may also use the convexity to get this; cf. [20, Lemma 2.7.10]). We use the auxiliary function

$$w = \log H - \log u \tag{7.3}$$

and deduce from (3.9), (3.33), the concavity of F, and

$$u_{;i} = \lambda h^k_i r_{;k} \tag{7.4}$$

that, at a maximal point of w,

$$\begin{aligned} 0 \leq \mathcal{L}w &= \frac{1}{H}\mathcal{L}H - \frac{1}{u}\mathcal{L}u \\ &\leq c + \frac{c}{H} - \frac{2n}{FH}\|A\|^2 \\ &\leq c + \frac{c}{H} - \frac{2}{F}H. \end{aligned} \tag{7.5}$$

Since F is bounded from above by Proposition 4.1, we get an upper bound of H from above. $\square$

We use the previous result to get bounds from below on F.

Lemma 7.2. *Under the assumptions of Lemma* 7.1, *there exists a constant $0 < c = c(n, \sup r_0, \inf r_0, \lambda)$ such that*

$$F \geq c. \tag{7.6}$$

Proof. We use the same method as in [36, Prop. 5.3] and bound the auxiliary function

$$z = -\log F + f(r), \tag{7.7}$$

where

$$f(r) = -\log(\lambda' - \alpha), \qquad 0 < \alpha < \frac{1}{2}\lambda'(\sup r_0). \tag{7.8}$$

Since $\lambda'' = -\lambda$, it is direct to check that

$$1 - f'\frac{\lambda'}{\lambda} = -\frac{\alpha}{\lambda' - \alpha}, \qquad f'^2 + f'\frac{\lambda'}{\lambda} - f'' = 0. \tag{7.9}$$

From the convexity, the 1-homogeneity of F, and Lemma 7.1, we see that

$$\frac{n}{F^2}F^{ij}h_{ik}h^k_j \leq \frac{nH}{F} \leq \frac{c}{F}. \tag{7.10}$$

Using (3.8), (3.10), and (7.10),

$$\begin{aligned}\mathcal{L}z &= -\frac{1}{F}\mathcal{L}F - \frac{n}{F^4}F^{ij}F_{;i}F_{;j} + f'\mathcal{L}r - f''\frac{n}{F^2}F^{ij}r_{;i}r_{;j} \\ &\leq \frac{n}{F^2}F^{ij}(\log F)_{;i}(\log F)_{;j} + \frac{c}{F} + c + \frac{n}{F^2}F^{ij}g_{ij} \\ &\quad + f'\frac{c}{F} - f'\frac{n\lambda'}{\lambda F^2}F^{ij}g_{ij} + f'\frac{n\lambda'}{\lambda F^2}F^{ij}r_{;i}r_{;j} - f''\frac{n}{F^2}F^{ij}r_{;i}r_{;j}.\end{aligned} \tag{7.11}$$

At a maximal point of z, we use $(\log F)_{;i} = f'r_{;i}$ and (7.9) to obtain

$$\begin{aligned}0 \leq \mathcal{L}z &\leq \frac{n}{F^2}F^{ij}r_{;i}r_{;j}\left(f'^2 + f'\frac{\lambda'}{\lambda} - f''\right) + \frac{n}{F^2}F^{ij}g_{ij}\left(1 - f'\frac{\lambda'}{\lambda}\right) + \frac{c}{F} + c + f'\frac{c}{F} \\ &= -\frac{\alpha}{\lambda' - \alpha}\frac{n}{F^2}F^{ij}g_{ij} + \frac{c}{F} + c \\ &< 0\end{aligned} \tag{7.12}$$

if F is small enough since $F^{ij}g_{ij} \geq n$. ☐

Now we finish the a priori estimates in the spherical case.

Proposition 7.3. *Let $x_0(M)$ be the embedding of a closed n-dimensional manifold M in $\mathbb{S}^{n+1}$ such that $x_0(M)$ is strictly convex. Let*

$$F = n\frac{H_k}{H_{k-1}}. \tag{7.13}$$

Then any solution x of (3.1) with initial embedding x_0 exists for all positive times with uniform C^∞-estimates.

Proof. We have uniform C^2 bounds from Proposition 6.1 and Lemma 7.1. Due to Lemma 7.2, we know that the principal curvatures range within a compact subset of the domain on definition of F. Hence we have the uniform parabolicity of the operator $\mathcal{L}$. Due to the concavity of the operator, we can apply the regularity theory of Krylov and Safonov [35] to deduce $C^{2,\alpha}$ bounds and, in turn, C^∞ bounds using Schauder theory. Thus we can extend the flow beyond any finite T. ☐

7.2. **The general case.** We provide the bounds on the principal curvatures and on the curvature function from below in case of mild assumptions on the warping factor.

Proposition 7.4. *Let $a, b \in \mathbb{R}$, and let $(N, \bar{g})$ be the warped space $((a,b)\times\mathbb{S}^n, dr^2 + \lambda^2(r)\sigma)$ with $\lambda > 0$, $\lambda' > 0$, and $\lambda'' \geq 0$. Let $F \in C^\infty(\Gamma)$ be a positive, 1-homogeneous, strictly monotone, and concave curvature function, and let $x_0(M)$ be the embedding of a closed n-dimensional manifold M in N such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$, and such that $\kappa \in \Gamma$ for all n-tupels of principal curvatures along $x_0(M)$. Then along any solution x of* (3.1) *with initial embedding x_0, there exists a positive constant $c = c(n, \sup r_0, \inf r_0, \lambda)$ such that*

$$F \geq c. \tag{7.14}$$

Remark 7.5. Proposition 7.4 is the only place where we use $\lambda'' \geq 0$ for proving Theorem 1.3.

Proof. We deduce the evolution of the function $\partial_t\varphi$, where φ is defined as in (5.2). Recall that it holds from (5.3) that

$$\begin{aligned}\partial_t\varphi &= \frac{1}{\lambda}\left(\frac{n}{F\left(\frac{\lambda'}{\lambda v}\delta^j_i - \frac{1}{\lambda v}\tilde{g}^{jk}\varphi_{ki}\right)} - \frac{u}{\lambda'}\right)v \\ &= \frac{nv^2}{F(\lambda'\delta^j_i - \tilde{g}^{jk}\varphi_{ki})} - \frac{1}{\lambda'} =: G(\varphi, \hat{\nabla}\varphi, \hat{\nabla}^2\varphi),\end{aligned} \tag{7.15}$$

where $\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$. Differentiation gives

$$\partial_t(\partial_t\varphi) = G^{ij}(\partial_t\varphi)_{ij} + G^{\varphi_p}(\partial_t\varphi)_p + G^{\varphi}\partial_t\varphi. \tag{7.16}$$

From (5.4) we obtain
(7.17)

$$G^\varphi \leq -\frac{n^2v^2\lambda''\lambda}{F^2} + \frac{\lambda''\lambda}{\lambda'^2} = -\frac{\lambda''\lambda}{v^2}\frac{n^2v^4}{F^2} + \frac{\lambda''\lambda}{\lambda'^2} = -\frac{\lambda''\lambda}{v^2}\left(\partial_t\varphi + \frac{1}{\lambda'}\right)^2 + \frac{\lambda''\lambda}{\lambda'^2}.$$

Since we already have $v \leq c$ due to Proposition 5.1, the third order leading term is dominating with a nonpositive sign. The maximum principle gives an upper bound for $\partial_t\varphi$ and hence the result. ∎

Proposition 7.6. *Under the assumptions of Proposition* 7.4*, there exists a positive constant $c = c(n, \sup r_0, \inf r_0, \lambda)$ such that*

$$\|A\|^2 \leq c. \tag{7.18}$$

Proof. In applying the maximum principle to the evolution of (h^i_j), we proceed as in the proof of Proposition 6.1. Define

$$\phi = \sup\{h_{ij}\eta^i\eta^j : g_{ij}\eta^i\eta^j = 1\}, \tag{7.19}$$

and suppose that the function

$$w = \log\phi + f(u) + \alpha r \tag{7.20}$$

attains a maximum at (t_0, ξ_0), $t_0 < T_0$, where f is defined by

$$f(u) = -\log(u - \beta), \tag{7.21}$$

where $\beta = \frac{1}{2}\min u$. Note that

$$1 + f'u = \frac{-\beta}{u-\beta} < 0. \tag{7.22}$$

Using normal coordinates around (t_0, ξ_0), with

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \leq \cdots \leq \kappa_n \tag{7.23}$$

at (t_0, ξ_0) and using (3.8), (3.9), and (3.22), we may pretend that the evolution equation of w at the point (t_0, ξ_0) is given by

$$\begin{aligned} \mathcal{L}w \leq{}& \frac{n}{F^2}\frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})(h_{nk;n})^2(h_n^n)^{-1} + c + \frac{c}{\kappa_n} + \frac{n}{F^2}F^{kl}h_{rk}h_l^r \\ &- \frac{2n}{F}\kappa_n + \frac{c(1+\kappa_n^{-1})}{F^2}F^{ij}g_{ij} + \frac{n}{F^2}F^{ij}(\log h_n^n)_{;i}(\log h_n^n)_{;j} \\ &+ \frac{n}{F^2}\left(F^{kl}h_{rk}h_l^r - \frac{1}{n}F^2\right)f'u - f'\frac{\lambda''\lambda}{\lambda'^2}\|\nabla r\|^2 u + c|f'| \\ &- f''\frac{n}{F^2}F^{ij}u_{;i}u_{;j} + \frac{\alpha c}{F} - \alpha\frac{\lambda}{\lambda'} - \frac{n\alpha\lambda'}{\lambda F^2}F^{ij}(g_{ij} - r_{;i}r_{;j}), \end{aligned} \tag{7.24}$$

where we used a trick that already appeared in [14, proof of Prop. 6.3] and in a similar fashion in [19, Thm. 9.7]—namely, that due to the concavity of F it holds that

$$F^{kl,rs}\eta_{kl}\eta_{rs} \leq \sum_{k\neq l}\frac{F^{kk} - F^{ll}}{\kappa_k - \kappa_l}\eta_{kl}^2 \leq \frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})\eta_{nk}^2 \tag{7.25}$$

for all symmetric matrices (η_{kl}); cf. [20, Lemma 2.1.14]. Furthermore, we have

$$F^{nn} \leq \cdots \leq F^{11}; \tag{7.26}$$

cf. [13, Lemma 2]. In order to estimate (7.24), we distinguish between two cases.

Case 1*:* $\kappa_1 < -\epsilon_1\kappa_n$, $0 < \epsilon_1 < \frac{1}{2}$. Then

$$F^{ij}h_{ik}h_j^k \geq F^{11}\kappa_1^2 \geq \frac{1}{n}F^{ij}g_{ij}\kappa_1^2 \geq \frac{1}{n}F^{ij}g_{ij}\epsilon_1^2\kappa_n^2. \tag{7.27}$$

We use $\nabla w = 0$ to estimate

$$\frac{n}{F^2}F^{ij}(\log h_n^n)_{;i}(\log h_n^n)_{;j} = f'^2\frac{n}{F^2}F^{ij}u_{;i}u_{;j} + f'\frac{2n\alpha}{F^2}F^{ij}u_{;i}r_{;j} + \frac{n\alpha^2}{F^2}F^{ij}r_{;i}r_{;j}. \tag{7.28}$$

If κ_n is sufficiently large, in this case (7.24) becomes

$$\begin{aligned} \mathcal{L}w \leq{}& \frac{1}{F^2}F^{ij}g_{ij}(\epsilon_1^2\kappa_n^2(1+f'u) + (c + |f'|\alpha)\kappa_n + c\alpha^2 + c) + c(|f'|+1) \\ &- \frac{2n}{F}(\kappa_n - \alpha c) - \alpha\frac{\lambda}{\lambda'} - \frac{n}{F^2}F^{ij}u_{;i}u_{;j}(f'' - f'^2), \end{aligned} \tag{7.29}$$

which is negative for large κ_n after fixing $\alpha_0 = \alpha_0(M_0, \sup r_0, \inf r_0, \lambda)$ large enough to ensure that

$$c(|f'|+1) - \alpha_0\frac{\lambda}{\lambda'} < 0. \tag{7.30}$$

We also use $1 + f'u \leq c < 0$ and $f'' - f'^2 = 0$. Hence in this case any $\alpha \geq \alpha_0$ yields an upper bound for κ_n.

Case 2: $\kappa_1 \geq -\epsilon_1 \kappa_n$. Then

(7.31)

$$\begin{aligned}\frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nk;n})^2(h^n_n)^{-1} &\leq \frac{2}{1+\epsilon_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nk;n})^2(h^n_n)^{-2}\\ &\leq \frac{2}{1+\epsilon_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nn;k})^2(h^n_n)^{-2} + c(\epsilon_1)\sum_{k=1}^n (F^{kk} - F^{nn})\kappa_n^{-2}\\ &\quad + \frac{4}{1+\epsilon_1}\sum_{k=1}^n (F^{nn} - F^{kk})h_{nn;k}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;n} x^\gamma_{;n} x^\delta_{;k}(h^n_n)^{-2}\\ &\leq \frac{2}{1+2\epsilon_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nn;k})^2(h^n_n)^{-2} + c(\epsilon_1)\sum_{k=1}^n (F^{kk} - F^{nn})\kappa_n^{-2},\end{aligned}$$

where we used the Codazzi equation (2.16) and the Cauchy–Schwarz inequality. We deduce further

(7.32)

$$\begin{aligned}&F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} + \frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nk;n})^2(h^n_n)^{-1}\\ &\quad\leq \frac{2}{1+2\epsilon_1}\sum_{k=1}^n F^{nn}(\log h^n_n)^2_{;k} - \frac{1-2\epsilon_1}{1+2\epsilon_1}\sum_{k=1}^n F^{kk}(\log h^n_n)^2_{;k} + c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2}\\ &\quad\leq \sum_{k=1}^n F^{nn}(\log h^n_n)^2_{;k} + c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2}\\ &\quad= c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2} + f'^2 F^{nn}\|\nabla u\|^2 + 2\alpha f' F^{nn}\langle\nabla u, \nabla r\rangle + \alpha^2 F^{nn}\|\nabla r\|^2.\end{aligned}$$

We plug this into (7.24) and obtain for large κ_n

(7.33)

$$\begin{aligned}\mathcal{L}w &\leq c + \frac{n}{F^2}F^{nn}\kappa_n^2(1+f'u) - \frac{2n}{F}(\kappa_n - \alpha c) + \frac{1}{F^2}F^{ij}g_{ij}\left(c + c(\epsilon_1) - \frac{n\alpha\lambda'}{v^2\lambda}\right)\\ &\quad - f''\frac{n}{F^2}F^{ij}u_{;i}u_{;j} - \alpha\frac{\lambda}{\lambda'} + f'^2\frac{n}{F^2}F^{nn}\|\nabla u\|^2 + \frac{2n\alpha f'}{F^2}F^{nn}\langle\nabla u, \nabla r\rangle\\ &\quad + \frac{n\alpha^2}{F^2}F^{nn}\|\nabla r\|^2\\ &\leq \frac{n}{F^2}F^{nn}\left(\kappa_n^2(1+f'u) + 2\alpha|f'|c\kappa_n + \alpha^2\|\nabla r\|^2\right) - \frac{2n}{F}(\kappa_n - \alpha c)\\ &\quad + c - \alpha\frac{\lambda}{\lambda'} + \frac{1}{F^2}F^{ij}g_{ij}\left(c + c(\epsilon_1) - \frac{n\alpha\lambda'}{v^2\lambda}\right)\\ &< 0\end{aligned}$$

after possibly enlarging α even further (compared to Case 1) and for large κ_n. This completes the proof. □

As in Proposition 7.3, we conclude the following.

Proposition 7.7. *Under the assumptions of Theorem* 1.3*, the flow* (3.1) *exists for all times with uniform* C^∞*-estimates.*

8. Proofs of the main theorems

We give the final arguments to complete the proofs concerning the flow results and start with the spherical case.

Proof of Theorem 1.1. In order to complete the proof of Theorem 1.1 with the help of Proposition 7.3, all that we have to show is that each subsequential limit is a sphere independent of the subsequence as $t \to \infty$.

The evolution of the weighted enclosed volume

$$V(t) = \int_{\Omega_t} \lambda' dN \tag{8.1}$$

is

$$\dot{V}(t) = \int_{M_t} \left(\frac{n\lambda'}{F} - u \right) d\mu_t \geq \int_{M_t} \left(\frac{n\lambda'}{H} - u \right) d\mu_t \geq 0. \tag{8.2}$$

The first inequality is due to the concavity of F, which implies $F \leq H$ [20, Lemma 2.2.20], and the second one is due to Brendle's Heintze–Karcher-type inequality [6, eq. (4)]. That is, V is increasing. Since V is obviously bounded, we have

$$\int_0^\infty \int_{M_t} \left(\frac{n\lambda'}{H} - u \right) d\mu_t dt < \infty, \tag{8.3}$$

and hence

$$\int_{M_t} \left(\frac{n\lambda'}{H} - u \right) d\mu_t \to 0. \tag{8.4}$$

So any convergent subsequence of M_t must converge to a sphere due to the characterization of the limiting case in the Heintze–Karcher inequality. Due to the spherical barriers, this sphere is unique, and we conclude the proof of the theorem. □

Now we turn to the other case and prove Theorem 1.3.

Proof of Theorem 1.3. Again it suffices to prove that there exists a subsequence that converges to a sphere. If no subsequence converges to a geodesic sphere, then there cannot be any subsequence for which $\|\nabla r\| \to 0$. Hence there exists a positive constant c such that, for all times $t > 0$, we have

$$\max_{M_t} \|\nabla r\|^2 \geq c. \tag{8.5}$$

The area evolves according to

$$\frac{d}{dt}|M_t| = \int_{M_t} \mathcal{F}H \geq \int_{M_t} \left(n - \frac{Hu}{\lambda'} \right) = \int_{M_t} \frac{\operatorname{div}(\lambda \nabla r)}{\lambda'} = \int_{M_t} \frac{\lambda'' \lambda}{\lambda'^2} \|\nabla r\|^2 \geq 0. \tag{8.6}$$

The inequality in (8.6) is again due to $F \leq H$. The last two equalities in (8.6) follow from the fact that $\operatorname{div}(\lambda \nabla r) = n\lambda' - Hu$ and integration by parts, respectively.

Due to the C^1-estimates, the area is bounded, and hence, because of $\lambda'' \geq 0$, every subsequential limit $M_t \to \tilde{M}$ must satisfy

$$\int_{\tilde{M}} \frac{\lambda''\lambda}{\lambda'^2}\|\nabla r\|^2 = 0, \tag{8.7}$$

whence

$$\lambda''\|\nabla r\|^2 = 0 \tag{8.8}$$

throughout any subsequential limit. For all $t > 0$ let

$$\xi_t := \operatorname{argmax}_{M_t} \|\nabla r\|^2. \tag{8.9}$$

We obtain that

$$\lambda''(\xi_t) \to 0, \quad \text{as} \quad t \to \infty, \tag{8.10}$$

for otherwise we reach a contradiction to (8.5) and (8.8). From (5.7) we obtain, at the points (t, ξ_t),

$$\begin{aligned}\mathcal{L}|\hat{\nabla}\varphi|^2 &\leq -2G^{ij}\sigma_{ij}|\hat{\nabla}\varphi|^2 + 2G^{ij}\varphi_i\varphi_j + 2G^{\varphi}|\hat{\nabla}\varphi|^2 \\ &\leq -\frac{2nv^2}{F^2}F^i_k\tilde{g}^{kj}\sigma_{ij}|\hat{\nabla}\varphi|^2 + \frac{2nv^2}{F^2}F^i_k\tilde{g}^{kj}\varphi_i\varphi_j + c\lambda''|\hat{\nabla}\varphi|^2 \\ &\leq -\epsilon|\hat{\nabla}\varphi|^2\end{aligned} \tag{8.11}$$

for some suitable $\epsilon > 0$. Thus $|\hat{\nabla}\varphi|^2$ actually has to decay exponentially, and we obtain a contradiction to (8.5). $\square$

9. Geometric inequalities

In this section we complete the proof of the geometric inequalities. First of all, along the flow $\frac{d}{dt}x = \mathcal{F}\nu$, we have the following variational formulas.

Proposition 9.1. *Let $M_t \subset N$ be a family of closed hypersurfaces evolving by $\frac{d}{dt}x = \mathcal{F}\nu$. Denote by Ω_t the enclosed domain by M_t and $\{a\} \times \mathbb{S}^n$. Then*

$$\frac{d}{dt}\int_{\Omega_t} f = \int_{M_t} f\mathcal{F} \qquad \forall f \in C^\infty(N), \tag{9.1}$$

and

$$\frac{d}{dt}|M_t| = \int_{M_t} H\mathcal{F}. \tag{9.2}$$

If $\bar{\Delta}\lambda'\bar{g} - \bar{\nabla}^2\lambda' + \lambda'\overline{\mathrm{Rc}} = 0$, then

$$\frac{d}{dt}\int_{M_t} H\lambda' = \int_{M_t} (2\sigma_2\lambda' + 2H\langle\bar{\nabla}\lambda', \nu\rangle)\mathcal{F}. \tag{9.3}$$

Proof. The first and second ones are well known and were already used in section 8. We compute the third one:

$$\begin{aligned}\frac{d}{dt}\int_{M_t} H\lambda' &= \int_{M_t} \lambda'(-\Delta\mathcal{F} - \mathcal{F}|A|^2 - \mathcal{F}\overline{\mathrm{Rc}}(\nu,\nu)) \\ &\quad + \int_{M_t} \left(H\left\langle\bar\nabla\lambda',\nu\right\rangle + H^2\lambda'\right)\mathcal{F} \\ &= \int_{M_t} -(\bar\Delta\lambda' - \bar\nabla^2\lambda'(\nu,\nu) - H\left\langle\bar\nabla\lambda',\nu\right\rangle + \lambda'\overline{\mathrm{Rc}}(\nu,\nu))\mathcal{F} \\ &\quad + \int_{M_t} \left(H\left\langle\bar\nabla\lambda',\nu\right\rangle + (H^2 - |A|^2)\lambda'\right)\mathcal{F} \\ &= \int_{M_t} 2\sigma_2\lambda'\mathcal{F} + 2H\left\langle\bar\nabla\lambda',\nu\right\rangle\mathcal{F}.\end{aligned} \tag{9.4}$$

□

Proposition 9.2. *Let* $\Sigma \subset N$ *be a closed hypersurface. If* Σ *is star shaped and* $\frac{\lambda''}{\lambda} + \frac{1-\lambda'^2}{\lambda^2} \geq 0$, *then*

$$\int_\Sigma (n-1)H\lambda' \leq \int_\Sigma 2\sigma_2 u. \tag{9.5}$$

Proof. Multiplying σ_2^{ij} by (2.31), summing over i,j, integrating over Σ, and using

$$\nabla_i\sigma_2^{ij} = h^i_{i;m}g^{mj} - h^{ij}{}_{;i} = -\overline{\mathrm{Rc}}(\nu, x_{;m})g^{mj}, \tag{9.6}$$

we have

$$\begin{aligned}\int_\Sigma (n-1)H\lambda' - 2\sigma_2 u &= \int_\Sigma \sigma_2^{ij}(\lambda r_{;j})_{;i} \\ &= \int_\Sigma \lambda\overline{\mathrm{Rc}}(\nu, x_{;m})r_;{}^m \\ &= \int_\Sigma -(n-1)\left[\frac{\lambda''}{\lambda} + \frac{1-\lambda'^2}{\lambda^2}\right]\lambda\|\nabla r\|^2\left\langle\partial_r,\nu\right\rangle \\ &\leq 0,\end{aligned} \tag{9.7}$$

where we used the star-shapedness and (2.22). □

Now we choose the flow as

$$\frac{d}{dt}x = \left(\frac{n}{H} - \frac{u}{\lambda'}\right)\nu. \tag{9.8}$$

Along this flow the area $|M_t|$ is nondecreasing and the quantity

$$\int_{M_t} H\lambda' d\mu_t - 2n\int_{\Omega_t}\frac{\lambda'\lambda''}{\lambda}dN \tag{9.9}$$

is nonincreasing.

Proposition 9.3. *Under the assumptions of Theorem* 1.5, *let* $M_t \subset N$ *be a family of closed star-shaped hypersurfaces evolving by* (9.8). *Then*

$$\frac{d}{dt}\int_{\Omega_t}\lambda' dN \geq 0, \tag{9.10}$$

$$\frac{d}{dt}|M_t| \geq 0, \tag{9.11}$$

and

$$\frac{d}{dt}\left(\int_{M_t} H\lambda' d\mu_t - 2n\int_{\Omega_t} \frac{\lambda'\lambda''}{\lambda} dN\right) \leq 0. \tag{9.12}$$

Proof. We first note that all of the assumptions in Propositions 9.1 and 9.2 are satisfied by anti–de Sitter Schwarzschild space and hyperbolic space. Also, the Heintze–Karcher-type inequality holds for anti–de Sitter Schwarzschild space and hyperbolic space. Thus inequality (9.10) is proved in the same way as the proof of Theorem 1.1 in section 8. Inequality (9.11) was proved in the proof of Theorem 1.3 in section 8. Next we show (9.12). From (9.3) and (9.1) we have

(9.13)

$$\begin{aligned}
\frac{d}{dt}\left(\int_{M_t} H\lambda' - 2n\int_{\Omega_t} \frac{\lambda'\lambda''}{\lambda}\right) &= \int_{M_t}\left(2\sigma_2\lambda' + 2H\left\langle\bar\nabla\lambda',\nu\right\rangle - 2n\frac{\lambda'\lambda''}{\lambda}\right)\left(\frac{n}{H}-\frac{u}{\lambda'}\right)\\
&= \int_{M_t} 2\sigma_2\lambda'\left(\frac{n}{H}-\frac{u}{\lambda'}\right) + \int_{M_t}\frac{\lambda''}{\lambda}\left(2Hu-2n\lambda'\right)\left(\frac{n}{H}-\frac{u}{\lambda'}\right)\\
&\leq \int_{M_t}\left((n-1)H\lambda' - 2\sigma_2 u\right) - \int_{M_t} 2H\frac{\lambda'\lambda''}{\lambda}\left(\frac{n}{H}-\frac{u}{\lambda'}\right)^2 \leq 0.
\end{aligned}$$

In the second equality, we used $\left\langle\bar\nabla\lambda',\nu\right\rangle = \frac{\lambda''}{\lambda}u$, and in the last two inequalities we used Newton–Maclaurin inequality (9.5) and $\lambda'' \geq 0$. □

The inequalities in Theorem 1.5 follow immediately from the monotonicity in Proposition 9.3 and the convergence result of the flow. The classification of the equality case follows easily by checking the equality in (9.13). □

Acknowledgments

The authors are grateful to Professor Pengfei Guan for his comments and for sending us their private note. The authors would also like to thank Professor Guofang Wang for bringing the paper [16] to their attention.

References

[1] Nicholas D. Alikakos and Alexandre Freire, *The normalized mean curvature flow for a small bubble in a Riemannian manifold*, J. Differential Geom. **64** (2003), no. 2, 247–303. MR2029906

[2] Ben Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), no. 2, 151–171, DOI 10.1007/BF01191340. MR1385524

[3] Ben Andrews, *Contraction of convex hypersurfaces in Riemannian spaces*, J. Differential Geom. **39** (1994), no. 2, 407–431. MR1267897

[4] Ben Andrews, *Pinching estimates and motion of hypersurfaces by curvature functions*, J. Reine Angew. Math. **608** (2007), 17–33, DOI 10.1515/CRELLE.2007.051. MR2339467

[5] Kenneth A. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes, vol. 20, Princeton University Press, Princeton, NJ, 1978. MR485012

[6] Simon Brendle, *Constant mean curvature surfaces in warped product manifolds*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 247–269, DOI 10.1007/s10240-012-0047-5. MR3090261

[7] Simon Brendle, Pengfei Guan, and Junfang Li, *An inverse curvature type hypersurface flow in* $\mathbb{H}^{n+1}$ (private note).

[8] Simon Brendle, Pei-Ken Hung, and Mu-Tao Wang, *A Minkowski inequality for hypersurfaces in the anti–de Sitter–Schwarzschild manifold*, Comm. Pure Appl. Math. **69** (2016), no. 1, 124–144, DOI 10.1002/cpa.21556. MR3433631

[9] Esther Cabezas-Rivas and Vicente Miquel, *Volume preserving mean curvature flow in the hyperbolic space*, Indiana Univ. Math. J. **56** (2007), no. 5, 2061–2086, DOI 10.1512/iumj.2007.56.3060. MR2359723

[10] Bennett Chow, *Deforming convex hypersurfaces by the nth root of the Gaussian curvature*, J. Differential Geom. **22** (1985), no. 1, 117–138. MR826427

[11] Bennett Chow, *Deforming convex hypersurfaces by the square root of the scalar curvature*, Invent. Math. **87** (1987), no. 1, 63–82, DOI 10.1007/BF01389153. MR862712

[12] Levi Lopes de Lima and Frederico Girão, *An Alexandrov-Fenchel-type inequality in hyperbolic space with an application to a Penrose inequality*, Ann. Henri Poincaré **17** (2016), no. 4, 979–1002, DOI 10.1007/s00023-015-0414-0. MR3472630

[13] Klaus Ecker and Gerhard Huisken, *Immersed hypersurfaces with constant Weingarten curvature*, Math. Ann. **283** (1989), no. 2, 329–332, DOI 10.1007/BF01446438. MR980601

[14] Christian Enz, *The scalar curvature flow in Lorentzian manifolds*, Adv. Calc. Var. **1** (2008), no. 3, 323–343, DOI 10.1515/ACV.2008.014. MR2458241

[15] Yuxin Ge, Guofang Wang, and Jie Wu, *Hyperbolic Alexandrov-Fenchel quermassintegral inequalities II*, J. Differential Geom. **98** (2014), no. 2, 237–260. MR3263518

[16] Yuxin Ge, Guofang Wang, and Jie Wu, *The GBC mass for asymptotically hyperbolic manifolds*, Math. Z. **281** (2015), no. 1-2, 257–297, DOI 10.1007/s00209-015-1483-y. MR3384870

[17] Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), no. 1, 299–314. MR1064876

[18] Claus Gerhardt, *Closed Weingarten hypersurfaces in space forms*, Geometric analysis and the calculus of variations, Int. Press, Cambridge, MA, 1996, pp. 71–97. MR1449403

[19] Claus Gerhardt, *Hypersurfaces of prescribed scalar curvature in Lorentzian manifolds*, J. Reine Angew. Math. **554** (2003), 157–199, DOI 10.1515/crll.2003.003. MR1952172

[20] Claus Gerhardt, *Curvature problems*, Series in Geometry and Topology, vol. 39, International Press, Somerville, MA, 2006. MR2284727

[21] Claus Gerhardt, *Inverse curvature flows in hyperbolic space*, J. Differential Geom. **89** (2011), no. 3, 487–527. MR2879249

[22] Claus Gerhardt, *Non-scale-invariant inverse curvature flows in Euclidean space*, Calc. Var. Partial Differential Equations **49** (2014), no. 1-2, 471–489, DOI 10.1007/s00526-012-0589-x. MR3148124

[23] Claus Gerhardt, *Curvature flows in the sphere*, J. Differential Geom. **100** (2015), no. 2, 301–347. MR3343834

[24] Frederico Girão and Neilha M. Pinheiro, *An Alexandrov-Fenchel-type inequality for hypersurfaces in the sphere*, Ann. Global Anal. Geom. **52** (2017), no. 4, 413–424, DOI 10.1007/s10455-017-9562-4. MR3735905

[25] Pengfei Guan and Junfang Li, *The quermassintegral inequalities for k-convex starshaped domains*, Adv. Math. **221** (2009), no. 5, 1725–1732, DOI 10.1016/j.aim.2009.03.005. MR2522433

[26] Pengfei Guan and Junfang Li, *A mean curvature type flow in space forms*, Int. Math. Res. Not. IMRN **13** (2015), 4716–4740, DOI 10.1093/imrn/rnu081. MR3439091

[27] Pengfei Guan and Junfang Li, *A fully-nonlinear flow and quermassintegral inequalities (in Chinese)*, Sci. Sin. Math. **48** (2018), no. 1, 147–156, DOI 10.1360/N012017-00009.

[28] Pengfei Guan, Junfang Li, and Mu Tao Wang, *A volume preserving flow and the isoperimetric problem in warped product spaces*, Trans. Am. Math. Soc. **372** (2019), 2777–2798, DOI 10.1090/tran/7661.

[29] Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266. MR772132

[30] Gerhard Huisken, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. **84** (1986), no. 3, 463–480, DOI 10.1007/BF01388742. MR837523

[31] Gerhard Huisken, *The volume preserving mean curvature flow*, J. Reine Angew. Math. **382** (1987), 35–48, DOI 10.1515/crll.1987.382.35. MR921165

[32] Gerhard Huisken and Tom Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437. MR1916951

[33] Gerhard Huisken and Carlo Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. **183** (1999), no. 1, 45–70, DOI 10.1007/BF02392946. MR1719551

[34] Gerhard Huisken and Carlo Sinestrari, *Mean curvature flow with surgeries of two-convex hypersurfaces*, Invent. Math. **175** (2009), no. 1, 137–221, DOI 10.1007/s00222-008-0148-4. MR2461428

[35] N. V. Krylov, *Nonlinear elliptic and parabolic equations of the second order*, Mathematics and its Applications (Soviet Series), vol. 7, D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian by P. L. Buzytsky [P. L. Buzytskiĭ]. MR901759

[36] Matthias Makowski and Julian Scheuer, *Rigidity results, inverse curvature flows and Alexandrov-Fenchel type inequalities in the sphere*, Asian J. Math. **20** (2016), no. 5, 869–892, DOI 10.4310/AJM.2016.v20.n5.a2. MR3622318

[37] James McCoy, *The surface area preserving mean curvature flow*, Asian J. Math. **7** (2003), no. 1, 7–30, DOI 10.4310/AJM.2003.v7.n1.a2. MR2015239

[38] James A. McCoy, *Mixed volume preserving curvature flows*, Calc. Var. Partial Differential Equations **24** (2005), no. 2, 131–154, DOI 10.1007/s00526-004-0316-3. MR2164924

[39] Julian Scheuer, *Non-scale-invariant inverse curvature flows in hyperbolic space*, Calc. Var. Partial Differential Equations **53** (2015), no. 1-2, 91–123, DOI 10.1007/s00526-014-0742-9. MR3336314

[40] Julian Scheuer, *The inverse mean curvature flow in warped cylinders of non-positive radial curvature*, Adv. Math. **306** (2017), 1130–1163, DOI 10.1016/j.aim.2016.11.003. MR3581327

[41] Julian Scheuer, *Isotropic functions revisited*, Arch. Math. (Basel) **110** (2018), no. 6, 591–604, DOI 10.1007/s00013-018-1162-4. MR3803748

[42] John I. E. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), no. 3, 355–372, DOI 10.1007/BF02571249. MR1082861

[43] John I. E. Urbas, *An expansion of convex hypersurfaces*, J. Differential Geom. **33** (1991), no. 1, 91–125. MR1085136

[44] Guofang Wang and Chao Xia, *Isoperimetric type problems and Alexandrov-Fenchel type inequalities in the hyperbolic space*, Adv. Math. **259** (2014), 532–556, DOI 10.1016/j.aim.2014.01.024. MR3197666

[45] Yong Wei and Changwei Xiong, *Inequalities of Alexandrov-Fenchel type for convex hypersurfaces in hyperbolic space and in the sphere*, Pacific J. Math. **277** (2015), no. 1, 219–239, DOI 10.2140/pjm.2015.277.219. MR3393689

[46] Chao Xia, *A Minkowski type inequality in space forms*, Calc. Var. Partial Differential Equations **55** (2016), no. 4, Art. 96, 8, DOI 10.1007/s00526-016-1037-0. MR3523663

Albert-Ludwigs-Universität, Mathematisches Institut, Abteilung Reine Mathematik, Ernst-Zermelo-Straße 1, 79104 Freiburg, Germany

Email address: julian.scheuer@math.uni-freiburg.de

School of Mathematical Sciences, Xiamen University, 361005 Xiamen, People's Republic of China

Email address: chaoxia@xmu.edu.cn