Ilya Gekhtman, Giulio Tiozzo

*Entropy and drift for Gibbs measures on geometrically finite manifolds*

Transactions of the American Mathematical Society

DOI: 10.1090/tran/8036

**Accepted Manuscript**

This is a preliminary PDF of the author-produced manuscript that has been peer-reviewed and accepted for publication. It has not been copyedited, proofread, or finalized by AMS Production staff. Once the accepted manuscript has been copyedited, proofread, and finalized by AMS Production staff, the article will be published in electronic form as a “Recently Published Article” before being placed in an issue. That electronically published article will become the Version of Record.

This preliminary version is available to AMS members prior to publication of the Version of Record, and in limited cases it is also made accessible to everyone one year after the publication date of the Version of Record.

The Version of Record is accessible to everyone five years after publication in an issue.
ENTROPY AND DRIFT FOR GIBBS MEASURES ON GEOMETRICALLY FINITE MANIFOLDS

ILYA GEKHTMAN AND GIULIO TIOZZO

ABSTRACT. We prove a generalization of the fundamental inequality of Guivarc’h relating entropy, drift and critical exponent to Gibbs measures on geometrically finite quotients of $\text{CAT}(-1)$ metric spaces. For random walks with finite superexponential moment, we show that the equality is achieved if and only if the Gibbs density is equivalent to the hitting measure. As a corollary, if the action is not convex cocompact, any hitting measure is singular to any Gibbs density.

1. INTRODUCTION

Let $M$ be a manifold of negative curvature. Then the boundary at infinity $\partial X$ of its universal cover $X = \tilde{M}$ carries two types of measures (see e.g. [29]):

- on the one hand, Gibbs measures on the unit tangent bundle capture the asymptotic distribution of weighted periodic orbits for the geodesic flow. These include the Bowen-Margulis measure, the Liouville measure, and the harmonic measure associated to Brownian motion. To each of these can be associated a pair of measures on the boundary called Gibbs densities. For the Bowen-Margulis measure these conditionals are classical Patterson-Sullivan measures and for the Liouville measure they are in the Lebesgue measure class (see [36], [6], [30], [34], [37], [33], [25], [35], [8], among others).
- on the other hand, one can run a random walk on the fundamental group $\Gamma$ of $M$, which acts by isometries on $X$. This determines a measure on the boundary $\partial X$ which is called the harmonic measure or hitting measure.

In this paper, we will compare these two classes of measures when $M$ is a geometrically finite manifold of pinched negative curvature (and more generally, a geometrically finite quotient of a $\text{CAT}(-1)$ space).

Several numerical invariants have been introduced to capture the global dynamical and geometric properties of a random walk on a group. Namely, let $\mu$ be a probability measure on $\Gamma$, and define a random walk

$$\omega_n := g_1 \cdots g_n$$

where the $(g_i)$ are i.i.d. with distribution $\mu$. Let $o \in X$ be a basepoint. Then one defines:

1. the entropy $h_\mu$, introduced by Avez [1],

$$h_\mu := \lim_{n \to \infty} \frac{1}{n} \sum_{g} -\mu^n(g) \log \mu^n(g);$$

This is a pre-publication version of this article, which may differ from the final published version. Copyright restrictions may apply.
(2) the drift $\ell_\mu$ of the random walk

$$\ell_\mu := \lim_{n \to \infty} \frac{d(o, \omega_n o)}{n}$$

where the limit exists a.s. and is independent of $\omega_n$ and $o$;

(3) the critical exponent $v$ of the action

$$v := \limsup_{R \to \infty} \frac{1}{R} \log \# \{ g \in \Gamma : d(o, go) \leq R \}.$$ 

These quantities are related via the inequality

(1) $h_\mu \leq \ell_\mu v$

due to Guivarc’h [22] (see also Vershik [41], who calls it the fundamental inequality).

Let us remark that $\frac{h}{v}$ is the Hausdorff dimension of the harmonic measure [40]. On the other hand, to a random walk on $\Gamma$ we can associate a Green metric $d_G$ on $\Gamma$, defined in [2], with $d_G(g, h)$ defined as the negative logarithm of the probability that a random path starting at $g$ ever hits $h$. A measure $\mu$ is generating $\Gamma$ if the semigroup generated by the support of $\mu$ equals $\Gamma$.

In order to define a Gibbs measure, one is given a potential, i.e. a Hölder continuous, $\Gamma$-invariant function $F : T^1 X \to \mathbb{R}$. Then one defines the topological pressure as

$$v_F := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{n-1 \leq d(o, go) \leq n} e^{\int_0^{\text{go}} F}$$

and a Gibbs measure $m_F$ is a probability measure on $T^1 M$ whose pressure equals the topological pressure (if one exists). We may lift $m_F$ to a Radon measure $\tilde{m}_F$ on $T^1 X \cong (\partial X \times \partial X(\Delta)) \times \mathbb{R}$, and this defines a pair of measures on $\partial X$, known as Gibbs densities.

In order to account for the potential, we introduce a new notion of drift. We define the following distance (which we call $F$-ake distance, as it does not satisfy any of the usual properties of a distance)

$$d_F(x, y) := \int_x^y F \, dt,$$

where the integral is taken along the geodesic from $x$ to $y$. Then, we define the $F$-ake drift as

$$\ell_{F, \mu} := \lim_{n \to \infty} \frac{d_F(o, \omega_n o)}{n}$$

where, under suitable hypotheses, the limit exists almost surely and is constant.

We show that hitting measures and Gibbs densities are in the same measure class if and only if we have a relation between the dynamical quantities defined above. Moreover, this holds if and only if the Green metric, the space metric, and the $F$-ake metric are related as follows.

**Theorem 1.1.** Let $M$ be a geometrically finite manifold of pinched negative curvature and let $\Gamma := \pi_1(M)$. Let $\kappa_F$ be a Gibbs density for a Hölder potential $F$, and let $\nu_\mu$ be the hitting measure for a random walk driven by a measure $\mu$ generating $\Gamma$ with finite superexponential moment. Then we have the inequality:

$$h_\mu \leq \ell_\mu v - \ell_{F, \mu}.$$ 

Moreover, the following conditions are equivalent.
(1) The equality
\[ h_\mu = \ell_\mu v_F - \ell_{F,\mu} \]
holds.
(2) The measures \( \nu_\mu \) and \( \kappa_F \) are in the same measure class.
(3) For any basepoint \( o \in X \), there exists \( C \geq 0 \) such that
\[ |d_G(e, g) - v_F d(o, go) + d_F(o, go)| \leq C \]
for every \( g \in \Gamma \).

In fact, we do not need \( X \) to be a manifold, as the result still holds when \( X \) is a proper \( CAT(-1) \) space (even though the Hölder condition becomes slightly more technical). See Theorem 6.9.

In the case when \( \Gamma \) is not convex cocompact, it is not too hard to show that (3) cannot hold, yielding the following.

**Corollary 1.2.** If the action is not convex cocompact, then any Gibbs density \( \kappa_F \) is mutually singular to any hitting measure \( \nu_\mu \).

In particular, if the derivatives of the curvature are uniformly bounded, then the geometric potential \( F := -\frac{1}{n} \log J^{au} \) (where \( J^{au} \) is the Jacobian of the geodesic flow in the unstable direction) is Hölder continuous and the Gibbs measure \( m_F \) is the Liouville measure, hence we obtain that no hitting measure is in the same class as the Lebesgue measure.

Theorem 1.1 addresses a question of Paulin-Pollicott-Shapira ([35], page 9). Our results Theorem 1.1 and Corollary 1.2 are new even in the case where \( F = 0 \), i.e. when \( \kappa_F \) is the quasiconformal or Patterson-Sullivan measure. In this case, we need not assume \( X \) is \( CAT(0) \): it need only be a proper geodesic Gromov hyperbolic space. We thus give the complete proof in this context in Theorem 4.1.

When in addition \( \Gamma \rhd X \) is a convex cocompact action of a word hyperbolic group and the measure \( \mu \) is symmetric, the latter result is proved by Blachère-Haïssinsky-Mathieu in [5]. The authors there also prove that if \( \Gamma \rhd X \) is an action of a hyperbolic group which is not convex cocompact then the hitting and Patterson-Sullivan measures are singular. In particular this is true for finite covolume Fuchsian groups with cusps, a fact also obtained by Guivarc’h-Le Jan [23], Deroin-Kleptsyn-Navas [13], and Gadre-Maher-Tiozzo [18]. Note that a lattice \( \Gamma < \text{Isom} \mathbb{H}^2 \) is a hyperbolic group, while this is not true for lattices in \( \text{Isom} \mathbb{H}^n, n \geq 3 \).

For symmetric random walks on geometrically finite but not convex cocompact isometry groups of Gromov hyperbolic spaces, the singularity of harmonic and Patterson-Sullivan measures was obtained by Gekhtman-Gerasimov-Potyagailo-Yang [20]. Note these groups need not be hyperbolic. As a corollary (Corollary 4.2) of Theorem 4.1, we obtain the corresponding result for asymmetric random walks.

Note finally that for lattices in constant negative curvature the Patterson-Sullivan measure lies in the Lebesgue measure class, while this need not be the case in variable curvature.

In the opposite direction, Connell-Muchnik ([10], [9]) show that for cocompact isometry groups of \( CAT(-1) \) spaces any Gibbs state on the boundary of a \( CAT(-1) \) space is a hitting measure for some random walk with finite first moment.

If one replaces the random walk by Brownian motion, Ledrappier [28] proved that harmonic measures coincide with the Patterson-Sullivan measure if and only if an
analogue of the fundamental inequality is satisfied; moreover, in dimension 2 these two measures coincide if and only if the curvature is constant. The corresponding question in higher dimensions is a well-known open problem.

We note that the hitting measure $\nu$ is a conditional measure for a geodesic flow invariant measure on $T^1 X$, called the harmonic invariant measure associated to the random walk (in analogy to the harmonic measure associated to Brownian motion). In turn, this measure induces a finite flow invariant measure on $T^1 M$. The construction is due to Kaimanovich [26] for convex-cocompact manifolds and to Gekhtman-Gerasimov-Potyagailo-Yang [20] for geometrically finite ones. Moreover, axes of loxodromic elements associated to typical random walk trajectories equidistribute with respect to this measure (see [19]).

Our results give conditions for equivalence of a Gibbs measure and a harmonic invariant measure, and in particular they imply:

**Corollary 1.3.** If $\Gamma$ is geometrically finite but not convex cocompact, then any harmonic invariant measure is singular with respect to any Gibbs measure on $T^1 M$.

The Guivarc’h inequality has also been studied for word metrics. In this context, Gouëzel, Mathéus and Maucourant proved that the inequality (1) is strict for any superexponential moment generating random walk on a word-hyperbolic group which is not virtually free. Dussaule-Gekhtman [15] extended this result to large classes of relatively hyperbolic groups, including finite covolume isometry groups of pinched negatively curved manifolds and geometrically finite Kleinian groups.

**Acknowledgements.** We thank the Fields Institute for its support during the semester on "Teichmüller theory and its connections to geometry, topology and dynamics". Moreover, we thank the referee for useful comments on the first draft. G.T. is partially supported by NSERC and the Alfred P. Sloan Foundation.

## 2. Background

### 2.1. Notation.** For two quantities $f$ and $g$ we write $f \asymp_C g$ if $\frac{1}{C} f \leq g \leq C f$ and $f \asymp_{+,C} g$ if $f - C \leq g \leq f + C$. We write $f \asymp g$ if $f \asymp_C g$ for some constant $C$ and similarly for $f \asymp_{+,C} g$. Also, whenever $f \leq C g$ (resp. $f \leq C + g$) for some constant $C$, we will use the notation $f \asymp g$ (resp. $f \asymp_{+,+} g$).

### 2.2. Geometrically Finite Actions.** Let $(X, d)$ be a proper geodesic Gromov hyperbolic space and $\Gamma < \text{Isom}(X, d)$ a properly discontinuous group of isometries.

The set $\Lambda$ of accumulation points in the Gromov boundary $\partial X$ of any orbit $\Gamma x$ ($x \in X$) is called the limit set of the action $\Gamma \curvearrowright X$. A point $\zeta \in \Lambda$ is called conical if for every geodesic ray $\gamma$ converging to $\zeta$ and every $x \in X$ there is some $D > 0$ such that $\gamma$ has infinite intersection with the $D$-neighborhood of the orbit $\Gamma o$. A point $\zeta \in \Lambda$ is called bounded parabolic if its stabilizer $\text{Stab}(\zeta)$ in $\Gamma$ is infinite and acts cocompactly and properly discontinuously on $\Lambda \setminus \{x\}$. The action $\Gamma \curvearrowright X$ is said to be geometrically finite if every point of $\Lambda$ is either conical or bounded parabolic.

Let $\Gamma \curvearrowright X$ be a nonelementary geometrically finite action. Let $v = v_{\Gamma,X}$ be the critical exponent of $\Gamma$ with respect to the action:

$$v := \inf \left\{ s : \sum_{g \in \Gamma} e^{-sd(o,g o)} < \infty \right\} = \liminf_{R \to \infty} \frac{1}{R} \log |\Gamma o \cap B_R(o)|.$$
We assume $v < \infty$.

2.3. **Busemann functions.** For $x, y, z \in X$ let us define the Busemann function as

$$
\beta_z(x, y) := d_X(x, z) - d_X(y, z)
$$

and its extension to the boundary as

$$
\beta_{\xi}(x, y) := \liminf_{z \in \partial X, z \to \xi} \beta_z(x, y)
$$

where $\xi \in \partial X$. Moreover, the Gromov product of $x, y$ based at $z$ is

$$
\rho_z(x, y) := \frac{d_X(x, z) + d_X(z, y) - d_X(y, x)}{2}
$$

and for $\xi, \zeta \in \partial X$ we define it as

$$
\rho_{\xi, \zeta}(x, y) := \liminf_{z \in X, z \to \xi, y \to \zeta} \rho_z(x, y).
$$

2.4. **Quasiconformal measures.** Fix a basepoint $o \in X$. A probability measure $\kappa$ on the limit set $\Lambda \subset \partial X$ is called quasiconformal of dimension $s$ for $\Gamma \rhd X$ if for any $g \in \Gamma$ and a.e. $\zeta \in \Lambda$

$$
\frac{dg\kappa}{d\kappa}(\zeta) = e^{s\beta_{\zeta}(o, go)},
$$

where the implicit constant depends on the basepoint $o$ but not on $g$ and $\zeta$.

If the growth rate is $v < \infty$ there necessarily exists a quasiconformal measure of dimension $v$ [11, Theorem 5.4]. Moreover, quasiconformal measures of dimension $s > v$ give zero weight to conical limit points [31, Proposition 2.12], and hence are atomic. If $\Gamma$ is of divergence type, a $v$-dimensional quasiconformal measure is unique up to bounded density, ergodic, and gives full weight to conical limit points [31, Corollary 3.14]. Otherwise, any quasiconformal measure gives zero weight to conical limit points [31, Proposition 2.12].

We call a quasiconformal measure of dimension $v$ a Patterson-Sullivan measure.

2.5. **Random walks and the Green metric.** Let $\mu$ be a probability measure on $\Gamma$. Assume that the support of $\mu$ generates $\Gamma$ as a semigroup. Assume furthermore that $\mu$ has finite superexponential moment with respect to some (equivalently every) word metric $\| \cdot \|$ on $\Gamma$: that is,

$$
\sum_{g \in \Gamma} e^{c\|g\|} \mu(g) < \infty
$$

for all $c > 1$.

The Green function $G : \Gamma \times \Gamma \to \mathbb{R}$ associated to $(\Gamma, \mu)$ is defined to be the total weight $G(x, y)$ of all paths between $x$ and $y$. Letting $d_G(x, y) := -\log \frac{G(x, y)}{G(e, e)}$ we obtain a (possibly asymmetric) metric on $\Gamma$, called the Green metric.

Let $P$ be the measure on sample paths induced by $\mu$. For $P$-almost every $\omega \in \Gamma^{\mathbb{N}}$ the quantities

$$
\ell_\mu := \lim_{n \to \infty} \frac{d(o, \omega_n o)}{n}
$$

and

$$
h_\mu := \lim_{n \to \infty} -\frac{\log \mu^* n(\omega_n)}{n}
$$
are defined and are independent of \( \omega \). They are called respectively the drift and asymptotic entropy of the random walk.

By [20, Theorem 1.3], conical limit points are in one-to-one correspondence with a subset of the Martin boundary, which is the horofunction boundary of \( d_G \). This means if \( (g_n) \) is a sequence in \( \Gamma \) converging to a conical limit point \( \xi \in \partial X \), then for every \( g \), the quotient \( \frac{G(g,g_n)}{G(e,g_n)} \) converges to some limit \( K_\xi(g) \). In particular, if \( g, g' \) are fixed, then \( \frac{G(g,g_n)}{G(e,g_n)} \) converges to \( \frac{K_\xi(g)}{K_\xi(g')} \). Define then

\[
\beta^G_\xi(g,g') := \lim_{n \to \infty} \left( d_G(g,g_n) - d_G(g',g_n) \right) = -\log \frac{K_\xi(g)}{K_\xi(g')}
\]

which can be considered as Busemann functions for the Green metric.

Let \( \nu \) be the unique \( \mu \)-stationary probability on \( \partial X \). The measure \( \nu \) is necessarily ergodic, has no atoms, and is supported on conical limit points. It satisfies a conformal-type property with respect to the Green metric:

\[
\frac{d\nu_g}{d\nu}(\xi) = K_\xi(g) = e^{-\beta^g_\xi(g,e)}
\]

for any \( g \in \Gamma \) and \( \nu_g \)-almost every conical limit point \( \xi \), see e.g. [42, Theorem 24.10].

2.6. Comparing shadows. For \( r > 0 \) and \( x, y \in X \) the shadow \( Sh_r(x, y) \) consists of all points \( \zeta \in \partial X \) such that some geodesic ray from \( x \) to \( \zeta \) intersects \( B_r(y) \). The following analogue of Sullivan’s classical shadow lemma is due to Coornaert [11].

**Proposition 2.1.** Let \( \kappa \) be a quasiconformal measure for \( \Gamma \curvearrowright X \). For large enough \( r > 0 \) we have \( \kappa(Sh_r(o,go)) = e^{-\nu(d(o,go))} \) where the implied constant depends only on \( r \), \( o \) and the quasiconformality constant.

Let \( \Lambda \subset \partial X \) be the limit set of \( \Gamma \). For \( D > 0 \) let \( \Lambda_D \subset \Lambda \) consist of \( \zeta \) such that any geodesic ray in \( X \) converging to \( \zeta \) intersects the \( D \)-neighborhood of \( \Gamma_0 \) infinitely many times. The set \( \Lambda_D \) is \( \Gamma \)-invariant, and \( \bigcup_{D>0} \Lambda_D \) is precisely the set of conical points of \( \Lambda \). Thus, we have:

**Lemma 2.2.** Any \( \Gamma \) quasi-invariant ergodic measure on \( \partial X \) which gives full weight to conical limit points of \( \Lambda \) gives full weight to \( \Lambda_D \) for large enough \( D \).

We will prove a shadow lemma for the \( \mu \)-harmonic measure \( \nu \).

**Proposition 2.3.** For large enough \( r > 0 \) we have \( \nu(Sh_r(o,go)) = e^{-d_G(e,g)} \) where the implied constant depends only on \( r \) and \( o \).

The following is a re-formulation of the deviation inequalities of Gekhtman-Gerasimov-Potyagailo-Yang [20, Corollary 1.4].

**Proposition 2.4.** For each \( o \in X \) and \( D > 0 \) there is an \( A > 0 \) such that for all \( g_1, g_2, g_3 \in \Gamma \) such that \( g_2o \) lies within distance \( D \) of a geodesic \( [g_1o, g_3o] \) we have

\[
d_G(g_1, g_2) + d_G(g_2, g_3) \leq d_G(g_1, g_3) + A.
\]

**Proof of Proposition 2.3.** Let \( r > 0 \) be such that the complement of \( \Lambda_r \) has \( \nu \)-measure zero. Note we have by eq. (2)

\[
\nu(Sh_r(o,go)) = g^{-1} \nu(Sh_r(g^{-1}o,o)) = \int_{Sh_r(g^{-1}o,o)} e^{-\beta^g_\xi(g^{-1},e)} d\nu(\zeta).
\]
Consider a point \( \zeta \in \text{Sh}_r(g^{-1}o, o) \cap \Lambda_r \). By definition, any geodesic ray \([g^{-1}o, \zeta]\) in \( \text{Sh}_o \) contains a point in \( B_r(o) \) and there is a sequence \( g_n \in \Gamma \) with \( d(g_n o, o) \to \infty \) and \( d(g_n o, [o, \zeta]) < r \). Then by Proposition 2.4 we have for each \( n \):

\[
d_G(g^{-1}, e) - A \leq d_G(g^{-1}, g_n) - d_G(e, g_n)
\]

where \( A \) depends only on \( r \). Taking limits as \( g_n \to \zeta \) and by the triangle inequality we obtain

\[
d_G(g^{-1}, e) - A \leq \beta_G(g^{-1}, e) \leq d_G(g^{-1}, e).
\]

Fix a metric \( \rho \) on \( \partial X \). Let \( 0 < c < \text{diam}(\Lambda, \rho)/100 \). By [11, Lemma 6.3], there is an \( r_0 > 0 \) such that for any \( r > r_0 \) and \( g \in \Gamma \) the complement \( \partial X \setminus \text{Sh}_r(g^{-1}o, o) \) is contained in a \( \rho \)-ball of radius \( c \). Consequently, \( \text{Sh}_r(g^{-1}o, o) \) must contain a \( \rho \)-ball of radius \( c \) centered at a point of \( \Lambda \). Since \( \nu \) has full support on \( \partial X \) there is a constant \( t > 0 \) such that any such ball has \( \nu \)-measure at least \( t \). Consequently, we have \( 1 \geq \nu(\text{Sh}_r(g^{-1}o, o)) \geq t > 0 \) for all \( g \in \Gamma \). This completes the proof. \( \square \)

3. A differentiation theorem

Unlike in the hyperbolic group case, the harmonic measure \( \nu \) is not known to be doubling for the visual metric on \( \partial X \). See Tanaka’s [40, Question 4.1] for a discussion. However, we can still prove a Lebesgue differentiation-type theorem.

The arguments in the next two sections are very similar to those in [15], where an analogue of Theorem 4.1 is proved for word metrics on relatively hyperbolic groups, although the geometric setting is different.

Throughout this section we assume the hypotheses of Theorem 4.1.

**Proposition 3.1.** Let \( \nu \) be a \( \Gamma \) quasi-invariant ergodic measure on \( \partial X \) supported on conical points. Assume furthermore that for a constant \( C > 0 \) and all large enough \( r \) we have \( \nu(\text{Sh}_r(o, go)) \lesssim C \nu(\text{Sh}_r(o, go)) \) for all \( g \in \Gamma \). Let \( \kappa \) be any finite Borel measure on \( X \). Then the following holds for large enough \( r > 0 \).

a) If \( \nu \) and \( \kappa \) are mutually singular then for \( \nu \)-almost every \( \xi \in \partial X \) we have

\[
\lim_{\xi \in \text{Sh}_r(o, go)} \frac{\kappa(\text{Sh}_r(o, go))}{\nu(\text{Sh}_r(o, go))} = 0.
\]

b) If \( \nu \) and \( \kappa \) are equivalent then for \( \nu \)-almost every \( \xi \in \partial X \) we have

\[
\lim_{\xi \in \text{Sh}_r(o, go)} \frac{\log \kappa(\text{Sh}_r(o, go))}{\log \nu(\text{Sh}_r(o, go))} = 1.
\]

To prove this proposition we will need the notion of Vitali relation, which is a generalization of coverings by balls in doubling metric spaces. See Federer’s book [16, Sections 2.8 and 2.9] for background on Vitali relations and their application to differentiation in metric spaces. Our main source is [31], who use them to study quasiconformal measures for divergence type groups of isometries of Gromov hyperbolic spaces.

Let \( \Lambda \) be a metric space. A covering relation is a subset of the set of all pairs \((\xi, S)\) such that \( \xi \in S \subset \Lambda \). A covering relation \( \mathcal{C} \) is said to be fine at \( \xi \in \Lambda \) if there exists a sequence \( S_n \) of subsets of \( \Lambda \) with \((\xi, S_n) \in \mathcal{C}\) and such that the diameter of \( S_n \) converges to zero.

For a covering relation \( \mathcal{C} \) and any measurable subset \( E \subset \Lambda \), define \( \mathcal{C}(E) \) to be the collection of subsets \( S \subset \Lambda \) such that \((\xi, S) \in \mathcal{C}\) for some \( \xi \in E \).
A covering relation \( \mathcal{V} \) is said to be a Vitali relation for a finite measure \( \mu \) on \( \Lambda \) if it is fine at every point of \( \Lambda \) and if the following holds: if \( C \subset \mathcal{V} \) is fine at every point of a measurable subset \( E \subset \Lambda \) then \( C(E) \) has a countable disjoint subfamily \( \{S_n\} \subset C(E) \) such that \( \mu(E \setminus \bigcup_{n=1}^{\infty} S_n) = 0 \).

For a covering relation \( \mathcal{V} \) and a function \( f \) on \( \Lambda \) let us denote
\[
(\mathcal{V}) \limsup_{S \to x} f := \limsup_{\epsilon \to 0} \{ f(x) : (x, S) \in \mathcal{V}, x \in S, \text{diam}(S) < \epsilon \}.
\]
Similarly we define \( (\mathcal{V}) \liminf_{S \to x} f \), and if the two limits are equal we denote its common value as \( (\mathcal{V}) \lim_{S \to x} f \).

We will use the following criterion to guarantee a covering relation is Vitali.

Proposition 3.2. [16, Theorem 2.8.17] Let \( \mathcal{V} \) be a covering relation on \( \Lambda \) such that each \( S \in \mathcal{V}(\Lambda) \) is a closed bounded subset and \( \mathcal{V} \) is fine at every point of \( \Lambda \). Let \( \lambda \) be a measure on \( \Lambda \) such that \( \lambda(A) > 0 \) for all \( A \in \mathcal{V}(\Lambda) \). For a positive function \( f \) on \( \mathcal{V}(\Lambda) \), \( S \in \mathcal{V}(\Lambda) \), and a constant \( \tau > 1 \) define \( \tilde{S} \) to be the union of all \( S' \in \mathcal{V}(\Lambda) \) which have nonempty intersection with \( S \) and satisfy \( f(S') \leq \tau f(S) \). Suppose that for \( \lambda \cdot \text{almost every } \xi \in \Lambda \) we have
\[
\limsup_{S \to \xi} \left( f(S) + \frac{\lambda(\tilde{S})}{\lambda(S)} \right) < \infty.
\]
Then the relation \( \mathcal{V} \) is Vitali for \( \lambda \).

Let \( D \) be large enough so that the complement of \( \Lambda_D \) has \( \nu \) measure zero, and large enough for all \( r \geq D \) to satisfy Proposition 2.3.

Lemma 3.3. Define the covering relation
\[
\mathcal{V} := \{(\xi, Sh_{2D}(o,go) \cap \Lambda_D) \mid \xi \in Sh_{2D}(o,go) \cap \Lambda_D \}.
\]
Then \( \mathcal{V} \) is a Vitali relation for \( (\Lambda_D, \nu) \).

Proof. This is shown in [31, Lemma 4.5] for quasiconformal measures, and the proof is essentially the same in our setting. Indeed, by definition of \( \Lambda_D \) this relation is fine at every point of \( \Lambda_D \). Furthermore, by the thin triangles property \( Sh_{2D}(o,go) \cap \Lambda_D \) is contained in \( Sh_{D}(o,go) \) for another constant \( D' \) (see proof of [31, Lemma 4.5]). Thus, letting \( f(Sh_{2D}(o,go) \cap \Lambda_D) := e^{-\nu(d(o,go))} \) and any \( \tau > 1 \) we see using the fact that
\[
\nu(Sh_{D}(o,go))/\nu(Sh_{2D}(o,go)) = 1
\]
(implied by Proposition 2.3) that \( \mathcal{V} \) satisfies the hypothesis of Proposition 3.2. Hence it is a Vitali relation.

The following is obtained by combining Theorems 2.9.5 and 2.9.7 of [16].

Proposition 3.4. Let \( \Lambda \) be a metric space, \( \kappa_1 \) a finite Borel measure on \( \Lambda \) and \( \mathcal{V} \) a Vitali relation for \( \kappa_1 \). Let \( \kappa_2 \) be any finite Borel measure on \( \Lambda \). Define a new Borel measure \( \psi(\kappa_1, \kappa_2) \) by
\[
\psi(\kappa_1, \kappa_2)(A) := \inf\{\kappa_2(B) : B \text{ Borel, } \kappa_1(B \Delta A) = 0\}.
\]
This measure is absolutely continuous with respect to \( \kappa_1 \). The limit
\[
D(\kappa_1, \kappa_2, \mathcal{V}, x) := (\mathcal{V}) \lim_{S \to x} \frac{\kappa_2(S)}{\kappa_1(S)}
\]
exists for \( \kappa_1 \cdot \text{almost every } x \) and equals the Radon-Nikodym derivative \( \frac{d\psi(\kappa_1, \kappa_2)}{d\kappa_1} \).
As a corollary we obtain:

**Corollary 3.5.** a) If the $\kappa_i$ are mutually singular, then

$$\lim_{S \to \zeta} \frac{\kappa_2(S)}{\kappa_1(S)} = 0$$

for $\kappa_1$-almost every $\zeta \in \Lambda$.

b) If the $\kappa_i$ are equivalent and non-atomic, then

$$\lim_{S \to \zeta} \frac{\log \kappa_2(S)}{\log \kappa_1(S)} = 1$$

for $\kappa_1$-almost every $\zeta \in \Lambda$.

**Proof.** If $\kappa_1$ and $\kappa_2$ are mutually singular, then by definition $\psi(\kappa_1, \kappa_2) = 0$. Together with Proposition 3.4, this proves a).

For b), assume $\kappa_1$ and $\kappa_2$ are equivalent. By Proposition 3.4 we have

$$D(\kappa_1, \kappa_2, \nu, \zeta) = \lim_{S \to \zeta} \frac{\kappa_2(S)}{\kappa_1(S)} > 0$$

for $\kappa_1$-almost every $\zeta$. Thus, since $\kappa_1$ is non-atomic, we have

$$\lim_{S \to \zeta} \frac{\log \kappa_2(S)}{\log \kappa_1(S)} = 1 + \lim_{S \to \zeta} \frac{\log \kappa_2(S)}{\log \kappa_1(S)} = 1.$$

Applying this corollary to the Vitali relation $V$ defined above and noting that $\nu(\Lambda^\gamma_\nu) = 0$ completes the proof of Proposition 3.1.

4. Entropy and drift in hyperbolic spaces

We will start with the proof of our main result in the case $F = 0$, i.e., the Patterson-Sullivan measure. In this case, we do not require the space $X$ to be $CAT(-1)$, but only $\delta$-hyperbolic. We prove the following.

**Theorem 4.1.** Let $X$ be a $\delta$-hyperbolic, proper metric space, let $\Gamma$ be a geometrically finite group of isometries of $X$, and let $o \in X$ be a basepoint.

Let $\mu$ be a measure on $\Gamma$ with finite superexponential moment, let $\nu$ be its corresponding hitting measure, and let $\kappa$ be a Patterson-Sullivan measure on $\partial X$.

Then the following conditions are equivalent.

1. The equality $h_\mu = \ell_{\mu \nu}$ holds.
2. The measures $\nu$ and $\kappa$ are in the same measure class.
3. The measures $\nu$ and $\kappa$ are in the same measure class with Radon-Nikodym derivatives bounded from above and below.
4. There exists $C \geq 0$ such that for every $g \in \Gamma$,

$$|d_G(e, g) - \nu d(o, go)| \leq C.$$

**Corollary 4.2.** If $\Gamma \simeq X$ is not convex cocompact then $\kappa$ and $\nu$ are mutually singular.

**Proof.** By Theorem 4.1, if $\kappa$ and $\nu$ are in the same measure class then $d_G$ and $d$ are quasi-isometric. On the other hand, the Green metric is quasi-isometric to the word metric for any random walk on a non-amenable group with finite exponential moment (see Proposition 7.8 in the appendix). Thus, the orbit map from the Cayley graph to $X$ must be a quasi-isometry, which is impossible in the presence of parabolics [39].
Let $\nu$ be the $\mu$-stationary measure on $\partial X$ and let $\kappa$ be a $\nu$-dimensional quasi-conformal measure. In this section we prove that $h = lv$ if and only if $\nu$ and $\kappa$ are mutually absolutely continuous. Let $r > 0$ be large enough to satisfy Proposition 2.3, 2.1, and 3.1.

For a sample path $\omega$ let $\omega_n$ be its $n$-th position. Define then

$$\phi_n = \phi_n(\omega) := \frac{\kappa(Sh_r(o, \omega_n o))}{\nu(Sh_r(o, \omega_n o))}.$$ 

Let $\psi_n := \log \phi_n$. Notice that the expectation of $\phi_n$ is given by

$$E(\phi_n) = \sum_{g \in \Gamma} \mu^{\#n}(g) \frac{\kappa(Sh_r(o, go))}{\nu(Sh_r(o, go))}.$$ 

**Proposition 4.3.** There exists $C_1 > 0$ such that for any $N \geq 1$ we have

$$1 \frac{1}{N} \sum_{n=1}^{N} E(\phi_n) \leq C_1.$$ 

**Proof.** Consider $n, N$ with $1 \leq n \leq N$. We will first show that there is some $k > 0$ such that the quantity

$$R_k := \sum_{g \in \Gamma : d(o, go) \geq kN} \frac{\kappa(Sh_r(o, go))}{\nu(Sh_r(o, go))} \mu^{\#n}(g)$$

is bounded independently of $n, N$.

Let $\|\cdot\|$ be any word norm on $\Gamma$. By the shadow lemma for harmonic measure (Proposition 2.3)

$$(3) \quad \nu(Sh_r(o, go)) \asymp G(e, g) = \sum_{n=1}^{\infty} \mu^{\#n}(g).$$

Furthermore, since the Green distance and word metric are quasi-isometric (see for example [24, Lemma 4.2]), and $\kappa$ is a finite measure,

$$\frac{\kappa(Sh_r(o, go))}{\nu(Sh_r(o, go))} \leq e^{c\|g\|}$$

for a constant $c$. Also, we have $d(o, go) \leq t\|g\|$ for a constant $t > 1$. We obtain:

$$R_k \lesssim \sum_{g \in \Gamma : \|g\| \geq 1} \frac{\kappa(Sh_r(o, go))}{\nu(Sh_r(o, go))} \mu^{\#n}(g) \lesssim \sum_{m \geq t^{-1}kN} e^{cm} \sum_{\|g\|=m} \mu^{\#n}(g) \lesssim \sum_{m \geq t^{-1}kN} e^{cm} P(\|\omega_n\| \geq m).$$

Since $\mu$ has finite superexponential moment, we can apply the exponential Chebyshev inequality with exponent $2c$ to obtain

$$R_k \lesssim \sum_{m \geq t^{-1}kN} e^{-cm} E(e^{2c\|\omega_n\|}).$$

Since $n \leq N$ we have

$$\|\omega_n\| \leq \sum_{j=0}^{N-1} \|\omega_j^{-1}\omega_{j+1}\|$$

from which we obtain, since the $\omega_j^{-1}\omega_{j+1}$ are independent random variables,

$$E(e^{2c\|\omega_n\|}) \leq E_0^N$$
where $E_0 = \sum_{g \in \Gamma} e^{2c||g||} \mu(g)$. Choosing $k \geq \frac{1}{c} \log E_0$ we thus obtain

$$R_k \leq \sum_{m \geq t^1} \sum_{g \in \Gamma} e^{-cm} E(e^{2c||g||}) \leq e^{-c^{-1}kN} E_0^N \leq 1$$

giving us the desired estimate for $R_k$.

Now, we will show that the quantity

$$P_N := \frac{1}{N} \sum_{n=1}^{\infty} \sum_{g \in \Gamma} \frac{\kappa(Sh_r(o, go))}{\nu(Sh_r(o, go))} e^{\nu n} \mu^{*n}(g),$$

where $B_{kN} := \{ g \in \Gamma : d(o, go) \leq kN \}$, is bounded independently of $N$. Together with the estimate on $R_k$ this will prove the proposition. Interchanging the order of summation we get, using (3),

$$P_N = \frac{1}{N} \sum_{g \in \Gamma} \sum_{n=1}^{\infty} \frac{\mu^{*n}(g)}{\nu(Sh_r(o, go))} \kappa(Sh_r(o, go)) \leq \frac{1}{N} \sum_{g \in \Gamma} \kappa(Sh_r(o, go)).$$

By [43, Theorem 1.9], for any $a > 0$, we have

$$|\{ g \in \Gamma : n - a < d(o, go) \leq n \}| \leq e^{vn}.$$

Consequently, if we denote $A_{r,R} := \{ g \in \Gamma : r \leq d(o, go) \leq R \}$, we get

$$\sum_{g \in \Gamma} \kappa(Sh_r(o, go)) = \sum_{n=1}^{\infty} \sum_{A_{k(n-1),kn}} \kappa(Sh_r(o, go))$$

and by Proposition 2.1

$$\leq \sum_{n=1}^{\infty} \sum_{A_{k(n-1),kn}} e^{-vkn}$$

$$\leq \sum_{n=1}^{\infty} e^{vkn} e^{-vkn} \leq N.$$

The estimate for $P_N$ follows.

The following will be proved in the appendix.

**Proposition 4.4.** Assume $\Gamma \curvearrowright X$ is a nonelementary action on a proper geodesic Gromov hyperbolic space and $\mu$ a generating probability measure on $\Gamma$ with finite exponential moment. Then, for each $o \in X$ there exists $C > 0$ such that for each $0 \leq k \leq n$ and any $a > 1$ we have

$$P(d(\omega_k o, [o, \omega_n o]) > a) \leq Ce^{-a/C}$$

and

$$P(d(\omega_k o, [o, \omega_{2k}]) > a) \leq Ce^{-a/C},$$

where $[o, \omega_n o]$ and $[o, \omega_{2k}]$ are any geodesics connecting the respective endpoints.

We now deduce the following proposition.

**Proposition 4.5.** There exists $C_2 > 0$ such that the sequence $E(\psi_n) + C_2$ is subadditive and $\psi_n/n$ converges to $h - lv$ almost surely and in expectation.
\textbf{Proof.} By the shadow lemmas (see Proposition 2.1 and Proposition 2.3),
\[ \frac{\psi_n}{n} = \frac{d_G(\omega_n, e)}{n} - \frac{d(o, \omega_n o)}{n} + O(1/n). \]
According to [4, Theorem 1.1], the term \( \frac{1}{n}d_G(e, \omega_n) \) almost surely converges to \( h \) whenever \( \mu \) has finite entropy \( h_\mu \), which is implied by finite first moment. In other words, entropy is equal to the drift of \( d_G \). Thus, \( \psi_n/n \) converges to \( h - lv \) almost surely and in expectation.

Let \( m, n \geq 1 \). The shadow lemmas for the Patterson-Sullivan measure (Proposition 2.1) and for the harmonic measure (Proposition 2.3) yield
\[ \psi_n = \log \kappa(Sh_r(o, \omega_n o)) - \log \nu(Sh_r(o, \omega_n o)) = -vd(o, \omega_n o) + d_G(o, \omega_n o) + O(1) \]
and the triangle inequality for \( d_G \) implies that
\[ E(\psi_{n+m}) - E(\psi_n) - E(\psi_m) \leq vE(d(o, \omega_n o) + d(o, \omega_m o) - d(o, \omega_{n+m} o)) + O(1) = \]
and by shift-invariance on the space of increments and \( \delta \)-hyperbolicity,
\[ = vE(d(o, \omega_n o) + d(o, \omega_n o, \omega_{n+m} o) - d(o, \omega_{n+m} o)) \leq vE(2d(\omega_n o, [o, \omega_{n+m} o])) + O(1). \]
Proposition 4.4 implies that the last expression is bounded by a constant \( C_2 \), independent of \( n \) and \( m \), so that \( E(\psi_n) + C_2 \) is sub-additive.

\textbf{Proposition 4.6.} Let \( r \) be large enough for the conclusion of Proposition 2.3 to hold. Then
\[ a) \text{ If } \kappa \text{ and } \nu \text{ are not equivalent, then } \phi_n \text{ tends to } 0 \text{ in probability.} \]
\[ b) \text{ If } \kappa \text{ and } \nu \text{ are equivalent, then } \frac{\log \kappa(Sh_r(o, \omega_n o))}{\log \nu(Sh_r(o, \omega_n o))} \text{ tends to } 1 \text{ in probability.} \]

\textbf{Proof.} Recall, \( \nu \) is always ergodic with respect to the action of \( \Gamma \) on \( \partial X \) and gives full weight to conical points. On the other hand, \( \kappa \) is ergodic, gives full weight to conical points when \( \Gamma \) is divergence type and gives full weight to parabolic points when \( \Gamma \) is convergence type. Thus, in either case, if the two measures are not equivalent, they are mutually singular. The result now follows by combining Proposition 3.1 and Proposition 4.4. We give the details below.

Let \( \alpha, c > 0 \). By Proposition 4.4 we have \( P(\omega_\infty \notin Sh_D(o, \omega_n o)) \leq F(D) \) independently of \( n \) where \( F(D) \to 0 \) as \( D \to \infty \). Fix \( D \) so that
\[ P(\omega_\infty \notin Sh_D(o, \omega_n o)) \leq \alpha \]
for all \( n \). By Proposition 3.1 a) we have, for \( \nu \)-almost every \( \xi \),
\[ \lim_{t \to \infty} \sup_{\xi \in Sh_D(o, go)} \frac{\kappa(Sh_D(o, go))}{\nu(Sh_D(o, go))} = 0. \]
The shadow lemma for \( \nu \) (Proposition 2.3) shows that \( \nu(Sh_D(o, go)) \leq C\nu(Sh_r(o, go)) \) where \( C \) depends only on \( r, D \). Thus the quantity
\[ R_t(\xi) := \sup_{\xi \in Sh_D(o, go)} \frac{\kappa(Sh_r(o, go))}{\nu(Sh_r(o, go))} \]
converges to 0 as \( t \to \infty \) for \( \nu \)-almost every \( \xi \). Furthermore, for almost every sample path \( \omega \), we have \( \|\omega_n\| \to \infty \). Thus, by Egorov’s theorem, we may choose a subset \( E \subset \Gamma^N \) of sample paths with \( P(E^c) < \alpha \) and such that \( \|\omega_n\| \to \infty \) and \( R_n(\omega_\infty) \to 0 \) uniformly over \( \omega \in E \). It follows that \( R_{|\omega_n|}(\omega_\infty) \to 0 \) uniformly over \( \omega \in E \). This means that for large enough \( n \) (depending on \( c \)), the conditions \( \omega \in E \)
and $\kappa(\text{Sh}_{e}(a, \omega_n, o)) \geq c > 0$ imply $\omega_X \not\in \text{Sh}_D(a, \omega_n, o)$. The latter has probability at most $\alpha$, so we get
\[ P(\phi_n \geq c) \leq P(E^c) + P(\omega_X \not\in \text{Sh}_D(a, \omega_n, o)) \leq 2\alpha. \]
As $\alpha, c > 0$ were chosen arbitrarily we get $P(\phi_n \geq c) \to 0$ as $n \to \infty$ for each $c > 0$ so $\phi_n \to 0$ in probability.

b) This time, we define for each $D > 0$
\[ R_t(\xi) := \sup_{\xi \in \text{Sh}_D(a, go)} \left| \frac{\log \kappa(\text{Sh}_{e}(a, \omega_n, o))}{\log \nu(\text{Sh}_{e}(a, \omega_n, o))} - 1 \right|. \]
Using b) of Proposition 3.1 we obtain that for each $D$, $\lim_{t \to \infty} R_t(\xi) = 1$ for $\nu$-almost every $\xi$. The proof is then similar to a).

\[ \square \]

Remark 4.1. The only properties of $\kappa$ used are that $\Gamma$ preserves its measure class and acts ergodically on $(\mathcal{F}, \kappa)$.

We are now ready to prove the following.

**Theorem 4.7.** The measures $\kappa$ and $\nu$ are equivalent if and only if $h = lv$.

**Proof.** Assume that the Patterson-Sullivan and the harmonic measures are not equivalent. Let $\beta > 0$. Let $A_n$ be the event $\{\phi_n \geq \beta\}$ and $B_n = A_n^c$. For every $n$,
\[ E(\psi_n) = \int_{A_n} \psi_n dP + \int_{B_n} \psi_n dP. \]
According to Proposition 4.6, $\phi_n$ converges to 0 in probability. Thus, there exists $n_0$ such that for every $n \geq n_0$, $P(B_n) \geq 1 - \beta$. In particular,
\[ \int_{B_n} \psi_n dP \leq P(B_n) \log \beta \leq (1 - \beta) \log \beta. \]
Let $C_1$ be the constant in Proposition 4.3. Jensen’s inequality shows that
\[ \int_{A_n} \psi_n dP = P(A_n) \int_{A_n} \log \phi_n \frac{dP}{P(A_n)} \leq P(A_n) \log \left( \int_{A_n} \phi_n \frac{dP}{P(A_n)} \right). \]
Rewrite the right-hand side as
\[ P(A_n) \log \left( \int_{A_n} \phi_n \frac{dP}{P(A_n)} \right) = -P(A_n) \log P(A_n) + P(A_n) \log \left( \int_{A_n} \phi_n dP \right). \]
The function $x \mapsto x \log x$ is first decreasing then increasing on $[0, 1]$, so if $\beta$ is small enough, $-P(A_n) \log P(A_n) \leq -\beta \log \beta$. Moreover,
\[ P(A_n) \log \left( \int_{A_n} \phi_n dP \right) \leq \beta \sup (0, \log E(\phi_n)). \]
We thus have
\[ \int_{A_n} \psi_n dP \leq \beta \log \frac{1}{\beta} + \beta \sup (0, \log E(\phi_n)). \]
According to Proposition 4.3, $\liminf E(\phi_n) \leq 2C_1$, so that there exists $p \geq n_0$ such that $E(\phi_p) \leq 2C_1$. In particular, for every small enough $\beta$, we can find $p$ such that
\[ E(\psi_p) \leq (1 - \beta) \log \beta + \beta \log \frac{1}{\beta} + \beta |\log 2C_1|. \]
The right-hand side tends to $-\infty$ when $\beta$ goes to 0. If $\beta$ is small enough, we thus have for some $p$
\begin{equation*}
E(\psi_p) + C_2 \leq -1,
\end{equation*}
where $C_2$ is the constant in Proposition 4.5. Since $E(\psi_p) + C_2$ is sub-additive, we have
\begin{equation*}
\frac{1}{k}(E(\psi_{kp}) + C_2) \leq E(\psi_p) + C_2 \leq -1.
\end{equation*}
Finally, $\frac{1}{kp}E(\psi_{kp})$ converges to $h_\mu - \ell_\mu v$, so letting $k$ tend to infinity, we get
\begin{equation*}
h_\mu - \ell_\mu v \leq -\frac{1}{p} < 0.
\end{equation*}
Thus, $h_\mu < \ell_\mu v$.

Conversely, suppose the measures are equivalent. By the shadow lemma for the Patterson-Sullivan measure $\kappa$ we have
\begin{equation*}
-\log \kappa(Sh_r(o, go)) \to v
\end{equation*}
as $d(o, go) \to \infty$ and in particular
\begin{equation*}
-\log \kappa(Sh_r(o, \omega_n o)) \to v
\end{equation*}
for almost every sample path. Furthermore, for a.e. sample path we have
\begin{equation*}
\lim_{n \to \infty} -\log \nu(Sh_r(o, \omega_n o)) = \lim_{n \to \infty} \frac{dg(e, \omega_n)}{d(o, \omega_n o)} = \lim_{n \to \infty} \frac{dG(e, \omega_n)/n}{d(o, \omega_n o)/n} = h_\mu.
\end{equation*}
Thus, almost surely,
\begin{equation*}
\log \kappa(Sh_r(o, \omega_n o)) \to \frac{v \ell_\mu}{h_\mu}.
\end{equation*}
As the measures are equivalent we have by Proposition 3.1 b)
\begin{equation*}
\log \kappa(Sh_r(o, \omega_n o)) \to 1
\end{equation*}
amost surely, which ensures that $h_\mu = \ell_\mu v$.

The following result is a consequence of Proposition 4.7. Indeed, notice that $h$ and $l$ are the same for the measure $\mu$ and the reflected measure $\tilde{\mu}$.

**Corollary 4.8.** Let $\tilde{\nu}$ be the harmonic measure for the reflected random walk $\tilde{\mu}$. Then $\tilde{\nu}$ is equivalent to $\kappa$ whenever $\nu$ is equivalent to $\kappa$.

5. Equivalence of measures and equivalence of metrics

As before, $\Gamma \acts X$ is a nonelementary geometrically finite action on a proper geodesic Gromov hyperbolic space and $\mu$ a generating probability measure on $\Gamma$ with finite superexponential moment.

In this section we prove the following.

**Proposition 5.1.** If the harmonic measure $\nu$ and its reflection $\tilde{\nu}$ are both equivalent to the Patterson-Sullivan measure $\kappa$, then the Radon-Nikodym derivative $\frac{d\kappa}{d\nu}$ is bounded away from 0 and infinity.

This will use the following general lemma.
Lemma 5.2. Let $Z$ be a compact metrizable space and let $G$ act by homeomorphisms on $Z$. Let $\nu_1, \nu_2, \kappa_1, \kappa_2$ be Borel probability measures with full support on $Z$ and with $\nu_i$ equivalent to $\kappa_i$ for $i = 1, 2$.

Assume $G$ preserves the measure class of $\nu_i$ for $i = 1, 2$ and acts ergodically on $(Z \times Z, \nu_1 \otimes \nu_2)$ and $(Z \times Z, \kappa_1 \otimes \kappa_2)$. Suppose there are positive, bounded away from 0, measurable functions $f_\nu, f_\kappa : Z \times Z \setminus \text{Diag} \to \mathbb{R}$, bounded on compact subsets of $Z \times Z \setminus \text{Diag}$ such that $m_\nu = f_\nu \nu_1 \otimes \nu_2$ and $m_\kappa = f_\kappa \kappa_1 \otimes \kappa_2$ are $G$-invariant ergodic Radon measures on $Z \times Z \setminus \text{Diag}$. Then for each $i = 1, 2$, $\frac{dm_\nu}{dm_\kappa}$ is bounded away from 0 and $\infty$.

Proof. Since $\nu_i$ and $\kappa_i$ are equivalent, we have $d\nu_i = J_i d\kappa_i$ for a measurable positive function $J_i$. We want to show $J_i$ is $\kappa_i$-essentially bounded.

Since $m_\nu$ and $m_\kappa$ are $G$-invariant ergodic measures, either they are mutually singular or they are scalar multiples of each other. Thus, the assumption $d\nu_i = J_i d\kappa_i$ implies they are scalar multiples of each other. Without loss of generality, we can assume that they coincide. Note $dm_\nu(a, b) = J_1(a)J_2(b)f_\nu(a, b)d\kappa_1(a)d\kappa_2(b)$ so we have $J_1(a)J_2(b) = f_\kappa(a, b)/f_\nu(a, b)$ for $\nu_1 \otimes \nu_2$-almost all $(a, b)$.

Let $U, V$ be disjoint closed subsets in $Z$ with nonempty interior. There is a $p \in V$ such that $J_1(a)J_2(p) = f_\kappa(a, p)/f_\nu(a, p)$ for $\nu_1$-almost all $a \in U$. Dividing and noting that the $f_\kappa$ and $f_\nu$ are positive and bounded away from 0 and infinity on $U \times V$, we see that $C_{\nu_1}^{-1} \leq J_1(a)/J_1(a') \leq C_{\nu}$ for $\nu_1$-almost all $a, a' \in U$. Thus, $J_1$ is $\nu_1$-essentially bounded on any closed subset $U$ whose complement has nonempty interior. Covering $Z$ by two such sets, we see that $J_1$ is essentially bounded. The same argument applies to $J_2$. \hfill $\Box$

Proof of Proposition 5.1. The $\Gamma$-action on $\nu \times \bar{\nu}$ is ergodic (see [27, Theorem 6.3]) and since $\nu$ and $\bar{\nu}$ are both equivalent to $\kappa$, the $\Gamma$-action is also ergodic for $\kappa \otimes \kappa$.

To complete the proof of Proposition 5.1 we just need to show that $\kappa \otimes \kappa$ and $\nu \otimes \bar{\nu}$ can both be scaled by functions $f_\kappa$ and $f_\nu$ to obtain $\Gamma$-invariant Radon measures on $(\partial^2 X \setminus \text{Diag})$.

For the harmonic measure $\nu$, we may take $f_\nu$ to be the Naim kernel defined for distinct conical points $\zeta, \xi$ as

$$\Theta(\zeta, \xi) := \lim_{s \to 1^-} \lim_{H \to 1^-} \frac{G(e, h)}{G(e, g)} = \lim_{s \to 1^-} \frac{K_\zeta(g)}{G(e, g)}.$$  

The construction is done in [20, Corollary 10.3].

For the Patterson-Sullivan measure $\kappa$, we may define a measure $m'$ on $(\partial X \times \partial X) \setminus \text{Diag}$ by

$$dm'(\zeta, \xi) := e^{2\nu_\kappa(\zeta, \xi)} \, d\kappa(\zeta) \, d\kappa(\xi)$$

where we recall $\rho^X_\kappa$ is the Gromov product. By [11, Corollary 9.4] this measure is $\Gamma$ quasi-invariant with uniformly bounded Radon-Nikodym cocycle. Hence, by a general fact in ergodic theory the Radon-Nikodym cocycle is also a coboundary (see [17], Proposition 1). Thus, there exists a $\Gamma$-invariant measure $m$ on $(\partial X \times \partial X) \setminus \text{Diag}$ in the same measure class as $m'$. In other words, one can take $f_\kappa$ to be within a bounded multiplicative constant of $e^{2\nu_\kappa(\zeta, \xi)}$. This completes the proof of Proposition 5.1. \hfill $\Box$

We are now ready to prove:
Proposition 5.3. If the harmonic measure and the Patterson-Sullivan measure are equivalent, then \(|d_G(g, g') - vd(go, go')|\) is uniformly bounded independently of \(g, g' \in \Gamma\).

Proof. It follows from Proposition 5.1 and Corollary 4.8 that if \(\kappa\) and \(\nu\) are equivalent, their respective Radon-Nikodym derivatives are bounded away from 0 and infinity. In particular, for any Borel set \(A \subset \partial X\) we have \(C^{-1}\nu(A) \leq \kappa(A) \leq C\nu(A)\), for a constant \(C > 0\). The shadow lemmas for the Patterson-Sullivan and the harmonic measures show that 
\(\kappa(Sh_r(o, go)) = e^{-vd(o, go)}\) and 
\(\nu(Sh_r(o, go)) = e^{-d_G(e, go)}\).

It follows that \(|d_G(e, g) - vd(o, go)| \leq C\) for some uniform \(C\). Since both distances are invariant by left multiplication, we have \(|d_G(g, g') - vd(go, go')| \leq C\) for any \(g, g'\).

6. Gibbs Measures

Let us now generalize the previous results to Gibbs measures. We begin by stating the relevant definitions.

Let \(X\) be a proper, geodesically complete CAT\((-1)\) space and \(\Gamma < \text{Isom}(X)\) be a nonelementary group of isometries. Let \(SX\) be the space of parameterized geodesic lines in \(X\) (that is, of geodesic embeddings \(\gamma: \mathbb{R} \to X\), which can be identified with \((\partial X \times \partial X \setminus \Delta) \times \mathbb{R}\). The unit tangent bundle \(T^1X\) is defined to be the quotient of \(SX\) obtained by identifying geodesic lines which agree on some interval around 0. See [35] for more details.

Let \(\pi: T^1X \to X\) be the projection map and \(\iota: T^1X \to T^1X\) be the direction reversing involution. Let \(F: T^1X \to \mathbb{R}\) be a \(\Gamma\)-invariant function, called a potential. For a potential \(F\), let \(\bar{F} = F \circ \iota\). Finally, given two points \(x, y \in X\), we denote as \(\int_x^y F\) the integral of \(F\) along the geodesic from \(x\) to \(y\).

The following definition is from [8, Definition 3.4].

Definition 6.1. The potential \(F: T^1X \to \mathbb{R}\) satisfies the Hölder-control (HC) property if:

(a) There exists \(c_1 > 0\) and \(c_2 \in (0, 1)\) such that for all \(x, y, x', y' \in X\) with 
\[d(x, x'), d(y, y') \leq 1\] we have 
\[\left| \int_x^{x'} F\ dt - \int_x^{x'} F\ dt \right| \leq (c_1 + \max_{\pi^{-1}(B(x,d(x,x')))}(\bar{F}))d(x, x')^{c_2} + (c_1 + \max_{\pi^{-1}(B(y,d(y,y')))}(\bar{F}))d(y, y')^{c_2}\]

(b) The potential \(F\) has subexponential growth: for each \(a > 1\) there is a \(b > 0\) such that 
\[|F(x) - F(y)| \leq ba^{d(\pi(x), \pi(y))}\]

The HC property is satisfied, for instance, by any Hölder potential when \(X\) is a contractible manifold of pinched negative curvature [8, Proposition 3.5]. From now on, \(F\) will be assumed to satisfy the HC property.

Define the \(F\)-ake metric between two points \(x, y \in X\) as 
\[d_F(x, y) := \int_x^y F\ dt.\]

We now define the topological pressure of \(F\) as 
\[v_F := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{g \in S_n} e^{d_F(o, go)}.\]
where $S_n := \{g \in \Gamma : n - 1 \leq d(o, go) \leq n\}$. We assume $v_F < \infty$. Given a boundary point $\zeta \in \partial X$, let us define the Gibbs cocyle $\beta^F_\zeta$ as

$$\beta^F_\zeta(x, y) := \lim_{z \to \zeta} (d_F(x, z) - d_F(y, z)).$$

The limit exists by [8, Proposition 3.10]. Note that by definition we have the cocycle property

$$\beta^F_\zeta(x, z) = \beta^F_\zeta(x, y) + \beta^F_\zeta(y, z)$$

for any $x, y, z \in X$, $\zeta \in \partial X$. Fix a basepoint $o \in X$. The associated Gibbs density is a $\Gamma$ quasi-invariant probability measure $\kappa_F$ on $\partial X$ such that

$$\frac{dg_{\kappa_F}}{d\kappa_F}(\zeta) = \exp(-\beta^F_\zeta(o, go) + v_F \beta_\zeta(o, go)).$$

The unit tangent bundle of $X$ is defined as $T^1X := (\partial X \times \partial X \setminus \Delta) \times \mathbb{R}$ and on it there is a natural action of the geodesic flow. When there exists a flow invariant probability measure $m_F$ on $T^1M := T^1X / \Gamma$ realizing the topological pressure $v_F$, its lift $\tilde{m}_F$ to $T^1X$ is the unique, up to scaling, flow invariant measure equivalent to $\kappa_F \times \kappa_F$. On the other hand, when the latter construction projects to an infinite measure on $T^1M$, no finite measure on $T^1M$ realizing the topological pressure exists. The pair $(\Gamma, F)$ is said to be of divergence type if the series

$$Q_{\Gamma, F}(s) := \sum_{g \in \Gamma} e^{v_\Gamma(F - s)}$$

diverges at its critical exponent $s = v_F$. In that case, there is a unique Gibbs density $\kappa_F$, and it is obtained as the weak limit of measures $Q_{\Gamma, F}(s)^{-1} \sum_{g \in \Gamma} e^{v_\Gamma(F - s)} \delta_{go}$ as $s \to v_F$. This is in particular the case when there exists a probability measure $m_F$ on $T^1M$ realizing the topological pressure $v_F$. See [8], [35] for details.

When $(\Gamma, F)$ is of divergence type, $\kappa_F$ is $\Gamma$ ergodic and gives full weight to conical limit points. Otherwise, $\kappa_F$ gives zero weight to conical limit points [8, Theorem 4.5].

Let us start with a few consequences of the HC property.

**Lemma 6.2.** If $F$ satisfies the (HC) property then for all and $x, y, z \in X$ we have

$$|d_F(x, z) - d_F(y, z)| \leq (c_1 + \max_{\pi^{-1}(B(x, d(x, y)))} |F|)(d(x, y) + 1)$$

and

$$|d_F(x, z) - d_F(z, y)| \leq (c_1 + \max_{\pi^{-1}(B(x, d(x, y)))} |F|)(d(x, y) + 1).$$

**Proof.** Let $N := [d(x, y)]$ and pick $p_0, p_1, \ldots, p_N$ points on $[x, y]$ with $p_0 = x$, $p_N = y$ and $d(p_i, p_{i+1}) \leq 1$ for $0 \leq i \leq N - 1$. Then by Definition 6.1 a) we have

$$|d_F(p_i, z) - d_F(p_{i+1}, z)| \leq c_1 + \max_{\pi^{-1}(B(x, d(x, y)))} |F|,$$

hence

$$|d_F(x, z) - d_F(y, z)| \leq (c_1 + \max_{\pi^{-1}(B(x, d(x, y)))} |F|)(d(x, y) + 1).$$

The second inequality is proved identically. 

The following statement is essentially the same as [8, Proposition 3.10(4)], but we give its proof for completeness.

---

1 The comparison with [8] is given by the formula $C_\zeta(x, y) = -\beta^F_\zeta(x, y) + v_F \beta_\zeta(x, y)$. 

---

This is a pre-publication version of this article, which may differ from the final published version. Copyright restrictions may apply.
Proposition 6.3. Let $F$ be a potential which satisfies the (HC) property. Then there exists $c_1 > 0$ such that for all $r > 0$, $x, y \in X$ and $\xi \in Sh_r(x, y)$ we have

$$|\beta^F_\xi(x, y) - d_F(x, y)| \leq 2(c_1 + \max_{\pi^{-1}(B(y, r))} |F|)(r + 1).$$

Proof. Suppose $\xi \in Sh_r(x, y)$. Let $p$ be the closest point to $y$ on $[x, \xi]$, so that $d(p, y) \leq r$. Then $\beta^F_\xi(x, p) = d_F(x, p)$ and so by the cocycle property of $\beta_F(\cdot, \cdot)$ we have:

$$|\beta^F_\xi(x, y) - d_F(x, y)| = |\beta^F_\xi(x, y) - \beta^F_\xi(x, p) + d_F(x, p) - d_F(x, y)|$$

$$\leq |\beta^F_\xi(p, y)| + |d_F(x, p) - d_F(x, y)|.\tag{5}$$

In view of Lemma 6.2 each of the two terms is bounded by

$$\left(c_1 + \max_{\pi^{-1}(B(y, r))} |F|\right)(r + 1),$$

completing the proof. \hfill \Box

The following version of the shadow lemma for $\kappa_F$ is proved in [8, Lemma 4.2].

Proposition 6.4. Let $F : T^1X \to \mathbb{R}$ be a potential which satisfies the (HC) condition, and let $o \in X$. Then:

(a) For large enough $r > 0$ we have for any $g \in \Gamma$

$$\kappa_F(Sh_r(o, go)) \approx e^{d_F(o, go) - v_Fd(o, go)},$$

where the implied constant depends only on $r, o$.

(b) There exists a constant $C$ such that for any $n$

$$\sum_{g \in S_n} e^{d_F(o, go) - v_Fd(o, go)} \leq C,$$

where $S_n := \{g \in \Gamma : n - 1 \leq d(o, go) \leq n\}$.

6.1. Existence of the F-ake drift. We will now show:

Theorem 6.5. Let $\Gamma$ be a countable group of isometries of a CAT(-1) metric space $X$, let $\mu$ be a probability measure on $\Gamma$ with finite exponential moment, and let $F : T^1X \to \mathbb{R}$ be a potential satisfying the (HC) property. Then for any $o \in X$ and for almost every sample path $\omega$ the limit

$$l_{F, \mu} := \lim_{n \to \infty} \frac{d_F(o, \omega_n o)}{n}$$

exists and is finite and independent of the sample path.

We will need the following.

Lemma 6.6. For all large enough $t > 0$ one has the estimate

$$P(|\beta^F_{\omega_n}(o, \omega_n o) - d_F(o, \omega_n o)| > t) \leq 1/t^4.$$

Proof. Proposition 6.3 and the subexponential growth of the potential $F$ provides a subexponentially growing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|\beta^F_{\omega}(o, go) - d_F(o, go)| \leq f(D)$ whenever $\xi \in Sh_D(o, go)$. Proposition 4.4 provides a $b > 1$ such that for all large enough $D$ (independent of $n$) and all $n$ we have $P(\omega_n \notin Sh_D(o, \omega_n o)) \leq b^{-D}$. On the other hand, for large enough $D$ we have $f(D) \leq b^{D/4}$. Consequently $P(|\beta^F_{\omega}(o, \omega_n o) - d_F(o, \omega_n o)| > b^{D/4}) \leq b^{-D}$. Letting $t = b^{D/4}$ completes the proof. \hfill \Box
For each $n$ define $\beta_n(\omega) := \beta_{\omega_n}^F(o, \omega_n o)$.

**Lemma 6.7.** Each $\beta_n(\omega)$ is $P$-integrable.

**Proof.** Let $\sigma$ be the left shift on the space of increments. Note,

$$\beta_{n+m}(\omega) = \beta_n(\omega) + \beta_m(\sigma^n \omega).$$

Thus, the $\sigma$ invariance of $P$ implies that $\int \beta_n dP = n \int \beta_1 dP$. and so it suffices to show $\beta_1$ is $P$-integrable. For this, it suffices to prove the integrability of:

(a) $\omega \rightarrow |d_F(o, \omega_1 o)|$
(b) $\omega \rightarrow |\beta_{\omega_n}^F(o, \omega_1 o) - d_F(o, \omega_1 o)|$.

The second is an immediate consequence of Lemma 6.6. For the first, note that Definition 6.1 implies for any $\alpha > 1$

$$|d_F(o, \omega_1 o)| \leq a d(o, \omega_1 o) \leq a K |\omega_1|$$

for a constant $K$, so that the integrability follows from the exponential moment assumption on $\mu$.

$\square$

**Proof of Theorem 6.5.** For a sample path $\omega = (\omega_n)$ converging to $\omega_\infty \in \mathcal{E}X$ we have

$$\beta_{\omega_n}^F(o, \omega_{n+m} o) = \beta_{\omega_n}^F(o, \omega_n o) + \beta_{\omega_n}^F(\omega_n o, \omega_{n+m} o) =$$

so we get

$$\beta_{n+m}(\omega) = \beta_n(\omega) + \beta_m(\sigma^n \omega)$$

where $\sigma$ is the left shift on the space of increments. Thus, Kingman’s subadditive ergodic theorem and the ergodicity of $\sigma$ imply that $\beta_{\omega_n}^F(o, \omega_n)/n$ almost surely converges as $n \rightarrow \infty$ to a constant $\ell_{F, \mu}$ independent of $\omega$.

The result will now follow by a Borel-Cantelli type argument. Indeed, let $A_n(\omega) = |\beta_{\omega_n}^F(o, \omega_n o) - d_F(o, \omega_n o)|$. Then Lemma 6.6 implies $P(A_n(\omega) > \sqrt{n}) \leq 1/n^2$ for all large enough $n$. Consequently by the summability of $1/n^2$, Borel-Cantelli implies that the set of sample paths $\omega$ such that $A_n(\omega) > \sqrt{n}$ for infinitely many $n$ has measure 0. Thus, for almost every sample path $\omega$ we have $\lim_{n \rightarrow \infty} A_n(\omega_n)/n = 0$, which implies $d_F(o, \omega_n o)/n \rightarrow \ell_{F, \mu}$.

$\square$

**6.2. The Guivarc’h inequality for Gibbs measures.**

**Theorem 6.8.** Let $\Gamma$ be a countable group of isometries of a $\text{CAT}(-1)$ metric space $X$ and let $\mu$ be a probability measure on $\Gamma$ with finite exponential moment. Then we have

$$h_\mu \leq \ell_{\mu} v_F - \ell_{F, \mu}.$$ 

**Proof.** Fix $\epsilon > 0$ and $n \geq 1$. Define $S_k := \{ g : d(o, go) \in [cn(k-1), cnk) \}$ for any $k \geq 1$.

$$\sum_k \mu^\alpha(S_k) cn(k-1) \leq \sum_g \mu^\alpha(g) d(o, go) \leq \sum_k \mu^\alpha(S_k) cnk$$

so by dividing by $n$ and taking the limit as $n \rightarrow \infty$ we have

$$\ell_{\mu} \leq \liminf_n \sum_k \mu^\alpha(S_k) ekn \leq \limsup_n \sum_k \mu^\alpha(S_k) ekn \leq \ell_{\mu} + \epsilon.$$ 

For any measure $\mu$ (not necessarily a probability) one has by Jensen’s inequality

$$\sum_{g \in A} \mu(g) \log \left( \frac{e^{d_F(o, go)}}{\mu(g)} \right) \leq \mu(A) \log \left( \sum_{g \in A} e^{d_F(o, go)} \right) - \mu(A) \log \mu(A)$$

This is a pre-publication version of this article, which may differ from the final published version. Copyright restrictions may apply.
so, if we denote $H(\mu) := -\sum g \mu(g) \log \mu(g)$, we obtain

$$H(\mu^n) + \sum_g \mu^n(g) d_{\mathcal{F}}(o, go) \leq \sum_k \mu^n(S_k) \log \left( \sum_{g \in S_k} e^{d_{\mathcal{F}}(o, go)} \right) - \sum_k \mu^n(S_k) \log \mu^n(S_k).$$

Now, by definition of critical exponent there exists $C_\varepsilon$ such that

$$\sum_{d(o, go) \leq R} e^{d_{\mathcal{F}}(o, go)} \leq C_\varepsilon e^{(v_F + \varepsilon) R} \quad \text{for any } R \geq 0$$

so

$$\log \left( \sum_{g \in S_k} e^{d_{\mathcal{F}}(o, go)} \right) \leq (v_F + \varepsilon) kn + \log C_\varepsilon.$$ 

Moreover,

$$- \sum_k \mu^n(S_k) \log \mu^n(S_k) = - \sum_{\mu^n(S_k) \leq \frac{1}{e^k}} \mu^n(S_k) \log \mu^n(S_k) - \sum_{\mu^n(S_k) > \frac{1}{e^k}} \mu^n(S_k) \log \mu^n(S_k) \leq$$

$$\leq \sum_{k=1}^\infty \frac{2 \log k + 1}{ek^2} + \sum_{k=1}^\infty \mu^n(S_k)(2 \log k + 1) \leq C$$

is bounded independently of $n$. Hence

$$h_\mu + \ell_{F,\mu} = \lim_{n \to \infty} \frac{H(\mu^n) + E_{\mu^n}[d_{\mathcal{F}}(o, go)]}{n} \leq$$

$$\leq (v_F + \varepsilon) \limsup_n \sum_k \mu^n(S_k) ek \leq (v_F + \varepsilon)(\ell_\mu + \varepsilon)$$

so the claim follows by taking $\varepsilon \to 0$.

**Remark 6.1.** Let us point out that in the previous proof we used that $\mu$ has finite exponential moment only to make sure that

\begin{equation}
\lim_{n \to \infty} E_{\mu^n}[d_{\mathcal{F}}(o, go)] = \ell_{F,\mu}
\end{equation}

exists. Hence, the Guivarc'h inequality for Gibbs measures holds as long as $\mu$ has finite first moment and (7) is true. Moreover, the assumption that $X$ is $\text{CAT}(-1)$ is also not strictly needed, as the essential property is that $\ell_{F,\mu}$ exists.

### 6.3. Entropy and drift for Gibbs measures.

We will now prove the following analogue of Theorem 4.1 for Gibbs states, which is a generalization of Theorem 1.1 in the Introduction.

**Theorem 6.9.** Let $X$ be a proper $\text{CAT}(-1)$ metric space, and let $\Gamma$ be a geometrically finite group of isometries of $X$. Let $\mu$ be a probability measure generating $\Gamma$ with finite superexponential moment, and let $\nu_\mu$ be the hitting measure of the random walk driven by $\mu$. Let $F : T^1 X \to \mathbb{R}$ be a potential which satisfies the (HC) property, and let $\kappa_F$ be the corresponding Gibbs density. Then

$$h_\mu \leq \ell_{\mu} v_F - \ell_{F,\mu}.$$ 

Moreover, the following conditions are equivalent.

1. The equality

$$h_\mu = \ell_{\mu} v_F - \ell_{F,\mu}$$

holds.
(2) The measures \( \nu_\mu \) and \( \kappa_F \) are in the same measure class.
(3) The measures \( \nu_\mu \) and \( \kappa_F \) are in the same measure class with Radon-Nikodym derivatives bounded from above and below.
(4) For any basepoint \( o \in X \), there exists \( C \geq 0 \) such that for every \( g \in \Gamma \),
\[
|d_G(e, g) - v_Fd(o, go) + d_F(o, go)| \leq C.
\]

Let us start with the proof. For a sample path \( \omega \) let \( \omega_n \) be its \( n \)-th position. Define then
\[
\phi_n = \phi_n(\omega) = \frac{\kappa_F(Sh_r(o, \omega_n o))}{\nu(Sh_r(o, \omega_n o))}.
\]
Let \( \psi_n = \log \phi_n \). We have the following analogue of Proposition 4.3.

**Proposition 6.10.** There exists \( C_1 > 0 \) such that for any \( N \geq 1 \) we have
\[
\frac{1}{N} \sum_{n=1}^{N} E(\phi_n) \leq C_1.
\]

**Proof.** The proof is the same as that of Proposition 4.3 except in the end one uses Proposition 6.4 to conclude that
\[
\sum_{g : d(o, go) \leq kN} \kappa_F(Sh_r(o, go)) \leq N.
\]

**Lemma 6.11.** There is a function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) of subexponential asymptotic growth (i.e. such that \( \lim_{r \to \infty} f(r)/c^r = 0 \) for all \( c > 1 \)) such that \( d(g_1 o, [o, g_2 o]) < D \) implies
\[
|d_F(o, g_2 o) - d_F(o, g_1 o) - d_F(g_1 o, g_2 o)| \leq f(D)
\]
for all \( g_1, g_2 \in \Gamma \).

**Proof.** Let \( p \in [o, g_2 o] \cap N_D(g_1 o) \). Then
\[
|d_F(o, g_2 o) - d_F(o, g_1 o) - d_F(g_1 o, g_2 o)| \leq |d_F(o, g_2 o) - d_F(o, p) - d_F(p, g_2 o)| + |d_F(o, p) - d_F(o, g_1 o)| + |d_F(p, g_2 o) - d_F(g_1 o, g_2 o)|.
\]
The first summand is simply zero since \( o, p, g_2 o \) lie on a geodesic in that order. Moreover, by Lemma 6.2 and the \( \Gamma \)-invariance of \( F \), the second summand is bounded by \((D + 1)(c_1 + \max_{B_D(o)} |F|)\), and the same is true for the third summand. Furthermore, the quantity \( D \max_{B_D(o)} |F| \) is subexponential in \( D \) by Definition 6.1 b).

**Proposition 6.12.** There exists \( C_2 > 0 \) such that the sequence \( E(\psi_n) + C_2 \) is subadditive and \( \psi_n/n \) converges to \( h_\mu - \ell_\mu v_F + \ell_{F, \mu} \) almost surely and in expectation.

**Proof.** By the shadow lemmas Proposition 6.4 and Proposition 2.3,
\[
\psi_n = d_G(e, \omega_n) - v_F d(o, \omega_n o) + d_F(o, \omega_n o).
\]
Thus, \( \psi_n/n \) converges to \( h_\mu - \ell_\mu v_F + \ell_{F, \mu} \) almost surely and in expectation. Note that, since \( d_G \) satisfies the triangle inequality,
\[
\psi_{n+m} - \psi_n - \psi_m \leq v_F(d(o, \omega_n o) + d(o, \omega_m o) - d(o, \omega_{n+m} o)) + d_F(o, \omega_n o) - d_F(o, \omega_m o) - d_F(o, \omega_{n+m} o)\]
Furthermore,
\[
E(d(o, \omega_n o) + d(o, \omega_m o) - d(o, \omega_{n+m} o)) =
\]
where the expectation is uniformly bounded by Proposition 4.4. Now, it remains to bound \( E(d_F(o, \omega_{n+m}o) - d_F(o, \omega_n o) - d_F(o, \omega_m o)) \).

Lemma 6.11 provides a subexponential function \( f \) such that \( d(\omega_n o, [o, \omega_{n+m} o]) < D \) implies \( |d_F(o, \omega_{n+m} o) - d_F(o, \omega_n o) - d_F(o, \omega_m o)| < f(D) \) for all \( g_1, g_2 \in \Gamma \). Thus, Proposition 4.4 implies that there is a constant \( C \) with

\[
P(|d_F(o, \omega_{n+m} o) - d_F(o, \omega_n o) - d_F(o, \omega_m o)| > f(D)) < Ce^{-D/C}
\]

for all \( n, m \geq 1 \) and \( D > 0 \). Since \( f \) has subexponential growth, this implies \( E(d_F(o, \omega_{n+m} o) - d_F(o, \omega_n o) - d_F(o, \omega_m o)) \) is bounded above independently of \( m, n \), completing the proof. \( \square \)

**Proposition 6.13.** Under the hypotheses of Theorem 6.9, let \( \kappa_F \) be the Gibbs density for the potential \( F \) and \( \nu_\mu \) the hitting measure for the random walk driven by \( \mu \). Then:

a) if \( \kappa_F \) and \( \nu_\mu \) are not equivalent, then \( \phi_n \) tends to 0 in probability;

b) if \( \kappa_F \) and \( \nu_\mu \) are equivalent then \( \frac{\log \kappa_F(\mathcal{S}_r(o, \omega_n o))}{\log \nu_\mu(\mathcal{S}_r(o, \omega_n o))} \) tends to 1 in probability.

**Proof.** Identical to the proof of Proposition 4.6. \( \square \)

**Theorem 6.14.** The measures \( \kappa_F \) and \( \nu_\mu \) are equivalent if and only if

\[
h_\mu = \ell_\mu v_F - l_{F, \mu}.
\]

**Proof.** Same as proof of Theorem 4.7. \( \square \)

Recall that \( \bar{\mu} \) is the reflection of \( \mu \), and we denote as \( \bar{F} = F \circ \ell \) the reflected potential.

**Corollary 6.15.** The measure \( \nu_{\bar{\mu}} \) is equivalent to \( \kappa_{\bar{F}} \) if and only if \( \nu_\mu \) is equivalent to \( \kappa_F \).

**Proof.** Note that \( h_{\bar{\mu}} = h_\mu, \ell_{\bar{\mu}} = \ell_\mu \) and \( v_{\bar{F}} = v_F \). Furthermore, \( l_{\bar{F}, \bar{\mu}} = l_{F, \mu} \). Consequently \( h_{\bar{\mu}} = \ell_\mu v_F - l_{F, \mu} \) if and only if \( h_{\bar{\mu}} = \ell_{\bar{\mu}} v_{\bar{F}} - l_{\bar{F}, \bar{\mu}} \). Theorem 6.14 now implies the result. \( \square \)

**Proposition 6.16.** If \( \nu_\mu \) is equivalent to \( \kappa_F \), then the Radon-Nikodym derivative \( d\kappa_F/d\nu_\mu \) is bounded away from 0 and infinity.

**Proof.** If \( \nu_\mu \) is equivalent to \( \kappa_F \) then \( \nu_{\bar{\mu}} \) is equivalent to \( \kappa_{\bar{F}} \). We need to show that \( \kappa_F \circ \kappa_{\bar{F}} \) can be scaled by a bounded function \( f \) to give a \( \Gamma \)-invariant Radon measure on \( \mathcal{S}^2 X \). This is done in [8, Equation 4.4]. The proof is now the same as that of Proposition 5.1. \( \square \)

**Proposition 6.17.** If \( \kappa_F \) and \( \nu_\mu \) are equivalent, then

\[
|d_G(g, g') - v_F d(g, g'o) + d_F(g, go')|
\]

is uniformly bounded independently of \( g, g' \in \Gamma \).
Proof. If $\kappa_F$ and $\nu_\mu$ are equivalent then their Radon-Nikodym derivative is bounded away from 0 and infinity. Consequently, the ratio satisfies

$$C^{-1} \leq \frac{\kappa_F(Sh_r(o,g_0))}{\nu_\mu(Sh_r(o,g_0))} \leq C$$

for some $C > 0$ independent of $g$. The shadow lemmas Propositions 2.3 and 6.4 now imply the result, together with the fact that all metrics we use are $\Gamma$-invariant. \qed

6.4. Growth of parabolics and singularity of harmonic measure. We will prove the following.

Proposition 6.18. Let $P \leq \Gamma$ be a parabolic subgroup. There are $c > 1$, $D > 0$ such that $|g| \geq Dd(o,g_0)$ for all $g \in P$.

Together with Theorem 6.9 this will imply:

Corollary 6.19. If $\Gamma \hookrightarrow X$ has parabolics then $\kappa_F$ and $\nu_\mu$ are mutually singular.

It remains to prove Proposition 6.18. First, recall that Osin [32, Proposition 2.27] showed that if $\Gamma$ is finitely generated so are the stabilizers of any parabolic point of $\partial X$, called maximal parabolic subgroups. Choose a symmetric finite generating set for $P$ and let $\|\cdot\|_P$ be the associated word metric on $P$. The following can be found in Drutu-Sapir [14] or Gerasimov-Potyagailo [21, Corollary 3.9].

Lemma 6.20. A maximal parabolic subgroup $P \leq \Gamma$ is quasi-convex. In particular, $\|g\|_P \approx \|g\|_F$ for all $g \in P$.

The following can be found in Bridson-Haefliger [7, Proposition I.8.25] in the context of CAT(0) spaces.

Lemma 6.21. Let $P$ be a maximal parabolic subgroup which stabilizes $\zeta \in \partial X$. Then for any $g \in P$ and $x, y \in X$ we have $\beta_\zeta(x,gy) = \beta_\zeta(x,y)$.

For $x \in X$ and $\zeta \in \partial X$ the horosphere $H_\zeta(x)$ through $x$ centered at $\zeta$ is defined to be the set of $z \in X$ with $\beta_\zeta(x,z) = 0$. The associated (open) horoball $B(x,\zeta,t)$ is defined to be the set of $z \in X$ with $\beta_\zeta(z,x) < -t$.

Proposition 6.22. There exists a $K > 0$ depending only on the hyperbolicity constant of $X$ such that if $\beta_\zeta(x,z) = 0$ and $d(x,z) \geq 2K + 4r$ then $B(x,\zeta,r)$ contains the ball of radius $d(x,z)/2 - K - 2r$ around the midpoint of $[x,z]$.

Proof. Let $\alpha$ be a geodesic in $X$ from $x$ to $\zeta$. By definition, the pair $(H_\zeta(x),x)$ is the pointed Gromov-Hausdorff limit of the pairs $(S(\alpha(n),n),x)$, where $S(\alpha(n),n)$ is the sphere of radius $n$ centered at $\alpha(n)$. Furthermore, $B(x,\zeta,r)$ is the limit of balls $B(\alpha(n+r),n)$. Thus, it suffices to show that for large enough $t$, for any $z \in S(\alpha(t),t)$, $B(\alpha(t+r),t)$ contains the ball of radius $d(x,z)/2 - K - 2r$ around the midpoint of $[x,z]$. Consider the geodesic triangle with vertices $x,z,\alpha(t)$. By Gromov hyperbolicity there is a $p \in X$ which is within the hyperbolicity constant $\delta$ of all three sides of the triangle. Then $t = d(x,\alpha(t)) \approx d(x,p) + d(p,\alpha(t))$ and $t = d(z,\alpha(t)) \approx d(z,p) + d(p,\alpha(t))$. Thus, $d(x,p) \approx d(z,p) \approx d(x,p)$ so $p$ is within $3\delta$ of a midpoint $q$ of $[x,z]$. Thus,

$$d(q,\alpha(t)) \approx t - d(x,q) = t - d(x,z)/2.$$  

It follows that $d(q,\alpha(t+r)) \leq t + r + 5\delta - d(x,z)/2$. The result follows with $K = 6\delta$. \qed
We are now ready to prove Proposition 6.18. Indeed, by hyperbolicity of $X$ (see e.g. [7, Proposition III.H.1.6]) there are $c > 1, D > 1$ such that for any $x, z, y$ along an $X$ geodesic (in that order) any path from $x$ to $y$ disjoint from $B_R(z)$ has length at least $Dc^R$. Consider a maximal parabolic subgroup $P$ and $g \in P$.

Let $S_P$ be a finite generating set for $P$, and $T := \max_{p \in S_P} d(o, po)$. Let $p_1, \ldots, p_n$ be an $(P, S_P)$ geodesic from $p_0 = e$ to $p_n = g$. For each $i$, let $\gamma_i$ be a geodesic in $X$ from $p_i o$ to $p_{i+1} o$, and let $\gamma$ be the concatenation of the $\gamma_i$.

Then $\gamma$ is a path in $X$ from $o$ to $go$ outside of $B(o, \zeta, T)$ of length at most $T \| g \|_P \approx \| g \|$. In particular, this path does not intersect the ball of radius $d(o, go)/2 - K - 2T$ about the midpoint of $[o, go]$. It follows that $\| g \|$ is bounded below by a constant times $D e^{d(o, go)/2 - K - 2T}$ completing the proof.

7. Appendix

7.1. Exponential deviation estimates. In this section we prove Proposition 4.4.

We assume $\Gamma \sim X$ is a nonelementary action on a proper geodesic Gromov hyperbolic space. Furthermore, $\mu$ is a probability measure on $\Gamma$ with finite exponential moment and support generating $\Gamma$ as a semigroup.

**Proposition 7.1** (Proposition 4.4). Let $\nu$ be the $\mu$-stationary measure on $\partial X$. For each $o \in X$ there exists a $C > 0$ such that for each $0 \leq k \leq n$ and $a > 1$ we have

$$P(d(\omega_k o, [o, \omega_n o]) > a) \leq C e^{-a/C}$$

and

$$P(d(\omega_k o, [o, \omega_n o]) > a) \leq C e^{-a/C}$$

where $[o, \omega_n o]$ and $[o, \omega_x]$ are any geodesics connecting the respective endpoints.

To prove the proposition we will need the following lemmas.

**Lemma 7.2.** [3, Remark 4.4] There is a $t > 0$ such that

$$\sup_{y \in \partial X} \int_{x \in \partial X} e^{t \rho_o(x,y)} \, d\nu(x) < \infty.$$ 

**Lemma 7.3.** The same is true if the supremum is taken over all $y \in \Gamma o \cup \partial X$.

**Proof.** Note for $o, o' \in X$ and $x \in X \cup \partial X$ and $g \in \Gamma$ we have

$$|\rho_o(x, go') - \rho_o(x, go)| \lesssim 2d(o, o').$$

It thus suffices to prove the lemma for any particular basepoint $o \in X$. Since $X$ is a proper, Gromov hyperbolic space, it has a bi-infinite geodesic $\alpha$. We assume without loss of generality that $o \in \alpha$. This means any $go$ lies on a bi-infinite geodesic $g\alpha$.

We claim that for any $y \in X$, $x \in \partial X$ and any bi-infinite geodesic $\alpha$ containing $y$ we have

$$\rho_o(x, y) \lesssim 2 \max(\rho_o(x, \alpha_+), \rho_o(x, \alpha_-)).$$

To that end, let $p_+$ and $p_-$ be points on $(x, \alpha_+)$ and $(x, \alpha_-)$, respectively, at minimal distance from $o$. By Gromov hyperbolicity, each $p_\pm$ is within the hyperbolicity constant $\delta$ of either $[y, x]$ or $[y, \alpha_\pm] \subset (\alpha_-, \alpha_)$. If $p_+$ is within $\delta$ of $[y, x]$ then

$$d(o, [y, x]) \leq d(o, p_+) + \delta = d(o, [x, \alpha_+]) + \delta$$

and so $\rho_o(x, y) \lesssim \rho_o(x, \alpha_+)$. Similarly, if $p_-$ is within $\delta$ of $[y, x]$, then $\rho_o(x, y) \lesssim \rho_o(x, \alpha_-)$. We are left to consider the case where each $p_\pm$ is within $\delta$ of some
Lemma 7.4. Let $D = \max(d(p_+, o), d(p_-, o))$. Then $d(p_+, p_-) \leq 2D$ and so $d(q_+, q_-) \leq 2D + 2\delta$. Hence, at least one of $q_\pm$, say $q_+$, is within $D + \delta$ of $y$. Thus $p_+$ is within $D + 2\delta$ of $y$. Consequently,

$$d(o, [y, x]) \leq d(o, p_+) + d(p_+, y) \leq 2D + 2\delta.$$ 

Hence $\rho_o(x, y) \leq D$, proving the claim.

Now, by the claim we have

$$\int_{x \in \mathbb{X}} e^{t \rho_o(x, go)} d\nu(x) \leq \max \left( \int_{x \in \mathbb{X}} e^{2t \rho_o(x, go)} d\nu(x), \int_{x \in \mathbb{X}} e^{2t \rho_o(x, g\alpha - )} d\nu(x) \right)$$

so we conclude using Lemma 7.2. $\square$

To simplify notation, we will from now on denote $|g| := d(o, go)$. The following lemma is due to Sunderland.

Lemma 7.4. [38, Criterion 11] There is an $N_0 > 0$ such that for all $N \geq N_0$ there is a $t(N) > 0$ and an $\epsilon > 0$ such that for $0 < t < t(N)$ we have

$$\sup_{g \in \Gamma} E(e^{-t(|g| |N| - |g|)}) \leq 1 - \epsilon.$$ 

This implies:

Lemma 7.5. There is a $t_0 > 0$ and $C > 0$ such that

$$E(e^{-t(|\omega_{N+k}| - |\omega_k|)}) \leq Ce^{-N/C}$$

for all $N, k \geq 0$ and $0 < t < t_0$.

Proof. Let $N_0$ be given by Lemma 7.4. It suffices to prove for some $t > 0$ the two following conditions:

1. $\sup_{k \geq 0} E(e^{-t(|\omega_{N+k}| - |\omega_k|)}) < \infty$ for $N \leq N_0$;
2. $E(e^{-t(|\omega_{N+k}| - |\omega_k|)}) \leq (1 - \epsilon)E(e^{-t(|\omega_{N+k}| - |\omega_k|)})$ for $N, k \geq 0,$

since then the claim follows by induction.

For the first claim, note that $|\omega_{N+k}| - |\omega_k| \geq -|\omega_k^{-}\omega_{N+k}|$, which has the same distribution as $-|\omega_N|$. Thus,

$$E(e^{-t(|\omega_{N+k}| - |\omega_k|)}) \leq E(e^{t|\omega_N|}) \leq E(e^{t|\omega_1|})^N \leq E(e^{t|\omega_1|})^{N_0}$$

which for small enough $t$ is finite by the exponential moment assumption.

We now prove the second claim. Indeed,

$$E(e^{-t(|\omega_{N+k}| - |\omega_k|)}) = E(e^{-t(|\omega_{N+k}| - |\omega_k|)}e^{-t(|\omega_{N+k+N_0+k}| - |\omega_{N+k+N_0+k}|)}).$$

Conditioning on $\omega_{N+k} = h, \omega_k = g$ the last expression becomes

$$\sum_{g, h \in \Gamma} e^{-t(|h| - |g|)}E(e^{-t(|h\omega_{N_0}| - |h|)})P(\omega_{N+k} = h, \omega_k = g).$$

By Lemma 7.4, we have $E(e^{-t(|h\omega_{N_0}| - |h|)}) \leq (1 - \epsilon)$ for any $t < t(N_0)$ and thus

$$E(e^{-t(|\omega_{N+k}| - |\omega_k|)}) \leq (1 - \epsilon)E(e^{-t(|\omega_{N+k}| - |\omega_k|)}).$$

$\square$

Lemma 7.6. There is a $C > 0$ such that for all $0 \leq k \leq n$ we have

$$P(d(\omega_n, o) - d(\omega_k, o) < (n - k)/C) \leq Ce^{-(n-k)/C}.$$
Proof. The exponential Markov inequality implies for any \( D > 0 \), any \( t > 0 \) and any \( N, k \geq 0 \):
\[
P(|\omega_{N+k} - \omega_k| < DN) = P(e^{-t(|\omega_{N+k} - \omega_k|)} > e^{-tDN})
\]
hence by Lemma 7.5, by taking \( t < \min\{1, t_0\} \) and \( D = \frac{1}{2C} \),
\[
P(|\omega_{N+k} - \omega_k| < DN) \leq e^{tDN} e^{-t(|\omega_{N+k} - \omega_k|)} 
\leq Ce^{tDN - N/C} \leq 2Ce^{-N/(2C)},
\]
so the claim follows by setting \( N = n - k \) and replacing \( C \) by \( 2C \). \( \square \)

**Lemma 7.7.** There is a \( C > 0 \) such that for all \( 0 \leq k \leq n \) and \( a > 0 \) we have
\[
P(d(\omega_k, o) - d(\omega_n, o) > a) \leq Ce^{-a/C}.\]
Proof. Let \( t > 0 \) be smaller than the \( t_0 \) in Lemma 7.5 and also small enough so that \( E(e^{t|\omega|}) < \infty \). Let \( D = E(e^{t|\omega|}) \).

Suppose \( n - k \leq ta/(2\log D) \). Then we have
\[
E(e^{t(|\omega_k| - |\omega_n|)}) \leq E(e^{t|\omega_k|}) \leq E(e^{t|\omega_n|}) \leq D^{n-k}.
\]
Consequently by the Markov inequality
\[
P(|\omega_k| - |\omega_n| > a) = P(e^{t(|\omega_k| - |\omega_n|)} > e^{ta}) \leq e^{-ta} E(e^{t(|\omega_k| - |\omega_n|)} \leq e^{-ta} D^{n-k} \leq e^{-ta/2}.
\]
On the other hand, if \( n - k \geq Ka \), then Lemma 7.6 implies that
\[
P(d(\omega_k, o) - d(\omega_n, o) < -a) \leq P(d(\omega_k, o) - d(\omega_n, o) < (n - k)/C) \leq Ce^{-(n-k)/C} \leq Ce^{-Ka/C}.
\]
\( \square \)

**Proof of Proposition 7.1.** We first prove the second statement of Proposition 7.1. By Lemma 7.3, we obtain a \( C > 0 \) such that
\[
\nu(\zeta \in \partial X : \rho_o(\gamma, \zeta) > a) \leq Ce^{-a/C}
\]
for any \( g \in \Gamma \) and any \( a > 0 \). Therefore, for each \( n \) we get:
\[
P(\rho_{\omega_n, o}(\omega_x) > a) = \sum_{g \in \Gamma} P(\rho_{\omega_n, o}(\omega_x) > a | \omega_n = g) P(\omega_n = g)
\]
\[
= \sum_{g \in \Gamma} P(\rho_o(\gamma^{-1}, \omega_x) > a) P(\omega_n = g) \leq Ce^{-a/C}.
\]
We now prove the first statement of Proposition 7.1. Consider \( 0 \leq k \leq n \) and \( a > 1 \). The events \( d(\omega_k, o) < a, d(\omega_k, o) < a \) and \( d(\omega_n, o) - d(\omega_k, o) > -a \) all have probability at least \( 1 - Ce^{-a/C} \), where \( C \) does not depend on \( k, n \). Suppose these events hold. Let \( p_k, p_n \in [o, \omega_x] \) be at minimal distance from \( \omega_k, o \) and \( \omega_n, o \), respectively. Then by the triangle inequality \( d(o, p_n) - d(o, p_k) > -3a \).
If \( d(o, p_n) > d(o, p_k) \) then \( p_k \) belongs to \([o, p_n]\), so the fellow traveling property in Gromov hyperbolic spaces [7, III.1.15] implies
\[
d(K, o, \omega_n, o) \leq Kd(\omega_n, o, p_n) + K \leq Ka + K
\]
for a constant \( K \) which only depends on the hyperbolicity constant of \( X \), and without loss of generality we can assume \( K \geq 2 \). On the other hand, if \( d(o, p_n) \leq d(o, p_k) \leq d(o, p_n) + 3a \) then
\[
d(p_k, o, \omega_n, o) \leq 3a + d(\omega_n, o, p_n) \leq 4a.
\]
In either case, we have 
\[ \frac{d(o, o_n) - a}{K_a + K} \leq (2 + a)K, \]
where we used \( a \geq 1 \) and \( K \geq 2 \). Thus we obtain 
\[ P(d(o, o_n) > (2 + a)K) \leq 3Ce^{-a/C}, \]
completing, up to suitably renaming the constants, the proof of Proposition 7.1. \( \square \)

7.2. The Green metric for non-symmetric measures. We will prove the following.

**Proposition 7.8.** Let \( \mu \) be a generating probability measure with finite exponential moment on a nonamenable group \( \Gamma \). Let \( G \) be the associated Green’s function. Then there is a constant \( C > 0 \) such that
\[ e^{-C\|x^{-1}y\|/C} \leq G(x, y) \leq Ce^{-\|x^{-1}y\|/C} \]
for any \( x, y \in \Gamma \). In other words, the Green metric \( d_G(x, y) = -\log \frac{G(x, y)}{G(e, e)} \) is quasi-isometric to the word metric.

The lower bound is immediate from the Harnack inequality. We thus just need to consider the upper bound. This was proved for symmetric measures on nonelementary hyperbolic groups in [5, Proposition 3.6] and [24, Lemma 4.2]. The proof there carries over without modification for symmetric measures on nonamenable groups. However, for non-symmetric measures an extra argument is required.

**Lemma 7.9.** [12, Proposition IV.4] Let \( \mu \) and \( \mu_0 \) be two generating probability measures on a group \( \Gamma \). Assume that \( \mu_0 \) is symmetric and that there exists \( C > 0 \) such that \( \mu_0(g) \leq C\mu(g) \) and \( \mu_0^n(g) \leq Ce^{-n/C} \) for all \( g \in \Gamma \). Then there is a \( C' > 0 \) such that \( \mu^n(g) \leq C'e^{-n/C'} \) for all \( g \in \Gamma \).

**Lemma 7.10.** Let \( \mu \) be a generating measure on a nonamenable group \( \Gamma \). There is a constant \( C > 0 \) such that for all \( n \geq 0 \) and \( g \in \Gamma \) we have \( \mu^n(g) \leq Ce^{-Cn} \).

**Proof.** We first prove the lemma for symmetric measures. This is in fact done in [5, Lemma 3.6] (where it is stated for hyperbolic groups, but the proof only uses amenability). We repeat it for completeness. Since \( \Gamma \) is nonamenable, the spectral radius of the random walk is less than 1, so we have \( \mu^n(e) \leq Ce^{-Cn} \) for all \( n \). Indeed, in the symmetric case by the Cauchy-Schwarz inequality
\[ \mu^{2n}(x) = \sum_{y \in \Gamma} \mu^n(y)\mu^n(y^{-1}x) \leq \sqrt{\sum_{y \in \Gamma} \mu^n(y)^2} \sqrt{\sum_{y \in \Gamma} \mu^n(y)^2} = \sum_{y \in \Gamma} \mu^n(y)^2. \]

By the symmetry of \( \mu \) we have
\[ \sum_{y \in \Gamma} \mu^n(y)^2 = \sum_{y \in \Gamma} \mu^n(y)\mu^n(y^{-1}) = \mu^{2n}(e). \]

Thus, \( \mu^{2n}(x) \leq \mu^{2n}(e) \). Similarly,
\[ \mu^{2n+1}(x) = \sum_{y \in \Gamma} \mu(y)\mu^{2n}(y^{-1}x) \leq \sum_{y \in \Gamma} \mu(y)\mu^{2n}(e) = \mu^{2n}(e). \]

Thus, we have proved the lemma for symmetric measures.

Now, suppose \( \mu \) is not symmetric. Let \( \mu_0 \) be any symmetric generating measure on \( \Gamma \) with finite support. Let \( K > 0 \) be an odd number such that \( \bigcup_{i=1}^{K} \text{supp}(\mu^i) \) contains \( \text{supp}(\mu_0) \). Let \( \tilde{\mu} := \frac{\sum_{i=1}^{K} \mu^i}{K} \). Then we have \( \tilde{\mu}(g) \geq c\mu_0(g) \) for every \( g \), for
hence there exists a constant $c$ for any $\mu$.

Let $M(K, n)$ be the coefficient of $x^{(K+1)/2}$ in the polynomial $(x + x^2 + \cdots + x^K)^n$. Then for all $g \in \Gamma$ we have $\tilde{\mu}(g) \leq M(K, n)\mu^{(K+1)/2}(g)$. Note that $\frac{M(K, n)}{K^{n/2}}$ equals the probability $P(X_1 + \cdots + X_n = \frac{n(K+1)}{2})$ where $(X_i)$ are i.i.d. random variables uniform on $\{1, 2, \ldots, K\}$. Moreover, $X_1$ has mean $\frac{K+1}{2}$. Now, by the central limit theorem

$$\lim_{n \to \infty} \sqrt{n}P(X_1 + \cdots + X_K = n(K+1)/2) > 0$$

hence there exists a constant $c_1$ such that

$$\frac{M(K, n)}{K^{n/2}} \leq c_1 \sqrt{n}$$

for any $n$. Thus,

$$\mu^{(K+1)/2}(g) \leq \frac{K^n}{M(K, n)}\tilde{\mu}(g) \leq c_1 \sqrt{n}e^{-n/c} \leq C^n e^{-n/c}$$

for a constant $C^n$. Furthermore, for $i \leq (K + 1)/2$ we have

$$\mu^{i+(K+1)n/2}(g) = \sum_{h \in \Gamma} \mu^{(K+1)n/2}(h^{-1}g)\mu^i(h) \leq C^n e^{-n/c}.$$ 

This completes the proof. $\square$

The proof of exponential decay of Green’s function now follows as in [5, Proposition 3.6] or [24, Lemma 4.2]. We reproduce it for completeness.

We assumed that $E_{\mu} \exp(\lambda\|g\|) = E < \infty$ for a given $r > 0$. Note, for any $k \leq n$

$$\left|\omega_k\right| \leq \sum_{i=1}^{n-1} \left|\omega_i^{-1}\omega_{i+1}\right|$$

and the increments $\omega_i^{-1}\omega_{i+1}$ are independent random variables and all follow the same law as $\omega_1$. Therefore for any $b > 0$ the exponential Chebyshev inequality implies

$$P\left(\sup_{1 \leq k \leq n} \|\omega_k\| \geq nb\right) \leq e^{-nb} E \left[\exp\left( r \sup_{1 \leq k \leq n} \|\omega_k\|\right)\right] \leq e^{-nb} E^n = e^{(-rb - \log E)n}.$$ 

We choose $b$ large enough so that $c = rb - \log E > 0$. Then

$$G(c, g) = \sum_{k=1}^{\infty} \mu^k(g) = \sum_{1 \leq k \leq |g|/b} \mu^k(g) + \sum_{k > |g|/b} \mu^k(g).$$

By Lemma 7.10 we have for the second summand the bound:

$$\sum_{k > |g|/b} \mu^k(g) \leq \sum_{k > |g|/b} Ce^{-k/C} \leq De^{-|g|/D}$$

for some constant $D > 0$. Meanwhile, for the first summand we have the bound

$$\sum_{1 \leq k \leq |g|/b} \mu^k(g) \leq \frac{|g|}{b} \sup_{1 \leq k \leq |g|/b} \mu^k(g) \leq \frac{|g|}{b} P\left(\sup_{1 \leq k \leq |g|/b} \|\omega_k\| \geq \|g\|\right) \leq \|g\|E^{-c|g|/b}.$$ 

As both summands decay exponentially in $|g|$, the proof is complete.
References

[8] Anne Broise-Alamichel, Jouni Parkkonen, and Frédéric Paulin. Equidistribution and counting under equilibrium states in negatively curved spaces and graphs of groups. Applications to non-archimedean diophantine approximation. hal-01421211, 2016.

University of Toronto, 40 St George St, Toronto ON, Canada
E-mail address: ilyagekh@gmail.com

University of Toronto, 40 St George St, Toronto ON, Canada
E-mail address: tiozzo@math.utoronto.ca