Project: Making Pancakes.

Prerequisites: Completion of Section 11.7 of *Differential Equations: Techniques, Theory, and Applications* by MacCluer, Bourdon, and Kriete.

I have a stainless steel frying pan whose circular base fits perfectly over one of the circular heating disks (HD) on my ceramic range-top. Assume one fine morning I've put the coffee on and am ready to make pancakes. I slide my frying pan over HD, which I assume provides a constant temperature of 180 degrees Celsius, and I immediately cover the bottom of my pan with pancake batter. Assume the pan (as well as the batter) is at room temperature: 20 degrees Celsius. Assume the pancake batter starts to cook when the cooking surface of the frying pan reaches 170 degrees Celsius. The base of my pan is 1/2 cm thick.

Your mission: I want you to obtain a good estimate on the amount of time for the batter to begin cooking.

Envision the 1/2 cm thick base of the pan as a stainless-steel "rod" with "left-endpoint" held at 180 degrees and "right-endpoint" insulated (by the pancake batter). This "rod" is quite short (1/2 cm in length) and quite wide (its cross section is the circular base of the pan). Since the lateral surface area of the rod is small, assume little heat is lost through its lateral surface; that is, assume the rod is laterally insulated.

If we let $u(x,t)$ be the temperature of the rod at position x and time t, then $u(x,t)$ satisfies the boundary-initial value problem

$$\begin{aligned} u_t(x,t) &= \kappa u_{xx}(x,t), \quad 0 < x < 1/2, \quad t > 0 \\ u(0,t) &= 180, \;\; u_x(1/2,t) = 0, \quad t > 0 \\ u(x,0) &= 20, \quad 0 < x < 1/2. \end{aligned} \tag{1}$$

The constant κ is the thermal diffusivity of stainless steel, which we will take to be $\kappa = 0.04$ cm^2/sec.

(a) Find the steady-state temperature $S(x)$ of the rod. Recall that a steady-state solution is one that does not change with time, so the solution depends only on x, not t. While it may be intuitively obvious to you that $S(x) = 180$, verify this is correct by determining the ODE and boundary conditions that $S(x)$ must satisfy and then solving the resulting BVP.

(b) Check that $u(x,t)$ is a solution to (1) if and only if the function $q(x,t)$ defined by $q(x,t) = u(x,t) - 180$ satisfies

$$\begin{aligned} q_t(x,t) &= \kappa q_{xx}(x,t), \quad 0 < x < 1/2, \quad t > 0 \\ q(0,t) &= 0, \;\; q_x(1/2,t) = 0, \quad t > 0 \\ q(x,0) &= -160, \quad 0 < x < 1/2. \end{aligned} \tag{2}$$

Solve this boundary-initial value problem for $q(x,t)$. If you have worked Exercise 10(a) in Section 11.2 you may find that result helpful here. You will also need the result in Exercise 25 of Section 11.6. If you have done that exercise, you may just use the result. If you have not done the exercise, work through it in the particular setting we have here, where $f(x) = -160$ and $L = 1/2$ (where the computations are particularly simple).

Once you have found $q(x,t)$, then $u(x,t) = q(x,t)+180$ is the desired solution to the boundary-initial value problem in Equation (1).

(c) Denote by $\text{Approx}_m(x,t)$ the approximation to $u(x,t)$ obtained as

$$180 + \text{ the first } m \text{ nonzero terms in the series for } q(x,t).$$

Use the 3D plotting capability of your CAS to plot $\text{Approx}_{100}(x,t)$ for $0 < x < 1/2$ and $0 < t < 20$. The relevant plot commands in *Mathematica* are illustrated in MakingPancakesMathematica.nb.

(d) Now its time to accomplish your mission. When does that pancake start to cook? We want to find t so that $u(1/2,t) = 170$. Let's try an approach that is easy to carry out by hand. Solve

$$\text{Approx}_1(1/2,t) = 170$$

by hand; note that $\text{Approx}_1(1/2,t)$ is just 180+ the first term of the series for $q(1/2,t)$. Call your solution t_0.

(e) How good is the approximate time t_0 to begin cooking that you obtained in (d)? In this part you will show that

$$|u(1/2,t_0) - \text{Approx}_1(1/2,t_0)| < 10^{-9}, \tag{3}$$

so that the approximation is very good indeed. Obtain this estimate by using the following theorem from Calculus (called the Alternating Series Theorem):

Theorem 0.1. *Suppose the series* $\sum(-1)^{k+1}c_k$ *satisfies* $c_k > 0$, $c_{k+1} \le c_k$ *and* $\lim_{k\to\infty} c_k = 0$. *Then the series converges, and if its sum is approximated by the sum of the first* m *terms of the series, the error is no larger than the absolute value of the* $(m+1)$*st term of the series.*

Write $u(1/2,t_0)$ as an infinite series and verify that this series satisfies the conditions of the theorem. Apply the theorem to obtain the estimate in Equation (3).