are printed on foolscap (one side only), with the usual machine spacing of one-sixth of an inch.

These tables were prepared about 1933–34, while I was engaged on producing the Br. Ass. Adv. Sci., *Mathematical Tables*, v. 6, *Bessel Function*, part I, 1936, in which these values appear (with their second differences) to 10D. The part up to $x=15\cdot5$ was formed by subtabulating to tenths the 12-figure table of Ernst Meissel, as given originally in the *Berliner Abhandlungen* for 1888, and reprinted in A. Gray and G. B. Mathews, *A Treatise on Bessel Functions*, London, Macmillan, 1895; 2nd ed. by Gray and T. M. Macrobert, 1922. For the range $x=15\cdot5$ to $x=25$, the sources were a manuscript table lent by H. T. Davis, original calculations based on Meissel's table of $J_n(x)$ for integral values of n and x (in Gray and Mathews), and comparison with Hayashi's *Tafeln der Besselschen, Theta, Kugel- und anderer Funktionen*, Berlin, Springer, 1930. The latter contains 22 errors in this range.

The tables were a stepping stone to the published 10-figure tables mentioned. They are, however, being retained in case these values should ever be required to more decimals.

L. J. C.

MECHANICAL AIDS TO COMPUTATION

2[Z].—S. Lilley, "Mathematical Machines," *Nature*, v. 149, 25 Apr. 1942, p. 462–465.

This is a pleasantly written survey article, beginning with the "rise of modern arithmetic" (based on material in Stevin's decimal arithmetic of 1585, and Napier's logarithms of 1614), then on to discussion of "future trends." The whole concludes with an excellent selected bibliography. Its a good article for the uninformed reader, desiring to get a general idea of achievements up to the present, especially if he reads also about half of the score of sources (1914–40) which have been brought to his attention.

The earliest calculating machines of Pascal, Morland, and Leibniz, discussed in MAC **1**, are merely mentioned, as well as several others dating from the eighteenth century. Taken together such machines specializing in the operations of addition, subtraction, multiplication, and division. But each of these machines was a failure, because "industrial revolution" had not caused the development of techniques for producing gears of adequate exactness and durability. Such a revolution took place in Great Britain in the first half of the nineteenth century when problems of power and its transmission, arising from the steam engine, were effectively handled. In this period Charles Babbage[1] (1792–1871), mathematician and scientific mechanician, was the only one appreciating economic trends and attempting to employ mathematics to assist in this work. His remarkable Difference Engine (invented in 1812), and, the incomplete "Analytical Engine" (1833+), of which the Hollerith punch-card machine is the modern counterpart, were products of his genius.

We are told that with the possible exception of the arithmometer (1820) designed and introduced by Charles X. Thomas of Colmar, no successful machine was produced until the 1880's, when the continually growing demand, coupled with the accurate machine tools which were then available caused an extremely rapid development of many efficient machines. The Comptometer (1877) invented by D. E. Felt of Chicago, was the first successful Key-operated machine; and many other successful machines were made about the same time. The German Brunsviga Calculating Machine,[2] based on the invention of a Russian engineer, first appeared in 1892, and the completion of the 20000th machine was celebrated in 1912. During the past fifty years no fundamental change has occurred in the ordinary calculating machine, although a multitude of detailed improvements have increased its speed and efficiency. Especially in the late nineteenth and early twentieth centuries did many new specialized machines appear for rapidly solving particular types of problems. Reference may be made to two of these. The National Accounting Machine[3] is very similar to the one planned by Babbage, and has been extensively used in computing the British *Nautical Almanac*. W. J. Eckert tells us[4] that the first extensive use of the early Hollerith in astronomy was made by L. J. Comrie. He used it for building a table from successive differences, and for adding large numbers of harmonic terms.[5] Modern Hollerith machines seem to be capable of solving an almost endless variety of mathematical problems. For many of these problems tables have been put on punched cards; see Eckert (*l.c.*) p. 39–42, and under "C. *Mils*" in the introductory article of this issue. The Hollerith system was evolved in the U. S. Census Bureau and was used in 1911 in connection with the British census.

In recent years the mechanical solution of differential equations has become a matter of great importance. As long ago as 1876 Kelvin indicated the type of machine[6] for theoretically solving linear differential equations, but mechanical difficulties prevented him from making a model. These difficulties were not overcome until 1931, when Vannevar Bush, now president of the Carnegie Institution of Washington, built the first Differential Analyzer, at the Massachusetts Institute of Technology.[7]

Problems of wave motion arising in innumerable problems of physics led to the construction of many machines such as Kelvin's Tide Predicting Machine[6] (1876) and various Harmonic Analyzers for finding the coefficients in Fourier expansions of periodic functions. Among harmonic analyzers are those of P. Boucherot (1893) in France; O. Henrici (1894), A. Sharp (1894), and G. U. Yule (1894) in England; A. A. Michelson and S. W. Stratton (1898) at the University of Chicago; O. Mader (1909) in Germany. Wave problems derived from radio and quantum mechanics have inspired the construction in this country of harmonic analyzer and synthesizers such as have been described by F. W. Kranz (1927), H. C. Montgomery (1938), and S. L. Brown (1939).

And finally Mr. Lilley refers to important problems in which two out of many other types of machines[8] are used; one type is for solving sets of simultaneous linear equations, and the other type for deriving the solutions of integral equations.

For engineering readers it may be noted that there is a very popular article by E. W. Crew, "Calculating Machines," in *Engineering*, v. 162, 19 Dec. 1941, p. 438–441. It is stated that this article was reprinted from Institution of Electrical Engineers, *Students' Quart. Jn.*, which is not to be found in the principal libraries of this country.

R. C. A.

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[1] The article misleads in baldly referring to Babbage as "Lucasian professor of mathematics at Cambridge," when he held this title for only the eleven years 1828–39, but delivered no lectures.

[2] For the first published account of the application of a commercial calculating machine to mechanical integration see L. J. Comrie, "On the application of the Brunsviga-Dupla calculating machine to double summation with finite differences," R. Astr. So., *Mo. Notices*, v. 88, 1928, p. 447–459. The Brunsviga-Dupla has since been superseded by the Brunsviga 20, the Burroughs, and the National. See also L. J. Comrie, (a) "The application of the Brunsviga Twin 13Z calculating machine to the Hartmann formula for the reduction of prismatic spectograms," *The Observatory*, v. 60, 1937, p. 70–73; (b) *On the Application of the Brunsviga Twin 13Z Calculating Machine to Artillery Survey*, London, Scientific Computing Service, 1938. This latter describes new and original methods of solving problems arising in surveys where rectangular coordinates are used.

[3] L. J. Comrie, "Inverse interpolation" and "Scientific applications of the National Accounting Machine," R. Statist. So., *Jn.*, v. 99, 1936, suppl., v. 3, p. 87–94, 94–114.

[4] W. J. Eckert, *Punched Card Methods in Scientific Computation*, New York, Columbia University, 1940. ix, 136 p.

[5] L. J. Comrie, (a) "On the construction of tables by interpolation," R. Astr. So., *Mo. Notices*, v. 88, 1928, p. 506–523; (b) "The application of the Hollerith tabulating machine to Brown's Tables of the Moon," R. Astr. So., *Mo. Notices*, v. 92, 1932, p. 694–707; (c) "Application of Hollerith equipment to an agricultural investigation," R. Statistical So., *Jn.*, v. 100, 1937, *Suppl.*, v. 4, p. 210. This last item has a full description of the equipment and the way in which it may be used for forming sums of products. See also Comrie's *The Hollerith and Powers Tabulating Machines*, London, printed for private circulation, 1933, 48 p.; based on lectures, many illustrations.

[6] An account of Kelvin's machines is given in W. Thomson and P. G. Tait, *Treatise on Natural Philosophy*, new ed., v. 1, pt. 1, Cambridge, 1879, app. B', p. 479–508.

[7] V. Bush, "The differential analyzer. A new machine for solving differential equations," Franklin Inst., *Jn.*, v. 212, 1931, p. 447–488, well documented. The original machine has been greatly extended and improved. The best simple account of the differential analyzer was by D. R. Hartree in "The mechanical integration of differential equations," *Math. Gazatte*, v. 22, 1938, p. 342–363, with five admirable plates, reproductions of photographs. There are also interesting "References" especially to "Applications."

[8] V. Bush, "Recent progress in analysing machines," *Proc. Fourth Intern. Congress for Applied Math., Cambridge, England . . . 1934*, Cambridge, 1935, p. 3–23. Admirable review of progress to 1934.

3[Z].—L. J. Comrie, "Calculating machines," Appendix III in L. R. Connor, *Statistics in Theory and Practice*, London, Pitman, 1938, p. 349–371.

In these 23 pages, with 14 illustrations of machines, is an admirable and condensed account of the main types of current machines, with some indication of the arithmetical field in which each might most advantageously be used. The machines are discussed under three headings: (a) Adding and listing machines; (b) Calculating machines; (c) Punched card sorting and tabulating machines.

Under the first heading the following machines are mentioned: Full Keyboard (Burroughs, Continental, National, and Victor); Ten-key (Sundstrand, Remington, Burroughs Typewriter Accounting); Key-driven (Burroughs Calculator, Felt and Tarrant Comptometer, and the Plus); Typewriter Accounting (Elliott Fisher, Mercedes-Euklid, Remington, Smith Premier, and Underwood); Multi-register (Burroughs Typewriter Bookeeping, Sunstrand, National Cash Register Analysis); National Accounting Machine.

The term calculating machine is properly applied to a machine which caters primarily for multiplication and division. Examples listed of such machines are the hand-operated Brunsviga; the electrically operated Mercedes-Euklid (German), Madas (Swiss), Archimedes, Marchant (American), Facit (Swedish) and Monroe (American); and the direct-multiplication machines, Millionaire, certain models of the Burroughs Bookkeeping machine, and the Hollerith multiplying punch.

Under the third heading are listed, Hollerith Sorter, Hollerith Electric Rolling Total Tabulator, and Hollerith Multiplying Punch.

The whole concludes with an annotated reference list, some use of which was made in MAC **2**.

Neither Mr. Lilley nor Mr. Comrie refer to the electric Fridén Automatic Calculator made in California; it has been on the market since 1934. A small condenser in the machine prevents the lights of the room being affected by its use; it is fast; and it is completely automatic.

4[Z].—(i) C. E. Shannon, "Mathematical theory of the differential analyzer," *Jn. Math. Phys.* (M. I. T), v. 20, 1941, p. 337–354.
(ii) H. S. W. Massey, J. Wylie, R. A. Buckingham, R. Sullivan, "A small scale differential analyser—its construction and operation," Irish Acad., *Proc.*, v. 45A, no. 1, Oct. 1938, 21 p.+5 plates.

The possibilities of applying modern technical developments in machines and electronics to both elementary and advanced calculations are very great. To date these possibilities remain relatively unexplored. An exception is the development by V. Bush in 1931 of the differential analyzer,—a mathematical tool which may well be regarded as the most extraordinary of our time. To a mathematician the most remarkable feature of the machine is its ability to solve non-linear differential equations (or systems) as readily as linear ones. Another interesting feature is that the machine can introduce functional relations automatically, provided these functions are themselves solutions of algebraic differential equations with constant coefficients. The main limitations of the differential analyzer are: (a) Only the one-point boundary problem can be solved directly. The two-point boundary problem must be solved by trial, by successively adjusting the initial conditions at one point until the solution given by the machine satisfies the given conditions at the second point. This objection would be overcome if the analyzer could be improved by increasing its rate of speed (or by simplifying the machine to the point where one could have several such analyzers to operate simultaneously). (b) The machine fails near a singular point where any of the variables or derivatives is unbounded.

In paper (i) the author points out a third limitation of a more technical nature. The analyzer consists of a number of units called integrators, adders, multipliers, etc. which are interconnected in a manner determined by the differential equation being solved. These units are unidirectional in operation, that is, one part of the unit will accept information from other units of the machine (input side) and another part will dispense information (output side), but the input and output sides cannot be interchanged. For instance, the integrators cannot be run as differentiators by interchanging the roles of the input and output sides; the machine can integrate an empirical curve but cannot differentiate one.

This limitation is not a serious one, for the author shows that any differential equation of the form $f(x, y, y', \cdots, y^{(n)})=0$ can be solved provided only that the function f is not hypertranscendental in character, that is, provided f considered as a function of any of its arguments satisfies a differential equation of the form $P(x, y, y', \cdots, y^{(m)})=0$ where P is a polynomial. The solution can be accomplished with a finite, but sufficiently large, number of units, and without any information supplied to the machine by the operator beyond the connections between the units demanded by the particular problem. Furthermore, even though f is hypertranscendental, it will still be possible to approximate the solution in the sense that a function y can be obtained which makes the left member of the given equation uniformly less than a given positive number ϵ. Or, the operator can supply the hypertranscendental function to the machine.

That differential analyzers are not in general use is due to their prohibitive cost, being of the order of many thousands of dollars for the larger machines. In paper (ii) there is described in detail the construction of a small scale analyzer which, while relatively inexpensive, is claimed to have an average accuracy of one half percent. The authors have reduced the cost of the machine by limiting the number of integrators to four, by eliminating certain refinements such as backlash compensators, automatic speed controls, etc., and by using standard parts for the construction. The resulting cost of materials is given as about £50, but the time required to assemble the parts must have been considerable. One wonders if the construction of machines following the authors design might not be brought within the range of the abilities and interests of groups of amateur hobbyists if fostered by mathematicians after the fashion set by astronomers in encouraging the construction and use of reflecting telescopes. That the completed machines are in the nature of glorified toys must have been in Hartree's mind when he in 1935 succeeded in making a demonstration model out of toy meccano parts.[1]

P. W. KETCHUM

NOTES

4. GIFFORD AND C. G. S. TABLES.—In RMT **77** the improvements in the Table of *Natural Sines and Cosines* published by the Coast and Geodetic Survey, as compared with Emma Gifford's volume, on which it was mainly based, were not made sufficiently clear. The Gifford volume is defective in that
(a) The arrangement of sines throughout the quadrant so that sines and cosines of a given angle have to be sought in different places.
(b) Consecutive values are in rows rather than in columns.
(c) The number of errors of more than a unit in the last decimal place is very large. The C. G. S. table is a notable improvement by virtue of
(a′–b′) Its semi-quadrantal arrangement, in columns, of the sines on one page and the cosines on the opposite page. (That an arrangement with sines and cosines on the same page would have been still better can hardly be gainsaid.)
(c′) The number of errors of more than a unit in the last decimal place being almost negligible. One may add (d) cost of Gifford 40 shillings, (d′) cost of C. G. S. 1.75 dollars.

R. C. A.

5. COAST AND GEODETIC SURVEY VERSUS PETERS.—In MTE **1**, p. 25, lines 6–12 from the bottom, six entries in C. G. S., *Natural Sines and Cosines* (RMT **77**) were called into question by quoting the corresponding results in Peters' *Eight-figure Table of the Trigonometrical Functions* (RMT **78**). On

[1] D. R. Hartree and A. Porter, "The construction and operation of a model differential analyser," Manchester Lit. and Phil. So., *Mem. and Proc.*, v. 79, 1935, p. 51–71, +2 plates.—EDITOR.