TABLE IV—Brocot's Tables of Gear Ratios

page	N	D	for	read	authority
148	3	106	·02830 187	··· 189	J
148	3	86	Omitted	·03488 372	M; J
148	3	70	Omitted	·04285 714	M; J
149	7	79	·08860 760	··· 759	J
150	9	67	·13432 856	··· 836	J
151	13	72	·18055 555	··· 556	J
152	22	101	·21789 178	·21782 178	J
153	28	103	·27184 465	··· 466	J
154	32	113	·28318 585	··· 584	J
155	38	107	·35513 919	·35514 019	J
156	33	85	·38823 530	··· 529	J
156	46	117	·39316 293	··· 239	J
156	41	[100]	D = 00	D = 100	M
157	41	97	·42268 042	··· 041	J
157	46	103	Omitted	·44660 194	M; J
157	31	72	·43055 555	··· 556	J
157	39	79	·49367 087	··· 089	J
157	50	101	·49504 951	··· 950	J
159	41	79	Omitted	·51898 734	M; J
161	67	105	·63803 524	·63809 524	J
162	59	89	·66291 135	·66292 135	J
165	62	79	·78481 083	··· 013	J
166	61	73	·83561 484	··· 644	J
166	98	117	·83760 601	··· 684	J
167	95	[119]	D = 119	D = 109	M
168	44	[87]	D = 87	D = 47	M
168	107	112	·95535 710	··· 714	J

L. J. C.

In *Math. Gazette*, v. 26, Dec. 1942, p. 226–230, J. C. P. Miller has an article entitled "The decimal subdivision of the degree," which is also a review of Buckingham's book. Many of the facts stated above were first published in this article; for example, besides the 7 errors credited to M in Table IV, 12 more of the others were also published in his own review. —EDITOR.

UNPUBLISHED MATHEMATICAL TABLES

In *MTAC*, p. 27, we referred to an unpublished ms. of the late A. J. C. CUNNINGHAM giving the complete factorization of $n^2 + 1$ for $1 \leqq n \leqq 15{,}000$. Through L. J. Comrie we were informed by a letter, dated 5 May 1943, from A. E. Western, custodian of the Cunningham mss. of the London Mathematical Society, that this ms., as well as others, and many of the Society's books, housed in the library of University College, London, were destroyed by an enemy air raid.

4[L].—PROJECT FOR COMPUTATION OF MATHEMATICAL TABLES, *Spherical Bessel Functions.* Ms. in possession of the Project.

The Spherical Bessel Functions

$$Q_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)$$

occur in a wide variety of problems of wave motion, potential theory, heat conduction and quantum mechanics. The Project's preliminary manuscript is of the functions $Q_n(x)$ for $n = 0, \pm1, \pm2, \cdots, \pm21$ and $x = [0(0.01)10; 8S–10S]$, with second and fourth central differences. It is contemplated to extend this table for values of n ranging from -20 to -35, $n = 20$ to $n = 35$ and for $x = [10(0.1)30$; about 7S].

Tables related to $Q_n(x)$ are contained in the Br. Ass. Adv. Sci., *Reports*, for 1907, 1909, 1914, 1916, 1922 and 1925. In these reports the tabulated functions are given for $x = 0.1(0.1)2, 1(1)20, 10(10)100, 100(100)1000$ for various values of n ranging from -22 to 44. Most of the entries are given either to 7S or to 12D. Compare *MTAC*, p. 70–73.

Hayashi's *Tafeln der Besellschen, Theta-, Kugel- und anderer Funktionen* (Berlin, Springer, 1930) contains tables of $J_{\pm 1/2}$ and $J_{\pm 3/2}$ with most entries given to 12D. An abridged table of these functions is reproduced in his *Fünfstellige Funktionentafeln* (Berlin, Springer, 1930) as well as in Jahnke and Emde's *Funktionentafeln mit Formeln und Kurven*, third rev. ed., Leipzig, Teubner, 1938.

Lommel's big article, Bay. Akad. Wiss., *Math. Natur. Abt., Abh.*, v. 15, 1886, p. 531–663, contains tables of $J_{n+1/2}$ to six decimals for integral arguments up to 50, with n depending upon the argument. Part of this table is reproduced with considerable additions in G. N. Watson's *Treatise on the Theory of Bessel Functions*, Cambridge, 1922.

With respect to the interval of the argument (.01), the number of significant figures in the entries and the range of the order n, the table of $Q_n(x)$ reported here supersedes existing tables.

A. N. LOWAN

5[L].—PROJECT FOR COMPUTATION OF MATHEMATICAL TABLES, *Bessel Functions of Fractional Order*. Ms. in possession of the Project.

Bessel Functions $J_\nu(x)$ and $I_\nu(x)$ for the fractional orders $\nu = \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{4}$ and $\pm\frac{3}{4}$ are encountered in many problems of electrical engineering pertaining to problems of propagation and diffraction of electromagnetic waves. The Project's preliminary manuscript is of these functions for $x = [0(0.01)25; 10D]$. The manuscript contains also the values of $P(x)$ and $Q(x)$ in the asymptotic expansion of these functions in the form

$$P(x)\cos(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + Q(x)\sin(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)$$

for x ranging from 25 to about 30,000, at various intervals depending on the ease of interpolation, to 10D. The manuscript is also available for the functions $I_\nu(x)$ where $\nu = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}$ and $\frac{3}{4}$ and $x = [0(0.01)10; 10D]$, $x = [10(0.01)25; 10S]$ as well as for the functions $I_{-\nu}(x)$ where $\nu = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}$ and $\frac{3}{4}$ and $x = [0(0.01)10; 10D]$, $x = [0(0.01)13; 10S]$ and values of $e^{-x}I_{\pm\nu}$ to 10D, obtained from the asymptotic expansion. Modified central differences of various orders are included in the manuscript. In addition to the above, thirteen-place values of the functions $J_{\pm\nu}(x)$ and $I_\nu(x)$ are available for the integral arguments 10, 11 $\cdots$ 25. For these arguments the first fifteen derivatives have also been computed to a number of places adequate for the subsequent subtabulation of the functions to about 12D. For the arguments $x = 10, 11, 12$, and 13, the values of $I_{-\nu}(x)$ were computed to 13S. For these arguments the first fifteen derivatives have also been computed to a number of places adequate for the subsequent subtabulation of the functions to about 12S.

Bessel functions of various fractional orders are given by Dinnik, Bursian and Karas. The reader is referred to RMT **11**, **59**, and **72**. Additional tabular material on these functions is contained in the article by G. N. Watson in the R. So. London, *Proc.*, v. 94A, 1918, p. 204. Some of the existing tables of J_ν go up to $x = 8$; others up to about 15, at intervals of 0.1, the entries being given mostly to 4S.

With respect to the range and interval of the argument and the number of significant figures, the tables of $J_\nu(x)$ recorded here, supersede practically all existing tables. Except for Bursian and Dinnik's short tables of I_ν, the tables of $I_\nu(x)$ reported here are almost all new. The values of $P(x)$ and $Q(x)$ in the asymptotic expressions above mentioned, are entirely new.

A. N. LOWAN

6[I, K].—*Tables of Lagrangian interpolation coefficients*, Preliminary Ms. prepared by, and in possession of the PROJECT FOR COMPUTATION OF MATHEMATICAL TABLES, 50 Church St., New York City.

The ranges and intervals of the various tables included in the manuscript are as follows:

Three-point interpolants,	$x = [-1(.0001)1]$
Four-point interpolants,	$x = [-1(.001)0(.0001)1(.001)2]$
Five-point interpolants,	$x = [-2(.001)2]$
Six-point interpolants,	$x = [-2(.01)0(.001)1(.01)3]$
Seven-point interpolants,	$x = [-3(.01)-1(.001)1(.01)3]$
Eight-point interpolants,	$x = [-3(.1)0(.001)1(.1)4]$
Nine-point interpolants,	$x = [-4(.1)4]$
Ten-point interpolants,	$x = [-4(.1)5]$
Eleven-point interpolants,	$x = [-5(.1)5]$.

The tabulated entries are either exact or given to 10D. Interpolants make it possible to perform interpolation using the actual entries in the table without recourse to tabulated differences. (It should be noted that n-point interpolants correspond to approximation by a polynomial of degree $n - 1$.)

The only other printed tables of interpolants available in the literature are:

(1). E. V. HUNTINGTON, "Tables of Lagrangian coefficients for interpolation without differences,"[1] Am. Acad. Arts & Sci., *Proc.*, v. 63, 1929, p. 421–437; this table lists four-point and six-point interpolants at intervals of .01, exact values or 8D, for interpolating in the various parts of a table. It contains also a short table for cumulative subtabulation to 1/5th of the interval.

(2). T. L. KELLEY, *The Kelley Statistical Tables*, New York, Macmillan, 1938; these tables list four-point interpolants at intervals of .001 and six-point interpolants at intervals of .01 to 10D, as well as a few eight-point interpolants to 11D.

(3). ORDNANCE DEPARTMENT, *Table of Lagrangean Interpolation Coefficients*, Washington, D. C., June 1941. 40 p. 20.3 × 26.7 cm.; not available for public distribution. Five-point interpolants, prepared from the above-mentioned preliminary ten-place ms. table, for $x = [0(0.001)2.000; 7D]$. In September 1942 this was reprinted, for general distribution, by the Marchant Calculating Machine Co., Oakland, Cal., 23 p. 21.5 × 28.1 cm.

For a theoretical discussion of interpolants, the following references may be mentioned:

G. RUTLEDGE, "Fundamental table for Lagrangian coefficients," *Jour. Math. & Phys.*, M.I.T., v. 8, 1929, p. 1–12; G. RUTLEDGE and P. CROUT, "Tables and methods of extending tables for interpolation without differences," *ibid.*, v. 9, 1930, p. 166–180 (basic values for the computation of Lagrangean coefficients of various orders); and K. PEARSON, *On the Construction of Tables and on Interpolation, Part I* (*Tracts for Computers*, no. II), Cambridge, Univ. Press, 1920, p. 22–56. In this last we find short tables of coefficients in 4, 5, 6, 7, 8, 9, 10, and 11-point interpolants, with considerable expository material.

A. N. LOWAN

[1] A survey to investigate the reliability of these "Tables" will be the basis of a report. —EDITOR.

7[D, L].—*Table of* $\sin^{-1} x$. Preliminary Ms. prepared by, and in possession of, the PROJECT FOR COMPUTATION OF MATHEMATICAL TABLES, 50 Church St., New York City.

This is a table of $\sin^{-1} x$ for $x = [0(.001).99(.00001)1; 12D]$, with second and fourth central differences. The extent to which this table will supersede existing tables will be apparent from the following list of previous tables of $\sin^{-1} x$ for real values of x:

(1) J. M. PEIRCE. *Three and Four Place Table of Logarithmic and Trigonometric functions.* Boston, 1871; later reprints or editions, 1874, 1876; also *Mathematical Tables*, 1879; later

reprints or editions, 1880, 1886, 1891, 1896, and 1903; varying formats. These tables contain $\sin^{-1} x$ to hundredths of a degree for arguments $\log x$ at intervals of .01 or less.

(2) H. SCHUBERT. *Fünfstellige Tafeln und Gegentafeln für logarithmisches und trigonometrisches Rechnen.* Leipzig, 1897, p. 126–141. This table lists $\sin^{-1} x$ for x = [0(.001).89(.0001)-.999(.00001).9999(.000001)1; 5D].

(3) K. HAYASHI. *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion,* Berlin, Springer, 1926. Includes $\sin^{-1} x$ for x = [0(.00001).001; 20D], [.001(.0001).0999; 10D], [.1(.001).999; 7D].

(4) K. HAYASHI. *Fünfstellige Funktionentafeln,* Berlin, Springer, 1930. Lists values of $\sin^{-1} x$ for x = [0(.01)1; 5D].

(5) L. M. MILNE-THOMSON and L. J. COMRIE. *Standard Four-Figure Mathematical Tables,* Editions *A* and *B*, London, MacMillan, 1931. Contains $\sin^{-1} x$ for x = [0(.001).99(.0001)1; 4S] in radians with first differences.

(6) R. A. DAVIS. *Table of Natural Sines and Radians.* Oakland, California, Marchant Calculating Machine Co., Nov. 1941, 8 p. 21.5 × 28 cm. This table lists $\sin^{-1} x$ for x = [0(.001)1; 6D], with differences.

Short tables of $\sin^{-1} x$ for complex values of x may be found in E. JAHNKE and F. EMDE, *Funktionentafeln mit Formeln und Kurven,* second ed., Leipzig, 1933, p. 68; and much longer ones in R. HAWELKA and F. EMDE, *Vierstellige Tafeln der Kreis- und Hyperbelfunktionen sowie ihrer Emkehrfunktionen im Komplexen,* Brunswick, 1931, p. 21–41.

The fundamental importance of tables of $\sin^{-1} x$ requires no elaboration.

A. N. LOWAN

8[A].—HERBERT ELLIS SALZER (1915–), *Tetrahedral Numbers.* Ms. of the first thousand tetrahedral numbers in possession of the author, and of the Brown University Library. 6 p.

The tetrahedral numbers $n(n + 1)(n + 2)/6$ or ${}_{(n+2)}C_3$ were calculated for n = 1(1)1000. For a discussion of their history and properties, see L. E. Dickson, *History of the Theory of Numbers,* v. 2, Washington, D. C., 1919, chap. 1. In addition to the theorems about tetrahedral numbers which are mentioned by Dickson, the author adds the following original empirical theorem: Every square of an integer is the sum of 4 (or less) positive tetrahedral numbers. (It has been verified for the first 200 squares.)

The tetrahedral numbers have these well-known properties: (1) The sum of the n^{th} and $(n + 1)^{\text{th}}$ is equal to the sum of the squares of the first $(n + 1)$ integers. (2) The difference between the n^{th} and $(n - 1)^{\text{th}}$ entry is equal to the n^{th} triangular number, $n(n + 1)/2$. (The first 20,000 triangular numbers were printed in E. de Joncourt's *De Natura et praeclaro usu simplicissimae speciei Numerorum Trigonalium,* 1762). (3) The square of the difference between the n^{th} and $(n - 1)^{\text{th}}$ entry is equal to the sum of the first n cubes.

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9[A].—EDWARD BRIND ESCOTT (1868–), *Amicable Numbers.* Ms. in possession of Mr. Escott and in the Library of Brown University.

This Ms. contains a list of the 391 Amicable Numbers discovered during the past 2500 years. Two numbers are called amicable if each equals the sum of the aliquot divisors of the other. Iamblichus attributes to Pythagoras (*c.*540 B.C.) the discovery of the first pair of amicable numbers 220 $(=2^2 \cdot 5 \cdot 11)$ and 284 $(=2^2 \cdot 71)$. The next two pairs were discovered by Fermat and Descartes. Euler added 59 pairs in the next century but even to the end of the nineteenth century only 66 pairs had been discovered. Details in this regard may be found in L. E. Dickson, *History of the Theory of Numbers,* v. 1, Washington, 1919, p. 38–50.

The complete record, of discoveries, with approximate dates, is as follows:

Pythagoras	1 (540 B.C.)	L. E. Dickson	2 (1911)
Fermat	1 (1636)	T. E. Mason[1]	14 (1921)
Descartes	1 (1638)	P. Poulet[2]	68 (1929)
Euler	59 (1747–50)	A. Gérardin[3]	5 (1929?)
Legendre	1 (1830)	E. B. Escott[4]	235 (1934)
B. N. I. Paganini	1 (1867)	B. H. Brown[5]	1 (1939)
P. Seelhoff	2 (1884)	Total (May 1943)	391

E. B. Escott,
1019 South East Ave., Oak Park, Ill.

[1] T. E. Mason, "On amicable numbers and their generalizations," *Amer. Math. Mo.*, v. 28, 1921, p. 195–200.

[2] P. Poulet, *La Chasse aux Nombres. Fascicule I. Parfait, Amiables et Extensions*, Brussels, 1929, p. 28–51. The 156 pairs of amicable numbers, known at this time, are here classified, and include the 68 new pairs found by Poulet.

[3] All of these pairs are in Poulet's list of 1929. It is possible that their discovery was announced earlier in Gérardin's periodical *Sphinx Oedipe*.

[4] P. Poulet, "De nouveaux amiables," *Sphinx*, v. 4, 1934, p. 134–135, Poulet here states that Mr. Escott had sent him 322 pairs of amicable numbers; he prints 21 pairs discovered by Mr. Escott, and all but one of the 42 numbers are less than 10^9. The other 214 pairs have not yet been published.

[5] B. H. Brown, "A new pair of amicable numbers," *Amer. Math. Mo.* v. 46, 1939, p. 345.

MECHANICAL AIDS TO COMPUTATION

5[X].—R. E. Beard, "The construction of a small-scale differential analyser and its application to the calculation of actuarial functions," Institute of Actuaries, *Jn.*, v. 71, 1942, p. 193–227 + 3 plates.

The author adopts the principles of the differential analyzer as conceived by Kelvin, Bush, and Hartree to construct another small-scale machine for experimental work in finding solutions of differential equations that arise in actuarial science. The article contains a description of its mechanical principles, operation, and application to actuarial functions. At the end is a detailed 16-page abstract of the discussion following the presentation of the author's paper.

The most important unit in the machine is the integrator where $(1/a)\int y\,dx$ is obtained from two mutually perpendicular wheels which touch each other. When the first wheel turns through angle dx, the rim of the second wheel (of radius a) at variable distance y from the center of the first wheel, will turn through distance ydx, corresponding to the angle ydx/a. Use is made of a torque amplifier to increase the friction; this is necessary for proper operation. Adequate description of the adding units, input and output tables and counting devices precedes the discussion of the operation of the machine.

The first stage in the operation consists of the drawing of a diagram showing the required units properly connected. Vannevar Bush's notation is followed. The reader can learn at a glance the notation for the integrator, input and output table, adding unit and right- or left-hand gears, after which he can understand all the diagrams. Initially the author gives a clear description of the set-up for simple integration and its application to the "circle test" of the machine, i.e., the integration of $y'' = -(1/k^2)y$. Finally a description of an elegant device to obtain the integral of a product without getting the product itself, completes the general introduction to the use of the analyzer.

The calculation of interest functions v^n and $(1 + i)^n$ is reduced to $e^{-n\delta}$ and $e^{n\delta}$ or an equation of the form $dy = \frac{y}{C}dt$. One immediately obtains $\int f(t)v^t dt$ by integration of a product. For $f(t) = 1$, we get $\bar{a}_{\overline{t}|}$. For mortality calculations the machine solves $\frac{d}{dt}\,{}_tp_x = -\,{}_tp_x\mu_{x+t}$, where ${}_tp_x$ is the probability of survival. A detailed discussion of joint-life functions and